A Coalgebraic Approach to Infinite Games

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Abstract. We study infinite outcomes of two-player games, and strategies therein, finding both to be captured by a categorical limit. Outcomes of strategies generalise traces in labelled transition systems, allowing us to link our work to, and even unify, various coalgebraic approaches to infinite trace semantics. We obtain largest homomorphism characterisations of outcomes in both strategies and games. For practical applications, we show how infinite outcomes can be approximated by a least fixed point, which may be viewed as maximally permissive controller synthesis.

Keywords: Two-player game · Coalgebra · Trace semantics

1 Introduction

Two-player games are ubiquitous throughout theoretical computer science – they are deeply connected to logic, algorithms, automata, complexity and verification. Our interest in two-player games stems from their use in automated verification, where *controller synthesis* can be reduced to solving an infinite game between a controller and its environment. Typical objectives for the controller include parity objectives, in which case synthesising winning controller strategies involves some form of (nested) fixed point computation. Our ultimate goal is to give a general account of infinite two-player games, that both covers several types of games of interest in verification (qualitative *and* quantitative), and lends itself well to the study of such fixed point computations.

To achieve this level of generality, one can model two-player games as coalgebras. Previous work [19] established a link between strategies in two-player games and coalgebraic trace semantics: it was shown in loc. cit. that an existing coalgebraic formalism for finite trace semantics [12] can be applied to two-player games, and the semantics recovers the outcomes of *finitely completing* controller strategies. The resulting least fixpoint characterisation of the (completed) outcome of a game can be exploited to compute completing strategies inductively.

The caveat with the work [19] is that it only deals with *finite* behaviour, whereas for the purpose of verification and synthesis system behaviour is viewed as infinite: many verification tasks deal with properties that must hold *forever*, e.g. safety properties, and these can be more naturally expressed when system behaviour is viewed as infinite. The present work extends loc. cit. to also include *infinite* plays and strategies.

Tackling this problem using coalgebraic machinery means that our results have the potential to be generalised to other types of games (like the work [19], which is already applicable to more than one type of game). It also allows us to make explicit the similarities with single-player models, such as labelled transition systems. Conversely, it is valuable to explore whether coalgebraic techniques can be used to capture infinite traces in this richer setting. One of the contributions of our work is to link existing coalgebraic approaches to infinite trace semantics [13,5,15,21], and to show that any of them can be applied to games.

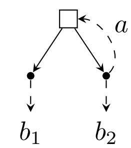
To the best of our knowledge, infinite outcomes of games have received little attention even outside the coalgebraic literature, with the closest results being work on (alternating-time) temporal logic [8], wherein *path effectivity models* and the notion of *limit-closure* are used to describe game outcomes.

Related Work The work of Hasuo, Jacobs and Sokolova [12] gives a method to obtain finite trace semantics for systems with branching behaviour, by modelling such systems as coalgebras. The typical model is a labelled transition system, in which the trace semantics of a state is given by the set of *traces* (finite sequences of observations) that can be observed from that state. To capture such linear-time semantics with coalgebra, one has to move to a category whose morphisms only preserve the linear-time semantics. In the classical case of labelled transition systems, this means moving from the category of sets and functions (wherein transitions systems can be naturally modelled as coalgebras) to the category **Rel** of sets and relations. This works smoothly for *finite* traces since, under suitable assumptions, the domain of finite traces forms both an initial algebra and a terminal coalgebra. On the other hand, the domain of *infinite* traces is much less well-behaved (as witnessed by several existing approaches to infinite trace semantics [13,5,21], none of which matches the simplicity of [12]).

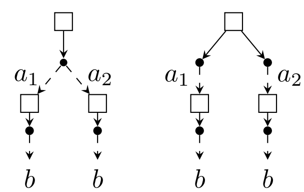
Previous work [19] showed that the above method can be applied to the setting of two-player games. Games have a more complex branching structure than transition systems, given the alternation between controller and environment moves. As a consequence, the resulting trace map has a different type: it assigns to each state a *set of subsets* of traces. The main result of loc. cit. states that the trace semantics assigned to each state is the set of precisely those subsets of finite traces which a controller strategy can force. In this setting, the trace map comes "for free": it is the unique coalgebra homomorphism.

Our Setting We focus purely on two non-deterministic players whose turns alternate (unlike [19], where probabilistic opponents are also considered). This is to simplify matters, as even in the classical case of labelled transition systems there are several, and previously unrelated, coalgebraic frameworks which can capture infinite traces [13,5,21,18].

Our main focus is on exploring the interplay between different definitions of *strategy outcome* and *game outcome*. The outcome of a controller strategy from a controller state is the set of plays that can arise when the controller follows said strategy from that state. A *play* records the controller states visited and the observable outcomes of basic interactions; the latter come in two forms: continuing or terminating. The outcome of a game at a controller state is the set of outcomes of controller strategies starting in that state. We give an example to demonstrate.



We denote controller states with squares and environment states with thick black dots. In the small example, we have a single controller state, and two environment states. There is a set $A = \{a\}$ of continuing observations and a set $B = \{b_1, b_2\}$ of terminating observations. Our coalgebraic model of games abstracts the environment, leaving just a set X of controller states (in this example, a singleton $\{x\}$). The game corresponds to a set $\{\{b_1\}, \{b_2, (a, x)\}\}$. This represents that the controller can either choose to terminate with a b_1, or let the environment choose to either terminate with a b_2, or to output a and remain in state x. There are an infinite number of controller strategies, each with their own outcome; they correspond to when the controller chooses to go left. For example, the strategy which chooses to go right first, and then left if the environment chooses to go back to x, has an outcome $\{xb_2, xaxb_1\}$. The outcome of the game is the set of all outcomes of strategies: $\{\{xb_1\}, \{xb_2, xaxb_1\}, \{xb_2, xaxb_2, xaxaxb_1\}, \ldots, \{xaxa\ldots\}\}$.



We make a few assumptions for technical reasons, that also make practical sense. Firstly, we assume that the environment cannot deadlock, so any strategy has a non-empty outcome (note that there *can* be no strategies, i.e. the outcome of a game at a state can be empty). Secondly, our games are *convex*: the controller has strategies where choices are left undetermined. In technical terms, this means that all sets of subsets are closed under arbitrary union. In the first example, this means that we have an element $\{b_1, b_2, (a, x)\}$ which corresponds to the controller leaving the decision to go left or right undetermined. Our reason for assuming convexity is threefold. Firstly, it is key to obtaining a monad structure, allowing us to associatively iterate games. Secondly, it is a natural assumption for the induced trace relation, for example it gives a subset inclusion between the semantics of the top two states in the games above. Intuitively, the right game is better for the controller than the left, as the environment choice in the left game becomes a controller choice in the right game. Without convexity, the traces at the top states in these games are incomparable. Finally, convexity ensures that a *maximally permissive strategy* exists from every state, which allows strategy refinement to be performed in an inductive fashion (see work such as [20,2,3]). Other categorical formulations of infinite games (e.g. [10]) make the stronger assumptions that sets of subsets are *upwards closed*.

Road Map Section 2.1 introduces game-theoretic notions, and how they are modelled using coalgebras. Section 2.2 discusses *linear functors*, which model the kind of observations made in the games we consider. Section 2.3 introduces the monads we use. In Section 2.4, we recall some facts about **Rel**, and adjunctions therein. Section 2.5 introduces some terminology related to cones. We finish the preliminaries in Section 2.6, by discussing related work on infinite traces.

We begin in Section 3.1 by showing how to define the outcome of a strategy; we find that the most natural definition is as a limit in **Set**. This idea is novel, but a similar definition was suggested in [18, Remark 4] for labelled transition systems. Furthermore, we can see the outcome of a strategy as a colimit in a corresponding subcategory of relations, by duality. It is natural to consider whether outcomes form colimits in the full category **Rel** (we will see that a strategy is naturally a chain in **Rel**). In Section 3.2, inspired by [16,17], we find that a strategy outcome is not a colimit in **Rel**, but that a weaker universal property, stating that there are largest mediating maps out the cocone, does exist. The following subsections, 3.3 and 3.4, discuss using other coalgebraic approaches, [13,21] and [5] respectively, to capture outcomes of strategies. Section 3.5 presents our main result in Section 3, Theorem 1, giving a correspondence between homomorphisms into the terminal coalgebra (lifted to **Rel**) and cones in **Set**. To establish this correspondence, we impose that cones are jointly monic and componentwise epi, to make them resemble sets of infinite paths. We profit from this result in Section 3.6, by showing that all our definitions of strategy outcome are equivalent. Furthermore we offer a final characterisation, which employs the notion of *limit closure*, linking to work in temporal logic [6]. Roughly, a subset of paths is limit closed when it is determined by its finite length approximations.

Outcomes of games are studied in Section 4. Section 4.1 provides a definition of game outcomes as limits in **Set**. This involves obtaining a cochain in **Set**, built from unfolding the coalgebra structure in the Kleisli category of the game monad. Proposition 15 explains how cones over this cochain in **Set** are collections of strategies in the game. Section 4.2 shows that we can take the largest homomorphism in a Kleisli homset, similarly to [13,21], and can build cones from them. In Section 4.3 we discuss how the homomorphism approach can *not* be used to give the outcome of a game in Example 3. We rectify this in Proposition 20 by restricting the homomorphisms to ones whose image contains limit closed subsets only, and then axiomatise cones to establish order-preserving isomorphism between such cones and limit-closed homomorphisms (Theorem 2). Section 4.4 shows how to approximate infinite game outcomes with a least fixed point. This resembles permissive controller strategy refinement, where it is assumed that a maximally permissive controller strategy exists, which can then be refined in a backwards fashion (see approaches such as [20,2,3]).

Section 5 provides a comparison between the monad we use to model games, and the monotone neighbourhood monad, used in other coalgebraic approaches to games. We conclude the paper with a discussion of future work in Section 6.

Contributions

1. Give definitions of outcomes of both strategies and games as limits.
2. Show that several existing approaches to infinite traces [13,5,18,21] coincide in the case of labelled transition systems.

3. Show that the largest mediating map approach to infinite traces in [5] only provides the expected result when intermediate states are recorded.
4. Show that the largest homomorphism approach of [13,21] is not suitable for defining outcomes of games, giving evidence for the superiority of the limit-based definition.
5. Obtain a greatest fixed-point characterisation of infinite outcomes in games by suitably restricting the homomorphisms considered. The new homomorphisms can be viewed as computing sets of strategies.
6. Theoretically ground permissive controller strategy synthesis as a least fixed point computation, and show how it approximates infinite strategies.

2 Preliminaries

2.1 Games, Plays, and Strategies

Given a set X, $P(X)$ denotes the powerset of X, and $Q(X)$ denotes the non-empty powerset of X. We let $\mathcal{G}(X)$ denote the set of sets of non-empty subsets which are closed under union, $\mathcal{G}(X) := \{\mathcal{U} \subseteq Q(X) \mid \forall \mathcal{V} \subseteq \mathcal{U} : \bigcup \mathcal{V} \in \mathcal{U}\}$.

A *controller-versus-environment game*, or simply a *game*, is a function $\delta : X \to \mathcal{G}(A \times X + B)$, where $+$ denotes the disjoint union of sets. The game is from the controllers perspective; each set $U \in c(x)$ is a controller move. The set $U \subseteq A \times X + B$ collects all valid environment moves following the controller move. These are either pairs of a *continuing observation* from A and a controller state, or *terminating observations* in B.

For technical reasons, we work with a coalgebra $\gamma : X \to \mathcal{G}(X \times (A \times X + B))$ built from δ, which maps $x \overset{\gamma}{\mapsto} \{\{(x, u) \mid u \in U\} \mid U \in \delta(x)\}$. This records state information, which is a useful assumption that is exploited throughout our work. The use of $\mathcal{G}(X)$, rather than $PQ(X)$, make our games *convex*: for any non-empty collection of controller moves $U_i \in \gamma(x)$, there is a move which consists of the union $\bigcup_{i \in I} U_i$. As discussed in the introduction, this is a common assumption which allows for maximally permissive controllers to be synthesised. Furthermore, it comes "for free" out of the categorical theory of weak distributive laws [7,9], which give that $\mathcal{G}$ is a *composite monad*, giving us nice mathematical properties which, for instance, allow us to compose games associatively.

An *n-step partial play* is an element of $(XA)^n X$ such that for each prefix $x_0 a_0 \cdots x_i a_{i+1} x_{i+1}$ we have that $a_{i+1} x_{i+1} \in U \in c(x_i)$. A *maximal play* on (X, γ) is an element of $z \in (XA)^\omega + (XA)^* XB$ such that each prefix is a partial play, and if $z = x_0 \ldots x_n b$, then $b \in U \in \gamma(x_n)$.

A *strategy* is a partial function $\sigma : (XA)^* X \to Q(A \times X + B)$ taking partial plays to successors: $\sigma(x_0 a_0 \ldots x_n) \in c(x_n)$. We say a partial play $x_0 a_0 \ldots x_n$ *conforms* to a strategy σ when $x_{i+1} \in \sigma(x_0 a_0 \ldots x_i)$ for all $i < n$. Analogously, we say a maximal play conforms to σ, when every partial prefix of the play conforms to σ, and if the maximal play has shape τb, then $b \in \sigma(\tau)$. We impose the condition that a strategy is defined over *exactly* the partial plays which conform to it (the standard approach is to define it on *at least* the partials plays which conform to it). The *n-step partial outcome of a strategy* is the set of all partial n-step plays and maximal plays of length less than n which a strategy can force. The *outcome of a strategy* is the set of maximal plays that a strategy can force. Finally, the *outcome of a game* at a state is the set of all outcomes of strategies which start in x (are defined only over the singleton play $\{x\}$).

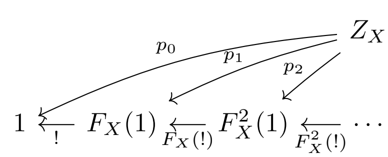
Games generalise labelled transition systems (LTSs) in two ways, corresponding to the two players (essentially coming from the two monad morphisms discussed in Section 2.3). Either, we take each move $\{u\} \in \gamma(x)$ from a transition $x \to u$ in an LTS (and close under convexity), or we take a single move $U = \{u \mid x \to u\} \in \gamma(x)$ at each state $x \in X$. The first method shows how games generalise LTSs: subsets of executions in the LTS give rise to strategies in the game. Using the second method, we see that there is a single memoryless strategy on a game constructed in this way, thus it makes sense to consider strategies as a generalisation of LTSs (in our work, ones that do not deadlock).

2.2 Linear Functors

The collection of *linear* functors captures the observable behaviour of plays in the games we consider. We use $[-]$ to denote a constant functor, and 1 to denote the terminal object in **Set** (a singleton). The collection can be specified as the least collection such that $[1]$ is linear, if F_i is linear then so is $\coprod_{i \in I} F_i$, and if F is linear then so is $F \times \mathsf{id}_{\mathbf{Set}}$. The general form we shall use here is $[A] \times (-) + [B]$, which shall be referred to as F throughout the paper.

To capture plays in games, we modify F to also capture states. This is achieved by defining a new linear functor $F_X := [X] \times F$, allowing plays in a game, with state space X, to also record intermediate states.

The terminal coalgebra for F is given by iterating the terminal sequence in **Set**, and taking the limit, denoted by Z, of the chain. We use (Z_X, ζ_X) to denote the terminal coalgebra for F_X. Using the standard characterisation of limits in **Set**, we obtain:



$$Z_X \cong \{(z_0, z_1, z_2, \dots) \mid z_n \in F_X^n(1),\ z_n = F_X^n(!)(z_{n+1})\} \cong (XA)^\omega + (XA)^* XB$$

Stream Operations We will require fine-grained control over elements in Z_X. We use the projection $\pi_1 : F_X(X) \to X$. The map $F_X^n(\pi_1) : F_X^{n+1}(X) \to F_X^n(X)$ provides us a function which forgets the final observable behaviour. For succinctness, we define an operation $\star : F(Y) \times Z \to F(Z)$ which maps $(ay, z) \mapsto az$ and $(b, z) \mapsto b$. We also use a map $\mathsf{snip}_n^m : F^m(X) \to F^n(X)$ for $n \leq m$, which snips off the first $m - n$ observations: $a_1 \dots a_m x \mapsto a_{m-n+1} \dots a_m x$, when we have a terminating observation $b \in B$ in the first $m - n$ indexes, we return that b.

2.3 Powerset Monads

Recall that $T : \mathbf{Set} \to \mathbf{Set}$ is a *monad* when it comes equipped with natural transformations $\mu^T : TT \to T$ and $\eta^T : 1 \to T$ which allow functions of the shape $X \to T(Y)$ to be composed associatively (and for $\eta_X : X \to T(X)$ to be the unit of composition). The category where this composition takes place is called the *Kleisli category* of the monad, and will be denoted $\mathbf{Kl}(T)$.

Two monads that we use frequently are the powerset monad (P, μ^P, η^P) and the non-empty powerset monad (Q, μ^Q, η^Q). It is well-known that $\mathbf{Kl}(P) \cong \mathbf{Rel}$, the category of sets with relations between them. We also have that $\mathbf{Kl}(Q) \cong \mathbf{Rel}_{\mathrm{lt}}$, the category of sets and *left-total relations*. We also have *Eilenberg-Moore categories* $\mathbf{EM}(P) \cong \mathbf{CJSL}$ (*complete join semi-lattices*), and $\mathbf{EM}(Q) \cong \mathbf{AJSL}$ (*affine join semi-lattices*), where only non-empty joins are required to exist.

We use $\nrightarrow$ to denote relations, aka morphisms in $\mathbf{Rel}$ and $\mathbf{Rel}_{\mathrm{lt}}$. We will use $\odot$ to denote relation composition, i.e. composition in both $\mathbf{Rel}$ and $\mathbf{Rel}_{\mathrm{lt}}$. An element in a category is a morphism from 1. Thus, given an element $U : 1 \nrightarrow X$, we can write $f \odot U : 1 \nrightarrow Y$ for some $f : X \nrightarrow Y$. Defining $f \odot U$ for all such U is equivalent to defining the *Kleisli extension* of a monad, which is equivalent data to the multiplication and unit.

We use $\overline{(-)} : \mathbf{Set} \to \mathbf{Rel}$ for the left adjoint of the Kleisli adjunction, it lets us view a function as a relation by mapping $f \mapsto \eta^P \circ f$. We also use, for some $\mathbf{Set}$ endofunctor G, $\overline{G} : \mathbf{Rel} \to \mathbf{Rel}$ to denote a *lifting* of G to $\mathbf{Rel}$, meaning that it commutes with $\overline{(-)}$, i.e. $\overline{G}(\overline{f}) = \overline{G(f)}$.

The Game Monad Recall from Section 2.1, that $\mathcal{G}(X)$ is the set of sets of non-empty subsets of X which are closed under arbitrary union. It is easy to see that $\mathcal{G}$ is a functor $\mathbf{Set} \to \mathbf{Set}$ with $\mathcal{G}(f)(\mathcal{U}) = \{\{f(x) \mid x \in U\} \mid U \in \mathcal{U}\}$. It turns out it is a monad formed by composing P with Q with a *weak distributive law* $\delta : QP \to PQ$. We do not go into details here, but present a novel description of the Kleisli extension in the following proposition. We denote morphisms in $\mathbf{Kl}(\mathcal{G})$ with $\nrightarrow\!\!\!\!\nrightarrow$, and composition with $\circledcirc$.

Proposition 1. *Let $f : X \nrightarrow\!\!\!\!\nrightarrow Y$, we have $f \circledcirc \mathcal{U} = \{\bigcup_{x \in U} V_x \mid U \in \mathcal{U}, \forall x \in U : V_x \in f(x)\}$. The unit of $\mathcal{G}$ maps $x \mapsto \{\{x\}\}$.*

Denote the left adjoint of the Kleisli adjunction by $\overline{\overline{(-)}} : \mathbf{Set} \to \mathbf{Kl}(\mathcal{G})$, and a lifting of a $\mathbf{Set}$ endofunctor G, with $\overline{\overline{G}} : \mathbf{Kl}(\mathcal{G}) \to \mathbf{Kl}(\mathcal{G})$. There is a natural transformation $\mathsf{cl} : PQ \to \mathcal{G}$ that closes a set of subsets under arbitrary non-empty unions. We have monad morphisms $\eta^P : Q \to \mathcal{G}$ and $\mathsf{cl} \circ P(\eta^Q) : P \to \mathcal{G}$.

2.4 Relations

Recall that $\mathbf{Rel}$ is a dagger category, it is equipped with an involution $(-)^\dagger : \mathbf{Rel}^{op} \to \mathbf{Rel}$. This maps a morphism $r : C \nrightarrow D$ to a morphism $r^\dagger : D \nrightarrow C$, defined as $r^\dagger(d) := \{c \in C \mid d \in r(x)\}$.

Definition 1. *Let $r, r' : C \nrightarrow D$ and $l, l' : D \nrightarrow C$. We say that*

- *l and r form an adjunction $l \dashv r$ iff $l \odot r \sqsubseteq \mathsf{id}_C$ and $\mathsf{id}_D \sqsubseteq r \odot l$*
- *$l \dashv r$ is a reflection iff $l \dashv r$ and $\mathsf{id}_C \sqsubseteq l \odot r$*

- *$l \dashv r$ is a coreflection iff $l \dashv r$ and $r \odot l \sqsubseteq \mathsf{id}_D$*
- *l is left total iff $\forall c \in C, \exists d \in D : c \in l(d)$*
- *l is deterministic iff $\forall d \in D, \forall c, c' \in C : c \in l(d)$ and $c' \in l(d) \implies c = c'$*
- *l is functional iff l is left total and deterministic*
- *r is right total iff $\forall d \in D, \exists c \in C : d \in r(c)$*
- *r is separating iff $\forall c, c' \in C, \forall d \in D : d \in r(c)$ and $d \in r(c') \implies c = c'$*
- *r is cofunctional iff r is right total and separating*

Proposition 2. *Let $l : D \nrightarrow C$ and $r : C \nrightarrow D$. We have $l \dashv r$ iff l is functional iff r is cofunctional. We have that $l \dashv r$ is a coreflection iff l is functional and right total iff l is a surjective function iff r is cofunctional and left total.*

Thus, $(-)^\dagger$ restricts to isomorphisms $\mathbf{Rel}_{\mathrm{co}}^{\mathrm{op}} \cong \mathbf{Set}$ and $\mathbf{Rel}_{\mathrm{co+lt}}^{\mathrm{op}} \cong \mathbf{Set}_{\mathrm{surj}}$.

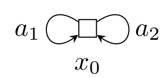
2.5 Cones

Let $D : \mathbb{I} \to \mathbf{Set}$. A cone $c : [C] \to D$ is *monic* when the collection $c_i : C \to D(i)$ is *jointly monic*. This means $\forall x, x' \in C : (\forall i \in \mathbb{I} : c_i(x) = c_i(x')) \implies x = x'$.

Similarly, if $D : \mathbb{I} \to \mathbf{Rel}$, a cocone $c : D \to [C]$ is *deterministic* when the collection $c_i : D(i) \to C$ is *jointly deterministic*: for any indexed collection $y_i \in D(i)$, and $x, x' \in C$, if $x \in c_i(y_i)$ and $x' \in c_i(y_i)$ for all $i \in \mathbb{I}$, then $x = x'$.

A *cone morphism* $f : (C, c) \to (C', c')$ is a morphism $f : C \to C'$ such that $c' \circ [f] = c$. We say that (C', c') is *larger than* (C, c) when there is a mono $C' \rightarrowtail C$ which is a cone morphism. We use this order to talk about order-preserving isomorphisms to and from collections of cones.

2.6 Infinite Traces



The work in [13] is the first to give a coalgebraic account of infinite traces in labelled transition systems. Recall that the central challenge in infinite traces is that we do not have a unique coalgebra morphism $(X, \gamma) \nrightarrow (Z, \bar{\zeta})$ in $\mathbf{Rel}$. This is exemplified in the transition system on the right, where there are non-canonical morphisms such as $x_0 \mapsto \{w \in \{a_1, a_2\}^\omega \mid a_1$ occurs in w infinitely often$\}$.

To remedy this, the solution in [13] is to take the *largest homomorphism*. This is equivalent to taking a join in the complete lattice $\mathbf{Rel}(X, Z)$.

This viewpoint is elaborated in [21], where they work with the operator $\Phi : (X \nrightarrow Z) \to (X \nrightarrow Z)$ mapping $f \mapsto \overline{\zeta^{-1}} \odot f \odot \gamma$. Given an order structure $\sqsubseteq$ on morphisms, a lax coalgebra morphism $f : (X, \gamma) \to (Y, \delta)$ is one s.t. $F(f) \circ \gamma \sqsubseteq \delta \circ f$, and an oplax coalgebra morphism is one s.t. $F(f) \circ \gamma \sqsupseteq \delta \circ f$. There is a correspondence between fixed points/pre-fixed points/post-fixed points of Φ and homomorphisms/lax homomorphisms/oplax homomorphisms $(X, \gamma) \nrightarrow (Z, \bar{\zeta})$. By Φ being a monotone operator (this follows from the enrichment of $\mathbf{Rel}$, and F being locally monotone) we can apply the Knaster-Tarski theorem to obtain a complete lattice of homomorphisms, where the smallest homomorphism is the least lax homomorphism, and the largest homomorphism is the largest oplax homomorphism. This perspective shift will be present throughout our work.

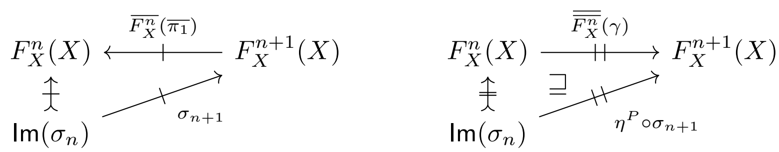
Largest Mediating Map Approach We briefly recall the approach in [5], which uses that the limiting cone $(Z, p_n : Z \to F^n(1))$ lifts into **Rel** and forms a *weak limit* over $1 \overset{\bar{!}}{\nleftarrow} F(1) \overset{\overline{F}(\bar{!})}{\nleftarrow} \cdots$. We can build a cone over this final sequence by unfolding a coalgebra structure $X \nrightarrow F(X)$: $X \overset{\gamma_n}{\nrightarrow} F^n(X) \overset{\overline{F^n}(\bar{!})}{\nrightarrow} F^n(1)$. Furthermore, we can take *largest maps* into Z which commute with the cone projections, again because **Rel** is enriched in **CJSL**. This serves as an alternative definition of infinite traces in [5]. We will discuss in Section 3.4 why this approach works better for executions, where states are recorded too, rather than traces.

3 Strategies

3.1 Definition

To begin, recall the definition of a strategy in [19], as a chain of maps in $\mathbf{Rel}_{\mathrm{lt}}$.

Definition 2 ([19]). *Let $\gamma : X \nrightarrow \mathcal{G}F_X(X)$. A* strategy σ in (X, γ) *consists of a family of morphisms $\{\sigma_n\}_{n\in\omega}$ in $\mathbf{Rel}_{\mathrm{lt}}$, with $\sigma_0 : 1 \nrightarrow X$ and $\sigma_{n+1} : \mathsf{Im}(\sigma_n) \nrightarrow F_X^{n+1}(X)$, satisfying the following conditions:*



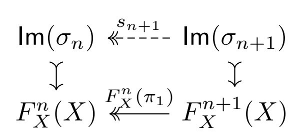
for all $n \in \omega$. We say that a strategy σ starts in x *when $\sigma_0(*) = \{x\}$, and use $\Sigma_\gamma(x)$ to denote the set of such strategies.*

This definition coincides with the definition of strategy presented in Section 2.1. The left diagram ensures strategies *extend* partial plays, and the right diagram ensures that moves chosen are from the game. Note in this definition the strategy is defined over exactly the partial plays it forces.

The Outcome of a Strategy In Section 2.1, we have seen a definition of outcome of a strategy σ: all the plays which conform to σ. We can make this definition categorical with a limit in **Set**. We prove that the chain $1 \overset{\sigma_0}{\nrightarrow} \mathsf{Im}(\sigma_0) \overset{\sigma_1}{\nrightarrow} \cdots$ in $\mathbf{Rel}_{\mathrm{lt}}$, lives in the subcategory $\mathbf{Rel}_{\mathrm{co+lt}}$. So by the isomorphism $\mathbf{Rel}_{\mathrm{co+lt}}^{\mathrm{op}} \cong \mathbf{Set}_{\mathrm{surj}}$ from Proposition 2, we have a surjective cochain a **Set**.

In fact, we have a nice characterisation of morphisms in this cochain in **Set**. We summarise this in the proposition below, note we abuse notation and use σ_{n+1} to refer to the relation after restricting its codomain to $\mathsf{Im}(\sigma_{n+1})$.

Proposition 3. *Each $\sigma_{n+1} : \mathsf{Im}(\sigma_n) \nrightarrow \mathsf{Im}(\sigma_{n+1})$ is right total, separating, and left total. Furthermore, the corresponding surjective function $s_{n+1} : \mathsf{Im}(\sigma_{n+1}) \twoheadrightarrow \mathsf{Im}(\sigma_n)$ is a restriction of $F_X^n(\pi_1) : F_X^{n+1}(X) \to F_X^n(X)$ to $\mathsf{Im}(\sigma_{n+1}) \to \mathsf{Im}(\sigma_n)$.*



Definition 3. *Denote the functor which picks out the chain* $\mathsf{Im}(\sigma_0) \overset{\sigma_1}{\nrightarrow} \dots$ *with* $D : \omega \to \mathbf{Rel}$. *The outcome of a strategy is the limit* $(\mathsf{Out}(\sigma), \pi)$ *of* $D^\dagger = (-)^\dagger \circ D : \omega \to \mathbf{Set}$*, the cochain* $\mathsf{Im}(\sigma_0) \leftarrow \mathsf{Im}(\sigma_1) \leftarrow \cdots$.

$\mathsf{Out}(\sigma)$ is isomorphic to a set with elements $(u_0, u_1, \dots)$ with $u_n \in \mathsf{Im}(\sigma_n)$ and $u_{n+1} \in \sigma_n(u_n)$ for all $n \in \omega$, thus precisely the plays which conform to the strategy are captured.

Notice that a limit in **Set** is a colimit in $\mathbf{Rel}_{\mathrm{co}}$, so $\mathsf{Out}(\sigma)$ is a colimit in this subcategory of relations. This leads to a question of whether $\mathsf{Out}(\sigma)$ is a colimit in **Rel**, which we tackle now.

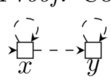
3.2 Colimits in Rel

This section is more technical than the rest, and is not directly relied upon in the rest of paper, so some readers may wish to skip to Section 3.3.

We start with an observation: $\mathsf{Out}(\sigma)$ is not a colimit in **Rel**. The counterexample we present is inspired by a counterexample in [16,17], where it is shown that in general colimits of ω-chains do not exist in **Rel**.

The flavour of this counterexample can be summarised by the slogan: "measuring" finite prefixes of plays does not determine measurements on infinite plays. Note that $\pi^\dagger$ refers to the cocone legs of $\mathsf{Out}(\sigma)$ in **Rel**. These have components $\pi_n^\dagger : \mathsf{Im}(\sigma_n) \nrightarrow \mathsf{Out}(\sigma)$, and extend an n-step prefix to an infinite play.

Counterexample 1 *The cocone* $(\mathsf{Out}(\sigma), \pi^\dagger)$ *is not a colimit in* **Rel**.

Proof. Consider the (memoryless) strategy (in a game with a single observation):


. We compute that $\mathsf{Out}(\sigma) = x^\omega + x^* y^\omega$, with $\pi_n^\dagger$ mapping: $x^n \mapsto \{\, x^k y^\omega \mid k \geq n \,\} \cup \{x^\omega\}$ and $x^k y^{n-k} \mapsto \{x^k y^\omega\}$ for $0 \leq k < n$. Define a cocone $C := \{0,1\}$ with $c_n \colon \mathsf{Im}(\sigma_n) \nrightarrow C$ mapping $x^n \mapsto \{0,1\}$ and $x^k y^{n-k} \mapsto \{1\}$ for $0 \leq k < n$. We claim there are two mediating maps $\mathsf{Out}(\sigma) \nrightarrow C$: the first maps $x^\omega \mapsto \{0\}$ and $x^\omega \mapsto \{0,1\}$, the second maps $x^* y^\omega \mapsto \{1\}$ and $x^* y^\omega \mapsto \{1\}$.

Despite this, one can show that $\mathsf{Out}(\sigma)$ has a 2-categorical universal property. This is again inspired by [16,17], however we work in a less general setting which allows to give us a short proof. We refer the reader to loc. cit. for more discussion, where this flavour of colimit is termed a *lax coop adjoint colimit.* Our presentation is in terms of adjunctions to keep it self-contained.

An *oplax cocone* $c : D \to [C]$ has legs $c_n : \mathsf{Im}(\sigma_n) \nrightarrow C$, which commute oplaxly: $c_{n+1} \sqsubseteq \sigma_{n+1} \odot c_n$, the category of these objects is denoted $\mathbf{Oplax}(D, [C])$.

Proposition 4. *We have an adjunction:* $\mathbf{Rel}(\mathsf{Out}(\sigma), C) \underset{!}{\overset{[-]\circ\pi^\dagger}{\rightleftarrows}} \mathbf{Oplax}(D, [C])$ ($\bot$) *for all sets* C. *We can restrict the right side to strict cocones (as* $\pi^\dagger$ *is strict), and obtain a reflection:* $\mathbf{Rel}(\mathsf{Out}(\sigma), C) \underset{!}{\overset{[-]\circ\pi^\dagger}{\rightleftarrows}} \mathbf{Nat}(D, [C])$ ($\bot$) *with* $[!(c)] \circ \pi^\dagger = c$.

Proof. $[-] \circ \pi^\dagger$ is a strict cocone by calculation. As **Rel** is enriched in **CJSL** we have $[-] \circ \pi^\dagger$ preserves joins (and is hence a functor), thus it has a right adjoint by the Adjoint Functor Theorem: for some oplax cocone $c : D \to [C]$: $!(c) = \bigsqcup\{h : \mathsf{Out}(\sigma) \nrightarrow C \mid [h] \circ \pi^\dagger \sqsubseteq c\}$. We prove $[!(c)] \circ \pi^\dagger = c$ in Proposition 6.

By the above proof, for some (possibly oplax) cocone (C, c), there exists a canonical mediating map h out of $(\mathsf{Out}(\sigma), \pi^\dagger)$, which is the largest map which "laxly commutes" ($[h] \circ \pi^\dagger \sqsubseteq c$) with the cone (C, c). Next, we give a second description of ! as a countable intersection. It allows us to prove the bottom adjunction is a reflection (although the characterisation holds in both adjunctions).

Proposition 5. *The right adjoint ! be defined as* $!(c)((u_n)_{n\in\omega}) = \bigcap_{n\in\omega} c_n(u_n)$.

With this characterisation we can prove the counit is an equality, meaning that any strict cocone $c : D \to [C]$ factors through $!(c)$ strictly (not just laxly).

Proposition 6. *Let* $c : D \to [C]$ *be a cocone, we have* $[!(c)] \circ \pi^\dagger = c$.

3.3 Homomorphism Approach

As described in Section 2.6, there is a nice view of infinite traces in a labelled transition system as a largest homomorphism (a greatest fixed point). Given strategies generalise transition systems, this view can be applied to the outcomes of strategies too, with some work. Our key insight is to use the *unravelling* of a strategy: we equip the set of finite prefixes which conform to the strategy with a coalgebra structure (this technique is familiar to temporal logicians).

We collect prefixes occurring along $\mathsf{Im}(\sigma_0) \nrightarrow \mathsf{Im}(\sigma_1) \nrightarrow \cdots$ in $\mathsf{Pref}(\sigma) := \coprod_{n\in\omega} \mathsf{Im}(\sigma_n)$, a coproduct in **Set** (and **Rel**). Recall that the strategy chain resides in $\mathbf{Rel}_{\mathrm{lt}}$, so the following coalgebra structure is a left-total relation.

Definition 4. *Define* $\mathsf{unravel}(\sigma) : \mathsf{Pref}(\sigma) \nrightarrow F_X(\mathsf{Pref}(\sigma))$ *by each component*

$$\mathsf{unravel}(\sigma)_n := \mathsf{Im}(\sigma_n) \xrightarrow{\sigma_{n+1}} \mathsf{Im}(\sigma_{n+1}) \xrightarrow{\overline{\mathsf{peek}_{n+1}}} F_X(\mathsf{Im}(\sigma_{n+1})) \xrightarrow{\overline{F_X}(\overline{\mathsf{in}_{n+1}})} F_X(\mathsf{Pref}(\sigma))$$

where $\mathsf{peek}_n := (\mathsf{Im}(\sigma_n) \xrightarrow{\langle \mathsf{snip}_1^n, \mathsf{id}\rangle} F_X(X) \times \mathsf{Im}(\sigma_n) \xrightarrow{\star} F_X(\mathsf{Im}(\sigma_n)))$.

That is, the states of the coalgebra are partial plays, and the transition map relates an n-step partial play to all the $n+1$-partial plays which extend it.

We can now use the approach in Section 2.6 to obtain an infinite trace semantics for the $\overline{F_X}$-coalgebra $(\mathsf{Pref}(\sigma), \mathsf{unravel}(\sigma))$.

Proposition 7. *Let* $\Phi : (\mathsf{Pref}(\sigma) \nrightarrow Z_X) \to (\mathsf{Pref}(\sigma) \nrightarrow Z_X)$ *be defined on some* $h : \mathsf{Pref}(\sigma) \nrightarrow Z_X$ *as mapping* $h \mapsto \overline{\zeta^{-1}} \odot \overline{F_X}(h) \odot \mathsf{unravel}(\sigma)$. *This operator is a monotone operator on a complete lattice.*

Let $u \in \mathsf{Im}(\sigma_n)$, *we have* $\Phi(h)(u) = \bigcup_{u' \in \sigma_{n+1}(u)} \mathsf{snip}_1^{n+1}(u') \star h(u')$.

So we have another potential definition of the outcome of a strategy: as $h(*)$, where h the largest homomorphism $h : (\mathsf{Pref}(\sigma), \mathsf{unravel}(\sigma)) \nrightarrow (Z_X, \overline{\zeta_X})$. In Section 3.6, we prove that this definition is equivalent to Definition 3.

3.4 Largest Mediating Map

The weak limit approach in [5] gives us another possible definition of the infinite outcome of a strategy. Let $\sigma_{0n} : 1 \nrightarrow \mathsf{Im}(\sigma_n)$ denote $\sigma_n \circ \cdots \circ \sigma_0$. We can lift the final sequence of F_X into **Rel**, and take a cone with projections

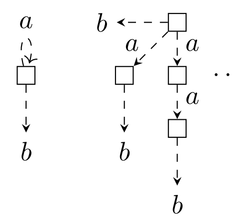
$$1 \overset{\sigma_{0n}}{\nrightarrow} \mathsf{Im}(\sigma_n) \rightarrowtail\!\!\!\!\nrightarrow F_X^n(X) \overset{\overline{F_X}(\bar{!})}{\nrightarrow} F_X^n(1) \ .$$

Proposition 8. *$(1, \overline{F_X}(\bar{!}) \odot \sigma_{0n})$ forms a cone over $1 \overset{\bar{!}}{\nleftarrow} F_X(1) \overset{\overline{F_X}(\bar{!})}{\nleftarrow} \cdots$ in* **Rel**.

The greatest cone morphism $1 \nrightarrow Z_X$ gives a subset of Z_X, which we can also take as the outcome of a strategy. We show in Section 3.6 that this is equivalent to Definition 3. This is somewhat surprising, when you consider that this approach gives too many traces when you do not record intermediate states.

Example 1. Consider the two memoryless strategies depicted on the right.

The largest mediating morphism $1 \nrightarrow Z$, takes the greatest subset which agrees with the finite approximations provided by the cone in Proposition 8. In both strategies we see that $a^*b + a^\omega$ is the largest mediating map $1 \nrightarrow Z$. The right strategy should have as an outcome a^*b: it is impossible for a^ω to be observed.



3.5 Correspondence Result

We present a correspondence result which identifies conditions under which cones in **Set**, cocones in **Rel**, coalgebra morphisms $\mathsf{Pref}(\sigma) \nrightarrow Z_X$, and mediating maps into Z_X viewed as a weak limit in **Rel**, coincide.

To begin, recall that $\mathbf{Rel}_{\mathrm{co+lt}}$, the category of sets with left-total, right-total, and separating relations, is dually equivalent to $\mathbf{Set}_{\mathrm{surj}}$, the category of sets and surjective functions. We can restrict D to have type $\omega \to \mathbf{Rel}_{\mathrm{co+lt}}$, by Proposition 3. We see a cone $[C] \to D^\dagger$ in **Set** as representing a collection of infinite plays, via sequences of partial plays they induce. We do not want duplicate copies of infinite plays, so we impose that our cones are monic (i.e. the cone legs are jointly monic). The dual condition to this in **Rel** is that cocone legs are *jointly deterministic*, this is defined in Section 2.5.

Proposition 9. *Let $c^\dagger : D \to [C]$ be a cocone in $\mathbf{Rel}_{\mathrm{co}}$. We have that $c^\dagger$ is deterministic iff c is monic.*

We also want the cone to be "full", in the sense that every finite prefix should be accounted for, this is captured by cone legs being surjective functions (the dual condition in **Rel** is legs being left total).

It turns out that the most laborious direction is verifying that cocones give us homomorphisms. This is because a cocone leg $c_n^\dagger : \mathsf{Im}(\sigma_n) \nrightarrow C$ (morally) extends a prefix to an infinite play, whereas a homomorphism $h : \mathsf{Pref}(\sigma) \nrightarrow Z_X$, gives the plays which conform to the strategy which complete the prefix. To help with this bookkeeping issue, we introduce a map which interrogates a cone C to obtain n-step prefixes with corresponding suffixes.

Definition 5. *Let $(C, c : [C] \to D^\dagger)$ be a cone in* **Set**. *$(Z_X, p_n : Z_X \to F_X^n(1))$ is a limiting cone for the final sequence of F_X and $F_X : \mathbf{Set} \to \mathbf{Set}$ preserves limits; so $F_X^n(Z_X)$ with projections $F_X^n(p_m) : F_X^n(Z_X) \to F_X^{n+m}(1)$ is a limiting cone for all $n \in \omega$. We define a map $\mathsf{obs}_n : C \to F_X^n(Z_X)$ as the unique cone morphism from C with legs $C \xrightarrow{c_{n+m}} \mathsf{Im}(\sigma_{n+m}) \rightarrowtail F_X^{n+m}(X) \xrightarrow{F_X^{n+m}(!)} F_X^{n+m}(1)$ into $F_X^n(Z_X)$.*

Theorem 1. *Let $\sigma \in \Sigma_\gamma(x)$ be a strategy, the following are equivalent:*

1. *$\overline{F_X}$-coalgebra morphisms $(\mathsf{Pref}(\sigma), \mathsf{unravel}(\sigma)) \nrightarrow (Z_X, \overline{\zeta_X})$*
2. *Fixed points of $\Phi(\sigma) : (\mathsf{Pref}(\sigma) \nrightarrow Z_X) \to (\mathsf{Pref}(\sigma) \nrightarrow Z_X)$*
3. *Mediating maps $(1, \overline{F_X^n}(\overline{!}) \odot \sigma_{0n})_{n\in\omega} \nrightarrow (Z_X, \overline{p_n})_{n\in\omega}$ in* **Rel**
4. *Deterministic cocones $D \to [C]$ in $\mathbf{Rel}_{\mathrm{co+lt}}$*
5. *Monic cones $[C] \to D^\dagger$ in $\mathbf{Set}_{\mathrm{surj}}$.*

Furthermore, these translations are order preserving.

Proof (Sketch). **(1)** $\iff$ **(2)** See Section 2.6. **(4)** $\iff$ **(5)** Proposition 9.

(1) $\implies$ **(3)** A coalgebra morphism $h : \mathsf{Pref}(\sigma) \nrightarrow Z_X$ has a component $h_0 : 1 \cong \mathsf{Im}(\sigma_0) \nrightarrow Z_X$, which defines a mediating map into $(Z_X, \overline{p_n})_{n\in\omega}$ in **Rel**.

(3) $\implies$ **(5)** Take a mediating map $f : 1 \nrightarrow Z_X$. The set $\mathsf{Im}(f)$ with cone legs $Z_X \xrightarrow{p_{n+1}} F_X^{n+1}(1) \xrightarrow{F_X^n(\pi_1)} F_X^n(X)$ (restricted to a map $\mathsf{Im}(f) \to \mathsf{Im}(\sigma_n)$) form a deterministic cone in $\mathbf{Set}_{\mathrm{surj}}$.

(4) $\implies$ **(1)** Given a deterministic cocone $(C, c_n^\dagger : \mathsf{Im}(\sigma_n) \nrightarrow C)$, the morphism $[\overline{\mathsf{snip}_0^n} \odot \overline{\mathsf{obs}}_n \odot c_n^\dagger]_{n\in\omega}$ is a $\overline{F_X}$-coalgebra morphism.

3.6 Equivalent Characterisations of Strategy Outcomes

Theorem 1 says that homomorphisms $\mathsf{Pref}(\sigma) \nrightarrow Z_X$ give rise to cones over the strategy chain. We have not established that defining a strategy outcome as a limit is equivalent to defining it as the greatest homomorphism. We do know that the greatest homomorphism will be the largest monic cone, as the translation is order preserving. Thus, all we must establish is that the largest monic cone is the limiting cone. This is provided by the following proposition, which we have not found a proof of in the literature, but imagine is known.

Proposition 10. *Take a diagram $D : \mathbb{I} \to \mathbf{Set}$, a largest monic cone is a limiting cone.*

This is an important characterisation of the largest monic cone, there is also one for the greatest homomorphism, which we discuss now. It involves the concept of *limit closure*, which arose when studying the semantics of temporal logics [1,6]. We present a definition here for executions, i.e. elements of Z_X.

Definition 6. *Let $U \subseteq Z_X$, we say that U is* limit closed *precisely when for any $z \in Z_X$, if $(\forall n \in \omega, \exists \rho \in Z_X : z_n \star \rho \in U)$ then $z \in U$.*

A subset of Z_X which forms a cone in $\mathbf{Set}_{\mathrm{surj}}$ over the strategy chain, is maximal iff it is limit-closed. We can exploit this to prove the following result.

Proposition 11. *Let $h : \mathsf{Pref}(\sigma) \nrightarrow Z_X$ be a homomorphism for some $\sigma \in \Sigma_c(x)$, h is the greatest homomorphism iff $h(x)$ is limit closed.*

Corollary 1. *We identify (under the transformations in Theorem 1): the $\overline{F_X}$-coalgebra homomorphism with a limit-closed image; the greatest $\overline{F_X}$-coalgebra homomorphism; the greatest fixed point of Φ; the greatest mediating map into $(Z_X, \overline{p_n})_{n\in\omega}$; the largest deterministic cocone for D; the largest monic cone for $D^\dagger$; and the limiting cone for $D^\dagger$.*

Thus, any of these can serve as a definition for the outcome of a strategy.

Remark 1. This result connects various coalgebraic approaches to infinite trace semantics in the literature, in the case of the powerset monad with a simple behaviour functor. The greatest homomorphism, like in [13,21], is equivalent to taking the largest mediating map, like in [5]. But also, applying the approach to trace semantics in [15,18] to a memoryless strategy, we will obtain the sequence $(\mathsf{Im}(\sigma_n))_{n\in\omega}$ which are called *pretraces* in loc. cit. (we include the states in the behaviour). In [15, Remark 4], it is suggested to take a limit over a sequence of spans which are the *immediate prefix relations*. This is similar to the idea we use here, as we take a limit over $\mathsf{Im}(\sigma_0) \xleftarrow{s_0} \mathsf{Im}(\sigma_1) \xleftarrow{s_1} \cdots$. We keep more data, as n-step executions are recorded rather than just traces, which allows a more economic presentation using functions rather than relations (this can be seen in Proposition 3, with the key property being that strategies are separating). We will see shortly that this idea can also be applied to outcomes in games.

4 Games

4.1 Definition of Outcome

Recall that a game is a coalgebra $\gamma : X \nrightarrow F_X(X)$, i.e. a function $X \to \mathcal{G}F_X(X)$.

Our first task is to define the *outcome of a game*. In [19], we defined the (finite) outcome of a game via a map $X \to \mathcal{G}(Z_X)$ (for a slightly different monad $\mathcal{G}$ subject to certain finiteness restrictions), assigning to each state those subsets of $(XA)^*XB$ (i.e. sets of finite completed plays) which can be forced by some controller strategy. It was shown in loc. cit. that this map arises as an instance of the coalgebraic framework for finite trace semantics in [12]. Here the situation is more subtle, since we are dealing with infinite trace semantics, for which no general account via initiality or finality exists.

A functor describing how we can drop morphisms from $\mathbf{Kl}(\mathcal{G})$ into $\mathbf{Rel}$ will be required. Recall that $\mathcal{G}$ is built from a weak distributive law, and while normally we automatically get a functor $\mathbf{Kl}(\mathcal{G}) \to \mathbf{Rel}$, here we do not[1].

Proposition 12. *There is a functor $K : \mathbf{Kl}(\mathcal{G}) \to \mathbf{Rel}$ mapping $f : X \nrightarrow Y$ to a relation $QX \nrightarrow QY$, it is defined as $K(f)(U) := \{ \bigcup_{x\in U} V_x \mid \forall x \in U : V_x \in f(x)\}$. For some $g : X \to Y$, we have $K(\bar{\bar{g}}) = \overline{Q}(\bar{g}) = \overline{Q(g)}$.*

[1] For standard distributive laws, $\overline{Q}$ is a monad on $\mathbf{Kl}(P)$ and $\mathbf{Kl}(\overline{Q}) \cong \mathbf{Kl}(PQ)$; however, here $\overline{Q}$ is only a semi-monad.

The only definition for the outcome of a game that we could use out-of-the-box is the greatest homomorphism approach of [13], however as we will show in Section 4.2, this does not give the expected outcome. For now, we show how to apply the limit approach of Section 3.1 to games.

Our starting point is the unfolding of γ in $\mathbf{Kl}(\mathcal{G})$: $X \overset{\gamma}{\twoheadrightarrow} F_X(X) \overset{\overline{\overline{F_X}}(\gamma)}{\twoheadrightarrow} \cdots$

We first drop this chain into **Rel**: $X \overset{\gamma}{\rightarrow} QF_X(X) \xrightarrow{K(\overline{\overline{F_X}}(\gamma))} QF_X^2(X) \xrightarrow{K(\overline{\overline{F_X^2}}(\gamma))} \cdots$ (γ is readily seen as a morphism in **Rel**, for the rest we use K). Define $\gamma_n : X \rightarrow QF_X^n X$ by $\gamma_1 = \gamma$ and $\gamma_{n+1} = K(\overline{\overline{F_X^n}}(\gamma)) \odot \gamma_n$. We then take the image of these maps (in **Rel**): we let $\mathsf{Im}(\gamma_0) = X$, and $e_n^\dagger : \mathsf{Im}(\gamma_n) \rightarrow \mathsf{Im}(\gamma_{n+1})$ be $K(\overline{\overline{F_X^n}}(\gamma))$ suitably restricted. Hence, we have a chain $\mathsf{Im}(\gamma_0) \overset{e_0^\dagger}{\rightarrow} \mathsf{Im}(\gamma_1) \overset{e_1^\dagger}{\rightarrow} \cdots$ The notation is justified by the following proposition.

Proposition 13. $\mathsf{Im}(\gamma_n) = \bigcup_{x \in X} \gamma_n(x)$.

Notice also, that each $e_n^\dagger$ is right total and separating. Like in Proposition 3, separation follows because we are recording state information. Note that, unlike before, these maps are not left-total (due to possible controller deadlocking).

Proposition 14. *$e_n^\dagger : \mathsf{Im}(\gamma_n) \rightarrow \mathsf{Im}(\gamma_{n+1})$ is separating. The left adjoint $e_n : \mathsf{Im}(\gamma_{n+1}) \to \mathsf{Im}(\gamma_n)$ restricts $QF_X^n(\pi_1) : QF_X^{n+1}(X) \to QF_X^n(X)$.*

We are again left with a cochain in **Set**: $\mathsf{Im}(\gamma_0) \overset{e_0}{\longleftarrow} \mathsf{Im}(\gamma_1) \overset{e_1}{\longleftarrow} \cdots$.

Definition 7. *The* outcome of the game γ*, denoted* $\mathsf{Out}(\gamma)$*, is the limit of the above cochain in* **Set**.

Strategy outcomes can now be recovered as special cones over this cochain.

Proposition 15. *Cones from* 1 *over the cochain* $\mathsf{Im}(\gamma_0) \leftarrow \mathsf{Im}(\gamma_1) \leftarrow$ *are in bijection with strategies. We have* $\mathsf{Out}(\gamma) \cong \{\mathsf{Out}(\sigma) \mid x \in X, \sigma \in \Sigma_\gamma(x)\}$.

Recall that the cones over $\mathsf{Im}(\gamma_0) \leftarrow \mathsf{Im}(\gamma_1) \leftarrow \cdots$ are equivalently cocones under $\mathsf{Im}(\gamma_0) \rightarrow \mathsf{Im}(\gamma_1) \rightarrow \cdots$ with right-total and separating legs (but not necessarily left total, again because of deadlocking).

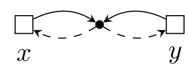
4.2 Homomorphism Approach

We discuss the connection with the largest homomorphism approach to infinite traces in [13,21]. The first thing we must establish is that the operator $\Psi : (X \twoheadrightarrow Z_X) \to (X \twoheadrightarrow Z_X)$ built from $f \mapsto \overline{\overline{\zeta_X^{-1}}} \circledcirc \overline{\overline{F_X}}(f) \circledcirc \gamma$ is monotone. This follows from $\mathbf{Kl}(\mathcal{G})$ being **Pos**-enriched and $\overline{\overline{F_X}}$ being locally monotone. Note that the conditions of [21, Proposition 4.1] are met, meaning that we can give a constructive proof of the existence of a greatest fixed point, using the transfinite induction proof in [4]. We present the version using the non-constructive Knaster-Tarksi Theorem here. Note it was shown in [19] that $\mathbf{Kl}(\mathcal{G})$ is not **CJSL**-enriched.

Proposition 16. $\mathbf{Kl}(\mathcal{G})$ *is* $\mathbf{Pos}$*-enriched,* $\overline{\overline{\overline{F_X}}}$ *is locally monotone (w.r.t. this enrichment), and* $\mathbf{Kl}(\mathcal{G})(X,Y)$ *is a complete lattice (inducing the same order as the enrichment).*

So we are in a position where we have least and greatest homomorphisms $(X,\gamma) \nrightarrow (Z_X, \overline{\overline{\overline{\zeta_X}}})$. It turns out that the greatest homomorphism gives too many sets of subsets, as the next example demonstrates.

Example 2. There is a homomorphism: $x \mapsto \{x{\cdot}\mathsf{GF}x, x{\cdot}\mathsf{GF}y, Z_X\}$ and $y \mapsto \{y\cdot\mathsf{GF}x, y\cdot\mathsf{GF}y, Z_X\}$ in the right game, where $A = \{*\}$. Here, $\mathsf{GF}x \subseteq Z_X$ is the set of infinite streams over $\{x,y\}$ that contain infinitely many xs.



Notice that in our example, the problematic subsets are not limit-closed. Restricting homomorphisms to only contain limit-closed subsets of plays, lets us recover $\mathsf{Out}(\gamma)$ as the largest limit-closed homomorphism. This is proven in the next subsection. For now, we state that all homomorphisms give rise to cones.

Lemma 1. *For any oplax coalgebra morphism* $f : (X,\gamma) \nrightarrow (Z_X, \overline{\overline{\overline{\zeta_X}}})$*, we have:* $\overline{\overline{\overline{F_X^n}}}(\overline{\overline{\pi_1}}) \circledcirc \overline{\overline{p_{n+1}}} \circledcirc f \sqsubseteq \gamma_n$ *for all* $n \in \omega$*.*

Proposition 17. *Let* $f : X \nrightarrow Z_X$ *be a coalgebra morphism. The set* $\mathsf{Im}(f)$ *with legs* $Q(Z_X) \xrightarrow{Q(p_{n+1})} Q(F_X^{n+1}(1)) \xrightarrow{QF_X^n(\pi_1)} QF_X^n(X)$ *restricted to maps* $\mathsf{Im}(f) \to \mathsf{Im}(\gamma_n)$ *form a cone over* $\mathsf{Im}(\gamma_0) \leftarrow \mathsf{Im}(\gamma_1) \leftarrow \cdots$*.*

4.3 Cones to Homomorphisms

We now answer the question: when does a cone over $\mathsf{Im}(\gamma_0) \leftarrow \mathsf{Im}(\gamma_1) \leftarrow \cdots$, give rise to a homomorphism $(X,\gamma) \nrightarrow (Z_X, \overline{\overline{\overline{\zeta_X}}})$? In Section 3, the cones were "full" in the sense they had epic components, however here we cannot assume that the legs are surjective, because the controller can deadlock.

Instead, we axiomatise cones directly. We introduce the following terminology for a cone (C,c): say $y \in C$ is *anchored* over a state $x \in X$, when $c_0(y) = x$. The set of elements of C which are anchored over x is denoted $C(x)$. We extend this terminology to some $u \in F_X(X)$, and notate the subset of C whose elements are anchored at u with $C(u) = \{y \in C \mid \mathsf{snip}_0^1(u) = c_0(y)\}$. Note that it only makes sense to talk about $C(u)$ when u is a partial play (ends in a state).

Definition 8. *Let* (C,c) *be a cone over* $\mathsf{Im}(\gamma_0) \leftarrow \mathsf{Im}(\gamma_1) \leftarrow \cdots$*. We say that* (C,c) *is closed under decomposition if when* $y \in C$

$$\exists\{y_u \in C(u)\}_{u\in c_1(y)}, \forall n \in \omega : \bigcup_{u\in c_1(y)} c_n(y_u) = Q(\mathsf{snip}_n^{n+1}) \circ c_{n+1}(y)$$

Say (C,c) *is closed under composition, if when* $U \in \gamma(x)$ *and* $\{y_u \in C(u)\}_{u\in U}$*:*

$$\exists y \in C(x), \forall n \in \omega : c_{n+1}(y) = \bigcup_{u\in U} u \star c_n(y)$$

Say a cone is convex closed if for any indexing set I and state $x \in X$:

$$\{y_i \in C(x)\}_{i \in I} \implies \exists y \in C, \forall n \in \omega : c_n(y) = \bigcup_{i \in I} c_n(y_i)$$

We say a cone is homomorphic precisely when it is closed under decomposition, composition, and convexity.

Proposition 18. *The construction in Proposition 17, from homomorphisms to cones, yield monic homomorphic cones.*

Proposition 19. *Given a cone (C, c) closed under convexity, define a morphism $f : X \nrightarrow\!\!\!\!\rightarrow Z_X$ as $f(x) := \{\mathsf{lim}(\{x\} \leftarrow c_1(y) \leftarrow \cdots) \mid y \in C(x)\}$.*

If (C, c) is closed under decomposition, then f is an oplax coalgebra morphism. If (C, c) is closed under composition, then f is a lax coalgebra morphism. Thus, if (C, c) is homomorphic, then f is a coalgebra morphism.

We now have constructions from monic homomorphic cones to coalgebra morphisms, and back. These constructions are not mutual inverses, as we show in the example below. Note that mapping a monic homomorphic cone to a coalgebra morphism and back, does yield the original cone.

Example 3. Recall Example 2. There is a homomorphism $x \mapsto \{x \cdot \mathsf{GF}x\}$ and $y \mapsto \{y \cdot \mathsf{GF}y\}$. This maps into a cone with two elements:

$$(x, \{xx, xy\}, \{xxx, xyx, xxy, xyy\}, \dots), (y, \{yx, yy\}, \{yxx, yyx, yxy, yyy\}, \dots)$$

Mapping this back to a homomorphism yields $x \mapsto \{x \cdot Z_X\}$ and $y \mapsto \{y \cdot Z_X\}$.

To obtain a bijection, we restrict the homomorphisms to *limit-closed homomorphisms*, i.e. those whose underlying $f : X \to \mathcal{G}(Z_X)$ is such that every $V \in f(x)$ is limit closed. Let $\mathcal{L}(Z_X)$ denote the union-closed sets of limit-closed subsets of executions: $\mathcal{L}(Z_X) := \{\mathcal{V} \in \mathcal{G}(Z_X) \mid \forall V \in \mathcal{V} : V \text{ is limit closed}\}$.

Proposition 20. *The operator $\Psi : (X \nrightarrow\!\!\!\!\rightarrow Z_X) \to (X \nrightarrow\!\!\!\!\rightarrow Z_X)$ restricts to $(X \to \mathcal{L}(Z_X)) \to (X \to \mathcal{L}(Z_X))$. The set $\mathcal{L}(Z_X)$ is a complete lattice. Thus the collection of limit-closed homomorphisms form a complete lattice.*

Theorem 2. *There is an order-preserving isomorphism between monic homomorphic cones and limit-closed homomorphisms.*

Theorem 2 gives us that the largest monic homomorphic cone is equivalently the largest limit-closed homomorphism $X \nrightarrow\!\!\!\!\rightarrow Z_X$. To show this is the limit of the cochain $\mathsf{Im}(\gamma_0) \leftarrow \mathsf{Im}(\sigma_1) \cdots$, we would like to employ Proposition 10, but for this we need a final proposition which states that the largest monic homomorphic cone is the largest monic cone.

Proposition 21. *Any monic cone can be closed under decomposition, composition and convexity, and turned into a homomorphic monic cone.*

Thus, the largest homomorphic monic cone is the limiting cone, meaning we have a secondary definition of the infinite outcome of a game: as the largest limit-closed homomorphism.

Remark 2. The homomorphism definition of infinite outcome generalises the definition of infinite traces in transition systems given in [21]. Recall that a one-player model $X \to PF_X(X)$ embeds into a game via $\mathsf{cl} \circ P(\eta^Q)$. It is possible to show that the induced embedding $\mathbf{Rel}(X, Z_X)$ into $\mathbf{Kl}(\mathcal{G})(X, Z_X)$ is monotone and preserves fixed points (this is almost an instance of [14, Proposition 4.11]), and maps into limit-closed morphisms. Thus the largest limit-closed homomorphism comes from the greatest homomorphism $X \nrightarrow Z_X$ in **Rel**.

4.4 Computing Strategies from Below

Finally, we tackle approximating strategies from below in the lattice $X \to \mathcal{G}(Z_X)$. Define $\mathsf{prune}_\gamma : P(X) \to P(X)$ as mapping U to $\{x \in X \mid \exists V \in \gamma(x) : \mathsf{snip}_0^1(V) \subseteq U\}$. Let Y be the set of states in the greatest fixed point of prune_γ. The $\mathcal{G}F_X$-coalgebra (X, γ) restricts to a coalgebra (Y, δ), where $\delta : Y \to \mathcal{G}F_X(Y)$ maps $x \mapsto \{V \in \gamma(x) \mid \mathsf{snip}_0^1(V) \subseteq Y\}$. By construction, this coalgebra is serial: $\delta(y) \neq \emptyset$ for any $y \in Y$. We now define $\underline{\delta} : Y \nrightarrow F_X(Y)$ as mapping $x \mapsto \bigcup \delta(x)$. We take $Z_{\gamma,x}$ to be all the valid traces in $\underline{\delta}$ from x. The following proposition states that $Z_{\gamma,x}$ is the outcome of the most permissive strategy in γ at x.

Proposition 22. *If $x \in Y$, there exists a strategy $\tau \in \Sigma_\gamma(x)$ with $\mathsf{Out}(\tau)(x) = Z_{\gamma,x}$. Moreover, for all $\sigma \in \Sigma_\gamma(x)$, $\mathsf{Out}(\sigma) \subseteq \mathsf{Out}(\tau)$.*

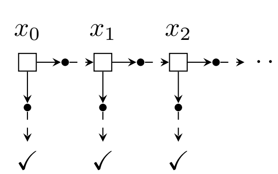
A simple corollary is that $x \in Y$ precisely if there is a strategy from x in γ.

Proposition 23. *Let $\mathcal{X}_{\gamma,x} := \{\mathcal{U} \in \mathcal{G}(X) \mid Z_{\gamma,x} \in \mathcal{U}\}$. The set of dependent functions $(x \in X) \to \mathcal{X}_{\gamma,x}$ is a complete lattice, with bottom $\bot_\gamma(x) = \{Z_{\gamma,x}\}$. Furthermore, Ψ restricts to $\Xi_\gamma : ((x \in X) \to \mathcal{X}_{\gamma,x}) \to ((x \in X) \to \mathcal{X}_{\gamma,x})$.*

Lemma 2. *Take a prefix point f of Ξ_γ, we have that for all $x \in X$, for any $\sigma \in \Sigma_c(x)$, and $n \in \omega$, that $\bigcup_{u \in \mathsf{Im}(\sigma_n)} u \star Z_{\gamma,\mathsf{snip}_0^n(u)} \in f(x)$.*

Proposition 24. *Let $l : (x \in X) \to \mathcal{X}_{\gamma,x}$ be the least fixed point of Ξ. Denote closing $(l(x), \subseteq)$ under meets (intersection) of descending ω-chains of sets with $\mathsf{close}(l(x))$. We have that $\mathsf{close}(l(x)) = \mathsf{Out}(\gamma)(x)$.*

Note in the least fixed point of Ξ_γ we do not get all strategies, i.e. we do really need to close.



Example 4. Consider the game on the right. The least fixed point does not contain the set $\{x_0x_1x_2\cdots\}$ at x_0. We can show this formally by verifying that $x_n \mapsto \mathsf{Out}(\gamma)(x_n) \setminus \{x_nx_{n+1}\dots\}$ is a prefix point of Ξ_γ. We have $x_0 \dots x_n \cdot \{x_{n+1}\checkmark, x_{n+1}x_{n+2}\checkmark, \dots\}$ in the least fixed point, so close adds the meet:

$$\bigcap_{n \in \omega} x_0 \dots x_n \cdot \{x_{n+1}\checkmark, x_{n+1}x_{n+2}\checkmark, \dots\} = \{x_0x_1 \dots\}$$

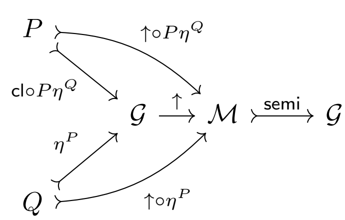
5 The Monotone Neighbourhood Monad

As we mentioned in the introduction, previous work in coalgebra [10,11,14] models the branching structure in a game with the *monotone neighbourhood monad* $\mathcal{M} : \mathbf{Set} \to \mathbf{Set}$. The analogous variant to our $\mathcal{G}$ maps a set X to the set of upwards closed subsets of X, i.e. fixed points of $\uparrow_X : PQ(X) \to PQ(X)$. Its unit $\eta^{\mathcal{M}}_X$ maps $x \mapsto \uparrow \{\{x\}\}$. Many aspects of our approach (especially in Section 3) will apply immediately to $\mathcal{M}F_X$-coalgebras, however some parts do not. In particular, the clear analogue of the functor $K : \mathbf{Kl}(\mathcal{G}) \to \mathbf{Rel}$ in Proposition 12 will not preserve units, and hence only be a *semi-functor*.

What is clear, is that the two monads are closely related. The composition in $\mathbf{Kl}(\mathcal{M})$ can in fact be written with an identical expression as Proposition 1, although can be simplified a little from upwards closure. Let $f : X \to \mathcal{M}(Y)$ and $\mathcal{U} \subseteq Q(X)$, we have $f \circledcirc^{\mathcal{M}} \mathcal{U} = \{V \subseteq X \mid \exists U \in \mathcal{U}, \forall x \in U : V \in f(x)\}$.

If we restrict $\uparrow_X$ to $\mathcal{G}(X) \to \mathcal{M}(X)$, we obtain a monad morphism (notice that this does not have injective components). Conversely, the inclusion $\mathcal{M}(X) \rightarrowtail \mathcal{G}(X)$ is a *semi-monad* morphism: it doesn't commute with the monads units. This semi-monad morphism allows us to map from $\mathcal{M}F_X$-coalgebras to $\mathcal{G}F_X$-coalgebras without destroying any information.

Proposition 25. *We have the following monad morphisms, where* $\rightarrowtail$ *denotes an injective monad morphism. The arrow labelled* semi *is only a semi-monad morphism. The two triangles in the diagram commute. Furthermore, we have that* $\uparrow_X \dashv \mathsf{semi}_X$ *is a coreflection.*



6 Future Work

We have presented a coalgebraic account of infinite outcomes in strategies and games. We showed that the outcomes of both are naturally captured as limits in **Set**, and can equivalently be characterised as largest homomorphisms. We also showed how infinite outcomes can be approximated using least fixed points.

Several avenues for future work remain. First, we aim to extend our results to games with probabilistic and weighted opponents by varying the underlying monad. It would also be interesting to investigate whether a unified framework can capture outcomes of games modelled both by our game monad and by the monotone neighbourhood monad. Finally, a natural next step is to generalise the coalgebraic approach to parity automata developed in [22] to the setting of games.

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