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Combinatorial Relative Algebraic K-theory and Exterior Power Operations

by

Jane Turner

MMath

ORCiD: 0000-0003-2359-7770

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Abstract

Faculty of Social Sciences
School of Mathematics

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Grayson has presented a conjectural combinatorial presentation of the higher relative algebraic K-groups. We prove this description to be correct. In proving this result, we provide a concrete description of Grayson's isomorphism between his combinatorial presentation of K_1 and the classical definition of K_1. We construct exterior power operations on Grayson's presentation of the relative algebraic K-groups, and prove that these operations satisfy the axioms of a λ-ring modified to allow nonunital rings. Kasprowski and Winges have provided a proof that the canonical homomorphism from Nenashev's presentation of K_1 to Grayson's presentation of K_1 is an isomorphism. We prove this isomorphism is compatible with the isomorphisms from these groups to the classical definition of K_1. Harris has provided an elementary proof that the homomorphism from Bass' presentation of K_1 to Grayson's presentation of K_1 is an isomorphism in certain circumstances. We use this result to prove the analogous result for relative K_0.

Contents

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I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;

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3. Where I have consulted the published work of others, this is always clearly attributed;

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Acknowledgements

Chapter 1

Introduction

K-theory is a very broad field that has applications in topology, algebraic geometry and beyond. In proving his generalisation of the Riemann-Roch theorem [Gro68], Alexander Grothendieck has introduced the concept of the group $K(X)$ (now called the Grothendieck group and denoted $K_0(X)$) of an algebraic variety X, which is the abelian group generated by isomorphism classes of vector bundles on X, where there is a relation $[V] = [V'] + [V'']$ whenever there is a short exact sequence of vector bundles:

$$0 \to V' \to V \to V'' \to 0.$$

Suitable algebraic definitions of K_1 and K_2 arose before Daniel Quillen defined K_n of a so-called exact category, a generalisation of an abelian category, by assigning a topological space $K\mathcal{A}$ to every exact category $\mathcal{A}$ using his Q-construction, so that $K_n(\mathcal{A}) := \pi_n K\mathcal{A}$ [Qui73]. While a useful generalisation, this definition proved difficult to compute, and so various methods have been developed to aid computation. One such method is using the localisation sequence: Given a short exact sequence of abelian categories (in the sense of lemma 12.10.6 in [Sta22])

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0,$$

the induced sequence of topological spaces

$$K\mathcal{A} \to K\mathcal{B} \to K\mathcal{C}$$

becomes a homotopy fibration sequence, and so there arises a long exact sequence

$$\cdots \to K_{n+1}\mathcal{C} \to K_n\mathcal{A} \to K_n\mathcal{B} \to K_n\mathcal{C} \to K_{n-1}\mathcal{A} \to \cdots$$

More relevant to this thesis is the definition of the relative K-groups. Namely, given an exact functor $F : \mathcal{M} \to \mathcal{N}$ between exact categories $\mathcal{M}$ and $\mathcal{N}$ we define the K-groups

of F via the homotopy fiber of the induced map $KF : K\mathcal{M} \to K\mathcal{N}$, i.e.

$$\text{hofib}(KF) \to K\mathcal{M} \to K\mathcal{N}$$

so that we obtain $K_n[F] := \pi_n\text{hofib}(KF)$. Along with this, there is a long exact sequence:

$$\cdots \to K_{n+1}\mathcal{N} \to K_n[F] \to K_n\mathcal{M} \to K_n\mathcal{N} \to K_{n-1}[F] \to \cdots$$

Daniel Grayson has provided a presentation of the higher K-groups using n-dimensional binary acyclic multicomplexes that does not require any homotopy theory [Gra12], and has given a conjectural presentation of the relative K-groups in a similar manner [Gra16].

In his work on the generalisation of the Riemann-Roch theorem, Alexander Grothendieck also introduced the formalism of a λ-ring, motivated by the fact that given a short exact sequence

$$0 \to V' \to V \to V'' \to 0$$

of vector bundles over a smooth scheme X, there is an equality

$$[\Lambda^k V] = \sum_{i=0}^{k}[\Lambda^i V' \otimes \Lambda^{k-i} V'']$$

in $K_0 X$, where $\Lambda^k V$ denotes kth exterior power of V for $k \geq 0$. These exterior powers induce operations $\lambda^k : K_0 X \to K_0 X$ for $k \geq 0$ satisfying the property

$$\lambda^k(x+y) = \sum_{i=0}^{k} \lambda^i(x)\lambda^{k-i}(y).$$

In general, Grothendieck defined a λ-ring as a ring L together with operations $\lambda^k : L \to L$ for $k \geq 0$ satisfying the following properties valid for all $x, y \in L$ and $k, l \geq 0$.

- $\lambda^0(x) = 1$

- $\lambda^1(x) = x$

- $\lambda^k(1) = 0$ if $k > 2$

- $\lambda^k(x+y) = \sum_{i=0}^{k} \lambda^i(x)\lambda^{k-i}(y)$

- $\lambda^k(xy) = P_k(\lambda^1(x), ..., \lambda^k(x), \lambda^1(y), ..., \lambda^k(y))$

- $(\lambda^k \circ \lambda^l)(x) = P_{k,l}(\lambda^1(x), ..., \lambda^{kl}(x))$

where P_k and $P_{k,l}$ are certain universal integral polynomials [FL85, I, §1].

The main results of this thesis are as follows.

As mentioned above, in Sections 1 and 2 of [Gra16], Grayson gives a conjectural presentation, which we denote by $K_n^{\text{Gr}}[F]$, of the higher relative K-group $K_n[F]$ in terms of explicit generators and relations. We confirm this:

Theorem A (Theorem 3.2.4). *The groups $K_n^{\text{Gr}}[F]$ and $K_n[F]$ are isomorphic.*

For $n = 0$, in Theorem 3.2.1 we construct an explicit map

$$\Phi : K_0^{\text{Gr}}[F] \to K_0[F].$$

An essential step in the proof that Φ is an isomorphism is the explicit description of the isomorphism $K_1^{\text{Gr}}\mathcal{N} \to K_1\mathcal{N}$ given in [Gra12, Corollary 5.10] which is of independent interest, proven in Theorem 3.2.3. Using the key results of [Gra12], we deduce an isomorphism for $n \geq 1$. In [Gra16, Conjecture 1.6], Grayson presents a related conjecture for K-theory spaces which this thesis does not address.

When F is, for example, the base change functor $\mathbf{Proj}_f(R) \to \mathbf{Proj}_f(S)$ arising from some ring homomorphism $f : R \to S$ between commutative rings with unity R and S, we extend the simplicial construction introduced in [HKT17] to define exterior power operations λ^k on the relative K-groups $K_n^{\text{Gr}}[F]$. Contrary to the construction in [HKT17], given a class $[X] \in K_0^{\text{Gr}}[F]$, the element $\lambda^k[X]$ is not simply defined as the class $[\Lambda^k X]$, but this definition needs to be modified in an a priori surprising way, see Lemma 4.2.5. We then prove

Theorem B (Theorem 4.3.5 and Theorem 4.3.4). *The relative K-group $K_n^{\text{Gr}}[F]$ together with the maps*

$$\lambda^k : K_n^{\text{Gr}}[F] \to K_n^{\text{Gr}}[F]$$

for $k \geq 1$ is a λ-ring. Furthermore, this λ-ring structure is compatible with the λ-ring structure on $K_n^{\text{Gr}}\mathcal{M}$ and $K_{n+1}^{\text{Gr}}\mathcal{N}$ via the long exact sequence

$$K_{n+1}^{\text{Gr}}\mathcal{M} \to K_{n+1}^{\text{Gr}}\mathcal{N} \to K_n^{\text{Gr}}[F] \to K_n^{\text{Gr}}\mathcal{M} \to K_n^{\text{Gr}}\mathcal{N}$$

given by Grayson in [Gra16, Corollary 2.3]. Finally, when $n \geq 1$, the map λ^k is a homomorphism.

Proving $K_n^{\text{Gr}}[F]$ together with its operations satisfy the addition and composition axioms is done in a way comparable to the analogous results for $K_n^{\text{Gr}}\mathcal{N}$ in [HKT17, Theorem 6.2, and Theorem 8.18]. The product axiom simply follows from the surprising result that the product axiom of a λ-ring follows from the addition and composition axioms inductively via comparing two different decompositions of the element $\lambda^k(\lambda^2(x + y))$, see Theorem 4.3.2.

We now give a description of the contents of this thesis section by section:

In Section 2.1, we recall the basic definitions and initial results about exact categories required for the discussion of algebraic K-theory, including the additional structure of weak equivalences introduced by Waldhausen.

In Section 2.2, we recall the G-construction of Gillet and Grayson [GG87], a generalisation of this construction for exact categories with weak equivalences introduced by Gunnarsson, Schwänzl, Vogt, and Waldhausen [Gun+92], and the combinatorial description given by Grayson in [Gra12] and [Gra16] of the higher algebraic K-groups and of higher relative algebraic K-groups.

In Section 3.1, we recall the constructions by Hyman Bass [Bas68] of $K_1^{\mathrm{Bass}}\mathcal{M}$ and $K_0^{\mathrm{Bass}}[F]$, construct a homomorphism $\Phi : K_0^{\mathrm{Bass}}[F] \to K_0^{\mathrm{Gr}}[F]$, and prove the following theorem:

Theorem C. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are split exact categories, and $F : \mathcal{M} \to \mathcal{N}$ is a cofinal exact functor. Then the homomorphism $\Phi : K_0^{\mathrm{Bass}}[F] \to K_0^{\mathrm{Gr}}[F]$ is an isomorphism.*

In Section 3.2, we prove Theorem A, namely that Grayson's conjectural presentation of $K_n^{\mathrm{Gr}}[F]$ is isomorphic to the classically defined $K_n[F]$. We provide an explicit map $\Phi : K_0^{\mathrm{Gr}}[F] \to K_0[F]$, and in proving that this map is an isomorphism we also provide a concrete description (see Theorem 3.2.3) of Grayson's isomorphism $K_1^{\mathrm{Gr}}\mathcal{N} \to K_1\mathcal{N}$ [Gra12, Corollary 5.10]. We also recall the presentation $K_1^{\mathrm{Nen}}\mathcal{N}$ given by Nenashev [Nen98] and prove the following theorem answering a question implicitly raised in [Gra12, Remark 8.1].

Theorem D. *The following diagram commutes:*

$$
\begin{array}{ccc}
K_1^{\mathrm{Nen}}\mathcal{N} & \longrightarrow & K_1^{\mathrm{Gr}}\mathcal{N} \\
& \searrow & \downarrow \\
& & K_1\mathcal{N}
\end{array}
$$

where the map $K_1^{\mathrm{Nen}}\mathcal{N} \to K_1^{\mathrm{Gr}}\mathcal{N}$ is the natural one shown to be an isomorphism in [KKW20], and the map $K_1^{\mathrm{Nen}}\mathcal{N} \to K_1\mathcal{N}$ was given in [Nen98].

In Section 4.1, we recall the process of endowing various constructions of $K_1\mathcal{M}$ and $K_0[F]$ with products, when such products are well-defined/nonzero, and we define a product on $K_0^{\mathrm{Gr}}[F]$, making it a commutative ring without a unit.

In Section 4.2, we recall the process of endowing various constructions of $K_1\mathcal{M}$ and $K_0[F]$ with exterior power operations, and we construct exterior power operations on $K_0^{\mathrm{Gr}}[F]$, proving that the usual sum relation applies.

In Section 4.3, we prove our exterior power operations on $K_0^{\mathrm{Gr}}[F]$ have the structure of a λ-ring (without a unit), and generalise this result to $K_n^{\mathrm{Gr}}[F]$, thus finishing the proof of Theorem B.

Chapter 2

Definitions of K-theory

2.1 Categorical Background

For this entire thesis, we will denote arbitrary categories with script letters (i.e. $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ...). For certain named categories we will instead use bold letters (i.e. **Set**, **Ab**, **Grp**,...). In this section, we will recall the basic definitions of exact categories needed to discuss K-theory.

We say that a locally small category $\mathcal{A}$, together with the structure of an abelian group on each hom-set $\text{Hom}_{\mathcal{A}}(A, B)$ is *additive* if composition distributes over the group operation and $\mathcal{A}$ admits all finite products (equivalently given the first condition $\mathcal{A}$ admits all finite coproducts, since every product is a coproduct and vice-versa).

Example 2.1.1. The following are examples of additive categories:

- The category R-**Mod** of modules over a ring R is additive.

- The category of representations $\textbf{Rep}_k(G)$ of a group G over a field k is additive.

- For every additive category $\mathcal{A}$, the arrow category $\mathcal{A}^{\rightarrow}$ is an additive category in the obvious way.

- Given a non-trivial ring R, we can view R as a category with one object where arrows correspond to elements of R. This is not an additive category, as there is no zero object (the only object has at least the morphisms 0 and 1).

A functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories $\mathcal{A}$ and $\mathcal{B}$ is called *additive* if it is an abelian group homomorphism on the hom-sets of $\mathcal{A}$. Equivalently, F is additive if it preserves all product (or coproduct) diagrams. If $\mathcal{A}$ is a subcategory of $\mathcal{B}$ (and $\mathcal{A}$ and $\mathcal{B}$ are both additive), we say that $\mathcal{A}$ is an *additive subcategory* of $\mathcal{B}$ if the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{B}$ is additive.

An additive category $\mathcal{A}$ is an *abelian category* if it admits all kernels and cokernels, and every monomorphism is the kernel of some other morphism and every epimorphism is the cokernel of some other morphism.

Example 2.1.2. Many additive categories we study are also abelian, in particular:

- The categories **Ab**, k-**Vect** and R-**Mod** are abelian.

- If $\mathcal{C}$ is a small category and $\mathcal{A}$ is an abelian category, then the category $\mathcal{A}^{\mathcal{C}}$ of all functors from $\mathcal{C}$ to $\mathcal{A}$ is abelian. In particular, the arrow category $\mathcal{A}^{\rightarrow}$ is the category of functors from the category with two objects and one non-trivial morphism to $\mathcal{A}$, and so is abelian whenever $\mathcal{A}$ is.

- The category of representations $\mathbf{Rep}_k(G)$ of a group G over a field k is abelian.

- The category of torsion-free abelian groups is additive but not abelian, as the monomorphism $\times 2 : \mathbb{Z} \to \mathbb{Z}$ is not the kernel of any morphism.

A sequence

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

in an abelian category $\mathcal{A}$, is *exact at B* if the composition $g \circ f$ is equal to the zero morphism $A \to C$ and the canonical isomorphism $\mathrm{im}(f) \to \ker(g)$ is an isomorphism.

In an additive category $\mathcal{A}$, a pair of composable morphisms

$$M' \xrightarrow{\ \phi\ } M \xrightarrow{\ \psi\ } M''$$

is called a *kernel-cokernel pair* (or *short exact sequence*) if ϕ is the kernel of ψ and ψ is the cokernel of ϕ. In an abelian category it is equivalent to say that the sequence is a kernel-cokernel pair if the sequence

$$0 \longrightarrow M' \xrightarrow{\ \phi\ } M \xrightarrow{\ \psi\ } M'' \longrightarrow 0$$

is exact at M', M and M''.

Example 2.1.3. In any additive category $\mathcal{A}$, any sequence of the form

$$A \xrightarrow{\ \begin{bmatrix} 1 & 0 \end{bmatrix}\ } A \oplus B \xrightarrow{\ \begin{bmatrix} 0 \\ 1 \end{bmatrix}\ } B$$

is a kernel-cokernel pair. We call any sequence isomorphic to such a sequence *split exact* and denote the class of split exact sequences in $\mathcal{A}^{\rightarrow\rightarrow}$ by $\mathcal{E}_{\min}$.

Given abelian categories $\mathcal{A}$ and $\mathcal{B}$, we say that a functor $F : \mathcal{A} \to \mathcal{B}$ is an *exact functor* if whenever there is a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathcal{A}$, then

$$0 \longrightarrow FA \xrightarrow{F(f)} FB \xrightarrow{F(g)} FC \longrightarrow 0$$

is a short exact sequence in $\mathcal{B}$. Further, we say that F is:

1. *left exact* if $F(\sigma)$ is exact at FA and FB

2. *right exact* if $F(\sigma)$ is exact at FB and FC

3. *half exact* if $F(\sigma)$ is exact at FB

whenever $\sigma : 0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{A}$.

Let $\mathcal{A}$ be an additive category, and $\mathcal{E} \subseteq \mathcal{A}^{\to\to}$ be a collection of kernel-cokernel pairs in $\mathcal{A}$. We say that a morphism $f : A \to B$ is an *admissible monomorphism* if it appears as the first morphism in a kernel-cokernel pair in $\mathcal{E}$, and we denote it with an arrow with a tail $f : A \rightarrowtail B$. Dually, we say that $g : B \to C$ is an *admissible epimorphism* if it appears as the second morphism in a kernel-cokernel pair in $\mathcal{E}$, and we denote it with an arrow with two heads $g : B \twoheadrightarrow C$. The pair $(\mathcal{A}, \mathcal{E})$ is called an *exact category* if the class $\mathcal{E}$ is closed under isomorphisms and satisfies the following axioms:

1. For every object $A \in \mathcal{A}$, the identity morphism 1_A is admissible monic.

2. For every object $A \in \mathcal{A}$, the identity morphism 1_A is admissible epic.

3. The class of admissible monics is closed under composition.

4. The class of admissible epics is closed under composition.

5. Given an admissible monic $f : A \rightarrowtail B$ and any morphism $g : A \to C$, the following pushout exists:

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & B \\
{\scriptstyle g}\big\downarrow & & \big\downarrow \\
C & \xdashrightarrow{\;h\;} & D
\end{array}
$$

and the morphism $h : C \rightarrowtail D$ is admissible monic.

6. Given an admissible epic $\phi : Z \twoheadrightarrow W$ and any morphism $\psi : Y \to W$, the following pullback exists:

$$
\begin{array}{ccc}
X & \xdashrightarrow{\;\chi\;} & Y \\
\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\
Z & \xrightarrow{\;\phi\;} & W
\end{array}
$$

and the morphism $\chi : X \to Y$ is admissible epic.

Given exact categories $(\mathcal{A}, \mathcal{E}), (\mathcal{B}, \mathcal{F})$ such that $\mathcal{B}$ is an additive subcategory of $\mathcal{A}$, we say that $(\mathcal{B}, \mathcal{F})$ is an *exact subcategory of* $(\mathcal{A}, \mathcal{E})$ if $\mathcal{F} \subseteq \mathcal{E}$.

Remark 2.1.4. In any exact category $(\mathcal{A}, \mathcal{E})$, every split exact sequence

$$A \xrightarrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} A \oplus B \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} B$$

is contained in $\mathcal{E}$, namely, $\mathcal{E}_{\min} \subseteq \mathcal{E}$.

Remark 2.1.5. Note that the definition of an exact category is dependent on the choice of the exact structure. For instance, an abelian category $\mathcal{A}$ admits the exact structure $\mathcal{E}_{\max}$ of all kernel-cokernel pairs in $\mathcal{A}$, but also the exact structure $\mathcal{E}_{\min}$ of all split exact sequences in $\mathcal{A}$. These structures are not in general equal.

Remark 2.1.6. The concept of an exact category is self-dual, in the sense that if $(\mathcal{A}, \mathcal{E})$ is an exact category, then so is $(\mathcal{A}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}})$.

Example 2.1.7. We have a large collection of examples of exact categories, such as the following:

- Every abelian category $\mathcal{A}$ together with the collection $\mathcal{E}_{\max}$ of all kernel-cokernel pairs in $\mathcal{A}$ is an exact category.

- Every additive category $\mathcal{A}$ is an exact category with the collection $\mathcal{E}_{\min}$ of all split exact sequences.

- For an exact category $(\mathcal{A}, \mathcal{E})$, the category $C\mathcal{A}$ of chain complexes (defined in Section 2.2) is an exact category with exact structure $C\mathcal{E}$. Naturally, the same is true of the category of cochain complexes in $\mathcal{A}$.

- For any exact category $(\mathcal{A}, \mathcal{E})$, the category $\mathcal{A}^2$ of pairs of objects in $\mathcal{A}$ (with morphisms given by pairs of morphisms) is an exact category with exact structure given by $\mathcal{E}^2$.

An exact category $(\mathcal{A}, \mathcal{E})$ is called *split exact* if every short exact sequence in $\mathcal{E}$ splits.

Given exact categories $(\mathcal{A}, \mathcal{E})$ and $(\mathcal{B}, \mathcal{F})$, a functor $F : \mathcal{A} \to \mathcal{B}$ is an *exact functor* if it is an additive functor and further $F(\mathcal{E}) \subseteq \mathcal{F}$. We say that F *reflects exactness* if given $\sigma \in \mathcal{A}^{\to\to}$, if $F(\sigma)$ is in $\mathcal{F}$, then σ is in $\mathcal{E}$.

Given an exact category $(\mathcal{B}, \mathcal{E})$ and a full additive subcategory $\mathcal{A} \subseteq \mathcal{B}$, we say that $\mathcal{A}$ is *closed under extensions* if whenever there is a short exact sequence

$$A' \rightarrowtail B \twoheadrightarrow A''$$

in $\mathcal{B}$ with $A', A'' \in \mathcal{A}$, there is an object $A \in \mathcal{A}$ such that $B \cong A$.

An additive category $\mathcal{A}$ is called *idempotent complete* if for every idempotent morphism $p : A \to A$ (i.e. $p \circ p = p$) there is a decomposition $A \cong K \oplus I$ of A such that p corresponds to $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1_I \end{smallmatrix} \right]$.

Example 2.1.8. Given the ring $R = \mathbb{Q} \times \mathbb{Q}$, we have that the category of free modules over R is an exact category that is not idempotent complete, as there is no decomposition of the above form of for the idempotent homomorphism $1 \times 0 : R \to R$

Given an exact category $(\mathcal{A}, \mathcal{E})$, a *category with weak equivalences in $\mathcal{A}$* is a subcategory $w\mathcal{A}$ of $\mathcal{A}$ satisfying the following three axioms:

1. The isomorphisms in $\mathcal{A}$ are contained in $w\mathcal{A}$ (in particular $w\mathcal{A}$ contains all the objects of $\mathcal{A}$).

2. If there is a commutative diagram

$$
\begin{array}{ccccc}
C & \xleftarrow{\ \phi\ } & A & \xrightarrowtail{\ \psi\ } & B \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
C' & \xleftarrow{\ \phi'\ } & A' & \xrightarrowtail{\ \psi'\ } & B'
\end{array}
$$

   in $\mathcal{A}$, where f, g and h are in $w\mathcal{A}$ and ψ and ψ' are admissible monomorphisms then the induced map on the pushouts of each row is also in $w\mathcal{A}$.

3. If there is a commutative diagram

$$
\begin{array}{ccccc}
A & \xrightarrowtail{\ \phi\ } & B & \xtwoheadrightarrow{\ \psi\ } & C \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
A' & \xrightarrowtail{\ \phi'\ } & B' & \xtwoheadrightarrow{\ \psi'\ } & C'
\end{array}
$$

   in $\mathcal{A}$ with short exact rows, where the morphisms f and h are in $w\mathcal{A}$, then g is also in $w\mathcal{A}$.

We call the maps in $w\mathcal{A}$ *weak equivalences*, and we call the triple $(\mathcal{A}, \mathcal{E}, w\mathcal{A})$ a *Waldhausen exact category*. We also denote by $\mathcal{A}^w$ the full subcategory of $\mathcal{A}$ consisting of objects A such that $0 \rightarrowtail A$ is in $w\mathcal{A}$.

Remark 2.1.9. Given any Waldhausen exact category $(\mathcal{A}, \mathcal{E}, w\mathcal{A})$, the subcategory $\mathcal{A}^w$ is an exact subcategory of $(\mathcal{A}, \mathcal{E})$ since it is closed under extensions by the third axiom above, and we denote its exact structure by $\mathcal{E}^w$.

Background in this section is adapted from [Büh10] and [Gra12].

2.2   Constructions of K-theory

From this section onward, we will suppress the choice of exact structure for exact categories and simply denote exact categories by the underlying additive category.

Given an essentially small (or equivalently under the axiom of choice, skeletally small) exact category $\mathcal{A}$, the *Grothendieck group of $\mathcal{A}$* is presented as the abelian group generated by the isomorphism classes of $\mathcal{A}$ modulo a relation $[M] = [M'] + [M'']$ whenever there is a short exact sequence

$$M' \rightarrowtail M \twoheadrightarrow M''$$

in $\mathcal{A}$. We denote the Grothendieck group of $\mathcal{A}$ by $K_0\mathcal{A}$.

Let $(\mathcal{A}, w\mathcal{A})$ be a Waldhausen exact category. We define the *Grothendieck group of $(\mathcal{A}, w\mathcal{A})$* as the abelian group generated by the isomorphism classes of objects in $\mathcal{A}$ modulo the following relations:

1. $[M] = [M']$ whenever there is a weak equivalence $M \to M'$,

2. $[M] = [M'] + [M'']$ whenever $M' \rightarrowtail M \twoheadrightarrow M''$ is a short exact sequence in $\mathcal{A}$.

In order for this definition to be well-defined, we must assume the class of weak equivalences forms a set.

Suppose we have a commutative monoid $(M, +)$, then we define the *group completion of M* in the following way: we take the monoid $M \times M$, where addition is given by

$$(a, b) + (c, d) = (a + c, b + d).$$

We define an equivalence relation $\sim$ on $M \times M$ such that $(a, b) \sim (c, d)$ whenever there is some element $m \in M$ such that $a + d + m = b + c + m$. We then define the group completion as the quotient $M \times M / \sim$, where a standard argument shows this is in fact a group.

This definition allows us to characterise the Grothendieck group of a split exact category. Given a split exact category $\mathcal{A}$, the set of isomorphism classes of $\mathcal{A}$, together with direct sum, forms a monoid with identity given by the zero object. Then the *Grothendieck group of $\mathcal{A}$* is the group completion of this monoid.

The following construction of K-theory involves the construction of a topological space using so-called simplicial objects, which we will recall now. Let Δ denote the *simplex category*, whose objects are the totally ordered sets

$$[n] = \{0 < 1 < \dots < n - 1 < n\}, \, n \in \mathbb{N}$$

and whose morphisms are the non-decreasing functions. Given any category $\mathcal{C}$, a *simplicial object in $\mathcal{C}$* is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. The category of such simplicial objects in $\mathcal{C}$ is denoted by $\mathcal{C}^{\Delta^{\mathrm{op}}}$. For $X \in \mathcal{C}^{\Delta^{\mathrm{op}}}$, we denote the image of $[n]$ by X_n.

Given an exact category $\mathcal{A}$, following [GG87], we will define the simplicial category $G_\bullet\mathcal{A}$ as follows. Given some $n \geq 0$, the objects of the category $G_n\mathcal{A}$ are diagrams in $\mathcal{A}$ of the following form:

$$
\begin{array}{ccccccccc}
A_0 & \rightarrowtail & A_1 & \rightarrowtail & A_2 & \rightarrowtail & \cdots & \rightarrowtail & A_n \\
& & & & & & & & \\
B_0 & \rightarrowtail & B_1 & \rightarrowtail & B_2 & \rightarrowtail & \cdots & \rightarrowtail & B_n \\
& & & & & & & & \\
& & C_{01} & \rightarrowtail & C_{02} & \rightarrowtail & \cdots & \rightarrowtail & C_{0n} \\
& & & & C_{12} & \rightarrowtail & \cdots & \rightarrowtail & C_{1n} \\
& & & & & \ddots & & & \vdots \\
& & & & & & & & C_{nn}
\end{array}
$$

where $A_i \rightarrowtail A_j \twoheadrightarrow C_{ij}$ and $B_i \rightarrowtail B_j \twoheadrightarrow C_{ij}$ are exact for all $0 \leq i < j \leq n$. In other words, an object of $G_n\mathcal{A}$ is a pair of n-step admissible filtrations in $\mathcal{A}$ (A_i and B_i) which agree on all successive subquotients, including the data of the choice of said subquotients. A morphism in this category is a map of such diagrams. If $\mathcal{A}$ is equipped with a subcategory $w\mathcal{A}$ of weak equivalences in the sense of [Gra12], we can equip $G_n\mathcal{A}$ with a subcategory $wG_n\mathcal{A}$ of weak equivalences consisting of morphisms that are level-wise weak equivalences in $w\mathcal{A}$. These subcategories themselves form a simplicial category $wG_\bullet\mathcal{A}$, which we can use to create a topological space by taking the nerve to form a bisimplicial set which we then take the diagonal of and apply geometric realisation, which we denote as follows:

$$
|wG\mathcal{A}| := |\mathrm{diag}(N_\bullet wG_\bullet\mathcal{A})|.
$$

This construction is outlined in [Gun+92], where it is verified that when the weak equivalences are just the isomorphisms $i\mathcal{A} \subseteq \mathcal{A}$, the space $|iG\mathcal{A}|$ is homotopy equivalent to the classically defined *K*-theory space of $\mathcal{A}$. For the purposes of this thesis, we will use this space as the model for the *K*-theory space, namely:

$$
K\mathcal{A} := |iG\mathcal{A}|.
$$

We will now also give meaning to the nth K-group of $\mathcal{A}$, which we define as the nth homotopy group of the K-theory space of $\mathcal{A}$, namely:
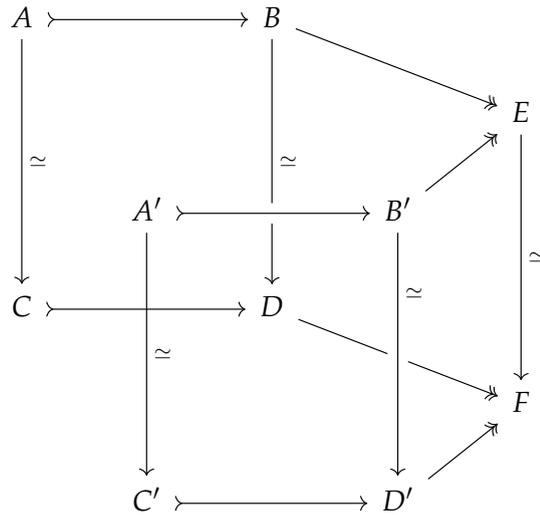
$$K_n\mathcal{A} := \pi_n K\mathcal{A}.$$

In [Gun+92, Theorem 2.6], they also verify that this construction is homotopy equivalent to the loop space of the space obtained from Waldhausen's $S_\bullet$ construction given in [Wal85], namely:

$$|wG\mathcal{A}| \simeq |wS_\bullet\mathcal{A}|$$

and so we will write $Kw\mathcal{A} := |wG\mathcal{A}|$ for the K-theory space of any exact category $\mathcal{A}$ with weak equivalences $w\mathcal{A}$. This result allows us to apply the result given in [Wei13, Theorem V.2.2] that the K-theory of the category of bounded chain complexes with weak equivalences given by quasi-isomorphisms is the same as regular K-theory, namely:

$$K\mathcal{A} \simeq KqC\mathcal{A}.$$

Remark 2.2.1. A 1-simplex of the space $K\mathcal{A} = |iG\mathcal{A}|$ (namely, an element of $N_1 wG_1\mathcal{A}$) consists of the following data:

for this 1-simplex we write $\left(\begin{smallmatrix}A\\A'\end{smallmatrix}\right) \to \left(\begin{smallmatrix}D\\D'\end{smallmatrix}\right)$. Now, using this data, we write

$$\left(\begin{smallmatrix}A\\A'\end{smallmatrix}\right) \rightarrowtail \left(\begin{smallmatrix}B\\B'\end{smallmatrix}\right)$$

for the 1-simplex of $Kw\mathcal{A}$ arising from degenerating an element of $N_0 wG_1\mathcal{A}$, and we write

$$\left(\begin{smallmatrix}A\\A'\end{smallmatrix}\right) \xrightarrow{\simeq} \left(\begin{smallmatrix}C\\C'\end{smallmatrix}\right)$$

for the 1-simplex in $Kw\mathcal{A}$ arising from degenerating an element of $N_1 wG_0\mathcal{A}$. It turns out that any 1-simplex is homotopic to a concatenation of one of each of the above two

kinds of 1-simplices. This homotopy is denoted by the following diagram:

$$
\begin{array}{ccc}
\binom{A}{A'} & \rightarrowtail & \binom{B}{B'} \\
\downarrow \simeq & \searrow & \downarrow \simeq \\
\binom{C}{C'} & \rightarrowtail & \binom{D}{D'}
\end{array}
$$

where the two triangles bound 2-simplices arising from face and degeneracy maps in the bisimplicial set to obtain non-degenerate simplices in the diagonal. Setting $\binom{A}{A'} = \binom{B}{B'} = \binom{C}{C'}$, we obtain the following diagram:

$$
\binom{A}{A'} \underset{\longrightarrow}{\overset{\simeq}{\longrightarrow}} \binom{D}{D'}
$$

giving us a homotopy between 1-simplices of different types between two set 0-simplices.

Given an exact functor $F : \mathcal{M} \to \mathcal{N}$ between exact categories $\mathcal{M}$ and $\mathcal{N}$ with weak equivalences $w\mathcal{M}$ and $v\mathcal{N}$ such that $F(w\mathcal{M}) \subseteq v\mathcal{M}$, we define the relative K-theory space of F as the homotopy fiber of the map $Kw\mathcal{M} \to Kv\mathcal{N}$ induced by F, which we denote as follows:

$$
K[w\mathcal{M} \xrightarrow{F} v\mathcal{N}] := \mathrm{hofib}(Kw\mathcal{M} \xrightarrow{KF} Kv\mathcal{N}).
$$

Note that this is the same notation as used in [Gra12], but we use the homotopy fiber rather than the cofiber as we are not in the situation of spectra. Unless explicitly stated, we will operate in the situation where the weak equivalences are just isomorphisms. Similarly to the absolute case, taking the nth homotopy group of this space we define the nth relative K-group of F as

$$
K_n[F] := \pi_n K[F].
$$

Since $K_n[F]$ is defined using homotopy groups of a homotopy fiber, we obtain a long exact sequence

$$
\cdots \longrightarrow K_{n+1}\mathcal{M} \longrightarrow K_{n+1}\mathcal{N} \longrightarrow K_n[F] \longrightarrow K_n\mathcal{M} \longrightarrow K_n\mathcal{N} \longrightarrow \cdots
$$

arising from the homotopy fiber sequence

$$
K[F] \longrightarrow K\mathcal{M} \longrightarrow K\mathcal{N}
$$

We will now outline the presentation of $K_n\mathcal{N}$ for any exact category $\mathcal{N}$ given $n \geq 1$ outlined in [Gra12], which relies on the definition of chain complexes. A *chain complex*

in an additive category $\mathcal{A}$ is a diagram $(A_\bullet, a_\bullet)$:

$$\cdots \longrightarrow A_{n+1} \xrightarrow{a_{n+1}} A_n \xrightarrow{a_n} A_{n-1} \longrightarrow \cdots$$

subject to the condition that $a_n \circ a_{n+1} = 0$ for every $n \in \mathbb{Z}$. We call $A_\bullet$ the *objects of* $(A_\bullet, a_\bullet)$, and we call $a_\bullet$ the *differential of* $(A_\bullet, a_\bullet)$. We will often simply write $(A_\bullet, a_\bullet)$ as A when it is clear that A is a chain complex. A morphism of chain complexes is called a *chain map*, and is a map of such diagrams. In particular, given chain complexes $(A_\bullet, a_\bullet)$ and $(B_\bullet, b_\bullet)$, a graded morphism $f_\bullet : A_\bullet \to B_\bullet$ is a chain map if the following diagram commutes:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & A_{n+1} & \xrightarrow{a_{n+1}} & A_n & \xrightarrow{a_n} & A_{n-1} & \longrightarrow \cdots \\
& \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & \\
\cdots \longrightarrow & B_{n+1} & \xrightarrow{b_{n+1}} & B_n & \xrightarrow{b_n} & B_{n-1} & \longrightarrow \cdots
\end{array}$$

In other words, for every $n \in \mathbb{Z}$, we have that $f_{n-1} \circ a_n = b_n \circ f_n$. We say that a chain complex $(A_\bullet, d_\bullet)$ is *bounded* if $A_n = 0$ for all but finitely many $n \in \mathbb{Z}$. We say that A is *supported in non-negative degrees* if $A_n = 0$ for $n < 0$. We will denote by $C\mathcal{A}$ the category of bounded chain complexes in $\mathcal{A}$ that are supported in non-negative degrees. If $\mathcal{A}$ is equipped with the structure of an exact category there is a natural exact category structure that can we can give $C\mathcal{A}$, where admissible monics and admissible epics are given pointwise. Note that this definition is non-standard, but we will show that using this definition will yield isomorphic K-groups.

Given a bounded chain complex $A = (A_\bullet, a_\bullet)$ in an exact category $\mathcal{A}$, the *Euler characteristic* $\chi(A)$ is the alternating sum

$$\chi(A) = \sum_{n \in \mathbb{Z}} (-1)^n [A_n]$$

in $K_0 \mathcal{A}$.

The definition of the category of chain complexes in an exact category $\mathcal{A}$ can be iterated. Namely, we can build a chain complex in the category $C\mathcal{A}$. We call such a complex a *double complex*, which we can equivalently describe as a commutative

diagram of the following form, where every row and column is a complex:

$$
\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & A_{n+1,m+1} & \longrightarrow & A_{n,m+1} & \longrightarrow & A_{n-1,m+1} & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & A_{n+1,m} & \longrightarrow & A_{n,m} & \longrightarrow & A_{n-1,m} & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & A_{n+1,m-1} & \longrightarrow & A_{n,m-1} & \longrightarrow & A_{n-1,m-1} & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
& \vdots & & \vdots & & \vdots &
\end{array}
$$

Naturally, this definition can be iterated to create an *n-dimensional multicomplex*. In particular, a 1-dimensional multicomplex is a chain complex, and a $(n+1)$-dimensional multicomplex is a chain complex in the category of n-dimensional multicomplexes. We denote the category of bounded n-dimensional multicomplexes in $\mathcal{A}$ by $C^n \mathcal{A}$.

Given a chain map $f : (A_\bullet, a_\bullet) \to (B_\bullet, b_\bullet)$ between chain complexes $(A_\bullet, a_\bullet)$ and $(B_\bullet, b_\bullet)$, the *mapping cone* $\mathrm{cone}(f)$ is the chain complex given by

$$
\mathrm{cone}(f)_n := A_{n-1} \oplus B_n \text{ with differential } d_n^f := \begin{bmatrix} -a_{n-1} & 0 \\ -f_{n-1} & b_n \end{bmatrix}.
$$

A fairly straightforward computation verifies that the mapping cone is a complex, and it is fairly easy to see that the mapping cone is a functor from $(C\mathcal{A})^\to$ to $C\mathcal{A}$.

The *shift functor*

$$
-[1] : C\mathcal{A} \to C\mathcal{A}
$$

is defined such that $A[1] := \mathrm{cone}(A \to 0)$. More explicitly, $A[1]$ is the complex with objects $(A[1])_n = A_{n+1}$ and differentials $(a[1])_n = -a_{n+1}$. If f is a chain map, its shift is given by $f[1]_n := f_{n+1}$. Clearly, the shift functor is an additive automorphism of $C\mathcal{A}$.

A chain map $f : A \to B$ is *chain homotopic to zero*, or *null homotopic* if there exist morphisms $h_n : A_n \to B_{n+1}$ such that $f_n = a_{n+1} \circ h_n + h_{n+1} \circ b_n$, see the following diagram:

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & A_{n+1} & \xrightarrow{a_{n+1}} & A_n & \xrightarrow{a_n} & A_{n-1} & \longrightarrow & \cdots \\
& {\scriptstyle f_{n+1}}\downarrow & {\scriptstyle h_n}\swarrow & {\scriptstyle f_n}\downarrow & {\scriptstyle h_{n-1}}\swarrow & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots \longrightarrow & B_{n+1} & \xrightarrow[b_{n+1}]{} & B_n & \xrightarrow[b_n]{} & B_{n-1} & \longrightarrow & \cdots
\end{array}
$$

We call $h : A[1] \to B$ a *chain homotopy*, and we say that a chain complex A is *null homotopic* if the chain map 1_A is chain homotopic to zero.

Two chain maps $f, g : A \to B$ are *homotopic* if $f - g$ is chain homotopic to zero, in which case we write $f \simeq g$. We say that two complexes are *homotopy equivalent* if there are chain maps $f : A \to B$ and $g : B \to A$ such that $g \circ f \simeq 1_A$ and $f \circ g \simeq 1_B$. In this case, we write $A \simeq B$.

When $\mathcal{A}$ is equipped with an exact structure, a chain complex A in $C\mathcal{A}$ is called *acyclic* or *long exact* if each differential factors as $A_n \twoheadrightarrow Z_n \rightarrowtail A_{n-1}$ in $\mathcal{A}$ such that $Z_{n+1} \rightarrowtail A_n \twoheadrightarrow Z_n$ is a short exact sequence. We can picture this setup in the following diagram:

$$
\begin{array}{ccccccccc}
 & & Z_{n+1} & & & & & & Z_{n-1} \\
 & \nearrow & & \searrow & & & & \nearrow & \\
\cdots \longrightarrow & A_{n+1} & \longrightarrow & & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \cdots \\
 \nearrow & & & & & \searrow & \nearrow & & \\
Z_{n+2} & & & & & & Z_n & &
\end{array}
$$

A chain map $f : A \to B$ is called a *quasi-isomorphism* if its mapping cone is homotopy equivalent to an acyclic complex. We have a subcategory of weak equivalences $qC\mathcal{A}$ of $C\mathcal{A}$ whose morphisms are all quasi-isomorphisms in $C\mathcal{A}$. We will denote the category of chain complexes A for which $0 \rightarrowtail A$ is a quasi-isomorphism by $C^q\mathcal{A}$. This category is then precisely the subcategory of acyclic chain complexes in $C\mathcal{A}$.

A *binary chain complex* in an additive category $\mathcal{A}$ is a triple $(A_\bullet, a_\bullet, a'_\bullet)$ where $a_\bullet$ and $a'_\bullet$ both form differentials of $A_\bullet$ to create chain complexes. A binary complex $(A_\bullet, a_\bullet, a'_\bullet)$ can be realised as a diagram

$$
\cdots \rightrightarrows A_{n+1} \underset{a'_{n+1}}{\overset{a_{n+1}}{\rightrightarrows}} A_n \underset{a'_n}{\overset{a_n}{\rightrightarrows}} A_{n-1} \rightrightarrows \cdots
$$

together with the condition that $a_{n+1} \circ a_n = 0$ and $a'_{n+1} \circ a'_n = 0$. A morphism of binary complexes is a chain map for both differentials, i.e. a map

$$
f_\bullet : (A_\bullet, a_\bullet, a'_\bullet) \to (B_\bullet, b_\bullet, b'_\bullet)
$$

such that $f_{n-1} \circ a_n = b_n \circ f_n$ and $f_{n-1} \circ a'_n = b'_n \circ f_n$. A binary complex $(A_\bullet, a_\bullet, a'_\bullet)$ is *bounded* if $A_n = 0$ for all but finitely many $n \in \mathbb{Z}$. We denote the category of bounded binary complexes in $\mathcal{A}$ by $B\mathcal{A}$. We define *n-dimensional binary chain multicomplexes* in the same way we do for *n*-dimensional chain complexes, and we denote the category of bounded *n*-dimensional binary chain multicomplexes in $\mathcal{A}$ by $B^n\mathcal{A}$. Assuming

further that $\mathcal{A}$ is an exact category, we have the following exact functors:

$$\Delta : C\mathcal{A} \to B\mathcal{A} \text{ defined by } (A, a) \mapsto (A, a, a)$$
$$\top : B\mathcal{A} \to C\mathcal{A} \text{ defined by } (A, a, a') \mapsto (A, a)$$
$$\bot : B\mathcal{A} \to C\mathcal{A} \text{ defined by } (A, a, a') \mapsto (A, a')$$
$$\tau : B\mathcal{A} \to B\mathcal{A} \text{ defined by } (A, a, a') \mapsto (A, a', a)$$
$$\mathbb{I} := (\top, \bot) : B\mathcal{A} \to C\mathcal{A}^2 \text{ defined by } (A, a, a') \mapsto ((A, a), (A, a'))$$

where we are careful to note here that by $C\mathcal{A}^2$ we mean the category of pairs of chain complexes in $\mathcal{A}$, rather than the category of double complexes, which as stated previously we denote by $C^2\mathcal{A}$.

A binary chain complex $A = (A, a, a')$ is called *acyclic* if both $\top A$ and $\bot A$ are acyclic as chain complexes. We denote the category of bounded acyclic chain complexes in $\mathcal{A}$ by $B^q\mathcal{A}$. Since $B^q\mathcal{A}$ can be endowed with an exact structure inherited from the exact structure of $\mathcal{A}$, this definition again iterates to obtain *bounded n-dimensional acyclic binary chain complexes* in $\mathcal{A}$, the category of which we denote by $(B^q)^n\mathcal{A}$. We say that a binary complex A is *diagonal* if $\top A = \bot A$, i.e. that A is in the image of $\Delta : C\mathcal{A} \to B\mathcal{A}$. Further, we say that an n-dimensional binary chain multicomplex is *diagonal* if any pair of its differentials in the same direction are equal. Given a binary chain map $f : A \to B$ in $B\mathcal{A}$, we say that f is a *homotopy equivalence* (or *quasi-isomorphism*) if both $\top f : \top A \to \top B$ and $\bot f : \bot A \to \bot B$ are homotopy equivalences (resp. quasi-isomorphisms) in $C\mathcal{A}$. We then define $qB\mathcal{A}$ in the same way as we have for $qC\mathcal{A}$.

Using these binary complexes, we can now outline the presentation of $K_n\mathcal{A}$ given in [Gra12]. Given $n \geq 1$, the group $K_n^{\mathrm{Gr}}\mathcal{A}$ has the following presentation: $K_n^{\mathrm{Gr}}\mathcal{A}$ is the abelian group generated by the isomorphism classes of $(B^q)^n\mathcal{A}$ and has two families of relations:

1. Whenever there is a short exact sequence

$$0 \to N' \to N \to N'' \to 0$$

   in $(B^q)^n\mathcal{A}$, there is a relation $[N] = [N'] + [N'']$.

2. Whenever an object D is an object in $(B^q)^n\mathcal{A}$ that is in the image of some functor Δ (i.e. its top and bottom differentials agree in at least one direction), there is a relation $[D] = 0$. We call such an object *diagonal*.

In [Gra16], Grayson gives a conjectural presentation of the relative algebraic K-groups. That is, given an exact functor $F : M \to \mathcal{N}$, Grayson defines categories $C[F]$ and $B[F]$ in the following way: the objects of $C[F]$ consists of triples (M, N, u) with $M \in C\mathcal{M}, N \in C\mathcal{N}$, and $u : FM \to N$ is a quasi-isomorphism. A morphism

$(M', N', u') \to (M, N, u)$ consists of a pair (ϕ, ψ) of morphisms $\phi : M' \to M$ and $\psi : N' \to N$ such that $u \circ F\phi = \psi \circ u'$, in other words the following diagram commutes:

$$
\begin{array}{ccc}
FM' & \xrightarrow{F\phi} & FM \\
\downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} \\
N' & \xrightarrow{\psi} & N
\end{array}
$$

Similarly, the objects of $B[F]$ consist of triples (M, N, u) with $M \in C\mathcal{M}^2$ and $N \in B\mathcal{N}$, and $u : FM \to \mathbb{I}N$ is a quasi-isomorphism. A morphism $(M', N', u') \to (M, N, u)$ is a pair (ϕ, ψ) of morphisms $\phi : M' \to M$ and $\psi : N' \to N$ such that $u \circ F\phi = \mathbb{I}\psi \circ u'$, in other words the following diagram commutes:

$$
\begin{array}{ccc}
FM' & \xrightarrow{F\phi} & FM \\
\downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} \\
\mathbb{I}N' & \xrightarrow{\mathbb{I}\psi} & \mathbb{I}N
\end{array}
$$

The categories $C[F]$ and $B[F]$ are exact, with admissible monics and epics given pointwise, and can be equipped with subcategories of weak equivalences. A morphism $(\phi, \psi) : (M', N', u') \to (M, N, u)$ in $C[F]$ is a weak equivalence if $\phi : M' \to M$ is a quasi-isomorphism and $\psi : N' \to N$ is an isomorphism. The same is true of morphisms in $B[F]$. We denote the subcategory of weak equivalences in $C[F]$ (resp. $B[F]$) by $pC[F]$ (resp. $pB[F]$). Following [Gra16], we then denote Grayson's presentation of $K_0[F]$ by $K_0^{\mathrm{Gr}}[F]$, which is defined as follows:

$$
K_0^{\mathrm{Gr}}[F] := \mathrm{coker}(K_0(C[F], pC[F]) \xrightarrow{\Delta} K_0(B[F], pB[F]))
$$

where $\Delta : C[F] \to B[F]$ is the functor

$$
(A, (N, d), u) \mapsto \left(\left({\textstyle \begin{smallmatrix} A \\ A \end{smallmatrix}} \right), \left(N, {\textstyle \begin{smallmatrix} d \\ d \end{smallmatrix}} \right), \left({\textstyle \begin{smallmatrix} u \\ u \end{smallmatrix}} \right) \right).
$$

In order to introduce Grayson's presentation of the higher relative K-groups of F, we first need to give meaning to a split cube in a given category, which we will define below:

Definition 2.2.2. Given a category $\mathcal{A}$, an object $[f]$ of the arrow category $\mathcal{A}^{\to}$ together with a choice of a map g such that $g \circ f = 1$ is called *split*, and we say that g is a splitting of f. We will now give meaning to the category $Q^n \mathcal{A}$ of so-called *split n-cubes in $\mathcal{A}$*. Firstly, $Q^0 \mathcal{A} := \mathcal{A}$, i.e. a split 0-cube in $\mathcal{A}$ is an object of $\mathcal{A}$. A split $(n+1)$-cube in $\mathcal{A}$, namely an object of $Q^{n+1}\mathcal{A}$, is a split object $[A' \to A]$ in the arrow category $(Q^n \mathcal{A})^{\to}$ of split n-cubes in $\mathcal{A}$. A morphism in $Q^{n+1}\mathcal{A}$ is a map $[A' \to A] \to [B' \to B]$ which commutes with the chosen splittings.

Given a functor $G : \mathcal{A} \to \mathbf{Ab}$ from $\mathcal{A}$ to the category of abelian groups, we can recursively extend G to a functor $G_{n+1} : Q^{n+1}\mathcal{A} \to \mathbf{Ab}$ from the category of split $(n+1)$-cubes to $\mathbf{Ab}$. Given a split $(n+1)$-cube $[A' \to A]$ in $Q^{n+1}\mathcal{A}$, we define

$$G_{n+1}[A' \to A] := \mathrm{coker}(G_n A' \to G_n A),$$

or equivalently

$$G_{n+1}[A' \to A] = \ker(G_n A \to G_n A')$$

where the map $G_n A \to G_n A'$ is induced by the choice of splitting of the map $A' \to A$.

The categories of n-dimensional multicomplexes, together with diagonal functors Δ provide a key example of a split n-cube in the category of exact categories, which we will denote by $\Omega^n \mathcal{M}$. Namely, we take as vertices the categories $D_1 D_2 \cdots D_n \mathcal{M}$, where $D_i \in \{C^q, B^q\}$ for each $i = 1, ..., n$, and every morphism is the relevant functor Δ. We take as splittings the corresponding functor $\perp$. This is an arbitrary choice, as we could have instead taken $\top$ as a splitting. As an example, the following is a diagram representing this split n-cube in the case of $n = 2$:

$$
\begin{array}{ccc}
(C^q)^2 \mathcal{M} & \xrightarrow{\ \Delta\ } & C^q B^q \mathcal{M} \\
\Delta \downarrow & & \downarrow \Delta \\
B^q C^q \mathcal{M} & \xrightarrow{\ \Delta\ } & (B^q)^2 \mathcal{M}
\end{array}
$$

Using the construction discussed above with the functor $G = K_0$ from the category of exact categories to the category of Abelian groups applied to the split n-cube of exact categories we have just introduced we obtain the group we will denote by $K_n^{\mathrm{Gr}}(\mathcal{M})$ introduced in [Gra12]. This group can be generated by isomorphism classes of objects in the category $(B^q)^n \mathcal{M}$ subject to a relation $[M] = [M'] + [M'']$ whenever there is a short exact sequence

$$M' \rightarrowtail M \twoheadrightarrow M''$$

and a relation $[D] = 0$ whenever D is a diagonal object of $(B^q)^n \mathcal{M}$, namely, whenever D is in the image of any of the above functors labelled by Δ.

Now once again recall our setup of an exact functor $F : \mathcal{M} \to \mathcal{N}$ between exact categories, and consider the object $[F : \mathcal{M} \to \mathcal{N}] =: [F]$ in the arrow category of the category of exact categories associated to F. For $n \geq 0$, we denote $\Omega^n[F]$ by the split n-cube in the arrow category of the category of exact categories whose vertices are the arrows $[D_1 D_2 \cdots D_n \mathcal{M} \xrightarrow{F} D_1 D_2 \cdots D_n \mathcal{N}]$ with $D_i \in \{C^q, B^q\}$ for $i = 1, ..., n$, with morphisms given by Δ and splittings $\perp$. As before in the absolute case, this is an arbitrary choice where we could have chosen $\top$ instead. As this definition is fairly

involved we will give two examples to make it clearer. Firstly, for $n = 1$, we will give the following diagram to represent $\Omega^1[F]$:

$$
\begin{array}{ccc}
& \left[C^q\mathcal{M} \xrightarrow{F} C^q\mathcal{N} \right] & \\
\perp \left(\begin{smallmatrix} \nearrow \\ \\ \searrow \end{smallmatrix} \right. \Big\downarrow \Delta & & \Delta \Big\downarrow \left. \begin{smallmatrix} \nwarrow \\ \\ \searrow \end{smallmatrix} \right) \perp \\
& \left[B^q\mathcal{M} \xrightarrow{F} B^q\mathcal{N} \right] &
\end{array}
$$

And now for $n = 2$, we have the following diagram to represent $\Omega^2[F]$:

Considering $K_0^{\mathrm{Gr}}[F]$, introduced at the end of section 1, as a functor from $\mathbf{Exact}^{\to}$ to $\mathbf{Ab}$, we again recursively extend K_0^{Gr} to a functor $Q^n\mathbf{Exact}^{\to} \to \mathbf{Ab}$. Applying this functor to the object $\Omega^n[F]$ of $Q^n\mathbf{Exact}^{\to}$, we obtain the group $K_n^{\mathrm{Gr}}[F]$.

This definition is a mouthful, so we will slowly unwrap it for $n = 1$. First, we apply $K_0^{\mathrm{Gr}}[-]$ to the vertices of the split 1-cube $\Omega^1[F]$. Then, $K_1^{\mathrm{Gr}}[F]$ is the cokernel of the induced homomorphism between the two vertices described below:

$$
\begin{array}{c}
K_0^{\mathrm{Gr}} \left[C^q\mathcal{M} \xrightarrow{F} C^q\mathcal{N} \right] \\
\Big\downarrow K_0^{\mathrm{Gr}}(\Delta) \\
K_0^{\mathrm{Gr}} \left[B^q\mathcal{M} \xrightarrow{F} B^q\mathcal{N} \right]
\end{array}
$$

More explicitly, since $K_0^{\mathrm{Gr}}[-]$ is also defined using cokernels, we obtain $K_1^{\mathrm{Gr}}[F]$ by taking cokernels in both directions of the following commutative diagram:

$$
\begin{array}{ccc}
K_0 pC \left[C^q\mathcal{M} \xrightarrow{F} C^q\mathcal{N} \right] & \xrightarrow{K_0(\Delta)} & K_0 pB \left[C^q\mathcal{M} \xrightarrow{F} C^q\mathcal{N} \right] \\
\Big\downarrow K_0(\Delta) & & \Big\downarrow K_0(\Delta) \\
K_0 pC \left[B^q\mathcal{M} \xrightarrow{F} B^q\mathcal{N} \right] & \xrightarrow{K_0(\Delta)} & K_0 pB \left[B^q\mathcal{M} \xrightarrow{F} B^q\mathcal{N} \right]
\end{array}
$$

Since in all the above arrows Δ is split by $\perp$, we obtain the same group by taking kernels in both directions of the following commutative diagram:

$$K_0 pC \left[C^q \mathcal{M} \xrightarrow{F} C^q \mathcal{N} \right] \xleftarrow{K_0(\perp)} K_0 pB \left[C^q \mathcal{M} \xrightarrow{F} C^q \mathcal{N} \right]$$

$$\uparrow K_0(\perp) \qquad\qquad\qquad \uparrow K_0(\perp)$$

$$K_0 pC \left[B^q \mathcal{M} \xrightarrow{F} B^q \mathcal{N} \right] \xleftarrow{K_0(\perp)} K_0 pB \left[B^q \mathcal{M} \xrightarrow{F} B^q \mathcal{N} \right]$$

In [Gra16, Corollary 2.3], Grayson gives us a long exact sequence

$$\cdots \longrightarrow K_{n+1}^{\mathrm{Gr}} \mathcal{M} \longrightarrow K_{n+1}^{\mathrm{Gr}} \mathcal{N} \longrightarrow K_n^{\mathrm{Gr}}[F] \longrightarrow K_n^{\mathrm{Gr}} \mathcal{M} \longrightarrow K_n^{\mathrm{Gr}} \mathcal{N} \longrightarrow \cdots$$

The map $K_0^{\mathrm{Gr}}[F] \to K_0 \mathcal{M}$ is given by $[(M, N, u)] \mapsto \chi(\top M) - \chi(\perp M)$, and the map $K_1^{\mathrm{Gr}} \mathcal{N} \to K_0^{\mathrm{Gr}}[F]$ is given by $[N] \mapsto [(0, N, 0)]$.

For the proof of our next result we must first recall the definition of a so-called *binary ladder* defined in [KKW20]. Given an exact category $\mathcal{N}$, a binary ladder is a quadruple (A, B, s, t) consisting of binary complexes A, B in $B\mathcal{N}$ together with isomorphisms $s : \top A \to \top B$ and $t : \perp A \to \perp B$. Given any binary ladder (A, B, s, t) whose complexes are acyclic, we have the following identity in $K_1^{\mathrm{Gr}} \mathcal{N}$ shown in the proof of [Gra16, Lemma 8.5]:

$$[B] - [A] = \sum_{i=0}^{\infty} (-1)^i \left[A_i \begin{array}{c} \xrightarrow{s_i} \\ \xrightarrow{t_i} \end{array} B_i \right]$$

Proposition 2.2.3. *Given an object*

$$X = \left(\left({}^A_B \right), \left(N, {}^d_{d'} \right), \left({}^u_v \right) \right)$$

of $B[F]$, we have that $[\tau X] = -[X]$ in $K_0^{\mathrm{Gr}}[F]$.

Proof. We will show that the element $[X \oplus \tau X]$ is equal to zero. To this end, note that the map $K_0^{\mathrm{Gr}}[F] \to K_0 \mathcal{M}$ sends $[X \oplus \tau X]$ to 0. Hence, by the proof of [Gra16, Lemma 6.3], $[X \oplus \tau X]$ is equal to the image of the class $\left[\mathrm{Cone} \left({}^{u \oplus v}_{s \circ (u \oplus v)} \right) \right] \in K_1^{\mathrm{Gr}} \mathcal{N}$, where here we denote by s the automorphism

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : N \oplus N \to N \oplus N.$$

The binary complex $\mathrm{Cone} \left({}^{u \oplus v}_{s \circ (u \oplus v)} \right)$ is defined as the binary complex with graded object $FA[1] \oplus FB[1] \oplus N \oplus N$, whose top differential (resp. bottom differential) is the same as the differential of the complex $\mathrm{cone}(u \oplus v)$ (resp. $\mathrm{cone}(s \circ (u \oplus v))$). Then we

have the binary ladder

$$\left(\mathrm{Cone} \begin{pmatrix} u \oplus v \\ s \circ (u \oplus v) \end{pmatrix}, \mathrm{Cone} \begin{pmatrix} u \oplus v \\ u \oplus v \end{pmatrix}, 1_{FA \oplus FB} \oplus 1_{N \oplus N}, 1_{FA \oplus FB} \oplus s \right)$$

and thus we have the following chain of equalities, where we note $\mathrm{Cone} \begin{pmatrix} u \oplus v \\ u \oplus v \end{pmatrix}$ is diagonal:

$$\left[\mathrm{Cone} \begin{pmatrix} u \oplus v \\ s \circ (u \oplus v) \end{pmatrix} \right] = \left[\mathrm{Cone} \begin{pmatrix} u \oplus v \\ s \circ (u \oplus v) \end{pmatrix} \right] - \left[\mathrm{Cone} \begin{pmatrix} u \oplus v \\ u \oplus v \end{pmatrix} \right]$$

$$= \sum_{i=0}^{\infty} (-1)^i \left[FA_{i-1} \oplus FB_{i-1} \oplus N_i \oplus N_i \xrightarrow[1 \oplus s]{1 \oplus 1} FA_{i-1} \oplus FB_{i-1} \oplus N_i \oplus N_i \right]$$

$$= \sum_{i=0}^{\infty} (-1)^i \left[N_i \oplus N_i \xrightarrow[s]{1} N_i \oplus N_i \right]$$

note that the binary complexes $FA_{i-1} \oplus FB_{i-1} \xrightarrow[1]{1} FA_{i-1} \oplus FB_{i-1}$ are diagonal, thus the third equality holds. Since u is a quasi-isomorphism, the Euler characteristic of $\mathrm{cone}(u)$ vanishes, and thus the Euler characteristic of N is the same as of FA. The same is then true of the images of those elements under the homomorphism $K_0 \mathcal{N} \to K_1^{\mathrm{Gr}} \mathcal{N}$ induced by the exact functor $P \mapsto \left(P \oplus P \xrightarrow[s]{1} P \oplus P \right)$. Thus the above alternating sum is equal to

$$\sum_{i=0}^{\infty} (-1)^i \left[FA_i \oplus FA_i \xrightarrow[s]{1} FA_i \oplus FA_i \right]$$

which is in the image of the map $K_1^{\mathrm{Gr}} \mathcal{M} \to K_1^{\mathrm{Gr}} \mathcal{N}$, and thus is sent to the class of 0 in $K_0^{\mathrm{Gr}}[F]$. Since we have already shown that it is equal to the class of $[X \oplus \tau X]$, and thus we have proven the claim. $\square$

Chapter 3

Elementary results

3.1 Bass K-theory

We will now discuss the K_1 and relative K_0 groups as defined by Hyman Bass in [Bas68] and show in a purely algebraic way that Grayson's $K_0^{\mathrm{Gr}}[F]$ is isomorphic to Bass' relative K_0 in the case when the categories $\mathcal{M}$ and $\mathcal{N}$ are split exact and the functor $F : \mathcal{M} \to \mathcal{N}$ is cofinal. To begin this section however, we only assume that $\mathcal{M}$ and $\mathcal{N}$ are exact categories and that $F : \mathcal{M} \to \mathcal{N}$ is an exact functor.

Let $\mathrm{Aut}(\mathcal{M})$ denote the category of pairs (A, α), where A is an object of $\mathcal{M}$ and $\alpha : A \to A$ is an automorphism. A morphism $(A, \alpha) \to (B, \beta)$ in $\mathrm{Aut}(\mathcal{M})$ is a morphism $\phi : A \to B$ such that $\phi \circ \alpha = \beta \circ \phi$, in other words the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \alpha\ } & A \\
\downarrow{\phi} & & \downarrow{\phi} \\
B & \xrightarrow{\ \beta\ } & B
\end{array}
$$

The category $\mathrm{Aut}(\mathcal{M})$ inherits an exact structure from $\mathcal{M}$ pointwise, and we have the following exact functors, with notation emulating those we use with binary complexes:

$$\Delta : \mathcal{M} \to \mathrm{Aut}(\mathcal{M}) \text{ given by } A \mapsto (A, 1_A)$$
$$\perp : \mathrm{Aut}(\mathcal{M}) \text{ given by } (A, \alpha) \mapsto A$$
$$\tau : \mathrm{Aut}(\mathcal{M}) \text{ given by } (A, \alpha) \mapsto (A, \alpha^{-1}).$$

Bass presents the group we will denote by $K_1^{\mathrm{Bass}}\mathcal{M}$ as the grothendieck group $K_0 \mathrm{Aut}(\mathcal{M})$ with the additional relation

$$[A, \beta \circ \alpha] = [A, \beta] + [A, \alpha]$$

whenever there is an object A of $\mathcal{M}$ with automorphisms $\alpha, \beta : A \to A$.

Objects in the image of the functor $\Delta : \mathcal{M} \to \mathrm{Aut}(\mathcal{M})$ are sent to zero in $K_1^{\mathrm{Bass}}(\mathcal{M})$, since given an object A, we have that $1_A \circ 1_A = 1_A$, and so $[A, 1_A] = [A, 1_A] + [A, 1_A] = 0$. Further, given an automorphism $\alpha : A \to A$, since $\alpha \circ \alpha^{-1} = 1_A$ we have that $0 = [A, 1_A] = [A, \alpha] + [A, \alpha^{-1}]$ and so $[\tau(A, \alpha)] = [A, \alpha^{-1}] = -[A, \alpha]$, and so τ allows us to compute inverses in $K_1^{\mathrm{Bass}}(\mathcal{M})$ on the level of objects in $\mathrm{Aut}(\mathcal{M})$.

Now, to approach Bass' presentation of the relative K_0 group of an exact functor $F : \mathcal{M} \to \mathcal{N}$ between exact categories $\mathcal{M}$ and $\mathcal{N}$, we will outline the category $\mathrm{co}(F)$ whose objects consist of triples of the form (A, α, B), where A and B are objects in $\mathcal{M}$ and $\alpha : FA \to FB$ is an isomorphism in $\mathcal{N}$. A morphism $(A, \alpha, B) \to (C, \gamma, D)$ in $\mathrm{co}(F)$ consists of a pair (f, g) of morphisms $f : A \to C$ and $g : B \to D$ in $\mathcal{M}$ such that $Fg \circ \alpha = \gamma \circ Ff$, in other words the following diagram commutes:

$$
\begin{array}{ccc}
FA & \xrightarrow{\ \alpha\ } & FB \\
\downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Fg} \\
FC & \xrightarrow{\ \gamma\ } & FD
\end{array}
$$

The category $\mathrm{co}(F)$ is an exact category (where admissible monics and epics are pointwise), and we have the following exact functors

$$
\Delta : \mathcal{M} \to \mathrm{co}(F) \text{ given by } A \mapsto (A, 1_{FA}, A)
$$
$$
\top : \mathrm{co}(F) \to \mathcal{M} \text{ given by } (A, \alpha, B) \mapsto A
$$
$$
\bot : \mathrm{co}(F) \to \mathcal{M} \text{ given by } (A, \alpha, B) \mapsto B
$$
$$
\mathbb{I} = (\top, \bot) : \mathrm{co}(F) \to \mathcal{M}^2 \text{ given by } (A, \alpha, B) \mapsto (A, B)
$$
$$
\tau : \mathrm{co}(F) \to \mathrm{co}(F) \text{ given by } (A, \alpha, B) \mapsto (B, \alpha^{-1}, A).
$$

Bass presents the group we will denote by $K_0^{\mathrm{Bass}}[F]$ as the Grothendieck group $K_0\mathrm{co}(F)$ together with the additional relation

$$
[A, \beta \circ \alpha, C] = [A, \alpha, B] + [B, \beta, C]
$$

wherever there are objects A, B, C in $\mathcal{M}$ with isomorphisms $\alpha : FA \to FB$ and $\beta : FB \to FC$ in $\mathcal{N}$. As above, we have that $[A, 1_{FA}, A] = [\Delta A] = 0$ and $[\tau(A, \alpha, B)] = [B, \alpha^{-1}, A] = -[A, \alpha, B]$ whenever we have objects A, B in $\mathcal{M}$ and an isomorphism $\alpha : FA \to FB$.

An exact functor $F : \mathcal{M} \to \mathcal{N}$ is called *cofinal* if for every object N in $\mathcal{N}$ there exists objects N' in $\mathcal{N}$ and M in $\mathcal{M}$ such that $N \oplus N' \cong FM$. Given such a cofinal exact

functor $F : \mathcal{M} \to \mathcal{N}$, in [Bas68, Theorem VII.5.3] Bass gives us an exact sequence

$$K_1^{\text{Bass}}\mathcal{M} \longrightarrow K_1^{\text{Bass}}\mathcal{N} \longrightarrow K_0^{\text{Bass}}[F] \longrightarrow K_0\mathcal{M} \longrightarrow K_0\mathcal{N}$$

where the homomorphism $K_0^{\text{Bass}}[F] \to K_0\mathcal{M}$ is given by $[A, \alpha, B] \mapsto [A] - [B]$, and the homomorphism $K_1^{\text{Bass}}\mathcal{N} \to K_0^{\text{Bass}}[F]$ can be described as follows: Given an object (A, α) of $\text{Aut}(\mathcal{N})$, by cofinality we have objects A' in $\mathcal{N}$ and B in $\mathcal{M}$ such that $A \oplus A' \cong FB$, so that we have

$$[A, \alpha] = [A \oplus A', \alpha \oplus 1] = [FB, \beta]$$

in $K_1^{\text{Bass}}\mathcal{N}$, where β is the automorphism of FB induced by $\alpha \oplus 1$. The image of this element in $K_0^{\text{Bass}}[F]$ is $[B, \beta, B]$

We will now introduce a functor from $\text{co}(F)$ to $B[F]$ that will extend to a homomorphism from $K_0^{\text{Bass}}[F]$ to $K_0^{\text{Gr}}[F]$.

Definition 3.1.1. We define the functor $\Psi : \text{co}(F) \to B[F]$ given by

$$\Psi(A, \alpha, B) := \left(\left(\begin{smallmatrix}A\\B\end{smallmatrix}\right), FB, \left(\begin{smallmatrix}\alpha\\1\end{smallmatrix}\right) \right),$$

where $\left(\begin{smallmatrix}A\\B\end{smallmatrix}\right)$ is considered as a pair of complexes in $\mathcal{M}$ supported in degree 0, and FB is considered as a binary complex in $\mathcal{N}$ supported in degree 0. The functor Ψ acts on morphisms in the obvious way.

Before we extend this to a homomorphism $K_0^{\text{Bass}}[F] \to K_0^{\text{Gr}}[F]$, we need to do some computations using the images of (A, α, B) in $B[F]$. In the following few lemmas we will be denoting objects X of $B[F]$ supported in degrees 0 and 1 in the following way:

$$\left[\left(A_1 \xrightarrow{\ a\ } A_0 \ , \ B_1 \xrightarrow{\ b\ } B_0 \right), \left(N_1 \xrightarrow[\ d'\ ]{\ d\ } N_0 \right), ((u_1; u_0), (v_1; v_0)) \right]$$

where we write the quasi-isomorphisms in an order that agrees with the ordering of the objects in the complexes.

Lemma 3.1.2. *Given an object (A, α, B) of $\text{co}(F)$, we have that the element $[\Psi(A, \alpha, B)]$ is equal to the element*

$$\left[\left(0 \xrightarrow{\ 0\ } 0 \ , \ A \xrightarrow{\ 0\ } B \right), \left(FA \xrightarrow[\ 0\ ]{\ \alpha\ } FB \right), ((0; 0), (1; 1)) \right]$$

in $K_0^{\text{Gr}}[F]$.

Proof. We have the following short exact sequence in $B[F]$, whose first component is $\Psi(A, \alpha, B)$ by definition:

$$
\begin{array}{ccc}
((0 \xrightarrow{0} A, 0 \xrightarrow{0} B), (0 \underset{0}{\overset{0}{\rightrightarrows}} FB), ((0;\alpha), (0;1))) \\
\Big\downarrow{\scriptstyle 0} \quad {\scriptstyle 1}\Big\downarrow \Big\downarrow{\scriptstyle 0} \qquad \Big\downarrow{\scriptstyle 1} \quad \Big\downarrow{\scriptstyle 0} \qquad\quad \Big\downarrow{\scriptstyle 1} \\
((A \xrightarrow{1} A, A \xrightarrow{0} B), (FA \underset{0}{\overset{\alpha}{\rightrightarrows}} FB), ((1;\alpha), (1;1))) \\
\Big\downarrow{\scriptstyle 1} \quad {\scriptstyle 0}\Big\downarrow \Big\downarrow{\scriptstyle 1} \qquad \Big\downarrow{\scriptstyle 0} \quad \Big\downarrow{\scriptstyle 1} \qquad\quad \Big\downarrow{\scriptstyle 0} \\
((A \xrightarrow{0} 0, A \xrightarrow{0} 0), (FA \underset{0}{\overset{0}{\rightrightarrows}} 0), ((1;0), (1;0)))
\end{array}
$$

A brief computation verifies that these maps are indeed morphisms in $B[F]$ and since the bottom triple is in the image of Δ, we have that $[\Psi(A, \alpha, B)]$ is equal to the middle triple in $K_0^{\mathrm{Gr}}[F]$. To show the required identity, notice that the following arrow is in $pB[F]$:

$$
\begin{array}{ccc}
((0 \xrightarrow{0} 0, A \xrightarrow{0} B), (FA \underset{0}{\overset{\alpha}{\rightrightarrows}} FB), ((0;0), (1;1))) \\
\Big\downarrow{\scriptstyle 0} \qquad {\scriptstyle 0}\Big\downarrow \Big\downarrow{\scriptstyle 1} \qquad \Big\downarrow{\scriptstyle 1} \quad \Big\downarrow{\scriptstyle 1} \qquad\quad \Big\downarrow{\scriptstyle 1} \\
((A \xrightarrow{1} A, A \xrightarrow{0} B), (FA \underset{0}{\overset{\alpha}{\rightrightarrows}} FB), ((1;\alpha), (1;1))).
\end{array}
$$

$\square$

Proposition 3.1.3. *The functor* $\Psi : \mathrm{co}(F) \to B[F]$ *induces a well-defined homomorphism* $\Phi : K_0^{\mathrm{Bass}}[F] \to K_0^{\mathrm{Gr}}[F]$.

Proof. It is obvious that Ψ is exact, and thus extends to a homomorphism

$$
K_0 \mathrm{co}(F) \to K_0^{\mathrm{Gr}}[F] \text{ given by } [A, \alpha, B] \mapsto [\Psi(A, \alpha, B)]
$$

and so it remains to check for every object A, B, C in $\mathcal{M}$ with isomorphisms $\alpha : FA \to FB$ and $\beta : FB \to FC$ that

$$
[\Psi(A, \beta \circ \alpha, C)] = [\Psi(A, \alpha, B)] + [\Psi(B, \beta, C)]
$$

in $K_0^{\mathrm{Gr}}[F]$. To demonstrate this, we apply lemma 3.1.2 to the object (B, β, C) and notice that the following morphism is in $pB[F]$:

$$
\begin{array}{ccc}
((0 \xrightarrow{0} 0, B \xrightarrow{0} C), (FB \underset{0}{\overset{\beta}{\rightrightarrows}} FC), ((0;0), (1;1))) \\
\Big\downarrow{\scriptstyle 0} \qquad {\scriptstyle 0}\Big\downarrow \Big\downarrow{\scriptstyle 1} \qquad \Big\downarrow{\scriptstyle 1} \quad \Big\downarrow{\scriptstyle 1} \qquad\quad \Big\downarrow{\scriptstyle 1} \\
((A \xrightarrow{1} A, B \xrightarrow{0} C), (FB \underset{0}{\overset{\beta}{\rightrightarrows}} FC), ((\alpha; \beta \circ \alpha), (1;1))).
\end{array}
$$

Now we see that we have the following short exact sequence in $B[F]$ obtained from a naïve filtration:

$$((0 \xrightarrow{0} A, 0 \xrightarrow{0} C), (0 \underset{0}{\overset{0}{\rightrightarrows}} FC), ((0; \beta \circ \alpha), (0; 1)))$$
$$\Big\downarrow 0 \quad 1\Big\downarrow \Big\downarrow 0 \qquad \Big\downarrow 1 \quad \Big\downarrow 0 \qquad \qquad \Big\downarrow 1$$
$$((A \xrightarrow{1} A, B \xrightarrow{0} C), (FB \underset{0}{\overset{\beta}{\rightrightarrows}} FC), ((\alpha; \beta \circ \alpha), (1; 1)))$$
$$\Big\downarrow 1 \quad 0\Big\downarrow \Big\downarrow 1 \qquad \Big\downarrow 0 \quad \Big\downarrow 1 \qquad \qquad \Big\downarrow 0$$
$$((A \xrightarrow{0} 0, B \xrightarrow{0} 0), (FB \underset{0}{\overset{0}{\rightrightarrows}} 0), ((\alpha; 0), (1; 0))),$$

and so we have the following equation in $K_0^{\mathrm{Gr}}[F]$:

$$[\Psi(B, \beta, C)] = [\Psi(A, \beta \circ \alpha, C)] + [\Psi(A, \alpha, B)[1]]$$
$$= [\Psi(A, \beta \circ \alpha, C)] - [\Psi(A, \alpha, B)],$$

where the second equality holds from [Gra16, Lemma 6.1], and thus the statement holds. $\qquad \square$

Theorem 3.1.4. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are split exact categories, and $F : \mathcal{M} \to \mathcal{N}$ is a cofinal exact functor. Then the homomorphism $K_0^{\mathrm{Bass}}[F] \to K_0^{\mathrm{Gr}}[F]$ induced by Ψ is an isomorphism.*

Proof. We have two long exact sequences, one featuring $K_0^{\mathrm{Bass}}[F]$ introduced in [Bas68, Theorem VII.5.3], and one featuring $K_0^{\mathrm{Gr}}[F]$ introduced at the end of Section 2.2. We wish to show that our homomorphism induced by Ψ is compatible with these sequences, namely that the following diagram commutes:

$$
\begin{array}{ccccccccc}
K_1^{\mathrm{Bass}}\mathcal{M} & \longrightarrow & K_1^{\mathrm{Bass}}\mathcal{N} & \longrightarrow & K_0^{\mathrm{Bass}}[F] & \longrightarrow & K_0\mathcal{M} & \longrightarrow & K_0\mathcal{N} \\
\Big\downarrow \cong & & \Big\downarrow \cong & & \Big\downarrow & & \Big\| & & \Big\| \\
K_1^{\mathrm{Gr}}\mathcal{M} & \longrightarrow & K_1^{\mathrm{Gr}}\mathcal{N} & \longrightarrow & K_0^{\mathrm{Gr}}[F] & \longrightarrow & K_0\mathcal{M} & \longrightarrow & K_0\mathcal{N}
\end{array}
$$

where the homomorphism

$$K_1^{\mathrm{Bass}}\mathcal{M} \to K_1^{\mathrm{Gr}}\mathcal{M} \text{ given by } [A, \alpha] \mapsto \left[A \underset{1}{\overset{\alpha}{\rightrightarrows}} A \right]$$

was shown to be an isomorphism for every split exact category $\mathcal{M}$ in [Har15, Theorem 2.24]. We note here that by the 5 lemma, Theorem 3.1.4 is true if this diagram commutes, which we will prove now.

First, we will show that the third square in the diagram commutes, as this step is easier. Recall that the map $K_0^{\mathrm{Bass}}[F] \to K_0\mathcal{M}$ is given by

$$[A, \alpha, B] \mapsto [A] - [B],$$

and the map $K_0^{\text{Gr}}[F] \to K_0 \mathcal{M}$ is given by

$$[M, N, u] \mapsto \chi(\top M) - \chi(\bot M),$$

and so it is clear that the third square commutes.

Now to show that the second square commutes, suppose that we have an element $[A, \alpha]$ of $K_1^{\text{Bass}} \mathcal{N}$. Recall that this element is mapped to the element $[B, \beta, B]$ in $K_0^{\text{Bass}}[F]$, where B is an object of $\mathcal{M}$ such that $FB \cong A \oplus A'$ for some A' in $\mathcal{N}$, and β is induced by $\alpha \oplus 1$. We then have the following exact sequence in $B[F]$ arising from a naïve filtration:

$$
\begin{array}{c}
((0 \xrightarrow{0} B, 0 \xrightarrow{0} B), (0 \underset{0}{\overset{0}{\rightrightarrows}} FB), ((0; \beta), (0; 1))) \\
\Big\downarrow{\scriptstyle 0} \quad {\scriptstyle 1}\Big\downarrow \Big\downarrow{\scriptstyle 0} \quad \Big\downarrow{\scriptstyle 1} \ \Big\downarrow{\scriptstyle 0} \qquad \Big\downarrow{\scriptstyle 1} \\
((B \xrightarrow{1} B, B \xrightarrow{1} B), (FB \underset{1}{\overset{\beta}{\rightrightarrows}} FB), ((1; \beta), (1; 1))) \\
\Big\downarrow{\scriptstyle 1} \quad {\scriptstyle 0}\Big\downarrow \Big\downarrow{\scriptstyle 1} \quad \Big\downarrow{\scriptstyle 0} \ \Big\downarrow{\scriptstyle 1} \qquad \Big\downarrow{\scriptstyle 0} \\
((B \xrightarrow{0} 0, B \xrightarrow{0} 0), (FB \underset{0}{\overset{0}{\rightrightarrows}} 0), ((1; 0), (1; 0))).
\end{array}
$$

Clearly, since the bottom row is diagonal, the first two rows are equal in $K_0^{\text{Gr}}[F]$, and since the first component of the middle row is acyclic the map

$$
\begin{array}{c}
((0 \longrightarrow 0, 0 \longrightarrow 0), (FB \underset{1}{\overset{\beta}{\rightrightarrows}} FB), ((0; 0), (0; 0))) \\
\Big\downarrow \qquad \Big\downarrow\Big\downarrow \qquad \Big\downarrow \quad \Big\downarrow{\scriptstyle 1} \qquad \Big\downarrow{\scriptstyle 1} \\
((B \xrightarrow{1} B, B \xrightarrow{1} B), (FB \underset{1}{\overset{\beta}{\rightrightarrows}} FB), ((1; \beta), (1; 1))).
\end{array}
$$

is in $pB[F]$ and thus the source of this arrow is equal to $[\Phi(B, \beta, B)]$ in $K_0^{\text{Gra}}[F]$. On the other hand, since the homomorphism $K_1^{\text{Gr}} \mathcal{N} \to K_0^{\text{Gr}}[F]$ is induced by the functor $N \mapsto (0, N, 0)$, this is equal to the image of $[FB, \beta]$ when mapped first via the isomorphism $K_1^{\text{Bass}} \mathcal{N} \to K_1^{\text{Gr}} \mathcal{N}$ and then by the homomorphism $K_1^{\text{Gr}} \mathcal{N} \to K_0^{\text{Gr}}[F]$. Hence, the second square commutes, and thus the diagram commutes. $\qquad\square$

3.2   Grayson's Relative K-groups are isomorphic to the classically defined groups

This section is taken from [Tur25, §3].

Recall our setup of an exact functor $F : \mathcal{M} \to \mathcal{N}$ and we now define a homomorphism

$$\Phi : K_0^{\text{Gr}}[F] \to K_0[q\mathcal{C}\mathcal{M} \xrightarrow{F} q\mathcal{C}\mathcal{N}] \cong K_0[F]$$

which we will prove to be an isomorphism. Recall given a continuous map $f : X \to Y$ between pointed spaces with base points x_0 and y_0, the *homotopy fiber of f* is the topological space

$$\text{hofib}(f) := \{(x, \alpha) \in X \times Y^I \mid \alpha(0) = y_0, \alpha(1) = f(x)\}.$$

If Y is the geometric realisation of a simplicial set, an element of Y^I (i.e. a path in Y) can be given by the concatenation of a zig-zag of 1-simplices:

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \cdots \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet$$

Given an object $X = (M, N, u)$ of $B[F]$, i.e. a generator of $K_0^{\text{Gr}}[F]$, we define $\Phi[X]$ to be the connected component of the point

$$\left(\left(\begin{smallmatrix}\top M\\\bot M\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \longmapsto \left(\begin{smallmatrix}\top N_{[0,0]}\\\bot N_{[0,0]}\end{smallmatrix}\right) \longmapsto \left(\begin{smallmatrix}\top N_{[0,1]}\\\bot N_{[0,1]}\end{smallmatrix}\right) \longmapsto \cdots \longmapsto \left(\begin{smallmatrix}\top N\\\bot N\end{smallmatrix}\right) \xleftarrow{\cong} \left(\begin{smallmatrix}\top FM\\\bot FM\end{smallmatrix}\right) \right)$$

in $K[qC\mathcal{M} \xrightarrow{F} qC\mathcal{N}]$, where the 1-simplex labeled by "$\xrightarrow{\cong}$" in the second component is given by $(\top u, \bot u)$ and the other 1-simplices labeled by "$\rightarrowtail$" arise from the naïve filtration of $\top N$ and $\bot N$, where for a chain complex A, we denote by $A_{[a,b]}$ the subcomplex of A supported on the interval $[a, b]$.

Theorem 3.2.1. *The assignment*

$$X \mapsto \Phi(X)$$

induces a well-defined isomorphism

$$K_0^{\text{Gr}}[F] \to K_0[qC\mathcal{M} \xrightarrow{F} qC\mathcal{N}].$$

The proof of this theorem requires the following result. Recall given an exact category with weak equivalences $w\mathcal{N}$ we have the exact category with weak equivalences $w\mathcal{E}\mathcal{N}$ whose objects consist of short exact sequences in $\mathcal{M}$ and whose arrows are commutative diagrams, where weak equivalences are inherited pointwise from $w\mathcal{N}$. We have canonical exact functors $s, t, q : w\mathcal{E}\mathcal{N} \to w\mathcal{N}$ which send a short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ to its source, A, its target, B, and its quotient, C, respectively. We have the following corollary of the additivity theorem applied to the homotopy fiber:

Lemma 3.2.2. *Given an exact functor between exact categories with weak equivalences $F : w\mathcal{A} \to w\mathcal{B}$, the induced map*

$$(s, q) : K[w\mathcal{E}\mathcal{A} \xrightarrow{F} w\mathcal{E}\mathcal{B}] \to K[w\mathcal{A}^2 \xrightarrow{F \times F} w\mathcal{B}^2]$$

is a homotopy equivalence.

Proof. This follows from the additivity theorem [Gun+92, Theorem 2.10] applied to the second and third vertical arrows in the following diagram:

$$
\begin{array}{ccccc}
K[w\mathcal{E}\mathcal{A} \xrightarrow{F} w\mathcal{E}\mathcal{B}] & \longrightarrow & Kw\mathcal{E}\mathcal{A} & \xrightarrow{F} & Kw\mathcal{E}\mathcal{B} \\
\downarrow{\scriptstyle (s,q)} & & \downarrow{\scriptstyle (s,q)} & & \downarrow{\scriptstyle (s,q)} \\
K[w\mathcal{A}^2 \xrightarrow{F\times F} w\mathcal{B}^2] & \longrightarrow & Kw\mathcal{A}^2 & \xrightarrow{F\times F} & Kw\mathcal{B}^2
\end{array}
$$

$\square$

Below is the proof of Theorem 3.2.1:

Proof. First, we will show that this is well-defined, and so we need to check that it is compatible with the relations of $K_0^{\mathrm{Gr}}[F]$. To this end, to show additivity, suppose we have a short exact sequence

$$
(M', N', u') \rightarrowtail (M, N, u) \twoheadrightarrow (M'', N'', u'')
$$

in $B[F]$. Taking n to be the smallest index such that $N_i' = N_i = N_i'' = 0$ for all $i > n$, we have the point

$$
\left(
\begin{array}{ccccccc}
\binom{\top M'}{\bot M'} & \binom{0}{0} \rightarrowtail \binom{\top N'_{[0,0]}}{\bot N'_{[0,0]}} \rightarrowtail \cdots \rightarrowtail \binom{\top N'_{[0,n]}}{\bot N'_{[0,n]}} \xleftarrow{\simeq} \binom{\top FM'}{\bot FM'} \\
\downarrow \quad\quad \downarrow \quad\quad\quad \downarrow \quad\quad\quad\quad\quad\quad \downarrow \quad\quad\quad \downarrow \\
\binom{\top M}{\bot M} \,,\, \binom{0}{0} \rightarrowtail \binom{\top N_{[0,0]}}{\bot N_{[0,0]}} \rightarrowtail \cdots \rightarrowtail \binom{\top N_{[0,n]}}{\bot N_{[0,n]}} \xleftarrow{\simeq} \binom{\top FM}{\bot FM} \\
\downarrow \quad\quad \downarrow \quad\quad\quad \downarrow \quad\quad\quad\quad\quad\quad \downarrow \quad\quad\quad \downarrow \\
\binom{\top M''}{\bot M''} & \binom{0}{0} \rightarrowtail \binom{\top N''_{[0,0]}}{\bot N''_{[0,0]}} \rightarrowtail \cdots \rightarrowtail \binom{\top N''_{[0,n]}}{\bot N''_{[0,n]}} \xleftarrow{\simeq} \binom{\top FM''}{\bot FM''}
\end{array}
\right)
$$

in $K[q\mathcal{E}C\mathcal{M} \xrightarrow{F} q\mathcal{E}C\mathcal{N}]$. By Lemma 3.2.2, replacing in this point the middle row by the direct sum of the top and bottom rows does not change the connected component it belongs to. Thus, since $KqC\mathcal{M}$ and $KqC\mathcal{N}$ are H spaces with direct sum and thus $K[qC\mathcal{M} \xrightarrow{F} qC\mathcal{N}]$ is a H-space induced by direct sum, applying the exact functor Φ yields

$$
\Phi(M, N, u) = \Phi(M', N', u') + \Phi(M'', N'', u''),
$$

and so Φ is compatible with short exact sequences.

Now suppose we have a diagonal object $X = (\Delta M, \Delta N, \Delta u)$ of $B[F]$. Then we have the following path in $K[qC\mathcal{M} \xrightarrow{F} qC\mathcal{N}]$:

$$
\left(
\begin{array}{c}
\binom{0}{0} \quad \binom{0}{0} = \binom{0}{0} = \cdots = \binom{0}{0} = \binom{0}{0} \\
, \\
\binom{M}{M} \quad \binom{0}{0} \rightarrowtail \binom{N_{[0,0]}}{N_{[0,0]}} \rightarrowtail \cdots \rightarrowtail \binom{N}{N} \xleftarrow{\simeq} \binom{FM}{FM}
\end{array}
\right)
$$

where the arrow $\binom{0}{0} === \binom{0}{0}$ is the obvious degenerate arrow, and there are obvious 2-simplices arising from the fact that every 1-simplex in $\Phi(\Delta M, \Delta N, \Delta u)$ is diagonal in the obvious sense (see the proof of [GG87, Theorem 3.1]).

Finally, suppose that we have a map $\varphi : (M, N, u) \to (M', N', u')$ in $pB[F]$, namely the maps $\top M \to \top M'$ and $\bot M \to \bot M'$ are quasi-isomorphisms in $C\mathcal{M}$ and the map $N \to N'$ is an isomorphism in $B\mathcal{N}$. Then φ gives us the path

$$
\left(
\begin{array}{c}
\binom{\top M}{\bot M} \quad \binom{0}{0} \rightarrowtail \binom{\top N_{[0,0]}}{\bot N_{[0,0]}} \rightarrowtail \cdots \rightarrowtail \binom{\top N_{[0,n]}}{\bot N_{[0,n]}} \xleftarrow{\simeq} \binom{\top FM}{\bot FM} \\
\downarrow\simeq \quad , \quad \downarrow\simeq \quad \downarrow\simeq \quad \downarrow\simeq \quad \downarrow\simeq \\
\binom{\top M'}{\bot M'} \quad \binom{0}{0} \rightarrowtail \binom{\top N'_{[0,0]}}{\bot N'_{[0,0]}} \rightarrowtail \cdots \rightarrowtail \binom{\top N'_{[0,n]}}{\bot N'_{[0,n]}} \xleftarrow{\simeq} \binom{\top FM'}{\bot FM'}
\end{array}
\right)
$$

in $K[qC\mathcal{M} \xrightarrow{F} qC\mathcal{N}]$ from the point corresponding to the top row to the point corresponding to the bottom row, where

$$n = \max\{|\mathrm{supp}(N)|, |\mathrm{supp}(N')|\},$$

and every triangle in the above diagram bounds a 2-simplex, and so $\Phi[M, N, u] = \Phi[M', N', u']$, hence Φ is a well-defined homomorphism.

To show Φ is an isomorphism by the 5-lemma, it suffices to show that the following diagram with exact rows and vertical isomorphisms as indicated is commutative:

$$
\begin{array}{ccccccccc}
K_1^{\mathrm{Gr}}\mathcal{M} & \longrightarrow & K_1^{\mathrm{Gr}}\mathcal{N} & \longrightarrow & K_0^{\mathrm{Gr}}[F] & \longrightarrow & K_0\mathcal{M} & \longrightarrow & K_0\mathcal{N} \\
\downarrow\cong & & \downarrow\cong & & \downarrow\Phi & & \downarrow\cong & & \downarrow\cong \\
K_1 qC\mathcal{M} & \longrightarrow & K_1 qC\mathcal{N} & \longrightarrow & K_0[qC\mathcal{M} \xrightarrow{F} qC\mathcal{N}] & \longrightarrow & K_0 qC\mathcal{M} & \longrightarrow & K_0 qC\mathcal{N}.
\end{array}
$$

The top row of this diagram is defined to be the exact sequence established by Grayson in [Gra16, Corollary 1.9], and the bottom row is the exact sequence arising from a homotopy fiber. The two vertical isomorphisms on the left are described by Grayson in [Gra12, Corollary 7.4], and the two on the right are given by $[M] - [N] \mapsto [\frac{M}{N}]$, viewing objects M and N as complexes concentrated in degree 0.

The first and fourth squares commute due to F being an exact functor, and the third square commutes as an element $[M, N, u]$ is sent to $\chi(\top M) - \chi(\bot M)$ in $K_0\mathcal{M}$ both via the horizontal map and via going around the square with the inverse of the isomorphism $K_0\mathcal{M} \to K_0qC\mathcal{M}$, which is given in [Wei13, Theorem V.2.2]. It remains to show that the second square commutes. Given an object N of $B^q\mathcal{N}$, we will prove in Theorem 3.2.3 below that the element $[N]$ of $K_1^{\mathrm{Gr}}(\mathcal{N})$ is sent to the element of $K_1qC\mathcal{N}$ corresponding to the loop

$$\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrowtail \begin{pmatrix} \top N_{[0,0]} \\ \bot N_{[0,0]} \end{pmatrix} \rightarrowtail \cdots \rightarrowtail \begin{pmatrix} \top N \\ \bot N \end{pmatrix} \xleftarrow{\simeq} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

where again the 1-simplices labelled by "$\rightarrowtail$" arise from the naïve filtration of $\top N$ and $\bot N$. This is then sent to the element of $K_0[F]$ arising from the point given by:

$$\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrowtail \begin{pmatrix} \top N_{[0,0]} \\ \bot N_{[0,0]} \end{pmatrix} \rightarrowtail \cdots \rightarrowtail \begin{pmatrix} \top N \\ \bot N \end{pmatrix} \xleftarrow{\simeq} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

On the other hand, $[N]$ is sent to the element $[0, N, 0]$ of $K_0^{\mathrm{Gr}}[F]$, which Φ sends to the same element as described above, and so the second square commutes. $\square$

Theorem 3.2.3. *Given an object N of $B^q\mathcal{N}$, the isomorphism*

$$K_1^{\mathrm{Gr}}(\mathcal{N}) \to K_1qC\mathcal{N}$$

given in [Gra12] sends the element $[N] - [\Delta\bot N]$ to the element corresponding to the loop

$$\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrowtail \begin{pmatrix} \top N_{[0,0]} \\ \bot N_{[0,0]} \end{pmatrix} \rightarrowtail \cdots \rightarrowtail \begin{pmatrix} \top N \\ \bot N \end{pmatrix} \xleftarrow{\simeq} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \qquad (*)$$

where the arrows labelled by "$\rightarrowtail$" arise from the naïve filtration of $\top N$ and $\bot N$ and the arrow labelled "$\xleftarrow{\simeq}$" arises from the quasi-isomorphism $0 \to N$.

Before we proceed with the proof of Theorem 3.2.3, we will present a discussion of the homotopy equivalence given in [Gra12, Corollary 5.10], with the caveat that we take as definition $K\Omega\mathcal{N} := K[iB^q\mathcal{N} \xrightarrow{\bot} iC^q\mathcal{N}]$. By a standard argument, this is equivalent to the formulation given in [Gra12], however this formulation will allow us to work entirely with spaces and homotopy fibers rather than spectra and cofibers. Grayson's homotopy equivalence is given by the following chain of equivalences:

Let $C^\chi\mathcal{N}$ denote the subcategory of $C\mathcal{N}$ consisting of complexes with vanishing Euler characteristic, which inherits quasi-isomorphisms as weak equivalences. Then [Gra12, Remark 5.8] tells us that the inclusion $C^\chi\mathcal{N} \hookrightarrow C\mathcal{N}$ induces a homotopy fibration sequence:

$$KqC^\chi\mathcal{N} \to KqC\mathcal{N} \to \text{``}K_0\mathcal{N}\text{''}$$

where "$K_0\mathcal{N}$" denotes the Eilenberg-MacLane space whose only nonvanishing homotopy group is $K_0\mathcal{N}$ at π_0. After applying Ω to this sequence, we obtain that the inclusion $C^\chi\mathcal{N} \hookrightarrow C\mathcal{N}$ induces a homotopy equivalence

$$\Omega KqC^\chi\mathcal{N} \to \Omega KqC\mathcal{N} \tag{3.1}$$

Let $bB\mathcal{N} \supset qB\mathcal{N}$ be the subcategory of weak equivalences of $B\mathcal{N}$ consisting of maps f such that $\perp f$ is a map in $qC\mathcal{N}$. Then $B^b\mathcal{N}$, which we define as the subcategory of $B\mathcal{N}$ where objects N are such that $\perp N$ is acyclic, inherits quasi-isomorphisms as a subcategory of weak equivalences, denoted by $qB^b\mathcal{N}$, and [Gra12, Theorem 5.9] shows that the exact functor $\top : qB^b\mathcal{N} \to qC^\chi\mathcal{N}$ induces a homotopy equivalence

$$KqB^b\mathcal{N} \to KqC^\chi\mathcal{N}. \tag{3.2}$$

Waldhausen's fibration theorem [Wal85] gives a homotopy fiber sequence

$$KqB^b\mathcal{N} \to KqB\mathcal{N} \to KbB\mathcal{N}.$$

Following [Gra12, Theorem 4.8], the map $\perp : KbB\mathcal{N} \to KqC\mathcal{N}$ is a homotopy equivalence. From here, observe the following diagram:

$$
\begin{array}{ccccc}
KqB^b\mathcal{N} & \longrightarrow & KqB\mathcal{N} & \longrightarrow & KbB\mathcal{N} \\
\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \perp} \\
K[qB\mathcal{N} \xrightarrow{\perp} qC\mathcal{N}] & \longrightarrow & KqB\mathcal{N} & \xrightarrow{\perp} & KqC\mathcal{N}
\end{array}
$$

The rows of this diagram are homotopy fiber sequences, and the map

$$KqB^b\mathcal{N} \to K[qB\mathcal{N} \xrightarrow{\perp} qC\mathcal{N}] \tag{3.3}$$

is induced by the middle vertical arrow, and since the middle and right vertical arrows are homotopy equivalences, as is the left vertical arrow.

Here, we diverge in our treatment of this chain of homotopy equivalences, though the map we will arrive at will obviously be the same as the one given in [Gra12]. By Waldhausen's fibration theorem, we have the following pair of fibration sequences:

$$\Omega KqB\mathcal{N} \to KiB^q\mathcal{N} \to KiB\mathcal{N} \to KqB\mathcal{N}$$
$$\Omega KqC\mathcal{N} \to KiC^q\mathcal{N} \to KiC\mathcal{N} \to KqC\mathcal{N}$$

Assembling these sequences with maps induced by $\perp$, we obtain the following diagram:

$$
\begin{array}{ccccc}
\Omega K[qB\mathcal{N} \xrightarrow{\perp} qC\mathcal{N}] & \dashrightarrow & K\Omega\mathcal{N} & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
\Omega KqB\mathcal{N} & \xrightarrow{\ \partial\ } & KiB^q\mathcal{N} & \longrightarrow & KiB\mathcal{N} \\
\downarrow{\scriptstyle\perp} & & \downarrow{\scriptstyle\perp} & & \downarrow{\scriptstyle\perp} \\
\Omega KqC\mathcal{N} & \xrightarrow{\ \partial\ } & KiC^q\mathcal{N} & \longrightarrow & KiC\mathcal{N}
\end{array}
$$

The columns of this diagram are homotopy fibration sequences, and the dashed arrow is hence induced by the map $\partial : \Omega KqB\mathcal{N} \to KiB^q\mathcal{N}$. Since the second and third rows are also homotopy fiber sequences, it follows by the nine lemma that the first row is a homotopy fiber sequence. Hence, the map

$$
\Omega K[qB\mathcal{N} \xrightarrow{\perp} qC\mathcal{N}] \to K\Omega\mathcal{N} \tag{3.4}
$$

is a homotopy equivalence.

Now we will present the proof of Theorem 3.2.3:

Proof. We apply π_0 to the above discussed chain of homotopy equivalences to obtain a chain of isomorphisms

$$
\pi_0 K\Omega\mathcal{N} \xleftarrow{(3.4)} K_1[qB\mathcal{N} \xrightarrow{\perp} qC\mathcal{N}] \xleftarrow{(3.3)} K_1 qB^b\mathcal{N} \xrightarrow{(3.2)} K_1 qC^{\chi}\mathcal{N} \xrightarrow{(3.1)} K_1 qC\mathcal{N},
$$

where $\pi_0 K\Omega\mathcal{N} := \pi_0 K[iB^q\mathcal{N} \xrightarrow{\perp} iC^q\mathcal{N}] \cong K_1^{\mathrm{Gr}}\mathcal{N}$ since Δ is split by $\perp$ and so

$$
\mathrm{coker}(\Delta : K_0 C^q\mathcal{N} \to K_0 B^q\mathcal{N}) \cong \ker(\perp : K_0 B^q\mathcal{N} \to K_0 C^q\mathcal{N}).
$$

We will find an element of the group $K_1 qB^b\mathcal{N}$ and show that it is sent on the one hand to the element $[N] - [\Delta\perp N]$ of $K_1^{\mathrm{Gr}}(\mathcal{N})$ and on the other hand to the element $(*)$ of $K_1 qC\mathcal{N}$. Since the above chain of homotopy equivalences is the one given in [Gra12], this proves the statement. To this end, we introduce the exact functor $H : B\mathcal{N} \to B\mathcal{N}$ such that given a binary complex A, we have:

$$
\top HA = \perp A[1] \oplus \top A
$$

$$
\perp HA = \mathrm{cone}(\perp A) = \mathrm{cone}(\perp A \xrightarrow{1} \perp A).
$$

Then consider the element of $K_1 qB^b\mathcal{N}$ corresponding to the loop

$$
\begin{pmatrix}0\\0\end{pmatrix} \rightarrowtail \begin{pmatrix}H(N_{[0,0]})\\H(\Delta\perp N_{[0,0]})\end{pmatrix} \rightarrowtail \cdots \rightarrowtail \begin{pmatrix}HN\\H\Delta\perp N\end{pmatrix} \xleftarrow{\simeq} \begin{pmatrix}0\\0\end{pmatrix}, \tag{**}
$$

where the arrows labelled "$\rightarrowtail$" arise from applying H to the naïve filtration of N, and the arrow labelled "$\xleftarrow{\simeq}$" is a weak equivalence by construction of H. Now we will apply the following isomorphisms to this element

$$K_1 q B^b \mathcal{N} \xrightarrow{(3.2)} K_1 q C^x \mathcal{N} \xrightarrow{(3.1)} K_1 q C \mathcal{N}$$

obtaining the element of $K_1 q C \mathcal{N}$ corresponding to the loop

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrowtail \begin{pmatrix} \bot N_{[0,0]}[1] \oplus \top N_{[0,0]} \\ \bot N_{[0,0]}[1] \oplus \bot N_{[0,0]} \end{pmatrix} \rightarrowtail \cdots \rightarrowtail \begin{pmatrix} \bot N[1] \oplus \top N \\ \bot N[1] \oplus \bot N \end{pmatrix} \xleftarrow{\simeq} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We will now show that this element is equal to $(*)$. Observe the following diagram:

$$
\begin{array}{ccccccccc}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} & \rightarrowtail & \begin{pmatrix} \top N_{[0,0]} \\ \bot N_{[0,0]} \end{pmatrix} & \rightarrowtail & \cdots & \rightarrowtail & \begin{pmatrix} \top N \\ \bot N \end{pmatrix} & \xleftarrow{\simeq} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\downarrow & \searrow & \downarrow & \searrow & & & \downarrow & \swarrow & \downarrow \\
\begin{pmatrix} 0 \\ 0 \end{pmatrix} & \rightarrowtail & \begin{pmatrix} \bot N_{[0,0]}[1] \oplus \top N_{[0,0]} \\ \bot N_{[0,0]}[1] \oplus \bot N_{[0,0]} \end{pmatrix} & \rightarrowtail & \cdots & \rightarrowtail & \begin{pmatrix} \bot N[1] \oplus \top N \\ \bot N[1] \oplus \bot N \end{pmatrix} & \xleftarrow{\simeq} & \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{array}
$$

The diagonal 1-simplices labelled by "$\rightarrowtail$" are given by the following short exact sequences, which are the direct sum of the short exact sequence $0 \rightarrowtail \bot N_{[0,i]}[1] \twoheadrightarrow \bot N_{[0,i]}[1]$ with the ith step of the naïve filtration of $\top N$ and $\bot N$ respectively:

$$\top N_{[0,i-1]} \rightarrowtail \top N_{[0,i]} \oplus \bot N_{[0,i]}[1] \twoheadrightarrow N_i[i] \oplus \bot N_{[0,i]}[1]$$

$$\bot N_{[0,i-1]} \rightarrowtail \bot N_{[0,i]} \oplus \bot N_{[0,i]}[1] \twoheadrightarrow N_i[i] \oplus \bot N_{[0,i]}[1]$$

Using these sequences, one can easily verify that every triangle in the above diagram involving these arrows bound 2-simplices. The square at the right end of the diagram gains a diagonal arrow and two 2-simplices in the usual natural way for squares of this kind. Hence, this diagram creates a homotopy between the loops associated with the rows, and hence their corresponding elements of $K_1 q C \mathcal{N}$ are equal.

Now we return to the element $(**)$ of $K_1 q B^b \mathcal{N}$ and we apply the following isomorphisms:

$$K_1 q B^b \mathcal{N} \xrightarrow{(3.3)} K_1[q B \mathcal{N} \xrightarrow{\bot} q C \mathcal{N}] \xrightarrow{(3.4)} K_1^{\mathrm{Gr}}(\mathcal{N})$$

Recalling that (3.3) is induced by the identity on $K_1 q B \mathcal{N}$ and (3.4) is induced by $\partial : K_1 q B \mathcal{N} \to K_0 i B^q \mathcal{N}$, we see that $(**)$ is sent to $[HN] - [H\Delta \bot N]$ in $K_1^{\mathrm{Gr}} \mathcal{N}$, since ∂ sends the element of $K_1 q B \mathcal{N}$ corresponding to a loop of the form

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrowtail \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} \rightarrowtail \cdots \rightarrowtail \begin{pmatrix} A_n \\ B_n \end{pmatrix} \xleftarrow{\simeq} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

to the connected component of the point $\left(\begin{smallmatrix} A_n \\ B_n \end{smallmatrix}\right)$ in $K_0 i B^q \mathcal{N}$. It remains to prove that $[HN] - [H\Delta\perp N]$ is equal to $[N] - [\Delta\perp N]$ in $K_1^{\mathrm{Gr}}\mathcal{N}$. To do this, we have the following short exact sequences

$$N \overset{\binom{0}{1}}{\rightarrowtail} HN \overset{(1\,0)}{\twoheadrightarrow} \Delta\perp N[1]$$

$$\Delta\perp N \overset{\binom{0}{1}}{\rightarrowtail} H\Delta\perp N \overset{(1\,0)}{\twoheadrightarrow} \Delta\perp N[1]$$

which verify the equality and completes the proof. $\qquad\square$

Theorem 3.2.1 allows us to give the following alternative proof of Proposition 2.2.3:

Proof. We will show that $[X \oplus \tau X] = 0$. Applying Φ defined above to $[X \oplus \tau X]$, we obtain the element

$$\left(\left(\begin{smallmatrix} \top M\oplus\perp M \\ \perp M\oplus\top M \end{smallmatrix}\right), \ \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \top N_{[0,0]}\oplus\perp N_{[0,0]} \\ \perp N_{[0,0]}\oplus\top N_{[0,0]} \end{smallmatrix}\right) \mapsto \cdots \mapsto \left(\begin{smallmatrix} \top N\oplus\perp N \\ \perp N\oplus\top N \end{smallmatrix}\right) \overset{\simeq}{\leftarrow} \left(\begin{smallmatrix} \top FM\oplus\perp FM \\ \perp FM\oplus\top FM \end{smallmatrix}\right) \right)$$

of $K_0[qC\mathcal{M} \overset{F}{\to} qC\mathcal{N}]$. Observe the following diagram:

$$
\begin{array}{ccccccccc}
\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) & \rightarrowtail & \left(\begin{smallmatrix} \top N_{[0,0]}\oplus\perp N_{[0,0]} \\ \perp N_{[0,0]}\oplus\top N_{[0,0]} \end{smallmatrix}\right) & \rightarrowtail & \cdots & \rightarrowtail & \left(\begin{smallmatrix} \top N\oplus\perp N \\ \perp N\oplus\top N \end{smallmatrix}\right) & \overset{\simeq}{\leftarrow} & \left(\begin{smallmatrix} \top FM\oplus\perp FM \\ \perp FM\oplus\top FM \end{smallmatrix}\right) \\
\downarrow{\simeq} & & \downarrow{\simeq} & & & & \downarrow{\simeq} & {\simeq} & \downarrow{\simeq} \\
\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) & \rightarrowtail & \left(\begin{smallmatrix} \top N_{[0,0]}\oplus\perp N_{[0,0]} \\ \top N_{[0,0]}\oplus\perp N_{[0,0]} \end{smallmatrix}\right) & \rightarrowtail & \cdots & \rightarrowtail & \left(\begin{smallmatrix} \top N\oplus\perp N \\ \top N\oplus\perp N \end{smallmatrix}\right) & \overset{\simeq}{\leftarrow} & \left(\begin{smallmatrix} \top FM\oplus\perp FM \\ \top FM\oplus\perp FM \end{smallmatrix}\right)
\end{array}
$$

where the vertical arrows are given by the obvious isomorphisms, and each triangle bounds the usual 2-simplex in the obvious way. This diagram, together with the obvious path $\left(\begin{smallmatrix} \top M\oplus\perp M \\ \perp M\oplus\top M \end{smallmatrix}\right) \overset{\simeq}{\to} \left(\begin{smallmatrix} \top M\oplus\perp M \\ \top M\oplus\perp M \end{smallmatrix}\right)$ provides a homotopy equating $\Phi(X \oplus \tau X)$ with $\Phi(\Delta(\top X \oplus \perp X)) = 0$. Since Φ is an isomorphism, this completes the proof. $\qquad\square$

Theorem 3.2.4. *The groups $K_n^{\mathrm{Gr}}[F]$ and $K_n[F]$ are isomorphic for all $n \geq 0$.*

Proof. In Theorem 3.2.1, we prove the case for $n = 0$, so it remains to prove the statement for $n > 0$. Recall that (an equivalent formulation of) the definition of $K_n^{\mathrm{Gr}}[F]$ is an iterated kernel of the n-cube with vertices $K_0^{\mathrm{Gr}}[D_1 D_2 \cdots D_n F]$ (with $D_1, ..., D_n \in \{C^q, B^q\}$) and edges induced by $\perp$. Set the space X as the iterated homotopy fiber of the $(n+1)$-cube with vertices $K(D_1 D_2 \cdots D_n \mathcal{M})$ and $K(D_1 D_2 \cdots D_n \mathcal{N})$ with edges given by the maps induced by $\perp$ and F in the relevant locations. By a standard argument, $\pi_0 X$ is isomorphic to the iterated kernel of the n-cube with vertices $K_0[D_1 \cdots D_n F]$ and edges induced by $\perp$. Since $K_0[-] \cong K_0^{\mathrm{Gr}}[-]$ as functors, this group is isomorphic to $K_n^{\mathrm{Gr}}[F]$. Rearranging the order of taking

homotopy fibers, we see that

$$\pi_0 X \cong \pi_0 \text{hofib}(K\Omega^n M \xrightarrow{F} K\Omega^n \mathcal{N}).$$

Due to [Gra12, Corollary 7.1], we can complete the proof with the following chain of isomorphisms:

$$\begin{aligned} K_n^{\text{Gr}}[F] &\cong \pi_0 X \\ &\cong \pi_0 \text{hofib}(K\Omega^n \mathcal{M} \xrightarrow{F} K\Omega^n \mathcal{N}) \\ &\cong \pi_0 \text{hofib}(\Omega^n K\mathcal{M} \xrightarrow{F} \Omega^n K\mathcal{N}) \\ &\cong \pi_0 \Omega^n K[F] \\ &\cong K_n[F]. \end{aligned}$$

$\square$

In [Nen98], Nenashev introduced $K_1^{\text{Nen}} \mathcal{N}$, a presentation of K_1 of an exact category $\mathcal{N}$, that we will outline now. Denote by $B_{[0,2]}^q \mathcal{N}$ the category of binary short exact sequences

$$A \rightrightarrows B \rightrightarrows C$$

The group $K_1^{\text{Nen}} \mathcal{N}$ is obtained from $K_0 B_{[0,2]}^q \mathcal{N}$ by imposing two additional relations. Firstly, $[D] = 0$ whenever D is diagonal, and secondly, given a diagram

$$\begin{array}{ccccc} A' & \rightrightarrows & A & \rightrightarrows & A'' \\ \Downarrow & & \Downarrow & & \Downarrow \\ B' & \rightrightarrows & B & \rightrightarrows & B'' \\ \Downarrow & & \Downarrow & & \Downarrow \\ C' & \rightrightarrows & C & \rightrightarrows & C'' \end{array}$$

in $\mathcal{N}$ where the "top" arrows horizontally commute with the "top" arrows vertically (but not necessarily with the "bottom" arrows vertically) and vice-versa for "bottom" arrows, there is a relation

$$[r_1] - [r_2] + [r_3] = [c_1] - [c_2] + [c_3]$$

where r_i is the ith row and c_j is the jth column.

In [Nen98], Nenashev provided an isomorphism

$$K_1^{\text{Nen}} \mathcal{N} \xrightarrow{\cong} K_1 \mathcal{N}$$

which sends the element $\left[\ A\ \underset{j}{\overset{i}{\rightrightarrows}}\ B\ \underset{q}{\overset{p}{\rightrightarrows}}\ C\ \right]$ to the element in $K_1\mathcal{N}$

corresponding to the loop

$$\begin{pmatrix}0\\0\end{pmatrix} \rightarrowtail \begin{pmatrix}A\\A\end{pmatrix} \xrightarrow{(i;j)} \begin{pmatrix}B\\B\end{pmatrix} \leftarrowtail \begin{pmatrix}0\\0\end{pmatrix}.$$

where in addition to our usual notation for 1-simplices in $K\mathcal{N}$ we write $\begin{pmatrix}A\\A\end{pmatrix} \xrightarrow{(i;j)} \begin{pmatrix}B\\B\end{pmatrix}$ to give an explicit label to the morphisms. In other words, this notation denotes the following 1-simplex:

$$
\begin{array}{ccc}
A \xrightarrow{\ i\ } B & \searrow{}^{p} & \\
& & \Rightarrow C \\
A \xrightarrow{\ j\ } B & \nearrow{}_{q} &
\end{array}
$$

There is a natural inclusion functor $B^q_{[0,2]}\mathcal{N} \hookrightarrow B^q\mathcal{N}$, which was shown to induce an isomorphism $K_1^{\mathrm{Nen}}\mathcal{N} \to K_1^{\mathrm{Gr}}\mathcal{N}$ in [KKW20]. We will now show Nenashev's isomorphism is compatible with Grayson's isomorphism $K_1^{\mathrm{Gr}}\mathcal{N}$. In other words

Theorem 3.2.5. *The following square commutes:*

$$
\begin{array}{ccc}
K_1^{\mathrm{Nen}}\mathcal{N} & \xrightarrow{\ \cong\ } & K_1\mathcal{N} \\
\Big\downarrow{\cong} & & \Big\downarrow{\cong} \\
K_1^{\mathrm{Gr}}\mathcal{N} & \xrightarrow{\ \cong\ } & K_1 q C\mathcal{N}
\end{array}
$$

We write Grayson's isomorphism as the composition $K_1^{\mathrm{Gr}}\mathcal{N} \xrightarrow{\cong} K_1 q C\mathcal{N} \xleftarrow{\cong} K_1\mathcal{N}$, where the first isomorphism sends an element of $K_1^{\mathrm{Gr}}\mathcal{N}$ corresponding to the object (N, d, d') to the element of $K_1 q C\mathcal{N}$ corresponding to the loop

$$
\left(\begin{pmatrix}0\\0\end{pmatrix} \rightarrowtail \begin{pmatrix}\top N_{[0,0]}\\ \bot N_{[0,0]}\end{pmatrix} \rightarrowtail \cdots \rightarrowtail \begin{pmatrix}\top N\\ \bot N\end{pmatrix} \xleftarrow{\ \simeq\ } \begin{pmatrix}0\\0\end{pmatrix} \right)
$$

as defined in Theorem 3.2.3. We recall that the second isomorphism is induced by the inclusion $N \hookrightarrow C\mathcal{N}$ which considers an object as a complex concentrated in degree 0.

Proof. Suppose we have an element x of $K_1^{\mathrm{Nen}}\mathcal{N}$ corresponding to the binary short exact sequence

$$A\ \underset{j}{\overset{i}{\rightrightarrows}}\ B\ \underset{q}{\overset{p}{\rightrightarrows}}\ C$$

As described above, x is sent to the element of $K_1\mathcal{N}$ corresponding to the loop

$$\begin{pmatrix}0\\0\end{pmatrix} \rightarrowtail \begin{pmatrix}A\\A\end{pmatrix} \xrightarrow{(i;j)} \begin{pmatrix}B\\B\end{pmatrix} \leftarrowtail \begin{pmatrix}0\\0\end{pmatrix}$$
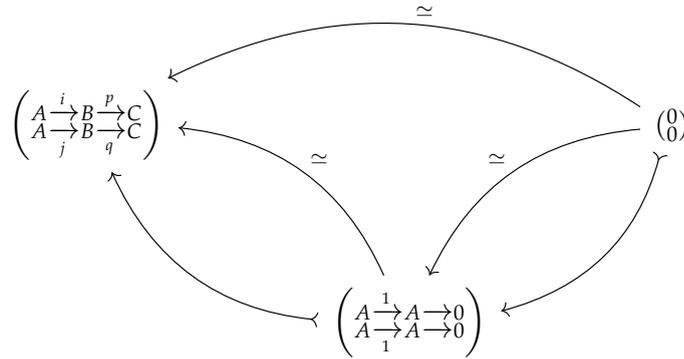
We also see x is sent via the composition

$$K_1^{\text{Nen}}\mathcal{N} \xrightarrow{\cong} K_1^{\text{Gr}}\mathcal{N} \xrightarrow{\cong} K_1 q\mathcal{CN}$$

to the element of $K_1 q\mathcal{CN}$ corresponding to the loop

$$\binom{0}{0} \rightarrowtail \binom{C}{C} \rightarrowtail \begin{pmatrix} B \xrightarrow{p} C \\ B \xrightarrow{q} C \end{pmatrix} \rightarrowtail \begin{pmatrix} A \xrightarrow{i} B \xrightarrow{p} C \\ A \xrightarrow{j} B \xrightarrow{q} C \end{pmatrix} \xleftarrow{\simeq} \binom{0}{0}$$

Observe that we have the following homotopy in $Kq\mathcal{CN}$:

where every space in the diagram can be filled in by a 2-simplex, as explained in Remark 2.2.1. This diagram provides a homotopy between the path

$$\begin{pmatrix} A \xrightarrow{i} B \xrightarrow{p} C \\ A \xrightarrow{j} B \xrightarrow{q} C \end{pmatrix} \xleftarrow{\simeq} \binom{0}{0}$$

and the path

$$\begin{pmatrix} A \xrightarrow{i} B \xrightarrow{p} C \\ A \xrightarrow{j} B \xrightarrow{q} C \end{pmatrix} \leftarrowtail \begin{pmatrix} A \xrightarrow{1} A \to 0 \\ A \xrightarrow{1} A \to 0 \end{pmatrix} \leftarrowtail \binom{0}{0} .$$

Thus, our element in $K_1 q\mathcal{CN}$ is equal to the element associated to the loop

$$\binom{0}{0} \rightarrowtail \binom{C}{C} \rightarrowtail \begin{pmatrix} B \xrightarrow{p} C \\ B \xrightarrow{q} C \end{pmatrix} \rightarrowtail \begin{pmatrix} A \xrightarrow{i} B \xrightarrow{p} C \\ A \xrightarrow{j} B \xrightarrow{q} C \end{pmatrix} \leftarrowtail \begin{pmatrix} A \xrightarrow{1} A \to 0 \\ A \xrightarrow{1} A \to 0 \end{pmatrix} \leftarrowtail \binom{0}{0}$$

The isomorphism

$$K_1 q\mathcal{CN} \to K_1 \mathcal{N}$$

is induced by the Euler characteristic $\chi : Kq\mathcal{CN} \to K\mathcal{N}$. We will now recall an outline of how Euler characteristic works for 0-simplices in this representation of the K-theory space. In particular:

$$\begin{pmatrix} \cdots \to M_2 \to M_1 \to M_0 \\ \cdots \to N_2 \to N_1 \to N_0 \end{pmatrix} \mapsto \begin{pmatrix} \cdots \oplus M_2 \oplus N_1 \oplus M_0 \\ \cdots \oplus N_2 \oplus M_1 \oplus N_0 \end{pmatrix} .$$

We then see that the element of $K_1\mathcal{N}$ that our loop is sent to is given by the following loop:

$$\begin{pmatrix}0\\0\end{pmatrix} \rightarrowtail \begin{pmatrix}C\\C\end{pmatrix} \rightarrowtail \begin{pmatrix}B\oplus C\\B\oplus C\end{pmatrix} \rightarrowtail \begin{pmatrix}A\oplus B\oplus C\\A\oplus B\oplus C\end{pmatrix} \xleftarrow{\left(\begin{pmatrix}1&0\\0&j\\0&0\end{pmatrix};\begin{pmatrix}1&0\\0&i\\0&0\end{pmatrix}\right)} \begin{pmatrix}A\oplus A\\A\oplus A\end{pmatrix} \leftarrowtail \begin{pmatrix}0\\0\end{pmatrix}$$

the first three arrows are diagonal in the sense that both filtrations associated with the 1-simplices are the same, and so they can be contracted. It is fairly easy to see that the remaining loop is the sum of the following two loops:

$$\begin{pmatrix}0\\0\end{pmatrix} \rightarrowtail \begin{pmatrix}A\oplus C\\A\oplus C\end{pmatrix} \leftarrowtail \begin{pmatrix}A\\A\end{pmatrix} \leftarrowtail \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\0\end{pmatrix} \rightarrowtail \begin{pmatrix}B\\B\end{pmatrix} \xleftarrow{(j;i)} \begin{pmatrix}A\\A\end{pmatrix} \leftarrowtail \begin{pmatrix}0\\0\end{pmatrix}$$

the first of which is obviously equal to 0 as the middle arrow is diagonal in the sense mentioned above, and the remaining loop is equal to the required loop, as reversing the direction of a loop and exchanging the "top" and "bottom" filtrations of the simplices both induce minus signs in the element, as shown in [GG87, §1].   □

Chapter 4

Additional Constructions

4.1 Products

From this point forward, we will assume our exact categories come equipped with a symmetric monoidal category structure that is compatible with the exact structure and distributive over direct sum. In other words, assume we have an exact category $\mathcal{M}$ equipped with a bi-exact functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ called a *tensor product*, an object I of $\mathcal{M}$ called the *unit*, and isomorphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$
$$\lambda_A : I \otimes A \to A$$
$$\rho_A : A \otimes I \to A$$
$$\sigma_{A,B} : A \otimes B \to B \otimes A$$
$$\delta_{A,B,C} : (A \otimes B) \oplus (A \otimes C) \to A \otimes (B \oplus C)$$

that are natural in A, B and C and satisfy certain coherence diagrams that guarantee that compositions of such isomorphisms commute in a natural way. We also assume functors are *strong symmetric monoidal exact* functors. In other words, given symmetric monoidal distributive exact categories $\mathcal{M}$ and $\mathcal{N}$ (with tensor products denoted by $\otimes$, units by $I_{\mathcal{M}}$ and $I_{\mathcal{N}}$ respectively, and corresponding isomorphisms $\alpha^{\mathcal{M}}, \alpha^{\mathcal{N}}$ etc) we have an exact functor $F : \mathcal{M} \to \mathcal{N}$ together with isomorphisms

$$\epsilon : I_{\mathcal{N}} \to F(I_{\mathcal{M}})$$
$$\mu_{A,B} : F(A) \otimes F(B) \to F(A \otimes B)$$

that are natural in A and B and satisfy certain natural coherence diagrams.

In this section, we will endow the elementary constructions of K-theory (in particular, those of Bass and Grayson) with products.

Given two objects (A, α) and (B, β) of $\mathrm{Aut}(\mathcal{M})$, we define their *tensor product* simply as $(A, \alpha) \otimes (B, \beta) := (A \otimes B, \alpha \otimes \beta)$. This is indeed endows $\mathrm{Aut}(\mathcal{M})$ with the structure of a symmetric monoidal category, where $(I, 1_I)$ acts as a multiplicative identity. This product endows $K_0\mathrm{Aut}(\mathcal{M})$ with the structure of a commutative ring with unity, and so it is natural to ask whether this allows us to define products on $K_1^{\mathrm{Bass}}\mathcal{M}$. In this case, since $(I, 1_I)$ is in the image of the functor $\Delta : \mathcal{M} \to \mathrm{Aut}(M)$, the multiplication on $K_1^{\mathrm{Bass}}(\mathcal{M})$ induced by the tensor product on $K_0\mathrm{Aut}(\mathcal{M})$ is trivial. It turns out that $K_1^{\mathrm{Bass}}\mathcal{M}$ does not admit a nontrivial ring structure using the tensor product defined on $\mathcal{M}$, however we do have the following construction: Consider the subgroup $\widetilde{I}$ of $K_0\mathrm{Aut}(\mathcal{M})$ generated by elements of the form

$$\langle A; \alpha, \beta \rangle := [A, \beta \circ \alpha] - [A, \beta] - [A, \alpha] + [A, 1].$$

We have an isomorphism $K_0\mathcal{M} \oplus K_1^{\mathrm{Bass}} \to K_0\mathrm{Aut}(\mathcal{M})/\widetilde{I}$ given by

$$(A, (B, \beta)) \mapsto [A \oplus B, 1 \oplus \beta].$$

The tensor product on $\mathrm{Aut}(\mathcal{M})$ does endow $K_0\mathrm{Aut}(\mathcal{M})/\widetilde{I}$ with a product, as shown in [Köc00, page 199]. Given x, y in $K_0\mathcal{M} \oplus K_1^{\mathrm{Bass}}(\mathcal{M})$ the resulting product xy is the usual product if $x, y \in K_0\mathcal{M}$, trivial if $x, y \in K_1^{\mathrm{Bass}}(\mathcal{M})$, and endows $K_1^{\mathrm{Bass}}(\mathcal{M})$ with a $K_0\mathcal{M}$-module structure if $x \in K_0\mathcal{M}, y \in K_1^{\mathrm{Bass}}(\mathcal{M})$.

We will now endow $K_0^{\mathrm{Bass}}[F]$ with a product. Given objects (A, α, B) and (A', α', B') of $\mathrm{co}(F)$, we define their tensor product simply as $(A, \alpha, B) \otimes (A', \alpha', B') := (A \otimes A', \alpha \otimes \alpha', B \otimes B')$. This tensor product, together with unit $(I_\mathcal{M}, I_\mathcal{N}, I_\mathcal{M})$, endows $\mathrm{co}(F)$ with the structure of a symmetric monoidal category, and thus induces a commutative ring structure on $K_0\mathrm{co}(F)$. As before, the unit $(I_\mathcal{M}, 1_{I_\mathcal{N}}, I_\mathcal{M})$ becomes zero in $K_0^{\mathrm{Bass}}[F]$, but in this case the situation can be remedied. Consider the group $\widetilde{K}_0\mathrm{co}(F)$ defined as the kernel of the homomorphism $K_0\perp : K_0\mathrm{co}(F) \to K_0\mathcal{M}$ given by $[A, \alpha, B] \mapsto [B]$. This subgroup becomes an ideal in $K_0\mathrm{co}(F)$.

Lemma 4.1.1. *The subgroup $\widetilde{I}$ of $K_0\mathrm{co}(F)$ generated by elements of the form*

$$\langle A, B, C; \alpha, \beta \rangle := [A, \beta \circ \alpha, C] - [B, \beta, C] - [A, \alpha, B] + [B, 1, B]$$

is an ideal of $\widetilde{K}_0\mathrm{co}(F)$, and the quotient $\widetilde{K}_0\mathrm{co}(F)/\widetilde{I}$ is isomorphic to $K_0^{\mathrm{Bass}}[F]$, thus giving the latter the structure of a nonunital commutative ring.

Proof. Suppose we have elements $\langle A, B, C; \alpha, \beta \rangle \in \widetilde{I}$ and $[D, \delta, E] \in K_0\mathrm{co}(F)$. Then we need to show that their product lies in $\widetilde{I}$, which we show using a modified version of a

proof given in [Köc00, page 199]:

$$[D, \delta, E]\langle A, B, C; \alpha, \beta \rangle = [D, \delta, E]([A, \beta\alpha, C] - [B, \beta, C] - [A, \alpha, B] + [B, 1, B])$$
$$= [D \otimes A, \delta \otimes \beta\alpha, E \otimes C] - [D \otimes B, \delta \otimes \beta, E \otimes C]$$
$$- [D \otimes A, \delta \otimes \alpha, E \otimes B] + [D \otimes B, \delta \otimes 1, E \otimes B]$$
$$= \langle D \otimes A, D \otimes B, E \otimes C; 1 \otimes \alpha, \delta \otimes \beta \rangle$$
$$- \langle D \otimes A, D \otimes B, E \otimes B; 1 \otimes \alpha, \delta \otimes 1 \rangle.$$

The remainder of the proof, namely that $\widetilde{K}_0 \mathrm{co}(F)/\widetilde{I} \cong K_0^{\mathrm{Bass}}[F]$, is easily verified, the details of which can be found in [Bas68]. □

This definition of products on $K_0^{\mathrm{Bass}}[F]$ is clean, but does not offer itself particularly nicely to computation. To this end, suppose we have objects (A, α, B) and (C, γ, D) of $\mathrm{co}(F)$. Their representatives in $\widetilde{K}_0 \mathrm{co}(F)$ are then $[A, \alpha, B] - [B, 1, B]$ and $[C, \gamma, D] - [D, 1, D]$ respectively, since $\perp : \mathrm{co}(F) \to \mathcal{M}$ is split by the functor $\mathcal{M} \to \mathrm{co}(F)$ given by $M \mapsto (M, 1, M)$. Then we have the following computation in $K_0^{\mathrm{Bass}}[F]$

$$([A, \alpha, B] - [B, 1, B])([C, \gamma, D] - [D, 1, D])$$
$$= [A \otimes C, \alpha \otimes \gamma, B \otimes D] - [A \otimes D, \alpha \otimes 1, B \otimes D]$$
$$- [B \otimes C, 1 \otimes \gamma, B \otimes D] + [B \otimes D, 1 \otimes 1, B \otimes D]$$
$$= [A \otimes C, 1 \otimes \gamma, A \otimes D] - [A \otimes D, 1 \otimes 1, A \otimes D]$$
$$- [B \otimes C, 1 \otimes \gamma, B \otimes D] + [B \otimes D, 1 \otimes 1, B \otimes D]$$
$$= [A \otimes C, 1 \otimes \gamma, A \otimes D] - [A \otimes D, 1 \otimes 1, A \otimes D]$$
$$+ [B \otimes D, 1 \otimes \gamma^{-1}, B \otimes C] - [B \otimes C, 1 \otimes 1, B \otimes C]$$

This element is equal to the equivalence class of the following single object of $\mathrm{co}(F)$:

$$\left(\begin{pmatrix} A \otimes C \\ \oplus \\ B \otimes D \end{pmatrix}, \begin{pmatrix} FA \otimes FC \xrightarrow{1 \otimes \gamma} FA \otimes FD \\ \oplus \qquad\qquad \oplus \\ FB \otimes FD \xrightarrow{1 \otimes \gamma^{-1}} FB \otimes FC \end{pmatrix}, \begin{pmatrix} A \otimes D \\ \oplus \\ B \otimes C \end{pmatrix} \right)$$

via a similar computation, we have that this element is also equal to the class of the object

$$\left(\begin{pmatrix} A \otimes C \\ \oplus \\ B \otimes D \end{pmatrix}, \begin{pmatrix} FA \otimes FC \xrightarrow{\alpha \otimes 1} FB \otimes FC \\ \oplus \qquad\qquad \oplus \\ FB \otimes FD \xrightarrow{\alpha^{-1} \otimes 1} FA \otimes FD \end{pmatrix}, \begin{pmatrix} B \otimes C \\ \oplus \\ A \otimes D \end{pmatrix} \right)$$

Lemma 4.1.2. *The product defined above on $K_0^{\text{Bass}}[F]$ makes the following diagram commute*

$$
\begin{array}{ccccccccc}
K_1^{\text{Bass}}\mathcal{M}^2 & \longrightarrow & K_1^{\text{Bass}}\mathcal{N}^2 & \longrightarrow & K_0^{\text{Bass}}[F]^2 & \longrightarrow & K_0\mathcal{M}^2 & \longrightarrow & K_0\mathcal{N}^2 \\
\downarrow {\scriptstyle 0} & & \downarrow {\scriptstyle 0} & & \downarrow {\scriptstyle \otimes} & & \downarrow {\scriptstyle \otimes} & & \downarrow {\scriptstyle \otimes} \\
K_1^{\text{Bass}}\mathcal{M} & \longrightarrow & K_1^{\text{Bass}}\mathcal{N} & \longrightarrow & K_0^{\text{Bass}}[F] & \longrightarrow & K_0\mathcal{M} & \longrightarrow & K_0\mathcal{N}
\end{array}
$$

where the rows are (induced by) the long exact sequence introduced in Section 3.1.

Proof. Since we assumed F is compatible with tensor products we only need to check compatibility with the homomorphisms involving $K_0^{\text{Bass}}[F]$. To this end, given objects $X = (A, \alpha, B)$ and $Y = (C, \gamma, D)$ in $\text{co}(F)$, we have the following computation in $K_0\mathcal{M}$:

$$
\begin{aligned}
[(A \otimes C) \oplus (B \otimes D)] &- [(A \otimes D) \oplus (B \otimes C)] \\
&= [A][C] + [B][D] - [A][D] - [B][C] \\
&= ([A] - [B])([C] - [D]) \\
&= ([\top X] - [\bot X])([\top Y] - [\bot Y]).
\end{aligned}
$$

which verifies that products are compatible with the outgoing homomorphism $K_0^{\text{Bass}}[F] \to K_0\mathcal{M}$. Now to verify that products are compatible with the incoming homomorphism $K_1^{\text{Bass}}\mathcal{N} \to K_0^{\text{Bass}}[F]$, since products are trivial in $K_1^{\text{Bass}}\mathcal{N}$, it suffices to show that products of elements of the form $[A, \alpha, A]$ are zero in $K_0^{\text{Bass}}[F]$. To this end, we have the following computation:

$$
\begin{aligned}
([A, \alpha, A] &- [A, 1, A])([B, \beta, B] - [B, 1, B]) \\
=&[A \otimes B, \alpha \otimes \beta, A \otimes B] - [A \otimes B, \alpha \otimes 1, A \otimes B] \\
&- [A \otimes B, 1 \otimes \beta, A \otimes B] + [A \otimes B, 1 \otimes 1, A \otimes B] \\
=&\langle A \otimes B, A \otimes B, A \otimes B; \alpha \otimes 1, 1 \otimes \beta \rangle \\
=&0
\end{aligned}
$$

since $\alpha \otimes \beta = (\alpha \otimes 1) \circ (1 \otimes \beta)$. $\qquad \square$

We will now define product structures on Grayson's presentations of absolute and relative K-theory. To that end, we need to recall the Dold-Kan correspondence. If $\mathcal{A}$ is an idempotent complete additive category, the Dold-Kan correspondence [Lur17, Theorem 1.2.3.7] gives us an equivalence of categories

$$
\mathcal{A}^{\Delta^{\text{op}}} \xrightleftharpoons[N]{\Gamma} C_{[0,\infty)}\mathcal{A},
$$

between $\mathcal{A}^{\Delta^{\text{op}}}$ (introduced in section 2.2) and $C_{[0,\infty)}\mathcal{A}$, the category of (not necessarily bounded) chain complexes supported in non-negative degrees.

Returning to the structure of an idempotent complete exact category $\mathcal{M}$ equipped with a tensor product, and given chain complexes A and B in $C\mathcal{M}$, following [HKT17] we define the *simplicial tensor product of A and B* as follows:

$$A \otimes_\Delta B := N(\text{diag}(\Gamma A \otimes \Gamma B))$$

where $\Gamma A \otimes \Gamma B$ is the bi-simplicial object in $\mathcal{M}$ with object $(\Gamma A)_i \otimes (\Gamma B)_j$ in the (i,j)th index. By the Eilenberg-Zilber theorem [DP61, Satz 2.9], $A \otimes_\Delta B$ and $\text{Tot}(A \otimes B)$ are homotopy equivalent. We use the former as our definition of products as they will be more obviously compatible with exterior powers defined in Section 4.2. We define the simplicial tensor product of binary complexes in $\mathcal{M}$ by viewing them as objects in $C\mathcal{M}^2$ and applying the simplicial tensor product pointwise. The resulting pair of chain complexes is then a binary complex again. We will note here that while this definition does induce a well-defined product in $K_0 B^q \mathcal{M}$, it was shown in [HKT17, Proposition 5.11] that this multiplication becomes trivial in $K_1^{\text{Gr}} \mathcal{M}$ (and similarly for $(B^q)^n \mathcal{M}$ and $K_n^{\text{Gr}} \mathcal{M}$).

Now for an exact functor $F : \mathcal{M} \to \mathcal{N}$ between idempotent complete exact categories equipped with tensor products such that F is compatible with those tensor products, it is natural to ask if the simplicial tensor product induces a tensor product in the categories $C[F]$ and $B[F]$. Indeed, given triples (M, N, u) and (M', N', u') in $C[F]$, we define their tensor product coordinate-wise

$$(M, N, u) \otimes_\Delta (M', N', u') := (M \otimes_\Delta M', N \otimes_\Delta N', u \otimes_\Delta u')$$

where $u \otimes_\Delta u' : F(M \otimes_\Delta M') \cong FM \otimes_\Delta FM' \to N \otimes_\Delta N'$ is again a quasi-isomorphism due to the Eilenberg-Zilber theorem [DP61]. Simplicial tensor products of objects in $B[F]$ are then defined in the obvious way. These products then clearly induce well-defined products on $K_0 C[F]$ and $K_0 pC[F]$ (similarly for $B[F]$).

Similarly to $K_0^{\text{Bass}}[F]$, the image of the identity element of $K_0 pC[F]$ becomes zero in $K_0 pB[F]$ after applying Δ, but we can identify $K_0^{\text{Gr}}[F]$ with the kernel of the homomorphism $K_0 \perp : K_0 pB[F] \to K_0 pC[F]$. With this identification, products in $K_0 pB[F]$ induce well-defined products on $K_0^{\text{Gr}}[F]$ (without a unit).

Lemma 4.1.3. *Products in $K_0^{\text{Gr}}[F]$ are compatible with the long exact sequence*

$$K_1^{\text{Gr}} \mathcal{M} \longrightarrow K_1^{\text{Gr}} \mathcal{N} \longrightarrow K_0^{\text{Gr}}[F] \longrightarrow K_0 \mathcal{M} \longrightarrow K_0 \mathcal{N}$$

given in Section 2.2.

Proof. First we will show compatibility with the outgoing map $K_0^{\text{Gr}}[F] \to K_0 \mathcal{M}$. Given representatives $X = (M, N, u)$ and $Y = (M', N', u')$ in $B[F]$ of elements x and y of

$K_0^{\mathrm{Gr}}[F]$, we have the following computation:

$$
\begin{aligned}
xy =&([X] - [\Delta \bot X])([Y] - [\Delta \bot Y]) \\
=&[X \otimes_\Delta Y] - [X \otimes_\Delta \Delta \bot Y] - [\Delta \bot X \otimes_\Delta Y] + [\Delta \bot X \otimes_\Delta \Delta \bot Y] \\
\mapsto&\chi(\top M \top M') - \chi(\bot M \bot M') - \chi(\top M \bot M') \\
&+ \chi(\bot M \bot M') - \chi(\bot M \top M') + \chi(\bot M \bot M') \\
=&\chi(\top M)\chi(\top M') - \chi(\top M)\chi(\bot M') - \chi(\bot M)\chi(\top M') + \chi(\bot M)\chi(\bot M') \\
=&(\chi(\top M) - \chi(\bot M))(\chi(\top M') - \chi(\bot M'))
\end{aligned}
$$

which is equal to the product of the images of x and y, where due to the Eilenberg-Zilber theorem $\chi(A \otimes_\Delta B) = \chi(A)\chi(B)$ for chain complexes A and B. Now to verify compatibility with the incoming map, recall that due to [HKT17, Proposition 5.11], the class of $[N \otimes_\Delta N']$ vanishes in $K_1^{\mathrm{Gr}}\mathcal{N}$ for any binary complexes N and N'. Thus the product on $K_1^{\mathrm{Gr}}\mathcal{N}$ given by identifying it with the kernel of the homomorphism $K_0 \bot : K_0 B^q \mathcal{N} \to K_0 C^q \mathcal{N}$ is on the one hand compatible with the incoming map $K_1^{\mathrm{Gr}}\mathcal{N} \to K_0^{\mathrm{Gr}}[F]$ but also vanishes. $\square$

4.2   Exterior power operations

In the case where $\mathcal{M}$ is the category $\mathbf{Proj}_{\mathrm{f}}(R)$ of finitely generated projective modules over a commutative ring R, we naturally have the structure of a symmetric monoidal category given by $\otimes$, but we also have *exterior powers* $\Lambda^k P$ for any projective R-module P. These tensor powers induce operations $\lambda^k : K_0\mathcal{M} \to K_0\mathcal{M}$ which makes $K_0\mathcal{M}$ into a so-called λ-ring, which we will outline now. A commutative (not necessarily unital) ring R together with operations $\lambda^k : R \to R$ for $k \geq 1$ is called a *pre-λ-ring* if $\lambda^1(x) = x$ for every $x \in R$ and the following holds for every $x, y \in R$ and for every $k \geq 1$

$$
\lambda^k(x + y) = \lambda^k(x) + \sum_{i=1}^{k-1} \lambda^{k-i}(x)\lambda^i(y) + \lambda^k(y).
$$

When R is unital, we can also set $\lambda^0(x) = 1$ for every $x \in R$. In this case we can rewrite the above formula as $\lambda^k(x + y) = \sum_{i=0}^{k} \lambda^{k-i}(x)\lambda^i(y)$. For nonunital R, we will abuse notation to allow $\lambda^0(x)$ in products, for example $\lambda^0(x)\lambda^k(y) = \lambda^k(y)$. The following are some examples of pre-λ-rings, given in [AT69]:

- The ring $\mathbb{Z}$ is a pre-λ-ring with operations $\lambda^k(n) = \binom{n}{k}$.

- The polynomial ring $\mathbb{Z}[x_1, x_2, ...]$ with indeterminates indexed by $\mathbb{N}$ is a pre-λ-ring, where the operations are determined by $\lambda^k(x_1) = x_k$ and $\lambda^k(n) = \binom{n}{k}$ for $n \in \mathbb{Z}$.

- The ring $K_0(R)$ for a ring R is a pre-λ-ring with operations given by exterior power operations.

Given any pre-λ-ring R, we have the following formula given in [Gra89, Equation 2.3] for every $x, y \in R$ and $k \geq 1$:

$$\lambda^k(x - y) = \sum_{\substack{a+b_1+\cdots+b_u=k \\ a,u\geq 0, b_1,\ldots,b_u\geq 1}} (-1)^u \lambda^a(x) \lambda^{b_1}(y) \cdots \lambda^{b_u}(y).$$

We will now define exterior powers on Bass' presentations of absolute and relative K-theory. Firstly, taking $\mathcal{M} = \mathbf{Proj}_f(R)$ for a commutative ring with unity R, if we have an object (A, α) of $\mathrm{Aut}(\mathcal{M})$, we define its kth exterior power as follows:

$$\lambda^k(A, \alpha) := (\Lambda^k A, \Lambda^k \alpha)$$

where $\lambda^k \alpha : \Lambda^k A \to \Lambda^k A$ is again an isomorphism. These operations naturally induce exterior power operations $\lambda^k : K_0 \mathrm{Aut}(\mathcal{M}) \to K_0 \mathrm{Aut}(\mathcal{M})$. Following our approach to defining products in the previous section, we define the group $\widetilde{K}_0 \mathrm{Aut}(\mathcal{M})$ as the kernel of the homomorphism $K_0 \bot : K_0 \mathrm{Aut}(\mathcal{M}) \to K_0 \mathcal{M}$ given by $[A, \alpha] \mapsto [A]$. This group naturally inherits exterior power operations from $K_0 \mathrm{Aut}(\mathcal{M})$. There is also a subgroup $\widetilde{I}$ generated by elements of the form

$$\langle A; \alpha, \beta \rangle := [A, \beta \circ \alpha] - [A, \beta] - [A, \alpha] + [A, 1_A],$$

and the quotient $\widetilde{K}_0 \mathrm{Aut}(\mathcal{M})/\widetilde{I}$ is isomorphic to $K_1^{\mathrm{Bass}}(\mathcal{M})$.

Lemma 4.2.1. *The above definition of exterior power operations on $\widetilde{K}_0 \mathrm{Aut}(\mathcal{M})$ induce well-defined homomorphisms $\lambda^k : K_1^{\mathrm{Bass}}(\mathcal{M}) \to K_1^{\mathrm{Bass}}(\mathcal{M})$.*

Proof. As explained in [Gra89, §2], the map

$$\lambda_t : K_0 \mathrm{Aut}(\mathcal{M}) \to (1 + t K_0 \mathrm{Aut}(\mathcal{M})[[t]])^\times$$

given by

$$(A, \alpha) \mapsto \sum_{k \geq 0} \lambda^k(A, \alpha) t^k$$

is a group homomorphism. Thus, if we show that λ^k sends an element $\langle A; \alpha, \beta \rangle \in \widetilde{I}$ to another element in $\widetilde{I}$, then λ_t induces a well-defined homomorphism on $K_1^{\mathrm{Bass}}(\mathcal{M})$ and thus we can recover the original statement. To this end, we suppose $X = [A, \beta \circ \alpha] + [A, 1_A]$ and $Y = [A, \beta] + [A, \alpha]$, and we have the following

computation in $\widetilde{K}_0\mathrm{Aut}(\mathcal{M})$:

$$\lambda^k\langle A; \alpha, \beta\rangle = \lambda^k(X - Y)$$

$$= \sum_{\substack{a+b_1+\cdots+b_u=k \\ a,u\geq 0, b_1,\ldots,b_u\geq 1}} (-1)^u \lambda^a(X)\lambda^{b_1}(Y)\cdots\lambda^{b_u}(Y)$$

$$= \sum_{\substack{a+b_1+\cdots+b_u=k \\ u\geq 0,a,b_1,\ldots,b_u\geq 1}} (-1)^u \lambda^a(X)\lambda^{b_1}(Y)\cdots\lambda^{b_u}(Y) + \sum_{\substack{b_1+\cdots+b_u=k \\ u,b_1,\ldots,b_u\geq 1}} (-1)^u \lambda^{b_1}(Y)\cdots\lambda^{b_u}(Y)$$

$$= \sum_{\substack{a+b_1+\cdots+b_u=k \\ u\geq 0,a,b_1,\ldots,b_u\geq 1}} (-1)^u \lambda^a(X)\lambda^{b_1}(Y)\cdots\lambda^{b_u}(Y) + \sum_{\substack{a+b_1+\cdots+b_u=k \\ u\geq 0,a,b_1,\ldots,b_u\geq 1}} (-1)^u \lambda^a(Y)\lambda^{b_1}(Y)\cdots\lambda^{b_u}(Y)$$

$$= \sum_{\substack{a+b_1+\cdots+b_u=k \\ u\geq 0,a,b_1,\ldots,b_u\geq 1}} (-1)^u \left(\lambda^a(X)\lambda^{b_1}(Y)\cdots\lambda^{b_u}(Y) - \lambda^a(Y)\lambda^{b_1}(Y)\cdots\lambda^{b_u}(Y)\right)$$

$$= \sum_{\substack{a+b_1+\cdots+b_u=k \\ u\geq 0,a,b_1,\ldots,b_u\geq 1}} (-1)^u \left(\lambda^a(X) - \lambda^a(Y)\right)\lambda^{b_1}(Y)\cdots\lambda^{b_u}(Y)$$

where we identify the index sets

$$\left\{\begin{smallmatrix} b_1+\cdots+b_u=k \\ u,b_1,\ldots,b_u\geq 1\end{smallmatrix}\right\} \sim \left\{\begin{smallmatrix} a+b_1+\cdots+b_u=k \\ u\geq 0,a,b_1,\ldots,b_u\geq 1\end{smallmatrix}\right\}.$$

Since $\widetilde{I}$ is an ideal of $K_0\mathrm{Aut}(\mathcal{M})$, it remains to show that $\lambda^a(X) - \lambda^a(Y)$ is of the desired form for each $a \geq 1$:

$$\lambda^a(X) - \lambda^a(Y) = \lambda^a([A, \beta\alpha] + [A, 1]) - \lambda^a([A, \beta] + [A, \alpha])$$

$$= \sum_{i=0}^{a} \lambda^i[A, \beta\alpha]\lambda^{a-i}[A, 1] - \sum_{i=0}^{a} \lambda^i[A, \beta]\lambda^{a-i}[A, \alpha]$$

$$= \sum_{i=0}^{a} \left(\lambda^i[A, \beta\alpha]\lambda^{a-i}[A, 1] - \lambda^i[A, \beta]\lambda^{a-i}[A, \alpha]\right)$$

we see that the ith summand of this expression is equal to

$$[\Lambda^i A \otimes \Lambda^{a-i}A, \Lambda^i(\beta\alpha) \otimes 1] - [\Lambda^i A \otimes \Lambda^{a-i}A, \Lambda^i\beta \otimes \Lambda^{a-i}\alpha]$$

$$= [\Lambda^i A \otimes \Lambda^{a-i}A, \Lambda^i\alpha \otimes 1] - [\Lambda^i A \otimes \Lambda^{a-i}A, 1 \otimes \Lambda^{a-i}\alpha]$$

$$+ \langle\Lambda^i A \otimes \Lambda^{a-i}A; \Lambda^i\alpha \otimes 1, \Lambda^i\beta \otimes 1\rangle - \langle\Lambda^i A \otimes \Lambda^{a-i}A; 1 \otimes \Lambda^{a-i}\alpha, \Lambda^i\beta \otimes 1\rangle$$

and so pairing the ith summand with the $(a - i)$th summand will yield an element of $\widetilde{I}$, and when a is even (and thus there is an odd number of terms), the $a/2$th summand will also be in $\widetilde{I}$ and thus it follows that λ^k is well defined. To show that λ^k is a homomorphism, recall that products vanish in $K_1^{\mathrm{Bass}}\mathcal{M}$, so given elements X and Y of $K_1^{\mathrm{Bass}}\mathcal{M}$, we have the computation:

$$\lambda^k(X + Y) = \sum_{i=0}^{k} \lambda^i X\lambda^{k-i}Y = \lambda^k X + \lambda^k Y.$$

$\square$

In order to discuss exterior powers in the context of relative *K*-theory, we assume we have a ring homomorphism $f : R \to S$ and we take $\mathcal{M} = \mathbf{Proj}_f(R), \mathcal{N} = \mathbf{Proj}_f(S)$ and $F : \mathcal{M} \to \mathcal{N}$ is given by the base change functor $M \mapsto M \otimes_R S$. Given an object (A, α, B) of co(F), we define the *k*th exterior power of (A, α, B) as follows:

$$\Lambda^k(A, \alpha, B) := (\Lambda^k A, \Lambda^k \alpha, \Lambda^k B)$$

where naturally $\Lambda^k \alpha : F \Lambda^k A \cong \Lambda^k FA \to \Lambda^k FB \cong F \Lambda^k B$ is again an isomorphism, recalling the canonical isomorphism $\Lambda_R^k M \otimes_R S \to \Lambda_S^k(M \otimes_R S)$. Similarly to above, these exterior powers induce exterior power operations $\lambda^k : K_0 \text{co}(F) \to K_0 \text{co}(F)$, and $\widetilde{K}_0 \text{co}(F)$ inherits these operations.

Lemma 4.2.2. *The above definition of exterior power operations on $\widetilde{K}_0 \text{co}(F)$ induce well-defined exterior power operations $\lambda^k : K_0^{\text{Bass}}[F] \to K_0^{\text{Bass}}[F]$.*

Proof. It suffices to show elements $\langle A, B, C; \alpha, \beta \rangle$ in the ideal $\widetilde{I}$ of $\widetilde{K}_0 \text{co}(F)$ introduced in Section 4.1 are sent via λ^k to another element of the ideal, as explained in the proof of *Lemma* 4.2.1. To prove this, we can apply the same argument used in the proof of Lemma 4.2.1, with care taken with the source and target of the relevant isomorphisms. $\square$

We will now outline a computation of the exterior power operations on $K_0^{\text{Bass}}[F]$. Given an object $X = (A, \alpha, B)$ representing an element $x = [X] - [\Delta \perp X]$ of $K_0^{\text{Bass}}[F]$,

we have the following computation for $k \geq 1$:

$$
\begin{aligned}
\lambda^k x =& \lambda^k([X] - [\Delta \perp X]) \\
=& \sum_{\substack{a+b_1+\cdots+b_u=k \\ a,u\geq 0, b_1,\ldots,b_u\geq 1}} (-1)^u \lambda^a[X]\lambda^{b_1}[\Delta\perp X]\cdots\lambda^{b_u}[\Delta\perp X] \\
&+ \sum_{\substack{b_1+\cdots+b_u=k \\ u,b_1,\ldots,b_u\geq 1}} (-1)^u \lambda^{b_1}[\Delta\perp X]\cdots\lambda^{b_u}[\Delta\perp X] \\
=& \sum_{\substack{a+b_1+\cdots+b_u=k \\ u\geq 0, a,b_1,\ldots,b_u\geq 1}} (-1)^u \lambda^a[X]\lambda^{b_1}[\Delta\perp X]\cdots\lambda^{b_u}[\Delta\perp X] + \lambda^k(-[\Delta\perp X]) \\
=& \sum_{\substack{a+b_1+\cdots+b_u=k \\ u\geq 0, a,b_1,\ldots,b_u\geq 1}} (-1)^u [\Lambda^a A, \Lambda^a\alpha, \Lambda^a B][\Lambda^{b_1}B, 1, \Lambda^{b_1}B]\cdots[\Lambda^{b_u}B, 1, \Lambda^{b_u}B] + \lambda^k(-[\Delta\perp X]) \\
=& \sum_{\substack{a+b_1+\cdots+b_u=k \\ u \text{ even} \\ u\geq 0, a,b_1,\ldots,b_u\geq 1}} [\Lambda^a A, \Lambda^a\alpha^{(-1)^u}, \Lambda^a B][\Lambda^{b_1}B, 1, \Lambda^{b_1}B]\cdots[\Lambda^{b_u}B, 1, \Lambda^{b_u}B] \\
&+ \sum_{\substack{a+b_1+\cdots+b_u=k \\ u \text{ odd} \\ u\geq 0, a,b_1,\ldots,b_u\geq 1}} [\Lambda^a B, \Lambda^a\alpha^{(-1)^u}, \Lambda^a A][\Lambda^{b_1}B, 1, \Lambda^{b_1}B]\cdots[\Lambda^{b_u}B, 1, \Lambda^{b_u}B] + \lambda^k(-[\Delta\perp X]) \\
=& [\Lambda^k_+(A,B), \widetilde{\Lambda}^k\alpha, \Lambda^k_-(A,B)] + \lambda^k(-[\Delta\perp X])
\end{aligned}
$$

where we define the following notation:

$$
\begin{aligned}
\Lambda^k_+(A,B) &:= \bigoplus_{\substack{a+b_1+\cdots+b_u=k \\ u \text{ even} \\ a,u\geq 0, b_1,\ldots,b_u\geq 1}} \Lambda^a A \otimes \Lambda^{b_1}B \otimes\cdots\otimes \Lambda^{b_u}B, \\
\Lambda^k_-(A,B) &:= \bigoplus_{\substack{a+b_1+\cdots+b_u=k \\ u \text{ odd} \\ a,u\geq 0, b_1,\ldots,b_u\geq 1}} \Lambda^a A \otimes \Lambda^{b_1}B \otimes\cdots\otimes \Lambda^{b_u}B \\
\widetilde{\Lambda}^k\alpha &:= \bigoplus_{\substack{a+b_1+\cdots+b_u=k \\ u\geq 0, a,b_1,\ldots,b_u\geq 1}} \Lambda^a\alpha^{(-1)^u} \otimes 1 \otimes\cdots\otimes 1
\end{aligned}
$$

Note we have one-to-one correspondences between index sets

$$
\left\{\begin{smallmatrix} a+b_1+\cdots+b_u=k \\ u \text{ even} \\ u\geq 0, a,b_1,\ldots,b_u\geq 1 \end{smallmatrix}\right\} \sim \left\{\begin{smallmatrix} b_1+\cdots+b_u=k \\ u \text{ odd} \\ u,b_1,\ldots,b_u\geq 1 \end{smallmatrix}\right\}
$$

and

$$
\left\{\begin{smallmatrix} a+b_1+\cdots+b_u=k \\ u \text{ odd} \\ u\geq 0, a,b_1,\ldots,b_u\geq 1 \end{smallmatrix}\right\} \sim \left\{\begin{smallmatrix} b_1+\cdots+b_u=k \\ u \text{ even} \\ u,b_1,\ldots,b_u\geq 1 \end{smallmatrix}\right\}.
$$

Proposition 4.2.3. *Exterior power operations on $K_1^{\text{Bass}}(\mathcal{M})$ and $K_0^{\text{Bass}}[F]$ are compatible with the usual long exact sequence, in other words the following diagram commutes:*

$$
\begin{array}{ccccccccc}
K_1^{\text{Bass}}\mathcal{M} & \longrightarrow & K_1^{\text{Bass}}\mathcal{N} & \longrightarrow & K_0^{\text{Bass}}[F] & \longrightarrow & K_0\mathcal{M} & \longrightarrow & K_0\mathcal{N} \\
\downarrow{\scriptstyle\lambda^k} & & \downarrow{\scriptstyle\lambda^k} & & \downarrow{\scriptstyle\lambda^k} & & \downarrow{\scriptstyle\lambda^k} & & \downarrow{\scriptstyle\lambda^k} \\
K_1^{\text{Bass}}\mathcal{M} & \longrightarrow & K_1^{\text{Bass}}\mathcal{N} & \longrightarrow & K_0^{\text{Bass}}[F] & \longrightarrow & K_0\mathcal{M} & \longrightarrow & K_0\mathcal{N}
\end{array}
$$

Proof. Since F is compatible with exterior powers, we need only verify the commutativity of the squares involving the incoming and outgoing maps of $K_0^{\text{Bass}}[F]$. First, we will discuss the incoming maps. Given an element $x \in K_1^{\text{Bass}}\mathcal{N}$, we pick a representative $X = (A, \alpha)$ in $\text{Aut}(\mathcal{N})$ such that $x = [A, \alpha] - [A, 1]$. Then we have objects A' in $\mathcal{N}$ and B in $\mathcal{M}$ such that $FB \cong A \oplus A'$. Setting $\beta := \alpha \oplus 1$, we have that the image of x in $K_0^{\text{Bass}}[F]$ is equal to $[B, \beta, B] - [B, 1, B]$. Then the preimage of $\lambda^k([B, \beta, B] - [B, 1, B])$ is $\lambda^k([FB, \beta] - [FB, 1])$ in $K_1^{\text{Bass}}\mathcal{N}$, which is clearly equal to $\lambda^k([A, \alpha] - [A, 1])$. Now to demonstrate compatibility with outgoing maps, suppose we have an element $y \in K_0^{\text{Bass}}[F]$, and pick a representative $[Y, \zeta, Z]$ of y in $\text{co}(F)$ such that $y = [Y, \zeta, Z] - [Z, 1, Z]$. Then we have that the image of $\lambda^k y$ in $K_0\mathcal{M}$ is

$$
\begin{aligned}
[\Lambda_+^k(Y, Z)] - [\Lambda_-^k(Y, Z)] &= \left[\bigoplus_{\substack{a+b_1+\cdots+b_u=k \\ u \text{ even} \\ a,u \geq 0, b_1,\ldots,b_u}} \Lambda^a Y \otimes \Lambda^{b_1} Z \otimes \cdots \otimes \Lambda^{b_u} Z \right] \\
&\quad - \left[\bigoplus_{\substack{a+b_1+\cdots+b_u=k \\ u \text{ odd} \\ a,u \geq 0, b_1,\ldots,b_u}} \Lambda^a Y \otimes \Lambda^{b_1} Z \otimes \cdots \otimes \Lambda^{b_u} Z \right] \\
&= \sum_{\substack{a+b_1+\cdots+b_u=k \\ a,u \geq 0, b_1,\ldots,b_u \geq 1}} (-1)^u \lambda^a[Y]\lambda^{b_1}[Z] \cdots \lambda^{b_u}[Z] \\
&= \lambda^k([Y] - [Z])
\end{aligned}
$$

as required. □

Now we will recall the construction to define exterior power operations on Grayson's higher algebraic K-groups. We note here that while we use the setup of categories of projective modules over commutative rings, these constructions also apply in contexts where the assumptions making Bass' K-theory isomorphic to the classical approach are not satisfied, such as using the category of vector bundles over a scheme, where we use the setup of assemblies of power operations [KZ21], and functors come with isomorphisms $F\Lambda^k X \cong \Lambda^k FX$. Given a chain complex A in $C\mathcal{M}$, the *kth exterior power*

of A is defined as in [HKT17, Proposition 3.1] as

$$\Lambda^k A := N\Lambda^k \Gamma A,$$

where as in defining products the functors N and Γ are the functors used in the Dold-Kan correspondence. Naturally, the exterior powers of a binary complex are obtained by considering the binary complex as a pair of complexes, applying this construction to both, and then returning to the binary complex view, since the resulting pair of complexes will once again have the same objects. This definition can be iterated in the same way as [HKT17, Definition 3.3] to apply to n-dimensional multicomplexes by viewing them as complexes in the category of $(n-1)$-dimensional multicomplexes and applying the above formula. As shown in [HKT17, Theorem 6.2], these exterior powers induce well-defined homomorphisms $\lambda^k : K_n^{\mathrm{Gr}}\mathcal{M} \to K_n^{\mathrm{Gr}}\mathcal{M}$.

Returning to the relative situation, we will now define exterior powers on $C[F]$ and $B[F]$. Given an object $X = (M, N, u)$ of $C[F]$, we define the *kth exterior power of X* as follows:

$$\Lambda^k X := (\Lambda^k M, \Lambda^k N, \Lambda^k u),$$

where $\Lambda^k u : F\Lambda^k M \cong \Lambda^k FM \to \Lambda^k N$ is again a quasi-isomorphism due to [KZ21, Proposition 1.9]. We do the natural thing when X is an object of $B[F]$.

Lemma 4.2.4. *Exterior powers on $C[F]$ and $B[F]$ give us well-defined operations on $K_0 C[F]$ and $K_0 B[F]$, and hence also on $K_0 p C[F]$ and $K_0 p B[F]$.*

Proof. We will show the operations are well-defined on $K_0 C[F]$, and it will follow that the filtration we will build in $C[F]$ also applies in $B[F]$ and hence proves Lemma 4.2.4 for $B[F]$. Suppose we have a short exact sequence

$$X' \rightarrowtail X \twoheadrightarrow X''$$

in $C[F]$, where $X' = (M', N', u')$, $X = (M, N, u)$ and $X'' = (M'', N'', u'')$. Then as usual we need to show the following formula:

$$[\Lambda^k X] = [\Lambda^k X'] + \sum_{i=1}^{k-1}[\Lambda^{k-i} X' \otimes \Lambda^i X''] + [\Lambda^k X'']$$

for every $k \geq 0$. We have filtrations

$$\Lambda^k M' \hookrightarrow \Lambda^{k-1}M' \wedge \Lambda^1 M \hookrightarrow \cdots \hookrightarrow \Lambda^1 M' \wedge \Lambda^{k-1}M \hookrightarrow \Lambda^k M,$$

$$\Lambda^k N' \hookrightarrow \Lambda^{k-1}N' \wedge \Lambda^1 N \hookrightarrow \cdots \hookrightarrow \Lambda^1 N' \wedge \Lambda^{k-1}N \hookrightarrow \Lambda^k N,$$

and short exact sequences

$$\Lambda^{k-i+1}M' \wedge \Lambda^{i-1}M \rightarrowtail \Lambda^{k-i}M' \wedge \Lambda^i M \twoheadrightarrow \Lambda^{k-i}M' \otimes \Lambda^i M'',$$

$$\Lambda^{k-i+1}N' \wedge \Lambda^{i-1}N \rightarrowtail \Lambda^{k-i}N' \wedge \Lambda^i N \twoheadrightarrow \Lambda^{k-i}N' \otimes \Lambda^i N'',$$

for $i = 1, ..., k$ as given in the proof of [HKT17, Theorem 6.2]. Then we have quasi-isomorphisms

$$\Lambda^{k-i}u' \wedge \Lambda^i u : \Lambda^{k-i}M' \wedge \Lambda^i M \to \Lambda^{k-i}N' \wedge \Lambda^i N$$

for $i = 1, ..., k$, defined using the Dold-Kan correspondence, i.e.

$$\Lambda^{k-i}u' \wedge \Lambda^i u := N(\Lambda^{k-i}\Gamma(u') \wedge \Lambda^i \Gamma(u)),$$

(see [HKT17] for details on the construction of the functor $-\Lambda-$ in this context) which commute with the maps in the above filtrations by definition and thus assemble the short exact sequences into short exact sequences in $C[F]$:

$$\Lambda^{k-i+1}X' \wedge \Lambda^{i-1}X \rightarrowtail \Lambda^{k-i}X' \wedge \Lambda^i X \twoheadrightarrow \Lambda^{k-i}X' \otimes \Lambda^i X'',$$

for $i = 1, ..., k$.

Now to show λ^k is well-defined on $K_0 pC[F]$, suppose we have a morphism

$$(\phi, \psi) : (M, N, u) \to (M', N', u')$$

in $pC[F]$, i.e. ϕ is a quasi-isomorphism and ψ is an isomorphism, then

$$\Lambda^k \phi : \Lambda^k M \to \Lambda^k M'$$

is also a quasi-isomorphism by [KZ21, Proposition 1.9], and

$$\Lambda^k \psi : \Lambda^k N \to \Lambda^k N'$$

is clearly an isomorphism, and it is clear that

$$\Lambda^k u' \circ F\Lambda^k \phi = \Lambda^k \psi \circ \Lambda^k u$$

and so

$$(\Lambda^k \phi, \Lambda^k \psi) : (\Lambda^k M, \Lambda^k N, \Lambda^k u) \to (\Lambda^k M', \Lambda^k N', \Lambda^k u')$$

is a morphism in $pC[F]$, and thus

$$\lambda^k[M, N, u] = \lambda^k[M', N', u']$$

in $K_0 pC[F]$ as required. $\square$

We remark here that exterior powers on $B[F]$ do induce well-defined exterior power operations on $K_0^{\mathrm{Gr}}[F]$, but these are not compatible with the long exact sequence

$$K_1^{\mathrm{Gr}}\mathcal{M} \longrightarrow K_1^{\mathrm{Gr}}\mathcal{N} \longrightarrow K_0^{\mathrm{Gr}}[F] \longrightarrow K_0\mathcal{M} \longrightarrow K_0\mathcal{N}$$

To see this, suppose we have an object $X = (M, N, u)$ of $B[F]$. Then applying the homomorphism $K_0^{\mathrm{Gr}}[F] \to K_0\mathcal{M}$ to the element $[\Lambda^k X]$ yields the element $\chi(\Lambda^k \top M) - \chi(\Lambda^k \bot M)$, which is clearly different from $\lambda^k(\chi(\top M) - \chi(\bot M))$. As before, this issue can be remedied by identifying $K_0^{\mathrm{Gr}}[F]$ with

$$\ker(\bot : K_0 pB[F] \to K_0 pC[F]).$$

Then, given an element $x \in K_0^{\mathrm{Gr}}[F]$ and an object $X = \left(\binom{A}{B}, \left(N, \frac{d}{d'}\right), \binom{u}{v}\right)$ of $B[F]$ representing x such that $x = [X] - [\Delta \bot X]$, we have the following computation:

$$
\begin{aligned}
\lambda^k x =& \lambda^k([X] - [\Delta\bot X]) \\
=& \sum_{\substack{a+b_1+\cdots+b_u=k \\ a,u\geq 0, b_1,\ldots,b_u\geq 1}} (-1)^u \lambda^a[X]\lambda^{b_1}[\Delta\bot X]\cdots\lambda^{b_u}[\Delta\bot X] \\
=& \sum_{\substack{a+b_1+\cdots+b_u=k \\ a,u\geq 0, b_1,\ldots,b_u\geq 1}} \left[\tau^u\Lambda^a X \otimes \Lambda^{b_1}\Delta\bot X \otimes \cdots \otimes \Lambda^{b_u}\Delta\bot X\right] \\
=& \sum_{\substack{a+b_1+\cdots+b_u=k \\ u\,\text{even} \\ a,u\geq 0, b_1,\ldots,b_u\geq 1}} \left[\Lambda^a X \otimes \Lambda^{b_1}\Delta\bot X \otimes \cdots \otimes \Lambda^{b_u}\Delta\bot X\right] \\
& + \sum_{\substack{a+b_1+\cdots+b_u=k \\ u\,\text{odd} \\ a,u\geq 0, b_1,\ldots,b_u\geq 1}} \left[\Lambda^a \tau X \otimes \Lambda^{b_1}\Delta\bot X \otimes \cdots \otimes \Lambda^{b_u}\Delta\bot X\right] \\
=& \left[\left(\begin{matrix}\Lambda^k_+(A,B) \\ \Lambda^k_-(A,B)\end{matrix}\right), \left(\widehat{\Lambda}^k N, \begin{matrix}\Lambda^k_+(d,d') \\ \Lambda^k_-(d,d')\end{matrix}\right), \left(\begin{matrix}\Lambda^k_+(u,v) \\ \Lambda^k_-(u,v)\end{matrix}\right)\right]
\end{aligned}
$$

where Λ_+ and Λ_- take the same meaning as earlier in this section, and

$$\widehat{\Lambda}^k N := \Lambda^k_+(N, N) \cong \Lambda^k_-(N, N).$$

From this point forward we refer to *the* exterior power operation on $K_0^{\mathrm{Gr}}[F]$ as the latter construction of exterior power operations.

Lemma 4.2.5. *Exterior power operations defined on $K_1^{\mathrm{Gr}}\mathcal{N}$ and $K_0^{\mathrm{Gr}}[F]$ are compatible with the usual long exact sequence. In other words, the following diagram commutes for each $k \geq 1$:*

$$
\begin{array}{ccccccccc}
K_1^{\mathrm{Gr}}\mathcal{M} & \longrightarrow & K_1^{\mathrm{Gr}}\mathcal{N} & \longrightarrow & K_0^{\mathrm{Gr}}[F] & \longrightarrow & K_0\mathcal{M} & \longrightarrow & K_0\mathcal{N} \\
\downarrow{\scriptstyle\lambda^k} & & \downarrow{\scriptstyle\lambda^k} & & \downarrow{\scriptstyle\lambda^k} & & \downarrow{\scriptstyle\lambda^k} & & \downarrow{\scriptstyle\lambda^k} \\
K_1^{\mathrm{Gr}}\mathcal{M} & \longrightarrow & K_1^{\mathrm{Gr}}\mathcal{N} & \longrightarrow & K_0^{\mathrm{Gr}}[F] & \longrightarrow & K_0\mathcal{M} & \longrightarrow & K_0\mathcal{N}
\end{array}
$$

Proof. Similar to the proof of Proposition 4.2.3, we need only prove compatibility with the incoming and outgoing morphisms of $K_0^{\text{Gr}}[F]$. We will first verify compatibility with the map $K_0^{\text{Gr}}[F] \to K_0 \mathcal{M}$. To this end, suppose we have an element $x \in K_0^{\text{Gr}}[F]$ and pick a representative $X = \left(\binom{A}{B}, \left(N, \frac{d}{d'} \right), \binom{u}{v} \right)$ in $B[F]$ such that $x = [X] - [\Delta \perp X]$. Then as before we have that

$$\lambda^k x = \left[\left(\begin{matrix} \Lambda_+^k(A,B) \\ \Lambda_-^k(A,B) \end{matrix} \right), \left(\widehat{\Lambda}^k N, \begin{matrix} \Lambda_+^k(d,d') \\ \Lambda_-^k(d,d') \end{matrix} \right), \left(\begin{matrix} \Lambda_+^k(u,v) \\ \Lambda_-^k(u,v) \end{matrix} \right) \right]$$

in $K_0^{\text{Gr}}[F]$, and we have that this element is sent to

$$[\Lambda_+^k(A,B)] - [\Lambda_-^k(A,B)] = \lambda^k([A] - [B])$$

in $K_0 \mathcal{M}$ as expected. To show compatibility with the incoming homomorphism $K_1^{\text{Gr}} \mathcal{N} \to K_0^{\text{Gr}}[F]$, we need to show the following equality in $K_0^{\text{Gr}}[F]$:

$$\left[0, \Lambda^k \left(N, \frac{d}{d'} \right), 0 \right] = \left[0, \left(\widehat{\Lambda}^k N, \begin{matrix} \Lambda_+^k(d,d') \\ \Lambda_-^k(d,d') \end{matrix} \right), 0 \right]$$

given any object $\left(N, \frac{d}{d'} \right)$ of $B^q \mathcal{N}$. This equality holds, since products of elements of the form $[0, N, 0]$ vanish in $K_0^{\text{Gr}}[F]$. $\qquad \square$

4.3   The λ-ring axioms

In this section, we recall the definition of a λ-ring, verify that the pre-λ-ring $K_0^{\text{Gr}}[F]$ is in fact a λ-ring, and construct exterior power operations on $K_n^{\text{Gr}}[F]$ that make it into a λ-ring. A commutative (not necessarily unital) ring L together with operations $\lambda^k : L \to L$ for each $k \geq 1$ is called a λ-ring if $\lambda^1 x = x$ for every $x \in L$ and the following holds for every $x, y \in L$ and every $k, l \geq 1$:

$$\lambda^k(x + y) = \lambda^k x + \sum_{i=1}^{k-1} \lambda^{k-i} x \lambda^i y + \lambda^k y$$
$$\lambda^k(xy) = P_k(\lambda^1 x, \lambda^2 x, ..., \lambda^k x, \lambda^1 y, \lambda^2 y, ..., \lambda^k y)$$
$$\lambda^k \lambda^l x = P_{k,l}(\lambda^1 x, \lambda^2 x, ..., \lambda^{kl} x)$$

for certain universal integral polynomials $P_k, P_{k,l}$ [FL85, I, §1]. In other words, a λ-ring is a pre-λ-ring subject to the two additional formulae for operations of a product and for the composition of operations. We will now briefly describe the universal polynomials P_k and $P_{k,l}$ and give some examples. Firstly, let $e_1, ..., e_k$ be the elementary symmetric polynomials in the variables $X_1, ..., X_k$ and let $f_1, ..., f_k$ be the elementary symmetric polynomials in the variables $Y_1, ..., Y_k$. Then P_k is the unique polynomial in

$2k$ variables such that $P_k(e_1, e_2, ..., e_k, f_1, f_2, ..., f_k)$ is the coefficient of t^k in the expression

$$\prod_{i,j=1}^{k} (1 + tX_i Y_j).$$

Next, if $e_1, ..., e_{kl}$ are the elementary symmetric polynomials in the variables $X_1, ..., X_{kl}$, then $P_{k,l}$ is defined as the unique polynomial in kl variables such that $P_{k,l}(e_1, e_2, ..., e_{kl})$ is the coefficient of t^k in the expression

$$\prod_{1 \leq i_1 < i_2 < ... < i_l \leq kl} (1 + tX_{i_1} X_{i_2} \cdots X_{i_l}).$$

The following are fairly easy to compute examples of these polynomials

$$P_1(e_1, f_1) = e_1 f_1$$
$$P_2(e_1, e_2, f_1, f_2) = e_1^2 f_2 + e_2 f_1^2 - 2e_2 f_2$$
$$P_{1,n}(e_1, ..., e_n) = P_{n,1}(e_1, ..., e_n) = e_n \text{ for every } n \geq 0$$
$$P_{2,2}(e_1, e_2, e_3, e_4) = e_1 e_3 - e_4$$

In order to prove that $K_0^{\mathrm{Gr}}[F]$ is a λ-ring, using the same setup as in Section 4.2, we will use methods similar to those used in [HKT17, Section 8]. In order to do this, we need some background on polynomial functors over the (commutative unital) ring R. First, we recall the *divided powers of degree d* of a module V over R, denoted by $\Gamma^d V$, which is defined as the submodule of the tensor power $V^{\otimes d}$ which is invariant under the action of the symmetric group $\mathfrak{S}_d$.

Given an integer $d \geq 1$ and a category $\mathcal{A}$ enriched in R-modules (i.e. a category where the hom-sets $\mathrm{Hom}_{\mathcal{A}}(A, B)$ are equipped with an R-module structure for all objects A, B and composition is R-bilinear). We define the category $\Gamma^d \mathcal{A}$, with the same objects as $\mathcal{A}$, and for objects A, B we set

$$\mathrm{Hom}_{\Gamma^d \mathcal{A}}(A, B) := \Gamma^d \mathrm{Hom}_{\mathcal{A}}(A, B).$$

Taking $\mathcal{A} = \mathbf{Proj}(R)$ as usual, we define a *strict polynomial functor* $\mathbf{Proj}(R) \to \mathbf{Proj}(R)$ *of degree d* as an R-linear functor $\Gamma^d \mathbf{Proj}(R) \to \mathbf{Proj}(R)$. We tend to identify a strict polynomial functor with the composition

$$\mathbf{Proj}(R) \xrightarrow{\gamma^d} \Gamma^d \mathbf{Proj}(R) \to \mathbf{Proj}(R)$$

where $\gamma^d : \mathbf{Proj}(R) \to \Gamma^d \mathbf{Proj}(R)$ is the functor that is the identity on objects and sends a morphism f to $f^{\otimes d}$. We denote the category of strict polynomial functors $\mathbf{Proj}(R) \to \mathbf{Proj}(R)$ by $\mathrm{Pol}_d(R)$. With the additional definition that $\mathrm{Pol}_0 \mathcal{M}$ is the

category of constant functors $\mathcal{M} \to \mathcal{M}$, we define the category

$$\mathrm{Pol}_{<\infty}(R) := \bigoplus_{d \geq 0} \mathrm{Pol}_d(R)$$

where the right hand side is the (full) subcategory of $\prod_{d \geq 0} \mathrm{Pol}_d(R)$ whose objects are finite products (i.e. only a finite number of terms are nonzero). If $X = (X_0, X_1, ..., X_d, 0, 0, ...)$ with $X_i \in \mathrm{Pol}_i(R)$ and $X_d \neq 0$ we say that X has strict polynomial degree d. We define a further subcategory $\mathrm{Pol}^0_{<\infty}(R)$ of functors that fix the zero object. Recall the base change functor $F : \mathbf{Proj}_f(R) \to \mathbf{Proj}_f(S)$ given by $M \mapsto M \otimes_R S$ obtained from a ring homomorphism $f : R \to S$. Note that for every integer $d \geq 0$, there is a natural isomorphism

$$\Gamma^d_R \mathrm{Hom}_{\mathbf{Proj}_f(R)}(V, W) \otimes_R S \xrightarrow{\cong} \Gamma^d_S \mathrm{Hom}_{\mathbf{Proj}_f(S)}(V \otimes_R S, W \otimes_R S)$$

Thus base change lifts to a functor $\Gamma^d \mathbf{Proj}_f(R) \to \Gamma^d \mathbf{Proj}_f(S)$. Further, we can obtain a right exact functor $\mathrm{Pol}_d(R) \to \mathrm{Pol}_d(S)$ in the following way: Given a polynomial functor $G \in \mathrm{Pol}_d(R)$, there is a projective resolution

$$\cdots \longrightarrow \Gamma^d \mathrm{Hom}(V_1, -) \longrightarrow \Gamma^d \mathrm{Hom}(V_0, -) \longrightarrow G \longrightarrow 0$$

as shown in [Kra21, Lemma 8.3.9]. The base change of G then has a projective resolution similar to above, with $\Gamma^d_R \mathrm{Hom}_{\mathbf{Proj}_f}(V_i, -)$ replaced by $\Gamma^d_S \mathrm{Hom}_{\mathbf{Proj}_f(S)}(V_i \otimes_R S, - \otimes_R S)$. This discussion of strict polynomial functors follows the definitions in [Kra21, §8.3] and [Tou13].

The proof of Lemma 4.2.4 gives us that $K_0 B[F]$ is a pre-λ-ring, and so we will use techniques inspired by the proof of [HKT17, Theorem 8.18] to show that it is in fact a λ-ring.

Theorem 4.3.1. *The pre-λ-ring $K_0 B[F]$ satisfies the composition axiom of λ-rings. In other words, given an element $x \in K_0 B[F]$, and $k, l \geq 1$, the following identity holds:*

$$(\lambda^k \circ \lambda^l)(x) = P_{k,l}(\lambda^1(x), \lambda^2(x), ..., \lambda^{kl}(x)),$$

where $P_{k,l}$ is defined above.

Proof. It suffices to prove the identity in the case where $x = [X]$, where X is an object of $B[F]$. By a standard argument, the category $\mathrm{End}(B[F])$ of endofunctors on $B[F]$ is an exact category with tensor product

$$\otimes : \mathrm{End}(B[F]) \times \mathrm{End}(B[F]) \to \mathrm{End}(B[F])$$
$$(G_1, G_2) \mapsto (V \mapsto G_1(V) \otimes G_2(V))$$

and exterior power $\Lambda^k : \text{End}(B[F]) \to \text{End}(B[F])$ defined by $\Lambda^k(G) := \Lambda^k \circ G$ for each $k \geq 1$. We will prove the stronger identity

$$[\Lambda^k \circ \Lambda^l(I)] = P_{k,l}([\Lambda^1(I)], [\Lambda^2(I)], ..., [\Lambda^{kl}(I)])$$

in $K_0\text{End}(B[F])$, where I is the identity functor. For any object X of $B[F]$, we have the homomorphism

$$K_0\text{End}(B[F]) \to K_0 B[F]$$
$$[G] \mapsto [G(X)]$$

which recovers the required formula from the stronger one. Recall from the proof of [HKT17, Theorem 8.18] that the stronger identity holds in $K_0\text{Pol}^0_{<\infty}(\mathbb{Z})$. It remains to construct an exact functor

$$\text{Pol}^0_{<\infty}(\mathbb{Z}) \to \text{End}(B[F])$$

that respects composition, tensor products and exterior powers, and sends the identity functor to the identity functor. To this end, given a polynomial functor G in $\text{Pol}^0_{<\infty}(\mathbb{Z})$ and an object $X = (M, N, u)$ of $B[F]$, we define

$$G(X) := (NG\Gamma(M), NG\Gamma(N), NG\Gamma(u))$$

where N and Γ are the relevant functors in the Dold-Kan correspondence (note that this Γ is different from the divided power functor Γ^d), and G is applied to R and S modules via base change as explained above, similar to the construction for $\Lambda^k : B[F] \to B[F]$. We note here that this operation is well-defined, as $NG\Gamma(u)$ is again a quasi-isomorphism, proved for $G = \Lambda^k$ in [KZ21, Proposition 1.9], and whose proof works for any polynomial functor G (This proof requires a scheme theoretic definition of polynomial functors which agrees with our definition in this context). This construction is functorial in G, compatible with tensor products and exterior powers by construction, and thus induces a homomorphism

$$K_0\text{Pol}^0_{<\infty}(\mathbb{Z}) \to K_0\text{End}(B[F]).$$

Thus, the above mentioned stronger identity holds in $K_0\text{End}(B[F])$. $\square$

It turns out that the above theorem is actually enough to prove that $K_0 B[F]$ is in fact a λ ring due to the following surprising result:

Theorem 4.3.2. *Let L be a pre-λ-ring which satisfies the composition axiom for all $x \in L$. Then L is a λ-ring.*

Proof. We will define polynomials $Q_k \in \mathbb{Z}[X_1, ..., X_{2k}; Y_1, ..., Y_{2k}]$ for $k \geq 1$ by induction on k as follows. Define $Q_1 := X_1 Y_1$ and

$$Q_k := P_{k,2}\left(X_1 + Y_1, X_2 + X_1 Y_1 + Y_2, ..., \sum_{i=0}^{2k} X_{2k-i} Y_i\right)$$
$$- \sum_{i=1}^{k} Q_{k-i} \sum_{j=0}^{i} P_{i-j,2}(X_1, ..., X_{2(i-j)}) P_{j,2}(Y_1, ..., Y_{2j}).$$

We will now show for all $x, y \in L$ that

$$\lambda^k(xy) = Q_k(\lambda^1(x), ..., \lambda^{2k}(x); \lambda^1(y), ..., \lambda^{2k}(y)).$$

We will prove this by induction on k. The base case $\lambda^1(xy) = xy = Q_1(\lambda^1(x), \lambda^2(y))$ follows easily by definition. Writing $\lambda^i x$ for $\lambda^i(x)$ etc., we obtain for $k \geq 2$:

$$\lambda^k \lambda^2(x+y) = P_{k,2}\left(\lambda^1(x+y), \lambda^2(x+y), ..., \lambda^{2k}(x+y)\right)$$
$$= P_{k,2}\left(x+y, \lambda^2 x + xy + \lambda^2 y, ..., \sum_{i=0}^{2k} \lambda^{2k-i} x \lambda^i y\right).$$

On the other hand, we have:

$$\lambda^k \lambda^2(x+y) = \lambda^k(\lambda^2 x + xy + \lambda^2 y)$$
$$= \lambda^k(xy) + \sum_{i=1}^{k} \lambda^{k-i}(xy) \lambda^i(\lambda^2 x + \lambda^2 y)$$
$$= \lambda^k(xy) + \sum_{i=1}^{k} \lambda^{k-i}(xy) \sum_{j=0}^{i} \lambda^{i-j} \lambda^2 x \lambda^j \lambda^2 y.$$

Now, for $i = 1, ..., k-1$, using the inductive hypothesis we obtain $\lambda^{k-i}(xy) = Q_{k-i}(\lambda^1 x, ..., \lambda^{2(k-i)} x; \lambda^1 y, ..., \lambda^{2(k-i)} y)$. Furthermore, for any $j \geq 1$, we have $\lambda^j \lambda^2 x = P_{j,2}(\lambda^1 x, ..., \lambda^{2j} x)$ and $\lambda^j \lambda^2 y = P_{j,2}(\lambda^1 y, ..., \lambda^{2j} y)$. The desired formula finally follows from this.

To finish the proof, we observe that the polynomial ring $\mathbb{Z}[X_1, X_2, ...; Y_1, Y_2, ...]$ in the infinitely many variables $X_1, X_2, ...$ and $Y_1, Y_2, ...$ can be equipped with a unique λ-ring structure such that $\lambda^k(X_1) = X_k$ and $\lambda^k(Y_1) = Y_k$. It is called the free λ-ring in two variables (though this universal property is not used here), and can be constructed similarly to the free λ-ring in one variable which has been introduced in [AT69, I, §2]. In particular, we have $\lambda^k(X_1 Y_1) = P_k$ by definition of a λ-ring and, by the above proof, $\lambda^k(X_1 Y_1) = Q_k$. Thus we obtain $Q_k = P_k$, and thus the result is true. $\square$

These results give us the following corollary:

Corollary 4.3.3. *The groups $K_0 pB[F]$ and $K_0^{\mathrm{Gr}}[F]$ are λ-rings.*

Proof. The rings $K_0 pB[F]$ and $K_0^{\mathrm{Gr}}[F]$ are obtained by applying relations and taking subobjects (i.e. the kernel of $\perp$). Since Λ^k is compatible with $\perp$ and maps in $pB[F]$ any identity involving λ^k in $K_0 B[F]$ also holds in $K_0 pB[F]$ and in $K_0^{\mathrm{Gr}}[F]$. $\qquad\square$

We will now outline the definition of exterior powers on the higher relative algebraic K-groups $K_n^{\mathrm{Gr}}[F]$ for $n \geq 1$. Recall from Section 2.2 that such a group is defined as the extension of the functor $K_0^{\mathrm{Gr}} : \mathbf{Exact}^{\rightarrow} \to \mathbf{Ab}$ to the split n cube $\Omega^n[F]$ in $Q^n \mathbf{Exact}^{\rightarrow}$ by applying cokernels in all directions. Since Δ is split by $\perp$, this group is canonically isomorphic to the group obtained by taking kernels in all directions of the same cube, but with arrows replaced with their splittings in the opposite direction.

Given an object $X = (M, N, u)$ of $B[(B^q)^n F]$, so that M is an object of $(C(B^q)^n \mathcal{M})^2$, N is an object of $B(B^q)^n \mathcal{N}$ and $u : FM \to \mathbb{I}N$ is a quasi-isomorphism in the category $(C(B^q)^n \mathcal{N})^2$, we define the kth exterior power of X as follows:

$$\Lambda^k X := (\Lambda^k M, \Lambda^k N, \Lambda^k u)$$

where Λ^k is defined on multicomplexes in an inductive process described in [HKT17, Definition 3.3]. Operations on binary multicomplexes are defined in the obvious way, and $\Lambda^k u$ is once again a quasi-isomorphism due to [KZ21, Corollary 1.8 and Proposition 1.9].

Theorem 4.3.4. *The operations $\Lambda^k : B[(B^q)^n F] \to B[(B^q)^n F]$ for $k \geq 1$ induce well-defined exterior power operations*

$$\lambda^k : K_0 B[(B^q)^n F] \to K_0 B[(B^q)^n F]$$

that turns $K_0 B[(B^q)^n F]$ into a λ-ring

Proof. The proof that exterior powers are well-defined and satisfy the addition axiom is nearly identical to Lemma 4.2.4. Proof of the composition axiom is identical to the proof of Theorem 4.3.1. Thus the theorem holds due to Theorem 4.3.2. $\qquad\square$

Theorem 4.3.5. *The ring $K_n^{\mathrm{Gr}}[F]$ together with operations $\lambda^k : K_n^{\mathrm{Gr}}[F] \to K_n^{\mathrm{Gr}}[F]$ for $k \geq 1$ induced by the above operations on $K_0 B[(B^q)^n F]$ is a λ-ring. When $n \geq 1$, multiplication becomes trivial and thus λ^k is a homomorphism. These exterior power operations are also compatible with the long exact sequence*

$$K_{n+1}^{\mathrm{Gr}} \mathcal{M} \to K_{n+1}^{\mathrm{Gr}} \mathcal{N} \to K_n^{\mathrm{Gr}}[F] \to K_n^{\mathrm{Gr}} \mathcal{M} \to K_n^{\mathrm{Gr}} \mathcal{N}$$

Proof. As $K_n^{\mathrm{Gr}}[F]$ is the iterated kernel of homomorphisms whose domain is $K_0 B[(B^q)^n F]$, which is a λ-ring by Theorem 4.3.4, together with an additional relation, any identity involving λ^k in $K_0 B[(B^q)^n F]$ holds in $K_n^{\mathrm{Gr}}[F]$ and thus $K_n^{\mathrm{Gr}}[F]$ is a λ-ring.

For $n \geq 1$, we consider the group $K_n B[F]$. By [HKT17, Proposition 5.11], products on $K_n B[F]$ (defined using [HKT17, Definition 5.4]) vanish, and thus on $\ker(\perp : K_n B[F] \to K_n C[F])$. We can identify this group with $K_n^{\mathrm{Gr}}[F]$, since we can think of an object of, say, the category $B[(B^q)^n F]$ as an object of the category $(B^q)^n B[F]$ and recall that the order of iterated kernels is arbitrary to the definition of $K_n^{\mathrm{Gr}}[F]$.

To show compatibility with the above mentioned long exact sequence, recall that we actually have an exact sequence of n-cubes of abelian groups

$$K_1^{\mathrm{Gr}} \Omega^n \mathcal{M} \to K_1^{\mathrm{Gr}} \Omega^n \mathcal{N} \to K_0^{\mathrm{Gr}} \Omega^n [F] \to K_0 \Omega^n \mathcal{M} \to K_0 \Omega^n \mathcal{N}$$

where here we do not mean the iterated kernel or cokernel, but the n-cube of abelian groups with the named functor applied to the relevant category at each vertex. Replacing homomorphisms induced by Δ with those induced by $\perp$ and taking iterated kernels we obtain the usual sequence

$$K_{n+1}^{\mathrm{Gr}} \mathcal{M} \to K_{n+1}^{\mathrm{Gr}} \mathcal{N} \to K_n^{\mathrm{Gr}} [F] \to K_n^{\mathrm{Gr}} \mathcal{M} \to K_n^{\mathrm{Gr}} \mathcal{N}.$$

This sequence is a levelwise subgroup of the exact sequence

$$K_1^{\mathrm{Gr}} (B^q)^n \mathcal{M} \to K_1^{\mathrm{Gr}} \mathcal{N} \to K_0^{\mathrm{Gr}} [(B^q)^n F] \to K_0 (B^q)^n \mathcal{M} \to K_0 (B^q)^n \mathcal{N},$$

and by construction all products and exterior power operations are compatible with this inclusion. Exterior powers are compatible with this sequence due to Lemma 4.2.5. $\square$

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