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ORIGINAL PAPER

Property R_∞ for groups with infinitely many ends

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Abstract

We show that an accessible group with infinitely many ends has property R_∞. That is, it has infinitely many twisted conjugacy classes for any twisting automorphism. We deduce that having property R_∞ is undecidable amongst finitely presented groups. We also show that the same is true for a wide class of relatively hyperbolic groups, filling in some of the gaps in the literature. Specifically, we show that a non-elementary, finitely presented relatively hyperbolic group with finitely generated peripheral subgroups which are not themselves relatively hyperbolic, has property R_∞.

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1 Introduction

Property R_∞ is a group theoretic property with connections to fixed point theory and which has been studied extensively by many authors. A non-exhaustive list might include [14, 16, 35, 36, 56] and [57].

It is a generalisation of the property of having infinitely many conjugacy classes. Instead, one asks that there are infinitely many twisted conjugacy classes, where one twists one side of the conjugacy by an automorphism. Property R_∞ then asks that a group have infinitely many twisted conjugacy classes for any automorphism - see Definition 2.1.5.

Our approach is to extend the techniques of [44], who proved (implicitly) that any non-elementary hyperbolic group has property R_∞. Our main theorem is that any accessible group with infinitely many ends has R_∞.

Theorem 5.1.18 *Any accessible group G with infinitely many ends has property R_∞.*

In particular, any finitely presented group with infinitely many ends has property R_∞.

We note that, intuitively, the ends of a group is the number of components of the group at infinity. More concretely, a group with infinitely many ends acts non-trivially on a simplicial tree with finite edge stabilisers. See Theorem 5.1.2 for more detail, or [17] for a more extensive reference source. In particular free products of groups have infinitely many ends (except for the case of the infinite Dihedral groups) and we start by first proving the result for free products.

In fact, from the result on free products we quickly deduce that the property of having R_∞ is undecidable amongst finitely presented groups.

Corollary 4.1.11 *The property of being R_∞ is undecidable amongst finitely presented groups.*

At this point we should mention the result [21, Theorem 3.3] where it is claimed that any non-elementary relatively hyperbolic group has property R_∞. The proof strategy is modelled on the proof in [44], but the technical details are all but absent and to that extent it is very hard to verify the proof given. Specifically, the technique used there is due to Paulin [51] and [52] where one takes a limit of hyperbolic spaces to produce a limiting $\mathbb{R}$-tree, extended by [1] to the relatively hyperbolic case.

Already there is an issue that the limiting tree is not projectively fixed by the automorphism, when using the results in [1] (which is required in the proof). The result [52, Theoreme A] does provide a projectively fixed $\mathbb{R}$-tree but that result is not proved in the relatively hyperbolic case and it seems that the result there would require that the peripheral subgroups be left invariant by the automorphism. In fact, this is a hypothesis that we require in our result about relatively hyperbolic groups in Theorem 6.1.1 and we suspect that it is an essential one for these proof techniques.

There are then various technical details missing in the treatment from [21]. For instance, finite generation of the parabolic subgroups is not mentioned in [21], although it does appear as a hypothesis in [1] and also appears in our Theorem 6.1.2. Leaving aside the invariance of the limiting tree, the next step in is then to invoke [44, Proposition 3.2] (and also [44, Proposition 3.1] in the case '$\lambda = 1$').

These Propositions have various hypotheses but the main ones for [44, Proposition 3.2] are that the limiting tree should be (i) Irreducible (ii) Have finite arc stabilisers and (iii) Admit finitely many orbits of directions at each branch point.

The first of these is very plausible but merely asserted in [21]. For the second, smallness is invoked but without due care. Certainly, the argument given in [51] is not sufficient to produce virtually cyclic stabilisers as one needs to worry about parabolic subgroups. In fact, [1, Theorem 1.2] specifically proves that arc stabilisers are elementary, by which they include the parabolic subgroups. While it is possible that this technical obstacle might be overcome with a specific construction - our Lemma 4.1.8 does prove that arc stabilisers are not parabolic when one takes a limit of edge-free trees - it really merits a careful argument. In short, this hypothesis on arc stabilisers is claimed in [21] but is not supported by the literature.

Finally, the only time (iii) is addressed is by reference to the paper [3] which does not seem to be relevant to that issue (that there are finitely many orbits of directions under the stabiliser of a branch point). It is possible that the paper meant was [2] as referenced by [44], but this is also not a correct reference for that fact; however, in [44] there are other ways of obtaining this result which do not seem available in the relatively hyperbolic case.

That said, the broad strokes of the argument are correct and we address these technical issues carefully to produce our main result about groups with infinitely many ends as well as a fairly general result about relatively hyperbolic groups.

Theorem 6.1.2 *Let G be a non-elementary finitely presented relatively hyperbolic group G whose peripheral subgroups are finitely generated but not relatively hyperbolic. Then G has property R_∞.*

Remark We note that this Theorem recovers the result of [44] that any non-elementary hyperbolic group has property R_∞, although we really use their argument except for the fact that we make use of JSJ decompositions at the final stage.

Our proof uses train track methods for free products so that we can refer to the literature to address the technical issues mentioned above. We then show that groups with infinitely many ends have R_∞ by extending the results for free products using the tree of cyclinders construction of Guirardel and Levitt, [38] and [39]. The idea here is that a group with infinitely many ends either admits a finite normal subgroup so that the quotient is a free product or admits a tree which is invariant under all automorphisms. See Theorem 5.1.17.

Finally, we prove the Theorem for relatively hyperbolic groups using the canonical JSJ splittings of [41].

The first version of this article was placed on the arXiv by HI, AM, WS and PW. We then received communications from FFF and Anthony Genevois independently observing that it is possible to broaden the results in our paper. We are hugely grateful for these observations and insights.

The most general way to do this seems to be via quasi-morphisms, and this is detailed in Appendix B. The most general statement of these results is as follows.

Corollary 8.1.6 *Let G be a finitely generated group satisfying one of the following properties.*

(1) G is a non-elementary hyperbolic group.
(2) G is non-elementary hyperbolic relative to a collection of finitely generated subgroups, none of which is relatively hyperbolic.
(3) G has infinitely many ends.
(4) G is a graph product of groups over a finite graph that does not decompose non-trivially as a join and is not reduced to a single vertex.
(5) G is a graph product of abelian groups and it is not virtually abelian.

Then G has property R_∞.

We note that the main body of this paper constructs translation length functions arising from trees which are constant on twisted conjugacy classes, whereas the method via quasi-morphisms construct such functions which are bounded on twisted conjugacy classes (albeit for all automorphisms at the same time). Nevertheless, the methods should be seen as analogous both in their overall strategy and in the details of their technical construction.

2 Twisted Conjugacy and the R_∞ Property

2.1 Twisted Conjugacy

Definition 2.1.1 Let G be a group and $\varphi \in \mathrm{Aut}(G)$ an automorphism of G. We define a relation $\sim_\varphi$ on G by $x \sim_\varphi y$ if and only if $\exists w \in G$ with $(w\varphi)xw^{-1} = y$. If $x \sim_\varphi y$, we say that x and y are twisted conjugates in G.

Remark We take the action of $\mathrm{Aut}(G)$ on G to be a right action, writing $g \cdot \varphi$ or $g\varphi$ for the image of $g \in G$ under the automorphism $\varphi \in \mathrm{Aut}(G)$.

Remark Note that there is an asymmetry in the definition of twisted conjugacy. The side on which we place the inverse makes no difference, but the side on which we put the automorphism does. However,

$$wx(w\varphi)^{-1} = (u\varphi^{-1})xu^{-1}, \text{ where } u = w\varphi.$$

This shows the 'left' and 'right' versions of twisted conjugacy are related at the cost of changing the automorphism to its inverse. Since the property of having R_∞ - Definition 2.1.3 - is about all possible automorphisms, it makes no difference which we choose.

The following is a standard fact and an easy exercise.

Lemma 2.1.2 *Twisted conjugacy ($\sim_\varphi$) is an equivalence relation on G.*

Definition 2.1.3 (R_∞) Let G be a group and $\varphi \in \mathrm{Aut}(G)$. We say that G has property $R_\infty(\varphi)$ if $\sim_\varphi$ has infinitely many equivalence classes in G. We say that G has property R_∞ if G has $R_\infty(\varphi)$ for all $\varphi \in \mathrm{Aut}(G)$.

Lemma 2.1.4 *If $\iota \in \mathrm{Inn}(G)$ then G has $R_\infty(\varphi)$ if and only if G has $R_\infty(\varphi\iota)$.*

Proof Let $x \in G$ and let $\iota_x \in \mathrm{Inn}(G)$ be the automorphism $g \mapsto x^{-1}gx = g^x$. Let $y, z \in G$. Then:

$$\begin{aligned} & y \sim_\varphi z \\ \Leftrightarrow\ & \exists w \in G \text{ such that } z = (w\varphi)yw^{-1} \\ \Leftrightarrow\ & \exists w \in G \text{ such that } xz = x(w\varphi)(x^{-1}x)yw^{-1} = (w\varphi)^x(xy)w^{-1} = (w\varphi\iota_x)(xy)w^{-1} \\ \Leftrightarrow\ & xy \sim_{\varphi\iota_x} xz. \end{aligned}$$

Thus we have a bijection between the equivalence classes of $\sim_\varphi$ and those of $\sim_{\varphi\iota_x}$. Hence G has $R_\infty(\varphi)$ if and only if G has $R_\infty(\varphi\iota_x)$. □

Definition 2.1.5 Let G be a group and let $\Phi \in \mathrm{Out}(G)$. We say that G has property $R_\infty(\Phi)$ if G has $R_\infty(\varphi)$ for some (and hence every) automorphism $\varphi \in \Phi$.

Remark By Lemma 2.1.4, this concept of $R_\infty(\Phi)$ for $\Phi \in \mathrm{Out}(G)$ is well-defined, and it follows that G has R_∞ (i.e. G has $R_\infty(\varphi)$ for all $\varphi \in \mathrm{Aut}(G)$) if and only if G has $R_\infty(\Phi)$ for all $\Phi \in \mathrm{Out}(G)$.

Remark As above, let $\Phi \in \mathrm{Out}(G)$ and fix $\varphi \in \Phi$. Following [44], we say that $\alpha, \beta \in \Phi$ are isogredient if and only if there exists $g \in G$ such that $\alpha = \iota_g \beta \iota_g^{-1}$, where ι_g denotes the inner automorphism with $\iota_g(h) = ghg^{-1}$ for all $h \in G$. We denote by $S(\Phi)$ the set of equivalence classes (called isogredience classes) of automorphisms representing Φ. We say G has property S_∞ if $S(\Phi)$ is infinite for every $\Phi \in \mathrm{Out}(G)$. As stated in the third paragraph of our introduction, we attribute the proof of property R_∞ for non-elementary hyperbolic groups primarily to [44]. Indeed, the authors show in Theorem 3.5 that any such group possesses property S_∞. But this, in turn, easily implies property R_∞, as we show below. This observation was also made by the paper [20]. Write $\alpha = \iota_r \varphi$ and $\beta = \iota_s \varphi$ for $r, s \in G$. We claim that α and β are isogredient if and only if $\overline{r} \sim_{\overline{\varphi}} \overline{s}$ in the quotient group $G/Z(G)$, where $\overline{\varphi}$ is the automorphism naturally induced from φ. In fact, α and β being isogredient means $\iota_r \varphi = \iota_g \iota_s \varphi \iota_g^{-1}$ for some $g \in G$. Since $\varphi \iota_g^{-1} = \iota_{\varphi(g)^{-1}} \varphi$, the former equation is equivalent to $\iota_r = \iota_{gs\varphi(g)^{-1}}$, or $gs\varphi(g)^{-1} r^{-1} \in Z(G)$, which in turn is clearly equivalent to saying that $\overline{r} \sim_{\overline{\varphi}} \overline{s}$ in $G/Z(G)$. Therefore, if the set $S(\Phi)$ is infinite, then $G/Z(G)$ has $R_\infty(\overline{\varphi})$, which implies G has $R_\infty(\varphi)$ (see Lemma 5.1.11). We conclude that property S_∞ for G implies property R_∞ for G.

2.2 Mapping Torus

Our argument about twisted conjugacy classes uses the standard technique of converting questions about twisted conjugacy in a group G to ones concerning genuine conjugacy in a mapping torus of G. We therefore recall the definition of a mapping torus.

Definition 2.2.1 *(Mapping torus)* Let G be a group and $\varphi \in \mathrm{Aut}(G)$. The mapping torus of φ is the semi-direct product $M_\varphi := G \rtimes_\varphi \mathbb{Z} = \langle G, t \mid g^t = g\varphi \ \forall g \in G \rangle$.

Remark Since M_φ is a semi-direct product, elements of M_φ have a standard form gt^k where $g \in G$ and $k \in \mathbb{Z}$ are unique. Note that for any $h \in G$ we have $t^{-1}h = h^t t^{-1} = (h\varphi)t^{-1}$, thus an element $t^k h$ can be written in the alternate standard form $(h \cdot \varphi^{-k})t^k$.

A key observation is that twisted conjugacy is realised as standard conjugacy in the mapping torus and that having $R_\infty(\varphi)$ amounts to there being infinitely many conjugacy classes of a certain type in the mapping torus.

Lemma 2.2.2 *Let G be a group and $\varphi \in \mathrm{Aut}(G)$. Then G has $R_\infty(\varphi)$ if and only if the set $\{tx \mid x \in G\}$ has infinitely many M_φ conjugacy classes.*

Proof Let $x, y \in G$. We have that tx and ty are conjugate in M_φ if and only if there exists $u = gt^k \in M_\varphi$ so that $ty = (tx)^u = u^{-1}(tx)u$. Observe that given $x \in G$ we can always find $h \in G$ so that $u = gt^k = (tx)^k h$. Then $(tx)^u = (tx)^{(tx)^k h} = (tx)^h$. Now for any $x, y \in G$, we have:

$$\begin{aligned} & tx \text{ and } ty \text{ are conjugate in } M_\varphi \\ \iff & \exists h \in G \text{ with } ty = (tx)^h = h^{-1}txh = t(h^{-1}\varphi)xh \\ \iff & \exists \hat{h} \in G \text{ with } y = (\hat{h}\varphi)x\hat{h}^{-1} \\ \iff & y \sim_\varphi x. \end{aligned}$$

□

3 Trees and Group Actions

3.1 Simplicial trees and $\mathbb{R}$-trees

We refer the reader to [15] for a treatment on $\mathbb{R}$-trees and to [12] for a more general textbook on Λ-trees. The definitions and results here are mainly based on [15] and [12]. We note that in the case $\Lambda = \mathbb{R}$ these concepts coincide, whereas the case $\Lambda = \mathbb{Z}$ corresponds to the case of a simplicial tree below.

Definition 3.1.1 A simplicial graph, Γ, is a 4-tuple, (V, E, σ, τ) where:

- V is a set called the vertices of T,
- $E \subseteq V \times V$ is the set of oriented edges of the graph,
- $\sigma : E \to V$ and $\tau : E \to V$ are incidence maps, defined by $\sigma(u, v) = u$ and $\tau(u, v) = v$.

Moreover, for every $e = (u, v) \in E$ we always have that $(v, u) \in E$; we call this the inverse edge, denoted, $\overline{e}$.

Definition 3.1.2 An edge path in a graph, $\Gamma = (V, E, \sigma, \tau)$ is a sequence of vertices, $v_0, v_1, \ldots, v_k$ where for each $0 \leqslant i \leqslant k-1$, $(v_i, v_{i+1}) \in E$. We allow the edge path to consist of a single vertex, v_0.

- The edge path is called trivial when it consists of a single vertex, and non-trivial otherwise.
- An edge path $v_0, v_1, \ldots, v_k$ is said to start at v_0 and end at v_k.
- A non-trivial edge path may also be described as a sequence of edges, $e_0 \ldots e_k$, where $\tau(e_i) = \sigma(e_{i+1})$ for $0 \leqslant i \leqslant k-1$.
- An edge path, $e_0 \ldots e_k$ is called reduced if, for all $0 \leqslant i \leqslant k-1$, $e_i \neq \overline{e_{i+1}}$. A trivial path is always considered reduced.

Definition 3.1.3 A simplicial tree is a simplicial graph where between any two vertices there is a unique reduced edge path starting at one and ending at the other.
For a tree, we let $[u, v]$ denote the unique reduced edge path from u to v; this is called the segment from u to v.

Definition 3.1.4 (Culler–Morgan [15]) An $\mathbb{R}$-tree T is a path-connected non-empty metric space so that for any points $x, y \in T$, there is a unique arc $[x, y] \subseteq T$ joining x and y, which is isometric to the interval $[0, d(x, y)] \subseteq \mathbb{R}$ (where d is the metric on T).

Equivalently, an $\mathbb{R}$-tree is a 0-hyperbolic geodesic metric space, in the sense of Gromov.

Definition 3.1.5 Given two points, u, v in an $\mathbb{R}$-tree T, we denote by $[u, v]$ the unique geodesic from u to v in T. This is called the segment from u to v.

Definition 3.1.6 Given an $\mathbb{R}$-tree T and a point $x \in T$, a direction at x is a connected component of $T - \{x\}$. We say a point $x \in T$ is a branch point if there are at least three directions at x.

Definition 3.1.7

- We say that an $\mathbb{R}$-tree is a metric simplicial tree (or simply a simplicial $\mathbb{R}$-tree) if the set of branch points is a discrete subset of the tree.
- If T is a simplicial $\mathbb{R}$-tree and V a discrete subset of T which includes all the branch points (but may include more points), then the edges are all the segments $[u, v]$ between elements of V where $[u, v] \cap V = \{u, v\}$. Any non-trivial segment between vertices may then be given as an edge-path, $e_1 \ldots e_k$ where each e_i is an edge.

- Conversely, a simplicial tree T given as a set of vertices and edges may be made into a simplicial $\mathbb{R}$-tree by assigning a positive length to each edge and making T into a metric space via the corresponding path metric.

Definition 3.1.8 Let T be an $\mathbb{R}$-tree. Then for any three points, $u, v, w \in T$, the Y-point, $Y(u, v, w) = y \in T$, is given by:

$$[u, v] \cap [u, w] = [u, y].$$

Lemma 3.1.9 ([12, Chapter 2, Lemma 1.2]) *Let T be an $\mathbb{R}$-tree. Then for any three points, $u, v, w \in T$, the Y-point is unique and does not depend on the order of the points. Moreover, $Y(u, v, w)$ is the point on $[u, v]$ whose distance from u is given by the Gromov product, $(v.w)_u = \frac{1}{2}(d_T(u, v) + d_T(u, w) - d_T(v, w))$.*

3.2 Group actions on $\mathbb{R}$-trees

Throughout this subsection, we will consider a group, G, acting isometrically on an $\mathbb{R}$-tree, T. We note that if one starts with a simplicial tree and a group action sending vertices to vertices and edges to edges then the process of making this a simplicial $\mathbb{R}$-tree described above allows one to extend the group action to an isometric action; that is, a group of automorphisms of a tree preserves the induced path metric.

Definition 3.2.1 ([15, p. 576 and Definition 1.4]) Suppose that G is a group acting isometrically on an $\mathbb{R}$-tree, T.

(i) For any $g \in G$ we define the translation length of g (with respect to T) to be,

$$\|g\|_T := \inf_{x \in T}\{d_T(x, xg)\}.$$

We write $\|g\|$ for $\|g\|_T$ if T is understood.

(ii) Define the characteristic set of g to be,

$$A_g = \{x \in T \ : \ d_T(x, xg) = \|g\|\}.$$

(iii) $g \in G$ is called ellipic if $\|g\|_T = 0$ and hyperbolic if $\|g\|_T > 0$.

Lemma 3.2.2 ([15, Lemma 1.3 (p.576)]) *Let G act isometrically on an $\mathbb{R}$-tree, T and let $g \in G$.*

(i) There exists an $x \in T$ such that $d_T(x, xg) = \|g\|$. That is, the infimum in Definition 3.2.1(i) is a minimum .

(ii) The set $A_g = \{x \in T \ d_T(x, xg) = \|g\| \ \}$ is non-empty (by the previous part). It is also a closed subtree of T, invariant under the action of g.

(iii) If g is elliptic, then A_g is the fixed point set of g, $\mathrm{Fix}(g)$.

(iv) If g is hyperbolic, then A_g is called the axis of g and is isometric to the real line. It is the smallest g-invariant subtree of T. The element g acts on A_g as a translation by the real number $\|g\|$.

(v) If g is hyperbolic and $0 \neq n \in \mathbb{Z}$, then $A_{g^n} = A_g$ and $\|g^n\| = |n|\|g\|$.

Lemma 3.2.3 *Let G act isometrically on an $\mathbb{R}$-tree T. Let $g \in G$ and $x \in T$ any point.*

(i) The midpoint of the segment $[x, xg]$ lies in A_g.

(ii) For any hyperbolic $g \in G$, we have $Y(xg^{-1}, x, xg) \in A_g$.

Proof (i) This is just [12, Chapter 3 (Lemma 1.1 for the elliptic case and Theorem 1.4 for the hyperbolic case)] .
(ii) This is [12, Chapter 3, Theorem 1.4]. ◻

Observation 3.2.4 It follows easily that $A_{g^h} = A_g \cdot h$. Hence if g and h commute then h preserves A_g. Further, if g and h commute and are both hyperbolic then $A_g = A_h$, since h preserves the line A_g but A_h is the smallest h-invariant subtree of T.

Definition 3.2.5 Let G act isometrically on an $\mathbb{R}$-tree, T. Then the length function of this action is the function, $l : G \to \mathbb{R}$ given by $l(g) = \|g\|_T$. This is also called the translation length function.

Definition 3.2.6 (Culler–Morgan [15])

Let T be an $\mathbb{R}$-tree equipped with an action of a group G by isometries. We say that T is:

- irreducible, if there is no point, line or end of T which is invariant under the action of G. Equivalently, there exist a pair of groups elements, $g, h \in G$ which are hyperbolic and whose axes meet in an arc of finite positive length (See [15, Theorem 2.7] for this equivalence).
- minimal, if there is no proper G-invariant subtree of T.
- non-trivial, if there is no global fixed point (i.e. no $x \in T$ such that $x \cdot G = x$).
- small, if for any non-trivial arc $[x, y]$, the pointwise stabiliser $\mathrm{Stab}([x, y]) = \{g \in G \ \ zg = z, \ \forall z \in [x, y]\} \leq G$ does not contain a free subgroup of rank 2.

Remark It is also true that the action of G on T is irreducible if and only if there are a pair of isometries whose axes are disjoint [15, Lemmas 2.1 and 1.5 and 1.6]. Concretely, if g, h are hyperbolic and $A_g \cap A_h$ meets in a segment of finite poisitive length then, for some large $n \in \mathbb{Z}$, the axes of g and g^{h^n} will be disjoint.

We shall also need the following.

Lemma 3.2.7 (Culler-Morgan [15], 1.5 and 1.6) *Let g, h be hyperbolic isometries of an $\mathbb{R}$-tree T whose axes A_g, A_h are disjoint. Then $\|gh\| = \|hg\| = \|g\| + \|h\| + 2d(A_g, A_h)$. In particular gh and hg are hyperbolic.*

Proposition 3.2.8 ([15, Proposition 3.1]) *If G acts on an $\mathbb{R}$-tree T and some element $g \in G$ acts hyperbolically, then T admits a unique, minimal G-invariant subtree. This subtree is exactly the union of all the hyperbolic axes of elements of G.*

Lemma 3.2.9 (Serre [55, Proposition 25 (p.63)]) *If G is finitely generated and acts isometrically on an $\mathbb{R}$-tree, T, then the action is non-trivial if and only if there exists a hyperbolic element in G. That is, if there exists some $g \in G$ with $\|g\|_T > 0$.*

Remark The result in [55] concerns simplicial trees, but the same proof works for $\mathbb{R}$-trees without change.

Definition 3.2.10 Let T be a tree. For points $x_1, \ldots, x_n$ we write $[x_1, \ldots, x_n]$ to mean that the unique segment from x_1 to x_n crosses the points $x_2, \ldots, x_{n-1}$ in the order given.

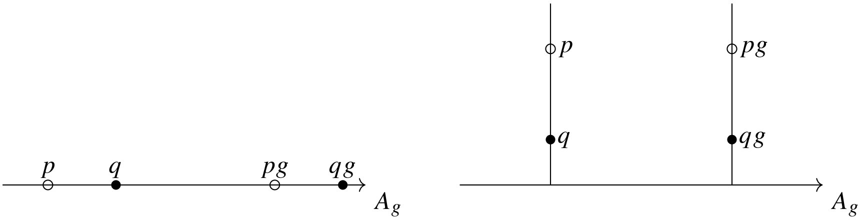
Lemma 3.2.11 (Circle-Dot Lemma) *Let G act isometrically on an $\mathbb{R}$-tree, T. Suppose we have distinct points, $p, q \in T$ such that $[p, q, pg, qg]$. That is, the segment from p to qg crosses the points q and pg in that order.*

Then g is hyperbolic and both p and q belong to the axis of g. In particular, $\|g\|_T = d(p, pg)$.

Remark We allow the possibility that $q = pg$ in this Lemma.

Proof We first show that g is not elliptic. We argue by contradiction; if g is elliptic then the midpoint of $[p, pg]$ is fixed by g by Lemmas 3.2.2(iii) and 3.2.3(i). Call this point w. Then $d_T(w, q) = d_T(w, qg)$. Since $w \in [p, pg]$, this forces $w \in [q, qg]$. But now w is both the midpoint of $[p, pg]$ and $[q, qg]$ which is impossible if $p \neq q$. Hence g is hyperbolic. Therefore, by Lemma 3.2.3(ii) it is enough to show that $p = Y(pg^{-1}, p, pg)$.

Indeed, since $[p, q, pg, qg]$ then $[pg^{-1}, qg^{-1}, p, q]$, and hence $[pg^{-1}, qg^{-1}, p, q, pg, qg]$. That is, $p \in [pg^{-1}, pg]$ and therefore $p = Y(pg^{-1}, p, pg)$, as required. □



On the axis:
○ — ● — ○ — ●

Not on the axis:
○ — ● — ● — ○

The Circle-Dot Lemma

We will also need the following,

Lemma 3.2.12 (Paulin's Lemma [50, Lemma 4.3]) *Let G act minimally, non-trivially, isometrically and irreducibly on an $\mathbb{R}$-tree, T. Then for any $a, b \in T$ there exists a hyperbolic element $g \in G$ whose axis contains the segment* $[a, b]$.

This allows us to deduce the R_∞ property when we have an action of the mapping torus on a tree. Thus the following Lemmas will be a key tool for deducing property R_∞. We note that the existence of the action required by this Lemma will be discussed in Corollary 3.5.5.

Lemma 3.2.13 *Let G be a group and let $\varphi \in \mathrm{Aut}(G)$. If M_φ acts on a tree T minimally, isometrically, irreducibly then G also acts minimally and irreducibly on T.*

Proof We can restrict the action to G and get an isometric action of G on T. We first claim that G admits a hyperbolic element with respect to this action.

Since M_φ acts irreducibly on T we get that (by the remark after Definition 3.2.6) we have two hyperbolic isometries, $m_1, m_2 \in M_\varphi$ whose axes are disjoint. If either of these are in G, then we have a hyperbolic isometry in G. Otherwise, we may write $m_1 = t^a u, m_2 = t^b v$, where $0 \neq a, b \in \mathbb{Z}$ and $u, v \in G$. Now consider, $m_1^b = t^{ab} g, m_2^{-a} = t^{-ab} h$ for some $g, h \in G$. By Lemma 3.2.2(iv) the axes of m_1 and m_1^b are equal, as are the axes of m_2 and m_2^{-a}. Hence the axes of $t^{ab} g$ and $t^{-ab} h$ are disjoint. Therefore, by Lemma 3.2.7, $(t^{ab} g)(t^{-ab} h) \in G$ is hyperbolic. This proves the claim.

Now, by Lemma 3.2.8, G admits a minimal G-invariant subtree. But since G is a normal subgroup of M_φ, this is also a M_φ-invariant subtree and so is the whole of T, as M_φ acts minimally. This proves that G also acts minimally on T.

Finally, to prove that G acts irreducibly on T notice that if there were an invariant line, then T would have to be a line by minimality, contradicting the fact that M_φ acts minimally and irreducibly. If G were to admit a fixed end, then any hyperbolic axes A_g, A_h of hyperbolic elements of G would intersect in this end. However, if $g \in G$ is hyperbolic and $m, \in M_\varphi$, then $g^m \in G$ is also hyperbolic and $A_{g^m} = (A_g)m$, showing that the end would need to be invariant by all elements of M_φ, again contradicting the fact that M_φ acts irreducibly. □

Lemma 3.2.14 *Let G be a group and let $\varphi \in \mathrm{Aut}(G)$. If M_φ acts on a tree T minimally, isometrically, irreducibly, then G has $R_\infty(\varphi)$.*

Proof Notice that by Lemma 3.2.13, G also acts on T minimally and irreducibly. In particular, Paulin's Lemma 3.2.12 applies to the action of G on T.

We will prove the Lemma by showing that we have infinitely many conjugacy classes of the form tx, where $x \in G$, and deduce the result by Lemma 2.2.2.

Case (i): First suppose that t acts hyperbolically on T. Choose a in the axis of t and let $b = at^2$. Then, by Paulin's Lemma, there exists a hyperbolic $g \in G$ whose axis contains the segment, $[a, b]$.

As the axes of t and g intersect non-trivially, we can assume (by replacing g with g^{-1} if needed) that t and g translate in the same direction along their intersection. In that case, using the Circle-Dot Lemma (Lemma 3.2.11) with $p = a, q = at$, we have that $\|tg^n\| = \|t\| + n\|g\|$ for all positive integers n. Hence these elements are all in distinct conjugacy classes, since translation length is a conjugacy invariant.

Case(ii)(a): Next suppose that t is elliptic and that $|\mathrm{Fix}(t)| > 1$. Here choose $a \neq b \in \mathrm{Fix}(t)$ and let $g \in G$ be a hyperbolic element whose axis contains $[a, b]$. By replacing g with a sufficiently large (but possibly negative) power, we may assume that $b \in [a, ag]$.

It is straightforward to verify using the Circle-Dot Lemma (Lemma 3.2.11) with $p = a, q = b$ that $\|tg^n\| = n\|g\|$ for all positive integers n and hence, again by Lemma 2.2.2, we are done.

Case(ii)(b): Lastly suppose that t is elliptic and fixes a unique point in the tree. Since the action is G-irreducible, G admits two hyperbolic elements whose axes are disjoint. In particular, we may find a hyperbolic $g \in G$ whose axis does not contain the unique fixed point of T.

Let p be the fixed point of t and let q be the point on the axis of g closest to p. The the Circle-Dot Lemma (3.2.11) gives us that $\|tg^n\| = 2d(p, q) + |n|\|g\|$. Hence these elements fall into infinitely many conjugacy classes. □

Remark Note that when using Lemma 3.2.14 we refer to Lemma 3.2.13 to deduce that the action of G on the same tree is also irreducible and minimal. In practice, however, we will start with a minimal irreducible action of G and extend to one on the mapping torus via Corollary 3.5.5. In that sense, Lemma 3.2.13 is redundant but we record it here for completeness.

3.3 Deformation spaces and reduced trees

In this section we wish to explore the concepts of a 'reduced' simplicial tree, which is different to that of an irreducible tree.

We start with the definition of a deformation space. This discussion is taken from [38], to which we refer the reader for a fuller account.

Definition 3.3.1 ([38, Definition 3.2]) Let T, S be two simplicial G-trees.

- We say that T and S have the same elliptic subgroups if for every subgroup H of G which fixes a point of T, H also fixes a point of S, and vice versa.
- T and S are then said to be in the same deformation space if they have the same elliptic subgroups.
- The deformation space $\mathcal{D}$ containing T is then the set of all G-trees (up to equivariant isometry) which have the same elliptic subgroups as T.

Remark Note that the notion of equivariant isometry makes sense since we can regard our simplicial trees T as simplicial $\mathbb{R}$-trees by assiging length 1 to each edge.

Definition 3.3.2 ([38, Definition 3.12]) Given a deformation space $\mathcal{D}$ and a collection of subgroups $\mathcal{A}$ of G (closed under conjugacy and passing to subgroups) we define $\mathcal{D}_{\mathcal{A}} \subseteq \mathcal{D}$ as the set of G-trees $T \in \mathcal{D}$ whose edge stabilizers belong to $\mathcal{A}$. We call $\mathcal{D}_{\mathcal{A}}$ a restricted deformation space, or simply a deformation space.

Proposition 3.3.3 ([38, Proposition 3.10 (1)]) *Let $\mathcal{D}$ be a deformation space. If one tree $T \in \mathcal{D}$ is irreducible, then all trees in $\mathcal{D}$ are irreducible.*

Definition 3.3.4 ([38, Definition 3.5]) A simplicial G-tree T is called reduced if whenever $e = (u, v)$ is an edge of T with $\mathrm{Stab}_G(e) = \mathrm{Stab}_G(v)$, then u and v are in the same G-orbit.

Observation 3.3.5 Note that any deformation space contains a reduced tree. See [38], Definitions 3.3, 3.4 and 3.5.

Lemma 3.3.6 *Let T be a simplicial reduced G-tree. Then T is also minimal.*

Proof We prove the contrapositive. So suppose that T is not minimal; we will deduce that it is not reduced.

We consider some proper, G-invariant subtree, S. Let $e = (u, v)$ be an edge of T such that $u \in S$ but $v \notin S$. Then $v \notin S \supseteq u.G$, as S is G-invariant and so u and v are not in the same G orbit. It then suffices to show that $\mathrm{Stab}_G(v) \subseteq \mathrm{Stab}_G(e)$ (as we always have that $\mathrm{Stab}_G(e) \subseteq \mathrm{Stab}_G(v)$).

Consider $g \in \mathrm{Stab}_G(v)$. Then the edge path, $e, \overline{e}g$ starts at u and ends at ug. Since neither e nor eg is an edge of S and $u, ug \in S$ are connected by a path in S, we must have that this path fails to be reduced (and $u = ug$). Thus $e = eg$, as required. □

Observation 3.3.7 [[38]] Let G be a finitely generated group acting on a simplicial tree, T.

(i) An edge e of T is called collapsible if collapsing e to a point produces a G-tree in the same deformation space as T, [38, Definition 3.3].
(ii) We can collapse all collapsible edges of T to produced a reduced G-tree, $r(T)$, the reduction of T. Any edge stabiliser of $r(T)$ stabilises an edge of T, [38, Definition 3.5].
(iii) T and $r(T)$ lie in the same deformation space. In particular, the action on T is non-trivial if and only if $r(T)$ is not a point, [38, Theorem 3.8] and Lemma 3.3.6.
(iv) If T is reduced and the edge stabilisers of T are finite, then all reduced trees in the same deformation space as T have the same edge stabilisers as T, [38, Proposition 7.1 (3) and Corollary 7.3]

3.4 Irreducible actions on trees

We write down a fairly simple criterion for an action of a group on a tree to be irreducible. This will be useful in what follows.

First we will need the following Lemma.

Lemma 3.4.1 *Suppose G is a finitely generated group and the derived subgroup $[G, G]$ is finite. Then G is virtually abelian.*

Proof Any element g of G has finitely many conjugates since $x^{-1}gx = gg^{-1}x^{-1}gx \in g[G, G]$. Therefore, the centraliser of any element has finite index. Hence the intersection of all the centralisers of the generators has finite index, as G is finitely generated. But this intersection is clearly central in G and hence abelian. Therefore G is virtually abelian. □

Proposition 3.4.2 *Let G be a finitely generated group acting minimally and non-trivially on a tree, T. Suppose further that the action is reducible. Then,*

(i) If T is simplicial and edge stabilisers are finite, then G is virtually cyclic and T is a line.

(ii) If T is an $\mathbb{R}$-tree and (pointwise) arc stabilisers are trivial, then G is virtually abelian and T is a line.

Proof If the action is reducible and non-trivial then we can, up to taking an index 2 subgroup of G, suppose that there is an invariant end. (See [15, Corollary 2.3 and Theorem 2.5 (ii)]; for a dihedral action take the index two subgroup corresponding to the orientation preserving isometries of $\mathbb{R}$.)

(i) Consider first the case where T is simplicial and edge stabilisers are finite.

Let R be a ray representing this invariant end. Invariance of the end means that $R \cap Rg$ is an infinite set (in fact, a subray) for every $g \in G$.

Next observe that for any vertex $p \in R$ there exist only finitely many pairs of vertices $x, y \in R$ such that p is the midpoint of $[x, y]$. This is because both x and y must be equidistant from p and be distinct. But now $d(x, p) = d(y, p) \leqslant d(\omega, p)$ where ω is the intial point of the ray, R. There are clearly only finitely many pairs satisfying this on the ray.

Armed with this observation, we proceed as follows. Note that for any $g \in G$, $R \cap Rg^{-1}$ is infinite. If we take any point $x \in R \cap Rg^{-1}$ then $x, xg \in R$. But the midpoint of $[x, xg]$ lies in $R \cap A_g$ by Lemma 3.2.3(i). Since we have infinitely many possible values for x this implies that $R \cap A_g$ is infinite for any $g \in G$.

Therefore if g is elliptic it fixes some subray of R (but potentially a different subray for each g) and if g is hyperbolic then the axis of g intersects R in a subray.

In any case it is then easy to see that the commutator of any two elements is elliptic with fixed point set containing an infinite subray of R. But now any finite set of commutators must fix some common infinite subray of R (one can take the intersection of the fixed subrays for each commutator). Hence any finite set of commutators are contained in some edge stabiliser; for instance the first edge of the common fixed subray. In particular, any finite set of commutators are contained in some edge stabiliser and so generate a finite group as edge stabilisers are finite.

While it is possible that each finite set of commutators is contained in a different edge stabiliser we will argue that the derived subgroup is finite (and contained in some edge stabiliser) as follows. The action of G on T is co-compact, since G is finitely generated. Hence there exists a constant, M, such that any edge stabiliser has order at most M. Therefore any finitely generated subgroup of $[G, G]$ has order bounded by M which implies that $[G, G]$ has order bounded by M. (As $[G, G]$ is countable and generated by commutators, we can realise it as a countable union of a chain of finitely generated subgroups. Since the order of these subgroups is bounded, the chain of subgroups stabilises and hence $[G, G]$ is finite.)

Therefore $[G, G]$ is finite and hence G is virtually abelian by Lemma 3.4.1. This implies that any two hyperbolic elements of G have positive powers which commute, since sufficiently large positive powers will lie in an abelian subgroup. Therefore, by Lemma 3.2.2 (v) and Observation 3.2.4, all hyperbolic elements have the same axis. This axis is now a G-invariant subtree of T by Propostion 3.2.8 and so is equal to T. Therefore T is a line.

To finish we note that some finite index subgroup of G is torsion-free abelian and acts on T with finite (and hence trivial) edge stabilisers. Thus the torsion-free subgroup of

G is both free and free abelian (and non-trivial since if G were finite then the action on T would be trivial). Therefore G is virtually infinite cyclic.

(ii) The case where T is an $\mathbb{R}$-tree with trivial arc stabilisers is similar but easier. In this case, once descending to an index two subgroup of G admitting an invariant end, we deduce that any two hyperbolic axes intersect in an infinite ray. Since arc stabilisers are trivial, this means that any two hyperbolic elements commute which implies that their axes are equal. Hence T is isometric to the real line and G is virtually abelian. □

3.5 Actions and automorphisms

Definition 3.5.1 Let G be a group, T a G-tree and $\varphi \in \mathrm{Aut}(G)$. Then φT is the same underlying tree but with the G-action twisted by φ. Specifically,

$$x \cdot_{(\varphi T)} g := x \cdot_T (g\varphi).$$

Equivalently, if the action on T is given by a homomorphism, $G \xrightarrow{\pi} Isom(T)$, then the action on φT is given by the pre-composition by φ; $G \xrightarrow{\varphi} G \xrightarrow{\pi} Isom(T)$.

Remark Since we are writing our action of $\mathrm{Aut}(G)$ on G as a right action, this action of $\mathrm{Aut}(G)$ on the space of G-trees is a left action. Were we to write automorphisms of G on the left, this would be a right action.

Definition 3.5.2 Let G be a group and $\varphi \in \mathrm{Aut}(G)$. A topological representative of φ is an equivariant map $f : T \to \varphi T$, from a G-tree, T, which sends vertices to vertices and edges to edge paths.

Observation 3.5.3 Note that $f : T \to \varphi T$ being G-equivariant means that $f(xg) = f(x)(g\varphi)$ for all $x \in T$, $g \in G$.

However, note that if f is a topological representative for φ, then f_w is a topological representative for $\varphi Ad(w)$, where $f_w(x) = f(x)w$, and $g(\varphi Ad(w)) = w^{-1}(g\varphi)w$. Thus having a topological representaive is really a property of outer automorphisms.

Theorem 3.5.4 (Culler–Morgan [15, Theorem 3.7 (p.586)]) *If T_1 and T_2 are two minimal, irreducible G-trees with $\|g\|_{T_1} = \|g\|_{T_2}$ for all $g \in G$, then there is a unique equivariant isometry, $f : T_1 \to T_2$.*

Remark Irreducible is a stronger hypothesis than needed but suffices for our purposes and makes the statement a little cleaner.

Corollary 3.5.5 *Let T be a minimal irreducible G-tree and $\Phi \in \mathrm{Out}(G)$. Then M_Φ has an isometric action on T which restricts to the G-action on T if and only if $\|g\Phi\|_T = \|g\|_T$ for all $g \in G$.*

Remark Note that the isomorphism class of M_Φ and the value of $\|g\Phi\|_T$ only depend on the outer automorphism class, so this makes sense. One can think of $g\Phi$ as a conjugacy class rather than a single element.

Proof Suppose first that M_Φ acts isometrically on T in a way which extends the G-action. Then, since translation length is a conjugacy invariant,

$$\|g\|_T = \|g^t\|_T = \|g\Phi\|_T \text{ for all } g \in G.$$

Conversely, suppose that $\|g\Phi\|_T = \|g\|_T$ for all $g \in G$. Then, by definition, $\|g\|_{\Phi T} = \|g\Phi\|_T$, which equals $\|g\|_T$ by hypothesis.

Choose some $\varphi \in \Phi$. Then, by Theorem 3.5.4, there is an isometry $f : T \to T$ which is equivariant when viewed as an isometry $f : T \to \varphi T$. But now we can extend the G action to an $M_\Phi = M_\varphi$ action by simply having t act as f. The only thing to check is that $t^{-1}gt$ and $g\varphi$ act in the same way on T. But this is readily checked to be equivalent to saying that $f : T \to \varphi T$ is equivariant. □

We can now state the main tools we will use to prove R_∞.

The first we have already stated in Lemma 3.2.14.

Proposition 3.5.6 (Levitt–Lustig [44, Proposition 3.1]) *Let G be a finitely generated group, fix $\Phi \in \mathrm{Out}(G)$. Let l be the length function of an irreducible action of G on an $\mathbb{R}$-tree. If $l \circ \Phi = l$, then G has $R_\infty(\Phi)$.*

Proposition 3.5.7 (Levitt–Lustig [44, Proposition 3.2]) *Let G be a finitely generated group, fix $\Phi \in \mathrm{Out}(G)$. Let l be the length function of an irreducible action of G on an $\mathbb{R}$-tree T and suppose $l \circ \Phi = \lambda l$ where $\lambda > 1$. If arc stabilisers of T are finite and the number of $\mathrm{Stab}(x)$-orbits of directions at branch points x are uniformly bounded, then G has $R_\infty(\Phi)$.*

4 Train Track Maps and free products

In this section we show that free products of groups have the R_∞ property using the machinery of (Relative) Train Track maps. These were first used in [6] in the context of free group automorphisms and then in [4] to define and analyse the stable (or forward) limit tree. The methods in the former paper were extended to free products in [13]. However, we are using the treatment in [25], [26] and [27] because of the more detailed relation to the stable limit tree. We also use some results from [42] to deal with the technical conditions mentioned in the introduction; specifically, this allows us to verify the condition that stabilisers of branch point act on the set of directions with finitely many orbits.

As always, the goal is to achieve condition R_∞ by looking at the action of the mapping torus on some tree and applying the argument of [44, Propositions 3.1 and 3.2].

4.1 Free factor systems

Definition 4.1.1 *(Free Factor System)* Let G be a group which splits as a free product. A free factor system of G is a pair $\mathcal{F} = (\{G_1, \ldots, G_k\}, r)$ such that $G = G_1 * \cdots * G_k * F_r$ where each G_i is non-trivial and F_r is the free group of rank r.

Following [38, Definitions 3.2 and 3.12] we make the following definition.

Definition 4.1.2 Let G be a group which splits as a free product and $\mathcal{F} = (\{G_1, \ldots, G_k\}, r)$ a free factor system for G. We denote by $D(\mathcal{F})$ the set of all simplicial G-trees (up to equivariant isometry) whose elliptic subgroups are subgroups of conjugates of the G_i. That is $T \in D(\mathcal{F})$ if and only if for every $H \leqslant G$, H fixes a point in T exactly when H is a subgroup of some conjugate of some G_i.

Further, we let $D_1(\mathcal{F})$ denote the subset of $D(\mathcal{F})$ consisting of those trees whose edge stabilisers are all trivial.

Definition 4.1.3 Let $\mathcal{F} = (\{G_1, \ldots, G_k\}, r)$ be a free factor system of some group G, and let $\varphi \in \mathrm{Aut}(G)$. We say that $\mathcal{F}$ is:

- a Grushko decomposition of G if each G_i is freely indecomposable and not infinite cyclic.
- proper if $k + r \geq 2$.
- maximal if for any proper free factor system $\mathcal{F}' = (\{G'_1, \ldots, G'_l\}, s)$ of G, there exists some i so that G_i is not contained in any conjugate of any G'_j.
- φ-invariant if for each i there is some j and some $g_j \in G$ so that $\varphi(G_i) = G_j^{g_j}$.

We say that φ is irreducible with respect to $\mathcal{F}$ if $\mathcal{F}$ is a maximal, proper, φ-invariant free factor system of G.

Remark Observe that a Grushko decomposition of a group G is φ-invariant for any $\varphi \in \mathrm{Aut}(G)$. See [47, III, Proposition 3.7] for a statement of Grushko's Theorem.

The goal is to produce a very well behaved topological representative (see Definition 3.5.2) for an automorphism of a free product.

Definition 4.1.4 Let T be a simplicial G-tree with finitely many (oriented) edge orbits, reprpresented by edges $e_1, \ldots, e_n$. Suppose that $f : T \to \varphi T$ is a topological representative. Then the transition matrix of f is the $n \times n$ matrix, M_f, whose (i, j)-entry is the number of times the edge path $f(e_i)$ crosses the orbit of e_j (in either direction).

Definition 4.1.5 Let T be a simplicial G-tree with finitely many orbits of edges and $f : T \to \varphi T$ a topological representative for some $\varphi \in \mathrm{Aut}(G)$. Then f is called a train track map if for every $k \geqslant 1$, f^k is locally injective on the interior of edges.

It is called a metric train track map if, in addition, T admits a G-equivariant path metric such that the length of $f(e)$ is λ times the length of e, where e is any edge of T for some fixed $\lambda \in \mathbb{R}_{\geq 0}$.

We refer the reader to [6] for the background on how irreducible matrices and their Perron–Frobenius eigenvalues are used in the theory of train tracks.

From a train track we can construct the (forward) stable limit tree as in [4], although we will base the following more closely on [29]. Namely, given a metric train track map, $f : T \to \varphi T$ with associated constant $\lambda > 1$, set $d_\infty(x, y) = \lim_{n\to\infty} \frac{d_T(f^n(x), f^n(y))}{\lambda^n}$, for $x, y \in T$. This converges due to monotonicity and is a pseudo-metric on T. Therefore on taking a suitable quotient, one obtains a metric space T_∞ with metric (which we will still name), d_∞.

It is straightforward to see that the quotient map, $T \to T_\infty$ is G-equivariant and 1-Lipschitz and that T_∞ is a 0-hyperbolic metric space on which G acts isometrically. In other words, T_∞ is an $\mathbb{R}$-tree on which G acts. Moreover, the train track property immediately implies that this quotient map is an isometry restricted to the edges of T.

It also follows quickly that for any $\gamma \in G$, $\|\gamma\|_{T_\infty} = \lim_{n\to\infty} \frac{\|\gamma\varphi^n\|_T}{\lambda^n}$. It is well known that given a (metric) train track there is a hyperbolic $g \in G$ such that f^k is injective on A_g for all $0 < k \in \mathbb{Z}$, [25, Corollary 8.12] (this is also found in [6]). Such a g is called a 'legal' group element. It now follows that if g is legal, $0 < \|g\|_{T_\infty} = \|g\|_T$.

One can now proceed as in [29, Lemma II.7], to see that T_∞ is a minimal and non-trivial G-tree. In fact we record more.

Theorem 4.1.6 *Let G be a non-trivial free product which is not the infinite dihedral group. Let $\Phi \in \mathrm{Out}(G)$.*

Then there exists a maximal, proper, φ-invariant free factor system $\mathcal{F}$ and a simplicial G-tree $T \in D_1(\mathcal{F})$ admitting a train track representative f for φ, with associated constant $\lambda \geqslant 1$. Furthermore:

(i) λ is the Perron–Frobenius eigenvalue of the transition matrix for f.
(ii) $\lambda = 1$ if and only if f is an isometry. In this case the mapping torus M_φ acts isometrically on T.
(iii) If $\lambda > 1$ then $\lim_{n\to\infty} \frac{\varphi^n T}{\lambda^n}$ is a G-tree, T_∞, called the stable limit tree for φ.
(iv) For all $g \in G$, $\|g\varphi\|_{T_\infty} = \lambda\|g\|_{T_\infty}$. In particular, there exists a G-equivariant homothety, $H : T_\infty \to \varphi T_\infty$ of stretching factor λ, by Theorem 3.5.4.
(v) The stable limit tree is a non-trivial, irreducible $\mathbb{R}$-tree.
(vi) If $1 \neq g \in G$ acts elliptically on T, then g fixes a unique point of T_∞. In particular, non-trivial elements of arc stabilisers of T_∞ act hyperbolically on T.
(vii) There exist finitely many segments, $f_1, \ldots, f_r$ in T_∞ whose union of G-orbits covers the whole of T_∞.

Proof As in [25, Corollary 8.25], there exists a maximal φ-invariant free factor system, $\mathcal{F}$ so that φ is $\mathcal{F}$-irreducible. Again using [25, Theorem 8.24], φ will admit a train track representative. Just as in [6], the transition matrix of this map has a Perron–Frobenius eigenvalue, $\lambda \geqslant 1$, which is equal to the Lipschitz constant of the train track map (in [6] one must choose lengths for the edges, whereas in [25] this is implicit in the definition. This is the distinction between topological and metric train track maps).

Moreover, the theory of irreducible matrices implies that $\lambda = 1$ if and only if the transition matrix is a permutation matrix, implying that f is an isometry.

This proves (i) and (ii). For (iii) and (iv) we refer the reader to [27, Lemma 2.14.1]. Point (v) is well known to experts but we can also deduce it from Lemma 4.1.8 and Theorem 3.4.2, since the only non-trivial virtually abelian free product is the infinite dihedral group. Point (vi) is [27, Lemma 2.13.6].

Finally, for part (vii), we argue as above (following [29], but see also [27, Section 2.13]). That is, since the quotient map $T \to T_\infty$ is an isometry restricted to edges of T we may take the f_i to be the isometric images of the edges of T. □

The following are also well-known to experts and the proofs are the same as in [29], adapted to the free product case.

Lemma 4.1.7 *For any $p \in \mathbb{N}$ there exists a constant L_p such that any arc σ in T_∞ whose length is greater than L_p, will contain at least p non-trivial, disjoint subsegments which are all in the same G-orbit.*

Proof By Theorem 4.1.6 (vii), there are finitely many arcs, $f_1, \ldots, f_r$ whose G-orbits cover T_∞. Moreover, any arc segment σ can be subdivided into subsegments, $\sigma = z_1 \cup \ldots \cup z_n$ such that for each i there is a group element g_i such that $z_i g_i$ is a subsegment of one of $f_1, \ldots, f_r$. As this is a subdivision, the length of σ is equal to the sum of the lengths of the z_i and two distinct z_i can intersect in at most a single point. (If numbered sequentially, z_i intersects precisely z_{i-1} and z_{i+1} in a single point and is disjoint from the rest).

Now, after possibly further subdividing, we may assume that for $i \neq j$ either $z_i g_i = z_j g_j$ or their intersection is at most a single point.

Next let us suppose that there is an integer p such that,

$$\max_i |\{j : z_i g_i = z_j g_j\}| \leqslant p.$$

It is then clear that the sum of the lengths of the $z_i g_i$ is at most $p(l_1 + \ldots + l_r)$, where l_j is the length of f_j. But since each g_i is an isometry, this means that the length of σ is also at most $p(l_1 + \ldots + l_r)$.

Therefore, if σ is any segment of length greater than $3p(l_1 + \ldots + l_r)$ then we can ensure that σ contains at least $3p$ subsegments, $c_1, \ldots, c_{3p}$ which are all in the same G orbit. Since each z_i meets at most z_{i-1} and z_{i+1}, we deduce that at least p of these must be disjoint. □

Lemma 4.1.8 *If G is finitely generated, then arc stabilisers in T_∞ are trivial.*

Remark The hypothesis that G be finitely generated is not needed here. It is used to deduce that the limit of small actions is small. However, this will always be true of a limit of edge-free actions and we include the hypothesis for convenience of referencing.

Remark Note that we are implicitly assuming that the λ from Theorem 4.1.6 is greater than 1.

Proof Consider some non-trivial arc c of T_∞ and let S be the pointwise stabiliser of c. By Theorem 4.1.6(vi), every non-trivial element of S acts on T as a hyperbolic isometry. In particular, S acts freely on the simplicial tree, T, and so is a free group. Thus, in order to show that S is trivial it is enough to show that S is finite.

We therefore argue by contradiction and suppose that S is infinite.

Since $T \in \mathcal{D}_1(\mathcal{F})$ then edge stabilisers in T are trivial, hence the action of G on T (and also on each $\varphi^n T$) is small. Since a limit of small actions is itself small, by Culler–Morgan [15, Theorem 5.3], S cannot contain a free subgroup of rank 2.

However, we have already noted that S is a free group and so for S to be infinite it would have to be an infinite cyclic group. Therefore the minimal S-invariant subtree of T is a line L on which S acts by translation. (Note that all the non-trivial elements of S have axis equal to L, by Lemma 3.2.2 (v)).

Let $\bar{S}$ be the (setwise) stabiliser of L. Then $\bar{S}$ is virtually cyclic as its minimal invariant tree is L, by Lemma 3.2.2 (iv). Moreover, it is also a maximal virtually cyclic group as any virtually cyclic group containing S must preserve L, since any hyperbolic element in such a subgroup has a power which lies in S and hence has axis equal to L. (Recall that an invariant subtree for a subgroup is the union of all the hyperbolic axes, by Proposition 3.2.8.)

Let $H : T_\infty \to \varphi T_\infty$ be the homothety described in Theorem 4.1.6(iv) and consider $H^k(c)$ for some k. Let p be the index of S in $\bar{S}$.

By Lemma 4.1.7, we may choose k sufficiently large so that $H^k(c)$ contains $p+1$ disjoint non-trivial arcs which are in the same G-orbit. Since H is equivariant, c will also contain $p+1$ disjoint arcs, $c_0, \ldots, c_p$ in the same G orbit. Hence there exist group elements, $g_1, \ldots, g_p$ such that $c_0 g_i = c_i$. (Set $g_0 = 1$.)

Let C_i denote the pointwise stabiliser of c_i. As argued above, each C_i is cyclic (since it is small and acts freely on T). Note that S is a subgroup of C_i and hence C_i is a subgroup of $\bar{S}$, since the latter is a maximal virtually cyclic subgroup of G.

Moreover, since $c_0 g_i = c_i$ we also have that $C_i = C_0^{g_i}$. But now $S \leqslant C_0^{g_i} \leqslant \bar{S}^{g_i}$. By maximality, $\bar{S} = \bar{S}^{g_i}$ for all i. However, any hyperbolic element of $\bar{S}$ has axis equal to L, whereas any hyperbolic element of $\bar{S}^{g_i}$ has axis equal to Lg_i. Hence $L = Lg_i$ and $g_i \in \bar{S}$ for all i. But since the c_i are disjoint, $g_0, \ldots, g_p$ are in distinct cosets of C_0 in $\bar{S}$, contradicting the fact that $S \leqslant C_0$ has index p in $\bar{S}$. This contradiction shows that S must in fact be trivial. □

Using this, we can bound the number of orbits of directions using the following result.

Proposition 4.1.9 (Horbez [42, Proposition 4.4])

Let G be a countable group, $\mathcal{F}$ a free factor system for G, and T an $\mathbb{R}$-tree equipped with an isometric G action such that any subgroup in $\mathcal{F}$ fixes a point in T. (The subgroups in $\mathcal{F}$ are elliptic in T).

Then if pointwise arc stabilisers in T are trivial, there is a uniform bound on the number of $\mathrm{Stab}(x)$*-orbits of directions at branch points x in T.*

Remark Horbez [42, Proposition 4.4] gives a bound on the 'index' of a tree T, a component of which is the sum over G-orbits of points x in T of the number of $\mathrm{Stab}(x)$-orbits of directions from x in T with trivial stabiliser. The proposition then follows immediately.

Combining this with Proposition 3.5.7 we get,

Theorem 4.1.10 *Let G be a finitely generated group which splits as a non-trivial free product of finitely many indecomposable groups. Then G has property R_∞.*

Remark The only times we need the hypothesis of being finitely generated are when we invoke the [15, Theorem 5.3] in Lemma 4.1.8 and [42, Proposition 4.4] in Proposition 4.1.9. However, both results are true in general without this hypothesis.

Proof The case where $G = D_\infty$ is known and can be dealt with separately (see Theorem 7.1.2).

In all other cases, every tree in $D_1(\mathcal{F})$, for any proper free factor system $\mathcal{F}$ is irreducible.

Consider some $\varphi \in \mathrm{Aut}(G)$. We now use Theorem 4.1.6. If $\lambda = 1$, we get an isometric action of M_φ on some $T \in D_1(\mathcal{F})$ and hence we are done by Proposition 3.5.6 (or Lemma 3.2.14). If $\lambda > 1$ we form the stable limit tree, T_∞.

All that is left is to verify the hypotheses of Proposition 3.5.7 in the case where $\lambda > 1$. This is done in Theorem 4.1.6, Lemma 4.1.8 and Proposition 4.1.9. □

Corollary 4.1.11 *The property of being R_∞ is undecidable amongst finitely presented groups.*

Proof It is well known [47, Chapter IV, Theorem 4.1] that the property of being trivial is undecidable amongst finitely presented groups. (It is clear that triviality is a Markov property).

Suppose we are given a finite presentation of a group, $G = \langle X \quad R \rangle$.

Then we can form the finitely presented group, $\Gamma = \mathbb{Z} * G = \langle X, t \quad R \rangle$, where t is a letter which does not appear in X.

The group Γ is isomorphic to $\mathbb{Z}$ precisely when G is trivial. In which case Γ would not have R_∞. ($\mathbb{Z}$ has exactly two twisted conjugacy classes with respect to the automorphism sending $n \mapsto -n$).

On the other hand, if G were non-trivial, then Γ would be a finitely generated group which splits as a non-trivial free product of finitely many indecomposable groups and so has R_∞ by Theorem 4.1.10.

Therefore, if we could decide R_∞ for Γ we could also decide triviality for G. □

5 R_∞ for Groups with Infinitely Many Ends

5.1 Accessible groups

In this section we extend the result to groups with infinitely many ends. We recall the definition of an end. (See also [17, IV, Definition 6.4], for an alternate but equivalent formulation.)

Definition 5.1.1 Let G be a finitely generated group with finite generating set S. Let Γ be the Cayley graph of G with respect to S. Then the number of ends of G is the supremum of the number of infinite components of the graphs, $\Gamma \setminus F$, where F ranges over all finite subgraphs of Γ.

It turns out that the number of ends does not depend on S and is an invariant of the group. In fact, more can be said via the classical Stallings Theorem on ends, which can be found in [17, IV, Theorem 6.10 and Theorem 6.12]

Theorem 5.1.2 *Let G be a finitely generated group.*

- *The number of ends of G is* $0, 1, 2$ *or* ∞.
- *If G has more than one end, then G acts non-trivially on a simplicial tree with finite edge stabilisers.*
- *The number of ends of G is 2 if and only if G has a infinite cyclic subgroup of finite index.*

Thus a group with infinitely many ends can be 'split' along finite subgroups. One could then ask whether the vertex stabilisers of such an action split further and whether this process terminates. The groups for which it does terminate are called *accessible*.

Definition 5.1.3 (See [54, p.189] and [17, IV, Definition 7.1]) Given a group G a simplicial G-tree T is called terminal if the vertex stabilisers of T have at most one end and all edge stabilisers are finite.

A finitely generated group G is called accessible if it admits a terminal G-tree.

Observation 5.1.4 We insist that an accessible group is finitely generated as in [54] but contrary to [17].

Note that the vertex groups of an accessible group G acting minimally on a terminal G-tree T are finitely generated, by [17, III, Lemma 8.1].

Definition 5.1.5 ([17, VI, Definition 4.1]) A group G is called almost finitely presented if it is of type FP_2 over $\mathbb{Z}_2$. In particular, every finitely presented group is almost finitely presented.

Remark Recall that a reduced G-tree (Definition 3.3.4) is also minimal by Lemma 3.3.6.

Theorem 5.1.6 ([17], VI, Theorem 6.3) *Let G be an almost finitely presented group. Then G has a reduced terminal G-tree T. Moreover, the action on T is co-compact.*

Proof The existence of a terminal G-tree follows from [17, VI, Theorem 6.3]. The fact that we can take the tree to be reduced follows from [38]. The action can then be taken to be minimal by Lemma 3.2.9 and Proposition 3.2.8 (as G is finitely generated). Finally, the action is co-compact, as is any action of a finitely generated group acting minimally on a simplicial tree. For instance, see [17, I, Proposition 4.13]. □

We will also need the following, which is just a restatement of the 'blowing up' construction given in [17, IV, Section 7].

Proposition 5.1.7 *Let H be an accessible group acting on a simplicial tree S with finite edge stabilisers. Then there exists a terminal H-tree, X so that every edge stabiliser of S is equal to some edge stabiliser of X.*

Moreover, if the action on S is non-trivial then so too is the action on X.

Proof The idea here is to 'blow up' S to produce X. Just as in Observation 5.1.4, all the vertex stabilisers in S are finitely generated and hence accessible, since H is. If we write H as a graph of groups using S, we can then replace each vertex group, H_v, with a graph of groups whose edge stabilisers are finite and whose vertex stabilisers have at most one end. In particular, since all edge stabilisers in S are finite, each edge group must fix a vertex in any incident H_v terminal tree and so we get a well defined graph of groups giving us the result. □

Proposition 5.1.8 *Suppose that a finitely generated group H acts minimally on a simplicial tree T with finite edge stabilisers and consider the reduction, $r(T)$ obtained from T by collapsing all collapsible edges. Then,*

(i) If T is not a point, then $r(T)$ is also not a point.
(ii) For any edge e of $r(T)$ there exists an edge e' of T such that $|\operatorname{Stab}_H(e')| = |\operatorname{Stab}_H(e)|$.

Proof Since T and $r(T)$ are in the same deformation space by Observation 3.3.7, if $r(T)$ is a point then the whole of H is elliptic in $r(T)$ and hence also in T, proving (i).

For (ii), we can simply regard the edge set of $r(T)$ as a subset of the edge set for T, from which the result immediately follows. □

Definition 5.1.9 A subgroup H of a group G is called characteristic if for every $\varphi \in \operatorname{Aut}(G)$ we have that $(H)\varphi \leq H$.

Observation 5.1.10 It is an easy exercise to see that if H is a characteristic subgroup of a group G, then the following all hold:

(1) For any $\varphi \in \operatorname{Aut}(G)$ we have $(H)\varphi = H$.
(2) H is a normal subgroup of G.
(3) Every automorphism of G induces an automorphism of G/H.
(4) If G has a unique subgroup K of a given order, then K is characteristic.

Lemma 5.1.11 *Suppose a group G has a characteristic subgroup N. If G/N has R_∞, then so too does G.*

Proof Let $\varphi \in \operatorname{Aut}(G)$. Since N is characteristic, φ induces an automorphism φ_* on G/N given by $(g \cdot N)\varphi_* = (g)\varphi \cdot N$. Given $xN, yN \in G/N$, we have that

$$\begin{aligned} &xN \sim_{\varphi_*} yN \\ \iff &\exists wN \in G/N \text{ such that } (wN)\varphi_* xN(wN)^{-1} = yN \\ \iff &\exists w \in G \text{ such that } (w\varphi)xw^{-1}N = yN \\ \iff &\exists w \in G \text{ and } \exists n \in N \text{ such that } (w\varphi)xw^{-1} = ny \\ \iff &\exists n \in N \text{ such that } x \sim_\varphi ny. \end{aligned}$$

In particular, if xN and yN belong to different $\sim_{\varphi_*}$ classes in G/N, then x and y belong to different $\sim_\varphi$ classes in G (since $1 \in N$). So if G/N has $R_\infty(\varphi_*)$, then we must have that G has $R_\infty(\varphi)$. This holds for all $\varphi \in \operatorname{Aut}(G)$, hence if G/N has R_∞ then so too must G. □

Lemma 5.1.12 *Let G be a group acting on a terminal minimal tree T, and let $N = \{g \in G | xg = x \ \forall x \in T\}$ be the kernel of this action. Then N is the maximal (normal and finite) subgroup of G. In particular, N is a finite characteristic subgroup of G.*

Proof First, observe that N must be a subgroup of every edge stabiliser of T. Since T is terminal, it has finite edge stabilisers, hence N must be a finite group. Since N is a kernel, then it must be normal.

We will now show that N is the maximal normal finite subgroup of G. Indeed, let K be another finite normal subgroup in G. Since K is finite, then it must fix some point p in the tree T. Let $g \in G$ and $k \in K$. Since K is normal then there exists $k' \in K$ with $g^{-1}k'g = k$. Then $(p \cdot g) \cdot k = p \cdot gk = p \cdot k'g = p \cdot G$. Thus K pointwise fixes $p \cdot G$, and so must also fix the convex hull of the points $p \cdot G$. Since T is assumed to be minimal, we must have that this convex hull is equal to T, and hence $K \leq N$. Hence N is the maximal normal finite subgroup of G.

Now by Observation 5.1.10 (4), we must have that N is characteristic in G. □

Corollary 5.1.13 *Let G be a group acting on a terminal minimal G-tree T, and let $N = \{g \in G | xg = x \ \forall x \in T\}$ be the kernel of this action. If G/N has R_∞, then so too does G.*

Proof This follows immediately from Lemmas 5.1.11 and 5.1.12. □

Remark Since N is finite and G is assumed to be finitely generated, then G is quasi-isometric to G/N. Thus G has infinitely many ends if and only if G/N has infinitely many ends.

Definition 5.1.14 Let G be an accessible group and T a reduced terminal G-tree.

(i) Let $q := q(G)$ be the order of the smallest edge stabiliser of T. By [38, Corollary 7.3] this does not depend on T.
(ii) Let $T(q)$ be the tree obtained from T by collapsing every edge whose stabiliser has order greater than q.
(iii) Let V_0 be the set of vertices $x \in T(q)$ for which there exist incident edges e_1 and e_2 with $\mathrm{Stab}(e_1) \neq \mathrm{Stab}(e_2)$ as subgroups of G. Let V_1 be the set of subgroups Y of G which occur as an edge stabiliser in $T(q)$. We construct a bipartite tree $T(q)_c$ whose vertex set is $V(T(q)_c) = V_0 \sqcup V_1$, and where there is an edge (x, Y) between some $x \in V_0$ and some $Y \in V_1$ if and only if there is some edge $e \in T(q)$ with endpoint x and stabiliser Y.

Remark Note that $T(q)_c$ is the tree of cylinders constructed by Guirardel and Levitt [39, Section 4.1], where the admissible relation is equality of edge stabilisers (as in [39, Example 3.6]).

We refer the reader to [39] for the construction of the tree of cylinders. However we note the following straightforward exercise.

Proposition 5.1.15 *Let G be a group acting on a simplicial tree T, subject to some admissable relation $\sim$ with corresponding tree of cylinders, T_c.*

(i) If the action on T is minimal, then so is the action on T_c
(ii) If the action on T is minimal and non-trivial, then the action on T_c is trivial exactly when all edges belong to a single cylinder. Equivalently, this is exactly when all edge groups are equivalent under $\sim$.
(iii) If the action on T_c is non-trivial and reducible, then so is the action on T.

Proof (i) Each vertex of T_c is either a vertex of T belonging to more than one cylinder (the V_0 vertices) or a cylinder in T (the V_1 vertices). It is straightforward to show that the action of G on T then induces an action of G on T_c. For minimality see [39, Section 4.1, p.13] or [37, Lemma 4.9].

(ii) If the action of G on T_c is minimal, then the action on T_c is trivial if and only if T_c is a single point, i.e. the vertex set of T_c consists of a single cylinder $Y \in V_1$.

(iii) We briefly argue that if T_c admits a fixed end or an invariant line, then so too must T. Suppose that T_c admits an invariant line L. Since T_c is bi-partite, we can look at the vertices of T_c which are vertices of T belonging to more than one cylinder. (That is, the vertices in V_0). Thus we can think of $L \cap V_0$ as a set of vertices of T. It is clear that the convex hull of these is a line, $\tilde{L}$ in T. The invariance of L implies the invariance of $\tilde{L}$. The case of an end is similar. □

Proposition 5.1.16 *Let G be an accessible group, T a reduced terminal G-tree and $T(q)$ the G-tree obtained from T as in Definition 5.1.14. Then,*

(i) Every edge stabiliser of $T(q)$ is finite of order q.
(ii) $T(q)$ is a reduced G-tree.
(iii) Every vertex stabiliser of $T(q)$ is accessible.
(iv) Suppose that H is a vertex stabiliser of $T(q)$ and $\varphi \in \mathrm{Aut}(G)$. Then $H\varphi$ also fixes a vertex of $T(q)$

Proof Let F denote the subforest of T consisting of all edges whose full stabiliser has order greater than q and their incident vertices. We can think of the edges of $T(q)$ as simply being the edges of T which are not in F, proving (i).

Further, note that F is G-invariant. Hence if C is a component of F and $g \in G$, then $C \cap Cg \neq \emptyset$ implies that $C = Cg$, since Cg must be another component of F. The vertex stabilisers of $T(q)$ are then either vertex stabilisers of T or stabilisers of components of F. It is then easy to see that $T(q)$ is reduced and so (ii) holds.

Next we want to argue that vertex stabilisers of $T(q)$ are finitely generated. This is clear if the vertex of $T(q)$ is equal to a vertex in T.

Otherwise, consider a component C of F and let H be its setwise stabiliser - this is a vertex stabiliser in $T(q)$. As above, $C \cap Cg \neq \emptyset$ implies that $C = Cg$. In particular, G-orbits on C must equal H-orbits and G-stabilisers in C must equal H stabilisers. Therefore the action of H on C is reduced and hence minimal, so C is the minimal H-invariant subtree T_H. Moreover, since G acts with finitely many orbits, so does H on T_H, and since G-stabilisers are finitely generated, so are H stabilisers in C. Hence H is finitely generated by the fundamental theorem of Bass–Serre Theory (see e.g. [17, I, Theorem 4.1]).

We also argue that each vertex stabiliser of $T(q)$ is accessible. If the vertex stabiliser of $T(q)$ equals the vertex stabiliser of some vertex in T, then it has at most one end and the result is clear. Otherwise, a vertex stabiliser H will be the setwise stabiliser of some component C of F. But the action of H on C is co-compact, has finite edge stabilisers and the vertex stabilisers have at most one end. This proves (iii).

Finally we prove (iv). First consider the action of H on T. If H is a vertex stabiliser of T, then it has at most one end and is finitely generated. Hence $H\varphi$ is also finitely generated and has at most one end. Therefore, since $T(q)$ has finite edge stabilisers, $H\varphi$ must fix a vertex of $T(q)$.

If H does not fix a vertex of T, then it is the setwise stabiliser of some component, C of F and, as argued above, is finitely generated and $C = T_H$, the minimal invariant subtree of H which is reduced and whose edge stabilisers have order greater than q.

Now, consider the action of $H\varphi$ on $T(q)$. Suppose that $H\varphi$ does not fix a vertex of $T(q)$, then as $H\varphi$ is finitely generated it admits an invariant subtree, S. By Proposition 5.1.7, $H\varphi$ admits a non-trivial terminal tree X whose edge stabilisers have order at most q (since G-edge stabilisers in $T(q)$ have order q).

Next we reduce X to obtain $r(X)$. By Proposition 5.1.8, the action of $H\varphi$ on $r(X)$ is reduced and non-trivial and edge stabilisers have order at most q. But since H is isomorphic to $H\varphi$, this gives us two reduced terminal H-trees with different edge stabilisers, contradicting Observation 3.3.7. This proves (iv). □

Theorem 5.1.17 *Let G be an accessible group with infinitely many ends and let N be the maximal normal finite subgroup of G. Then:*

(i) For any reduced terminal G-tree T the deformation space of $T(q)$ is $\mathrm{Aut}(G)$ *invariant.*
(ii) If G/N does not split as a non-trivial free product then the tree of cylinders $T(q)_c$ is a non-trivial, irreducible, minimal G-tree which is invariant under the action of $\mathrm{Aut}(G)$. *(Here the admissable relation is equality.)*

Proof Let $\mathcal{D}$ be a deformation space of G-trees and $\varphi \in \mathrm{Aut}(G)$. Then $\mathcal{D}\varphi$ consists of all simiplicial G-trees for which $H\varphi$ is elliptic for any $\mathcal{D}$-elliptic subgroup H.

Therefore, to prove (i), it is enough to show that for vertex stabiliser H of T_q and any automorphism φ of G that $H\varphi$ also stabilises a vertex of T_q. This is Proposition 5.1.16(iv). This proves (i).

To prove (ii), we first consider the general situation.

Note that by [39], equality is an admissable relation since all edge stabilisers in $T(q)$ have the same order. By Proposition 5.1.16, we know that $T(q)$ is reduced and hence minimal by Lemma 3.3.6. Therefore, $T(q)_c$ is minimal. If $T(q)_c$ were trivial, this would imply that all edge stabilisers are equivalent, which in our context means equal (by Proposition 5.1.15). In particular, any edge stabiliser would be normal and so G/N would act on the tree T_q with trivial edge stabilisers.

Hence, if G/N does not split as a free product, then the action of G on $T(q)_c$ is non-trivial. Furthermore, we know that the action of G on $T(q)$ is irreducible by Proposition 3.4.2 and hence the action on $T(q)_c$ is also irreducible by Proposition 5.1.15. The fact that $T(q)_c$ is invariant under the action of $\mathrm{Aut}(G)$ follows from the fact that the deformation space of $T(q)$ is $\mathrm{Aut}(G)$ invariant and [39, Proposition 4.11] . □

Theorem 5.1.18 *Any accessible group G with infinitely many ends has property R_∞.*

In particular, any finitely presented group with infinitely many ends has property R_∞.

Proof Let N be the maximal normal finite subgroup of G. If G/N is a non-trivial free product then G has R_∞ by Corollary 5.1.13 and Theorem 4.1.10.

Otherwise, $T(q)_c$ is a non-trivial, irreducible, minimal G-tree which is invariant under the action of $\mathrm{Aut}(G)$ by Theorem 5.1.17 and hence G has R_∞ by Proposition 3.5.6. □

6 Relatively hyperbolic groups

6.1 JSJ decompositions and invariant trees

Theorem 6.1.1 *Let G be a non-elementary finitely presented hyperbolic group relative to a family $\mathcal{P} = \{P_1, \ldots, P_n\}$ of finitely generated groups. Let* $\mathrm{Out}(G;\mathcal{P})$ *denote the subgroup of* $\mathrm{Out}(G)$ *preserving the conjugacy classes of the P_i, but allowing permutation of the P_i.*

Then G has infinitely many twisted conjugacy classes with respect to any $\Phi \in \mathrm{Out}(G;\mathcal{P})$.

Remark Note that our definition of $\mathrm{Out}(G;\mathcal{P})$ differs from that of [40] in that we are allowing permutations. However, this does not effect the invariance of the JSJ tree and so this is a benign change to make. We also note that the hypothesis that the P_i are finitely generated arises since we use the JSJ decomposition of [41], where that is required.

Proof In our context G being non-elementary means that it is not virtually cyclic and it is not equal to a P_i.

If G has infinitely many ends, we can invoke Theorem 5.1.18 to deduce that G has R_∞. If Φ has finite order, we can use the argument of Delzant as in [44, Proposition 3.3 and Lemma 3.4]. We note that the argument there is for hyperbolic, rather than relatively hyperbolic groups, but the same arguments work since relative hyperbolicity is a commensurability (in fact a quasi-isometry) invariant by [18, Theorem 1.2], and one can replace the arguments about infinite order elements with ones about loxodromic elements in the relative hyperbolic setting.

Thus the remaining case is where $\mathrm{Out}(G;\mathcal{P})$ has an element of infinite order and G is one-ended. By [41, Corollary 9.20], there is a canonical relative JSJ tree, T. The fact that it is relatively canonical means that it is $\mathrm{Out}(G;\mathcal{P})$ invariant. In particular, we are done by Proposition 3.2.14 and Corollary 3.5.5 as long as T is G-irreducible.

However, if T were reducible then either G fixes a point of T or G admits an invariant line or end. If G were to fix a point of T, then G would equal a vertex stabiliser. Since G is not elementary, the vertex stabiliser cannot be parabolic or virtually cyclic. And since $\mathrm{Out}(G;\mathcal{P})$ is infinite, it cannot be rigid. Hence it must be a flexible quadratically hanging group with finite fibre. Therefore G would be hyperbolic and we would be done by [44, Theorem 3.5]. (This case could also be dealt with more concretely.)

So we can assume that G does not fix a point of T and in particular T is not a point and has an edge. We proceed to argue that T does not admit an invariant end or line. Consider N the kernel of the action. If the action were reducible we would get that G/N is either infinite cyclic or infinite dihedral by [15, Corollary 2.3 and Theorem 2.5]. (We note that in [15], those results construct homomorphisms to $\mathbb{R}$ and $Isom(\mathbb{R})$, but in the context of a simplicial tree, these land in $\mathbb{Z}$ and $Isom(\mathbb{Z})$ instead.).

On the other hand, as in [41, Corollary 9.20], edge stabilisers of T are elementary meaning that they are either virtually cyclic or contained in (a conjugate of) a parabolic subgroup P_i. However, elementary subgroups are almost malnormal (for instance, see [40, Corollary 3.2]), which implies that N is finite and if T were reducible, then G would have 0 or 2 ends and hence be elementary. Therefore T is G-irreducible and G has infinitely many twisted conjugacy classes with respect to any $\Phi \in \mathrm{Out}(G;\mathcal{P})$. ◻

Theorem 6.1.2 *Let G be a non-elementary finitely presented relatively hyperbolic group G whose peripheral subgroups are finitely generated but not relatively hyperbolic. Then G has property* R_∞.

Proof Since the peripheral subgroups are not relatively hyperbolic, they are $\mathrm{Aut}(G)$ invariant by [49, Lemma 3.2] and so $\mathrm{Out}(G;\mathcal{P}) = \mathrm{Out}(G)$. Therefore G has property R_∞ by Theorem 6.1.1. ◻

7 Appendix A: R_∞ for the infinite Dihedral Group D_∞

The following result is well-known but we include it for completeness. See [36, Proposition 2.3].

Note that the infinite Dihedral group is the only group which can be written as a non-trivial free product but having finitely many ends (it has two ends).

7.1 The infinite dihedral group

The group D_∞ is the free product of two cyclic groups of order 2 and hence has presentation,

$$\langle x, y \mid x^2 = y^2 = 1\rangle.$$

However, we will use the alternate generating set $\{x, t = xy\}$, which gives the corresponding presentation,

$$\langle x, t \mid x^2 = 1, t^x = t^{-1}\rangle.$$

This second presentation expresses the fact that D_∞ is a semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}_2$, where the monodromy is given by the (unique) non-identity automorphism of $\mathbb{Z}$.

With this second description is is straightforward to see that every element of D_∞ can be written as either t^n or $t^n x = xt^{-n}$ for some $n \in \mathbb{Z}$, and that this representation is unique.

Lemma 7.1.1 *The outer automorphism group of the infinite dihedral group,* $D_\infty = \langle x, y \mid x^2 = y^2 = 1\rangle$ *has order* 2. *The non-identity outer automorphism has a representative given by the map:*

$$x \mapsto y, y \mapsto x.$$

Remark It is easy to see from the universal property of free products that the map above defines an endomorphism of D_∞ whose square is the identity. Hence it defines an automorphism.

Proof It is convenient to work with the alternate generating set, $x, t = xy$. First note that every nontrivial element of $\langle t\rangle$ has infinite order whereas every element of the coset $x\langle t\rangle$ has order 2. Therefore any automorphism of D_∞ must restrict to an automorphism of $\langle t\rangle$ and hence send t to $t^{\pm 1}$.

Notice that since $t^{-1} = t^x$ we may assume, up to inner automorphisms, that any automorphism fixes t. Also, since it has order 2, the image of x must therefore be xt^n for some $n \in \mathbb{Z}$. However, $(xt^n)^{t^k} = xt^{n+2k}$ and since $t^{t^k} = t$ we deduce that, up to inner automorphisms, any automorphism is represented by the maps $t \mapsto t, x \mapsto x$ and $t \mapsto t, x \mapsto xt$. It is a simple calculation to see that these maps are in the same outer automorphism classes as the identity map and the map given in the statement of this Lemma, respectively. □

Theorem 7.1.2 *The group D_∞ has the R_∞ property.*

Proof The outer automorphism group of D_∞ has order 2 by Lemma 7.1.1, so by Lemma 2.1.4, it is sufficient to check that two automorphisms which represent these classes have infinitely many twisted conjugacy classes.

Case 1: φ is the identity

When φ is the identity, twisted conjugacy is simply standard conjugacy. We then note that,

$$(t^n)^t = t^n \quad \text{and} \quad (t^n)^x = t^{-n}.$$

It follows that t^n and t^m are conjugate if and only if $|m| = |n|$. Hence, in this case, the infinitely many elements t^n, where n is a positive integer, are all in different (twisted) conjugacy classes.

Case 2: $t\varphi = t^{-1}, x\varphi = xt.$

This automorphism can be seen from the other presentation as the one which interchanges x and y. In particular it is not inner and so represents the other outer automorphism class. Here we note that

$$(t\varphi)(t^n x)t^{-1} = t^{-1}t^n xt^{-1} = t^n x \quad \text{and} \quad (x\varphi)(t^n x)x = xt^{n+1} = t^{-n-1}x.$$

It follows that $t^n x$ is twisted φ-conjugate to $t^m x$ if and only if $m = n$ or $m = -n - 1$. Hence in this case, the infinitely many elements $t^n x$ where n is a positive integer are all in different twisted conjugacy classes. □

8 Appendix B: Property R_∞ via quasimorphisms

8.1 Quasimorphisms

In this appendix we present a more indirect approach to prove property R_∞, which applies to the groups considered in this paper, and more.

Definition 8.1.1 Let G be a group, and $f : G \to \mathbb{R}$ a function. The *defect* of f is

$$D(f) := \sup_{x,y \in G} |f(xy) - f(x) - f(y)|.$$

If $D(f) < \infty$, we say that f is a *quasimorphism*. If moreover $f(x^n) = nf(x)$ for all $x \in G$ and all $n \in \mathbb{Z}$, we say that f is *homogeneous*.

Quasimorphisms are useful tools for the interactions of group theory with various subjects, such as bounded cohomology [28], stable commutator length [11], knot theory [48], symplectic geometry [53] and dynamics [34]. Quasimorphisms are abundant among groups with hyperbolic features: this originates from the Brooks construction for free groups [9], and culminated in the Bestvina–Fujiwara construction for acylindrically hyperbolic groups [5]. Here we are only concerned with quasimorphisms with an additional special property.

Definition 8.1.2 Let $f : G \to \mathbb{R}$ be a quasimorphism and $\varphi \in \mathrm{Aut}(G)$. We say that f is *φ-invariant* if $f(x\varphi) = f(x)$ for all $x \in G$. We say that f is Aut-*invariant* if it is φ-invariant for every $\varphi \in \mathrm{Aut}(G)$.

Every quasimorphism is at a bounded distance from a homogeneous one [11, Lemma 2.21], so it is common to restrict to those. Homogeneous quasimorphisms are easily seen to be conjugacy invariant [11, Section 2.2.3], therefore φ-invariance of a homogeneous quasimorphism is really a property of the corresponding outer automorphism.

Theorem 8.1.3 *Let G be a group, let $\varphi \in \mathrm{Aut}(G)$, and suppose that there exists a non-zero φ-invariant homogeneous quasimorphism on G. Then G has property $R_\infty(\varphi)$.*

Proof Let $f : G \to \mathbb{R}$ be as in the statement. Let $\approx_D$ denote an equality up to $D(f)$, which is finite by assumption. Then we estimate:

$$f(wx(w\varphi)^{-1}) \approx_D f(wx) + f((w\varphi)^{-1}) \approx_D f(w) + f(x) + f((w\varphi)^{-1}) = f(x),$$

where we used homogeneity and φ-invariance in the last equality. In other words:

$$x \sim_\varphi y \qquad \Longrightarrow \qquad |f(x) - f(y)| \leqslant 2D(f).$$

Now let $x \in G$ be such that $f(x) \neq 0$: this exists by assumption. Up to replacing x by a suitable power, using homogeneity, we may assume that $|f(x)| > 2D(f) \geqslant 0$. Then for all $i \neq j \in \mathbb{Z}$:

$$|f(x^i) - f(x^j)| = |i - j||f(x)| \geqslant |f(x)| > 2D(f),$$

and so x^i and x^j cannot be in the same $\sim_\varphi$-class. □

We immediately obtain:

Corollary 8.1.4 *If there exists a non-zero* Aut*-invariant homogeneous quasimorphism on* G*, then* G *has property* R_∞. □

Let us note that the proof is showing more: if there exists a non-zero φ-invariant homogeneous quasimorphism, then there exists $x \in G$ whose powers are in pairwise distinct $\sim_\varphi$ classes. Similarly, if there exists a non-zero Aut-invariant homogeneous quasimorphism, then there exists $x \in G$ whose powers are in pairwise distinct $\sim_\varphi$ classes, for all φ simultaneously.

Remark Of course homomorphisms are homogeneous quasimorphisms, therefore our result recovers [45, Theorem 5.4], which states that groups with non-zero Aut-invariant homomorphisms have property R_∞. There, it was applied to prove property R_∞ for a large family of Thompson-like groups, including the piecewise linear F-like groups of Bieri–Strebel [7], the piecewise projective groups of Lodha–Moore [46], and the braided Thompson group F_{br} [8]. In the first two cases, our more general criterion does not help: if G is a piecewise linear or piecewise projective group of the line, then every homogeneous quasimorphism is a homomorphism [10, 22]. In the third case, there exist many homogeneous quasimorphisms that are not homomorphisms [23], but the construction does not give information about Aut-invariance. The interest of our more general criterion is for applications to groups with hyperbolic features: already in free groups, it is easy to see that there exist no Aut-invariant homomorphisms, but Aut-invariant homogeneous quasimorphisms do exist [24].

Such quasimorphisms have been constructed in several cases. The most widely applicable criterion is the following:

Theorem 8.1.5 ([24, Theorem E]) *Let* G *be a group such that* $\mathrm{Inn}(G)$ *is infinite and* $\mathrm{Aut}(G)$ *is acylindrically hyperbolic. Then there exists an infinite-dimensional space of* Aut*-invariant homogeneous quasimorphisms.*

Acylindrical hyperbolicity of automorphism groups has been proved in several cases (in fact, it is still an open question whether it holds for *all* finitely generated acylindrically hyperbolic groups [30, Question 1.1]), and so we obtain the following list of examples of groups with property R_∞:

Corollary 8.1.6 *Let* G *be a finitely generated group satisfying one of the following properties.*

(1) G *is a non-elementary hyperbolic group.*
(2) G *is non-elementary hyperbolic relative to a collection of finitely generated subgroups, none of which is relatively hyperbolic.*
(3) G *has infinitely many ends.*
(4) G *is a graph product of groups over a finite graph that does not decompose non-trivially as a join and is not reduced to a single vertex.*
(5) G *is a graph product of abelian groups and it is not virtually abelian.*

Then G *has property* R_∞.

Proof Groups in the second item have acylindrically hyperbolic automorphism group [32, Theorem 1.3]. Therefore, they admit non-zero Aut-invariant quasimorphisms by Theorem 8.1.5, and hence have property R_∞ by Corollary 8.1.4. The groups in the first and third item are special cases (see also [30] and [32, Theorem 1.1]). Similarly, the groups in the fourth item have acylindrically hyperbolic automorphism group [19, Theorem A.27] (see also [33] and [31]), with the exception of the infinite dihedral group, which is treated separatedly in Theorem 7.1.2. The groups in the fifth item need not have acylindrically hyperbolic automorphism group (because we are allowing joins), but they still admit Aut-invariant homogeneous quasimorphisms [24, Theorem B(4)], so they have property R_∞ by Corollary 8.1.4. ☐

The first item recovers [44]. The second item recovers Theorem 6.1.2 (and removes the hypothesis of finite presentability). The third item recovers Theorem 5.1.18 (and removes the hypothesis of accessibility). The last item recovers the R_∞ property for non-abelian right angled Artin groups [57]. Corollary 8.1.4 has since been applied to prove property R_∞ for a large class of Artin groups, see [43, Corollary F].

Remark The same argument as Theorem 8.1.5 (see [24, Corollary 4.4] and the proof of [24, Theorem 5.1]) shows that, if $\mathrm{Inn}(G)$ is infinite, and $\langle \mathrm{Inn}(G), \varphi\rangle < \mathrm{Aut}(G)$ is acylindrically hyperbolic, then G admits an infinite-dimensional space of φ-invariant homogeneous quasimorphism. By Theorem 8.1.3, this statement is enough to deduce property $R_\infty(\varphi)$. It is possible that, for some groups, proving that $\langle \mathrm{Inn}(G), \varphi\rangle$ is acylindrically hyperbolic for all φ is easier than proving that $\mathrm{Aut}(G)$ itself is acylindrically hyperbolic.

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Declarations

Competing interests The authors declare no competing interests.

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