A Presentation for the Group of Pure Symmetric Outer Automorphisms of a Given Splitting of a Free Product

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Abstract

We give a concise presentation for the group of pure symmetric outer automorphisms of a given splitting of a free product $G_1 * \cdots * G_n$. These are the (outer) automorphisms which preserve the conjugacy classes of the free factors G_i. This is achieved by considering the action of these automorphisms on a particular subcomplex of 'Outer Space', which we show to be simply connected. We then apply a theorem of K. S. Brown to extract our presentation.

Introduction

The study of group presentations, especially finite ones, is a core part of geometric group theory, dating back to the work of M. Dehn in the 1910's. Providing such group presentations is not only necessary for such study, but interesting in and of itself. Automorphism groups of free groups, and more generally, of free products, are natural objects to consider in this area. Different presentations may display various desirable properties, such as having few generators or relations, or highlighting some structure of the group.

In 2008, H. Armstrong, B. Forrest, and K. Vogtmann [2] gave a finite presentation for $\mathrm{Aut}(F_r)$, the automorphism group of a free group of rank r. They achieved this by applying a theorem of Brown [6, Theorem 1] to a subcomplex of a version of M. Culler and K. Vogtmann's 'Auter Space' [9] on which $\mathrm{Aut}(F_r)$ acts 'nicely'.

While finite presentations for $\mathrm{Aut}(F_r)$ were already known (for example, see the works of J. Nielsen [19] from 1924, whose presentation demonstrates the surjectivity of the map to $GL_n(\mathbb{Z})$, and B. Neumann [18] from 1933, whose presentation had only 2 generators, but many relations), Armstrong, Forrest, and Vogtmann [2] gave a presentation whose generators are all involutions, with a relatively small number of relations, making it straightforward to comprehend and apply.

In 1986, J. McCool [16] gave a concise presentation for the subgroup of $\mathrm{Aut}(F_r)$ comprising automorphisms which map each generator to a conjugate of itself. McCool's presentation comprised $r^2 - r$ generators, but only three (families of) relations.

There is a longstanding trend of generalising results from automorphisms of free groups to automorphisms of free products. In the 1940's, D. I. Fouxe-Rabinovitch [10], [11] gave a finite presentation for the automorphism group of a free product $G = G_1 * \cdots * G_n * F_k$, where F_k is the free group of rank k and where each G_i is non-trivial, freely indecomposable, and not infinite cyclic (i.e. $G_i \ncong \mathbb{Z}$). N. D. Gilbert [12, Theorem 2.20] gave an equivalent presentation for $\mathrm{Aut}(G)$ in 1987 with fewer relations, using 'peak reduction' methods of J. H. C. Whitehead [21] adapted to the free product case by D. J. Collins and H. Zieschang [8].

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Main Result

We follow the methods of Armstrong, Forrest, and Vogtmann [2] to give a concise presentation for the group of pure symmetric outer automorphisms of a given splitting $G_1 * \cdots * G_n$ of free product G, denoted $\mathrm{Out}(G; G_1, \ldots, G_n)$. In our case, we have a 'strict fundamental domain' for the action of $\mathrm{Out}(G; G_1, \ldots, G_n)$, so can apply a more straightforward theorem of Brown [6, Theorem 3] to extract our presentation, given below:

Theorem 4.1.1. *Let $G_1 * \cdots * G_n$ be a free splitting of a group G where each G_i is non-trivial and $n \geqslant 5$. For $i \in [n] := \{1, \ldots, n\}$ and $j \in [n] - \{i\}$, let $f_{i_j} : G_i \to G_{i_j}$ be group isomorphisms, and for $g \in G_i$ let $\mathrm{Ad}_{G_i}(g)$ be the inner automorphism $x \mapsto gxg^{-1}$ of G_i. Then the group $\mathrm{Out}(G; G_1, \ldots, G_n)$ is generated by the $n(n-1)$ groups $G_{i_j} \cong G_i$ and $\Phi = \prod_{i=1}^{n} \mathrm{Aut}(G_i)$, subject to relations:*

1. $[f_{i_j}(g), f_{i_k}(h)] = 1 \ \forall g, h \in G_i$, *for all* $i \in [n]$, $j, k \in [n] - \{i\}$
2. $[f_{i_j}(g), f_{k_l}(h)] = 1 \ \forall g \in G_i, h \in G_k$, *for all distinct* $i, j, k, l \in [n]$
3. $[f_{j_k}(g), f_{i_j}(h) f_{i_k}(h)] = 1 \ \forall g \in G_j, h \in G_i$, *for all distinct* $i, j, k \in [n]$
4. $f_{i_{v_1}}(g) \ldots f_{i_{v_{n-1}}}(g) = \mathrm{Ad}_{G_i}(g^{-1}) \ \forall g \in G_i$, *for all* $i \in [n]$ *and* $\{v_1, \ldots, v_{n-1}\} = [n] - \{i\}$
5. $\varphi^{-1} f_{i_j}(g) \varphi = f_{i_j}(g\varphi) \ \forall g \in G_i$, *for all distinct* $i, j \in [n]$ *and all* $\varphi \in \Phi$

As well as all relations in G and Φ.

Note that here we assume $\mathrm{Aut}(G)$ acts on G on the right.

Corollary 4.1.2. *If a group G splits as a free product where the factor groups are non-trivial, freely indecomposable, not infinite cyclic, and pairwise non-isomorphic, then Theorem 4.1.1 gives a presentation for* $\mathrm{Out}(G)$.

Observation 0.0.1. In the case where some of the factor groups may be isomorphic, one may choose to study the symmetric automorphisms of the splitting. Then a finite direct product of symmetric groups, Π, acts on the splitting by permuting all possible isomorphic factors. The group of symmetric outer automorphisms of the splitting is then given by $\mathrm{Out}(G; G_1, \ldots, G_n) \rtimes \Pi$. While this is hard to see geometrically using the methods of this paper, it may be deduced algebraically.

The cases $n = 4$ and $n = 3$ are similar, and are given in Theorems 4.2.2 and 4.3.1 in Section 4.

If each of the groups G_i and $\mathrm{Aut}(G_i)$ is finitely presented, then one may extract a finite presentation for $\mathrm{Out}(G; G_1, \ldots, G_n)$ from this theorem by replacing each group G_{i_j} with a set of elements $\{f_{i_j}(g_1), \ldots, f_{i_j}(g_{m_i})\}$ such that the g_k's generate G_i, and replacing the group Φ with a generating set $\{\varphi_1, \ldots, \varphi_{m_\Phi}\}$. For conciseness, we do not make this more formal.

Our result may be considered to be a generalisation of McCool's presentation for the group of pure symmetric automorphisms of a free group.

Theorem 0.0.2 (McCool [16])**.** *Let $F_r = \langle x_1, \ldots, x_r \rangle$ be the free group on r generators. The group of pure symmetric automorphisms of F_r is generated by $r(r-1)$ elements $(x_i; x_j)$ (for $i, j \in \{1, \ldots, r\}$ and $i \neq j$), subject to commutation relations:*

1. $(x_i; x_j)(x_k; x_j) = (x_k; x_j)(x_i; x_j)$

2. $(x_i; x_j)(x_k; x_l) = (x_k; x_l)(x_i; x_j)$

3. $(x_i; x_j)(x_k; x_j)(x_i; x_k) = (x_i; x_k)(x_i; x_j)(x_k; x_j)$

(where i, j, k, l are assumed to be distinct).

Observe by comparing indices that these relations directly translate to our Relations 1–3. Our Relation 4 is an 'outer' relation so is not present in the automorphism group, and our Relation 5 describes automorphisms within a given factor, which are trivial in McCool's case.

In the case $n = 3$, we recover a special case of Gilbert's result [12, Theorem 2.20], given by D. J. Collins and N. D. Gilbert [7] in 1990, for three freely indecomposable, non-trivial, not infinite cyclic, pairwise non-isomorphic factors:

Theorem 0.0.3 (Collins–Gilbert [7, Proposition 4.1])**.** *For $G = X * Y * Z$ where each of X, Y, Z is freely indecomposable, non-trivial, not infinite cyclic, and where none of X, Y, Z are isomorphic to each other, we have that* $\mathrm{Out}(G)$ *is generated by*

$$\{(Y, x), (Z, y), (X, z), \varphi | x \in X, y \in Y, z \in Z, \varphi \in \Phi\}$$

where Φ is the set of factor automorphisms (see Definition 1.1.5), subject to relations:

- $(Y, x_1)(Y, x_2) = (Y, x_1 x_2)$
- $(Z, y_1)(Z, y_2) = (Z, y_1 y_2)$
- $(X, z_1)(X, z_2) = (X, z_1 z_2)$
- $\varphi^{-1}(Y, x)\varphi = (Y, x\varphi)$
- $\varphi^{-1}(Z, y)\varphi = (Z, y\varphi)$
- $\varphi^{-1}(X, z)\varphi = (X, z\varphi)$
- *All relations from* Φ

In particular, $\mathrm{Out}(G) \cong G \rtimes \Phi$.

Collins and Gilbert's result may be thought of as a presentation for the pure symmetric outer automorphisms preserving a free splitting structure $G_1 * G_2 * G_3$ where the only condition on the G_i's is that they are non-trivial. Thus our result may also be seen as both a special case of Gilbert's presentation [12, Theorem 2.20] and a generalisation of Collins and Gilbert's presentation [7, Proposition 4.1].

In future, we hope to generalise this further to free splittings of the form $G_1 * \cdots * G_n * F_k$ where automorphisms need not preserve the conjugacy classes of the generators for F_k. However this greatly increases the number of cells in our chosen subcomplex of Outer Space. Moreover the fundamental domain of the action ceases to be strict, meaning we can no longer apply the simplified version of Brown's theorem. These complications increase the complexity of the problem, though we hope that the end result will still be a pleasing presentation.

Methods and Techniques

To achieve our presentation, we choose a particular subcomplex of the 'Outer Space' for a free product introduced by Guirardel and Levitt in [13]. In order to study the symmetric automorphisms, we use the version where there is no free rank, so the Outer Space is similar to the poset complex introduced by D. McCullough and A. Miller in [17]. We work with the definition of the space provided by Guirardel and Levitt [13], since this interpolates between the Outer Spaces of Culler and Vogtmann [9] and of McCullough and Miller [17], which lends itself well to future work in the case of a splitting $G_1 * \cdots * G_n * F_k$.

We call our chosen complex $\mathcal{C}_n$, discussed in Section 2. In the cases $n = 3$ and $n = 4$, $\mathcal{C}_n$ is precisely the barycentric spine of Guirardel and Levitt's Outer Space for a free product whose Grushko decomposition has four non-isomorphic free factors and no free rank (see Section 1.3). Definition 2.2.3 details the construction of the complex $\mathcal{C}_n$ for $n \geqslant 5$.

In order to apply Brown's theorem [6, Theorem 3], we require that $\mathrm{Out}_{\mathfrak{S}}(G)$ acts cellularly on our complex $\mathcal{C}_n$ and with a strict fundamental domain, and that the complex $\mathcal{C}_n$ is both connected and simply connected. We will also need suitable presentations for $\mathrm{Out}_{\mathfrak{S}}(G)$-stabilisers of vertices (graphs of groups) in $\mathcal{C}_n$.

The action of $\mathrm{Out}_{\mathfrak{S}}(G)$ on $\mathcal{C}_n$ and its fundamental domain are studied in Section 2.3, and the connectedness of the complex $\mathcal{C}_n$ is Corollary 3.1.4 of Section 3.1.

Vertex stabilisers are studied in Section 2.4, where we combine techniques of Guirardel and Levitt [13] with those of H. Bass and R. Jiang [4] to procure presentations which are both concise and precise (Propositions 2.4.1, 2.4.2, and 2.4.3).

Showing that the complex $\mathcal{C}_n$ is simply connected is highly non-trivial and is delayed until the second half of the paper, comprising Sections 5 and 6. We give a brief overview of the idea of the proof below.

In 1928, P. Alexandroff [1] introduced the notion of a 'nerve complex' associated to a cover of a space. In ideal conditions, this shares many of the same topological properties as the original space, while often being a much simpler object to understand.

We apply a similar concept in Section 5, introducing the 'Space of Domains' (see Definition 5.1.4) as a way of recording intersection patterns of $\mathrm{Out}_{\mathfrak{S}}(G)$-images of the fundamental domain in $\mathcal{C}_n$. Unlike Alexandroff's nerve complex, we are only interested in 2-way and 3-way intersections.

We show in Proposition 5.3.5 that in order to prove simple connectivity of the complex $\mathcal{C}_n$, it suffices to show that the Space of Domains is simply connected (having already shown that our fundamental domain of the $\mathrm{Out}_{\mathfrak{S}}(G)$-action on $\mathcal{C}_n$ is simply connected in Theorem 3.2.11 of Section 3.2).

Finally in Section 6 we apply 'peak reduction' techniques as used by Collins and Zieschang [8] and Gilbert [12] to deduce that the Space of Domains is simply connected (Theorem 6.3.2).

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Contents

1 Preliminaries

1.1 Some Useful Definitions and Notation

We adapt the notation for automorphisms used by Gilbert [12, Section 1]. Throughout, we consider a group G which splits as a free product $G_1 * \cdots * G_n$, where each G_i is non-trivial and $n \geqslant 3$. We refer to each G_i as a factor group. Note that throughout, we adopt the convention for conjugation that $g^x = xgx^{-1}$. Recall then that $g^{xy} = (xy)g(xy)^{-1} = x(ygy^{-1})x^{-1} = (g^y)^x$.

Notation 1.1.1. Let G be a group. We denote by $\mathrm{Aut}(G)$ the group of automorphisms of G, that is, isomorphisms from G to itself. We say $\psi \in \mathrm{Aut}(G)$ is an inner automorphism if there exists $x \in G$ so that for all $g \in G$, $\psi(g) = g^x = xgx^{-1}$. The collection of inner automorphisms forms a normal subgroup, $\mathrm{Inn}(G)$, of $\mathrm{Aut}(G)$. We then define $\mathrm{Out}(G) := {}^{\mathrm{Aut}(G)}\!/_{\mathrm{Inn}(G)}$, and call this the outer automorphism group of G.

Definition 1.1.2 (Pure Symmetric Automorphism)**.** Let $G = G_1 * \cdots * G_n$ be a group which splits as a free product. We say $\psi \in \mathrm{Aut}(G)$ is a pure symmetric automorphism of the splitting $G_1 * \cdots * G_n$ if for each i there is some $g_i \in G$ such that $\psi(G_i) = G_i^{g_i} = g_i G_i g_i^{-1}$. We say $\hat{\psi} \in \mathrm{Out}(G)$ is a pure symmetric outer automorphim of the splitting if there is some $\psi \in \hat{\psi}$ which is a pure symmetric automorphism of the splitting.

Remark. It is easy to see that if ψ is a pure symmetric automorphism of some free splitting, and ι is an inner automorphism of the free product, then $\iota\psi$ is also a pure symmetric automorphism of the splitting. Thus the concept of 'pure symmetric outer automorphisms' is well-defined. It is not hard to verify that the collection of pure symmetric (outer) automorphisms forms a subgroup of $\mathrm{Aut}(G)$ (respectively, $\mathrm{Out}(G)$).

Notation 1.1.3. We denote by $\mathrm{Out}(G; G_1 * \cdots * G_n)$ the subgroup of $\mathrm{Out}(G)$ comprising pure symmetric outer automorphisms of the splitting $G_1 * \cdots * G_n$ of G. Given such a splitting, we may set $\mathfrak{S}$ to be the tuple $(G_1, \ldots, G_n)$ and let $\mathrm{Out}(G; G_1, \ldots, G_n) =: \mathrm{Out}_{\mathfrak{S}}(G)$, for brevity. We may similarly define $\mathrm{Aut}(G; G_1, \ldots, G_n)$ and $\mathrm{Aut}_{\mathfrak{S}}(G)$. We will sometimes refer to $\mathfrak{S}$ itself as the splitting, as opposed to the product $G_1 * \cdots * G_n$.

Observation 1.1.4. Given a splitting $G = G_1 * \cdots * G_n$ with $\mathfrak{S} = (G_1, \ldots, G_n)$, it is clear that $\mathrm{Inn}(G) \subseteq \mathrm{Aut}_{\mathfrak{S}}(G)$. Since $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$ then $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}_{\mathfrak{S}}(G)$, and it follows that ${}^{\mathrm{Aut}_{\mathfrak{S}}(G)}\!/_{\mathrm{Inn}(G)} \cong \mathrm{Out}_{\mathfrak{S}}(G)$, as one would expect.

Definition 1.1.5 (Factor Automorphism)**.** We say $\varphi \in \mathrm{Aut}(G; G_1, \ldots, G_n)$ is a factor automorphism if for each $i \in \{1, \ldots, n\}$, $\varphi|_{G_i}$ (that is, φ with domain restricted to the embedding of G_i in G) is an automorphism of G_i (i.e. $\varphi|_{G_i} \in \mathrm{Aut}(G_i)$). We will say $\hat{\varphi} \in \mathrm{Out}(G; G_1, \ldots, G_n)$ is a factor automorphism if $\hat{\varphi}$ has a representative $\varphi \in \mathrm{Aut}(G; G_1, \ldots, G_n)$ which is a factor automorphism. We will denote the set of factor automorphisms in $\mathrm{Out}(G; G_1, \ldots, G_n)$ by Φ.

The set of factor automorphisms Φ forms a subgroup of $\mathrm{Out}(G; G_1, \ldots, G_n)$, with $\Phi \cong \prod_{i=1}^{n} \mathrm{Aut}(G_i)$.

Notation 1.1.6. We write $\mathrm{Ad}_{G_i}(g)$ for the inner automorphism of G_i which conjugates each element of G_i by g (with $g \in G_i$), that is, $\mathrm{Ad}_{G_i}(g) : x \mapsto gxg^{-1}$. Since $\mathrm{Ad}_{G_i}(g) \in \mathrm{Inn}(G_i) \leqslant \mathrm{Aut}(G_i)$, then $\mathrm{Ad}_{G_i}(g) \in \Phi \leqslant \mathrm{Out}(G; G_1, \ldots, G_n)$. Note however that $\mathrm{Ad}_{G_i}(g)$ is **not** in $\mathrm{Inn}(G)$.

We will often abuse notation by writing ψ for both an automorphism in $\mathrm{Aut}(G)$ (or $\mathrm{Aut}(G; G_1, \ldots, G_n)$), and for the class it represents in $\mathrm{Out}(G)$ (or $\mathrm{Out}(G; G_1, \ldots, G_n)$).

Definition 1.1.7. Let $\mathfrak{G} = (G_1, \ldots, G_n)$ be the tuple associated to a group G which splits as a free product $G_1 * \cdots * G_n$, and let T be a finite tree on at least n vertices.
A free product $H_1 * \cdots * H_n$ is an $\mathfrak{G}$ free factor splitting for $G = G_1 * \cdots * G_n$ if for each i, there exists $g_i \in G$ so that $H_i = G_i^{g_i}$, and the subgroups $G_1^{g_1}, \ldots, G_n^{g_n}$ generate the group G. Note that by assumption $H_1 * \cdots * H_n \leqslant G$.
An $\mathfrak{G}$-labelling of T is an assignment of n vertex groups H_v to vertices $v \in V(T)$ so that $H_1 * \cdots * H_n$ is an $\mathfrak{G}$ free factor splitting for G.
Given an $\mathfrak{G}$-labelling $(H_1, \ldots, H_n)$ of T, we may consider the graph of groups $\mathbf{T} = (T, (H_1, \ldots, H_n))$ formed by associating the trivial group $\{1\}$ to any remaining vertices of T, and setting all edge groups to also be trivial.

Lemma 1.1.8. *Let G be a group with splitting $\mathfrak{G} = (G_1, \ldots, G_n)$ and let $H_1 * \cdots * H_n$ be an $\mathfrak{G}$ free factor splitting for G. Then there exists $\psi \in \mathrm{Aut}_{\mathfrak{G}}(G)$ with $(G_i)\psi = H_i$ for each i.*

Proof. Since $H_1 * \cdots * H_n$ is an $\mathfrak{G}$ free factor splitting for G, for each i there exists $g_i \in G$ so that $H_i = G_i^{g_i}$. Let $\psi_i : G_i \to H_i$ be the map $(x)\psi_i = g_i x g_i^{-1}$ $\forall x \in G_i$. Clearly, ψ_i is an isomorphism of (sub)groups. By the universal property of free products, these isomorphisms ψ_i extend to an endomorphism $\psi : G \to G$. Since $H_1 * \cdots * H_n$ is an $\mathfrak{G}$ free factor splitting of $G_1 * \cdots * G_n$, then G is generated by the subgroups $H_1, \ldots, H_n$, and so ψ is surjective. Repeating this process on the maps $\psi_i^{-1} : H_i \to G_i$, we recover a surjective homomorphism $\varphi : G \to G$, which composes with ψ to give the identity map. Thus φ is an inverse for ψ, and so $\psi \in \mathrm{Aut}(G)$. Moreover, ψ restricts to ψ_i on each G_i, that is, $(G_i)\psi = G_i^{g_i} = H_i$, and so $\psi \in \mathrm{Aut}_{\mathfrak{G}}(G)$, as required. ☐

Definition 1.1.9 (Whitehead Automorphism)**.** An automorphism in $\mathrm{Aut}(G; G_1, \ldots, G_n)$ which, for each j, either pointwise fixes G_j, or pointwise conjugates G_j by a given $x \in G$ is called a Whitehead automorphism. Given $x \in G_i$ and $A \subseteq \{G_1, \ldots, G_n\} - \{G_i\}$, we write (A, x) for the Whitehead automorphism which pointwise fixes any $G_j \notin A$, and pointwise conjugates by x any $G_j \in A$.

Given finite sequences $\mathbf{x} = (x_1, \ldots, x_k)$ with $x_1, \ldots, x_k \in G_i$ and $\mathbf{A} = (A_1, \ldots, A_k)$ with $A_1, \ldots, A_k \subseteq \{G_1, \ldots, G_n\} - \{G_i\}$ (with the A_j's pairwise disjoint), we write $(\mathbf{A}, \mathbf{x})$

for the composition $(A_1, x_1)\ldots(A_k, x_k)$ (which should be read from left to right, since we consider the action of $\mathrm{Aut}(G)$ on G to be a right action). We call such a map a multiple Whitehead automorphism.

An element $\hat{\psi} \in \mathrm{Out}(G; G_1, \ldots, G_n)$ will be called a (multiple) Whitehead automorphism if it has some representative $\psi \in \mathrm{Aut}(G; G_1, \ldots, G_n)$ which is a (multiple) Whitehead automorphism.

Remark. Our notation differs from that of Gilbert [12] in that we decide not to include the operating factor G_i (see below) in the set A.

More detail on Whitehead automorphisms, including relative Whitehead automorphisms, can be found in Section 5.5.

Notation 1.1.10. Given factor groups G_i and G_j, we will write G_{i_j} (sometimes abbreviated as i_j) for the group generated by automorphisms (G_j, x) where $x \in G_i$. We call G_i the operating factor and G_j the dependant factor. Additionally, given factor groups G_i and $G_{v_1}, \ldots, G_{v_k}$, we will write $i_{v_1 \ldots v_k}$ (or $G_{i_{v_1 \ldots v_k}}$) for the subgroup of $i_{v_1} \times \cdots \times i_{v_k}$ generated by the Whitehead automorphisms $(\{G_{v_1}, \ldots, G_{v_k}\}, x)$ where $x \in G_i$. We think of this as the diagonal subgroup, and denote this by $i_{v_1 \ldots v_k} \mathrel{\underline{\Diamond}} i_{v_1} \times \cdots \times i_{v_k}$.

Observation 1.1.11. We have a natural isomorphism $f_{i_j} : G_i \to G_{i_j}$ given by $f_{i_j}(x) = (G_j, x)$. Indeed, $G_j \cdot f_{i_j}(x) f_{i_j}(y) = G_j \cdot (G_j, x)(G_j, y) = x G_j x^{-1} \cdot (G_j, y) = xy G_j y^{-1} x^{-1} = G_j^{xy} = G_j \cdot f_{i_j}(xy)$.

1.2 Key Theorems

We will later make repeated use of the Seifert–van Kampen Theorem. As our simplicial complexes are all closed, and we usually only care about closed subcomplexes of these, we will use a 'closed version' of the theorem. Such a theorem can be found in some undergraduate Algebraic Topology notes, such as [22] delivered by H. Wilton at the University of Cambridge.

Theorem 1.2.1 (Seifert–Van Kampen (Closed Version))**.** *For closed sets A and B with A, B, and $A \cap B$ path-connected and such that there exist open sets $U \subset A$ and $V \subset B$ with $A \cap B$ a (strong) deformation retract of both U and V, we have that the diagram:*

$$\begin{array}{ccc} \pi_1(A \cap B) & \xrightarrow{i_{A*}} & \pi_1(A) \\ \big\downarrow{\scriptstyle i_{B*}} & & \big\downarrow{\scriptstyle j_{A*}} \\ \pi_1(B) & \xrightarrow{j_{B*}} & \pi_1(A \cup B) \end{array}$$

is a pushout, where $i_A : A \cap B \hookrightarrow A$, $i_B : A \cap B \hookrightarrow B$, $j_A : A \hookrightarrow A \cup B$, and $j_B : B \hookrightarrow A \cup B$ are inclusion maps. We will abuse notation and abbreviate this by writing:

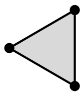
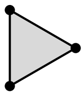
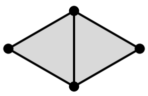
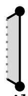
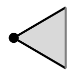
$$\pi_1(A \cup B) \cong \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$$

Remark. This closed version follows by noting that the diagram:

$$\begin{array}{ccc} \pi_1(U \cup V) & \longrightarrow & \pi_1(A \cup V) \\ \big\downarrow & & \big\downarrow \\ \pi_1(U \cup B) & \longrightarrow & \pi_1(A \cup B) \end{array}$$

is a pushout by the standard Seifert–van Kampen Theorem, where the corresponding components in the closed version are neighbourhood deformation retracts of those here, and hence have the same fundamental group.

Since our sets A, B, $A \cup B$, and $A \cap B$ will always be (finite) simplicial complexes, we will always have that $A \cap B$ is a neighbourhood deformation retract in both A and B. Indeed, we can take a union of open subsets of each simplex of A containing $A \cap B$, and similarly for B, and we will have open sets U and V satisfying this requirement. We illustrate this with an example:

Example 1. Let $A =$  and $B =$  be two (closed) simplices, and $X = A \cup B =$  a simplicial complex. Then in X we have that $A \cap B =$ . We can then take our sets $U \subseteq A$ and $V \subseteq B$ to $U =$  and $V =$ . Then $A - U =$  which is a closed set, hence U is open in A. Similarly, V is open in B, and it is clear that $A \cap B$ is a deformation retract of both U and V.

In [6], Brown presents a method for extracting a group presentation from its action on a CW complex. A streamlined version of this is given as Theorem 3 in [6] which holds when the action of the group on the complex has a strict fundamental domain:

Theorem 1.2.2 (Brown [6, Theorem 3]). *Let $\mathcal{G}$ act on a simply connected $\mathcal{G}$-CW complex X (without inversion on the 1-cells of X). Suppose there is a subcomplex W of X so that every cell of X is equivalent under the action of $\mathcal{G}$ to a unique cell of W. Then $\mathcal{G}$ is generated by the isotropy subgroups $\mathcal{G}_v$ ($v \in V(W)$) subject to edge relations $\iota_{o(e)}(g) = \iota_{t(e)}(g)$ for all $g \in \mathcal{G}_e$ ($e \in E(W)$) (where for any $e \in E(W)$, $\iota_{o(e)} : \mathcal{G}_e \to \mathcal{G}_{o(e)}$ and $\iota_{t(e)} : \mathcal{G}_e \to \mathcal{G}_{t(e)}$ are inclusions).*

It is this theorem that forms the basis of Section 4 in which we give a presentation for $\mathrm{Out}_{\mathfrak{G}}(G)$.

1.3 Outer Space for Free Products

In [13], Guirardel and Levitt give a description of a deformation space for certain free products $G = G_1 * \cdots * G_n * F_k$ on which $\mathrm{Out}(G)$ acts, allowing us to study properties of the outer automorphism group of a free product. They call this space $\mathcal{O}$, the 'Outer Space' (for a free product), and the projectivised space $\mathcal{PO}$.

The space $\mathcal{PO}$, while cellular, is not simplicial, due to 'missing' faces (faces 'at infinity'). To resolve this, we consider a construction called the barycentric spine of $\mathcal{PO}$ (denoted '$\mathcal{S}$'). This is obtained by taking the first barycentric subdivision of $\mathcal{PO}$, and then linearly retracting off the missing faces, to give a simplicial complex. This equates to taking the geometric realisation of the poset on the cells of $\mathcal{PO}$ given by $A < B$ if and only if A is a face of B.

Whilst their construction is defined for a Grushko decomposition (i.e. each factor group G_i is non-trivial, freely indecomposable, and not infinite cyclic), by considering instead the subgroup $\mathrm{Out}(G; G_1, \ldots, G_n, F_k)$ of $\mathrm{Out}(G)$ which preserves a given splitting of G, we can loosen these conditions. We may refer to this as a 'relative' Outer Space.

Since we are going to be interested in subcomplexes of the barycentric spine $\mathcal{S}$, we will now give an explicit description for it. We will restrict ourselves to the case where every factor group in the splitting of G acts elliptically (i.e. $k = 0$).

Points in the Barycentric Spine of Projectivised Outer Space

Let G be a group which splits as a free product $G_1 * \cdots * G_n$ where each G_i is non-trivial, and let $\mathfrak{G} = (G_1, \ldots, G_n)$ be the tuple associated to the splitting.

The barycentric spine $\mathcal{S}$ of $\mathcal{PO}$ is a simplicial complex whose 0-cells are graphs of groups Γ (with $\pi_1(\Gamma) \cong G$), as follows:

- The underlying graph structure of Γ is a tree
- Γ has one vertex with vertex group conjugate to G_i for each i
- All other vertex groups are trivial (vertices with trivial vertex group will be called 'trivial vertices')
- Any trivial vertex has valency at least 3
- All edge groups are trivial
- The vertex groups $G_1^{g_1}, \ldots, G_n^{g_n}$ generate the group G (that is, $G_1^{g_1} * \cdots * G_n^{g_n}$ is a free fractor splitting for G)

Note that two graphs of groups are equivalent if and only if they are isomorphic in the sense of Bass [3, Definition 2.1].

Via Bass–Serre Theory, we could equally consider points of $\mathcal{S}$ to be certain actions of G on trees T, up to equivariant isometry.

Structure of the Barycentric Spine $\mathcal{S}$ of $\mathcal{PO}$

Given two 0-cells Γ_1 and Γ_2 in our barycentric spine, we have a 1-cell $[\Gamma_1, \Gamma_2]$ whenever Γ_2 can be achieved by collapsing an edge or edges of Γ_1 (or vice versa).

Whenever a collection of 0-cells $\Gamma_1, \ldots, \Gamma_m$ form an m-clique in the 1-skeleton (that is, whenever the restriction of the 1-skeleton to the vertices $\Gamma_1, \ldots, \Gamma_m$ forms a complete graph), we have a unique $(m-1)$-cell $[\Gamma_1, \ldots, \Gamma_m]$.

Since the maximum number of edges such a graph of groups can have is $2n-3$ (when all non-trivial vertices have valency 1 and all trivial vertices have valency 3), and the minimum number of edges is $n-1$ (when there are no trivial vertices), then the dimension of the barycentric spine of projectivised Outer Space is $(2n-3)-(n-1) = n-2$. Since $\mathcal{PO}$ is contractible [13, Theorem 4.2 and Corollary 4.4], and $\mathcal{PO}$ deformation retracts onto $\mathcal{S}$, then so too is $\mathcal{S}$.

Action of $\mathrm{Out}_{\mathfrak{G}}(G)$ on $\mathcal{S}$

If we consider points of $\mathcal{S}$ to be actions $\rho : G \to \mathrm{Isom}(T)$, $\rho(g) = \rho_g : T \to T$, $\rho_g(x) = g \cdot_T x$ on G-trees T, then for $\varphi \in \mathrm{Aut}_{\mathfrak{G}}(G)$, the action on $\mathcal{S}$ is given by $(T, \rho) \cdot \varphi = (T, \varphi^{-1}\rho)$, that is, $g \cdot_{T\varphi} x = g\varphi^{-1} \cdot_T x$. This extends to a cellular action on $\mathcal{S}$. Since inner automorphisms act trivially on $\mathcal{S}$ then this defines an action of $\mathrm{Out}_{\mathfrak{G}}(G)$, where for any cell C of $\mathcal{S}$ and $\hat{\varphi} \in \mathrm{Out}_{\mathfrak{G}}(G)$, $C \cdot \hat{\varphi} = C \cdot \varphi$ for any automorphism $\varphi \in \hat{\varphi}$. In Section 2.3, we give a description of the action of $\mathrm{Out}_{\mathfrak{G}}(G)$ on our chosen subcomplex of $\mathcal{S}$ in terms of graphs of groups. Note that we have chosen notation so that G always acts on the left and $\mathrm{Aut}(G)$ always acts on the right.

We will be interested in finding $\mathrm{Out}_{\mathfrak{G}}(G)$-stabilisers of vertices in the barycentric spine. Considering points as actions of G on trees T, the stabiliser of a point T is precisely the group of automorphisms acting trivially on the quotient graph of groups $\Gamma = {}^{T}\!\diagup_{G}$. This is the subgroup denoted by Guirdardel and Levitt as $\mathrm{Out}_0^S(G)$.

If the vertex v_i of $\Gamma = {}^{T}\!\diagup_{G}$ represents the orbit of the vertex in T whose stabiliser is G_i, and μ_i is the valency of v_i in Γ, then $\mathrm{Out}_0^S(G)$ is isomorphic to the direct product $\prod_{i=1}^{n}(G_i^{\mu_i-1} \rtimes \mathrm{Aut}(G_i))$ (where $\mathrm{Aut}(G_i)$ is identified with its projection in $\mathrm{Aut}_{\mathfrak{G}}(G)$ (or $\mathrm{Out}_{\mathfrak{G}}(G)$)). The precise details of this are found in [13, Section 5]. We explore this more explicitly in Section 2.4.

2 The Complex $\mathcal{C}_n$

From now on, we fix a splitting $\mathfrak{G} = (G_1, \dots, G_n)$ of a group $G = G_1 * \cdots * G_n$, where each G_i is non-trivial. We will consider graphs of groups of G which respect the splitting $\mathfrak{G}$ — note that these will all be trees, as each factor group acts elliptically in the relative Bass–Serre tree.

The barycentric spine of the projectivised relative Outer Space for G with respect to $\mathfrak{G}$ has a 'reasonably sized' quotient under the action of $\mathrm{Out}_{\mathfrak{G}}(G)$ when $n = 3$ and $n = 4$ (4 vertices contributing to a total of 7 cells, and 32 vertices contributing to a total of 159 cells, respectively). As n grows, this quotient space quickly becomes unwieldy.

Definition 2.0.1. For $n = 3$ or $n = 4$, we define $\mathcal{C}_n$ to be the barycentric spine of Guirardel and Levitt's projectivised relative Outer Space associated to the splitting $\mathfrak{G}$ of G.

That is, $\mathcal{C}_3$ and $\mathcal{C}_4$ are the geometric realisations of the posets whose elements are simplices in the projectivised relative Outer Spaces for the splittings $G_1 * G_2 * G_3$ and $G_1 * G_2 * G_3 * G_4$, respectively, where $A \prec B$ if the simplex A is a face of the simplex B.

Lemma 2.0.2. *$\mathcal{C}_3$ and $\mathcal{C}_4$ are contractible. In particular, they are simply connected.*

Proof. This follows from contractibility of projectivised Outer Space, proven by Guirardel and Levitt [13, Theorem 4.2 and Corollary 4.4] , since projectivised Outer Space deformation retracts onto its spine. □

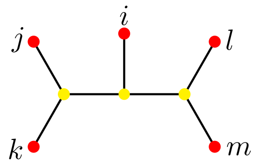
Our goal now is to construct a simplicial complex $\mathcal{C}_n$ for each $n \geqslant 5$ whose quotient under the action of $\mathrm{Out}_{\mathfrak{G}}(G)$ remains 'reasonably sized'.

2.1 Restricting to a Subcomplex of Outer Space

For the rest of this section, we assume $n \geqslant 5$. In general, the barycentric spine of the Outer Space for n factors will be $(n-2)$-dimensional.

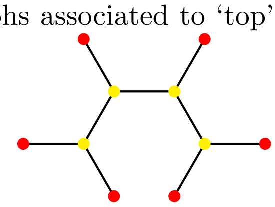
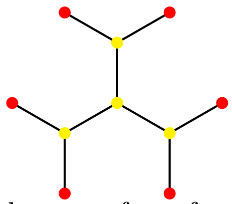
Since our graphs of groups are all trees, we will find that the stabilisers of higher-dimensional simplices in Outer Space are contained in the stabilisers of their faces. Hence restricting ourselves to lower dimensional simplices will not sacrifice information gathered from vertex stabilisers in the barycentric spine. We will thus restrict ourselves to the three lowest possible dimensions of simplex; then when we take the barycentyric spine of this restricted space, we will recover a 2-dimensional complex.

The lowest dimension of a simplex in Outer Space for n factors is $n-1$ (since our trees will have the minimal possible number of vertices, n, leading to $n-1$ edges). Thus we are interested in graphs of groups with $n-1$, n, or $n+1$ edges.

For $n = 5$, this means we just drop the top-dimensional simplices, which represent the graphs of groups of the form 

, where the labelled (red) vertices have vertex groups conjugate to the free factors of G, and the unlabelled (yellow) vertices have trivial vertex group (and all edge groups are trivial).

We will call such a graph (i.e. associated to a top-dimensional simplex) a maximal graph. Note that in our case, these are characterised by having precisely n leaves (all with non-trivial vertex group) and with all other vertices (each having trivial vertex group) having valency exactly 3.

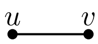
As n increases, so too does the number of maximal graphs associated to 'top' simplices. For $n = 6$, there are two maximal graph structures,  and . For $n = 7$ there are also two types of maximal graph, for $n = 8$ there are four, for $n = 9$ there are six, for $n = 10$ there are twelve, and for $n = 11$ there are eighteen. [1] Collapsing edges (passing to faces in the associated simplex in Outer Space) in each of these leads to a variety of structures.

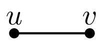
Definition 2.1.1. In a graph of groups, we say that an edge is collapsible if it has at least one trivial endpoint (that is, at least one endpoint whose vertex group is the trivial group).

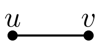
The process of replacing a collapsible edge (including its endpoints) by a single vertex whose vertex group is the free product of the vertex groups of the endpoints of said edge is called collapsing.

Given two graphs of groups T_1 and T_2, we will say T_2 is a collapse of T_1 if T_2 can be achieved as the result of successively collapsing edges of T_1.

Remark. Since a collapsible edge has at least one trivial endpoint, then one may think of the edge as collapsing to its other (potentially non-trivial) vertex.

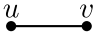
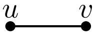
That is, if 

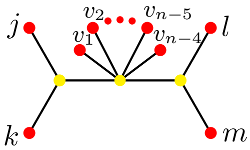
is a collapsible edge, with u being the trivial vertex and v having vertex group G_v (possibly also trivial), then in collapsing 

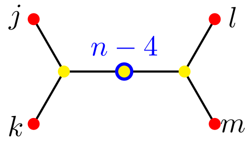
, we replace it with a vertex whose vertex group is equal to $\{1\} * G_v = G_v$. Thus we may think of collapsing 

as replacing it with the vertex v. Note that the new valency of v is equal to the old valency of v plus the valency of u minus 2.

[1]This is somewhat analogous to alkane chains in organic chemistry, and the various isomers for these (if we were to pretend that carbon could make only three bonds, and not four).

Alternatively, the collapse of such an edge  in T_1 may be thought of as a map $f : T_1 \to T_2$ sending  to  and acting as the identity on the rest of T_1. Collapses of multiple edges can be achieved by composing these maps.

Recall that we have already decided to limit ourselves to graphs with $n-1$, n, or $n+1$ edges. So we will restrict ourselves further to collapses of graphs of groups of the form 

, which we will 'abbreviate' as 

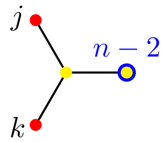
(where the '$n-4$' means we have suppressed $n-4$ leaves). We will use this method of abbreviation on a frequent basis. We will often refer to the blue-ringed vertex (with valency dependent on n) as the 'basepoint' of the graph.

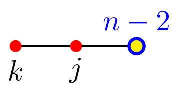
For $n = 5$ this is exactly as we have described, and results in taking the barycentric spine of the 3-skeleton of Outer Space. This graph shape provides a natural way to generalise to $n > 5$, without having to worry about the varying maximal graphs. Note that this means that for $n > 5$ there will be graphs of groups representing simplices of the 'correct' dimension (i.e. $n-1$, n, or $n+1$) in Outer Space which we do not include in our chosen complex.

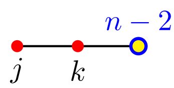
Our complex $\mathcal{C}_n$ will be the geometric realisation of the poset whose elements are the graphs of groups we have selected above, where the order is given by collapsing. We formalise this in the following subsection.

2.2 Points in the Complex

Table 1 summarises the graph shapes we will encounter, as well as a naming convention, the number we expect to see in a fundamental domain of the subcomplex we choose, and associated colours which are useful in drawing diagrams (though can largely be ignored).

In general, subscripts separated by a comma are ordered, whereas subscripts not separated by a comma are not ordered. So ρ_{jk} and ρ_{kj} both refer to the tree 

whereas $\beta_{j,k}$ and $\beta_{k,j}$ refer to distinct trees, 

and 

, respectively.

There is some additional symmetry from our trees, so we also have that $\sigma_{i,jk,lm} = \sigma_{i,lm,jk}$, $\varepsilon_{j,k,l,m} = \varepsilon_{l,m,j,k}$, and $C_{i,j,k,l,m} = C_{i,l,m,j,k}$. It is always assumed that, for example, $\{i, j, k, l, v_1, \ldots, v_{n-4}\} = \{1, \ldots, n\}$ as sets.

Recall from Definition 1.1.7 that an $\mathfrak{S}$-labelling is an assignement of vertex groups $H_1, \ldots, H_n$ to a tree T so that $\pi_1(\mathbf{T}) \cong G$ which respects the splitting $\mathfrak{S}$ of G. For trees in Table 1, vertex groups are only assigned to named (red) vertices.

Definition 2.2.1. Let $H = (H_{v_1}, \ldots, H_{v_n})$ and $H' = (H'_{v_1}, \ldots, H'_{v_n})$ be two $\mathfrak{S}$-labellings of a tree T from Table 1 with $\{v_1, \ldots, v_n\} \subseteq V(T)$. Then H and H' are equivalent as labellings (with respect to T) if for each $v \in V(T)$ (including trivial vertices) there exists $g_v \in G$ and (if v is not a trivial vertex) $\varphi_v \in \mathrm{Aut}(H_v)$ so that $H'_v = g_v(H_v)\varphi_v g_v^{-1}$, and moreover, for any edge $e \in E(T)$ we have $g_{o(e)}^{-1} g_{t(e)} \in H_{o(e)}$ (where $o(e)$ is the endpoint

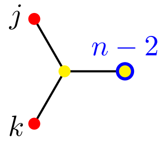
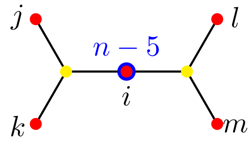
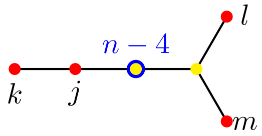
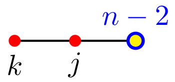
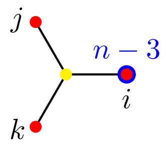
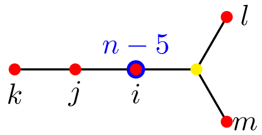
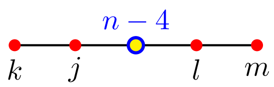
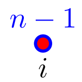
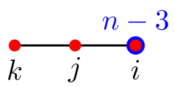
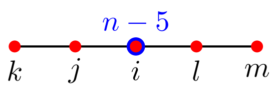
Tree	Name	No. per Domain	Colour
	ρ_{jk}	$\frac{n(n-1)}{2}$	
	$\sigma_{i,jk,lm}$	$\frac{n!}{8\times(n-5)!}$	
	$\tau_{j,k,lm}$	$\frac{n!}{2\times(n-4)!}$	
	α	1	
	$\beta_{j,k}$	$n(n-1)$	
	$\gamma_{i,jk}$	$\frac{n!}{2\times(n-3)!}$	
	$\delta_{i,j,k,lm}$	$\frac{n!}{2\times(n-5)!}$	
	$\varepsilon_{j,k,l,m}$	$\frac{n!}{2\times(n-4)!}$	
	A_i	n	
	$B_{i,j,k}$	$\frac{n!}{(n-3)!}$	
	$C_{i,j,k,l,m}$	$\frac{n!}{2\times(n-5)!}$	

Table 1: Points in the Subcomplex for $n \geqslant 5$

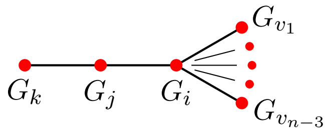
of e closest in T to the 'basepoint', and $t(e)$ is the further endpoint). If $o(e)$ does not have a vertex group assigned (i.e. $o(e)$ is a trivial vertex) then $g_{t(e)} = g_{o(e)}$.

Two graphs of groups $\mathbf{T_1}$ and $\mathbf{T_2}$ are equivalent if they each have underlying graph isomorphic to some graph T, and their labellings are equivalent (with respect to T). We denote this equivalence by $\mathbf{T_1} \simeq \mathbf{T_2}$.

Considering the fundamental group of a labelled tree $\mathbf{T} = (T, H)$ to be $\mathop{*}\limits_{i=1}^{n} H_{v_i}$, this equivalence induces an isomorphism $H_{v_1} * \cdots * H_{v_n} \to H'_{v_1} * \cdots * H'_{v_n}$. Some basic manipulation of notation shows that this notion of equivalence corresponds to taking isomorphism classes of graphs of groups described by Bass [3, Section 2].

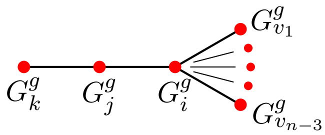
When considering $\mathcal{C}_n$, we assume a given splitting $\mathfrak{S}$ of our group G, and may simply refer to 'labellings' of trees.

Observation 2.2.2. If $\mathbf{T_1} \simeq \mathbf{T_2}$ are equivalent graphs of groups and f is a collapsing map of the underlying graph T, then $f(\mathbf{T_1}) \simeq f(\mathbf{T_1})$ are also equivalent.

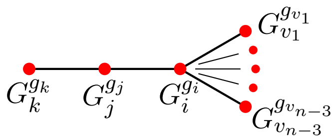
Example 2. Consider the labelled graph of groups $T :=$ 

.

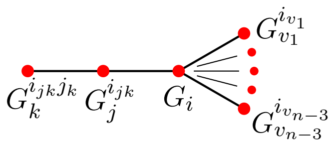
The following labelled graphs of groups are all equivalent to T:

1. 

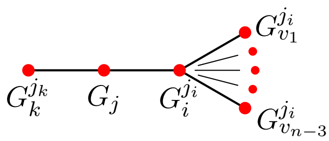
where $g \in G$ — since $gg^{-1} = 1 \in G_v$ for any v.

2. 

where each $g_v \in G_v$ — since for each v, $G_v \mapsto g_v G_v g_v^{-1}$ is an element of $\mathrm{Aut}(G_v)$.

3. 

where $i_{v_1}, \ldots, i_{v_{n-3}}, i_{jk} \in G_i$ and $j_k \in G_j$ — since $i \in G_i \Rightarrow 1^{-1}i \in G_i$ and ${i_{jk}}^{-1}(i_{jk}j_k) \in G_j$.

4. 

where $j_k, j_i \in G_j$ — this is achieved by combining 1. (with $g = j_i$), 2. (conjugating $G^{j_i}_j$ by $j_i^{-1} \in G_j$), and 3. (conjugating $G^{j_i}_k$ by $j_k j_i^{-1} \in G_j$) above.

In general, elements in the equivalence class of T all have the form shown in Figure 1, where $g \in G$, $g_v \in G_v$ for each v, $i_{v_1}, \ldots, i_{v_{n-3}}, i_{jk} \in G_i$, and $j_k \in G_j$.

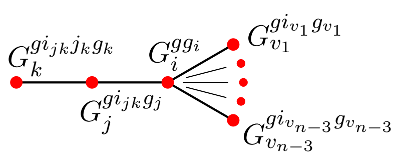
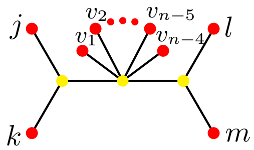


Figure 1: Equivalence Class of $\mathfrak{S}$-Labellings of T

We are now ready to define our complex.

Definition 2.2.3 (The Complex $\mathcal{C}_n$)**.** We build a (2-dimensional simplicial) complex called $\mathcal{C}_n$ as follows:

- Take one 0-simplex for each equivalence class of $\mathfrak{S}$-labellings of each tree in Table 1 (equivalently, take one 0-simplex for each equivalence class of $\mathfrak{S}$-labellings of each tree which is achieved by collapsing at least one edge of one of the trees 

for each of the $\frac{1}{2}\binom{n}{2}\binom{n-2}{2}$ unordered pairs of subsets $\{j, k\}$ and $\{l, m\}$ of $\{1, \ldots, n\}$).

- Given 0-simplices $[T_1]$ and $[T_2]$, insert a 1-simplex joining $[T_1]$ and $[T_2]$ if and only if some representative T_2 of $[T_2]$ is a collapse of some representative T_1 of $[T_1]$ or vice versa.

- Insert a 2-simplex wherever there is a 3-clique $[T_1]$—$[T_2]$—$[T_3]$—$[T_1]$ in the 1-skeleton.

We will often refer to simplices of $\mathcal{C}_n$ as cells. We will use these terms interchangeably. Additionally, we will sometimes refer to 0-cells as 'vertices', 1-cells as 'edges', and 2-cells as 'faces'.

Note that $\mathcal{C}_n$ is the barycentric spine of the subspace of Outer Space obtained by restricting to only simplices representing the above graph shapes. As such, we will sometimes refer to it as 'the/our complex', or 'the/our subcomplex'.

2.3 The Action of $\mathrm{Out}_{\mathfrak{S}}(G)$ on $\mathcal{C}_n$ and its Fundamental Domain $\mathcal{D}_n$

By considering the action of $\mathrm{Out}_{\mathfrak{S}}(G)$ on our complex $\mathcal{C}_n$, there is a natural idea of a quotient of $\mathcal{C}_n$ (two points are equivalent if they are in the same $\mathrm{Out}_{\mathfrak{S}}(G)$-orbit). We can then pick a 'fundamental domain' $\mathcal{D}_n$ for the action by choosing a lift of this quotient in $\mathcal{C}_n$.

Recall from Section 1.3 that $\mathrm{Out}_{\mathfrak{S}}(G)$ acts on the Spine of Outer Space. We restate this action in terms of graphs of groups (rather than G-trees), and verify that this is indeed an action on $\mathcal{C}_n$ when restricted to our chosen graphs. We begin by stating the action of $\mathrm{Out}_{\mathfrak{S}}(G)$ on the 0-skeleton $\mathcal{C}_n^{(0)}$ of $\mathcal{C}_n$, and then show this extends to an action on the full complex $\mathcal{C}_n$.

Definition 2.3.1 (Action of $\mathrm{Out}_{\mathfrak{S}}(G)$ on $\mathcal{C}_n^{(0)}$)**.** Let $[\psi] \in \mathrm{Out}_{\mathfrak{S}}(G)$ have representative $\psi \in \mathrm{Aut}_{\mathfrak{S}}(G)$ and let T be a point in $\mathcal{C}_n^{(0)}$ with $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$. Then $T \cdot [\psi]$ is a graph of groups with the same underlying graph as T and labelling $((H_1)\psi, \ldots, (H_n)\psi)$ (where $(H_i)\psi$ is given by the usual action of $\mathrm{Aut}(G)$ on G, noting that $H_i \leqslant G$).

Remark. Given $[\psi_1] = [\psi_2] \in \mathrm{Out}_{\mathfrak{S}}(G)$, there is some $\iota_g : x \mapsto gxg^{-1} \in \mathrm{Inn}(G)$ so that $\psi_2 = \psi_1\iota$. Then for any point $T \in \mathcal{C}_n^{(0)}$ with labelling $(H_1, \ldots, H_n)$, we have $(H_i)\psi_2 = ((H_i)\psi_1)^g$. As noted in 1 of Example 2, $(((H_1)\psi_1)^g, \ldots, ((H_n)\psi_1)^g)$ and $((H_1)\psi_1, \ldots, (H_n\psi_1))$ are equivalent as labellings. So we really do have that $T \cdot [\psi_1] = T \cdot [\psi_2]$ — that is, the action here is well-defined. As such, we will often write $T \cdot \psi$ (or even $T\psi$) for $T \cdot [\psi]$.

Lemma 2.3.2. *Let $S, T \in \mathcal{C}_n^{(0)}$ such that S is a collapse of T, and let $f : T \to S$ be the collapsing map. Let $\psi \in \mathrm{Out}_{\mathfrak{S}}(G)$. Then $f(T \cdot \psi) = f(T) \cdot \psi$.*

Proof. Suppose T as a graph has labelling $(H_1, \ldots, H_n)$. Recall from Definition 2.1.1 that permitted collapses do not alter vertex groups in any way. Thus $(H_1, \ldots, H_n)$ must also be a labelling for S. Now $T \cdot \psi$ is a graph of groups with the same underlying graph as T, and labelling $((H_1)\psi, \ldots, (H_n)\psi)$. Similarly, $S \cdot \psi$ has the same underlying graph as S, with labelling $((H_1)\psi, \ldots, (H_n)\psi)$. Since $T \cdot \psi$ has the same underlying graph as T, applying f to $T \cdot \psi$ yields a graph of groups whose underlying graph is the same as that of S, and has $((H_1)\psi, \ldots, (H_n)\psi)$ as a labelling. But this exactly describes the graph of groups $S \cdot \psi$. That is, $f(T \cdot \psi) = S \cdot \psi = f(T) \cdot \psi$. □

Since $\mathcal{C}_n$ is a simplicial complex, then any cell is uniquely determined by its vertices (0-cells). We will thus denote a cell by $[T_0, \ldots, T_k]$ where $T_0, \ldots, T_k$ are its vertices. Note that for us we will only ever have $k = 1$ or $k = 2$ (or $k = 0$).

Proposition 2.3.3. *Let* $\psi \in \mathrm{Out}_{\mathfrak{G}}(G)$*. If* $[T_0, \ldots, T_k]$ *is a cell in* $\mathcal{C}_n$*, then so is* $[T_0 \cdot \psi, \ldots, T_k \cdot \psi]$.

Proof. This is true by definition of the action for $k = 0$.

Let $[T_0, T_1]$ be an edge in $\mathcal{C}_n$. Then T_0 and T_1 are graphs of groups with T_1 a collapse of T_0 — say $f : T_0 \to T_1$ is the collapsing map. We know that $T_0 \cdot \psi$ is a point in $\mathcal{C}_n^{(0)}$, and since it has the same underlying graph as T_0, then so is $f(T_0 \cdot \psi)$. So we have an edge $[T_0 \cdot \psi, f(T_0 \cdot \psi)] \in \mathcal{C}_n$. But by Lemma 2.3.2, $f(T_0 \cdot \psi) = f(T_0) \cdot \psi = T_1 \cdot \psi$. So if $[T_0, T_1]$ is a cell in $\mathcal{C}_n$, then so is $[T_0 \cdot \psi, T_1 \cdot \psi]$.

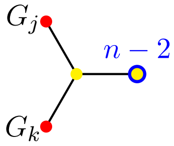
Now suppose $[T_0, T_1, T_2]$ is a 2-cell in $\mathcal{C}_n$. Then we must have a 3-clique $[T_0]$—$[T_1]$—$[T_2]$—$[T_0]$, so $[T_0, T_1]$, $[T_1, T_2]$, and $[T_0, T_2]$ are 1-cells in $\mathcal{C}_n$. Then $[T_0 \cdot \psi, T_1 \cdot \psi]$, $[T_1 \cdot \psi, T_2 \cdot \psi]$, and $[T_0 \cdot \psi, T_2 \cdot \psi]$ are 1-cells in $\mathcal{C}_n$ forming a 3-clique, hence by Definition 2.2.3 we must have a 2-cell $[T_0 \cdot \psi, T_1 \cdot \psi, T_2 \cdot \psi]$. □

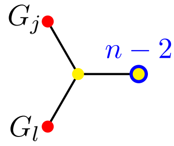
Definition 2.3.4 (Action of $\mathrm{Out}_{\mathfrak{G}}(G)$ on $\mathcal{C}_n$)**.** The action of an element $\psi \in \mathrm{Out}_{\mathfrak{G}}(G)$ on a k-cell $[T_0, \ldots, T_k]$ of $\mathcal{C}_n$ is defined to be:

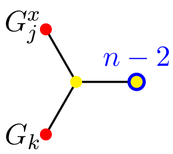
$$[T_0, \ldots, T_k] \cdot \psi := [T_0 \cdot \psi, \ldots, T_k \cdot \psi]$$

We now construct a fundamental domain for this action. The quotient space obtained from the action has one cell for each orbit of cells in $\mathcal{C}_n$. The obvious choice to make here is to take the lift to be the subcomplex supported by vertices which are all the graphs of groups (as listed in Table 1) whose vertex groups are precisely the factor groups $G_1, \ldots, G_n$. This is formalised below:

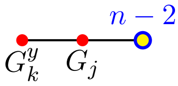
Definition 2.3.5 (Construction of $\mathcal{D}_n$)**.** We take the 0-skeleton $\mathcal{D}_n^{(0)}$ of $\mathcal{D}_n$ to be the set of graphs of groups T whose underlying graph is a tree from Table 1 so that, up to permuting the indices, T has a labelling $(G_1, \ldots, G_n)$. We now define $\mathcal{D}_n$ to be the subcomplex of $\mathcal{C}_n$ made up of all cells whose vertices are in $\mathcal{D}_n^{(0)}$.

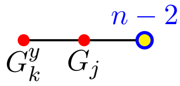
Example 3. Note that in our selection of graphs of groups, we still allow permutation of the vertex labels, just not conjugation. So 

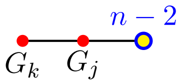
and 

are both in $\mathcal{D}_n^{(0)}$, while 

is not (for $x \notin G_j$, i.e. $G_j \neq G_j^x$ as sets).

Note however that (for $y \in G_j$) 

is in $\mathcal{D}_n^{(0)}$, since 

is equivalent to 

under Definition 2.2.1.

Notation 2.3.6. Given a vertex T in $\mathcal{C}_n$, we denote by $\mathrm{Stab}(T)$ the $\mathrm{Out}_{\mathfrak{G}}(G)$-stabiliser of T, that is, the set $\{\psi \in \mathrm{Out}_{\mathfrak{G}}(G) | T \simeq T \cdot \psi\}$, where $\simeq$ is the equivalence described in Definition 2.2.1. We will often abuse notation and write $T = S$ for $T \simeq S$.

Lemma 2.3.7. *Let T be a vertex in $\mathcal{C}_n$ (so T is a graph of groups) and let S be achieved by collapsing edges of T. Then* $\mathrm{Stab}(T) \subseteq \mathrm{Stab}(S)$.

Proof. Let $f : T \to S$ be the collapsing map, and let $\psi \in \mathrm{Stab}(T) \leqslant \mathrm{Out}_{\mathfrak{G}}(G)$. By Lemma 2.3.2, $S \cdot \psi = f(T) \cdot \psi = f(T \cdot \psi)$. Since $\psi \in \mathrm{Stab}(T)$ then $T \cdot \psi = T$, hence $S \cdot \psi = f(T) = S$. That is, $\psi \in \mathrm{Stab}(S)$. □

Proposition 2.3.8. *The subcomplex $\mathcal{D}_n$ of $\mathcal{C}_n$ described above is indeed a fundamental domain for the action of* $\mathrm{Out}_{\mathfrak{G}}(G)$.

Proof. We need to show that every orbit of cells in $\mathcal{C}_n$ is represented in $\mathcal{D}_n$. That is, if $C \in \mathcal{C}_n$ is a k-cell of $\mathcal{C}_n$ (for $k \in \{0, 1, 2\}$), then there is some $\psi \in \mathrm{Out}_{\mathfrak{G}}(G)$ so that $C \cdot \psi^{-1} \in \mathcal{D}_n$.

Let $(T, (H_1, \ldots, H_n))$ be a point in $\mathcal{C}_n^{(0)}$. Since $H_1 * \cdots * H_n$ is an $\mathfrak{G}$ free factor splitting of $G_1 * \cdots * G_n$, then by Lemma 1.1.8, there exists $\psi \in \mathrm{Aut}_{\mathfrak{G}}(G)$ so that for each i, $(G_i)\psi = H_i$. Then $(T, (H_1, \ldots, H_n)) \cdot [\psi^{-1}] = (T, (G_1, \ldots, G_n)) \in \mathcal{D}_n$, with $[\psi^{-1}] \in \mathrm{Out}_{\mathfrak{G}}(G)$ as required.

Now let $[\mathbf{T}, \mathbf{S}]$ be an edge in $\mathcal{C}_n$ (so $\mathbf{S}$ is a collapse of $\mathbf{T}$), and choose $\psi \in \mathrm{Out}_{\mathfrak{G}}(G)$ so that $\mathbf{T} \cdot \psi^{-1} \in \mathcal{D}_n$. Then $(G_1, \ldots, G_n)$ is an $\mathfrak{G}$-labelling for $\mathbf{T} \cdot \psi^{-1}$, and by Lemma 2.3.2, $\mathbf{S} \cdot \psi$ is a collapse of $\mathbf{T} \cdot \psi^{-1}$ and hence $(G_1, \ldots, G_n)$ is also an $\mathfrak{G}$-labelling for $\mathbf{S} \cdot \psi^{-1}$. That is, $[\mathbf{T} \cdot \psi^{-1}, \mathbf{S} \cdot \psi^{-1}]$ is an edge in $\mathcal{D}_n$.

Similarly, if $[\mathbf{T_0}, \mathbf{T_1}, \mathbf{T_2}]$ is a face in $\mathcal{C}_n$, then $\mathbf{T_2}$ is a collapse of $\mathbf{T_1}$, which in turn is a collapse of $\mathbf{T_0}$. Choosing $\psi \in \mathrm{Out}_{\mathfrak{G}}(G)$ with $\mathbf{T_0} \cdot \psi^{-1} \in \mathcal{D}_n$, the above argument then yields that $[\mathbf{T_0} \cdot \psi^{-1}, \mathbf{T_1} \cdot \psi^{-1}, \mathbf{T_2} \cdot \psi^{-1}]$ is a face in $\mathcal{D}_n$. □

Proposition 2.3.9. *The fundamental domain $\mathcal{D}_n$ described above is strict. That is, it contains precisely one representative of each vertex, edge, and face (2-cell) orbit.*

Proof. First, note that for two vertices to share an $\mathrm{Out}_{\mathfrak{G}}(G)$-orbit, they must have the same underlying graph structure. Moreover, since our automorphisms are pure symmetric (i.e. do not permute factor groups), they must have the same indexing of vertices. Since our fundamental domain was chosen to allow only one labelling for each distinct graph structure, this precisely means that each vertex of $\mathcal{D}_n$ is in a distinct orbit.

Now suppose we have two faces in the fundamental domain, $[T_0, T_1, T_2]$ and $[S_0, S_1, S_2]$, which are in the same orbit. Then their vertices are also in the same respective orbits (i.e. T_i and S_i share an orbit for each i). Since our fundamental domain contains only one representative of each vertex orbit, we must have that $T_i = S_i$ for each $i = 1, 2, 3$. But when we constructed $\mathcal{C}_n$, we inserted only one 2-cell for each 3-clique. That is, a face is uniquely determined by its vertices, so $[T_0, T_1, T_2] = [S_0, S_1, S_2]$.

The same argument applies to edges (cells with the form $[T_0, T_1]$). Hence no two cells of our fundamental domain are in the same orbit, that is, we have a strict fundamental domain. □

2.4 Stabilisers of Vertices in $\mathcal{D}_n$

To move through our complex $\mathcal{C}_n$, we consider ‘collapse–expansion’ paths, since two vertices (graphs of groups) are adjacent in $\mathcal{C}_n$ if and only if one is a collapse of the other. If $T_1 — T_2 — T_3$ is a path in $\mathcal{C}_n$ such that T_2 is a collapse of both T_1 and T_3, and T_1 and T_3 have the same underlying graph structure, then we will have that $T_3 = T_1 \cdot \psi$ for some $\psi \in \mathrm{Stab}(T_2)$. Thus understanding vertex stabilisers is key to understanding adjacency in $\mathcal{C}_n$. We will also need to understand vertex stabilisers in order to apply Brown’s Theorem (Theorem 1.2.2).

Recall that given a point $T = (T, (H_1, \ldots, H_n)) \in \mathcal{C}_n^{(0)}$, we have that $\psi \in \mathrm{Out}_{\mathfrak{G}}(G)$ is in the stabiliser $\mathrm{Stab}(T)$ of T if and only if $T \cdot \psi = T$, that is, $(H_1, \ldots, H_n)$ and $((H_1)\psi, \ldots, (H_n)\psi)$ are equivalent as labellings of T. Recall from Definition 2.2.1 that this means for each $i = 1, \ldots, n$ there exists $g_i \in G$ and $\varphi_i \in \mathrm{Aut}(H_i)$ so that $(H_i)\psi = \varphi_i(H_i)^{g_i}$, and moreover, for every edge $\overset{u}{\bullet}\!\longrightarrow\!\overset{v}{\bullet}$ of T we have $g_u^{-1} g_v \in H_u$.

We will only compute stabilisers of vertices in $\mathcal{D}_n$. However, if $T \cdot \chi \in \mathcal{C}_n$ (with $T \in \mathcal{D}_n$ and $\chi \in \mathrm{Out}_{\mathfrak{G}}(G)$), then $\mathrm{Stab}(T \cdot \chi) = \chi^{-1}\,\mathrm{Stab}(T)\chi = \mathrm{Stab}(T)^{\chi^{-1}}$. As such, we will assume for now that any graph of groups T has $(G_1, \ldots, G_n)$ as a labelling.

We will present several viewpoints on the stabiliser of a vertex, from Guirardel–Levitt [13] and Bass–Jiang [4], respectively. Note that Levitt [15] shows how these results are equivalent, and demonstrates how to translate between them.

The Guirardel–Levitt Approach

Recall from Definition 1.1.5 that $\Phi \leqslant \mathrm{Out}_{\mathfrak{G}}(G)$ is the group of factor automorphisms of $G_1 * \cdots * G_n$, with $\Phi = \prod_{i=1}^{n} \mathrm{Aut}(G_i)$.

Given a vertex v_i of a point (graph of groups) $T \in \mathcal{C}_n$, with vertex group G_{v_i} (assuming $G_{v_i} \neq \{1\}$, that is, v_i is not a trivial vertex), let μ_i be the valency of v_i in T.

In [13, Section 5], Guirardel and Levitt give the stabiliser of T in $\mathrm{Out}_{\mathfrak{G}}(G)$ as being isomorphic to:

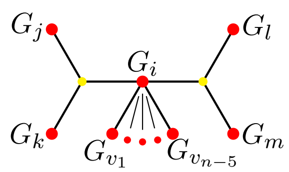
$$\prod_{i=1}^{n} \left(G_i^{\mu_i - 1} \rtimes \mathrm{Aut}(G_i) \right) = \left(\prod_{i=1}^{n} G_i^{\mu_i - 1} \right) \rtimes \Phi$$

where the semidirect product relation is given by the natural action of Φ on each G_i.

Using this, we recover Table 2 showing the stabilisers (up to isomorphism) of points in $\mathcal{D}_n$. Recall that the graph structures of these points are shown in Table 1.

This point of view corresponds to fixing a particular edge of a graph of groups T, and then twisting the remaining edges (outwards from the fixed edge). We demonstrate this with an example:

Example 4. Consider the graph of groups σ:



(the graph $\sigma_{i,jk,lm}$ with labelling $\hat{G} := (G_1, \ldots, G_n)$). We will compute the stabiliser of σ by ‘fixing’ an edge in the graph of groups.

Let $\hat{H} = \left(G_1^{h_1}, \ldots, G_n^{h_n} \right)$ be an arbitrary labelling in the equivalence class of $\hat{G}$ of labellings of σ. Note that by Definition 2.2.1 we must also have elements $h_{jk}, h_{lm} \in G$

Vertex	Stabiliser
ρ_{jk}	Φ
$\sigma_{i,jk,lm}$	$G_i^{n-4} \rtimes \Phi$
$\tau_{j,k,lm}$	$G_j \rtimes \Phi$
α	Φ
$\beta_{j,k}$	$G_j \rtimes \Phi$
$\gamma_{i,jk}$	$G_i^{n-3} \rtimes \Phi$
$\delta_{i,j,k,lm}$	$(G_i^{n-4} \times G_j) \rtimes \Phi$
$\varepsilon_{j,k,l,m}$	$(G_j \times G_l) \rtimes \Phi$
A_i	$G_i^{n-2} \rtimes \Phi$
$B_{i,j,k}$	$(G_i^{n-3} \times G_j) \rtimes \Phi$
$C_{i,j,k,l,m}$	$(G_i^{n-4} \times G_j \times G_l) \rtimes \Phi$

Table 2: Vertex Stabilisers (up to isomorphism) using Guirardel–Levitt

corresponding to the two trivial vertices of σ, and that $h_j = h_k = h_{jk}$ and $h_l = h_m = h_{lm}$. As noted in 1 of Example 2, inner automorphisms of G preserve equivalence classes of labellings. Thus the labelling $\hat{H}'$ achieved by replacing each $G_a^{h_a}$ by $(G_a^{h_a})^{h_{lm}^{-1}} = G_a^{h_{lm}^{-1}h_a}$ is equivalent to $\hat{H}$. Note that in this labelling, the conjugator of G_i is $h_{lm}^{-1}h_i = (h_i^{-1}h_{lm})^{-1} \in G_i$ by Definition 2.2.1. Since this is an (inner) factor automorphism of G_i, then the labelling $\hat{H}''$ achieved by replacing $G_i^{h_i h_{lm}^{-1}}$ in $\hat{H}'$ with simply G_i is also equivalent to $\hat{H}$. We have now essentially 'fixed' the edge i — lm (i.e. the edge which separates G_l and G_m from G_i and all the other vertex groups) in σ.

Note that for any a we have $h_{lm}^{-1}h_a = (h_i^{-1}h_{lm})^{-1}(h_i^{-1}h_a) \in G_i$. So aside from factor automorphisms (i.e. replacing G_a with $(G_a)\varphi$ for $\varphi \in \Phi$), the only freedom we have left is to 'twist' along the remaining edges incident to i (the vertex in σ whose vertex group is G_i); that is, given each remaining edge e incident to i, to conjugate all vertex groups separated from i by e by an element of G_i. Note that for $g_i \in G_i$ and $\varphi \in \Phi$, we have $(G_a^{g_i})\varphi = (G_a\varphi)^{(g_i)\varphi} = G_a^{(g_i)\varphi}$. We will let G_{i_v} denote the group of Whitehead automorphisms which conjugate the vertex group G_v by elements of G_i (for $v = v_1, \ldots, v_{n-5}$), and similarly denote by $G_{i_{jk}}$ the group of Whitehead automorphisms which conjugate the vertex groups G_j and G_k simultaneously by elements of G_i. Note that $G_{i_{jk}} \cong G_{i_v} \cong G_i$ (for $v = v_1, \ldots, v_{n-5}$). Since twists along edges from i happen independently of each other, we then have that $\mathrm{Stab}(\sigma) = (G_{i_{jk}} \times G_{i_{v_1}} \times \cdots \times G_{i_{v_{n-5}}}) \rtimes \Phi \cong G_i^{n-4} \rtimes \Phi$, as listed in Table 2.

Note that by choosing an edge to 'fix' in a graph of groups T and indexing the remaining edges as in the above example, we can similarly expand all the stabilisers listed in Table 2. While this works well for some graphs, it does lead to a lack of symmetry, and in graphs such as A_i, such a choice can feel entirely arbitrary.

Even in the above example, we could have chosen to fix the edge from G_i leading to G_j and G_k (or even an edge G_i — G_v) instead of the edge from G_i leading to G_l and G_m . This means we must have that $\mathrm{Stab}(\sigma) = (G_{i_{jk}} \times G_{i_{v_1}} \times \cdots \times G_{i_{v_{n-5}}}) \rtimes \Phi = (G_{i_{lm}} \times G_{i_{v_1}} \times \cdots \times G_{i_{v_{n-5}}}) \rtimes \Phi \left(= (G_{i_{jk}} \times G_{i_{lm}} \times G_{i_{v_2}} \times \cdots \times G_{i_{v_{n-5}}}) \rtimes \Phi\right)$ as subgroups of $\mathrm{Out}_{\mathfrak{G}}(G)$.

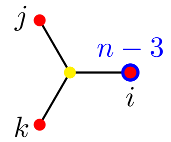
To see why this holds, let $g_i \in G_i$, let ι_{g_i} be the element of $\mathrm{Inn}(G)$ which conjugates every element of G by g_i, and for groups $G_{w_1}, \ldots, G_{w_a}$ ($\{w_1, \ldots, w_a\} \subseteq \{1, \ldots, n\}$) let

$(\{G_{w_1}, \dots, G_{w_k}\}, g_i)$ be the Whitehead automorphism which conjugates each G_w by g_i (for $w = w_1, \dots, w_a$). Then for $(\{G_l, G_m\}, g_i)$ an arbitrary element of $G_{i_{lm}}$, we have that $(\{G_l, G_m\}, g_i) = \left(\{G_i\}, g_i^{-1}\right)\left(\{G_j, G_k, G_{v_1}, \dots, G_{v_{n-5}}\}, g_i^{-1}\right)\iota_{g_i}$, where $\left(\{G_i\}, g_i^{-1}\right) \in \mathrm{Inn}(G_i) \leqslant \mathrm{Aut}(G_i) \leqslant \Phi$ and $\left(\{G_j, G_k, G_{v_1}, \dots, G_{v_{n-5}}\}, g_i^{-1}\right) \in G_{i_{jk}} \times G_{i_{v_1}} \times \cdots \times G_{i_{v_{n-5}}}$. That is, $G_{i_{lm}} \leqslant (G_{i_{jk}} \times G_{i_{v_1}} \times \cdots \times G_{i_{v_{n-5}}}) \rtimes \Phi$. One can similarly show that $G_{i_{jk}} \leqslant (G_{i_{lm}} \times G_{i_{v_1}} \times \cdots \times G_{i_{v_{n-5}}}) \rtimes \Phi$, as well as the inclusions required for the third claimed equality.

Thus while correct, and simple to write down, this method of determining stabilisers can obscure subgroups and other structure. To remedy this, we may consider initially fixing just a vertex, rather than an edge, in our graph of groups, or more generally, not 'fixing' anything at all.

The Bass–Jiang Approach

Given a vertex v_i of a point (graph of groups) $T \in \mathcal{C}_n^{(0)}$, with vertex group G_i (assuming $G_i \neq \{1\}$, that is, v_i is not a trivial vertex), let $E(v_i)$ be the set of edges of T with v_i as an endpoint. We will index these edges by the vertices they separate from v_i (so for example in



, the edges incident to the vertex i are indexed by $v_1, \dots, v_{n-3}$ and (jk)).

Bass and Jiang [4, Theorem 8.1] give a filtration explicitly describing the $\mathrm{Out}(G)$ stabiliser of a graph of groups. Since we are restricting to pure symmetric (outer) automorphisms, we have trivial edge stabilisers in our graphs of groups and no graph automorphisms (as our graphs of groups are trees, and we do not permit permutation of the vertex groups). So this filtration simplifies to a short exact sequence:

$$1 \longrightarrow \prod_{i=1}^{n} \left(\left(\prod_{e \in E(v_i)} G_{i_e} \right) \Big/ Z(G_i) \right) \longrightarrow \mathrm{Stab}(T) \longrightarrow \prod_{i=1}^{n} \mathrm{Out}(G_i) \longrightarrow 1$$

where $f_{i_e} : G_i \to G_{i_e}$ is an isomorphism and G_{i_e} is the group of Whitehead automorphisms which conjugate the vertex groups of vertices separated from v_i by e by elements of the vertex group G_i, and $Z(G_i)$ is the centre of G_i, with diagonal embedding. For brevity, given $T \in \mathcal{D}_n^{(0)}$ we will write M_T for the $\prod_{i=1}^{n} \left(\left(\prod_{e \in E(v_i)} G_{i_e} \right) \Big/ Z(G_i) \right)$ term of the above short exact sequence. This term corresponds to 'twisting' along each edge incident to each vertex in T.

Writing v for the vertex group G_v, and v_e for the automorphism group G_{v_e} (with the indexing described above), we recover Table 3, showing the M_T term of the Bass–Jiang short exact sequence for each tree T of Table 1.

Observe that for any $a = 2, \dots, n-1$ we can embed the group $i_{v_1 \dots v_a}$ diagonally into the direct product $i_{v_1} \times \cdots \times i_{v_a}$. We write $i_{v_1 \dots v_a} \underline{\diamond}\, i_{v_1} \times \cdots \times i_{v_a}$ to indicate that we consider $i_{v_1 \dots v_a}$ to be the diagonal subgroup of $i_{v_1} \times \cdots \times i_{v_a}$. If $g_i \in G_i$ then $f_{i_{v_1 \dots v_{n-1}}}(g_i) \in i_{v_1 \dots v_{n-1}}$ and $\iota_{g_i^{-1}}$ is the inner automorphism which conjugates all elements of G by g_i^{-1}. Now $f_{i_{v_1 \dots v_{n-1}}}(g_i)\iota_{g_i^{-1}}$ conjugates all elements of G_i by g_i^{-1} and

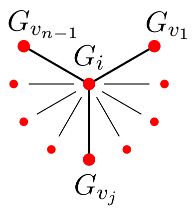
T	M_T
ρ_{jk}	$\prod_{v} \mathrm{Inn}(v)$
$\sigma_{i,jk,lm}$	$\prod_{v\neq i} \mathrm{Inn}(v) \times {}^{(i_{v_1} \times \cdots \times i_{v_{n-5}} \times i_{jk} \times i_{lm})}\big/ {}_{Z(i)}$
$\tau_{j,k,lm}$	$\prod_{v\neq j} \mathrm{Inn}(v) \times {}^{(j_{v_1 \ldots v_{n-4} lm} \times j_k)}\big/ {}_{Z(j)}$
α	$\prod_{v} \mathrm{Inn}(v)$
$\beta_{j,k}$	$\prod_{v\neq j} \mathrm{Inn}(v) \times {}^{(j_{v_1 \ldots v_{n-2}} \times j_k)}\big/ {}_{Z(j)}$
$\gamma_{i,jk}$	$\prod_{v\neq i} \mathrm{Inn}(v) \times {}^{(i_{v_1} \times \cdots \times i_{v_{n-3}} \times i_{jk})}\big/ {}_{Z(i)}$
$\delta_{i,j,k,lm}$	$\prod_{v\neq i,j} \mathrm{Inn}(v) \times {}^{(i_{v_1} \times \cdots \times i_{v_{n-5}} \times i_{jk} \times i_{lm})}\big/ {}_{Z(i)} \times {}^{(j_{v_1 \ldots v_{n-5} ilm} \times j_k)}\big/ {}_{Z(j)}$
$\varepsilon_{j,k,l,m}$	$\prod_{v\neq j,l} \mathrm{Inn}(v) \times {}^{(j_{v_1 \ldots v_{n-4} lm} \times j_k)}\big/ {}_{Z(j)} \times {}^{(l_{v_1 \ldots v_{n-4} jk} \times l_m)}\big/ {}_{Z(l)}$
A_i	$\prod_{v\neq i} \mathrm{Inn}(v) \times {}^{(i_{v_1} \times \ldots i_{v_{n-1}})}\big/ {}_{Z(i)}$
$B_{i,j,k}$	$\prod_{v\neq i,j} \mathrm{Inn}(v) \times {}^{(i_{v_1} \times \cdots \times i_{v_{n-3}} \times i_{jk})}\big/ {}_{Z(i)} \times {}^{(j_{iv_1 \ldots v_{n-3}} \times j_k)}\big/ {}_{Z(j)}$
$C_{i,j,k,l,m}$	$\prod_{v\neq i,j,l} \mathrm{Inn}(v) \times {}^{(i_{v_1} \times \cdots \times i_{v_{n-5}} \times i_{jk} \times i_{lm})}\big/ {}_{Z(i)} \times {}^{(j_{iv_1 \ldots v_{n-5} lm} \times j_k)}\big/ {}_{Z(j)} \times {}^{(l_{iv_1 \ldots v_{n-5} jk} \times l_m)}\big/ {}_{Z(l)}$

Table 3: M_T Terms of Vertices T from Bass–Jiang Short Exact Sequence

fixes all other elements of G. That is, $f_{i_{v_1 \cdots v_{n-1}}}(g_i)\iota_{g_i} \in \mathrm{Inn}(G_i)$. So we have that $i_{v_1 \ldots v_{n-1}} \mathrm{Inn}(G) = \mathrm{Inn}(G_i)\,\mathrm{Inn}(G)$ as cosets in $\mathrm{Out}_{\mathfrak{S}}(G)$.

More generally, if $A \sqcup B$ partitions $\{1, \ldots, n\} - \{i\}$ then $G_{i_A} = G_{i_B}$ in $\mathrm{Out}_{\mathfrak{S}}(G)$.

Example 5. Consider the graph of groups A:



(the graph A_i with labelling $\hat{G} := (G_1, \ldots, G_n)$). We explore two ways of determining $\mathrm{Stab}(A)$.

1. We will first deduce $\mathrm{Stab}(A)$ from the Bass–Jiang filtration. Note that this filtration allows us to compute the M_A term of the Bass–Jiang short exact sequence, rather than the stabiliser of A itself.

   We begin by considering 'twisting' from a vertex v_j (with vertex group G_{v_j}) for some $j = 1, \ldots, n-1$. Since this is a vertex of valency 1, we conjugate all other vertex groups by the same element g_j of G_{v_j}, to achieve a labelling (up to appropriate reordering) $\left(G_{v_j}, G_i^{g_j}, G_{v_1}^{g_j}, \ldots, G_{v_{n-1}}^{g_j}\right)$. However, by applying the inner automorphism $\iota_{g_j^{-1}} \in \mathrm{Inn}(G)$ which conjugates all elements of G by g_j^{-1}, we see that this is equivalent to the labelling $\left(G_{v_j}^{g_j^{-1}}, G_i, G_{v_1}, \ldots, G_{v_{n-1}}\right)$, which equates to having applied the inner factor automorphism which conjugates G_{v_j} by an element of itself. Thus twisting from a valency 1 vertex v_j simply yields $\mathrm{Inn}(G_{v_j})$ at this stage of the Bass–Jiang filtration.

We now consider twists from the vertex i with vertex group G_i. This has valency $n-1$, and so we get $n-1$ groups $i_{v_j} \cong G_i$, which equate to conjugating the vertex group G_{v_j} by elements of G_i. Since twisting along edges incident to i is independent of the order in which we twist them, this forms a direct product of groups. Note that if we were to twist all edges by the same element g_i of G_i, this is equivalent (up to an inner automorphism of G) to just conjugating G_i by g_i^{-1}. That is, $\mathrm{Inn}(G_i) = i_{v_1 \ldots v_{n-1}} \lessdot i_{v_1} \times \cdots \times i_{v_{n-1}}$, the diagonal subgroup. Moreover, if $g_i \in Z(G_i)$ is central in G_i, then conjugation of G_i by g_i^{-1} is the identity map on G_i, and so twisting along all edges incident to i by g_i is equivalent to the identity automorphism. Thus we must quotient out by the centre of G_i, embedded diagonally into $i_{v_1 \ldots v_{n-1}} \lessdot i_{v_1} \times \cdots \times i_{v_{n-1}}$.

Hence we have the short exact sequence:

$$1 \to \prod_{v \neq i} \mathrm{Inn}(v) \times {}^{(i_{v_1} \times \ldots i_{v_{n-1}})}\!\big/_{Z(i)} \to \mathrm{Stab}(A) \to \prod_{v \neq i} \mathrm{Out}(v) \times \mathrm{Out}(i) \to 1$$

We deduce from this that $\mathrm{Stab}(A)$ is generated by $\Phi = \prod_{a=1}^{n} \mathrm{Aut}(G_a)$ and groups $G_{i_a} \cong G_i$ for $a \in \{1, \ldots, n\} - \{i\}$, so that $[G_{i_a}, G_{i_b}] = 1$ (each element of G_{i_a} commutes with each element of G_{i_b}) for every $a, b \in \{1, \ldots, n\} - \{i\}$, and where $G_{i_{v_1 \ldots v_{n-1}}} \lessdot G_{i_{v_1}} \times \cdots \times G_{i_{v_{n-1}}}$ is the diagonal subgroup, we have ${}^{G_{i_{v_1 \ldots v_{n-1}}}}\!\big/_{Z(G_i)} = \mathrm{Inn}(G_i)$. We observe that for $a, b \neq i$, $[\mathrm{Aut}(G_a), G_{i_b}] = 1$. However, for $a \neq i$, $(\{G_a\}, g_i) \in G_{i_a}$, and $\varphi_i \in \mathrm{Aut}(G_i)$, we have $(\{G_a\}, g_i)\,\varphi_i = \varphi_i\,(\{G_a\}, \varphi_i(g_i))$. These relations on the given generators are enough to fully determine $\mathrm{Stab}(A)$ as a subgroup of $\mathrm{Out}(G)$.

2. Alternatively, we can calculate $\mathrm{Stab}(A)$ from Definition 2.2.1 by considering equivalent labellings on A and the automorphisms which lead to these. This corresponds to 'twisting outwards from i'. Note that in this method, we implicitly 'fix' the 'basepoint' i. Recall that A has labelling $\hat{G} := (G_1, \ldots, G_n)$, and let $\hat{H} := \left(G_1^{h_1}, \ldots, G_n^{h_n}\right)$ be an arbitrary labelling in the equivalence class of $\hat{G}$ of labellings of A.

   Observe that as in 1 of Example 2, we can apply the inner automorphism $\iota_{h_i^{-1}} \in \mathrm{Inn}(G)$ which conjugates each element of G by h_i^{-1}. Thus we obtain the labelling $\hat{H}' := \left(G_i, G_{v_1}^{h_i^{-1}h_{v_1}}, \ldots, G_{v_{n-1}}^{h_i^{-1}h_{v_{n-1}}}\right)$ which (up to appropriate reordering) is equivalent to $\hat{H}$.

   By Definition 2.2.1, we have that for each $a \in \{1, \ldots, n-1\}$, $h_i^{-1}h_{v_a} \in G_i$. Hence (aside from factor automorphisms) our only freedom in labellings is to conjugate each non-G_i vertex group by an element of G_i, i.e. to 'twist' along each of the edges incident to the vertex i (with vertex group G_i) in A.

   Thus $\mathrm{Stab}(A)$ is generated by Φ and by $n-1$ groups $G_{i_a} \cong G_i$. Note that this is exactly as determined above, and the same arguments can be made to determine relations, resulting in the same presentation for $\mathrm{Stab}(A)$.

While the Bass–Jiang approach deals with the removal of symmetry which occurs by making specific choices in the Guirardel–Levitt approach, we lose the ability to concisely write down stabilisers. As such, we do not wish to replace the Guirardel–Levitt approach with this one, but rather enhance it.

Vertex Stabilisers for Common Use

We will now detail presentations for the stabilisers of vertices in $\mathcal{D}_n$ (graphs of groups with structure listed in Table 1 and labelling $(G_1, \ldots, G_n)$) which will be useful throughout the paper, but especially in Sections 4.1 and 5.2. For brevity, we write i_j for the group $G_{i_j} \cong G_i$ of Whitehead automorphisms which conjugate the factor group G_j by elements of the factor group G_i. Recall that there is an isomorphism $f_{i_j} : G_i \to i_j$ given by $f_{i_j}(g) = (\{G_j\}, g)$. When the isomorphism is relevant, we may write $f_{i_j}(G_i)$ for the group i_j. We will consider the centre $Z(G_i)$ of G_i to be embedded in $i_{v_1 \ldots v_k} \leq i_{v_1} \times \cdots \times i_{v_k}$ via the isomorphisms f_{i_j}.

We divide the vertices of $\mathcal{D}_n$ into three categories, according to which method(s) we will use to compute their stabilisers.

First, are vertices which have graph structure well-suited to the Guirardel–Levitt approach:

Proposition 2.4.1. *As subgroups of* $\mathrm{Out}_{\mathfrak{S}}(G)$ *we have:*

- $\mathrm{Stab}(\rho_{jk}) = \mathrm{Stab}(\alpha) = \Phi$
- $\mathrm{Stab}(\tau_{j,k,lm}) = \mathrm{Stab}(\beta_{j,k}) = j_k \rtimes \Phi$
- $\mathrm{Stab}(\varepsilon_{j,k,l,m}) = (j_k \times l_m) \rtimes \Phi$

where the semidirect relation is given by $(\{G_k\}, g_j) \circ \varphi = \varphi \circ (\{G_k\}, (g)\varphi)$ *for any* j, k *with* $(\{G_k\}, g) \in j_k$ *and* $\varphi \in \Phi$*. In other words,* $\varphi^{-1} f_{j_k}(g)\varphi = f_{j_k}((g)\varphi)$ *for* $g \in G_j$*.*

Proof. These are lifted directly from Table 2, utilising Example 4 which follows it to index the groups of automorphisms by the vertices they act on. □

Our second category is that of vertices whose graph structures are well-suited to the Bass–Jiang approach. We write $i_{v_1 \ldots v_k}$ for the group of automorphims
$\{(\{G_{v_1}, \ldots, G_{v_k}\}, g) \,|\, g \in G_i\}$.

Proposition 2.4.2. *As subgroups of* $\mathrm{Out}_{\mathfrak{S}}(G)$ *we have:*

- $\mathrm{Stab}(\sigma_{i,jk,lm})$ *is generated by* $(i_{jk} \times i_{lm} \times i_{v_1} \times \cdots \times i_{v_{n-5}}) / Z(G_i)$ *and* Φ
- $\mathrm{Stab}(\gamma_{i,jk})$ *is generated by* $(i_{jk} \times i_{v_1} \times \cdots \times i_{v_{n-3}}) / Z(G_i)$ *and* Φ
- $\mathrm{Stab}(A_i)$ *is generated by* $(i_{v_1} \times \cdots \times i_{v_{n-1}}) / Z(G_i)$ *and* Φ

each subject to the relations $f_{i_{w_1}}(g) \ldots f_{i_{w_{n-1}}}(g) = \mathrm{Ad}_{G_i}(g^{-1})$ *(with* $\mathrm{Ad}_{G_i}(g^{-1})$ *as in Notation 1.1.6) and* $\varphi^{-1} f_{i_v}(g)\varphi = f_{i_v}((g)\varphi)$*, where* $\{w_1, \ldots, w_{n-1}\} = \{1, \ldots, n\} - \{i\}$*,* $v \in \{w_1, \ldots, w_{n-1}\}$*, and* $\varphi \in \Phi$*. That is,* $i_{w_1 \ldots w_{n-1}} / Z(G_i) = \mathrm{Inn}(G_i)$ *and* $f_{i_v}((G_i)\varphi)^{\varphi} = i_v$*.*

Proof. These are deduced from the M_T terms of the Bass–Jiang short exact sequences listed in Table 3. Example 5 explicitly details how to recover relations for $\mathrm{Stab}(A_i)$, and the others follow similarly. □

Finally, our third category is that of vertices whose graph structures are not well-suited to either approach, and we thus work directly from Definition 2.2.1:

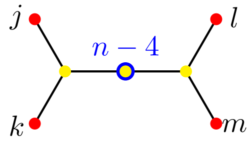
Proposition 2.4.3. *As subgroups of* $\mathrm{Out}_{\mathfrak{S}}(G)$ *we have:*

- $\mathrm{Stab}(\delta_{i,j,k,lm})$ *is generated by* j_k, ${(i_{jk} \times i_{lm} \times i_{v_1} \times \cdots \times i_{v_{n-5}})}/{Z(G_i)}$, *and* Φ, *subject to the relations:*
  1. ${i_{jklmv_1 \ldots v_{n-5}}}/{Z(G_i)} = \mathrm{Inn}(G_i)$
  2. $\left[j_k, {(i_{jk} \times i_{lm} \times i_{v_1} \times \cdots \times i_{v_{n-5}})}/{Z(G_i)}\right] = 1$
  3. $f_{j_k}((G_j)\varphi)^{\varphi} = j_k$ *and* $f_{i_x}((G_i)\varphi)^{\varphi} = i_x$ *for each* $x \in \{jk, lm, v_1, \ldots, v_{n-5}\}$
- $\mathrm{Stab}(B_{i,j,k})$ *is generated by* j_k, ${(i_{jk} \times i_{v_1} \times \cdots \times i_{v_{n-3}})}/{Z(G_i)}$, *and* Φ, *subject to the relations:*
  1. ${i_{jkv_1 \ldots v_{n-3}}}/{Z(G_i)} = \mathrm{Inn}(G_i)$
  2. $\left[j_k, {(i_{jk} \times i_{v_1} \times \cdots \times i_{v_{n-3}})}/{Z(G_i)}\right] = 1$
  3. $f_{j_k}((G_j)\varphi)^{\varphi} = j_k$ *and* $f_{i_x}((G_i)\varphi)^{\varphi} = i_x$ *for each* $x \in \{jk, v_1, \ldots, v_{n-3}\}$
- $\mathrm{Stab}(C_{i,j,k,l,m})$ *is generated by* j_k, l_m, ${(i_{jk} \times i_{lm} \times i_{v_1} \times \cdots \times i_{v_{n-5}})}/{Z(G_i)}$, *and* Φ, *subject to the relations:*
  1. ${i_{jklmv_1 \ldots v_{n-5}}}/{Z(G_i)} = \mathrm{Inn}(G_i)$
  2. $\left[a_b, {(i_{jk} \times i_{lm} \times i_{v_1} \times \cdots \times i_{v_{n-5}})}/{Z(G_i)}\right] = 1$ *for each of* $a_b = j_k$ *and* $a_b = l_m$
  3. $f_{j_k}((G_j)\varphi)^{\varphi} = j_k$, $f_{l_m}((G_l)\varphi)^{\varphi} = l_m$, *and* $f_{i_x}((G_i)\varphi)^{\varphi} = i_x$ *for each* $x \in \{jk, lm, v_1, \ldots, v_{n-5}\}$
  4. $[j_k, l_m] = 1$

Proof. We calculate these using Definition 2.2.1. The equivalence class of labellings for $B_{i,j,k}$ is explicitly described in Example 2. The classes for $\delta_{i,j,k,lm}$ and $C_{i,j,k,l,m}$ follow similarly. From here, deduction of the $\mathrm{Out}_{\mathfrak{S}}(G)$ stabilisers follows in much the same way as in the second part of Example 5. □

3 Properties of the Fundamental Domain $\mathcal{D}_n$

Before proving any structural statements about the fundamental domain $\mathcal{D}_n$, we provide some illustrations to aid in understanding this subcomplex. We describe some of the substructures found within the 1-skeleton $\mathcal{D}_n^{(1)}$ of the fundamental domain of $\mathcal{C}_n$. These will be particularly useful in determining edge inclusion relations in Theorem 4.1.1, as well as in showing that $\mathcal{D}_n$ is simply connected in Section 3.2.

The subcomplex of the fundamental domain obtained by restricting to collapses of a given graph 

will be referred to as a 'spike' of the fundamental domain.

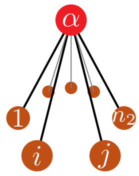


Figure 2: The α-A–Star

Figure 2 shows the α-A–Star. The circle with label 'i' represents the vertex A_i. The circle with label 'n_2' represents the vertex $A_{v_{n-2}}$. Where the 3 small dots at the back are, one should imagine "many", that is, that there are in fact $n-4$ A-vertices there. This structure appears precisely once in the fundamental domain, and is present in every 'spike' (of which there are $\frac{n!}{8(n-4)!}$).

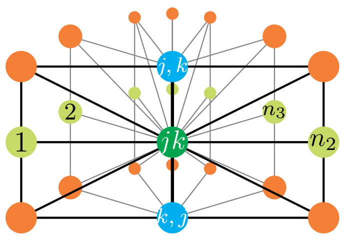


Figure 3: The ρ–Book

Figure 3 shows a ρ–Book (**the** ρ–Book associated to the graph ρ_{jk}). The circle with label '1' represents the vertex $\gamma_{v_1,jk}$. The circle with label 'jk' represents the vertex ρ_{jk}, and the circle with label 'j,k' represents the vertex $\beta_{j,k}$. The orange circle adjacent to both $\beta_{j,k}$ and $\gamma_{v_1,j,k}$ represents $B_{v_1,j,k}$. This structure appears $\frac{n(n-1)}{2}$ times in the fundamental domain. There are two ρ–Books per 'spike', and each ρ–Book appears in $\frac{(n-2)(n-3)}{2}$ spikes. Two distinct ρ–Books appear in only one spike together, and only if they are associated to ρ_{jk} and ρ_{lm} where j,k,l,m are all distinct.

Figure 4 shows a τ-ε–Box. It appears precisely once in each spike, and is unique to its spike. It has $n-4$ layers, each associated to a σ-vertex. A given layer will be called a σ–Slice. The cycle left when removing all σ–Slices is called a τ-ε–Square. The yellow circle with label 'n_5' represents $\sigma_{v_{n-5},jk,lm}$. The blue circle with label 'jk' represents $\tau_{j,k,lm}$, and the pink circle with label '$jklm$' represents $\varepsilon_{j,k,l,m}$.

The remainder of this section will be spent proving connectivity results (namely path connectivity and simple connectivity) for the fundamental domain $\mathcal{D}_n$, which will be key to proving respective theorems for the complex $\mathcal{C}_n$.

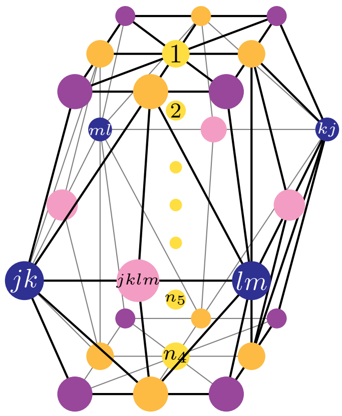


Figure 4: The τ-ε–Box

3.1 Connectedness of the Fundamental Domain

We first show that the fundamental domain $\mathcal{D}_n$ is (path) connected. We will do this by finding a path from an arbitrary vertex $T \in \mathcal{D}_n$ to the vertex $\alpha \in \mathcal{D}_n$ (see Table 1). Then any two arbitrary vertices of $\mathcal{D}_n$ will be connected via α.

Lemma 3.1.1. *The fundamental domain $\mathcal{D}_n$ of $\mathcal{C}_n$ is path connected.*

Proof. We refer to Table 1 for the naming convention of vertices in $\mathcal{D}_n$ (where each vertex group is precisely one of the factor groups $G_1, \ldots, G_n$). Note that α is adjacent to every ρ_{jk} and every A_i in $\mathcal{D}_n$. Every β and γ graph is a collapse of some ρ graph, hence any β or γ vertex in $\mathcal{D}_n$ is path connected to α. Additionally, every B graph is the collapse of some β graph, so B vertices are also connected. Note that $\sigma_{i,jk,lm}$ and $\tau_{j,k,lm}$ collapse to A_i and A_j, respectively (which are both adjacent to α). Any C graph is the collapse of some σ graph, and every δ and ε graph is the collapse of some τ graph. Hence any vertex in $\mathcal{D}_n$ has a path in the fundamental domain to α. Thus the fundamental domain is (path) connected. Explicit paths are listed in Table 4. □

α
$\rho_{jk} — \alpha$
$A_i — \alpha$
$\beta_{j,k} — \rho_{jk} — \alpha$
$\gamma_{i,jk} — \rho_{jk} — \alpha$
$B_{i,j,k} — \beta_{j,k} — \rho_{jk} — \alpha$
$\sigma_{i,jk,lm} — A_i — \alpha$
$\tau_{j,k,lm} — A_j — \alpha$
$C_{i,j,k,l,m} — \sigma_{i,jk,lm} — A_i — \alpha$
$\delta_{i,j,k,lm} — \tau_{j,k,lm} — A_j — \alpha$
$\varepsilon_{j,k,l,m} — \tau_{j,k,lm} — A_j — \alpha$

Table 4: Paths Between Vertices in the Fundamental Domain

Corollary 3.1.2. *Any vertex T in the complex $\mathcal{C}_n$ is connected via an edge path to some α graph.*

Proof. This follows by noting that T sits in at least one copy of the fundamental domain, and that the action of $\mathrm{Out}_{\mathfrak{S}}(G)$ preserves adjacency, so the above argument applies. □

In [12, Section 4], Gilbert gives a summary of Fouxe-Rabinovitch's presentation for $\mathrm{Aut}(G)$ described in [10] and [11]. Restricting to the pure symmetric automorphisms $\mathrm{Aut}_{\mathfrak{S}}(G)$ of a splitting $\mathfrak{S}$ of G, this states that $\mathrm{Aut}_{\mathfrak{S}}(G)$ is generated by factor automorphisms and Whitehead automorphisms only (see Definitions 1.1.5 and 1.1.9). We can use this to give a quick proof that our full complex $\mathcal{C}_n$ is path connected.

Proposition 3.1.3. *Any two α-graphs in the complex $\mathcal{C}_n$ are connected via a path which travels only via α and A graphs.*

Proof. Let α_0 be the α-graph in the fundamental domain $\mathcal{D}_n$ and let $\alpha_0 \cdot \hat{\psi}$ be an arbitrary α-graph in $\mathcal{C}_n$ (with $\hat{\psi} \in \mathrm{Out}_{\mathfrak{S}}(G)$ and $\psi \in \mathrm{Aut}_{\mathfrak{S}}(G)$ a representative for $\hat{\psi}$). By [10] and [11], transcribed in [12, Section 4], we can write ψ as $\psi_0\psi_1 \dots \psi_m$ for some $m \in \mathbb{N}$ where $\psi_0 \in \Phi$ is a factor automorphism, and for each $1 \leqslant i \leqslant m$, ψ_i is a Whitehead automorphism of the form (S_i, x_i) with $x_i \in G_{j(i)}$ for some $j(i) \in \{1, \dots, n\}$ and $S_i \subseteq \{G_1, \dots, G_n\} - \{G_{j(i)}\}$.

By Proposition 2.4.1, $\psi_0 \in \mathrm{Stab}(\alpha_0)$, that is, $\alpha_0 \cdot \psi_0 = \alpha_0$. We now write $\psi_1 \dots \psi_m = \psi_m(\psi_{m-1}^{\psi_m})(\psi_{m-2}^{\psi_{m-1}\psi_m}) \dots (\psi_2^{\psi_3 \dots \psi_m})(\psi_1^{\psi_2 \dots \psi_m})$. Observe that for each $1 \leqslant i < m$, $\psi_i^{\psi_{i+1} \dots \psi_m}$ acts on $\alpha_0 \cdot \psi_{i+1} \dots \psi_m$ to produce the graph $\alpha_0 \cdot \psi_i \dots \psi_m$ (and ψ_m acts on $\alpha_0 = \alpha_0 \cdot \psi_0$ producing $\alpha_0 \cdot \psi_m$). Moreover, if ψ_i has operating factor $G_{j(i)}$ and $A_{j(i)}$ is the A-graph in $\mathcal{D}_n$ whose central vertex has stabiliser $G_{j(i)}$, then by Proposition 2.4.2, $\psi_i \in \mathrm{Stab}(A_{j(i)})$ and thus $\psi_i^{\psi_{i+1} \dots \psi_m} \in \mathrm{Stab}(A_{j(i)} \cdot \psi_{i+1} \dots \psi_m)$ (and $\psi_m \in \mathrm{Stab}(A_{j(m)})$). Thus both the graphs $\alpha_0 \cdot \psi_i \dots \psi_m$ and $\alpha_0 \cdot \psi_{i+1} \dots \psi_m$ collapse to the graph $A_{j(i)} \cdot \psi_{i+1} \dots \psi_m$.

We therefore have a path $\alpha_0 — A_{j(m)} — \alpha_0 \cdot \psi_m — A_{j(m-1)} \cdot \psi_m — \alpha_0 \cdot \psi_{m-1}\psi_m — \dots — \alpha_0 \cdot \psi_2 \dots \psi_m — A_{j(1)} \cdot \psi_2 \dots \psi_m — \alpha_0 \cdot \psi_1 \dots \psi_m = \alpha_0 \cdot \psi_0\psi_1 \dots \psi_m = \alpha_0 \cdot \psi$, as required. ☐

We give an alternative proof of this in [14], which does not rely on already having a presentation for $\mathrm{Aut}_{\mathfrak{S}}(G)$ or $\mathrm{Out}_{\mathfrak{S}}(G)$, and rather uses the geometry of $\mathcal{C}_n$.

Corollary 3.1.4. *The complex $\mathcal{C}_n$ is (path) connected.*

Proof. This follows immediately by combining Corollary 3.1.2 with Proposition 3.1.3. ☐

3.2 Simple Connectivity of the Fundamental Domain

We will now show that the fundamental domain $\mathcal{D}_n$ of the space $\mathcal{C}_n$ (and hence each $\mathrm{Out}_{\mathfrak{S}}(G)$-image of $\mathcal{D}_n$) is simply connected. This will be the main result of this section, and is given as Theorem 3.2.11.

We will consider nested subcomplexes of $\mathcal{D}_n$, adding 'types' of 0-cell at each stage. We will show that the first of these subcomplexes is simply connected, and then apply a corollary of the Seifert–van Kampen Theorem to see that each successive subcomplex is also simply connected.

Corollary 3.2.1. *Let X and Y both be simply connected (simplicial) complexes. If we (suitably[2]) glue X and Y together along a path connected collection of edges, then $X \cup Y$ is simply connected.*

Proof. Note that in the complex $X \cup Y$, the subset $X \cap Y$ is precisely the collection of edges we have glued along. Since this is stipulated to be path connected, then by the

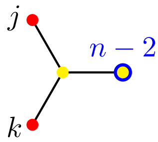
[2] i.e. so that $X \cup Y$ is still a simplicial complex

Seifert–van Kampen Theorem (Theorem 1.2.1), we have $\pi_1(X \cup Y) \cong \pi_1(X) *_{\pi_1(X\cap Y)} \pi_1(Y) = \{1\} *_{\pi_1(X\cap Y)} \{1\} = \{1\}$. □

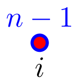
Recall that we describe a vertex $[T]$ of $\mathcal{C}_n$ as 'collapsing' to another vertex $[S]$ if a graph represented by $[T]$ has an edge (or edges) which can be collapsed to form a graph associated to $[S]$. That is, the vertices $[T]$ and $[S]$ are adjacent in $\mathcal{C}_n$.

Definition 3.2.2. Let $\mathcal{T}$ be a subset of the graph structures shown in Table 1. Denote by $\mathcal{D}_n[\mathcal{T}]$ the subcomplex of the fundamental domain of $\mathcal{C}_n$ obtained by restricting to simplices whose 0-cells are those associated to graph structures in $\mathcal{T}$.

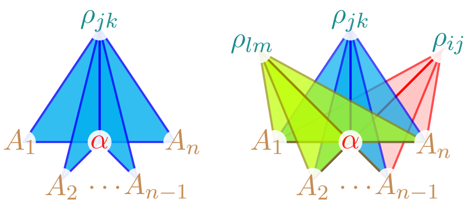
Lemma 3.2.3. *$\mathcal{D}_n[\{\alpha, \rho, A\}]$ is simply connected.*

Proof. Note that any ρ-vertex 

collapses to α, 

(by collapsing the edge whose endpoints are both trivial), and that α in turn collapses to any A-vertex 

(including $i = j$ or $i = k$). According to how we constructed the space $\mathcal{C}_n$, this means that for each pair (ρ, A) we have a 2-cell $[\rho, \alpha, A]$. So the subcomplex $\mathcal{D}_n[\{\alpha, \rho, A\}]$ comprises these 2-cells, glued along 'matching' edges. Given a particular ρ_{ij}-graph, we have a cone on a star at α (where the leaves of the star are the various A graphs). As we vary ρ, we get copies of this cone, all glued along the star formed by the α and A vertices.

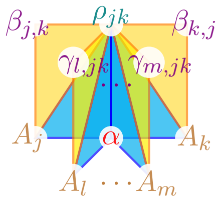


Clearly each cone is simply connected. Since the intersection of these cones is (the star based at α with leaves the A vertices, which is) path connected, we can iteratively apply Corollary 3.2.1 to 'add in' each cone (of which there are finitely many). Hence the structure $\mathcal{D}_n[\{\alpha, \rho, A\}]$ is simply connected. □

Lemma 3.2.4. *$\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma\}]$ is simply connected.*

Proof. By the previous lemma, we have that $\pi_1(\mathcal{D}_n[\{\alpha, \rho, A\}]) = \{1\}$. We will (iteratively) apply Corollary 3.2.1 to $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma\}]$ taking X to be $\mathcal{D}_n[\{\alpha, \rho, A\}]$ (or the union of this with successive Y's) and Y to be the neighbourhood in $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma\}]$ of β or γ.

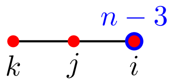
Each ρ graph collapses to 2 β graphs, and $n-2$ γ graphs. These βs and γs are unique to the given ρ (that is, two distinct ρ graphs cannot both collapse to the same β or γ. Specifically, ρ_{jk} collapses to $\beta_{j,k}$, $\beta_{k,j}$ and $\gamma_{i,jk}$ for $i \neq j, k$. In turn, $\beta_{j,k}$ collapses to A_j, $\beta_{k,j}$ to A_k, and $\gamma_{i,jk}$ to A_i. Thus the neighbourhood of $\beta_{j,k}$ (or $\gamma_{i,jk}$) is a 2-cell $[\rho_{jk}, \beta_{j,k}, A_j]$ (or $[\rho_{jk}, \gamma_{i,jk}, A_i]$, respectively). Note that any 2-cell is simply connected. The intersection of each of these neighbourhoods with any of the spaces X is an edge ρ — A (which is, in particular, path connected). So by Corollary 3.2.1, $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma\}]$ is simply connected.

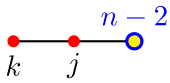


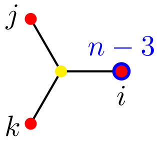
□

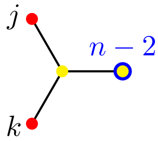
We now consider how to attach the B-vertices.

Lemma 3.2.5. *$\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B\}]$ is simply connected.*

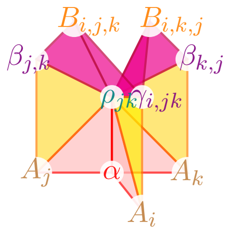
Proof. Each $B_{i,j,k}$ 

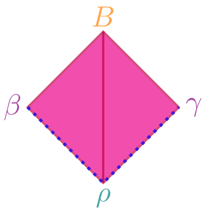
grows to a unique $\beta_{j,k}$ 

, a unique $\gamma_{i,jk}$ 

, and a unique ρ_{jk} 

. So $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B\}]$ is the subcomplex $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma\}]$ with the addition of 2-cells $[\rho_{jk}, \beta_{j,k}, B_{i,j,k}]$ and $[\rho_{jk}, \gamma_{i,jk}, B_{i,j,k}]$ for all possible values of i, j and k (by gluing along the ρ-β and ρ-γ edges that already exist, and additionally gluing our two 2-cells along the ρ-β edge they both share):

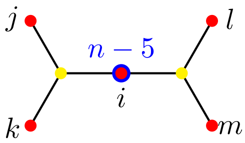


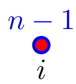
That is, given a specific ρ-vertex in our structure, for every β and γ we see adjacent to said ρ, we glue in a 'fin' 

along the dotted line. The edge path $\beta — \rho — \gamma$ is path connected, so by repeated applications of Corollary 3.2.1, we see that $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B\}]$ is simply connected. □

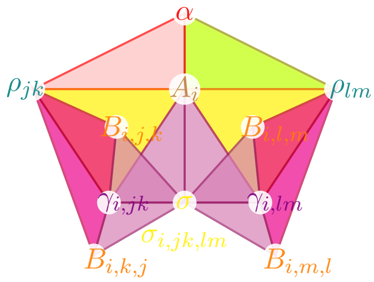
Lemma 3.2.6. *$\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma\}]$ is simply connected.*

Proof. Given a vertex $\sigma_{i,jk,lm}$ in $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma\}]$, with associated graph



, we see there is precisely one A-vertex it is adjacent to, 

. We also see that $\sigma_{i,jk,lm}$ collapses to two γ-vertices, $\gamma_{i,jk}$ and $\gamma_{i,lm}$. Further, $\gamma_{i,jk}$ collapses to both $B_{i,j,k}$ and $B_{i,k,j}$ (similarly for $\gamma_{i,lm}$). To create $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma\}]$, in $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B\}]$ at any A_i-vertex we attach two 2-cells $[\sigma_{i,jk,lm}, \gamma_{i,jk}, A_i]$ and $[\sigma_{i,jk,lm}, \gamma_{i,lm}, A_i]$ (glued to each other along the shared edge A_i — $\sigma_{i,jk,lm}$) wherever we see two vertices $\gamma_{i,jk}$ and $\gamma_{i,lm}$ adjacent to A_i with j, k, l and m (and i) distinct. We then glue in additional 2-cells of the form $[\sigma, \gamma, B]$ wherever we see a path σ — γ — B.

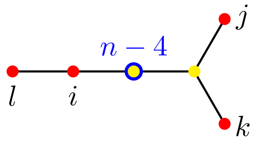


That is, each $\sigma_{i,jk,lm}$ has a simply connected neighbourhood, and the intersection of this neighbourhood with $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B\}]$ is the collection of edges $\gamma_{i,jk}$ — $B_{i,j,k}$, $\gamma_{i,jk}$ — $B_{i,k,j}$, $\gamma_{i,lm}$ — $B_{i,l,m}$, $\gamma_{i,lm}$ — $B_{i,m,l}$, $\gamma_{i,jk}$ — A_i, and $\gamma_{i,lm}$ — A_i, which is path conected. The result follows from Corollary 3.2.1. □

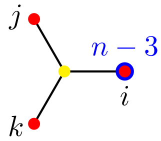
The process of adding in τ-vertices to our structure will be very similar to that for σ-vertices.

Lemma 3.2.7. *$\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau\}]$ is simply connected.*

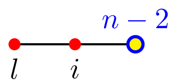
Proof. The vertex $\tau_{i,l,jk}$ in $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau\}]$ with graph



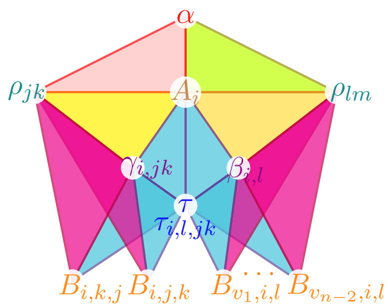
collapses to $\gamma_{i,jk}$



and $\beta_{i,l}$



. In turn, $\beta_{i,l}$ and $\gamma_{i,jk}$ both collapse to A_i. Additionally $\gamma_{i,jk}$ collapses to $B_{i,j,k}$ and $B_{i,k,j}$, and $\beta_{i,l}$ collapses to $n-2$ vertices of the form $B_{v,i,l}$ (for $v \neq i, l$). and in our structure so far, wherever we see a path $\gamma_{i,jk}$ — A_i — $\beta_{i,l}$ with i, j, k, l distinct, we glue in a pair of 2-cells of the form $[\tau, \gamma, A]$ and $[\tau, \beta, A]$ (glued together along their common edge A — τ). As before with σ, we must also glue in all possible 2-cells of the form $[\tau, \gamma, B]$ and $[\tau, \beta, B]$ as determined by the relative collapses of γ and β.

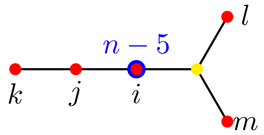


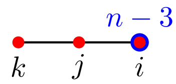
We have then identified the neighbourhood of $\tau_{i,l,jk}$ inside $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau\}]$, and found the intersection of this neighbourhood with $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma\}]$. By the previous lemma, $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma\}]$ is simply connected, and clearly the neighbourhood of τ (shown in pale blue) is simply connected. Moreover, the intersection of these subsets is the collection of edges $\gamma_{i,jk}$ — A_i, $\gamma_{i,jk}$ — $B_{i,j,k}$, $\gamma_{i,jk}$ — $B_{i,k,j}$, $\beta_{i,l}$ — A_i,

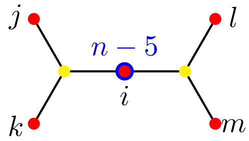
$\beta_{i,l} — B_{v,i,l}$ (for $v \neq i, j, k, l$). Since this is path connected, then by Corollary 3.2.1, $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau\}]$ is simply connected. $\square$

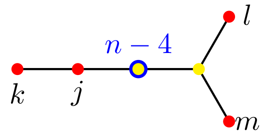
Lemma 3.2.8. *$\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta\}]$ is simply connected.*

Proof. Each δ in $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta\}]$ collapses to a unique B and 'grows' to a unique σ and a unique τ.

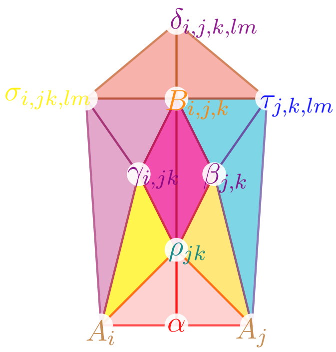
Precisely, $\delta_{i,j,k,lm}$ 

collapses to $B_{i,j,k}$ 

and grows to $\sigma_{i,jk,lm}$ 

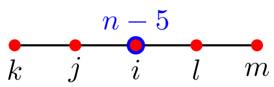
and $\tau_{j,k,lm}$ 

. As such, we see that a given $\sigma_{i,jk,lm}$ and $\tau_{j,k,lm}$ are both adjacent to $B_{i,j,k}$ (and share no other common B adjacency). So whenever we see a path $\sigma_{i,jk,lm} — B_{i,j,k} — \tau_{j,k,lm}$ in our structure, we will glue in 2-cells $[\sigma, \delta, B]$ and $[\tau, \delta, B]$ (gluing them along their shared δ-B edge).

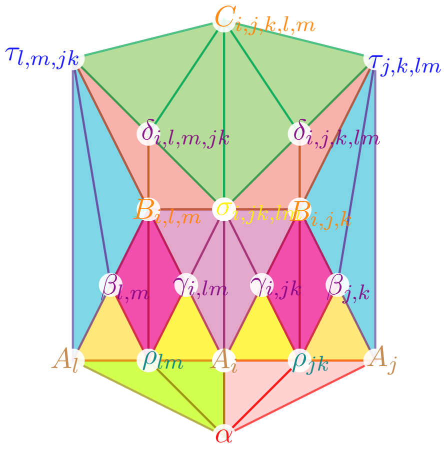


That is, each $\delta_{ij,k,lm}$ has a neighbourhood in $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta\}]$ comprising two 2-cells glued along a single edge. The intersection of this neighbourhood with $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau\}]$ is the edge path $\sigma_{i,jk,lm} — B_{i,j,k} — \tau_{j,k,lm}$. So successive applications of Corollary 3.2.1 tells us that $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta\}]$ is simply connected. $\square$

Lemma 3.2.9. *$\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta, C\}]$ is simply connected.*

Proof. Each $C_{i,j,k,l,m}$ 

grows to $\delta_{i,j,k,lm}$, $\delta_{i,l,m,jk}$, $\tau_{l,m,jk}$, $\tau_{j,k,lm}$, and $\sigma_{i,jk,lm}$.
So the neighbourhood of $C_{i,j,k,l,m}$ inside $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta, C\}]$ is four 2-cells $[\tau_{l,m,jk}, \delta_{i,l,m,jk}, C_{i,j,k,l,m}]$, $[\sigma_{i,jk,lm}, \delta_{i,l,m,jk}, C_{i,j,k,l,m}]$, $[\tau_{j,k,lm}, \delta_{i,j,k,lm}, C_{i,j,k,l,m}]$, and $[\sigma_{i,jk,lm}, \delta_{i,j,k,lm}, C_{i,j,k,l,m}]$, glued along their common edges.
The intersection of this neighbourhood with $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta\}]$ is the edge path $\tau_{l,m,jk} — \delta_{i,l,m,jk} — \sigma_{i,jk,lm} — \delta_{i,j,k,lm} — \tau_{j,k,lm}$.
By repeated applications of Corollary 3.2.1 (and by the previous lemma), $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta, C\}]$ is simply connected.

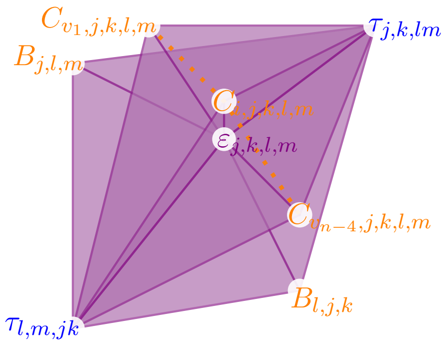


□

Finally, we add ε-vertices to our complex.

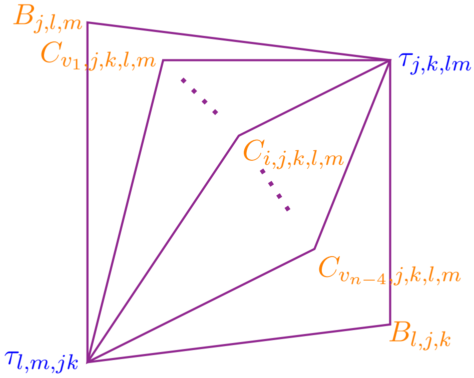
Lemma 3.2.10. *$\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta, C, \varepsilon\}]$ is simply connected.*

Proof. Note that the subcomplex neighbourhood around $\varepsilon_{j,k,l,m}$ in our fundamental domain (equal to $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta, C, \varepsilon\}]$) is:



That is, $\varepsilon_{j,k,l,m}$ grows to $\tau_{j,k,lm}$ and $\tau_{l,m,jk}$ and collapses to $B_{j,l,m}$, $B_{l,j,k}$, and $n-4$ vertices of the form $C_{v,j,k,l,m}$ (for $v \neq j,k,l,m$).

The intersection of this neighbourhood with $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta, C\}]$ is the boundary of the neighbourhood.



Since this is path connected, Corollary 3.2.1 applies, and $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta, C, \varepsilon\}]$ is simply connected. □

Finally, we have proved:

Theorem 3.2.11. *The fundamental domain $\mathcal{D}_n$ of the complex $\mathcal{C}_n$ is simply connected.*

Proof. By Lemma 3.2.10, $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta, C, \varepsilon\}]$ is simply connected. But $\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta, C, \varepsilon\}$ covers all of the graph structures in Table 1.
So by Definitions 3.2.2 and 2.3.5 $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta, C, \varepsilon\}]$ is precisely the fundamental domain $\mathcal{D}_n$ of $\mathcal{C}_n$. □

4 A Presentation for $\mathrm{Out}_{\mathfrak{S}}(G)$

This section is the main result of the paper.

We recall from Section 1 the theorem of Brown [6] which we will use to determine a presentation for $\mathrm{Out}_{\mathfrak{S}}(G)$ (where $G = G_1 * \cdots * G_n$, $\mathfrak{S} = (G_1, \ldots, G_n)$, and $\mathcal{G} = \mathrm{Out}_{\mathfrak{S}}(G)$):

Theorem 1.2.2 (Brown [6, Theorem 3])**.** *Let $\mathcal{G}$ act on a simply connected $\mathcal{G}$-CW complex X (without inversion on the 1-cells of X). Suppose there is a subcomplex W of X so that every cell of X is equivalent under the action of $\mathcal{G}$ to a unique cell of W. Then $\mathcal{G}$ is generated by the isotropy subgroups $\mathcal{G}_v$ ($v \in V(W)$) subject to edge relations $\iota_{o(e)}(g) = \iota_{t(e)}(g)$ for all $g \in \mathcal{G}_e$ ($e \in E(W)$) (where for any $e \in E(W)$, $\iota_{o(e)} : \mathcal{G}_e \to \mathcal{G}_{o(e)}$ and $\iota_{t(e)} : \mathcal{G}_e \to \mathcal{G}_{t(e)}$ are inclusions).*

The complex X we will use is $\mathcal{C}_n$, and the subcomplex W is $\mathcal{D}_n$. Since $\mathcal{C}_3$ and $\mathcal{C}_4$ are just the barycentric spine of Guirardel and Levitt's Outer Space (for $n = 3$ and $n = 4$ respectively), results satisfying the restrictions on X and W are assumed from [13]. It is highly non-trivial to show that $\mathcal{C}_n$ is simply connected for $n \geqslant 5$, so we delay the proof of this to Sections 5 and 6. The required result here is:

Corollary 6.3.3. *The space $\mathcal{C}_n$ (for $n \geqslant 5$) is simply connected.*

That $\mathcal{D}_n$ for $n \geqslant 5$ satisfies the strictness condition on W is the result of Propositions 2.3.8 and 2.3.9.

The isotropy subgroups $\mathcal{G}_v$ here are the vertex stabilisers $\mathrm{Stab}(T)$ for $T \in \mathcal{D}_n^{(0)}$, which are detailed in Propositions 2.4.1, 2.4.2, and 2.4.3.

Note that Lemma 2.3.7 implies that the edge relations in Brown's Theorem become: '$g = \iota_{o(e)}^{-1}\left(\iota_{t(e)}(g)\right)$ for all $g \in \mathcal{G}_{o(e)}$' (that is, that vertex stabilisers $\mathrm{Stab}(T)$ are identified with their natural images under inclusion in the stabilisers $\mathrm{Stab}(S)$ of any vertices S to which the original vertex T collapses).

Notation 4.0.1. We summarise the notational shorthand we have adopted thus far:

$[A, B] = 1$**:** For subgroups A and B, 'A commutes with B', in the sense that for all $a \in A$ and for all $b \in B$ we have $ab = ba$.

G_{i_j}**:** The group of (outer) automorphisms which act by conjugating all elements of the factor group G_j by an element of the factor group G_i.

f_{i_j}**:** The isomorphism $f_{i_j} : G_i \to G_{i_j}$ which maps an element $g \in G_i$ to the element in G_{i_j} which conjugates each element of G_j by g.

$G_{i_{v_1\dots v_k}}$: The group $\left\{ \left(f_{i_{v_1}}(g_i), \dots, f_{i_{v_k}}(g_i) \right) \middle| g_i \in G_i \right\}$ which is the diagonal subgroup of $f_{i_{v_1}}(G_i) \times \cdots \times f_{i_{v_k}}(G_i) = G_{i_{v_1}} \times \cdots \times G_{i_{v_k}}$.
We often denote this by $G_{i_{v_1\dots v_k}} \diamondsuit G_{i_{v_1}} \times \cdots \times G_{i_{v_k}}$.

$Z(G_i)$: The centre of the group G_i, i.e. the subgroup $\{g \in G_i \mid gh = hg \ \forall h \in G_i\}$. We will often identify $Z(G_i)$ with its images $f_{i_j}(Z(G_i))$.

$\mathrm{Aut}(G_i)$: Often considered to be the subgroup $\{(1, \dots, 1, \mathrm{Aut}(G_i), 1, \dots, 1)\}$ of $\Phi = \prod_{j=1}^{n} \mathrm{Aut}(G_j)$.

$\mathrm{Ad}_{G_i}(g)$: The element of $\mathrm{Inn}(G_i)$ which conjugates each element of G_i by g (where $g \in G_i$).

$G_{i_j}^{\varphi}$: The group of automorphisms $\left\{ \varphi \circ f_{i_j}(g) \circ \varphi^{-1} \ \middle| g \in G_i \right\}$ (where $\varphi \in \Phi$).

$\varphi(G_{i_j})$: The group of automorphisms $\left\{ f_{i_j}(\varphi(g)) \middle| g \in G_i \right\}$ (where $\varphi \in \Phi$).

We now split into cases dependent on the number n of factors in our splitting $G = G_1 * \cdots * G_n$.

4.1 The Case $n \geqslant 5$

We have all the pieces required to build our presentation for $\mathrm{Out}_{\mathfrak{S}}(G)$.

Theorem 4.1.1. *Let $G_1 * \cdots * G_n$ be a free splitting of a group G where each G_i is non-trivial and $n \geqslant 5$. For $i \in [n] := \{1, \dots, n\}$ and $j \in [n] - \{i\}$, let $f_{i_j} : G_i \to G_{i_j}$ be group isomorphisms, and for $g \in G_i$ let $\mathrm{Ad}_{G_i}(g)$ be the inner automorphism $x \mapsto gxg^{-1}$ of G_i. Then the group $\mathrm{Out}(G; G_1, \dots, G_n)$ is generated by the $n(n-1)$ groups $G_{i_j} \cong G_i$ and $\Phi = \prod_{i=1}^{n} \mathrm{Aut}(G_i)$, subject to relations:*

1. $[f_{i_j}(g), f_{i_k}(h)] = 1 \ \forall g, h \in G_i$, *for all* $i \in [n]$, $j, k \in [n] - \{i\}$
2. $[f_{i_j}(g), f_{k_l}(h)] = 1 \ \forall g \in G_i, h \in G_k$, *for all distinct* $i, j, k, l \in [n]$
3. $[f_{j_k}(g), f_{i_j}(h) f_{i_k}(h)] = 1 \ \forall g \in G_j, h \in G_i$, *for all distinct* $i, j, k \in [n]$
4. $f_{i_{v_1}}(g) \dots f_{i_{v_{n-1}}}(g) = \mathrm{Ad}_{G_i}(g^{-1}) \ \forall g \in G_i$, *for all* $i \in [n]$ *and* $\{v_1, \dots, v_{n-1}\} = [n] - \{i\}$
5. $\varphi^{-1} f_{i_j}(g) \varphi = f_{i_j}(g\varphi) \ \forall g \in G_i$, *for all distinct* $i, j \in [n]$ *and all* $\varphi \in \Phi$

As well as all relations in G and Φ.

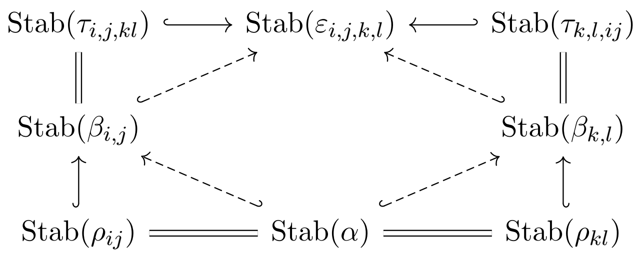
Proof. We apply Brown's Theorem (Theorem 1.2.2) to the fundamental domain $\mathcal{D}_n$ of the action of $\mathrm{Out}_{\mathfrak{S}}(G)$ on $\mathcal{C}_n$. As previously noted, Proposition 2.3.9 states that the strictness requirement on $\mathcal{D}_n$ to apply Brown's Theorem is satisfied. We also require $\mathcal{C}_n$ to be simply connected. We delay the proof of this until after this section. The desired result here is Corollary 6.3.3.

We now have that $\mathrm{Out}_{\mathfrak{S}}(G)$ is generated by $\left\{ \mathrm{Stab}(T) \middle| T \in \mathcal{D}_n^{(0)} \right\}$, such that if $[S, T]$ is an edge in $\mathcal{D}_n^{(1)}$ (i.e. $S, T \in \mathcal{D}_n^{(0)}$ with T a collapse of S) then we have inclusions $\mathrm{Stab}(S) \hookrightarrow \mathrm{Stab}(T)$. We use the descriptions of $\mathrm{Stab}(T)$ from Propositions 2.4.1, 2.4.2, and 2.4.3. We proceed by examining the structure of $\mathcal{D}_n^{(1)}$. Recall that graph shapes for $T \in \mathcal{D}_n^{(0)}$ are listed in Table 1.

We first observe that every ρ_{ij} collapses to α. Since abstractly, $\mathrm{Stab}(\rho_{ij}) = \mathrm{Stab}(\alpha)$ (Proposition 2.4.1), then each $\mathrm{Stab}(\rho_{ij})$ is identified with $\mathrm{Stab}(\alpha) = \Phi$ in $\mathrm{Out}_{\mathfrak{G}}(G)$. Similarly, for each i, j we have that every $\mathrm{Stab}(\tau_{i,j,kl})$ ($k, l \in [n] - \{i, j\}$) is identified with $\mathrm{Stab}(\beta_{i,j})$ in $\mathrm{Out}_{\mathfrak{G}}(G)$.

Since ρ_{jk} collapses to $\beta_{j,k}$, $\gamma_{i,jk}$, A_v, and $B_{i,j,k}$ (for any $v \in \{1, \ldots, n\}$ and any $i \notin \{j, k\}$), we immediately deduce that the Φ contribution from any $\mathrm{Stab}(\beta)$, $\mathrm{Stab}(\gamma)$, $\mathrm{Stab}(A)$, or $\mathrm{Stab}(B)$ is identified with $\mathrm{Stab}(\alpha)$.

We now consider the τ-ε–Square (Figure 4). Note that for each $\varepsilon_{i,j,k,l}$ there are precisely two τ graphs, $\tau_{i,j,kl}$ and $\tau_{k,l,ij}$, which collapse to $\varepsilon_{i,j,k,l}$. We may 'replace' $\tau_{i,j,kl}$ with $\beta_{i,j}$ and $\tau_{k,l,ij}$ with $\beta_{k,l}$, and recalling that ρ_{ij} collapses to $\beta_{i,j}$ and ρ_{kl} to $\beta_{k,l}$, also 'replace' ρ_{ij} and ρ_{kl} with α. We thus have a diagram:

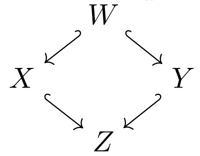


where the dashed inclusions are naturally induced by the 'replacements' we made. Note that $\mathrm{Stab}(\beta_{i,j})$ and $\mathrm{Stab}(\beta_{k,l})$ 'cover' $\mathrm{Stab}(\varepsilon_{i,j,k,l})$ in the sense that $\mathrm{Stab}(\varepsilon_{i,j,k,l}) \subseteq \mathrm{Stab}(\beta_{i,j}) \times \mathrm{Stab}(\beta_{k,l})$. Since $\mathrm{Stab}(\beta_{i,j}) \cap \mathrm{Stab}(\beta_{k,l}) = \Phi = \mathrm{Stab}(\alpha)$, the dashed inclusions form something akin to a pushout diagram, and we may conclude that $\mathrm{Stab}(\varepsilon) = \mathrm{Stab}(\beta_{i,j}) \times_\Phi \mathrm{Stab}(\beta_{k,l})$.

Next, we consider $\mathrm{Stab}(A_i)$. Observe that given $i \in [n]$, every $\beta_{i,j}$ for $j \in [n] - \{i\}$ collapses to A_i, thus we have $n-1$ inclusions $\mathrm{Stab}(\beta_{i,j}) \hookrightarrow \mathrm{Stab}(A_i)$. Then the G_{i_j} contribution from $\mathrm{Stab}(A_i)$ is identified in $\mathrm{Out}_{\mathfrak{G}}(G)$ with the G_{i_j} contribution from $\mathrm{Stab}(\beta_{i,j})$, and we can consider $\mathrm{Stab}(A_i)$ to be generated by $\mathrm{Stab}(\beta_{i v_1}) \times_\Phi \cdots \times_\Phi \mathrm{Stab}(\beta_{i v_{n-1}})$ subject to the relations in Proposition 2.4.2, as well as the relation $Z(G_{i_{v_1 \cdots v_{n-1}}}) = \{1\}$ (since the $\beta_{i,j}$'s 'cover' A_i).

A similar principle applies to $\mathrm{Stab}(\gamma_{i,jk})$. Given $i, j, k \in [n]$, we have that for any $l \in [n] - \{i, j, k\}$, the graph $\tau_{i,l,jk}$ collapses to $\gamma_{i,jk}$. Noting that $\mathrm{Stab}(\tau_{i,l,jk})$ is identified in $\mathrm{Out}_{\mathfrak{G}}(G)$ with $\mathrm{Stab}(\beta_{i,j})$, we have $n-3$ inclusions of the form $\mathrm{Stab}(\beta_{i,l}) \hookrightarrow \mathrm{Stab}(\gamma_{i,jk})$. Recall from Proposition 2.4.2 that abstractly, $\mathrm{Stab}(\gamma_{i,jk})$ is generated by $G_{i_{jk}} \times G_{i_{l_1}} \times \cdots \times G_{i_{l_{n-3}}} \big/ Z(G_i)$ and Φ. However, by manipulation of relations in $\mathrm{Stab}(\gamma_{i,jk})$ (or by considering the Guirardel–Levitt approach to computing stabilisers), we have that the $G_{i_{jk}}$ component is redundant as a generator. Specifically, for $f_{i_{jk}}(g_i) \in G_{i_{jk}}$ (with $g_i \in G_i$), we have that $f_{i_{jk}}(g_i) = \iota_{g_i} f_{i_{l_1 \ldots l_{n-3}}}(g_i^{-1}) f_i(g_i^{-1})$, where $\iota_{g_i} \in \mathrm{Inn}(G)$ conjugates every element of G by g_i, and $f_i : G_i \to \mathrm{Inn}(G_i)$ is the canonical homomorphism. Thus we can consider $\mathrm{Stab}(\gamma_{i,jk})$ to be generated by $\mathrm{Stab}(\beta_{i l_1}) \times_\Phi \cdots \times_\Phi \mathrm{Stab}(\beta_{i l_{n-3}})$ (with relations similar to $\mathrm{Stab}(A_i)$).

Given a 'top' vertex in $\mathcal{D}_n^{(0)}$ (i.e. a ρ, σ, or τ graph), we can reach a 'bottom' vertex (A, B, or C graph) by successively collapsing two edges. By changing the order in which we collapse these edges, we produce square (or 'diamond') diagrams of inclusions



where so long as X and Y 'cover' Z (in the sense that

$\mathrm{Stab}(Z) \subseteq \mathrm{Stab}(X) \times \mathrm{Stab}(Y)$), we will have $\mathrm{Stab}(Z) = \mathrm{Stab}(X) \times_{\mathrm{Stab}(W)} \mathrm{Stab}(Y)$, where $\mathrm{Stab}(W) = \mathrm{Stab}(X) \cap \mathrm{Stab}(Y)$. Figure 5 illustrates some such diagrams which are of particular use.

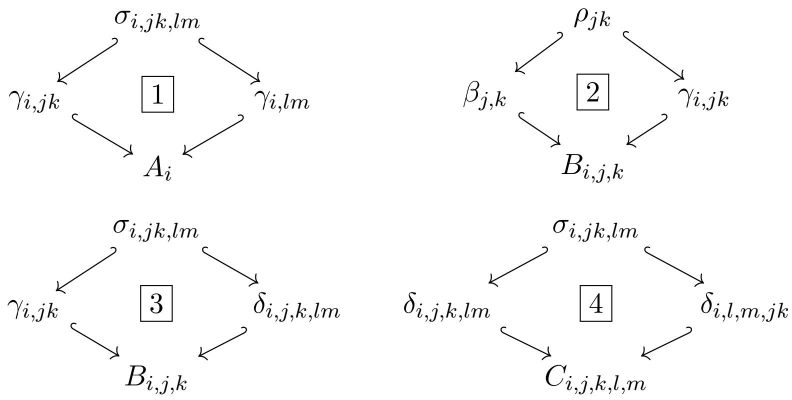


Figure 5: Inclusion Diagrams in $\mathcal{D}_n^{(1)}$

Diagram $\boxed{1}$ may be thought of as akin to a pullback diagram, in that $\sigma_{i,jk,lm}$ is uniquely determined by $\gamma_{i,jk}$ and $\gamma_{i,lm}$. Since $\mathrm{Stab}(A_i) \subseteq \mathrm{Stab}(\gamma_{i,jk}) \times \mathrm{Stab}(\gamma_{i,lm})$, we deduce that $\mathrm{Stab}(\sigma_{i,jk,lm}) = \mathrm{Stab}(\gamma_{i,jk}) \cap \mathrm{Stab}(\gamma_{i,lm})$ in $\mathrm{Out}_{\mathfrak{G}}(G)$. The remaining diagrams are more akin to pushouts than pullbacks.

Diagram $\boxed{2}$ is from the ρ–Book of Figure 3, and is enough to uniquely determine a given $B_{i,j,k}$. Recalling that $\mathrm{Stab}(\rho_{jk}) = \mathrm{Stab}(\alpha) = \Phi$ in $\mathrm{Out}_{\mathfrak{G}}(G)$, we conclude that $\mathrm{Stab}(B_{i,j,k}) = \mathrm{Stab}(\beta_{j,k}) \times_\Phi \mathrm{Stab}(\gamma_{i,jk})$. Diagram $\boxed{3}$ implies that $\mathrm{Stab}(B_{i,j,k}) = \mathrm{Stab}(\delta_{i,j,k,lm}) \times_{\mathrm{Stab}(\sigma_{i,jk,lm})} \mathrm{Stab}(\gamma_{i,jk})$, thus $\mathrm{Stab}(\beta_{j,k}) \times_\Phi \mathrm{Stab}(\gamma_{i,jk}) =$ $\mathrm{Stab}(\delta_{i,j,k,lm}) \times_{\mathrm{Stab}(\sigma_{i,jk,lm})} \mathrm{Stab}(\gamma_{i,jk})$. From this, we deduce that $\mathrm{Stab}(\delta_{i,j,k,lm}) = \mathrm{Stab}(\beta_{j,k}) \times_\Phi \mathrm{Stab}(\sigma_{i,jk,lm})$.

Diagram $\boxed{4}$ is from the σ–Slice of Figure 4, and is enough to uniquely determine a given $C_{i,j,k,l,m}$. We then have that:

$$\begin{aligned}\mathrm{Stab}(C_{i,j,k,l,m}) &= \mathrm{Stab}(\delta_{i,j,k,lm}) \times_{\mathrm{Stab}(\sigma_{i,jk,lm})} \mathrm{Stab}(\delta_{i,l,m,jk}) \\ &= (\mathrm{Stab}(\beta_{jk}) \times_\Phi \mathrm{Stab}(\sigma_{i,jk,lm})) \times_{\mathrm{Stab}(\sigma_{i,jk,lm})} (\mathrm{Stab}(\beta_{l,m}) \times_\Phi \mathrm{Stab}(\sigma_{i,jk,lm})) \\ &= \mathrm{Stab}(\beta_{j,k}) \times_\Phi \mathrm{Stab}(\beta_{l,m}) \times_\Phi \mathrm{Stab}(\sigma_{i,jk,lm})\end{aligned}$$

We have now shown that any $\mathrm{Stab}(T)$ for $T \in \mathcal{D}_n^{(0)}$ can be written in terms of $\Phi = \mathrm{Stab}(\alpha)$ and $\mathrm{Stab}(\beta_{i,j})$ (allowing i and j to vary over $\{1, \ldots, n\}$). Thus $\mathrm{Out}_{\mathfrak{G}}(G)$ is generated by $\{\mathrm{Stab}(\beta_{i,j}) \,|\, i \in [n], j \in [n] - \{i\}\}$, that is, $\mathrm{Out}_{\mathfrak{G}}(G)$ is generated by $\left\{ G_{i_j} \,\middle|\, i \in \{1, \ldots, n\}, j \in \{1, \ldots, n\} - \{i\} \right\} \cup \Phi$.

From $\mathrm{Stab}(A_i)$ (Proposition 2.4.2), we see that $[G_{i_j}, G_{i_k}] = 1$ and ${G_{i_{v_1 \cdots v_{n-1}}}}/{Z(G_i)} = \mathrm{Inn}(G_i) \cong {G_i}/{Z(G_i)}$. Using the isomorphisms f_{i_j} for preciseness, we recover Relations 1 and 4. Note that if $g \in Z(G_i)$ then Relation 4 gives $f_{i_{v_1}}(g) \ldots f_{i_{v_{n-1}}}(g) = 1$. We deduce from $\mathrm{Stab}(\varepsilon_{i,j,k,l})$ (Proposition 2.4.1) that $[G_{i_j}, G_{k_l}] = 1$, from $\mathrm{Stab}(B_{i,j,k})$ (Proposition 2.4.3) that $[G_{j_k}, G_{i_{jk}}] = 1$, and from $\mathrm{Stab}(\beta_{i,j})$ (Proposition 2.4.1) that $G_{i_j}^{\varphi^{-1}} = \varphi(G_{i_j})$. We now recover Relations 2,3, and 5 by substituting the appropriate isomorphisms f_{i_j} into the above formulae. All other relations found in vertex stabilisers are subsumed by these five. □

Remark. Note that n of these generators are 'redundant', in that for each $i \in [n]$ and $j \in [n] - \{i\}$, $G_{i_j} \leqslant (G_{i_{v_1}} \times \cdots \times G_{i_{v_{n-5}}}) \rtimes \Phi$. However, consistently choosing generators to remove without overcomplicating the relations is tricky, so we elect not to do this.

Corollary 4.1.2. *If a group G splits as a free product where the factor groups are non-trivial, freely indecomposable, not infinite cyclic, and pairwise non-isomorphic, then Theorem 4.1.1 gives a presentation for* $\mathrm{Out}(G)$*.*

Proof. Note that the splitting $G_1 * \cdots * G_n$ described is a Grushko decomposition for G, and so every automorphism must preserve the conjugacy classes of the factor groups. That is, $\mathrm{Out}(G; G_1, \ldots, G_n) = \mathrm{Out}(G)$. □

4.2 The Case $n = 4$

Let $G = G_1 * G_2 * G_3 * G_4$ be a free splitting of a group G, and let $\mathfrak{S} = (G_1, G_2, G_3, G_4)$.

By Definition 2.0.1, our complex $\mathcal{C}_4$ is the barycentric spine of Guirardel and Levitt's Outer Space relative to $\mathfrak{S}$, which we build by taking only the graph shapes from Table 1 which have at most four non-trivial (red) vertices. This leaves us with graph shapes ρ, α, β, γ, A, and B. Note however that for $\{i,j,k,l\} = \{1,2,3,4\}$, we have $\gamma_{i,kl} = \beta_{i,j}$. Also, $\rho_{ij} = \rho_{kl}$ and $B_{i,j,k} = B_{j,i,l}$. Note additionally that we did not take τ or ε graphs (despite these only displaying four non-trivial vertices) since the trivial 'basepoint' must have valency at least three here, implying at least one suppressed non-trivial vertex in each case.

Thus the vertex set of the fundamental domain $\mathcal{D}_4^{(0)}$ of $\mathcal{C}_4$ consists of $\frac{1}{2}\binom{4}{2} = 3$ ρ vertices, 1 α vertex, $4 \times 3 = 12$ β vertices, 4 A vertices, and $\frac{4!}{2} = 12$ B vertices. These form three ρ_{ij} 'spikes' (for $\{i,j\} \subseteq \{1,2,3,4\}$) in $\mathcal{D}_4^{(1)}$, shown in Figure 6, which are

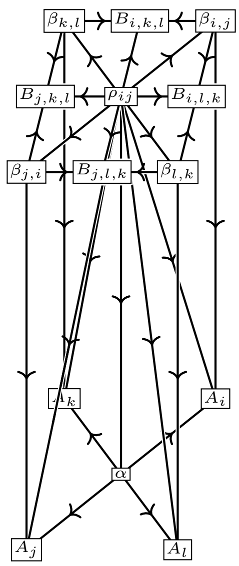
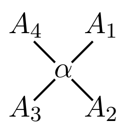


Figure 6: A 'Spike' in $\mathcal{D}_4^{(1)}$

identified along the α-A–Star



.

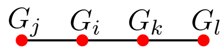
Proposition 4.2.1. *We have the following for $\{i,j,k,l\} = \{1,2,3,4\}$:*

1. $\mathrm{Stab}(\rho_{ij}) = \mathrm{Stab}(\alpha) = \Phi$

2. $\mathrm{Stab}(\beta_{i,j}) = G_{i_j} \rtimes \Phi$

3. $\mathrm{Stab}(A_i)$ *is generated by* $\left.(G_{i_j} \times G_{i_k} \times G_{i_l})\middle/ Z(G_i)\right.$ *and* Φ *with relations* $\left.G_{i_{jkl}}\middle/ Z(G_i)\right. = \mathrm{Inn}(G_i)$ *and* $(\{G_a\}, g_i) \circ \varphi = \varphi \circ (\{G_a\}, (g_i)\varphi)$ *for* $\varphi \in \Phi$ *and* $(\{G_a\}, g_i) \in i_a$ *for each* $a \in \{j, k, l\}$

4. $\mathrm{Stab}(B_{i,k,l}) = (G_{i_j} \times G_{k_l}) \rtimes \Phi$

Proof. The first three items are the same as the presentations listed for the $n \geqslant 5$ case. The arguments there also hold for $n = 4$. We use the Guirardel–Levitt approach (see Example 4) to compute $\mathrm{Stab}(B_{i,k,l})$, noting that $B_{i,k,l} =$ 

and fixing the edge between vertex groups G_i and G_k. □

Theorem 4.2.2. *Let* $G_1 * G_2 * G_3 * G_4$ *be a free splitting of a group* G *where each* G_i *is non-trivial, and let* $\mathfrak{G} = (G_1, G_2, G_3, G_4)$*. Writing* f_{i_j} *for the isomorphism* $G_i \to G_{i_j}$*,* $\mathrm{Out}_{\mathfrak{G}}(G)$ *is generated by the twelve groups* G_{i_j} *for* $i, j \in \{1, 2, 3, 4\}$ *distinct, and* $\Phi = \prod_{k=1}^{4} \mathrm{Aut}(G_k)$*, subject to relations:*

1. $\left[f_{i_j}(g), f_{i_k}(h)\right] = 1 \ \forall g, h \in G_i$

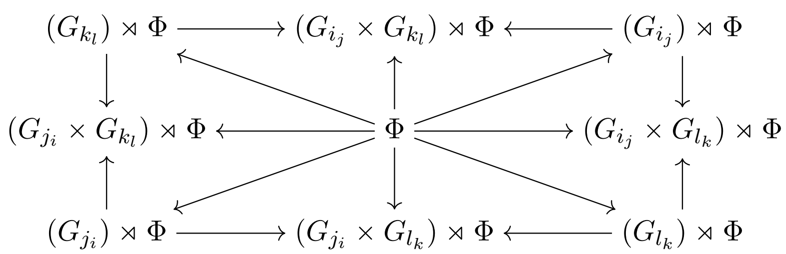
2. $\left[f_{i_j}(g), f_{k_l}(h)\right] = 1 \ \forall g \in G_i, h \in G_k$

3. $f_{i_j}(g) f_{i_k}(g) f_{i_l}(g) = \mathrm{Ad}_{G_i}(g^{-1}) \ \forall g \in G_i$

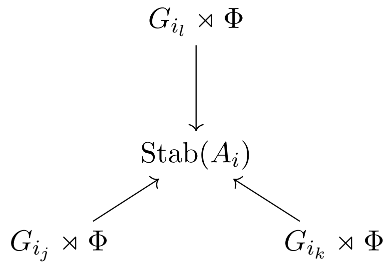
4. $\varphi^{-1} f_{i_j}(g) \varphi = f_{i_j}((g)\varphi) \ \forall g \in G_i$ *for all* $\varphi \in \Phi$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$*.*

Proof. We apply Theorem 1.2.2 to the complex $\mathcal{C}_4$. Note that Lemma 2.0.2 tells us $\mathcal{C}_4$ is simply connected, and it is not hard to deduce that the fundamental domain $\mathcal{D}_4$ meets the strictness requirements here. Then $\mathrm{Out}_{\mathfrak{G}}(G)$ is generated by the stabiliser groups described in Proposition 4.2.1 for each combination of $\{i, j, k, l\} = \{1, 2, 3, 4\}$. We now consider the edge relations from $\mathcal{D}_4$. Since each ρ_{ij} collapses to α, we have that $\mathrm{Stab}(\rho_{12})$, $\mathrm{Stab}(\rho_{13})$, $\mathrm{Stab}(\rho_{14})$, and $\mathrm{Stab}(\alpha)$ all generate the same subgroup Φ of $\mathrm{Out}(G)$. Moreover, since every vertex is a collapse of some ρ_{ij} then the Φ contribution from each stabiliser in Proposition 4.2.1 is identified with $\mathrm{Stab}(\alpha)$. Each 'spike' of $\mathcal{D}_4$ gives a diagram of inclusions:



and each A_i neighbourhood gives a further diagram of inclusions:



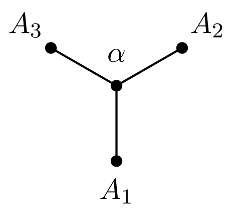
Hence any G_{i_j} contribution from $\mathrm{Stab}(A_i)$ or $\mathrm{Stab}(B_{i,k,l})$ is identified with the G_{i_j} contribution from $\beta_{i,j}$. Thus $\mathrm{Out}_{\mathfrak{S}}(G)$ is generated $\mathrm{Stab}(\beta_{i,j})$ for each pair (i,j) and α, with the Φ contribution from each $\mathrm{Stab}(\beta_{i,j})$ identified with $\mathrm{Stab}(\alpha)$. That is, $\mathrm{Out}_{\mathfrak{S}}(G)$ is generated by G_{i_j} for each pair (i,j) and Φ. The Relations 1–4 come from the groups in Proposition 4.2.1 (specifically, 1 and 3 are from $\mathrm{Stab}(A_i)$, 2 is from $\mathrm{Stab}(B_{i,k,l})$, and 4 is from $\mathrm{Stab}(\beta_{i,j})$). ☐

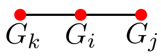
Observe that the Relation '$\left[f_{j_k}(g), f_{i_j}(h)f_{i_k}(h)\right] = 1$' from the case $n \geqslant 5$ may be deduced from Relations 2 and 3 here (since writing k_k for $\mathrm{Inn}(k)$ yields $k_{ij} = k_{kl}$). Thus Theorem 4.1.1 in fact holds for $n \geqslant 4$.

One should note that one of each of the groups $G_{i_j} \cong G_i$ is 'redundant' in the sense that only any 2 of the groups $G_{i_j}, G_{i_k}, G_{i_l}$ are independent of each other. It is possible to consistently choose 4 such groups to eliminate from the list of generators, but doing so would somewhat complicate the relations.

4.3 The Case $n = 3$

For $G = G_1 * G_2 * G_3$, the only graphs which respect the splitting $\mathfrak{S} = (G_1, G_2, G_3)$ are α and A graphs. Thus the fundamental domain $\mathcal{D}_3$ of our complex $\mathcal{C}_3$ is the following tripod:



where A_i is the graph 

for $i = 1,2,3$ and $\{j,k\} = \{1,2,3\} - \{i\}$.

As before, we have that $\mathrm{Stab}(\alpha) = \Phi = \mathrm{Aut}(G_1) \times \mathrm{Aut}(G_2) \times \mathrm{Aut}(G_3)$, and that for each $i \in \{1,2,3\}$ with $\{j,k\} = \{1,2,3\} - \{i\}$, $\mathrm{Stab}(A_i)$ is generated by $(G_{i_j} \times G_{i_k}) / Z(G_i)$ and Φ such that $G_{i_{jk}} / Z(G_i) = \mathrm{Inn}(G_i)$ and $(\{G_a\}, g_i) \circ \varphi = \varphi \circ (\{G_a\}, (g_i)\varphi)$ (for $a = j,k$) for all $(\{G_a\}, g_i) \in G_{i_a}$ and $\varphi \in \Phi$.

Theorem 4.3.1. *Let $G = G_1 * G_2 * G_3$ be a free splitting of a group G where each G_i is non-trivial, and let $\mathfrak{S} = (G_1, G_2, G_3)$. Then writing f_{i_j} for the isomorphism $G_i \to G_{i_j}$, $\mathrm{Out}_{\mathfrak{S}}(G)$ is generated by the six groups G_{i_j} for $i,j \in \{1,2,3\}$ distinct, and Φ, subject to relations:*

1. $\left[f_{i_j}(g), f_{i_k}(h)\right] = 1 \ \forall g,h \in G_i$
2. $f_{i_j}(g) f_{i_k}(g) = \mathrm{Ad}_{G_i}(g) \ \forall g \in G_i$

3. $\varphi^{-1}f_{i_j}(g)\varphi = f_{i_j}((g)\varphi)$ $\forall g \in G_i$ *for all* $\varphi \in \Phi$

for all $i = 1, 2, 3$ *and* $\{j, k\} = \{1, 2, 3\} - \{i\}$.

Proof. By Theorem 1.2.2, $\mathrm{Out}_{\mathfrak{G}}(G)$ is generated by $\mathrm{Stab}(\alpha)$, $\mathrm{Stab}(A_1)$, $\mathrm{Stab}(A_2)$, and $\mathrm{Stab}(A_3)$. The structure of $\mathcal{D}_3$ means that the Φ contribution from each $\mathrm{Stab}(A_i)$ is identified with $\mathrm{Stab}(\alpha)$. That is, $\mathrm{Out}_{\mathfrak{G}}(G) = \mathrm{Stab}(A_1) *_\Phi \mathrm{Stab}(A_2) *_\Phi \mathrm{Stab}(A_3)$. The result then follows by examining each $\mathrm{Stab}(A_i)$. □

Note that ${G_{i_{jk}}}/{Z(i)} = \mathrm{Inn}(G_i)$ means $G_{i_j}\,\mathrm{Inn}(G_i) = G_{i_k}\,\mathrm{Inn}(G_i)$ as cosets in $\mathrm{Stab}(A_i)$. Thus we can write $\mathrm{Stab}(A_i) = G_{i_j} \rtimes \Phi = G_{i_k} \rtimes \Phi$ (using the Guirardel–Levitt approach to computing stabilisers demonstrated in Example 4). Since the only relation between vertex groups here is the amalgamation over Φ, we can in this case obtain a much simpler presentation:

Corollary 4.3.2. *For* $G = G_1 * G_2 * G_3$ *as above, we have:*

$$\mathrm{Out}_{\mathfrak{G}}(G) = (G_{1_2} * G_{2_3} * G_{3_1}) \rtimes \Phi \cong G \rtimes \Phi$$

Proof. As noted above, in this case we have that $\mathrm{Out}_{\mathfrak{G}}(G) = \mathrm{Stab}(A_1) *_\Phi \mathrm{Stab}(A_2) *_\Phi \mathrm{Stab}(A_3)$. We now observe that $(G_{1_2} \rtimes \Phi) *_\Phi (G_{2_3} \rtimes \Phi) *_\Phi (G_{3_1} \rtimes \Phi) = (G_{1_2} * G_{2_3} * G_{3_1}) \rtimes \Phi$. □

If each of G_1, G_2, and G_3 is additionally freely indecomposable, not infinite cyclic, and pairwise non-isomorphic, then we have that $\mathrm{Out}_{\mathfrak{G}}(G) = \mathrm{Out}(G)$. In this case, this is exactly the presentation for $\mathrm{Out}(G)$ given by Collins and Gilbert [7, Propositions 4.1,4.2].

5 The Space of Domains

The rest of the paper will be spent proving that for $n \geqslant 5$, $\mathcal{C}_n$ is simply connected.

In order to study global properties of our complex $\mathcal{C}_n$, we will define a new space, akin to a nerve complex (first introduced by Alexandroff in [1]).

A nerve complex is an abstract simplicial complex built using information on the intersections within a family of sets. The sets we will choose are copies of the fundamental domain $\mathcal{D}_n$, which form a (closed) cover of $\mathcal{C}_n$. However, in order to keep the dimension low, we will only consider k-wise intersections of sets for $k \leqslant 3$.

It is not necessary to have background knowledge of nerve complexes in order to understand our space or arguments.

5.1 Defining the Space of Domains

We will call a subset of our complex a domain if it is of the form $\mathcal{D}_n \cdot \psi$ for some $\psi \in \mathrm{Out}_{\mathfrak{G}}(G)$, where we think of $\mathcal{D}_n$ as a set and $\mathcal{D}_n \cdot \psi = \{x \cdot \psi | x \in \mathcal{D}_n\}$. Since the action of $\mathrm{Out}_{\mathfrak{G}}(G)$ on $\mathcal{C}_n$ preserves adjacency, then $\mathcal{D}_n \cdot \psi$ has the same topological structure as $\mathcal{D}_n$.

Definition 5.1.1. The Graph of Domains is a graph whose vertex set contains one vertex for every domain in our complex, and whose edge set contains an edge joining distinct vertices u and v if and only if the intersection of the domains associated to u and v is non-empty.

Since the graph  (denoted α) occurs precisely once per domain, taking one vertex per domain equates to taking one vertex for every point of the form  in our complex $\mathcal{C}_n$. We will thus often denote vertices in the Graph of Domains by α. Now any two distinct vertices α_1 and α_2 in the Graph of Domains are joined by an edge precisely when the intersection of the domain containing the graph α_1 and the domain containing the graph α_2 is non-empty.

Note that the action of $\mathrm{Out}_{\mathfrak{G}}(G)$ on $\mathcal{C}_n$ induces a natural $\mathrm{Out}_{\mathfrak{G}}(G)$-action on the Graph of Domains.

Definition 5.1.2. The splitting associated to a domain $\mathcal{D}_n \cdot \psi$ is the labelling $((G_1)\psi, \dots, (G_n)\psi)$, which is equivalent to any labelling $(H_1, \dots, H_n)$ of the α-graph contained within the domain $\mathcal{D}_n \cdot \psi$.

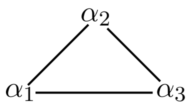
Note that $\psi \in \mathrm{Out}_{\mathfrak{G}}(G)$ is only unique up to factor automorphisms — that is, if $\varphi \in \Phi$ then $\mathcal{D}_n \cdot \psi = \mathcal{D}_n \cdot \varphi\psi$.

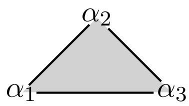
Corollary 5.1.3. *The Graph of Domains is (path) connected.*

Proof. This follows immediately from Proposition 3.1.3. □

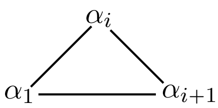
Remark. Note that the Graph of Domains is **not** locally finite.

To show our complex $\mathcal{C}_n$ is simply connected, we will need the following construction:

Definition 5.1.4. Wherever we have a 3-cycle 

in our Graph of Domains, we will insert a 2-simplex 

if and only if $\alpha_1 \cap \alpha_2 \cap \alpha_3 \neq \varnothing$ (in the complex $\mathcal{C}_n$). We call the resulting CW-complex the Space of Domains.

Note that given a cycle $\alpha_1 — \dots — \alpha_{n-1} — \alpha_n = \alpha_1$ with $\alpha_1 \cap \dots \cap \alpha_{n-1} \neq \varnothing$, we can split this up into 3-cycles 

for $i = 2, \dots, n-2$, with each $\alpha_1 \cap \alpha_i \cap \alpha_{i+1} \neq \varnothing$, so any such loop is contractible in our Space of Domains.

The idea behind this definition is that, since we have shown that the fundamental domain $\mathcal{D}_n$ is simply connected, then if we had that any pairwise intersection of domains is either empty or path-connected, then any non-trivial loop in $\mathcal{C}_n$ would be projected to a non-trivial loop in the Space of Domains.

In particular, if $\alpha_1 \cup \alpha_2$ were simply connected (assuming $\alpha_1 \cap \alpha_2 \neq \varnothing$) then there would be no non-trivial loops in $\mathcal{C}_n$ which would appear as a forwards-and-backwards traversal of an edge when projected to the Space of Domains, and similarly for $\alpha_1 \cup \alpha_2 \cup \alpha_3$ (or the 2-cell $\alpha_1 — \alpha_2 — \alpha_3 — \alpha_1$ in the Space of Domains).

Then to show that our space $\mathcal{C}_n$ is simply connected, it would suffice to show that our Space of Domains is simply connected.

Unfortunately, it will not be quite this simple, but the general idea will remain the same. We will formalise (and resolve) this in Sections 5.2 and 5.3.

5.2 Pairwise Intersections

Here we would hope to show that the intersection of two adjacent domains is path-connected.

Then we could deduce using the Seifert–van Kampen Theorem that the union of two adjacent domains is simply connected. This would ensure, for example, that there are no non-trivial loops in $\mathcal{C}_n$ of the form $\alpha_1 — A — \alpha_2 — A — \alpha_1$, and would justify the use of a single edge between adjacent vertices in our Graph of Domains.

As it turns out, not quite all such intersections are path-connected, but we show that the case where this does not hold can be circumvented. This is deduced in Propositions 5.2.10 and 5.2.11, the main results of this subsection.

To avoid confusion regarding domains and graphs within domains, we will temporarily break from the convention of naming domains α. So let $\aleph_1$ and $\aleph_2$ be two arbitrary domains. Assume $\aleph_1 \cap \aleph_2 \neq \varnothing$. Without loss of generality, we may assume $\aleph_1$ is the fundamental domain $\mathcal{D}_n$.

Observation 5.2.1. Note that if $\aleph_2 = \aleph_1 \cdot \psi$ then for $T \in \aleph_1$, we have $T \in \aleph_2$ if and only if $T = T' \cdot \psi$ for some $T' \in \aleph_1$. But each domain contains precisely one element of each orbit, so we must have $T = T'$. Then $\psi \in \mathrm{Stab}(T)$. Moreover, for $T \in \aleph_1$ and $\psi \in \mathrm{Stab}(T)$, we have $T = T \cdot \psi \in \aleph_1 \cdot \psi$. That is, for $T \in \aleph_1$, we have $T \in \aleph_2$ if and only if $\aleph_2 = \aleph_1 \cdot \psi$ for some $\psi \in \mathrm{Stab}(T)$.

Lemma 5.2.2. *Let T_1, T_2, and T_3 be vertices in the complex $\mathcal{C}_n$. If $T_1, T_2 \in \aleph_1 \cap \aleph_2$ and $T_3 \in \aleph_1$ with* $\mathrm{Stab}(T_1) \cap \mathrm{Stab}(T_2) \subseteq \mathrm{Stab}(T_3)$ *then $T_3 \in \aleph_1 \cap \aleph_2$.*

Proof. By Observation 5.2.1, $T \in \aleph_1 \cap \aleph_2$ if and only if $\aleph_2 = \aleph_1 \cdot \psi$ for some $\psi \in \mathrm{Stab}\, T$. Thus if $T_1, T_2 \in \aleph_1 \cap \aleph_2$ then $\aleph_2 = \aleph_1 \cdot \psi_1 = \aleph_1 \cdot \psi_2$ for some $\psi_1 \in \mathrm{Stab}\, T_1$ and $\psi_2 \in \mathrm{Stab}\, T_2$. Note then that $\aleph_1 = \aleph_1 \cdot \psi_1\psi_2^{-1}$, so $\psi_1\psi_2^{-1} \in \mathrm{Stab}(\aleph_1) = \mathrm{Stab}(\alpha) = \Phi \subseteq \mathrm{Stab}(T_1) \cap \mathrm{Stab}(T_2)$. In particular, $\psi_1 = (\psi_1\psi_2^{-1})\psi_2 \in \mathrm{Stab}(T_2)$, so $\aleph_2 = \aleph_1 \cdot \psi_1$ with $\psi_1 \in \mathrm{Stab}(T_1) \cap \mathrm{Stab}(T_2)$. Additionally, if for some $T_3 \in \aleph_1$ we have $\psi_1 \in \mathrm{Stab}(T_3)$, then $T_3 \in \aleph_1 \cap \aleph_2$. This last condition holds if (but not only if) $\mathrm{Stab}(T_1) \cap \mathrm{Stab}(T_2) \subseteq \mathrm{Stab}(T_3)$. ☐

Corollary 5.2.3. *Let T_1 and T_2 be vertices in the intersection $\aleph_1 \cap \aleph_2 \subseteq \mathcal{C}_n$. If* $\mathrm{Stab}(T_1) \cap \mathrm{Stab}(T_2) = \Phi$*, then $\aleph_1 \cap \aleph_2 = \aleph_1 = \aleph_2$.*

Proof. Recall from Proposition 2.4.1 that $\mathrm{Stab}(\alpha) = \Phi$. Thus if $\mathrm{Stab}(T_1) \cap \mathrm{Stab}(T_2) = \Phi$ for some $T_1, T_2 \in \aleph_1 \cap \aleph_2$, then $\alpha \in \aleph_1 \cap \aleph_2$. But each α vertex appears in exactly one domain, hence $\aleph_1 = \aleph_2$. ☐

Suppose T_1 and T_2 are distinct vertices in the complex $\mathcal{C}_n$. By Lemma 2.3.7, if T_1 is a collapse of T_2, then $\mathrm{Stab}(T_2) \subseteq \mathrm{Stab}(T_1)$, so $T_2 \in \aleph_1 \cap \aleph_2 \implies T_1 \in \aleph_1 \cap \aleph_2$. Since every graph collapses to at least one of A_i, $B_{i,j,k}$, or $C_{i,j,k,l,m}$ (for some i, j, k, l, m), then to show path connectivity of intersections, it suffices to find paths in the intersection $\aleph_1 \cap \aleph_2$ with endpoints as the following six cases:

1. $A_i — A_p$
2. $B_{i,j,k} — A_p$
3. $B_{i,j,k} — B_{p,q,r}$
4. $C_{i,j,k,l,m} — A_p$

5. $C_{i,j,k,l,m} — B_{p,q,r}$

6. $C_{i,j,k,l,m} — C_{p,q,r,s,t}$

(where $i, j, k, l, m, p, q, r, s, t$ need not be distinct, unless appearing together as indices of a single vertex.) In our proofs, we will assume the 'left' vertex has fixed indices, and allow the indices of the 'right' vertex to vary.

Writing i_j for the group G_{i_j}, we recall that the stabiliser of A_i is a quotient of $(i_{v_1} \times \cdots \times i_{v_{n-1}}) \rtimes \Phi$, the stabiliser of $B_{i,j,k}$ is a quotient of $(i_{jk} \times j_k \times i_{v_1} \times \cdots \times i_{v_{n-3}}) \rtimes \Phi$, and the stabiliser of $C_{i,j,k,l,m}$ is a quotient of $(i_{lm} \times l_m \times i_{jk} \times j_k \times i_{v_1} \times \cdots \times i_{v_{n-5}}) \rtimes \Phi$. In the group of automorphisms G_{i_j}, we call G_i the operating factor and G_j the dependent factor. We say a graph has operating and dependent factors if the same is true of its stabiliser. So A_i has one operating factor, $B_{i,j,k}$ has two, and $C_{i,j,k,l,m}$ has three distinct operating factors. In each case, only one operating factor has more than one dependent factor. We now proceed through the Cases 1–6:

Lemma 5.2.4 (Case 1). *If A_i and A_p are vertices in $\aleph_1 \cap \aleph_2$, then there is a path in $\aleph_1 \cap \aleph_2$ from A_i to A_p.*

Proof. We have $\mathrm{Stab}(A_i) \cap \mathrm{Stab}(A_p) \neq \Phi$ if and only if $i = p$. But each domain contains only one A_i-graph for each $i \in \{1, \ldots, n\}$. So either $\alpha \in \aleph_1 \cap \aleph_2$ (in which case $\alpha_1 = \alpha_2$), or $A_i = A_p$.

So if A_i and A_p are points in $\aleph_1 \cap \aleph_2$ for any $i, p \in \{1, \ldots, n\}$ then there is a path in $\aleph_1 \cap \aleph_2$ connecting them. □

Lemma 5.2.5 (Case 2). *If $B_{i,j,k}$ and A_p are vertices in $\aleph_1 \cap \aleph_2$, then there is a path in $\aleph_1 \cap \aleph_2$ from $B_{i,j,k}$ to A_p.*

Proof. If $p \notin \{i, j\}$ then $B_{i,j,k}$ and A_p share no common operating factors, hence $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(A_p) = \Phi$ and $\alpha \in \aleph_1 \cap \aleph_2$.

If $p = i$ then $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(A_p)$ contains only one operating factor, with $n - 3$ dependent factors. That is, $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(A_i) = (i_{v_1} \times \cdots \times i_{v_{n-3}}) \rtimes \Phi$ for $\{v_1, \ldots, v_{n-3}\} = \{1, \ldots, n\} - \{i, j, k\}$. This is precisely the stabiliser of $\gamma_{i,jk}$, hence $\gamma_{i,jk} \in \aleph_1 \cap \aleph_2$. Moreover, $\gamma_{i,jk}$ collapses to both A_i and $B_{i,j,k}$, so we have a path $B_{i,j,k} — \gamma_{i,jk} — A_i$.

For $p = j$, we have $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(A_j) = j_k \rtimes \Phi = \mathrm{Stab}(\beta_{j,k})$, and so $B_{i,j,k} — \beta_{j,k} — A_j$ is a path in $\aleph_1 \cap \aleph_2$.

So if $B_{i,j,k}$ and A_p are points in $\aleph_1 \cap \aleph_2$ for any $i, j, k, l \in \{1, \ldots, n\}$ then there is a path in $\aleph_1 \cap \aleph_2$ connecting them. □

Lemma 5.2.6 (Case 3). *If $B_{i,j,k}$ and $B_{p,q,r}$ are vertices in $\aleph_1 \cap \aleph_2$, then there is a path in $\aleph_1 \cap \aleph_2$ from $B_{i,j,k}$ to $B_{p,q,r}$.*

Proof. In order to have $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(B_{p,q,r}) \neq \Phi$, we must have that $\{i, j\} \cap \{p, q\} \neq \varnothing$. We will thus assume this holds.

If $\{i, j\} = \{p, q\}$ and additionally $r = k$, then either $B_{p,q,r} = B_{i,j,k}$ and we are done, or we have $B_{p,q,r} = B_{j,i,k}$, in which case $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(B_{p,q,r}) = \Phi$. So we may assume $r \neq k$ in this case.

If $p = j$ and $q = i$ (with $r \neq k$) then $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(B_{p,q,r}) = (i_r \times j_k) \rtimes \Phi = \mathrm{Stab}(\varepsilon_{i,r,j,k})$, and $B_{i,j,k} — \varepsilon_{i,r,j,k} — B_{j,i,r}$ is a path in $\aleph_1 \cap \aleph_2$.

If $p = i$ and $q = j$ (with $r \neq k$) then $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(B_{p,q,r}) = (i_{v_1} \times \cdots \times i_{v_{n-4}}) \rtimes \Phi$ where $\{i_{v_1}, \ldots, i_{v_{n-4}}\} = \{1, \ldots, n\} - \{i, j, k, r\}$. This is contained within $\mathrm{Stab}(A_i)$, hence

$A_i \in \aleph_1 \cap \aleph_2$. Then by Case 2 (Lemma 5.2.5), there is some path from $B_{i,j,k}$ to A_i and some path from A_i to $B_{i,j,r}$ in $\aleph_1 \cap \aleph_2$.

We will now consider $\{i,j\} \neq \{p,q\}$. Then $|\{i,j\} \cap \{p,q\}| = 1$ and so $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(B_{p,q,r})$ has at most one operating factor (with at most $n-4$ dependent factors). If there is no common operating factor, then $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(B_{p,q,r}) = \Phi$. Otherwise, $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(B_{p,q,r}) \subset A_v$ for some $v \in \{i,j,p,q\}$. Then we are reduced to Case 2.

So if $B_{i,j,k}$ and $B_{p,q,r}$ are points in $\aleph_1 \cap \aleph_2$ for any $i,j,k,p,q,r \in \{1,\ldots,n\}$ then there is a path in $\aleph_1 \cap \aleph_2$ connecting them. □

Lemma 5.2.7 (Case 4). *If $C_{i,j,k,l,m}$ and A_p are vertices in $\aleph_1 \cap \aleph_2$, then there is a path in $\aleph_1 \cap \aleph_2$ from $C_{i,j,k,l,m}$ to A_p.*

Proof. In order to satisfy $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(A_p) \neq \Phi$, we require that $p \in \{i,j,l\}$. Note that by symmetry, $C_{i,j,k,l,m} = C_{i,l,m,j,k}$, so we only need to consider one of $p = j$ and $p = l$.

If $p = j$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(A_p) = j_k \rtimes \Phi = \mathrm{Stab}(\tau_{j,k,lm})$. We have that $\tau_{j,k,lm}$ collapses to both A_j and $C_{i,j,k,l,m}$, so $C_{i,j,k,l,m} — \tau_{j,k,lm} — A_j$ is a path in $\aleph_1 \cap \aleph_2$.

If $p = i$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(A_p) = (i_{jk} \times i_{v_1} \times \cdots \times i_{v_{n-5}}) \rtimes \Phi$ (where $\{v_1,\ldots,v_{n-5}\} = \{1,\ldots,n\} - \{i,j,k,l,m\}$). This is contained in the stabiliser of $\sigma_{i,jk,lm}$, which is a graph that collapses to both A_i and $C_{i,j,k,l,m}$, hence $C_{i,j,k,l,m} — \sigma_{i,jk,lm} — A_i$ is a path in $\aleph_1 \cap \aleph_2$.

So if $C_{i,j,k,l,m}$ and A_p are points in $\aleph_1 \cap \aleph_2$ for any $i,j,k,l,m,p \in \{1,\ldots,n\}$ then there is a path in $\aleph_1 \cap \aleph_2$ connecting them. □

Lemma 5.2.8 (Case 5). *If $C_{i,j,k,l,m}$ and $B_{p,q,r}$ are vertices in $\aleph_1 \cap \aleph_2$, then there is a path in $\aleph_1 \cap \aleph_2$ from $C_{i,j,k,l,m}$ to $B_{p,q,r}$.*

Proof. To satisfy $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r}) \neq \Phi$, we require $\{p,q\} \cap \{i,j,l\} \neq \varnothing$.

Suppose $p = j$ and $q = l$. If $r = k$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r}) = \Phi$. If $r = m$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r}) = (j_k \times l_m) \rtimes \Phi = \mathrm{Stab}(\varepsilon_{j,k,l,m})$, and $\varepsilon_{j,k,l,m}$ collapses to both $B_{j,l,m}$ and $C_{i,j,k,l,m}$, so these points are connected by a path in $\aleph_1 \cap \aleph_2$. If $r \notin \{k,m\}$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r}) = j_k \rtimes \Phi \subset \mathrm{Stab}(A_j)$. Thus $A_j \in \aleph_1 \cap \aleph_2$. By Case 2 (Lemma 5.2.5) there is a path in $\aleph_1 \cap \aleph_2$ from $B_{j,l,r}$ to A_j, and by Case 4 (Lemma 5.2.7) there is a path in $\aleph_1 \cap \aleph_2$ from A_j to $C_{i,j,k,l,m}$.

By symmetry of $C_{i,j,k,l,m} = C_{i,l,m,j,k}$, we do not need to consider the case $p = l$ and $q = j$.

We may now assume $\{p,q\} \neq \{j,l\}$. Again by the symmetry of C-vertices, we need only consider $\{p,q\} \cap \{i,j\} \neq \varnothing$.

Suppose $p = i$ and $q = j$ (or $q = l$ by symmetry). If $r = k$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r})$ is either $(j_k \times i_{v_1} \times \cdots \times i_{v_{n-5}}) \rtimes \Phi$ or $(j_k \times i_{jk} \times i_{v_1} \times \cdots \times i_{v_{n-5}}) \rtimes \Phi$, in either case this is contained in $\mathrm{Stab}(\delta_{i,j,k,lm})$. Since $\delta_{i,j,k,lm}$ collapses to both $B_{i,j,k}$ and $C_{i,j,k,l,m}$ then this provides a path in $\aleph_1 \cap \aleph_2$. If $r \neq k$ then the only operating factor in $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r})$ is i, hence $A_i \in \aleph_1 \cap \aleph_2$ and thus by Cases 2 and 4 there is a path from $C_{i,j,k,l,m}$ to $B_{i,j,r}$ in $\aleph_1 \cap \aleph_2$.

Suppose $q = i$ and $p = j$ (or $p = l$ by symmetry). If $r = k$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r}) = \Phi$ and $\alpha \in \aleph_1 \cap \aleph_2$. If $r \in \{l,m\}$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r}) = j_k \rtimes \Phi \subset \mathrm{Stab}(A_j)$, hence we are reduced to Cases 2 and 4. If $r \notin \{k,l,m\}$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r}) = (i_r \times j_k) \rtimes \Phi \subset \mathrm{Stab}(B_{i,j,k})$. In the previous paragraph we showed there is a path in $\aleph_1 \cap \aleph_2$ from $C_{i,j,k,l,m}$ to $B_{i,j,k}$ via a δ-graph, and by Case 3 (Lemma 5.2.6), there is a path in $\aleph_1 \cap \aleph_2$ from $B_{i,j,k}$ to $B_{j,i,r} = B_{p,q,r}$.

If $|\{p,q\} \cap \{i,j,l\}| = 1$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r})$ has at most one operating factor (with at most $n-6$ dependent factors). So either $\alpha \in \aleph_1 \cap \aleph_2$, or there is some A-vertex in $\aleph_1 \cap \aleph_2$, and we are reduced to Cases 2 and 4.

So if $C_{i,j,k,l,m}$ and $B_{p,q,r}$ are points in $\aleph_1 \cap \aleph_2$ for any $i,j,k,l,m,p,q,r \in \{1,\ldots,n\}$ then there is a path in $\aleph_1 \cap \aleph_2$ connecting them. $\square$

Lemma 5.2.9 (Case 6)**.** *If $C_{i,j,k,l,m}$ and $C_{p,q,r,s,t}$ are vertices in $\aleph_1 \cap \aleph_2$, then there is a path in $\aleph_1 \cap \aleph_2$ from $C_{i,j,k,l,m}$ to $C_{p,q,r,s,t}$ if and only if $r \neq k \implies t \neq m$.*

Proof. Suppose we have $C_{i,j,k,l,m} \in \aleph_1 \cap \aleph_2$ and $C_{p,q,r,s,t} \in \aleph_1 \cap \aleph_2$.

If $\{p,q,s\} \cap \{i,j,l\} = \varnothing$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{p,q,r,s,t}) = \Phi$ and $\alpha \in \aleph_1 \cap \aleph_2$.

If $|\{p,q,s\} \cap \{i,j,l\}| = 1$ then we have at most one operating factor in $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{p,q,r,s,t})$ (with at most $n-7$ dependent factors), so either $\alpha \in \aleph_1 \cap \aleph_2$ (and we are done) or there is some A-vertex in $\aleph_1 \cap \aleph_2$, and we are reduced to Case 4 (Lemma 5.2.7).

Suppose $|\{p,q,s\} \cap \{i,j,l\}| = 2$. Then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{p,q,r,s,t})$ has at most two operating factors. If $p \neq i$ then all of these operating factors have at most one dependent factor. Since any B-vertex has two operating factors, each with at least one dependent, then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{p,q,r,s,t}) \subset \mathrm{Stab}(B)$ for some B-vertex. Then we are reduced to Case 5 (Lemma 5.2.8). If $p = i$ then we have at most one operating factor with at most $n-5$ dependent factors, and the possible other operating factor has at most one dependent factor. So again there is some B-vertex in $\aleph_1 \cap \aleph_2$, and by Case 5, there must be some path between our two C-vertices.

Now suppose $\{p,q,s\} = \{i,j,l\}$. If $p = i$ then by symmetry of C-vertices we may assume $q = j$ and $s = l$. If in addition we have $r = k$ and $t = m$ then $C_{p,q,r,s,t} = C_{i,j,k,l,m}$. So suppose $\{r,t\} \neq \{k,m\}$. If $|\{r,t\} \cap \{k,m\}| = 1$ then either $B_{i,j,k}$ or $B_{i,l,m}$ is in $\aleph_1 \cap \aleph_2$, which reduces us to Case 5. If $|\{r,t\} \cap \{k,m\}| = 0$ then $A_i \in \aleph_1 \cap \aleph_2$, and we are reduced to Case 4.

Finally, suppose $p = j$, $q = i$, and $s = l$. By permuting indices in accordance with the symmetries of $C_{i,j,k,l,m}$ and $C_{p,q,r,s,t}$, this covers all cases where $\{p,q,s\} = \{i,j,l\}$ and $p \neq i$. If $r = k$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{p,q,r,s,t}) \subseteq l_t \rtimes \Phi \subset \mathrm{Stab}(A_l)$, so by Case 4 we are done. If $r \neq k$ and $t \neq m$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{p,q,r,s,t}) \subseteq (j_k \times i_r) \rtimes \Phi \subset \mathrm{Stab}(B_{i,j,k})$, so by Case 5 we are done.

A problem arises when $r \neq k$ but $t = m$. Then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{p,q,r,s,t}) = \mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{j,i,r,l,m}) = (j_k \times l_m \times i_r) \rtimes \Phi$. The only graph T in our complex (besides $C_{i,j,k,l,m}$ and $C_{j,i,r,l,m}$) with $(j_k \times l_m \times i_r) \rtimes \Phi \subseteq \mathrm{Stab}(T)$ is $C_{l,i,r,j,k}$. So in this case $\aleph_1 \cap \aleph_2$ consists of three distinct non-adjacent points. Note that this is the only case where $\aleph_1 \cap \aleph_2$ is not path-connected.

So if $C_{i,j,k,l,m}$ and $C_{p,q,r,s,t}$ are points in $\aleph_1 \cap \aleph_2$ for any $i,j,k,l,m,p,q,r,s,t \in \{1,\ldots,n\}$, and we don't have that $r \neq k$ and $t = m$, then there is a path in $\aleph_1 \cap \aleph_2$ connecting $C_{p,q,r,s,t}$ to $C_{i,j,k,l,m}$. $\square$

Having dealt with all six cases, we may now conclude:

Proposition 5.2.10. *If $\aleph_1$ and $\aleph_2$ are two domains with $\aleph_1 \cap \aleph_2 \neq \varnothing$ such that $\aleph_1 \cap \aleph_2$ contains some vertex which is **not** of the form $C_{i,j,k,l,m}$, then $\aleph_1 \cap \aleph_2$ is path-connected.*

Proof. Note that in the proof of Lemma 5.2.9 we showed that if $C_{i,j,k,l,m} \in \aleph_1 \cap \aleph_2$ and $C_{p,q,r,s,t} \in \aleph_1 \cap \aleph_2$ then there exists some non-C vertex in $\aleph_1 \cap \aleph_2$ if and only if $r \neq k \implies t \neq m$. The result then follows from Lemmas 5.2.4, 5.2.5, 5.2.6, 5.2.7, 5.2.8, and 5.2.9. $\square$

Remark. Note that this statement is not saying that $\aleph_1 \cap \aleph_2$ cannot contain a vertex $C_{i,j,k,l,m}$ if it is to be path connected, just that it must also contain some other vertex as well which is not a C-vertex.

While it is not ideal that such an intersection containing only C-vertices is not path-connected, this can be handled via the following:

Proposition 5.2.11. *If $\aleph_1$ and $\aleph_2$ are two domains with $\aleph_1 \cap \aleph_2 \neq \varnothing$ such that $\aleph_1 \cap \aleph_2$ contains **only** vertices of the form $C_{i,j,k,l,m}$, then $\aleph_1 \cap \aleph_2 = \{C_{i,j,k,l,m}, C_{j,l,m,i,p}, C_{l,i,p,j,k}\}$ for some (distinct) $i, j, k, l, m, p \in \{1, \ldots, n\}$. Moreover, there exists a domain $\aleph_3$ with $\aleph_1 \cap \aleph_2 \subset \aleph_3$, such that $\aleph_1 \cup \aleph_2 \cup \aleph_3$ is simply connected.*

Proof. As noted in Case 6 (Lemma 5.2.9) above, if $\aleph_1 \cap \aleph_2$ does not contain any vertices except C-vertices, then there exist some distinct $i, j, k, l, m, p \in \{1, \ldots, n\}$ such that $\aleph_1 \cap \aleph_2 = \{C_{i,j,k,l,m}, C_{j,l,m,i,p}, C_{l,i,p,j,k}\}$. Further, we have that $\aleph_2 = \aleph_1 \psi$ for some $\psi \in (i_p \times j_k \times l_m) \rtimes \Phi$ (and $\psi \notin (i_p \times j_k) \rtimes \Phi \cup (j_k \times l_m) \rtimes \Phi \cup (l_m \times i_p) \rtimes \Phi$). That is, $\psi = (G_p, i_0)(G_k, j_0)(G_m, l_0)\phi$ for some $i_0 \in G_i$, $j_0 \in G_j$, $l_0 \in G_l$, and $\phi \in \Phi$. Define $\aleph_3$ to be $\aleph_1(G_p, i_0)$. Note that $(G_p, i_0) \in i_p \rtimes \Phi = \mathrm{Stab}(A_i)$, so we have $A_i \in \aleph_1 \cap \aleph_3$. Then $\aleph_2 = \aleph_1(G_p, i_0)(G_k, j_0)(G_m, l_0)\phi = \aleph_3(G_k, j_0)(G_m, l_0)\phi$. Note that $(G_k, j_0)(G_m, l_0)\phi \in (j_k \times l_m) \rtimes \Phi \subseteq \mathrm{Stab}(B_{j,l,m})$, so we have $B_{j,l,m} \in \aleph_2 \cap \aleph_3$. Moreover, $(G_p, i_0), (G_k, j_0)(G_m, l_0)\phi \in (i_p \times j_k \times l_m) \rtimes \Phi$. So $C_{i,j,k,l,m}, C_{j,l,m,i,p}, C_{l,i,p,j,k} \in \aleph_3$.

By Proposition 5.2.10, we have that both $\aleph_1 \cap \aleph_3$ and $\aleph_2 \cap \aleph_3$ are path-connected, and by Theorem 3.2.11, $\aleph_1$, $\aleph_2$, and $\aleph_3$ are each simply connected. Then by the Seifert–Van Kampen Theorem (Theorem 1.2.1), $\aleph_1 \cup \aleph_3$ and $\aleph_2 \cup \aleph_3$ are both simply connected. Then we can again apply the Seifert–Van Kampen Theorem to the sets $A = \aleph_1 \cup \aleph_3$ and $B = \aleph_2 \cup \aleph_3$. Since we have that $\pi_1(A) = \pi_1(B) = \{1\}$, and $A \cap B = (\aleph_1 \cap \aleph_2) \cup \aleph_3 = \aleph_3$ is path-connected by Corollary 3.1.2, then we get that $\pi_1(\aleph_1 \cup \aleph_2 \cup \aleph_3) = \pi_1(A \cup B) = \{1\}$. That is, $\aleph_1 \cup \aleph_2 \cup \aleph_3$ is simply connected. ☐

In plain language, this means if there is an edge in the Space of Domains which does not represent a simply connected subset of the complex $\mathcal{C}_n$, then it must be in the boundary of a 2-cell which **does** represent a simply connected subset of $\mathcal{C}_n$.

Observation 5.2.12. In fact, the proof of Proposition 5.2.11 shows that if $\aleph_1$ and $\aleph_2$ are two domains with $\aleph_1 \cap \aleph_2 \neq \varnothing$ such that $\aleph_1 \cap \aleph_2$ contains only vertices of the form $C_{i,j,k,l,m}$, then there exists a domain $\aleph_3$ with $\aleph_1 \cap \aleph_3$ containing a vertex of the form A_i and $\aleph_2 \cap \aleph_3$ containing a vertex of the form $B_{j,l,m}$, such that $\aleph_1 \cup \aleph_2 \cup \aleph_3$ is simply connected.

Corollary 5.2.13. *Let $\aleph_1$ and $\aleph_2$ be two domains so that $\aleph_1 \cap \aleph_2 \neq \varnothing$. Then there exists $U \subset \mathcal{C}_n$ with $\aleph_1 \cup \aleph_2 \subseteq U$ such that U is simply connected.*

Proof. Note that any pair of domains with non-empty intersection fall under precisely one of Proposition 5.2.10 or Proposition 5.2.11. In the second case, we take $U = \aleph_1 \cup \aleph_2 \cup \aleph_3$ with $\aleph_3$ as in Proposition 5.2.11. In the first case, we take $U = \aleph_1 \cup \aleph_2$. By Theorem 3.2.11 we have that each of $\aleph_1$ and $\aleph_2$ is simply connected, and since in this case $\aleph_1 \cap \aleph_2$ is path connected, then by the Seifert–van Kampen Theorem (Theorem 1.2.1), we must have that $\aleph_1 \cup \aleph_2$ is simply connected. ☐

5.3 A Map from the Space of Domains to the Complex $\mathcal{C}_n$

In this subsection, we will define a (continuous) map from the Space of Domains to the complex $\mathcal{C}_n$. This will allow us to conclude that simple connectivity of $\mathcal{C}_n$ can be deduced from simple connectivity of the Space of Domains, as desired.

Remark. If we had that pairwise intersections of domains were always path connected, then since each domain is simply connected, a theorem of Bjorner [5, Theorem 6] would tell us that if the Space of Domains is simply connected then so too is $\mathcal{C}_n$. However, since we do not have this, we must do some extra work to deduce this.

Definition 5.3.1. We define a map F from the 1-skeleton of the Space of Domains (equivalently, from the Graph of Domains) to $\mathcal{C}_n$ as follows:

- A vertex $\aleph$ in the 0-skeleton of the Space of Domains is mapped under F to the α-graph contained in the domain $\aleph \subseteq \mathcal{C}_n$.
- Let $\aleph_1 — \aleph_2$ be some edge in the Space of Domains, and set $\alpha_1 = F(\aleph_1)$ and $\alpha_2 = F(\aleph_2)$. Choose an edge path $\lambda_{12} : [0,1] \to \aleph_1 \cup \aleph_2 \subset \mathcal{C}_n$ with $\lambda_{12}(0) = \alpha_1$ and $\lambda_{12}(1) = \alpha_2$ and let $|\lambda_{12}|$ be its image. We define $F(\aleph_1 — \aleph_2) := |\lambda_{12}|$.
- Given λ_{12}, we require that λ_{21} (the chosen path in $\aleph_1 \cup \aleph_2$ from α_2 to α_1) be defined by $\lambda_{21}(t) := \lambda_{12}(1-t)$ for all $t \in [0,1]$; that is, that $F(\aleph_1 — \aleph_2) = F(\aleph_2 — \aleph_1)$.

Observation 5.3.2. Note that by Definition 5.1.4, if $\aleph_1 — \aleph_2$ is an edge in the Space of Domains, then we have that $\aleph_1 \cap \aleph_2 \neq \varnothing$ as a subset of $\mathcal{C}_n$. By Lemma 3.1.1, each of $\aleph_1$ and $\aleph_2$ is a path-connected subset of $\mathcal{C}_n$, hence so too is $\aleph_1 \cup \aleph_2 \subset \mathcal{C}_n$. Thus there must exist some path in $\aleph_1 \cup \aleph_2$ connecting $F(\aleph_1)$ and $F(\aleph_2)$.

Remark. If one wishes to be more explicit, given an edge $\aleph_1 — \aleph_2$, one may choose $T \in \aleph_1 \cap \aleph_2 \subset \mathcal{C}_n$ and $\psi, \varphi \in \mathrm{Out}_{\mathfrak{G}}(G)$ with $\aleph_1 = \mathcal{D}_n \cdot \psi$ and $\aleph_2 = \mathcal{D}_n \cdot \varphi$ so that $\psi^{-1}\varphi \in \mathrm{Stab}(T)$, where $\mathcal{D}_n$ is the fundamental domain. Denote the path in Table 4 from $T \cdot \psi^{-1} = T \cdot \varphi^{-1}$ to α_0 by p, where α_0 is the unique α-graph contained in $\mathcal{D}_n$, and set $\lambda_{12} := (p^{-1} \cdot \psi)(p \cdot \varphi)$. Then $F(\aleph_1 — \aleph_2) = |p| \cdot \psi \cup |p| \cdot \varphi$. It may well be that this construction leads to an $\mathrm{Out}_{\mathfrak{G}}(G)$-equivariant map, but one would need to carefully handle the choice of $T \in \aleph_1 \cap \aleph_2$ to make it so.

Lemma 5.3.3. *Any loop in $\mathcal{C}_n$ is homotopic to a loop which is the image under F of a loop in the Graph of Domains.*

Proof. Let $\lambda' = T_0' — T_1' — \cdots — T_{m'}' — T_0'$ be a loop in the complex $\mathcal{C}_n$. By Corollary 3.1.4, we can write down a path p in $\mathcal{C}_n$ from T_0' to the α graph in our fundamental domain (which we call α_0). By setting $\lambda = p^{-1}\lambda' p$ we now have a based loop in $\mathcal{C}_n$ (i.e. a loop containing the 'basepoint' α_0) which is homotopic to λ'. Say $\lambda = T_0 — T_1 — \cdots — T_m$, where $T_m = T_0 = \alpha_0$.

We will now describe how to associate a domain $\aleph_i$ to each vertex T_i in λ with $T_i \in \aleph_i$. This will allow us to construct a path in the Graph of Domains whose image under F we will show to be homotopic to λ.

First set $\aleph_0$ to be $\mathcal{D}_n$, the fundamental domain. If T_i is associated to $\aleph_i$ and $T_{i+1} \in \aleph_i$, we set $\aleph_{i+1} := \aleph_i$. In particular, this is the case whenever T_{i+1} is a collapse of T_i. Note that for any edge $S — T$ in $\mathcal{C}_n$ we have that either S is a collapse of T, or T is a collapse of S. Now suppose we have a domain $\aleph_i \ni T_i$ and $T_{i+1} \notin \aleph_i$. We must then have that T_i is a collapse of T_{i+1}. Choose any domain $\aleph_{i+1}$ containing T_{i+1}. Then $T_i \in \aleph_{i+1}$, hence $T_i — T_{i+1} \subseteq \aleph_{i+1}$, and $T_i \in \aleph_i \cap \aleph_{i+1}$.

For a given domain $\aleph_i$, let $\alpha_i \in \mathcal{C}_n$ be the unique α-graph contained in $\aleph_i$. Since $\aleph_i \cap \aleph_{i+1} \neq \varnothing$, then either $\aleph_i = \aleph_{i+1}$ or $\aleph_i — \aleph_{i+1}$ is an edge in the Graph of Domains. We define paths $\mu_{i,i+1}$ as follows: if $\aleph_i = \aleph_{i+1}$, set $\mu_{i,i+1}$ to be the constant path at $\alpha_i = \alpha_{i+1}$; otherwise, set $\mu_{i,i+1}$ to be the path $\lambda_{i,i+1}$ from α_i to α_{i+1} such that $F(\aleph_i — \aleph_{i+1}) = |\lambda_{i,i+1}|$.

Then the concatenation $\mu := \mu_{0,1}\mu_{1,2}\dots\mu_{m-1,m}$ is equal to the concatenation $F(\aleph_{\sigma(0)} — \aleph_{\sigma(1)})F(\aleph_{\sigma(1)} — \aleph_{\sigma(2)})\dots F(\aleph_{\sigma(k)} — \aleph_{\sigma(0)})$ which is $F(\aleph_{\sigma(0)} — \aleph_{\sigma(1)} — \aleph_{\sigma(2)} — \cdots — \aleph_{\sigma(k)} — \aleph_{\sigma(0)})$, where $\sigma(0) = 0$, and given $\sigma(i)$, $\sigma(i+1)$ is the next index such that $\alpha_{\sigma(i+1)} \neq \alpha_{\sigma(i)}$.

We now prove that λ (and hence also λ') is homotopic in $\mathcal{C}_n$ to μ.

Given T_i and its associated domain $\aleph_i$ with graph α_i, let ν_i be a path contained in $\aleph_i$ from T_i to α_i (to be explicit, one may take the correct $\mathrm{Out}(G)$-image of the relevant path listed in Table 4). Additionally, let e_i denote the (oriented) edge (path) $T_i — T_{i+1}$. By Corollary 5.2.13, there exists a simply connected neighbourhood $U_i \subset \mathcal{C}_n$ containing $\aleph_i \cup \aleph_{i+1}$ (if $\aleph_{i+1} = \aleph_i$, instead set $U_i = \aleph_i$, and note that by Theorem 3.2.11 this is simply connected).

Then the loop $\nu_i\mu_i\nu_{i+1}^{-1}\overline{e_i}$ is contained in U_i, hence said loop is contractible in $\mathcal{C}_n$. In other words, the edge $T_i — T_{i+1}$ is homotopic in $\mathcal{C}_n$ to the path $\nu_i\mu_i\nu_{i+1}^{-1}$. Since this holds for all i, it follows that λ is homotopic in $\mathcal{C}_n$ to μ, the image under F of a loop in the Graph of Domains. □

Lemma 5.3.4. *The map F described in Definition 5.3.1 extends to a continuous map $\bar{F}$ from the Space of Domains to $\mathcal{C}_n$.*

Proof. Let $[\aleph_1, \aleph_2, \aleph_3]$ be a face (2-cell) in the Space of Domains. Then as subsets of $\mathcal{C}_n$, we have that $\aleph_1 \cap \aleph_2 \cap \aleph_3 \neq \varnothing$. Let λ_{12}, λ_{23}, and λ_{31} be paths such that each λ_{ij} is an edge path in $\aleph_i \cup \aleph_j$ from α_i to α_j with $F(\aleph_i — \aleph_j) = |\lambda_{ij}|$ (for $i, j \in \{1, 2, 3\}$ distinct). To show that F extends to a map $\bar{F}$, it will suffice to show that the concatenated path $\lambda_{12}\lambda_{23}\lambda_{31}$ is the boundary of some simply connected subset of $\mathcal{C}_n$, that is, that the loop $\lambda_{12}\lambda_{23}\lambda_{31}$ is contractible in $\mathcal{C}_n$. We illustrate the following process in Figure 7.

Step 1: Since $[\aleph_1, \aleph_2, \aleph_3]$ is a 2-cell in the Space of Domains, then by Definition 5.1.4, there must exist some point $x \in \aleph_1 \cap \aleph_2 \cap \aleph_3 \subseteq \mathcal{C}_n$. Let λ_{03} be a path in $\aleph_3 \subseteq \mathcal{C}_n$ from x to α_3 and let $\lambda_{30} = \overline{\lambda_{03}}$ be its reverse path (i.e. the same set of edges, but read from α_3 to x). By Theorem 3.2.11, each domain is simply connected, so $\lambda_{12}\lambda_{23}\lambda_{30}\lambda_{03}\lambda_{31}$ is a path in $\mathcal{C}_n$ which is homotopic to the image $\lambda_{12}\lambda_{23}\lambda_{31}$ of the loop $\aleph_1 — \aleph_2 — \aleph_3 — \aleph_1$ in the Space of Domains.

Step 2: By Corollary 5.2.13, there exists a simply connected neighbourhood in $\mathcal{C}_n$ containing $\aleph_2 \cup \aleph_3$. Thus the subpath $\lambda_{23}\lambda_{30}$ from α_2 to x is contained within a simply connected subset of $\mathcal{C}_n$ containing $\aleph_2$, hence is homotopic to some path λ_2 from α_2 to x fully contained in $\aleph_2$.

Step 3: By the same reasoning as in Step 2, the subpath $\lambda_{03}\lambda_{31}$ from x to α_1 is homotopic in $\mathcal{C}_n$ to some path λ_1 from x to α_1 fully contained in $\aleph_1$. We have so far shown that $\lambda_{12}\lambda_{23}\lambda_{31}$ is homotopic in $\mathcal{C}_n$ to the loop $\lambda_{12}\lambda_2\lambda_1$.

Step 4: We have that $\lambda_{12}\lambda_2\lambda_1$ is a loop contained in $\aleph_1 \cup \aleph_2$. Again, by Corollary 5.2.13, the loop $\lambda_{12}\lambda_2\lambda_1$ is contained within a simply connected subset of $\mathcal{C}_n$, hence it must be contractible. □

Proposition 5.3.5. *Suppose that the Space of Domains is simply connected. Then so too is the complex $\mathcal{C}_n$.*

Proof. Let $\lambda : \mathbb{S}^1 \to \mathcal{C}_n$ be an arbitrary loop in $\mathcal{C}_n$. We temporarily denote the Space of Domains by $\mathcal{S}$, and its 1-skeleton (the Graph of Domains) by $\mathcal{S}^{(1)}$. By Lemma 5.3.3, λ is homotopic in $\mathcal{C}_n$ to some loop $\mu : \mathbb{S}^1 \to F(\mathcal{S}^{(1)})$ which lifts to a loop $M : \mathbb{S}^1 \to \mathcal{S}$

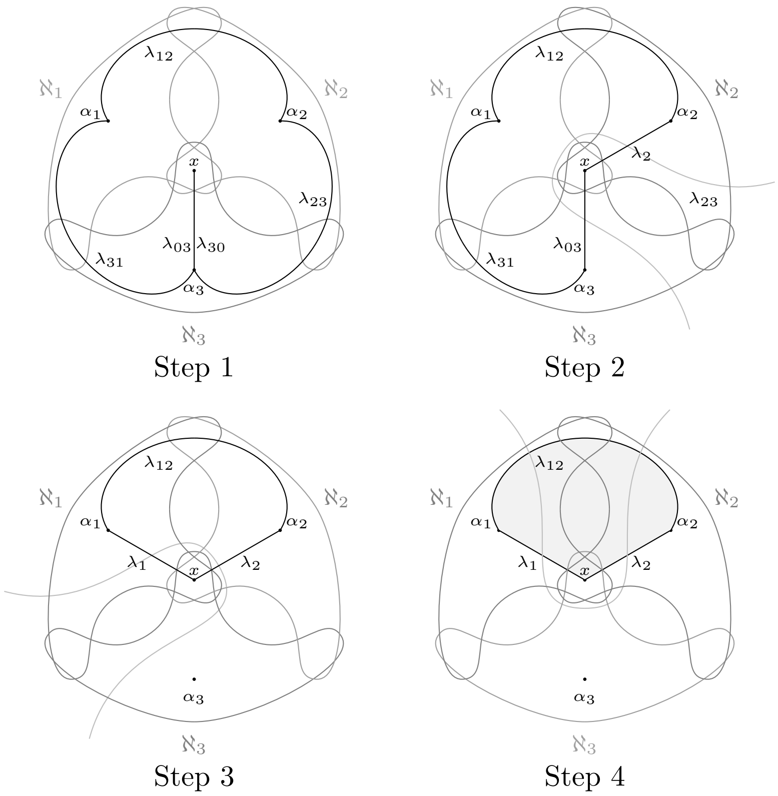


Figure 7: Illustration Contracting the Image Under F of a 3-Cycle

in the Space of Domains with $\mu = F \circ M$, where $F : \mathcal{S}^{(1)} \to \mathcal{C}_n$ is the map described in Definition 5.3.1. If $\mathcal{S}$ is simply connected, then the loop M is contractible. That is, there exists a continuous map $f : \mathbb{D}^2 \to \mathcal{S}$ so that $f|_{\mathbb{S}^1} = M$. By Lemma 5.3.4, F extends to a continuos map $\bar{F} : \mathcal{S} \to \mathcal{C}_n$. Now $\bar{F} \circ f : \mathbb{D}^2 \to \mathcal{C}_n$ is a continuous map, and $\bar{F} \circ f|_{\mathbb{S}^1} = F \circ M = \mu : \mathbb{S}^1 \to \mathcal{C}_n$. Thus μ is contractible, and hence λ is as well. □

5.4 Edges in the Space of Domains

Here we will consider adjacency in the Graph/Space of Domains. We will use α to represent both a vertex in the Space of Domains (i.e. a domain, as a subspace of $\mathcal{C}_n$), and a graph in a domain (i.e. a vertex in $\mathcal{C}_n^{(0)}$).

Let $\alpha_1 \text{---} \alpha_2$ be an edge in the Space of Domains, that is, let $\alpha_1 \subset \mathcal{C}_n$ and $\alpha_2 \subset \mathcal{C}_n$ be two domains such that $\alpha_1 \cap \alpha_2 \neq \varnothing$. Then there is some vertex $T \in \mathcal{C}_n$ in the intersection $\alpha_1 \cap \alpha_2 \subset \mathcal{C}_n$ and, as noted in Section 5.2, some $\varphi \in \operatorname{Stab}(T)$ such that $\alpha_2 = (\alpha_1)\varphi$.

Thus the edge $\alpha_1 \text{---} \alpha_2$ in the Space of Domains can be described in two ways: according to some vertex $T \in \alpha_1 \cap \alpha_2$, or according to some (pure symmetric outer) automorphism $\varphi \in \operatorname{Out}_{\mathfrak{S}}(G)$ satisfying $\alpha_2 = (\alpha_1)\varphi$ (and hence also $\alpha_1 = (\alpha_2)\varphi^{-1}$). We may then label the edge $\alpha_1 \text{---} \alpha_2$ by either $\alpha_1 \overset{T}{\text{---}} \alpha_2$ or $\alpha_1 \overset{\varphi}{\to} \alpha_2$, depending on our viewpoint.

This subsection considers the former viewpoint, i.e. points in the intersection $\alpha_1 \cap \alpha_2$. As such, this subsection can be considered the Space of Domains analogue to Section 5.2. The latter viewpoint (automorphisms satisfying $\alpha_2 = (\alpha_1)\varphi$) will be discussed in

Definition 5.4.1. We will say an edge $\alpha_1 — \alpha_2$ in the Graph/Space of Domains is of Type T if there is some tree T in the intersection $\alpha_1 \cap \alpha_2$ in the complex $\mathcal{C}_n$.

Note that edges can be of more than one Type. In particular, if an edge is of Type T_1, and T_2 is a collapse of T_1, then the edge is also of Type T_2. However an edge can be of Type T_1 and Type T_2 even if neither is a collapse of the other. Recall from Section 5.2 that if $T \in \alpha_1 \cap \alpha_2$, then at least one of A_i, $B_{i,j,k}$, or $C_{i,j,k,l,m}$ is in $\alpha_1 \cap \alpha_2$ for some $i, j, k, l, m \in \{1, \ldots, n\}$, hence every edge in the Space of Domains is at least one of Type A, Type B, or Type C.

Some earlier results may be summarised using this new terminology:

Proposition 3.1.3: Any two vertices in the Graph of Domains are connected via a path whose edges are all of Type A.

Proposition 5.2.10: If an edge $\alpha_1 — \alpha_2$ in the Graph of Domains is of Type A or Type B, then $\alpha_1 \cap \alpha_2$ is path connected in the complex $\mathcal{C}_n$.

Proposition 5.2.11: If an edge $\alpha_1 — \alpha_2$ in the Graph of Domains is of Type C but not of Type A or Type B, then $\alpha_1 \cap \alpha_2$ is not path connected in the complex $\mathcal{C}_n$. However, there exists a domain $\alpha_3 \supseteq \alpha_1 \cap \alpha_2$ so that $[\alpha_1, \alpha_2, \alpha_3]$ is a 2-cell in the Space of Domains and $\alpha_1 \cup \alpha_2 \cup \alpha_3$ is simply connected in the complex $\mathcal{C}_n$.

We will now deduce that simple connectivity of the Space of Domains can be proved considering only edges of Type A.

Proposition 5.4.2. *Suppose $\alpha_1 — \alpha_2$ is an edge in the Space of Domains of Type B but not of Type A. Then there exists some domain α_3 and edges $\alpha_1 — \alpha_3$ and $\alpha_3 — \alpha_2$ of Type A such that $\alpha_1 — \alpha_2$ is homotopic to $\alpha_1 — \alpha_3 — \alpha_2$ in the Space of Domains.*

Proof. Suppose $B_{i,j,k}$ is a B-graph in the intersection $\alpha_1 \cap \alpha_2 \subset \mathcal{C}_n$.
Let $\{i, j, k, v_1, \ldots, v_{n-3}\} = \{1, \ldots, n\}$ and let $\left(H_i, H_j, H_k, H_{v_1}, \ldots, H_{v_{n-3}}\right)$ be an $\mathfrak{S}$-labelling for the α-graph in the domain α_1 of $\mathcal{C}_n$. Then $\left(H_i, H_j, H_k, H_{v_1}, \ldots, H_{v_{n-3}}\right)$ is an $\mathfrak{S}$-labelling for $B_{i,j,k}$, and by Example 2, any equivalent labelling for $B_{i,j,k}$ must be of the form $\left(H_i^{gg_i}, H_j^{g i_{jk} g_j}, H_k^{g i_{jk} j_k g_k}, H_{v_1}^{g i_{v_1} g_{v_1}}, \ldots, H_{v_{n-3}}^{g i_{v_{n-3}} g_{v_{n-3}}}\right)$ for some $g \in G = H_1 * \ldots H_n$, $g_i \in H_i$, $g_j \in H_j$, $g_k \in H_k$, $j_k \in H_j$, $i_{jk} \in H_i$, with $g_v \in H_v$ and $i_v \in H_i$ for each $v = v_1, \ldots, v_{n-3}$.

We may now assume that the α-graph in α_2 has an $\mathfrak{S}$-labelling of the form $\left(H_i, H_j^{i_{jk}}, H_k^{i_{jk}} j_k, H_{v_1}^{i_{v_1}}, \ldots, H_{v_{n-3}}^{i_{v_{n-3}}}\right)$ (since both inner automorphisms and relative factor automorphisms stabilise α). Since the edge $\alpha_1 — \alpha_2$ is stipulated to not be of Type A, then we must have that $j_k \neq 1$ and $i_a \neq 1$ for some $a \in \{jk, v_1, \ldots, v_{n-3}\}$.

Let α_3 be the domain whose α-graph has $\mathfrak{S}$-labelling $\left(H_i, H_j^{i_{jk}}, H_k^{i_{jk}}, H_{v_1}^{i_{v_1}}, \ldots, H_{v_{n-3}}^{i_{v_{n-3}}}\right)$, and let A_i and A_j be the A-graphs in α_3 with central vertex H_i and $H_j^{i_{jk}}$, respectively. Observe that $A_i \in \alpha_3 \cap \alpha_1$, thus $\alpha_1 — \alpha_3$ is an edge of Type A in the Space of Domains. Further, note that since $\left(H_i, H_j^{i_{jk}}, H_k^{i_{jk}}, H_{v_1}^{i_{v_1}}, \ldots, H_{v_{n-3}}^{i_{v_{n-3}}}\right)$ is an $\mathfrak{S}$-labelling for A_j, then by Definition 2.2.1, so too is $\left(H_i, H_j^{i_{jk}}, \left(H_k^{i_{jk}}\right)^{\left(j_k^{i_{jk}}\right)}, H_{v_1}^{i_{v_1}}, \ldots, H_{v_{n-3}}^{i_{v_{n-3}}}\right)$. But

$\left(H_k^{i_{jk}}\right)^{\left(j_k^{i_{jk}}\right)} = H_k^{(i_{jk}^{-1} j_k i_{jk}) i_{jk}} = H_k^{i_{jk} j_k}$. Thus $A_j \in \alpha_3 \cap \alpha_2$, and so $\alpha_2 — \alpha_3$ is an edge of Type A in the Space of Domains.

Finally, since $B_{i,j,k} \in \alpha_1 \cap \alpha_2 \cap \alpha_3$ then $[\alpha_1, \alpha_2, \alpha_3]$ is a 2-cell in the Space of Domains, and so $\alpha_1 — \alpha_2$ is homotopic to the path $\alpha_1 — \alpha_3 — \alpha_2$ whose edges are both of Type A. □

Proposition 5.4.3. *Suppose $\alpha_1 — \alpha_2$ is an edge in the Space of Domains of Type C, but which is not of Type A or Type B. Then there exist domains α_3 and α_4 and edges $\alpha_1 — \alpha_3$, $\alpha_3 — \alpha_4$ and $\alpha_4 — \alpha_2$ of Type A such that $\alpha_1 — \alpha_2$ is homotopic in the Space of Domains to $\alpha_1 — \alpha_3 — \alpha_4 — \alpha_2$.*

Proof. By Proposition 5.2.11 (or rather, Observation 5.2.12), there exists a domain α_3 so that $\alpha_1 — \alpha_2$ is homotopic to $\alpha_1 — \alpha_3 — \alpha_2$, where $\alpha_1 — \alpha_3$ is an edge of Type A and $\alpha_3 — \alpha_2$ is an edge of Type B. Then by Proposition 5.4.2, there exists a domain α_4 with $\alpha_3 — \alpha_2$ homotopic to $\alpha_3 — \alpha_4 — \alpha_2$, where $\alpha_3 — \alpha_4$ and $\alpha_4 — \alpha_2$ are both edges of Type A. Now $\alpha_1 — \alpha_2$ is homotopic to $\alpha_1 — \alpha_3 — \alpha_4 — \alpha_2$. □

Corollary 5.4.4. *Any path in our Graph of Domains is homotopic (in the Space of Domains) to a path whose edges are all of Type A.*

Proof. This follows from Propositions 5.4.2 and 5.4.3, recalling that any edge in the Graph of Domains is at least one of Type A, Type B, or Type C. □

5.5 Relative Whitehead Automorphisms

In this subsection, we consider how to move through the Graph of Domains (i.e. how to move along edges in the Space of Domains). By Corollary 5.4.4, we need only consider edges of Type A.

In Definition 1.1.9, we define Whitehead automorphisms and multiple Whitehead automorphisms, which provide a convenient way to discuss elements of the stabilisers of vertices in the fundamental domain $\mathcal{D}_n$. These are the same (differing only in notation) as those used by Gilbert [12] and Collins and Zieschang [8]. However, to examine domains other than the fundamental domain (i.e. vertices in the Space of Domains), it will prove useful to discuss 'relative' Whitehead automorphisms, which may act on groups beyond just the factor groups G_i of the splitting $\mathfrak{S}$.

We will find that moving along edges of Type A in the Space of Domains is achieved by applying 'relative multiple Whitehead automorphisms', or equivalently, by collapsing and expanding edges of graphs of groups in $\mathcal{C}_n$ (i.e. travelling along $\alpha — A — \alpha$ paths in $\mathcal{C}_n$).

Definition 5.5.1 (Relative Whitehead Automorphism)**.** Let $H_1 * \cdots * H_n$ be an $\mathfrak{S}$ free factor splitting for $G = G_1 * \cdots * G_n$. A relative Whitehead automorphism (with respect to the splitting $H_1 * \cdots * H_n$) is a map ψ for which there exists $x \in H_i$ for some i and $A \subseteq \{H_1, \ldots, H_n\} - \{H_i\}$ so that ψ pointwise conjugates H_j by x for each $H_j \in A$, and pointwise fixes H_k for each $H_k \notin A$. We denote such a map ψ by (A, x). If $|A| = 1$, i.e. $A = \{H_j\}$ for some j, we may abuse notation and write (H_j, x) for $(\{H_j\}, x)$.

If $\mathbf{x} = (x_1, \ldots, x_k) \subset H_i$ for some i and $\mathbf{A} = (A_1, \ldots, A_k)$ where each $A_j \subseteq \{H_1, \ldots, H_n\} - \{H_i\}$ and $A_{j_1} \cap A_{j_2} = \varnothing$ for $j_1 \neq j_2$, then we denote by $(\mathbf{A}, \mathbf{x})$ the composition $(A_1, x_1) \ldots (A_k, x_k)$. Such a map is called a relative multiple Whitehead automorphism. We denote the union $A_1 \cup \cdots \cup A_k$ by $\hat{A}$, or, for longer expressions, by $\bigcup \mathbf{A}$.

The Whitehead automorphisms of Definition 1.1.9 may be thought of as relative Whitehead automorphisms with respect to the initial splitting $G_1 * \cdots * G_n$ of G. However it should be noted that they behave quite differently under composition.

Lemma 5.5.2. *Let $H_1 * \cdots * H_n$ be an $\mathfrak{S}$ free factor splitting for $G_1 * \cdots * G_n$. If $\psi \in \mathrm{Out}(G)$ is a relative Whitehead automorphism (with respect to $H_1 * \cdots * H_n$), then $\psi \in \mathrm{Out}_{\mathfrak{S}}(G)$.*

Proof. By Lemma 1.1.8, there exists $\varphi \in \mathrm{Out}_{\mathfrak{S}}(G)$ with $(G_i)\varphi = H_i = G_i^{g_i}$ for each i (for some $g_i \in G$). Since ψ is a relative Whitehead automorphism for $H_1 * \cdots * H_n$, for each i there exists $h_i \in G$ such that $(H_i)\psi = H_i^{h_i}$. Now let $\chi \in \mathrm{Out}_{\mathfrak{S}}(G)$ be such that for each i, $\chi : g \mapsto g^{(h_i)\varphi^{-1}}$ for all $g \in G_i$. Then for each i, $((G_i)\chi)\varphi = \left((h_i)\varphi^{-1} G_i (h_i^{-1})\varphi^{-1}\right)\varphi = h_i (G_i)\varphi h_i^{-1} = h_i g_i G_i g_i^{-1} h_i^{-1} = G_i^{h_i g_i}$, and the following diagram commutes:

$$\begin{array}{ccccc}
G_1 * \cdots * G_n & \xrightarrow{\ \varphi\ } & G_1^{g_1} * \cdots * G_n^{g_n} & = & H_1 * \cdots * H_n \\
\downarrow \chi & & \downarrow \psi & & \downarrow \psi \\
G_1^{(h_1)\varphi^{-1}} * \cdots * G_n^{(h_n)\varphi^{-1}} & \xrightarrow{\ \varphi\ } & G_1^{h_1 g_1} * \cdots * G_n^{h_n g_n} & = & H_1^{h_1} * \cdots * H_n^{h_n}
\end{array}$$

Thus in $\mathrm{Out}(G)$ we have $\psi = \varphi^{-1}\chi\varphi$ and since $\varphi, \chi \in \mathrm{Out}_{\mathfrak{S}}(G) \leqslant \mathrm{Out}(G)$ then $\psi \in \mathrm{Out}_{\mathfrak{S}}(G)$. ☐

Lemma 5.5.3. *If $\alpha_1 \,—\, \alpha_2$ is an edge in the Space of Domains of Type A, then there exists some relative multiple Whitehead automorphism $(\mathbf{A}, \mathbf{x})$ such that $\alpha_2 = \alpha_1 \cdot (\mathbf{A}, \mathbf{x})$.*

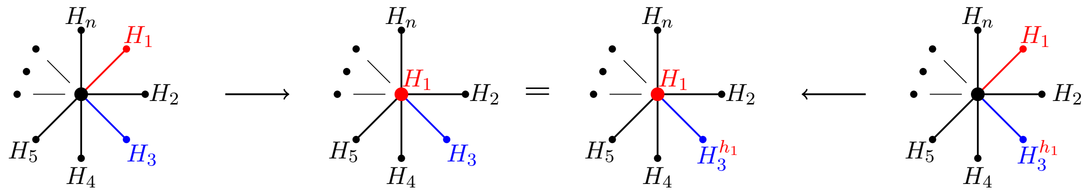
Proof. If $\alpha_1 \,—\, \alpha_2$ is an edge of Type A, then there is some A-graph $A_i \in \alpha_1 \cap \alpha_2 \subset \mathcal{C}_n$. Suppose the α-graph in the domain α_1 has $\mathfrak{S}$-labelling $(H_1, \ldots, H_n) = (G_1^{g_1}, \ldots, G_n^{g_n})$ for some $g_1, \ldots, g_n \in G$. Then so too does A_i, and the α-graph in α_2 must have labelling $((H_1)\chi, \ldots, (H_n)\chi)$ for some $[\chi] \in \mathrm{Stab}(A_i)$.

Let $[\psi] \in \mathrm{Out}_{\mathfrak{S}}(G)$ be such that $H_k = (G_k)\psi$ for all k (that is, $\alpha_1 = \mathcal{D}_n \cdot \psi$). If $\underline{A}_i$ is the A_i-graph in $\mathcal{D}_n$, the fundamental domain, then $\mathrm{Stab}(A_i) = \psi^{-1}\,\mathrm{Stab}(\underline{A}_i)\psi$. Recall from Proposition 2.4.2 and Example 5 that $\mathrm{Stab}(\underline{A}_i)$ comprises elements of the form $(G_{v_1}, y_1) \ldots (G_{v_{n-1}}, y_{n-1})\varphi$ where $y_1, \ldots, y_{n-1} \in G_i$ and $\varphi \in \Phi = \prod_{k=1}^{n} \mathrm{Aut}(G_k)$.

Since $\varphi \in \mathrm{Stab}(\underline{\alpha})$ (where $\underline{\alpha}$ is the α-graph in $\mathcal{D}_n$), then we may assume that $\alpha_2 \cdot \psi^{-1} = \mathcal{D}_n \cdot (G_{v_1}, y_1) \ldots (G_{v_{n-1}}, y_{n-1})$ for some $y_1, \ldots, y_{n-1} \in G_i$. Now for a given factor group G_j (with $j \neq i$), we have $(G_j)(G_j, y_j)\psi = (G_j^{y_j})\psi = ((G_j)\psi)^{((y_j)\psi)} = \left(G_j^{g_j}\right)^{\left(y_j^{g_i}\right)} = (H_j)\left(H_j, y_j^{g_i}\right)$. Observe that since $y_j \in G_i$ then $y_j^{g_i} \in G_i^{g_i} = H_i$.

Thus setting $x_j = y_j^{g_i}$ for all $j \neq i$, we have $\alpha_2 = \alpha_1 \cdot \psi^{-1}(G_{v_1}, y_1) \ldots (G_{v_{n-1}}, y_{n-1})\psi = \alpha_1 \cdot (H_{v_1}, x_1) \ldots (H_{v_{n-1}}, x_{v_{n-1}})$. Grouping together terms for which $x_j = x_k$, we may then write this in the form $\alpha_2 = \alpha_1 \cdot (\mathbf{A}, \mathbf{x})$ with $\hat{A} = \bigcup \mathbf{A} \subseteq \{H_1, \ldots, H_n\} - \{H_i\}$ and $\mathbf{x} \subseteq H_i$, as required. Moreover, $(\mathbf{A}, \mathbf{x}) \in \mathrm{Stab}(A_i)$, as one would expect. ☐

Example 6. Let α_1 be the α-graph with $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$, and let $h_1 \in H_1$. Then $\alpha_2 := \alpha_1 \cdot (H_3, h_1)$ is the α-graph with $\mathfrak{S}$-labelling $(H_1, H_2, H_3^{h_1}, H_4, \ldots, H_n)$. Moreover, $\alpha_1 \,—\, \alpha_2$ is an edge in the Space of Domains of Type A (writing α_i for the domain containing the α-graph α_i). We demonstrate this via the following collapse–expansion path in $\mathcal{C}_n$:



It is not hard to see using this example how to extend Lemma 5.5.3 to an "if and only if" statement. Additionally, we see that relative (multiple) Whitehead automorphisms must obey the relations of the stabiliser of the relevant A-graph in $\mathcal{C}_n$.

Observation 5.5.4. In order to apply this kind of geometric argument, we must ensure that all automorphisms are written relative to the domain on which they act. This includes automorphisms written within a composition. Thus, continuing Example 6, if $h_2 \in H_2$ say, we have $(H_3, h_2h_1) = (H_3, h_1)(H_3^{h_1}, h_2)$.

Lemma 5.5.5. *Let $(H_1, \ldots, H_n)$ be an $\mathfrak{S}$-labelling for some α-graph $\alpha \in \mathcal{C}_n$. Suppose $x \in H_i$ for some i, and $A, B \subseteq \hat{H} = \{H_1, \ldots, H_n\}$ with $H_i \notin A \cup B$. If $A = \{H_{a_1}, \ldots, H_{a_m}\}$, set $A^x := \{H_{a_1}^x, \ldots, H_{a_m}^x\}$.*

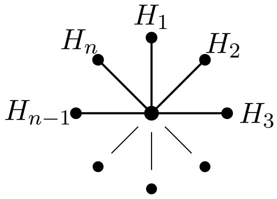
1. *For $x_1, x_2 \in H_i$, we have $\alpha \cdot (A, x_1)(A^{x_1}, x_2) = \alpha \cdot (A, x_2x_1)$.*
2. *We have $\alpha \cdot (A, x)(A^x, x^{-1}) = \alpha$. We will thus write $(A^x, x^{-1}) = (A, x)^{-1}$.*
3. *If $A \cap B = \varnothing$, then $\alpha \cdot (A, x)(B, x) = \alpha \cdot (B, x)(A, x)$, which we may write as $\alpha \cdot (A \cup B, x)$.*

Proof. Let α be the α-graph in $\mathcal{C}_n$ with $(H_1, \ldots, H_n)$ as a labelling. We partition $\{H_1, \ldots, H_n\}$ as $\{H_{a_1}, \ldots, H_{a_m}\} \cup \{H_{i_1}, \ldots, H_{i_s}\}$ where $m + s = n$ and $H_a \in A$ for each $a \in \{a_1, \ldots, a_m\}$.

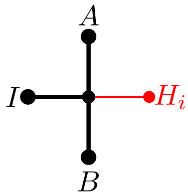
1. By Definition 2.3.1, we have that $\alpha \cdot (A, x_1)$ is the α-graph, say α_1, with labelling $(H_{a_1}^{x_1}, \ldots, H_{a_m}^{x_1}, H_{i_1}, \ldots, H_{i_s})$. Then $\alpha_1 \cdot (A^{x_1}, x_2)$ is the α-graph, say α_2, with labelling $((H_{a_1}^{x_1})^{x_2}, \ldots, (H_{a_m}^{x_1})^{x_2}, H_{i_1}, \ldots, H_{i_s}) = (H_{a_1}^{x_1x_2}, \ldots, H_{a_m}^{x_1x_2}, H_{i_1}, \ldots, H_{i_s})$. On the other hand, we clearly have that $\alpha \cdot (A, x_1x_2) = \alpha_2$.
2. This follows immediately from 1 by setting $x_1 = x$ and $x_2 = x^{-1}$ and noting that $xx^{-1} = 1$ and $(A, 1)$ is the identity for any A.
3. Since $A \cap B = \varnothing$, we will partition $\{H_1, \ldots, H_n\}$ as $\{H_{a_1}, \ldots, H_{a_p}\} \cup \{H_{b_1}, \ldots, H_{b_q}\} \cup \{H_{i_1}, \ldots, H_{i_r}\}$ where $p + q + r = n$, $H_a \in A$ for each $a \in \{a_1, \ldots, a_p\}$, and $H_b \in B$ for each $b \in \{b_1, \ldots, b_q\}$. Set $\alpha_1 := \alpha \cdot (A, x)$ and $\alpha_2 := \alpha \cdot (B, x)$. Then α_1 is the α-graph in $\mathcal{C}_n$ with labelling $(H_{a_1}^x, \ldots, H_{a_p}^x, H_{b_1}, \ldots, H_{b_q}, H_{i_1}, \ldots, H_{i_r})$, and α_2 is the α-graph in $\mathcal{C}_n$ with labelling $(H_{a_1}, \ldots, H_{a_p}, H_{b_1}^x, \ldots, H_{b_q}^x, H_{i_1}, \ldots, H_{i_r})$. Now $\alpha_3 := \alpha_1 \cdot (B, x)$ is the α-graph in $\mathcal{C}_n$ with labelling $(H_{a_1}^x, \ldots, H_{a_p}^x, H_{b_1}^x, \ldots, H_{b_q}^x, H_{i_1}, \ldots, H_{i_r})$. But $\alpha_4 := \alpha_2 \cdot (A, x)$ is also an α-graph in $\mathcal{C}_n$, with the same labelling as α_3. Thus α_3 and α_4 belong to the same $\mathrm{Out}(G)$-orbit of the fundamental domain, $\mathcal{D}_n$. Since $\mathcal{D}_n$ contains a unique α-graph, then we must have that $\alpha_3 = \alpha_4$. That is, $\alpha \cdot (A, x)(B, x) = \alpha \cdot (B, x)(A, x)$.

□

Remark. Let $\hat{H} = \{H_1, \ldots, H_n\}$. For $I = \hat{H} - (A \cup B \cup \{H_i\})$ with $A \cap B = \varnothing$ and $H_i \notin A \cup B$, we can 'partition' the α-graph with labelling $(H_1, \ldots, H_n)$



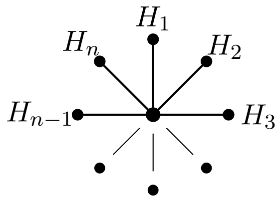
as



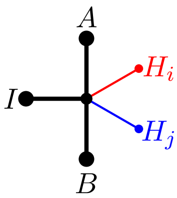
. One should think of this still as an α-graph, but 'abbreviated' — instead of drawing individual edges for each leaf H_j of A, we draw one wider edge (similarly for B and I).

Lemma 5.5.6. *Let $H_1 * \cdots * H_n$ be an $\mathfrak{G}$ free factor splitting for G, and let α be the α-graph (and the domain containing it) with $\mathfrak{G}$-labelling $(H_1, \ldots, H_n)$. Suppose there are elements $x \in H_i$ and $y \in H_j$ for some $i \neq j$, and subsets $A, B \subseteq \{H_1, \ldots, H_n\} - \{H_i, H_j\}$. If $A \cap B = \varnothing$, then $\alpha \cdot (A, x)(B, y) = \alpha \cdot (B, y)(A, x)$.*

Proof. We have that α is the graph of groups



. For $I = \{H_1, \ldots, H_n\} - (A \cup B \cup \{H_i, H_j\})$, we 'partition' α as



, noting that by construction $A \sqcup B \sqcup \{H_i, H_j\} \sqcup I$ forms a disjoint partition of the labelling $(H_1, \ldots, H_n)$ for α. Then the diagram in Figure 8 commutes.

That is, in the Space of Domains there is a loop

$$\begin{array}{ccc} \alpha & \xrightarrow{(A,x)} & \alpha \cdot (A,x) \\ \downarrow{\scriptstyle (B,y)} & & \downarrow{\scriptstyle (B,y)} \\ \alpha \cdot (B,y) & \xrightarrow{(A,x)} & \alpha \cdot (A,x)(B,y) \end{array}$$

which can be seen algebraically by considering the labellings of each α-graph. Thus we have $\alpha \cdot (A, x)(B, y) = \alpha \cdot (B, y)(A, x)$. □

Remark. With notation as in Lemma 5.5.6, if instead we have $H_j \in A$, then $\alpha \cdot (A, x)(B, y)$ is **not** well-defined geometrically — we would instead need to write $\alpha \cdot (A, x)(B, y^x)$. On the other hand, if $A \cap B \neq \varnothing$, say $B \subseteq A$, then $\alpha \cdot (A, x)(B, y)$ would also not be well-defined geometrically. Instead, we would write $\alpha \cdot (A, x)(B^x, y)$. Note that this differs from the notation used for non-relative Whitehead automorphisms, where if α_0 is the α-graph in $\mathcal{D}_n$ (i.e. with labelling $(G_1, \ldots, G_n)$) and $x, y \in G_n$, we would have that $\alpha_0 \cdot (G_1, x)(G_1, y)$ is the α-graph with labelling $(G_1^{yx}, G_2, \ldots, G_n)$. For the remainder of the paper, all automorphisms will be assumed to be relative (multiple) Whitehead automorphisms, unless otherwise specified.

Despite our relative (multiple) Whitehead automorphisms being different objects than the Whitehead automorphisms used by Gilbert [12], we borrow some notation introduced in [12, Section 2]:

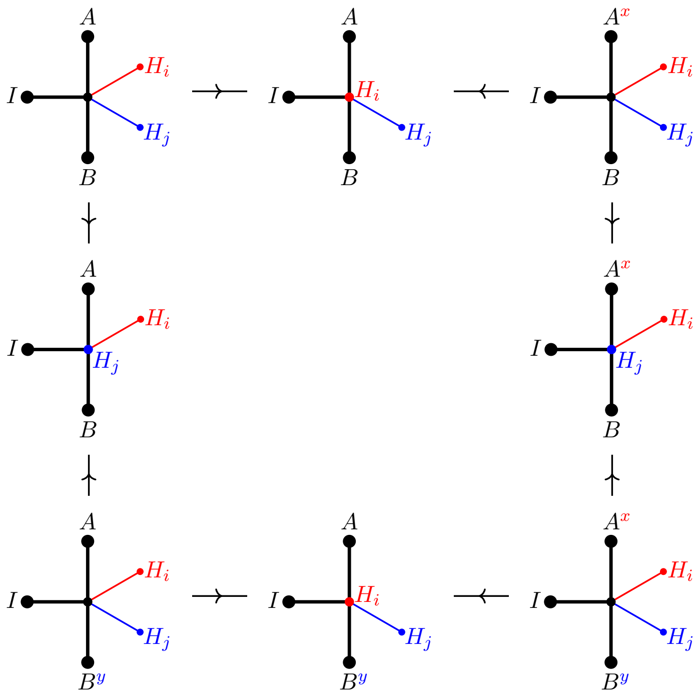


Figure 8: Commuting Diagram of α and A Graphs

Notation 5.5.7. Let $H_1 * \cdots * H_n$ be an $\mathfrak{S}$ free factor splitting for G and set $\hat{H} := \{H_1, \ldots, H_n\}$. Given subsets $A_1, \ldots, A_k \subseteq \hat{H}$ where $A_i \cap A_j = \varnothing$ for $i \neq j$, set $\mathbf{A} = (A_1, \ldots, A_k)$ and let $\hat{A} := A_1 \cup \cdots \cup A_k$. Suppose B is an arbitrary subset of $\hat{H} - \{H_i\}$, let $x_1 \ldots, x_k, y \in H_i$ (where $H_i \notin \hat{A}$) and set $\mathbf{x} = (x_1, \ldots, x_k)$. Also set $\bar{A} := (\hat{H} - \hat{A}) - \{H_i\}$. We will adopt the following notation:

- $(\mathbf{A} \cap B, \mathbf{x}) := (A_1 \cap B, x_1) \ldots (A_k \cap B, x_k)$
- $(\mathbf{A} - B, \mathbf{x}) := (A_1 - B, x_1) \ldots (A_k - B, x_k)$
- $$\begin{aligned}(\mathbf{A} +_j B, \mathbf{x}) :=& (A_1 - B, x_1) \ldots (A_{j-1} - B, x_{j-1})(A_j \cup B, x_j) \\ & (A_{j+1} - B, x_{j+1}) \ldots (A_k - B, x_k) \\ =& (A_1 - B, x_1) \ldots (A_j \cup B, x_j) \ldots (A_k - B, x_k)\end{aligned}$$
- $(\mathbf{A}, y\mathbf{x}) := (A_1, yx_1) \ldots (A_k, yx_k)$
- $(\mathbf{A}, \mathbf{x}y) := (A_1, x_1 y) \ldots (A_k, x_k y)$
- $$\begin{aligned}(\bar{\mathbf{A}}_j, \mathbf{x}) :=& (A_1, x_1) \ldots (A_{j-1}, x_{j-1})(\bar{A}, x_j)(A_{j+1}, x_{j+1}) \ldots (A_k, x_k) \\ =& (A_1, x_1) \ldots (\bar{A}, x_j) \ldots (A_k, x_k)\end{aligned}$$
- $$\begin{aligned}(\mathbf{A}, \tilde{\mathbf{x}}_j) :=& (A_1, x_1) \ldots (A_{j-1}, x_{j-1})(A_j, 1)(A_{j+1}, x_{j+1}) \ldots (A_k, x_k) \\ =& (A_1, x_1) \ldots (A_j, 1) \ldots (A_k, x_k)\end{aligned}$$
- $[\mathbf{A}]_j := A_j$, $[\mathbf{x}]_j := x_j$, and $[(\mathbf{A}, \mathbf{x})]_j := (A_j, x_j)$
- If $A = \{H_{a_1}, \ldots, H_{a_m}\}$ then $A^x := \{H^x_{a_1}, \ldots, H^x_{a_m}\}$ and we define $\mathbf{A}^{\mathbf{x}} := (A_1^{x_1}, \ldots, A_k^{x_k})$

Note that $(\mathbf{A} \cap B, \mathbf{x})$ and $(\mathbf{A} - B, \mathbf{x})$ may still be defined when $H_i \in B$.

We find that similar (though not identical) properties hold for us as are used by Gilbert [12]. In particular, part 4 of the following Proposition is adapted from [12, Lemma 2.10].

Proposition 5.5.8. *With the above notation, we have that:*

1. $(\mathbf{A}, \mathbf{x}) = (\mathbf{A} - B, \mathbf{x})(\mathbf{A} \cap B, \mathbf{x}) = (\mathbf{A} \cap B, \mathbf{x})(\mathbf{A} - B, \mathbf{x})$
2. $(\mathbf{A} +_j B, \mathbf{x}) = (\mathbf{A} - B, \mathbf{x})(B, x_j)$
3. $(\bar{\mathbf{A}}_j, x_j^{-1}\tilde{\mathbf{x}}_j) = (\mathbf{A}, x_j^{-1}\mathbf{x})(\bar{A}, x_j^{-1})$ *and* $(\bar{\mathbf{A}}_j, \tilde{\mathbf{x}}_j x_j^{-1}) = (\mathbf{A}, \mathbf{x}x_j^{-1})(\bar{A}, x_j^{-1})$
4. $(\mathbf{A}, \mathbf{x}) = (\mathbf{A} +_j B, \mathbf{x})((\bar{\mathbf{A}}_j \cap B)^{x_j}, \tilde{\mathbf{x}}_j x_j^{-1}) = (\bar{\mathbf{A}}_j \cap B, x_j^{-1}\tilde{\mathbf{x}}_j)\,((\mathbf{A} +_j B)', \mathbf{x})$ *where* $[(\mathbf{A} +_j B)']_a := [(\mathbf{A} +_j B)]_a = A_a - B$ *for* $a \in \{1, \dots, k\} - \{j\}$ *and*
$$[(\mathbf{A} +_j B)']_j := \Big([(\mathbf{A} +_j B)]_j - \bigcup(\bar{\mathbf{A}}_j \cap B)\Big) \cup \bigcup\left([(\mathbf{A} +_j B)]_j \cap (\bar{\mathbf{A}}_j \cap B)\right)^{x_j^{-1}\tilde{\mathbf{x}}_j}$$
$$= (A_j - B) \cup \bigcup(\mathbf{A} \cap B)^{x_j^{-1}\mathbf{x}} \cup (B - \hat{A})^{x_j^{-1}}$$

Proof. 1. Given arbitrary sets A and B, we have $A = (A - B) \cup (A \cap B)$, which is a disjoint partition of the set A. Now
$$\begin{aligned}(\mathbf{A}, \mathbf{x}) &= (A_1, x_1) \dots (A_k, x_k)\\ &= ((A_1 - B) \cup (A_1 \cap B), x_1) \dots ((A_k - B) \cup (A_k \cap B), x_k)\\ &= (A_1 - B, x_1)(A_1 \cap B, x_1) \dots (A_k - B, x_k)(A_k \cap B, x_k)\\ &= (A_1 - B, x_1) \dots (A_k - B, x_k)(A_1 \cap B, x_1) \dots (A_k \cap B, x_k)\\ &= (\mathbf{A} - B, \mathbf{x})(\mathbf{A} \cap B, \mathbf{x}).\end{aligned}$$
Moreover, we also have $A = (A \cap B) \cup (A - B)$, and a similar argument yields $(\mathbf{A}, \mathbf{x}) = (\mathbf{A} \cap B, \mathbf{x})(\mathbf{A} - B, \mathbf{x})$.

2. Given arbitrary sets A and B, we have $A \cup B = (A - B) \cup B$, which is a disjoint partition of the set $A \cup B$. Now
$$\begin{aligned}(\mathbf{A} +_j B, \mathbf{x}) &= (A_1 - B, x_1) \dots (A_j \cup B, x_j) \dots (A_k - B, x_k)\\ &= (A_1 - B, x_1) \dots ((A_j - B) \cup B, x_j) \dots (A_k - B, x_k)\\ &= (A_1 - B, x_1) \dots (A_j - B, x_j)(B, x_j) \dots (A_k - B, x_k)\\ &= (A_1 - B, x_1) \dots (A_j - B, x_j) \dots (A_k - B, x_k)(B, x_j)\\ &= (\mathbf{A} - B, \mathbf{x})(B, x_j).\end{aligned}$$

3. We have
$$\begin{aligned}(\bar{\mathbf{A}}_j, x_j^{-1}\tilde{\mathbf{x}}_j) &= (A_1, x_j^{-1}x_1) \dots (\bar{A}, x_j^{-1}1) \dots (A_k, x_j^{-1}x_k)\\ &= (A_1, x_j^{-1}x_1) \dots (A_j, 1)(\bar{A}, x_j^{-1}) \dots (A_k, x_j^{-1}x_k)\\ &= (A_1, x_j^{-1}x_1) \dots (A_j, 1) \dots (A_k, x_j^{-1}x_k)(\bar{A}, x_j^{-1})\\ &= (A_1, x_j^{-1}x_1) \dots (A_j, x_j^{-1}x_j) \dots (A_k, x_j^{-1}x_k)(\bar{A}, x_j^{-1})\\ &= (\mathbf{A}, x_j^{-1}\mathbf{x})(\bar{A}, x_j^{-1}).\end{aligned}$$
Similarly,
$$\begin{aligned}(\bar{\mathbf{A}}_j, \tilde{\mathbf{x}}_j x_j^{-1}) &= (A_1, x_1 x_j^{-1}) \dots (\bar{A}, x_j^{-1}) \dots (A_k, x_k x_j^{-1})\\ &= (A_1, x_1 x_j^{-1}) \dots (A_j, x_j x_j^{-1})(\bar{A}, x_j^{-1}) \dots (A_k, x_k x_j^{-1})\\ &= (A_1, x_1 x_j^{-1}) \dots (A_j, x_j x_j^{-1}) \dots (A_k, x_k x_j^{-1})(\bar{A}, x_j^{-1})\\ &= (\mathbf{A}, \mathbf{x}x_j^{-1})(\bar{A}, x_j^{-1}).\end{aligned}$$

4. By 2. and 3. above, we have
$$(\mathbf{A} +_j B, \mathbf{x})((\bar{\mathbf{A}}_j \cap B)^{x_j}, \tilde{\mathbf{x}}_j x_j^{-1})$$
$$= (\mathbf{A} - B, \mathbf{x})(B, x_j)((\mathbf{A} \cap B)^{x_j}, \mathbf{x}x_j^{-1})((\bar{A} \cap B)^{x_j}, x_j^{-1})$$
$$= (\mathbf{A} - B, \mathbf{x})(B \cap \hat{A}, x_j)(B - \hat{A}, x_j)((\mathbf{A} \cap B)^{x_j}, \mathbf{x}x_j^{-1})((B - \hat{A})^{x_j}, x_j^{-1})$$
$$= (\mathbf{A} - B, \mathbf{x})(\hat{A} \cap B, x_j)((\mathbf{A} \cap B)^{x_j}, \mathbf{x}x_j^{-1})(B - \hat{A}, x_j)((B - \hat{A})^{x_j}, x_j^{-1})$$
$$= (\mathbf{A} - B, \mathbf{x})(\mathbf{A} \cap B, \mathbf{x}x_j^{-1}x_j)(B - \hat{A}, x_j^{-1}x_j)$$
$$= (\mathbf{A} - B, \mathbf{x})(\mathbf{A} \cap B, \mathbf{x})$$
$$= (\mathbf{A}, \mathbf{x}).$$
Similarly,
$$(\bar{\mathbf{A}}_j \cap B, x_j^{-1}\tilde{\mathbf{x}}_j)((\mathbf{A} +_j B)', \mathbf{x})$$
$$= (\mathbf{A} \cap B, x_j^{-1}\mathbf{x})(B - \hat{A}, x_j^{-1})(\mathbf{A} - B, \mathbf{x})((\bigcup(\mathbf{A} \cap B)^{x_j^{-1}\mathbf{x}}) \cup (B - \hat{A})^{x_j^{-1}}), x_j)$$
$$= (\mathbf{A} - B, \mathbf{x})(\mathbf{A} \cap B, x_j^{-1}\mathbf{x})(B - \hat{A}, x_j^{-1})(\bigcup(\mathbf{A} \cap B)^{x_j^{-1}\mathbf{x}}, x_j)(B - \hat{A})^{x_j^{-1}}, x_j)$$
$$= (\mathbf{A} - B, \mathbf{x})(\mathbf{A} \cap B, x_j^{-1}\mathbf{x})(\bigcup(\mathbf{A} \cap B)^{x_j^{-1}\mathbf{x}}, x_j)(B - \hat{A}, x_j^{-1})(B - \hat{A})^{x_j^{-1}}, x_j)$$
$$= (\mathbf{A} - B, \mathbf{x})(\mathbf{A} \cap B, \mathbf{x})$$
$$= (\mathbf{A}, \mathbf{x}).$$

□

6 Peak Reduction in the Space of Domains

In this section we will prove that the Space of Domains is simply connected. We will do this via 'peak reduction'. The idea of this is that given a (based) loop in the Space of Domains, any 'peaks' in the loop can be reduced, until the loop is just the basepoint. This is roughly illustrated in Figure 9.

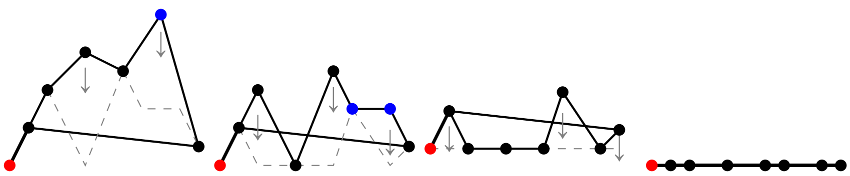


Figure 9: The Idea of 'Squashing Loops' by Reducing Peaks

We will follow the outline below, which is largely based on the method used by Gilbert [12, Section 2] (which in turned is based on the work of Collins and Zieschang [8, Section 2])[3]:

- Define a concept of 'height' of a vertex/domain α, and define a 'peak' of a loop in the Graph/Space of Domains
- By Corollary 5.4.4, we may solely consider loops comprising edges of Type A, so a 'peak' looks like $\alpha \overset{A}{—} \alpha \overset{A}{—} \alpha$

[3]Note however that the objects Gilbert as well as Collins and Zieschang study are words in a group, as opposed to geometric objects, thus while the overall structure of the idea is similar, the details vary greatly.

- Split into four cases of $\alpha \overset{A_i}{\text{---}} \alpha \overset{A_j}{\text{---}} \alpha$ for various conditions on i and j
- For a given path $\alpha \overset{A_i}{\text{---}} \alpha \overset{A_j}{\text{---}} \alpha$ show there is either a 4-cycle or 5-cycle in the Graph of Domains (whose edges are all of Type A) with $\alpha \overset{A_i}{\text{---}} \alpha \overset{A_j}{\text{---}} \alpha$ as a subpath
- Given such a loop in the Graph of Domains, show that it is contractible in the Space of Domains (that is, that $\alpha \overset{A_i}{\text{---}} \alpha \overset{A_j}{\text{---}} \alpha$ is homotopic to a path of length 2 or 3 with the same endpoints)
- Show that if $\alpha \overset{A_i}{\text{---}} \alpha \overset{A_j}{\text{---}} \alpha$ was a peak in some loop in the Space of Domains, then it is homotopic to a path whose 'middle' is 'smaller' than that of $\alpha \overset{A_i}{\text{---}} \alpha \overset{A_j}{\text{---}} \alpha$

Note that we are only interested in loops in the Space of Domains whose endpoints are vertices and who strictly follow edge paths (with no backtracking etc.). This is permissible, as any loop can be deformed into such a loop quite easily. It also means we can easily consider the loop in the Graph of Domains (which is just the one-skeleton of the Space of Domains).

6.1 Defining Height

Notation 6.1.1. Let α be an α-graph 

in $\mathcal{C}_n$ with $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$. We denote by $\hat{\alpha}$ the G-tree which is the universal cover relative to α (according to Serre [20]), that is, the G-tree satisfying (up to equivariant isometry) $\hat{\alpha}\diagup G = \alpha$ viewing α here as a quotient graph of groups (via Bass–Serre theory). We will label vertices of $\hat{\alpha}$ by their stabiliser in G, so, for example, $g \cdot G_i = G_i^g$ for $g \in G$ and $G_i \in \hat{\alpha}$. When edges in $\hat{\alpha}$ are given labels, we will write the action of G multiplicatively, so, for example, $g \cdot e = ge$.

Remark. Note that each labelling $(H_1', \ldots, H_n')$ in the equivalence class of $(H_1, \ldots, H_n)$ of labellings of α determines a lift of α in $\hat{\alpha}$ (by taking the convex hull of the vertices in $\hat{\alpha}$ with stabilisers $H_1', \ldots, H_n'$). Since this lift has the same 'shape' as α (deduced from equivalence of the labellings), we may consider α with its labelling $(H_1, \ldots, H_n)$ to be a subgraph of $\hat{\alpha}$, acting as a fundamental domain for the action of G.

Let α_0 be the fundamental domain $\mathcal{D}_n$ (the domain with the graph in Figure 10 at its centre). Let α be an arbitrary domain (with the graph in Figure 11 at its centre).

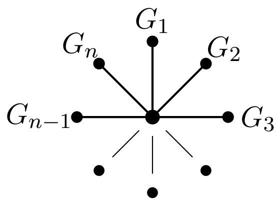


Figure 10: The α Graph at the Centre of the Fundamental Domain

Let $\mathcal{W}$ be the set of pairs $\{G_i, G_j\}$ of (distinct) elements of $\{G_1, \ldots, G_n\}$. Note that $|\mathcal{W}| = \binom{n}{2} = \frac{1}{2}n(n-1)$.

Set $|\{G_i, G_j\}|_\alpha$ to be the length of the edge path from the vertex labelled G_i to the vertex labelled G_j in $\hat{\alpha}$ (note that this is symmetric, so we don't need to worry about the order of our pair).

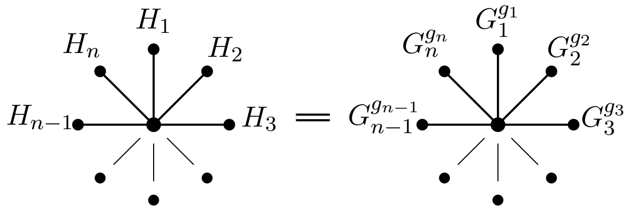


Figure 11: An Arbitrary α Graph

Definition 6.1.2. We define the height of the domain α to be

$$||\alpha|| := \sum_{w \in \mathcal{W}} (|w|_\alpha - 2)$$

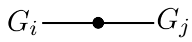
Note that this can only take (non-negative) integer values.

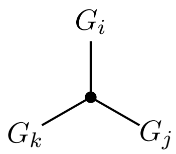
Lemma 6.1.3. *We have* $||\alpha|| = 0$ *if and only if* $\alpha = \alpha_0$.

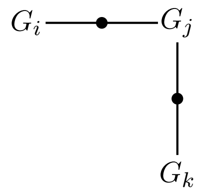
Proof. On the one hand, in $\hat{\alpha_0}$ we have that $|w|_{\alpha_0} = 2 \ \forall w \in \mathcal{W}$, so clearly $||\alpha_0|| = 0$.

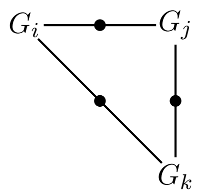
On the other hand, observe that for any $w \in \mathcal{W}$, $|w|_\alpha \geqslant 2 \ \forall \alpha$. So

$$\begin{aligned} ||\alpha|| = ||\alpha_0|| &\implies \sum_{w \in \mathcal{W}} (|w|_\alpha - 2) = 0 \\ &\implies \sum_{w \in \mathcal{W}} |w|_\alpha = 2|\mathcal{W}| \\ &\implies |w|_\alpha = 2 \quad \forall w \in \mathcal{W} \end{aligned}$$

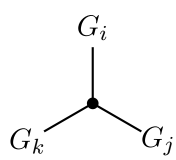
We claim that this implies $\alpha = \alpha_0$. Indeed, if there is some α such that for all $i, j \in \{1, \ldots, n\}$ we have $|\{G_i, G_j\}|_\alpha = 2$, then the path in $\hat{\alpha}$ between the vertex labelled G_i and the vertex labelled G_j must be 

(for each pair $\{G_i, G_j\}$). Suppose for some $i, j, k \in \{1 \ldots, n\}$ that $\hat{\alpha}$ does not contain the tripod 

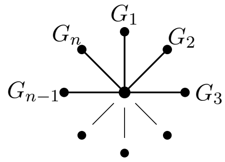
. Then $\hat{\alpha}$ must contain the path 

. But the only way for the length of (G_i, G_k) to be 2 now is to have a cycle 

. But $\hat{\alpha}$ is a tree, so this cannot happen.

Hence $\hat{\alpha}$ must contain every tripod of the form 

for all $i, j, k \in$

$\{1, \ldots, n\}$. But this precisely means that $\hat{\alpha}$ contains as a subgraph the star 

.

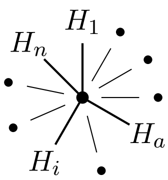
Thus if $|w|_\alpha = 2 \ \forall w \in \mathcal{W}$, then we must have $\alpha = \alpha_0$.

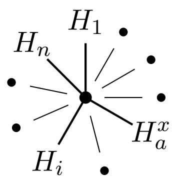
□

Note that we think of $\{G_i, G_j\}$ as both a pair of groups, and a pair of vertices in the universal cover of the graph of groups. Since the universal cover is a tree, there is a unique path from the vertex whose stabiliser is G_i to the vertex whose stabiliser is G_j, so we may also use $\{G_i, G_j\}$ to refer to the edge path connecting them (in a given $\hat{\alpha}$).

Definition 6.1.4. Let w be a sequence of edges forming a path in a G-tree $\hat{\alpha}$, and let u be any subpath of w. Denote by $\Lambda_w(u)$ the number of times the subword u (or some G-translation $z \cdot u$, or inverse $\overline{z \cdot u}$) appears in w. Define $|w|_\alpha$ to be the reduced path length of w in $\hat{\alpha}$.

Convention 6.1.5. Let α_1 be the α-graph with $\mathfrak{G}$-labelling $(H_1, \ldots, H_n)$, let $\psi \in \mathrm{Out}_{\mathfrak{G}}(G)$, and set $\alpha_2 := \alpha_1 \cdot \psi$. Recall that $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are the G-trees associated to α_1 and α_2, respectively, with vertices labelled by their G-stabiliser. Call the vertex with trivial stabiliser in the convex hull of the vertices $H_1, \ldots, H_n$ in $\hat{\alpha}_1$ 'v', and the vertex with trivial stabiliser in the convex hull of the vertices $(H_1)\psi, \ldots, (H_n)\psi$ in $\hat{\alpha}_2$ 'v''. We equivariantly label the edges of $\hat{\alpha}_1$ by assigning an edge $v \rightarrowtail H_j$ the label 'e_j', and its G-images 'xe_j' where $x \in G$. Similarly, we label the edges of $\hat{\alpha}_2$ of the form $v' \rightarrowtail (H_j)\psi$ 'f_j', and equivariantly extend this to a labelling of all the edges of $\hat{\alpha}_2$. We now define an equivariant map $\varphi_\psi : \hat{\alpha}_1 \to \hat{\alpha}_2$ with $(v)\varphi_\psi = v'$ so that the vertex in $\hat{\alpha}_1$ whose stabiliser is H_j is mapped to the vertex in $\hat{\alpha}_2$ whose stabiliser is H_j.

Example 7. Let α_1 be 

and $\alpha_2 = \alpha_1(H_a, x)$ (with $x \in H_i$) be 

.

Consider (the subgraphs of) $\hat{\alpha}_1$ and $\hat{\alpha}_2$ as illustrated in Figure 12, labelled according to Convention 6.1.5. Observe that $(e_a)\varphi_{(H_a,x)} = f_i(\overline{x^{-1}f_i})(x^{-1}f_a)$, where $x^{-1}f_i$ is the

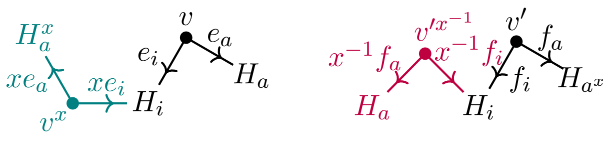


Figure 12: Subgraphs of $\hat{\alpha}_1$ (left) and $\hat{\alpha}_2$ (right)

image in $\hat{\alpha}_2$ of f_i under the action of x^{-1}, and $\overline{x^{-1}f_i}$ is its inverse edge (the same 1-cell, with opposite orientation), and for $j \neq a$ we have $(e_j)\varphi_{(H_a,x)} = f_j$. Moreover, we see that $(e_i(\overline{xe_i})(xe_a))\varphi_{(H_a,x)} = f_a$. Thus $\varphi_{(H_a,x)}$ expands some edge paths, contracts some edge paths, and does not change the length of other edge paths.

We now prove some technical lemmas. Unless otherwise stated, Λ_w will always concern the word w relating to α_1. The following lemmas (and proofs) are similar in structure to a lemma (and proof) of Collins and Zieschang [8, Lemma 1.5], in that we count 'subwords' to determine how an automorphism changes the height of a domain. However, since our arguments are applied to different objects, we recover quite different formulae.

Lemma 6.1.6. *Let α_1 and $\alpha_2 = \alpha_1(H_a, x)$ be as in Example 7, and let w be an edge path in $\hat{\alpha}_1$. Then*

$$|(w)\varphi_{(H_a,x)}|_{\alpha_2} = |w|_{\alpha_1} + 2\Lambda_w(e_a) - 2\Lambda_w(\bar{e_i}e_a) - 2\Lambda_w((x^{-1}e_i)\bar{e_i}e_a)$$

Proof. Given any word u, let $l(u)$ be its unreduced length. We may assume that w is a reduced word, that is if $w = d_1d_2\dots d_m$, then for any d_j, we have that $d_{j+1} \neq \overline{d_j}$. Then $l(w) = \sum_{k=1}^{n} \Lambda_w(e_k) = |w|_{\alpha_1} = m$. Since w is an edge path in $\hat{\alpha}_1$, then given a letter (edge) d_j, there is some $k \in \{1, \dots, n\}$ and some $y \in G$ so that either $d_j = ye_k$ or $d_j = \overline{ye_k}$. Equivariance of $\varphi_{(H_a,x)}$ means that $(ye_k)\varphi_{(H_a,x)} = y(e_k)\varphi_{(H_a,x)}$, and $(\overline{e_k})\varphi_{(H_a,x)} = \overline{(e_k)\varphi_{(H_a,x)}}$.

Let w' be the unreduced word $(w)\varphi_{(H_a,x)}$ in α_2. That is, $w' = (d_1)\varphi_{(H_a,x)}\dots(d_m)\varphi_{(H_a,x)}$ and $l(w') = l((d_1)\varphi_{(H_a,x)}) + \dots + l((d_m)\varphi_{(H_a,x)})$. We will say $d_j \simeq e_k$ if $d_j = ye_k$ or $d_j = \overline{ye_k}$ for some $k \in \{1, \dots, n\}$ and some $y \in G$. If $d_j \simeq e_k$ for some $k \neq a$ then $(d_j)\varphi_{(H_a,x)} \simeq f_k$ and $l((d_j)\varphi_{(H_a,x)}) = l(d_j) = 1$. If on the other hand $d_j \simeq e_a$, then $(d_j)\varphi_{(H_a,x)} \simeq f_i(\overline{x^{-1}f_i})(x^{-1}f_a)$, and $l((d_j)\varphi_{(H_a,x)}) = 3l(d_j)$. Thus:

$$l(w') = \sum_{k \neq a} \Lambda_w(e_k) + 3\Lambda(e_a) = l(w) + 2\Lambda(e_a) = |w|_{\alpha_1} + 2\Lambda_w(e_a)$$

We now consider reductions to w'. Note that $(\bar{e_i})\varphi_{(H_a,x)}(e_a)\varphi_{(H_a,x)} = \bar{f_i}f_i(\overline{x^{-1}f_i})(x^{-1}f_a)$, which reduces to $(\overline{x^{-1}f_i})(x^{-1}f_a)$. Let w'' be the result of applying all such reductions to w' (including inversions and G-translations of the subword $\bar{f_i}f_i$ resulting from images $(\bar{e_i}e_a)\varphi_{(H_a,x)})$. Then the length of $(\bar{e_i}e_a)\varphi_{(H_a,x)}$ (and its inversions and G-translations) is 2 less in w'' than it is in w'. Hence $l(w'') = l(w') - 2\Lambda_w(\bar{e_i}e_a)$.

We also have that $(x^{-1}e_i)\varphi_{(H_a,x)}(\bar{e_i}e_a)\varphi_{(H_a,x)} = (x^{-1}f_i)(\overline{x^{-1}f_i})(x^{-1}f_a)$, which reduces to $x^{-1}f_a$. Let w''' be the result of applying all such reductions to w'' (including inversions and G-translations of the subword $((x^{-1}e_i)\bar{e_i}e_a)\varphi_{(H_a,x)})$. Then the length of $((x^{-1}e_i)\bar{e_i}e_a)\varphi_{(H_a,x)}$ (and its inversions and G-translations) is 2 less in w''' than it is in w''. So $l(w''') = l(w'') - 2\Lambda_w((x^{-1}e_i)\bar{e_i}e_a)$. Since w was assumed to be reduced, there are no further reductions we can apply to w''', hence $l(w''') = |(w)\varphi_{(H_a,x)}|_{\alpha_2}$. We therefore have:

$$\begin{aligned}|(w)\varphi_{(H_a,x)}|_{\alpha_2} = l(w''') &= l(w'') - 2\Lambda_w((x^{-1}e_i)\bar{e_i}e_a) \\ &= l(w') - 2\Lambda_w(\bar{e_i}e_a) - 2\Lambda_w((x^{-1}e_i)\bar{e_i}e_a) \\ &= |w|_{\alpha_1} + 2\Lambda_w(e_a) - 2\Lambda_w(\bar{e_i}e_a) - 2\Lambda_w((x^{-1}e_i)\bar{e_i}e_a).\end{aligned}$$

□

Lemma 6.1.7. *Let α_1 be the α-graph with $\mathfrak{S}$-labelling $(H_1, \dots, H_n)$, let $x \in H_i$ for some i, and let $A \subseteq \{H_1, \dots, H_n\} - \{H_i\}$. Set $\alpha_2 := \alpha_1(A, x)$, label $\hat{\alpha}_1$ and $\hat{\alpha}_2$ according to Convention 6.1.5, and let $\varphi_{(A,x)} : \hat{\alpha}_1 \to \hat{\alpha}_2$ be the equivariant map described in Convention 6.1.5. Given an edge path w in $\hat{\alpha}_1$, we have:*

$$|(w)\varphi_{(A,x)}|_{\alpha_2} = |w|_{\alpha_1} + 2\sum_{a:H_a\in A}\left(\Lambda_w(e_a) - \Lambda_w(\bar{e_i}e_a) - \Lambda_w((x^{-1}e_i)\bar{e_i}e_a) - \sum_{b:H_b\in A-\{H_a\}}\Lambda_w(\bar{e_a}e_b)\right)$$

Proof. As in Lemma 6.1.6, we let $l(u)$ be the unreduced length of a given word u, and we assume that w is a reduced word $w = d_1 d_2 \dots d_m$ with $l(w) = m = |w|_{\alpha_1}$.

Let w' be the unreduced word $(w)\varphi_{(A,x)}$ in $\hat{\alpha}_2$. That is, $w' = (d_1)\varphi_{(A,x)} \dots (d_m)\varphi_{(A,x)}$ and $l(w') = l((d_1)\varphi_{(A,x)}) + \dots + l((d_m)\varphi_{(A,x)})$. Extended from Lemma 6.1.6, we have that $(e_a)\varphi_{(A,x)} = f_i(\overline{x^{-1}f_i})(x^{-1}f_a)$ for any a such that $H_a \in A$, and $(e_k)\varphi_{(A,x)} = f_k$ for any k such that $H_k \notin A$. Thus if $d_j \simeq e_a$ where $H_a \in A$ then $l((d_j)\varphi_{(A,x)}) = 3 = l(d_j) + 2$ and if $d_j \simeq e_k$ where $H_k \notin A$ then $l((d_j)\varphi_{(A,x)}) = 1 = l(d_j)$. Now:

$$l(w') = \sum_{j=1}^{m} l((d_j)\varphi_{(A,x)}) = 3\sum_{a:H_a\in A}\Lambda_w(e_a) + \sum_{k:H_k\notin A}\Lambda_w(e_k) = l(w) + 2\sum_{a:H_a\in A}\Lambda(e_a)$$

We now consider reductions to the word w'. As in Lemma 6.1.6, we have that for any a where $H_a \in A$, $(\overline{e_i})\varphi_{(A,x)}(e_a)\varphi_{(A,x)} = \overline{f_i} f_i(\overline{x^{-1}f_i})(x^{-1}f_a) = (\overline{x^{-1}f_i})(x^{-1}f_a)$ and $(x^{-1}e_i)\varphi_{(A,x)}(\overline{e_i}e_a)\varphi_{(A,x)} = (x^{-1}f_i)(\overline{x^{-1}f_i})(x^{-1}f_a) = x^{-1}f_a$. Let w'' be the result of applying all such reductions to w'. Then:

$$l(w'') = l(w') - 2\sum_{a:H_a\in A}\left(\Lambda_w(\overline{e_i}e_a) + \Lambda_w((x^{-1}e_i)\overline{e_i}e_a)\right)$$

Contrary to Lemma 6.1.6, w'' is not yet fully reduced. Indeed, observe that for any distinct a and b with $H_a, H_b \in A$, we have that $(\overline{e_a}e_b)\varphi_{(A,x)} =$ $(\overline{x^{-1}f_a})(x^{-1}f_i)\overline{f_i}f_i(\overline{x^{-1}f_i})(x^{-1}f_b) = (\overline{x^{-1}f_a})(x^{-1}f_b)$. Let w''' be the result of applying all such reductions to w'' (including inversions and G-translations). Then for each distinct a and b with $H_a, H_b \in A$, the length of $(\overline{e_a}e_b)\varphi_{(A,x)}$ is 4 less in w''' than it is in w''. Note that for any a and b we have $\Lambda_w(\overline{e_a}e_b) = \Lambda_w(\overline{\overline{e_a}e_b}) = \Lambda_w(\overline{e_b}e_a)$. Thus:

$$l(w''') = l(w'') - \frac{1}{2}\sum_{a:H_a\in A}\sum_{b:H_b\in A-\{H_a\}} 4\Lambda_w(\overline{e_a}e_b) = l(w'') - 2\sum_{a:H_a\in A}\sum_{b:H_b\in A-\{H_a\}}\Lambda_w(\overline{e_a}e_b)$$

Since w was assumed to be reduced, there are now no further reductions we can apply to w''', hence $l(w''') = |(w)\varphi_{(A,x)}|_{\alpha_2}$. We therefore have:

$$\begin{aligned}
&|(w)\varphi_{(A,x)}|_{\alpha_2} \\
=&l(w''') \\
=&l(w'') - 2\sum_{a:H_a\in A}\sum_{b:H_b\in A-\{H_a\}}\Lambda_w(\overline{e_a}e_b) \\
=&l(w') - 2\sum_{a:H_a\in A}\left(\Lambda_w(\overline{e_i}e_a) + \Lambda_w((x^{-1}e_i)\overline{e_i}e_a)\right) - 2\sum_{a:H_a\in A}\sum_{b:H_b\in A-\{H_a\}}\Lambda_w(\overline{e_a}e_b) \\
=&l(w) + 2\sum_{a:H_a\in A}\Lambda(e_a) - 2\sum_{a:H_a\in A}\left(\Lambda_w(\overline{e_i}e_a) + \Lambda_w((x^{-1}e_i)\overline{e_i}e_a) + \sum_{b:H_b\in A-\{H_a\}}\Lambda_w(\overline{e_a}e_b)\right) \\
=&|w|_{\alpha_1} + 2\sum_{a:H_a\in A}\left(\Lambda(e_a) - \Lambda_w(\overline{e_i}e_a) - \Lambda_w((x^{-1}e_i)\overline{e_i}e_a) - \sum_{b:H_b\in A-\{H_a\}}\Lambda_w(\overline{e_a}e_b)\right).
\end{aligned}$$

□

Lemma 6.1.8. *Let α_1 be the α-graph with $\mathfrak{S}$-labelling $(H_1, \dots, H_n)$, and let $(\mathbf{A}, \mathbf{x})$ be a relative multiple Whitehead automorphism with respect to α_1, where $\mathbf{x} \subset H_i$ for* *some i. For brevity, we will write $\sum_{H_a\in A_j}$ for $\sum_{a:H_a\in A_j}$ (etc.). If $\alpha_2 = \alpha_1(\mathbf{A}, \mathbf{x})$, then $||\alpha_2|| - ||\alpha_1||$ is equal to:*

$$2\sum_{w\in\mathcal{W}}\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\left(\Lambda_w(e_a) - \Lambda_w(\bar{e_i}e_a) - \Lambda_w((x_j^{-1}e_i)\bar{e_i}e_a) - \sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b) - \frac{1}{2}\sum_{H_c\in\hat{A}-A_j}\Lambda_w(\overline{e_a}e_c)\right)$$

Proof. We have that $(\mathbf{A}, \mathbf{x}) = (A_1, x_1)\dots(A_K, x_K)$ for some disjoint subsets $A_1, \dots, A_K \subset \{H_1, \dots, H_n\} - \{H_i\}$ and some distinct $x_1, \dots, x_K \in H_i$. We consider a word $w = d_1\dots d_m$ in $\hat{\alpha}_1$ and its unreduced image $w' = (d_1)\varphi_{(\mathbf{A},\mathbf{x})}\dots(d_m)\varphi_{(\mathbf{A},\mathbf{x})}$ in $\hat{\alpha}_2$. For any word u, let $l(u)$ be its unreduced length. As in Lemmas 6.1.6 and 6.1.7, we have that for any a with $H_a \in A_j \in \mathbf{A}$, $(e_a)\varphi_{(\mathbf{A},\mathbf{x})} = f_i(\overline{x_j^{-1}f_i})(x_j^{-1}f_a)$, and for any k with $H_k \notin \hat{A}$, $(e_k)\varphi_{(\mathbf{A},\mathbf{x})} = f_k$. Thus $l(w') = l(w) + 2\sum_{H_a\in\hat{A}}\Lambda_w(e_a) = |w|_{\alpha_1} + 2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\Lambda_w(e_a)$.

As in Lemma 6.1.7, each (A_j, x_j) leads to reductions of the forms:

(i) $(\bar{e_i}e_a)\varphi_{(\mathbf{A},\mathbf{x})} = \overline{f_i}f_i(\overline{x_j^{-1}f_i})(x_j^{-1}f_a) = (\overline{x_j^{-1}f_i})(x_j^{-1}f_a)$,

(ii) $((x_j^{-1}e_i)\bar{e_i}e_a)\varphi_{(\mathbf{A},\mathbf{x})} = (x_j^{-1}f_i)(\overline{x_j^{-1}f_i})(x_j^{-1}f_a) = x_j^{-1}f_a$, and

(iii) $(\overline{e_a}e_b)\varphi_{(\mathbf{A},\mathbf{x})} = (\overline{x_j^{-1}f_a})(x_j^{-1}f_i)\overline{f_i}f_i(\overline{x_j^{-1}f_i})(x_j^{-1}f_b) = (\overline{x_j^{-1}f_a})(x_j^{-1}f_b)$,

where a and b are such that H_a and H_b are distinct elements of A_j. Let w'' be the result of applying all such reductions to w', and observe then that:

$$\begin{aligned} l(w'') &= l(w') - 2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\left(\Lambda_w(\bar{e_i}e_a) + \Lambda_w((x_j^{-1}e_i)\bar{e_i}e_a) + \sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b)\right) \\ &= |w|_{\alpha_1} + 2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\left(\Lambda_w(e_a) - \Lambda_w(\bar{e_i}e_a) - \Lambda_w((x_j^{-1}e_i)\bar{e_i}e_a) - \sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b)\right) \end{aligned}$$

We now consider further reductions to w'' which come from interactions between distinct (A_j, x_j) and (A_k, x_k). Suppose that $H_a \in A_j$ and $H_c \in A_k$, and observe that

$$\begin{aligned} (\overline{e_a}e_c)\varphi_{(\mathbf{A},\mathbf{x})} &= (\overline{e_a})\varphi_{(\mathbf{A},\mathbf{x})}(e_c)\varphi_{(\mathbf{A},\mathbf{x})} \\ &= (\overline{x_j^{-1}f_a})(x_j^{-1}f_i)\overline{f_i}f_i(\overline{x_k^{-1}f_i})(x_k^{-1}f_c) \\ &= (\overline{x_j^{-1}f_a})(x_j^{-1f_i})(\overline{x_k^{-1}f_i})(x_k^{-1}f_c). \end{aligned}$$

Let w''' be the result of applying all such reductions to w'', and note that the length of $(\overline{e_a}e_c)\varphi_{(\mathbf{A},\mathbf{x})}$ is 2 less in w''' than it is in w''. Recall that for any a and c, $\Lambda_w(\overline{e_a}e_c) = \Lambda_w(\overline{e_c}e_a)$. Thus $l(w''') = l(w'') - \frac{1}{2}\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\sum_{H_c\in\hat{A}-A_j}2\Lambda_w(\overline{e_a}e_c)$.

Since w was assumed to be reduced, we now have that there are no further reductions to w'''. Thus:

$$\begin{aligned} &|(w)\varphi_{(\mathbf{A},\mathbf{x})}|_{\alpha_2} \\ =&l(w''') \\ =&l(w'') - 2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\sum_{H_c\in\hat{A}-A_j}\frac{1}{2}\Lambda_w(\overline{e_a}e_c) \end{aligned}$$

$$
\begin{aligned}
=&|w|_{\alpha_1} + 2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\left(\Lambda_w(e_a) - \Lambda_w(\overline{e_i}e_a) - \Lambda_w((x_j^{-1}e_i)\overline{e_i}e_a) - \sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b)\right)\\
&-2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\sum_{H_c\in\hat{A}-A_j}\frac{1}{2}\Lambda_w(\overline{e_a}e_c)\\
=&|w|_{\alpha_1} + 2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\Bigg(\Lambda_w(e_a) - \Lambda_w(\overline{e_i}e_a) - \Lambda_w((x_j^{-1}e_i)\overline{e_i}e_a) - \sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b)\\
&-\frac{1}{2}\sum_{H_c\in\hat{A}-A_j}\Lambda_w(\overline{e_a}e_c)\Bigg)
\end{aligned}
$$

Now:

$$
\begin{aligned}
||\alpha_2|| =&\sum_{w\in\mathcal{W}}(|w|_{\alpha_2}) - 2|\mathcal{W}|\\
=&\sum_{w\in\mathcal{W}}\Bigg(|w|_{\alpha_1} + 2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\Bigg(\Lambda_w(e_a) - \Lambda_w(\overline{e_i}e_a) - \Lambda_w((x_j^{-1}e_i)\overline{e_i}e_a) - \sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b)\\
&-\frac{1}{2}\sum_{H_c\in\hat{A}-A_j}\Lambda_w(\overline{e_a}e_c)\Bigg)\Bigg) - 2|\mathcal{W}|\\
=&2\sum_{w\in\mathcal{W}}\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\Bigg(\Lambda_w(e_a) - \Lambda_w(\overline{e_i}e_a) - \Lambda_w((x_j^{-1}e_i)\overline{e_i}e_a) - \sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b)\\
&-\frac{1}{2}\sum_{H_c\in\hat{A}-A_j}\Lambda_w(\overline{e_a}e_c)\Bigg) + \sum_{w\in\mathcal{W}}(|w|_{\alpha_1}) - 2|\mathcal{W}|\\
=&2\sum_{w\in\mathcal{W}}\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\Bigg(\Lambda_w(e_a) - \Lambda_w(\overline{e_i}e_a) - \Lambda_w((x_j^{-1}e_i)\overline{e_i}e_a) - \sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b)\\
&-\frac{1}{2}\sum_{H_c\in\hat{A}-A_j}\Lambda_w(\overline{e_a}e_c)\Bigg) + ||\alpha_1||.
\end{aligned}
$$

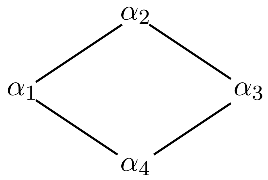
□

Remark. Since an edge path w in $\hat{\alpha}_1$ is uniquely defined by its endpoints, which are preseved by the map $\varphi_{(\mathbf{A},\mathbf{x})}$, we will often write $|w|_{\alpha_2}$ for $|(w)\varphi_{(\mathbf{A},\mathbf{x})}|_{\alpha_2}$.

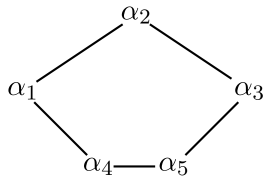
6.2 Reducible Peaks

Definition 6.2.1. We will say a path $\alpha_1 — \alpha_2 — \alpha_3$ in our Graph/Space of Domains is a peak if $||\alpha_2|| \geqslant ||\alpha_1||$ and $||\alpha_2|| \geqslant ||\alpha_3||$, and either $||\alpha_2|| > ||\alpha_1||$ or $||\alpha_2|| > ||\alpha_3||$ (or both). Equivalently, $\alpha_1 — \alpha_2 — \alpha_3$ is a peak if $||\alpha_2|| \geqslant \max(||\alpha_1||, ||\alpha_3||)$ and $||\alpha_2|| > \min(||\alpha_1||, ||\alpha_3||)$.

In this section, we claim that given a path $\alpha_1 — \alpha_2 — \alpha_3$, there exist domains α_4 and α_5 such that



or



forms a loop in the Graph of Domains.

Moreover, we claim that this loop is contractible in our Space of Domains. Further, we claim that if $\alpha_1 — \alpha_2 — \alpha_3$ was a peak, then $\alpha_1 — \alpha_4 — \alpha_3$ or $\alpha_1 — \alpha_4 — \alpha_5 — \alpha_3$ is a reduction; that is, $||\alpha_4|| < ||\alpha_2||$ and (if we are in the second of these cases) $||\alpha_5|| < ||\alpha_2||$. In other words, we are claiming that the peak is reducible.

Later in this section we will encounter many lemmas, divided into multiple cases. The series of lemmas in each case will roughly follow the structure outlined above. These will often correspond to lemmas used by Gilbert [12, Section 2], but as Gilbert is reducing (cyclic) words of G and we are reducing domains in $\mathcal{C}_n$, the proofs are quite different. Recall as well that the notation we use differs subtly to that used by Gilbert.

Definition 6.2.2. We say a peak $\alpha_1 — \alpha_2 — \alpha_3$ is reducible (or 'can be reduced') if the path $\alpha_1 — \alpha_2 — \alpha_3$ is homotopic (in the Space of Domains) to some path $\alpha_1 = \chi_0 — \chi_1 — \cdots — \chi_{k-1} — \chi_k = \alpha_3$ where $||\chi_i|| < ||\alpha_2||$ for every $1 \leqslant i \leqslant k-1$.

Proposition 6.2.3. *Suppose* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak in the Space of Domains (whose edges are both of Type A), where the α-graph in the domain α_2 has $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$. If there is some $i \in \{1, \ldots, n\}$ so that $\mathbf{x}, \mathbf{y} \subset H_i$, then this peak is reducible.*

This proposition corresponds to [12, Lemma 2.4].

Proof. If $(\mathbf{A}, \mathbf{x}) = (\mathbf{B}, \mathbf{y})$, then $\alpha_1 = \alpha_2(\mathbf{A}, \mathbf{x}) = \alpha_2(\mathbf{B}, \mathbf{y}) = \alpha_3$. Then our peak is really the loop $\alpha_1 — \alpha_2 — \alpha_1$. Since this is a forwards and backwards traversal of a single edge in the Space of Domains, then this is clearly contractible to the point α_1. By Definition 6.2.1, we must have that $||\alpha_1|| < ||\alpha_2||$. Thus the constant 'path' at α_1 is a reduction of the peak $\alpha_1 — \alpha_2 — \alpha_1$.

Now suppose $(\mathbf{A}, \mathbf{x}) \neq (\mathbf{B}, \mathbf{y})$. Since $\mathbf{x}, \mathbf{y} \subset H_i$, then the A-graph in the domain α_2 with central vertex group H_i (call it A_i) belongs to both $\alpha_2 \cap \alpha_1$ and $\alpha_2 \cap \alpha_3$. In particular, $A_i \in \alpha_1 \cap \alpha_3$, so there is an edge $[\alpha_1, \alpha_3]$ between α_1 and α_3 in the space of domains. Moreover, $A_i \in \alpha_1 \cap \alpha_2 \cap \alpha_3$, so there is a 2-cell $[\alpha_1, \alpha_2, \alpha_3]$ in the Space of Domains. Hence the path $\alpha_1 — \alpha_2 — \alpha_3$ is homotopic in the Space of Domains to the single edge path $\alpha_1 — \alpha_3$. The condition in Definition 6.2.2 is vacuously satisfied here, thus $\alpha_1 — \alpha_3$ is a reduction of the peak $\alpha_1 — \alpha_2 — \alpha_3$. ☐

From now on, we will be considering peaks of the form $\alpha_1 \overset{A_i}{—} \alpha_2 \overset{A_j}{—} \alpha_3$ where $i \neq j$, that is, paths $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ where $\mathbf{x}$ and $\mathbf{y}$ belong to different factor groups of the splitting associated to α_2. If the α-graph contained in the domain α_2 has $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$, let i and j be the (distinct) elements of $\{1, \ldots, n\}$ such that $\mathbf{x} \subset H_i$ and $\mathbf{y} \subset H_j$. Recall from Definition 5.5.1 that $\mathbf{A} = (A_1, \ldots, A_k)$, a disjoint partition of a subset of $\{H_1, \ldots, H_n\}$, and $\hat{A} := A_1 \cup \cdots \cup A_k$. Similarly, $\mathbf{B} = (B_1, \ldots, B_l)$ is another disjoint partition of some subset of $\{H_1, \ldots H_n\}$ and $\hat{B} := B_1 \cup \cdots \cup B_l$.

Observation 6.2.4. Note that we necessarily have $H_i \notin \hat{A}$ and $H_j \notin \hat{B}$. We adopt the four cases used by Gilbert [12, Lemma 2.12]:

Case 1: $H_i \notin \hat{B}$ and $H_j \notin \hat{A}$

Case 2: $H_i \in \hat{B}$ and $H_j \notin \hat{A}$

Case 3: $H_i \notin \hat{B}$ and $H_j \in \hat{A}$

Case 4: $H_i \in \hat{B}$ and $H_j \in \hat{A}$

As Cases 2 and 3 are symmetric, we will not consider Case 3 (since after renaming, this will be identical to Case 2). Additionally, if $H_j \in \hat{A}$ then there exists $A_p \in \mathbf{A}$ with $H_j \in A_p$, and if $H_i \in \hat{B}$, then there exists $B_q \in \mathbf{B}$ with $H_i \in B_q$. As Gilbert does in [12, Lemma 2.12], we further split the remaining cases as follows:

Case 1(a): $H_i \notin \hat{B}$ and $H_j \notin \hat{A}$, with $\hat{A} \cap \hat{B} = \varnothing$

Case 1(b): $H_i \notin \hat{B}$ and $H_j \notin \hat{A}$, with $\hat{A} \cap \hat{B} \neq \varnothing$

Case 2(a): $H_i \in \hat{B}$ (say $H_i \in B_q$) and $H_j \notin \hat{A}$, with $\hat{A} \subseteq B_q$

Case 2(b): $H_i \in \hat{B}$ (say $H_i \in B_q$) and $H_j \notin \hat{A}$, with $\hat{A} \nsubseteq B_q$

Case 4: $H_i \in \hat{B}$ and $H_j \in \hat{A}$ (say $H_i \in B_q$ and $H_j \in A_p$)

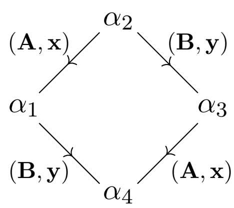
We now present a series of lemmas in order to prove that in each of the above cases, the peak $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is reducible.

Case 1(a): $H_i \notin \hat{B}$ and $H_j \notin \hat{A}$, with $\hat{A} \cap \hat{B} = \varnothing$

The lemmas for this case are adapted from [12, Lemma 2.6].

Lemma 6.2.5. *If $H_i \notin \hat{B}$ and $H_j \notin \hat{A}$ with $\hat{A} \cap \hat{B} = \varnothing$, then $(\mathbf{A},\mathbf{x})^{-1}(\mathbf{B},\mathbf{y}) = (\mathbf{B},\mathbf{y})(\mathbf{A},\mathbf{x})^{-1}$.*

That is, there exists a vertex α_4 in our Graph of Domains such that



is a loop.

Proof. By assumption, we have $\alpha_1 = \alpha_2(\mathbf{A},\mathbf{x})$. Since $H_j \notin \hat{A}$, then the $\mathfrak{S}$-labelling for the α-graph α_1 contained in the domain α_1 contains the group H_j. Thus as vertices of $\mathcal{C}_n$, we can collapse an edge of α_1 to achieve an A-graph with the group H_j at its centre. Then $(\mathbf{B},\mathbf{y})$ is in the stabiliser of this A-graph, meaning there is an edge in the Graph of Domains from α_1 to $\alpha_1(\mathbf{B},\mathbf{y}) = \alpha_2(\mathbf{A},\mathbf{x})(\mathbf{B},\mathbf{y})$, which we will call α_4.

Now by Definition 5.5.1 we can write $(\mathbf{A},\mathbf{x})(\mathbf{B},\mathbf{y})$ as $(A_1,x_1)\ldots(A_k,x_k)(B_1,y_1)\ldots(B_l,y_l)$ for some k and l in $\mathbb{N}$. So by repeated applications of Lemma 5.5.6, we have $(\mathbf{A},\mathbf{x})(\mathbf{B},\mathbf{y}) = (\mathbf{B},\mathbf{y})(\mathbf{A},\mathbf{x})$, noting that each A_a and B_b are pairwise disjoint.

Hence $\alpha_3 \cdot (\mathbf{A},\mathbf{x}) = \alpha_2 \cdot (\mathbf{B},\mathbf{y})(\mathbf{A},\mathbf{x}) = \alpha_2 \cdot (\mathbf{A},\mathbf{x})(\mathbf{B},\mathbf{y}) = \alpha_4$. □

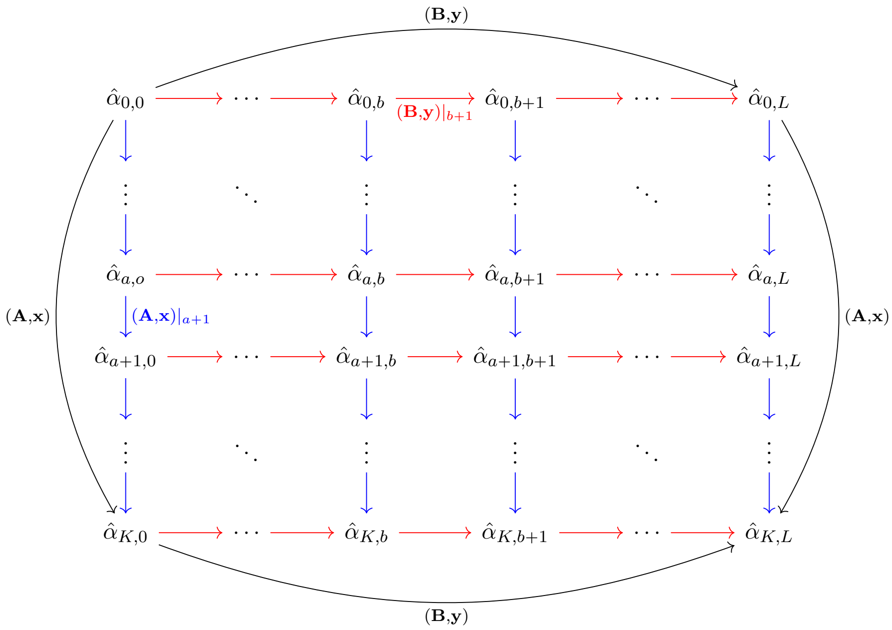
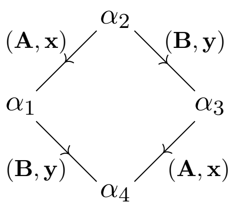


Figure 13: Lattice Describing $(A, x)(B, y) = (B, y)(A, x)$ in Case 1a

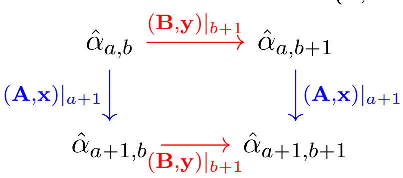
Lemma 6.2.6. *The loop*



described in Lemma 6.2.5 (with $\mathbf{x} \subset H_i$*,* $\mathbf{y} \subset H_j$*,* $H_i \notin \hat{B}$*,* $H_j \notin \hat{A}$*, and* $\hat{A} \cap \hat{B} = \varnothing$*) is contractible in our Space of Domains.*

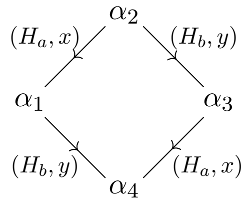
Proof. Let $\hat{A} = \{H_{A_1}, \ldots, H_{A_K}\}$ and $\hat{B} = \{H_{B_1}, \ldots, H_{B_L}\}$. For $a \in \{1, \ldots, K\}$, write $(\mathbf{A}, \mathbf{x})|_a := (\mathbf{A} \cap \{H_{A_a}\}, \mathbf{x})$. Set $\hat{\alpha}_{0,0} := \alpha_2$, and recursively define $\hat{\alpha}_{a+1,b} := \hat{\alpha}_{a,b}(\mathbf{A}, \mathbf{x})|_{a+1}$ and $\hat{\alpha}_{a,b+1} := \hat{\alpha}_{a,b}(\mathbf{B}, \mathbf{y})|_{b+1}$. Note then that $\alpha_1 = \hat{\alpha}_{K,0}$, $\alpha_3 = \hat{\alpha}_{0,L}$, and $\alpha_4 = \hat{\alpha}_{K,L}$. We can now build the lattice depicted in Figure 13.

Note that for each $a \in \{0, \ldots, K-1\}$ and $b \in \{0, \ldots, L-1\}$, the square



is such that the B-graph $B_{i,j,B_{b+1}}$ in the domain $\hat{\alpha}_{a,b}$ (and similarly the graph $B_{j,i,A_{a+1}}$) lives in the intersection $\hat{\alpha}_{a,b} \cap \hat{\alpha}_{a+1,b} \cap \hat{\alpha}_{a,b+1} \cap \hat{\alpha}_{a+1,b+1}$, hence this intersection is non-empty. By Definition 5.1.4, this means that the square is contractible in the Space of Domains. In the same way, loops $\hat{\alpha}_{0,b} - \cdots - \hat{\alpha}_{K,b} - \hat{\alpha}_{0,b}$ are contractible via the A-graph A_j in $\hat{\alpha}_{0,b}$, and similarly $\hat{\alpha}_{a,0} - \cdots - \hat{\alpha}_{a,L} - \hat{\alpha}_{a,0}$ via A_i. Since every 'cell' in our lattice is contractible in our Space of Domains, then so too is our initial loop (for which the lattice is akin to a tiling). □

Note that we only actually needed the first and last columns (or rows) of this lattice, since the graph B_{j,i,A_a} (or B_{i,j,B_b}) lies in the intersection of all the domains in row a (or column b).

Lemma 6.2.7. *Suppose we have a loop* 

where $x \in H_i$, $y \in H_j$, $H_a \neq H_b$, *and* $\{H_a, H_b\} \cap \{H_i, H_j\} = \varnothing$. *Then for any edge path* w *in* α_2, *we have* $|w|_{\alpha_4} - |w|_{\alpha_1} = |w|_{\alpha_3} - |w|_{\alpha_2}$.

Proof. Let w be a reduced edge path in $\hat{\alpha}_2$, the G-tree associated to the α-graph α_2 contained in the domain α_2. By assumption, α_2 is the (domain containing the) α-graph with $\mathfrak{S}$-labelling $(H_1, \ldots, H_n) = (H_a, H_b, H_i, H_j, H_{v_1}, \ldots, H_{v_{n-4}})$. It follows that α_4 is the (domain containing the) α-graph with $\mathfrak{S}$-labelling $(H_a^x, H_b^y, H_i, H_j, H_{v_1}, \ldots, H_{v_{n-4}})$, where $x \in H_i$ and $y \in H_j$. Suppose the edges in $\hat{\alpha}_2$ are labelled with e's, and the edges in $\hat{\alpha}_4$ are labelled with h's. Let $\varphi_{24} : \hat{\alpha}_2 \to \hat{\alpha}_4$ be the equivariant map described in Convention 6.1.5. Then $(e_a)\varphi_{24} = h_i(\overline{x^{-1}h_i})(x^{-1}h_a)$, $(e_b)\varphi_{24} = h_j(\overline{y^{-1}h_j})(y^{-1}h_b)$, and $(e_k)\varphi_{24} = h_k$ for any $k \neq a, b$.

Utilising the ideas from the proof of Lemma 6.1.6, we will let $l(u)$ be the unreduced length of a word u, and let w' be the unreduced word $(w)\varphi_{24}$ in $\hat{\alpha}_4$. Then $l(w') = l(w) + 2\Lambda_w(e_a) + 2\Lambda_w(e_b) = |w|_{\alpha_2} + 2\left(\Lambda_w(e_a) + \Lambda_w(e_b)\right)$.

Let w'' be the result of applying all reductions of the forms

$$\begin{aligned}
(\overline{e_i}e_a)\varphi &= \overline{h_i}h_i(\overline{x^{-1}h_i})(x^{-1}h_a) = (\overline{x^{-1}h_i})(x^{-1}h_a)\\
((e_i x^{-1})\overline{e_i}e_a)\varphi &= (x^{-1}h_i)(\overline{x^{-1}h_i})(x^{-1}h_a) = x^{-1}h_a\\
(\overline{e_j}e_b)\varphi &= \overline{h_j}h_j(\overline{y^{-1}h_j})(y^{-1}h_b) = (\overline{y^{-1}h_j})(y^{-1}h_b)\\
((e_j y^{-1})\overline{e_j}e_b)\varphi &= (y^{-1}h_j)(\overline{y^{-1}h_j})(y^{-1}h_b) = y^{-1}h_b,
\end{aligned}$$

as in the proof of Lemma 6.1.6. Then:

$$\begin{aligned}
l(w'') =& l(w') - 2\Lambda_w(\overline{e_i}e_a) - 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a) - 2\Lambda_w(\overline{e_j}e_b) - 2\Lambda_w((y^{-1}e_j)\overline{e_j}e_b)\\
=& |w|_{\alpha_2} + 2\big(\Lambda_w(e_a) + \Lambda_w(e_b) - \Lambda_w(\overline{e_i}e_a) - \Lambda_w(\overline{e_j}e_b) - \Lambda_w((x^{-1}e_i)\overline{e_i}e_a)\\
& -\Lambda_w((y^{-1}e_j)\overline{e_j}e_b)\big).
\end{aligned}$$

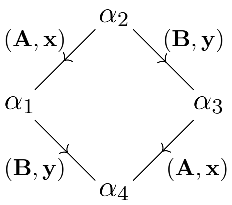
Since H_i, H_j, H_a, and H_b are all distinct, then there are no possible 'cross-reductions' to w'' as found in the proofs of Lemmas 6.1.7 and 6.1.8. Since w was assumed to be reduced in $\hat{\alpha}_2$, this now implies that w'' is reduced in $\hat{\alpha}_4$, hence $|w|_{\alpha_4} = l(w'')$.

Thus by Lemma 6.1.6, we have:

$$\begin{aligned}
& |w|_{\alpha_1} + |w|_{\alpha_3} - |w|_{\alpha_2}\\
=& |w|_{\alpha_2} + 2\Lambda_w(e_a) - 2\Lambda_w(\bar{e}_i e_a) - 2\Lambda_w((x^{-1}e_i)\bar{e}_i e_a)\\
& + |w|_{\alpha_2} + 2\Lambda_w(e_b) - 2\Lambda_w(\bar{e}_j e_b) - 2\Lambda_w((y^{-1}e_j)\bar{e}_j e_b) - |w|_{\alpha_2}\\
=& |w|_{\alpha_2} + 2\big(\Lambda_w(e_a) - \Lambda_w(\bar{e}_i e_a) - \Lambda_w((x^{-1}e_i)\bar{e}_i e_a) + \Lambda_w(e_b) - \Lambda_w(\bar{e}_j e_b) - \Lambda_w((y^{-1}e_j)\bar{e}_j e_b)\big)\\
=& |w|_{\alpha_4}.
\end{aligned}$$

□

Observation 6.2.8. Recall from the proof of Lemma 6.2.6 that we can write the loop



as a lattice, each 'cell' of which has the form of the loop in Lemma 6.2.7. Using the notation from Lemma 6.2.6, for any edge path w in $\alpha_2 = \hat{\alpha}_{0,0}$, we have that:

$$\begin{aligned}
&|w|_{\alpha_4} - |w|_{\alpha_1} \\
=&|w|_{\hat{\alpha}_{K,L}} - |w|_{\hat{\alpha}_{K,0}} \\
=&|w|_{\hat{\alpha}_{K,L}} - |w|_{\hat{\alpha}_{K,L-1}} + |w|_{\hat{\alpha}_{K,L-1}} - \cdots - |w|_{\hat{\alpha}_{K,1}} + |w|_{\hat{\alpha}_{K,1}} - |w|_{\hat{\alpha}_{K,0}} \\
=&|w|_{\hat{\alpha}_{K-1,L}} - |w|_{\hat{\alpha}_{K-1,L-1}} + |w|_{\hat{\alpha}_{K-1,L-1}} - \cdots - |w|_{\hat{\alpha}_{K-1,1}} + |w|_{\hat{\alpha}_{K-1,1}} - |w|_{\hat{\alpha}_{K-1,0}} \\
&\vdots \\
=&|w|_{\hat{\alpha}_{0,L}} - |w|_{\hat{\alpha}_{0,L-1}} + |w|_{\hat{\alpha}_{0,L-1}} - \cdots - |w|_{\hat{\alpha}_{0,1}} + |w|_{\hat{\alpha}_{0,1}} - |w|_{\hat{\alpha}_{0,0}} \\
=&|w|_{\hat{\alpha}_{0,L}} - |w|_{\hat{\alpha}_{0,0}} \\
=&|w|_{\alpha_3} - |w|_{\alpha_2}.
\end{aligned}$$

Lemma 6.2.9. *If $H_i \notin \hat{B}$ and $H_j \notin \hat{A}$ and $\hat{A} \cap \hat{B} = \varnothing$ then $||\alpha_4|| - ||\alpha_1|| = ||\alpha_3|| - ||\alpha_2||$, where $\alpha_4 = \alpha_1(\mathbf{A}, \mathbf{x})(\mathbf{B}, \mathbf{y})$.*

Proof. By Lemma 6.2.7 and the Observation 6.2.8,

$$\begin{aligned}
||\alpha_4|| - ||\alpha_1|| &= \sum_{w \in \mathcal{W}} \Big(|w|_{\alpha_4} - 2\Big) - \sum_{w \in \mathcal{W}} \Big(|w|_{\alpha_1} - 2\Big) \\
&= \sum_{w \in \mathcal{W}} \Big(|w|_{\alpha_4} - 2 - |w|_{\alpha_1} + 2\Big) \\
&= \sum_{w \in \mathcal{W}} \Big(|w|_{\alpha_3} - |w|_{\alpha_2}\Big) \\
&= \sum_{w \in \mathcal{W}} \Big(|w|_{\alpha_3} - 2\Big) - \sum_{w \in \mathcal{W}} \Big(|w|_{\alpha_2} - 2\Big) \\
&= ||\alpha_3|| - ||\alpha_2||.
\end{aligned}$$

□

Lemma 6.2.10. *If $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is a peak in the Graph of Domains, where $H_i \notin \hat{B}$ and $H_j \notin \hat{A}$ with $\hat{A} \cap \hat{B} = \varnothing$, then $||\alpha_2 \cdot (\mathbf{A}, \mathbf{x})(\mathbf{B}, \mathbf{y})|| < ||\alpha_2||$.*

Proof. By Lemma 6.2.9, $||\alpha_2(\mathbf{A}, \mathbf{x})(\mathbf{B}, \mathbf{y})|| = ||\alpha_4|| = ||\alpha_3|| + ||\alpha_1|| - ||\alpha_2||$. Since $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is a peak, then $||\alpha_2|| \geqslant \max(||\alpha_1||, ||\alpha_3||)$ and $||\alpha_2|| > \min(||\alpha_1||, ||\alpha_3||)$. Now:

$$\begin{aligned}
||\alpha_4|| &= \max(||\alpha_1||, ||\alpha_3||) + \min(||\alpha_1||, ||\alpha_3||) - ||\alpha_2|| \\
&< ||\alpha_2|| + ||\alpha_2|| - ||\alpha_2|| \\
&= ||\alpha_2||.
\end{aligned}$$

□

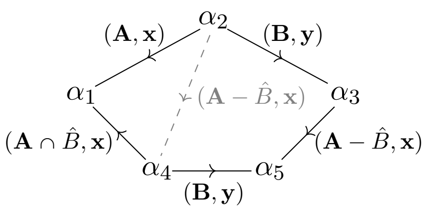
Proposition 6.2.11. *Let $H_1 * \cdots * H_n$ be an $\mathfrak{S}$ free factor splitting for G. Let $(\mathbf{A}, \mathbf{x})$ and $(\mathbf{B}, \mathbf{y})$ be relative multiple Whitehead automorphisms with $\mathbf{x} \subset H_i$ and $\mathbf{y} \subset H_j$ and $\hat{A}, \hat{B} \subset \{H_1, \ldots, H_n\} - \{H_i, H_j\}$ such that $\hat{A} \cap \hat{B} = \varnothing$. Let α_2 be the domain whose α-graph has $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$, and let $\alpha_1 = \alpha_2(\mathbf{A}, \mathbf{x})$ and $\alpha_3 = \alpha_2(\mathbf{B}, \mathbf{y})$. If $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is a peak, then it is reducible.*

Proof. By Lemmas 6.2.5 and 6.2.6, the path $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is homotopic in the Space of Domains to the path $\alpha_1 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_2(\mathbf{A}, \mathbf{x})(\mathbf{B}, \mathbf{y}) \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_3$, and by Lemma 6.2.10, $||\alpha_2(\mathbf{A}, \mathbf{x})(\mathbf{B}, \mathbf{y})|| < ||\alpha_2||$. Thus by Definition 6.2.2, the peak $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is reducible. □

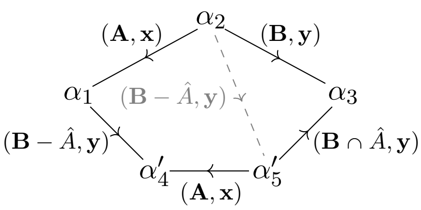
Case 1(b): $H_i \notin \hat{B}$ and $H_j \notin \hat{A}$, with $\hat{A} \cap \hat{B} \neq \varnothing$

Lemma 6.2.12. *If $H_i \notin \hat{B}$ and $H_j \notin \hat{A}$ with $\hat{A} \cap \hat{B} \neq \varnothing$, then $(\mathbf{A}, \mathbf{x})^{-1}(\mathbf{B}, \mathbf{y}) = (\mathbf{A} \cap \hat{B}, \mathbf{x})^{-1}(\mathbf{B}, \mathbf{y})(\mathbf{A} - \hat{B}, \mathbf{x})^{-1}$ and $(\mathbf{A}, \mathbf{x})^{-1}(\mathbf{B}, \mathbf{y}) = (\mathbf{B} - \hat{A}, \mathbf{y})(\mathbf{A}, \mathbf{x})^{-1}(\mathbf{B} \cap \hat{A}, \mathbf{y})$.*

That is, there exist vertices α_4, α_5, α_4', α_5' in our Graph of Domains such that



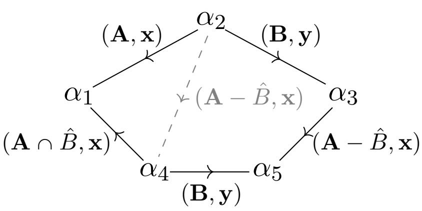
and



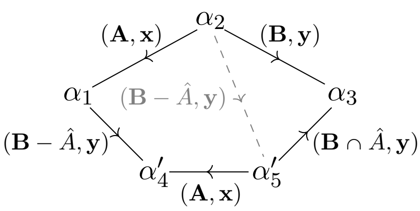
are both loops.

Proof. By Proposition 5.5.8 (1), $(\mathbf{A}, \mathbf{x})(\mathbf{A} \cap \hat{B}, \mathbf{x})^{-1} = (\mathbf{A} - \hat{B}, \mathbf{x})$. Note that $\widehat{\mathbf{A} - \hat{B}}$ and $\hat{B}$ are disjoint, and we have $H_i \notin \hat{B}$ and $H_j \notin \hat{A} - \hat{B}$. Thus by Lemma 6.2.5, $(\mathbf{A} - \hat{B}, \mathbf{x})(\mathbf{B}, \mathbf{y}) = (\mathbf{B}, \mathbf{y})(\mathbf{A} - \hat{B}, \mathbf{x})$. The second statement follows similarly, by appropriately switching A and B and x and y. □

Lemma 6.2.13. *The loops*



and



are contractible in our Space of Domains.

Proof. Note that the α_1–α_2–α_4 triangle can be 'filled' with an A_i graph (that is to say, since $(\mathbf{A}, \mathbf{x})$, $(\mathbf{A} \cap \hat{B}, \mathbf{x})$ and $(\mathbf{A} - \hat{B}, \mathbf{x})$ all live in the stabiliser of the A_i-graph with H_i at its centre in the domain α_2, then by Definition 5.1.4, we have a 2-cell $[\alpha_1, \alpha_2, \alpha_4]$). Similarly, the α_2–α_3–α_5' triangle can be 'filled' with an A_j graph. Now $(\widehat{\mathbf{A} - \hat{B}}) \cap \hat{B} = \varnothing = \hat{A} \cap (\widehat{B - \hat{A}})$, so by Lemma 6.2.6 (Case 1a) the squares $\alpha_4 — \alpha_2 — \alpha_3 — \alpha_5 — \alpha_4$ and $\alpha_1 — \alpha_2 — \alpha_5' — \alpha_4' — \alpha_1$ are both contractible. □

Lemma 6.2.14. *Let $H' = (H_1', \ldots, H_n')$, and suppose $\mathbf{u} \subset H_i'$, $\mathbf{v} \subset H_j'$, $\mathbf{C}, \mathbf{D} \subset \hat{H}'$ with $\hat{C} = \hat{D}$ and $H_i', H_j' \notin \hat{C}$. If $\|\alpha(\mathbf{C}, \mathbf{u})\| - \|\alpha\| \leqslant 0$ and $\|\alpha(\mathbf{D}, \mathbf{v})\| - \|\alpha\| \leqslant 0$ then $\|\alpha(\mathbf{C}, \mathbf{u})\| - \|\alpha\| = \|\alpha(\mathbf{D}, \mathbf{v})\| - \|\alpha\| = 0$.*

Proof. Suppose $\|\alpha(\mathbf{C},\mathbf{u})\| - \|\alpha\| \leqslant 0$ and $\|\alpha(\mathbf{D},\mathbf{v})\| - \|\alpha\| \leqslant 0$. By Lemma 6.1.8:

$$\|\alpha(\mathbf{C},\mathbf{u})\| - \|\alpha\| = 2\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\Bigg(\Lambda_w(e_c) - \Lambda_w(\overline{e_i}e_c) - \Lambda_w((u_k^{-1}e_i)\overline{e_i}e_c) - \sum_{H_a\in C_k-\{H_c\}}\Lambda_w(\overline{e_a}e_c) - \frac{1}{2}\sum_{H_a\in\hat{C}-C_k}\Lambda_w(\overline{e_a}e_c)\Bigg).$$

Note that we can write $\hat{C}-\{H_c\} = (\hat{C}-C_k)\sqcup(C_k-\{H_c\})$. Thus $\sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c) = \sum_{H_a\in\hat{C}-C_k}\Lambda_w(\overline{e_a}e_c) + \sum_{H_a\in C_k-\{H_c\}}\Lambda_w(\overline{e_a}e_c)$ for all c such that $H_c\in\hat{C}$.

Since Λ_w counts occurrences of subwords of w, we must have that $\Lambda_w((u_k^{-1}e_i)\overline{e_i}e_c) \leqslant \Lambda_w(\overline{e_i}e_c) \leqslant \Lambda_w(e_c)$ for every k such that $C_k\in\mathbf{C}$ and every c such that $H_c\in C_k$. Since e_i, e_j, and e_a (where $H_a\in\hat{C}$) are distinct, we must also have that

$$\Lambda_w(\overline{e_i}e_c) + \Lambda_w(\overline{e_j}e_c) + \sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c) \leqslant \Lambda_w(e_c) \tag{1}$$

holds for every c such that $H_c\in\hat{C}$. By assumption, $\|\alpha(\mathbf{C},\mathbf{u})\| - \|\alpha\| \leqslant 0$, that is:

$$\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\Bigg(\Lambda_w(e_c) - \sum_{H_a\in\hat{C}-C_k}\Lambda_w(\overline{e_a}e_c) - \frac{1}{2}\sum_{H_a\in C_k-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\Bigg) \leqslant \sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\Big(\Lambda_w(\overline{e_i}e_c) + \Lambda_w((u_k^{-1}e_i)\overline{e_i}e_c)\Big).$$

We now deduce the following system of inequalities:

$$\begin{aligned}
&\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Bigg(\Lambda_w(e_c) - \sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\Bigg) && (2)\\
=&\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\Bigg(\Lambda_w(e_c) - \sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\Bigg) && (3)\\
=&\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\Bigg(\Lambda_w(e_c) - \sum_{H_a\in\hat{C}-C_k}\Lambda_w(\overline{e_a}e_c) - \sum_{H_a\in C_k-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\Bigg) && (4)\\
\leqslant&\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\Bigg(\Lambda_w(e_c) - \sum_{H_a\in\hat{C}-C_k}\Lambda_w(\overline{e_a}e_c) - \frac{1}{2}\sum_{H_a\in C_k-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\Bigg) && (5)\\
\leqslant&\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\left(\Lambda_w(\overline{e_i}e_c) + \Lambda_w((e_iu_k^{-1})\overline{e_i}e_c)\right) && (6)\\
\leqslant&\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}(2\Lambda_w(\overline{e_i}e_c)) && (7)\\
=&2\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_i}e_c). && (8)
\end{aligned}$$

The same argument yields that $\sum_{w\in\mathcal{W}}\sum_{H_d\in\hat{D}}\Big(\Lambda_w(e_d) - \sum_{H_b\in\hat{D}-\{H_d\}}\Lambda_w(\overline{e_b}e_d)\Big) \leqslant 2\sum_{w\in\mathcal{W}}\sum_{H_d\in\hat{D}}\Lambda_w(\overline{e_j}e_d)$. Since it is assumed that $\hat{D}=\hat{C}$, we can rewrite this to give $\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Big(\Lambda_w(e_c) - \sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\Big) \leqslant 2\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_j}e_c)$.

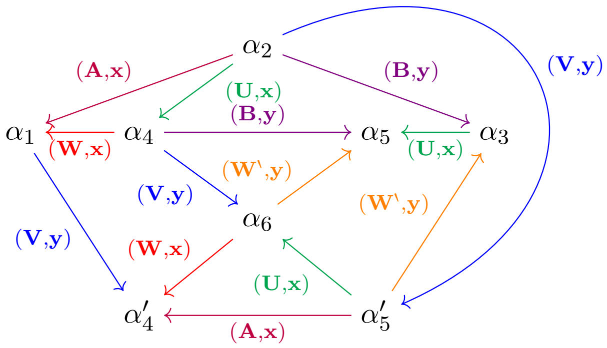


Figure 14: Commuting Diagram for Case 1b

Now $\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_i}e_c) \geqslant \frac{1}{2}\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Big(\Lambda_w(e_c) - \sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\Big)$ and $\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_j}e_c) \geqslant \frac{1}{2}\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Big(\Lambda_w(e_c) - \sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\Big)$. Summing these terms, and comparing with the inequality (1), gives that $\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_i}e_c) + \sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_j}e_c) = \sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Big(\Lambda_w(e_c) - \sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\Big)$. We must then in fact have that $\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_i}e_c) = \sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_j}e_c)$ $= \frac{1}{2}\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Big(\Lambda_w(e_c) - \sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\Big)$. This in turn forces each line of (2)–(8) to be an equality. In particular, $\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\Big(\Lambda_w(e_c) - \sum_{H_a\in\hat{C}-C_k}\Lambda_w(\overline{e_a}e_c) - \frac{1}{2}\sum_{H_a\in C_k-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\Big)$ $= \sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\big(\Lambda_w(\overline{e_i}e_c) + \Lambda_w((u_k^{-1}e_i)\overline{e_i}e_c)\big)$. That is, $\|\alpha(\mathbf{C},\mathbf{u})\| - \|\alpha\| = 0$. The same argument applies to see that we must also have $||\alpha(\mathbf{D},\mathbf{v})|| - ||\alpha|| = 0$. □

Lemma 6.2.15. *Suppose* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak. If* $H_i \notin \hat{B}$ *and* $H_j \notin \hat{A}$ *with* $\hat{A}\cap\hat{B} \neq \varnothing$*, then either* $||\alpha_2(\mathbf{A},\mathbf{x})(\mathbf{A}\cap\hat{B},\mathbf{x})^{-1}|| < ||\alpha_2||$ *and* $||\alpha_2(\mathbf{B},\mathbf{y})(\mathbf{A}-\hat{B},\mathbf{x})|| < ||\alpha_2||$ *, or* $||\alpha_2(\mathbf{A},\mathbf{x})(\mathbf{B}-\hat{A},\mathbf{y})|| < ||\alpha_2||$ *and* $||\alpha_2(\mathbf{B},\mathbf{y})(\mathbf{B}\cap\hat{A},\mathbf{y})^{-1}|| < ||\alpha_2||$ *.*

That is, $||\alpha_2|| > \max(||\alpha_4||, ||\alpha_5||)$ or $||\alpha_2|| > \max(||\alpha_4'||, ||\alpha_5'||)$.

Proof. Let $\alpha_6 = \alpha_2(\mathbf{A}-\hat{B},\mathbf{x})(\mathbf{B}-\hat{A},\mathbf{y})$. For brevity, set $\mathbf{U} = \mathbf{A}-\hat{B}$, $\mathbf{V} = \mathbf{B}-\hat{A}$, $\mathbf{W} = \mathbf{A}\cap\hat{B}$, and $\mathbf{W}^{\backprime} = \mathbf{B}\cap\hat{A}$. Note that $\hat{W} = \hat{W}^{\backprime} = \hat{A}\cap\hat{B}$, $\hat{U} = \hat{A}-\hat{B} = \hat{A}\cap\bar{\hat{B}}$, and $\hat{V} = \hat{B}-\hat{A} = \hat{B}\cap\bar{\hat{A}}$. Observe that $\hat{U}\cap\hat{V} = (\hat{A}-\hat{B})\cap(\hat{B}-\hat{A}) = (\hat{A}\cap\bar{\hat{B}})\cap(\hat{B}\cap\bar{\hat{A}}) = \hat{A}\cap\bar{\hat{A}}\cap\bar{\hat{B}}\cap\hat{B} = \varnothing\cap\varnothing = \varnothing$. We leave it to the reader to verify that we also have $\hat{U}\cap\hat{B} = \hat{U}\cap\hat{W} = \hat{V}\cap\hat{W} = \hat{V}\cap\hat{A} = \varnothing$; these follow in much the same way. Then by Proposition 5.5.8 (1), the diagram in Figure 14 commutes.

By applying Lemma 6.2.9 to each of the squares $\alpha_1 — \alpha_4 — \alpha_6 — \alpha_4' — \alpha_1$ and $\alpha_3 — \alpha_5' — \alpha_6 — \alpha_5 — \alpha_1$ (which both fall under Case 1a), we recover that $\|\alpha_1\| - \|\alpha_4\| = \|\alpha_4'\| - \|\alpha_6\|$ and $\|\alpha_3\| - \|\alpha_5'\| = \|\alpha_5\| - \|\alpha_6\|$. By considering the triangles $\alpha_1 — \alpha_2 — \alpha_4 — \alpha_1$ and $\alpha_3 — \alpha_2 — \alpha_5' — \alpha_3$, we see that $\|\alpha_1\| - \|\alpha_2\| = (\|\alpha_1\| - \|\alpha_4\|) + (\|\alpha_4\| - \|\alpha_2\|)$ and $\|\alpha_3\| - \|\alpha_2\| = (\|\alpha_3\| - \|\alpha_5'\|) + (\|\alpha_5'\| - \|\alpha_2\|)$. By Lemma 6.2.14, either $\max(\|\alpha_4'\| - \|\alpha_6\|, \|\alpha_5\| - \|\alpha_6\|) > 0$ or $\|\alpha_4'\| - \|\alpha_6\| = \|\alpha_5\| - \|\alpha_6\| = 0$. Since $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is a peak, then $\max(\|\alpha_1\| - \|\alpha_2\|, \|\alpha_3\| - \|\alpha_2\|) \leqslant 0$ and $\min(\|\alpha_1\| -$

$\|\alpha_2\|, \|\alpha_3\| - \|\alpha_2\|) < 0$.

We claim that $\min(\|\alpha_4\| - \|\alpha_2\|, \|\alpha_5'\| - \|\alpha_2\|) < 0$. If $\|\alpha_4'\| - \|\alpha_6\| = \|\alpha_5\| - \|\alpha_6\| = 0$, then:

$$\begin{aligned}&\min(\|\alpha_4\| - \|\alpha_2\|, \|\alpha_5'\| - \|\alpha_2\|)\\ =&\min\big((\|\alpha_1\| - \|\alpha_2\|) - (\|\alpha_1\| - \|\alpha_4\|), (\|\alpha_3\| - \|\alpha_2\|) - (\|\alpha_3\| - \|\alpha_5'\|)\big)\\ =&\min\big((\|\alpha_1\| - \|\alpha_2\|) - (\|\alpha_4'\| - \|\alpha_6\|), (\|\alpha_3\| - \|\alpha_2\|) - (\|\alpha_5\| - \|\alpha_6\|)\big)\\ =&\min(\|\alpha_1\| - \|\alpha_2\|, \|\alpha_3\| - \|\alpha_2\|)\\ <&0.\end{aligned}$$

On the other hand, if $\max(\|\alpha_4'\| - \|\alpha_6\|, \|\alpha_5\| - \|\alpha_6\|) > 0$ (without loss of generality, say $\|\alpha_4'\| - \|\alpha_6\| > 0$ — a symmetrically identical argument holds if $\|\alpha_5\| - \|\alpha_6\| > 0$), then:

$$\begin{aligned}\|\alpha_4\| - \|\alpha_2\| &= (\|\alpha_1\| - \|\alpha_2\|) - (\|\alpha_1\| - \|\alpha_4\|)\\ &= (\|\alpha_1\| - \|\alpha_2\|) - (\|\alpha_4'\| - \|\alpha_6\|)\\ &< \|\alpha_1\| - \|\alpha_2\|\\ &\leqslant 0.\end{aligned}$$

In either case, we have that $\min(\|\alpha_4\| - \|\alpha_2\|, \|\alpha_5'\| - \|\alpha_2\|) < 0$.

Without loss of generality, assume $\|\alpha_4\| - \|\alpha_2\| < 0$. Then $\alpha_4 — \alpha_2 — \alpha_3$ is a peak falling under Case 1a, and by Lemma 6.2.10, $\|\alpha_5\| < \|\alpha_2\|$. An identical (symmetric) argument holds if instead $\|\alpha_5'\| - \|\alpha_2\| < 0$. Thus $\min(\max(\|\alpha_4\|, \|\alpha_5\|), \max(\|\alpha_4'\|, \|\alpha_5'\|)) < \|\alpha_2\|$, as required. ☐

Proposition 6.2.16. *Let $H_1 * \ldots H_n$ be an $\mathfrak{S}$ free factor splitting for G. Let $(\mathbf{A}, \mathbf{x})$ and $(\mathbf{B}, \mathbf{y})$ be relative multiple Whitehead automorphisms with $\mathbf{x} \subset H_i$ and $\mathbf{y} \subset H_j$ and $\hat{A}, \hat{B} \subset \{H_1, \ldots, H_n\} - \{H_i, H_j\}$ such that $\hat{A} \cap \hat{B} \neq \varnothing$. Let α_2 be the domain whose α-graph has $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$, and let $\alpha_1 = \alpha_2(\mathbf{A}, \mathbf{x})$ and $\alpha_3 = \alpha_2(\mathbf{B}, \mathbf{y})$. If $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is a peak, then it is reducible.*

Proof. By Lemmas 6.2.12 and 6.2.13, the path $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is homotopic in the Space of Domains to each of the paths

$$\alpha_1 \xleftarrow{(\mathbf{A}\cap\hat{B},\mathbf{x})} \alpha_2(\mathbf{A},\mathbf{x})(\mathbf{A}\cap\hat{B},\mathbf{x})^{-1} \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_2(\mathbf{B},\mathbf{y})(\mathbf{A}-\hat{B},\mathbf{x}) \xleftarrow{(\mathbf{A}-\hat{B},\mathbf{x})} \alpha_3 \quad \text{and}$$

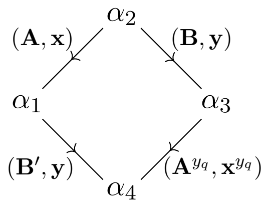
$$\alpha_1 \xrightarrow{(\mathbf{B}-\hat{A},\mathbf{y})} \alpha_2(\mathbf{A},\mathbf{x})(\mathbf{B}-\hat{A},\mathbf{y}) \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2(\mathbf{B},\mathbf{y})(\mathbf{B}\cap\hat{A},\mathbf{y})^{-1} \xrightarrow{(\mathbf{B}\cap\hat{A},\mathbf{y})} \alpha_3 \quad .$$

By Lemma 6.2.15, either $||\alpha_2(\mathbf{A},\mathbf{x})(\mathbf{A}\cap\hat{B},\mathbf{x})^{-1}|| < ||\alpha_2||$ and $||\alpha_2(\mathbf{B},\mathbf{y})(\mathbf{A} - \hat{B},\mathbf{x})|| < ||\alpha_2||$, or $||\alpha_2(\mathbf{A},\mathbf{x})(\mathbf{B}-\hat{A},\mathbf{y})|| < ||\alpha_2||$ and $||\alpha_2(\mathbf{B},\mathbf{y})(\mathbf{B}\cap\hat{A},\mathbf{y})^{-1}|| < ||\alpha_2||$ Thus by Definition 6.2.2, one of the above paths is a reduction for the peak $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$. ☐

Case 2(a): $H_i \in \hat{B}$ (say $H_i \in B_q$) and $H_j \notin \hat{A}$, with $\hat{A} \subseteq B_q$

The lemmas for this case are adapted from [12, Lemma 2.7].

Lemma 6.2.17. *If $H_i \in B_q$ and $H_j \notin \hat{A}$ with $\hat{A} \subseteq B_q$, then $(\mathbf{A},\mathbf{x})^{-1}(\mathbf{B},\mathbf{y}) = (\mathbf{B}',\mathbf{y})(\mathbf{A}^{y_q},\mathbf{x}^{y_q})^{-1}$ where $[\mathbf{B}']_b := B_b$ for $b \in \{1,\ldots,l\} - \{q\}$ and $[\mathbf{B}']_q := (B_q - \hat{A}) \cup (\widehat{B_q \cap \mathbf{A}})^{\mathbf{x}} = (B_q - \hat{A}) \cup \bigcup_{a=1}^{k} A_a^{x_a}$.*

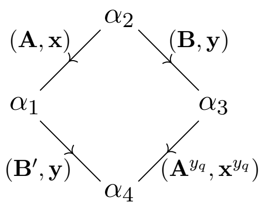
That is, there exists a vertex α_4 in our Graph of Domains such that 

is a loop.

Proof. We have that $B_q = (B_q - \hat{A}) \sqcup \hat{A}$. In particular, $\hat{A} \cap B_b = \varnothing$ for any $b \neq q$. By Lemma 5.5.5 (3) we have that $(B_q, y_q) = (B_q - \hat{A}, y_q)(\hat{A}, y_q)$. By (1) and (2) of Lemma 5.5.5, we have that $(\mathbf{A}^{y_q}, \mathbf{x}^{y_q}) = (\hat{A}^{y_q}, y_q^{-1})(\mathbf{A}^{y_q^{-1} y_q}, y_q \mathbf{x}) = (\hat{A}, y_q)^{-1}(\mathbf{A}, y_q \mathbf{x})$. Now

$$
\begin{aligned}
(\mathbf{A}, \mathbf{x})(\mathbf{B}', \mathbf{y}) &= (\mathbf{A}, \mathbf{x})(B_1, y_1) \dots ((B_q - \hat{A}) \cup \widehat{\mathbf{A}^{\mathbf{x}}}, y_q) \dots (B_l, y_l) \\
&= (\mathbf{A}, \mathbf{x})(B_1, y_1) \dots (B_q - \hat{A}, y_q)(\widehat{\mathbf{A}^{\mathbf{x}}}, y_q) \dots (B_l, y_l) \\
&= (B_1, y_1) \dots (B_q - \hat{A}, y_q) \dots (B_l, y_l)(\mathbf{A}, \mathbf{x})(\widehat{\mathbf{A}^{\mathbf{x}}}, y_q) \\
&= (\mathbf{B} - \hat{A}, \mathbf{y})(\mathbf{A}, y_q \mathbf{x}) \\
&= (\mathbf{B} - \hat{A}, \mathbf{y})(\hat{A}, y_q)(\mathbf{A}^{y_q}, \mathbf{x}^{y_q}) \\
&= (\mathbf{B}, \mathbf{y})(\mathbf{A}^{y_q}, \mathbf{x}^{y_q}).
\end{aligned}
$$

□

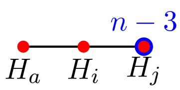
Lemma 6.2.18. *The loop* 

(where $H_i \in B_q$ and $H_j \notin \hat{A}$ with $\hat{A} \subseteq B_q$, and $\mathbf{B}'$ is given by $[\mathbf{B}']_b = B_b$ for $b \neq q$ and $[\mathbf{B}']_q = (B_q - \hat{A}) \cup \widehat{\mathbf{A}^{\mathbf{x}}}$) is contractible in our Space of Domains.

Proof. Let $\hat{A} = \{H_{A_1}, \dots, H_{A_K}\}$. Recall from Lemma 6.2.6 that we denote $(\mathbf{A}, \mathbf{x})|_a := (\mathbf{A} \cap \{H_a\}, \mathbf{x})$. Then there is some $x_{A_a} \in \mathbf{x}$ so that $(\mathbf{A}, \mathbf{x})|_a = (H_a, x_{A_a})$. Let $\tilde{\alpha}_{0,0} := \alpha_2$ and for $a = 1, \dots, K$, recursively define $\tilde{\alpha}_{a,0} := \tilde{\alpha}_{a-1,0}(\mathbf{A}, \mathbf{x})|_a$. Then for each a, $\tilde{\alpha}_{a,0}$ is the (domain whose) α-graph has $\mathfrak{S}$-labelling comprising the groups $H_{A_1}^{x_{A_1}}, \dots, H_{A_a}^{x_{A_a}}, H_{A_{a+1}}, \dots, H_{A_K}, H_{v_1}, \dots, H_{v_{n-K}}$. For each a, define $\mathbf{B^{a\prime}}$ by $[\mathbf{B^{a\prime}}]_b := B_b$ and $[\mathbf{B^{a\prime}}]_q := (B_q - \{H_{A_1}, \dots, H_{A_a}\}) \cup \{H_{A_1}^{x_{A_1}}, \dots, H_{A_a}^{x_{A_a}}\}$. Note that $\mathbf{B^{K\prime}} = \mathbf{B}'$. Again for each $a = 0, \dots, K$, we define $\tilde{\alpha}_{a,L} := \tilde{\alpha}_{a,0}(\mathbf{B^{a\prime}}, \mathbf{y})$. Then $\tilde{\alpha}_{a+1,L} = \tilde{\alpha}_{a,L}(H_{A_a}^{y_q}, x_{A_a}^{y_q})$. Observe that $\tilde{\alpha}_{K,0} = \alpha_1$, $\tilde{\alpha}_{0,L} = \alpha_3$, and $\tilde{\alpha}_{K,L} = \alpha_4$. Thus we have constructed a lattice as depicted in Figure 15. As in the proof of Lemma 6.2.6, the loops $\tilde{\alpha}_{0,0} — \dots — \tilde{\alpha}_{K,0} — \tilde{\alpha}_{0,0}$ and $\tilde{\alpha}_{0,L} — \dots — \tilde{\alpha}_{K,L} — \tilde{\alpha}_{0,L}$ are contractible via A_i-graphs living in domains $\tilde{\alpha}_{0,0}$ and $\tilde{\alpha}_{0,L}$, respectively. Also, for each $a \in \{0, \dots, K-1\}$ the square

$$
\begin{array}{ccc}
\tilde{\alpha}_{a,0} & \xrightarrow{(\mathbf{B^{a\prime}}, \mathbf{y})} & \tilde{\alpha}_{a,L} \\
{\scriptstyle (\mathbf{A},\mathbf{x})|_{a+1}} \downarrow & & \downarrow {\scriptstyle (H_{A_a}^{y_q}, x_{A_a}^{y_q})} \\
\tilde{\alpha}_{a+1,0} & \xrightarrow[(\mathbf{B^{a+1\prime}}, \mathbf{y})]{} & \tilde{\alpha}_{a+1,L}
\end{array}
$$

is contractible via the graph $B_{j,i,a}$ in the domain $\tilde{\alpha}_{a,0}$

(that is, the graph 

lives in the intersection $\tilde{\alpha}_{a,0} \cap \tilde{\alpha}_{a,L} \cap \tilde{\alpha}_{a+1,0} \cap \tilde{\alpha}_{a+1,L}$). □

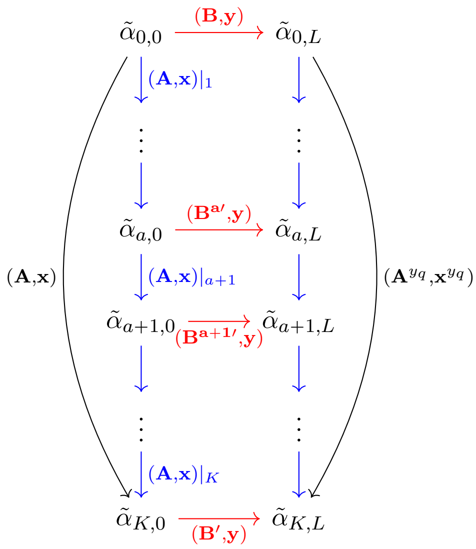
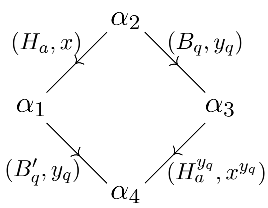


Figure 15: Lattice Describing $(A, x)(B', y) = (B, y)(A^{y_q}, x^{y_q})$ in Case 2a

Lemma 6.2.19. *Suppose we have a loop*



where $B'_q = \{H_a^x\} \cup (B_q - \{H_a\})$, *and* $y_q \in H_j \neq H_a$, $x \in H_i \in B_q$, *and* $H_a \in B_q$. *Then for any edge path* w *in* $\hat{\alpha}_2$, *we have* $|w|_{\alpha_4} - |w|_{\alpha_1} = |w|_{\alpha_3} - |w|_{\alpha_2}$.

Proof. Set $\tilde{B}_q := B_q - \{H_i, H_a\}$. By Lemma 6.1.6, we have that

$$|w|_{\alpha_1} = |w|_{\alpha_2} + 2\Lambda_w(e_a) - 2\Lambda_w(\overline{e_i}e_a) - 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a).$$

By Lemma 6.1.7, we have that

$$\begin{aligned}
|w|_{\alpha_3} =& |w|_{\alpha_2} + 2\sum_{H_b \in B_q}\left(\Lambda_w(e_b) - \Lambda_w(\overline{e_j}e_b) - \Lambda_w((y_q^{-1}e_j)\overline{e_j}e_b) - \sum_{H_c \in B_q - \{H_b\}}\Lambda_w(\overline{e_c}e_b)\right)\\
=& |w|_{\alpha_2} + 2\sum_{H_b \in \tilde{B}_q}\left(\Lambda_w(e_b) - \Lambda_w(\overline{e_j}e_b) - \Lambda_w((y_q^{-1}e_j)\overline{e_j}e_b) - \sum_{H_c \in \tilde{B}_q - \{H_b\}}\Lambda_w(\overline{e_c}e_b)\right)\\
&+ 2\Lambda_w(e_i) - 2\Lambda_w(\overline{e_j}e_i) - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_i) - 4\sum_{H_c \in \tilde{B}_q}\Lambda_w(\overline{e_c}e_i)\\
&+ 2\Lambda_w(e_a) - 2\Lambda_w(\overline{e_j}e_a) - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_a) - 4\sum_{H_c \in \tilde{B}_q}\Lambda_w(\overline{e_c}e_a)\\
&- 4\Lambda_w(\overline{e_i}e_a).
\end{aligned}$$

We will follow the methods used in Section 6.1 to compute $|w|_{\alpha_4}$.

Let $\varphi_{24} := \varphi_{(H_a,x)(B'_q,y_q)} : \hat{\alpha}_2 \to \hat{\alpha}_4$ be the equivariant map described in Convention 6.1.5. If the edges of $\hat{\alpha}_2$ are labelled by e's, and the edges of $\hat{\alpha}_4$ are labeled by h's, then we have:

$$\begin{aligned}
(e_a)\varphi_{24} &= h_j(\overline{y_q^{-1h_j}})(y_q^{-1}h_i)(\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a)\\
(e_i)\varphi_{24} &= h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_i)\\
(e_b)\varphi_{24} &= h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_b)\\
(e_k)\varphi_{24} &= h_k
\end{aligned}$$

for all b with $H_b \in \tilde{B}_q$, and for all k with $H_k \notin B_q$. In particular, since $H_j \notin B_q$, we have $(e_j)\varphi_{24} = h_j$.

Given a word (edge path) w in $\hat{\alpha}_2$, set w' to be the unreduced word $(w)\varphi_{24}$ in $\hat{\alpha}_4$. If $l(u)$ is the unreduced length of a given word u, then $l(w') = l(w) + 4\Lambda_w(e_a) + 2\Lambda_w(e_i) + 2\sum_{H_b\in\tilde{B}_q}\Lambda_w(e_b)$.

Observe that for any b with $H_b \in \tilde{B}_q$ we have $(\overline{e_j}e_b)\varphi_{24} = \overline{h_j}h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_b) = (\overline{y_q^{-1}h_j})(y_q^{-1}h_b)$. We also have that $(\overline{e_j}e_a)\varphi_{24} = \overline{h_j}h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_i)(\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a)$ $= (\overline{y_q^{-1}h_j})(y_q^{-1}h_i)(\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a)$ and $(\overline{e_j}e_i)\varphi_{24} = \overline{h_j}h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_i) = (\overline{y_q^{-1}h_j})(y_q^{-1}h_i)$. Let w'' be the result of performing all such reductions (i.e. of the form $(\overline{e_j}e_b)\varphi_{24}$ where $H_b \in B_q$) to w'. Then for each b with $H_b \in B_q$, we have that the length of $(\overline{e_j}e_b)\varphi_{24}$ is 2 less in w'' than it is in w'. Thus $l(w'') = l(w') - 2\sum_{H_b\in B_q}\Lambda_w(\overline{e_j}e_b) = l(w') - 2\Lambda_w(\overline{e_j}e_i) - 2\Lambda_w(\overline{e_j}e_a) - 2\sum_{H_b\in\tilde{B}_q}\Lambda_w(\overline{e_j}e_b)$.

We now observe that for any b with $H_b \in \tilde{B}_q$, we have $((y_q^{-1}e_j)\overline{e_j}e_b)\varphi_{24} = (y_q^{-1}h_j)(\overline{y_q^{-1}h_j})(y_q^{-1}h_b) = (y_q^{-1}h_b)$, and similarly, $((y_q^{-1}e_j)\overline{e_j}e_i)\varphi_{24} = (y_q^{-1}h_j)(\overline{y_q^{-1}h_j})(y_q^{-1}h_i) = (y_q^{-1}h_i)$. Also note that $((y_q^{-1}e_j)\overline{e_j}e_a)\varphi_{24} = (y_q^{-1}h_j)(\overline{y_q^{-1}h_j})(y_q^{-1}h_i)(\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a) = (y_q^{-1}h_i)(\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a)$. Let w''' be the result of applying all such reductions to w'', and note that for each b with $H_b \in B_q$, the length of $((y_q^{-1}e_j)\overline{e_j}e_b)\varphi_{24}$ is 2 less in w''' than it is in w''. Now $l(w''') = l(w'') - 2\sum_{H_b\in B_q}\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_b) = l(w'') - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_i) - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_a) - 2\sum_{H_b\in\tilde{B}_q}\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_b)$.

We now consider 'cross-reductions' between elements of B_q. Let b and c be such that $H_b \in \tilde{B}_q$ and $H_c \in \tilde{B}_q - \{H_b\}$. Then $(\overline{e_c}e_b)\varphi_{24} = (\overline{y_q^{-1}h_c})(y_q^{-1}h_j)\overline{h_j}h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_b) = (\overline{y_q^{-1}h_c})(y_q^{-1}h_b)$. Additionally, $(\overline{e_i}e_b)\varphi_{24} = (\overline{y_q^{-1}h_i})(y_q^{-1}h_j)\overline{h_j}h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_b) = (\overline{y_q^{-1}h_i})(y_q^{-1}h_b)$, and $(\overline{e_a}e_b)\varphi_{24} = (\overline{x^{-1}y_q^{-1}h_a})(x^{-1}y_q^{-1}h_i)(\overline{y_q^{-1}h_i})(y_q^{-1}h_j)\overline{h_j}h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_b)$ $= (\overline{x^{-1}y_q^{-1}h_a})(x^{-1}y_q^{-1}h_i)(\overline{y_q^{-1}h_i})(y_q^{-1}h_b)$. Finally, $(\overline{e_i}e_a)\varphi_{24} = (\overline{y_q^{-1}h_i})(y_q^{-1}h_j)\overline{h_j}h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_i)(\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a) = (\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a)$. Letting w'''' be the result of applying all such reductions to w''', we see that for b and c with $H_b, H_c \in \tilde{B}_q$ distinct, the length of $(\overline{e_c}e_b)\varphi_{24}$ is 4 less in w'''' than it is in w''', the lengths of $(\overline{e_i}e_b)\varphi_{24}$ and $(\overline{e_a}e_b)\varphi_{24}$ are each 4 less in w'''' than in w''', and the length of $(\overline{e_i}e_a)\varphi_{24}$ is 6 less in w'''' than it is in w'''. Observe that $\Lambda_w(\overline{e_c}e_b) = \Lambda_w(\overline{\overline{e_c}e_b}) = \Lambda_w(\overline{e_b}e_c)$. Thus $l(w'''') = l(w''') - 6\Lambda_w(\overline{e_i}e_a) - 4\sum_{H_b\in\tilde{B}_q}\Big(\Lambda_w(\overline{e_a}e_b) + \Lambda_w(\overline{e_i}e_b) + \frac{1}{2}\sum_{H_c\in\tilde{B}_q-\{H_b\}}\Lambda_w(\overline{e_c}e_b)\Big)$.

Assuming w was reduced to begin with, then there is only one final type of reduction we can apply to w''''. We have that $((x^{-1}e_i)\overline{e_i}e_a)\varphi = (x^{-1}h_j)(\overline{x^{-1}y_q^{-1}h_j})(x^{-1}y_q^{-1}h_i)(\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a) = (x^{-1}h_j)(\overline{x^{-1}y_q^{-1}h_j})(x^{-1}y_q^{-1}h_a)$. Letting w''''' be the result of applying all such reductions to w'''', we see that the length of $((x^{-1}e_i)\overline{e_i}e_a)\varphi$ is 2 less in w''''' than it is in w''''. Then $l(w''''') = l(w'''') - 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a)$.

Since w''''' and w are both fully reduced, we have:

$$
\begin{aligned}
&|w|_{\alpha_4} = |(w)\varphi_{24}|_{\alpha_4} = l(w''''') \\
=&l(w'''') - 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a) \\
=&l(w''') - 6\Lambda_w(\overline{e_i}e_a) - 4\sum_{H_b\in\tilde{B}_q}\left(\Lambda_w(\overline{e_a}e_b) + \Lambda_w(\overline{e_i}e_b) + \frac{1}{2}\sum_{H_c\in\tilde{B}_q-\{H_b\}}\Lambda_w(\overline{e_c}e_b)\right) - 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a) \\
=&l(w'') - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_i) - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_a) - 2\sum_{H_b\in\tilde{B}_q}\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_b) - 6\Lambda_w(\overline{e_i}e_a) \\
&- 4\sum_{H_b\in\tilde{B}_q}\left(\Lambda_w(\overline{e_a}e_b) + \Lambda_w(\overline{e_i}e_b) + \frac{1}{2}\sum_{H_c\in\tilde{B}_q-\{H_b\}}\Lambda_w(\overline{e_c}e_b)\right) - 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a) \\
=&l(w') - 2\Lambda_w(\overline{e_j}e_i) - 2\Lambda_w(\overline{e_j}e_a) - 2\sum_{H_b\in\tilde{B}_q}\Lambda_w(\overline{e_j}e_b) - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_i) - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_a) \\
&- 2\sum_{H_b\in\tilde{B}_q}\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_b) - 6\Lambda_w(\overline{e_i}e_a) - 4\sum_{H_b\in\tilde{B}_q}\left(\Lambda_w(\overline{e_a}e_b) + \Lambda_w(\overline{e_i}e_b) + \frac{1}{2}\sum_{H_c\in\tilde{B}_q-\{H_b\}}\Lambda_w(\overline{e_c}e_b)\right) \\
&- 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a) \\
=&l(w) + 4\Lambda_w(e_a) + 2\Lambda_w(e_i) + 2\sum_{H_b\in\tilde{B}_q}\Lambda_w(e_b) - 2\Lambda_w(\overline{e_j}e_i) - 2\Lambda_w(\overline{e_j}e_a) - 2\sum_{H_b\in\tilde{B}_q}\Lambda_w(\overline{e_j}e_b) \\
&- 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_i) - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_a) - 2\sum_{H_b\in\tilde{B}_q}\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_b) - 6\Lambda_w(\overline{e_i}e_a) \\
&- 4\sum_{H_b\in\tilde{B}_q}\left(\Lambda_w(\overline{e_a}e_b) + \Lambda_w(\overline{e_i}e_b) + \frac{1}{2}\sum_{H_c\in\tilde{B}_q-\{H_b\}}\Lambda_w(\overline{e_c}e_b)\right) - 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a) \\
=&|w|_{\alpha_2} + 2\Lambda_w(e_a) - 2\Lambda_w(\overline{e_i}e_a) - 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a) \\
&+ 2\sum_{H_b\in\tilde{B}_q}\left(\Lambda_w(e_b) - \Lambda_w(\overline{e_j}e_b) - \Lambda_w((y_q^{-1}e_j)\overline{e_j}e_b) - \sum_{H_c\in\tilde{B}_q-\{H_b\}}\Lambda_w(\overline{e_c}e_b)\right) \\
&+ 2\Lambda_w(e_i) - 2\Lambda_w(\overline{e_j}e_i) - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_i) - 4\sum_{H_c\in\tilde{B}_q}\Lambda_w(\overline{e_c}e_i) \\
&+ 2\Lambda_w(e_a) - 2\Lambda_w(\overline{e_j}e_a) - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_a) - 4\sum_{H_c\in\tilde{B}_q}\Lambda_w(\overline{e_c}e_a) \\
&- 4\Lambda_w(\overline{e_i}e_a) \\
=&|w|_{\alpha_1} + |w|_{\alpha_3} - |w|_{\alpha_2}.
\end{aligned}
$$

□

Lemma 6.2.20. *If* $\mathbf{x} \subset H_i \in B_q$ *and* $\mathbf{y} \subset H_j \notin \hat{A}$ *and* $\hat{A} \subseteq B_q$ *then* $||\alpha_4|| - ||\alpha_1|| = ||\alpha_3|| - ||\alpha_2||$, *where* $\alpha_1 = \alpha_2(\mathbf{A}, \mathbf{x})$, $\alpha_3 = \alpha_2(\mathbf{B}, \mathbf{y})$, *and* $\alpha_4 = \alpha_2(\mathbf{B}, \mathbf{y})(\mathbf{A}^{y_q}, \mathbf{x}^{y_q})$.

Proof. By Lemma 6.2.19,

$$||\alpha_4|| - ||\alpha_1|| = \sum_{w\in\mathcal{W}}\Big(|w|_{\alpha_4} - 2\Big) - \sum_{w\in\mathcal{W}}\Big(|w|_{\alpha_1} - 2\Big)$$

$$
\begin{aligned}
&= \sum_{w\in\mathcal{W}} \Big(|w|_{\alpha_4} - 2 - |w|_{\alpha_1} + 2\Big)\\
&= \sum_{w\in\mathcal{W}} \Big(|w|_{\alpha_3} - |w|_{\alpha_2}\Big)\\
&= \sum_{w\in\mathcal{W}} \Big(|w|_{\alpha_3} - 2\Big) - \sum_{w\in\mathcal{W}} \Big(|w|_{\alpha_2} - 2\Big)\\
&= ||\alpha_3|| - ||\alpha_2||.
\end{aligned}
$$

□

Lemma 6.2.21. *If* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak with* $H_i \in \hat{B}$, $H_j \notin \hat{A}$, *and* $\hat{A} \subseteq B_q$, *then* $||\alpha_2(\mathbf{B},\mathbf{y})(\mathbf{A}^{y_q},\mathbf{x}^{y_q})|| < ||\alpha_2||$.

Proof. By Lemma 6.2.20, $||\alpha_2(\mathbf{B},\mathbf{y})(\mathbf{A}^{y_q},\mathbf{x}^{y_q})|| = ||\alpha_4|| = ||\alpha_3|| + ||\alpha_1|| - ||\alpha_2||$. Since $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is a peak, then $||\alpha_2|| \geqslant \max(||\alpha_1||, ||\alpha_3||)$ and $||\alpha_2|| > \min(||\alpha_1||, ||\alpha_3||)$. Now:

$$
\begin{aligned}
||\alpha_4|| &= \max(||\alpha_1||, ||\alpha_3||) + \min(||\alpha_1||, ||\alpha_3||) - ||\alpha_2||\\
&< ||\alpha_2|| + ||\alpha_2|| - ||\alpha_2||\\
&= ||\alpha_2||.
\end{aligned}
$$

□

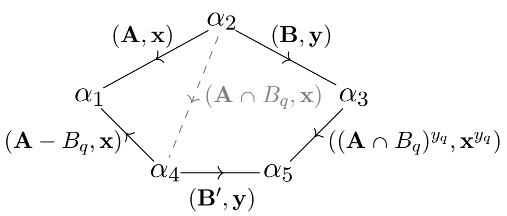
Proposition 6.2.22. *Let* $H_1 * \cdots * H_n$ *be an* $\mathfrak{S}$ *free factor splitting for* G. *Let* $(\mathbf{A},\mathbf{x})$ *and* $(\mathbf{B},\mathbf{y})$ *be relative multiple Whitehead automorphisms with* $\mathbf{x} \subset H_i$ *and* $\mathbf{y} \subset H_j$, $\hat{A} \subset \{H_1,\ldots,H_n\} - \{H_i,H_j\}$, $\hat{B} \subset \{H_1,\ldots,H_n\} - \{H_j\}$ *such that* $\{H_i\} \cup \hat{A} \subset B_q$ *for some* q. *Let* α_2 *be the domain whose* α*-graph has* $\mathfrak{S}$*-labelling* $(H_1,\ldots,H_n)$, *and let* $\alpha_1 = \alpha_2(\mathbf{A},\mathbf{x})$ *and* $\alpha_3 = \alpha_2(\mathbf{B},\mathbf{y})$. *If* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak, then it is reducible.*

Proof. By Lemmas 6.2.17 and 6.2.18, the path $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is homotopic in the Space of Domains to the path $\alpha_1 \xrightarrow{(\mathbf{B}',\mathbf{y})} \alpha_2(\mathbf{B},\mathbf{y})(\mathbf{A}^{y_q},\mathbf{x}^{y_q}) \xleftarrow{(\mathbf{A}^{y_q},\mathbf{x}^{y_q})} \alpha_3$ where $\mathbf{B}' = (B_1,\ldots,(B_q - \hat{A}) \cup \widehat{\mathbf{A}^{\mathbf{x}}},\ldots,B_k)$. By Lemma 6.2.21, $||\alpha_2(\mathbf{B},\mathbf{y})(\mathbf{A}^{y_q},\mathbf{x}^{y_q})|| < ||\alpha_2||$. Thus by Definition 6.2.2, the peak $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is reducible. □

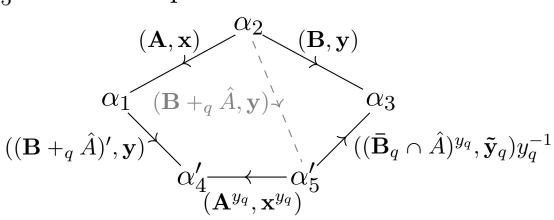
Case 2(b): $H_i \in \hat{B}$ **(say** $H_i \in B_q$**) and** $H_j \notin \hat{A}$**, with** $\hat{A} \nsubseteq B_q$

Lemma 6.2.23. *If* $H_i \in \hat{B}$ *and* $H_j \notin \hat{A}$ *with* $\hat{A} \nsubseteq B_q$, *then* $(\mathbf{A},\mathbf{x})^{-1}(\mathbf{B},\mathbf{y}) = (\mathbf{A} - B_q,\mathbf{x})^{-1}(\mathbf{B}',\mathbf{y})((\mathbf{A}\cap B_q)^{y_q},\mathbf{x}^{y_q})^{-1}$ *and* $(\mathbf{A},\mathbf{x})^{-1}(\mathbf{B},\mathbf{y}) = ((\mathbf{B}+_q\hat{A})',\mathbf{y})(\mathbf{A}^{y_q},\mathbf{x}^{y_q})^{-1}((\bar{\mathbf{B}}_q\cap \hat{A})^{y_q},\tilde{\mathbf{y}}_q y_q^{-1})$, *where* $[\mathbf{B}']_q := (B_q - \hat{A}) \cup \bigcup((\mathbf{A}\cap B_q)^{\mathbf{x}})$ *and* $[\mathbf{B}']_k := B_k$ *for* $k \neq q$, *and* $[(\mathbf{B} +_q \hat{A})']_q := ([(\mathbf{B} +_q \hat{A})]_q - \hat{A}) \cup \bigcup([(\mathbf{B} +_q \hat{A})]_q \cap \mathbf{A})^{\mathbf{x}} = (B_q - \hat{A}) \cup \bigcup \mathbf{A}^{\mathbf{x}}$ *and* $[(\mathbf{B} +_q \hat{A})']_k := [(\mathbf{B} +_q \hat{A})]_k = B_k - \hat{A}$ *for* $k \neq q$.

That is, there exist vertices α_4, α_5, α_4', α_5' in our Graph of Domains such that



and



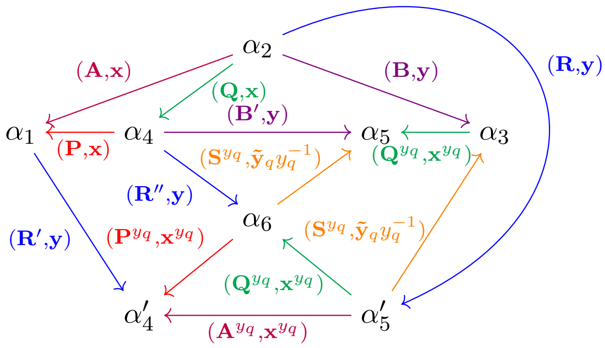
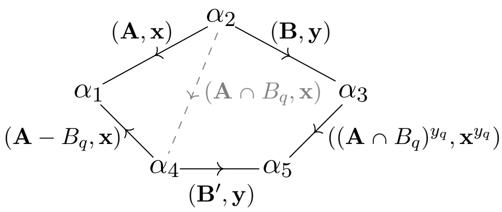


Figure 16: Commuting Diagram for Case 2b

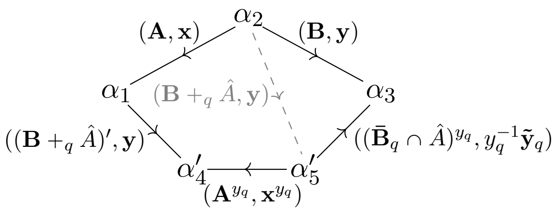
are both loops.

Proof. By Proposition 5.5.8 (1), $(\mathbf{A},\mathbf{x})(\mathbf{A}-B_q,\mathbf{x})^{-1} = (\mathbf{A}\cap B_q,\mathbf{x})$. Writing $\bigcup \mathbf{A} = \hat{A}$, we have that $\bigcup(\mathbf{A}\cap B_q) = \hat{A}\cap B_q \subseteq B_q$. Similarly, by Proposition 5.5.8 (4), $(\mathbf{B},\mathbf{y})((\bar{\mathbf{B}}_q\cap\hat{A})^{y_q},\tilde{\mathbf{y}}_q y_q^{-1})^{-1} = (\mathbf{B}+_q\hat{A},\mathbf{y})$. Note that $[\mathbf{B}+_q\hat{A}]_q = B_q\cup\hat{A}\supseteq\hat{A}$. We have now reduced both problems to the form required by Case 2a, so the result follows from Lemma 6.2.17. $\square$

Lemma 6.2.24. *The loops*



and



are both contractible in our Space of Domains.

Proof. As in Lemma 6.2.13, the α_1–α_2–α_4 triangle can be 'filled' with an A_i graph, and the α_2–α_3–α_5' triangle can be 'filled' with an A_j graph. Now $\hat{A}\cap B_q\subseteq B_q$ and $\hat{A}\subseteq[\mathbf{B}+_q\hat{A}]_q = B_q\cup\hat{A}$, so by Lemma 6.2.18 (Case 2a), the squares $\alpha_4 — \alpha_2 — \alpha_3 — \alpha_5 — \alpha_4$ and $\alpha_1 — \alpha_2 — \alpha_5' — \alpha_4' — \alpha_1$ are both contractible. $\square$

Lemma 6.2.25. *If* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak with* $H_i\in\hat{B}$, $H_j\notin\hat{A}$, *and* $\hat{A}\nsubseteq B_q$, *then either* $||\alpha_2(\mathbf{A},\mathbf{x})(\mathbf{A}-B_q,\mathbf{x})^{-1}|| < ||\alpha_2||$ *and* $||\alpha_2(\mathbf{A}\cap B_q,\mathbf{x})(\mathbf{B}',\mathbf{y})|| < ||\alpha_2||$ *, or* $||\alpha_2(\mathbf{A},\mathbf{x})((\mathbf{B}+_q\hat{A})',\mathbf{y})|| < ||\alpha_2||$ *and* $||\alpha_2(\mathbf{B}+_q\hat{A},\mathbf{y})|| < ||\alpha_2||$ *.*

That is, $||\alpha_2|| > \max(||\alpha_4||,||\alpha_5||)$ or $||\alpha_2|| > \max(||\alpha_4'||,||\alpha_5'||)$.

Proof. It will suffice to show that either $||\alpha_4|| < ||\alpha_2||$ or $||\alpha_5'|| < ||\alpha_2||$. The problem then reduces to Case 2a (Lemma 6.2.21).

Let $\alpha_6 = \alpha_2(\mathbf{A}\cap B_q,\mathbf{x})(\mathbf{B}+_q\hat{A},\mathbf{y})$. For brevity, let $\mathbf{P} = \mathbf{A}-B_q$, $\mathbf{Q} = \mathbf{A}\cap B_q$, $\mathbf{R} = \mathbf{B}+_q\hat{A}$, and $\mathbf{S} = \bar{\mathbf{B}}_q\cap\hat{A}$. For $b\neq q$ set $[\mathbf{R}'']_b = [\mathbf{R}']_b := B_b-\hat{A} = [\mathbf{R}]_b$, $[\mathbf{R}'']_q := (B_q-\hat{A})\sqcup(\widehat{B_q\cap\mathbf{A}})^{\mathbf{x}}\sqcup(\hat{A}-B_q)$, and $[\mathbf{R}']_q := (B_q-\hat{A})\sqcup(\widehat{\mathbf{A}})^{\mathbf{x}}$. Then the diagram in Figure 16 commutes.

Since $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is a peak, we have that $\min(\|\alpha_1\| - \|\alpha_2\|, \|\alpha_3\| - \|\alpha_2\|) < 0$ and $\max(\|\alpha_1\| - \|\alpha_2\|, \|\alpha_3\| - \|\alpha_2\|) \leqslant 0$.

Observe that $\widehat{\mathbf{Q}^{y_q}} \cap \widehat{\mathbf{S}^{y_q}} = (\hat{A} \cap B_q)^{y_q} \cap (\hat{A} - \hat{B})^{y_q} = \varnothing$. Additionally, $\tilde{\mathbf{y}}_q y_q^{-1} \in H_j \notin \hat{A}$ (so $\tilde{\mathbf{y}}_q y_q^{-1} \notin \widehat{\mathbf{Q}^{y_q}}$), and $\mathbf{x}^{y_q} \in H_i^{y_q} \in B_q^{y_q}$ (so $\mathbf{x}^{y_q} \notin \widehat{\mathbf{S}^{y_q}}$). Thus $\alpha_5' — \alpha_6 — \alpha_5 — \alpha_3 — \alpha_5'$ is a square falling under Case 1a, so by Lemma 6.2.9, we have that $\|\alpha_3\| - \|\alpha_5'\| = \|\alpha_5\| - \|\alpha_6\|$.

Also observe that $\hat{P} = \hat{A} - B_q \subseteq [\mathbf{R}'']_q$, $\mathbf{x} \in H_i \in B_q - \hat{A} \subset [\mathbf{R}'']_q$, and $\mathbf{y} \in H_j \notin \hat{A} - B_q = \hat{P}$. So $\alpha_4 — \alpha_6 — \alpha_4' — \alpha_1 — \alpha_4$ is a square falling under Case 2a, and by Lemma 6.2.20, we have that $\|\alpha_1\| - \|\alpha_4\| = \|\alpha_4'\| - \|\alpha_6\|$.

Since $\widehat{\mathbf{P}^{y_q}} = (\hat{A} - B_q)^{y_q} = \widehat{\mathbf{S}^{y_q}}$ (and $H_i^{y_q}, H_j^{y_q} \notin \widehat{\mathbf{P}^{y_q}}$) then by Lemma 6.2.14, we have that either $\max(\|\alpha_4'\| - \|\alpha_6\|, \|\alpha_5\| - \|\alpha_6\|) > 0$ or $\|\alpha_4'\| - \|\alpha_6\| = \|\alpha_5\| - \|\alpha_6\| = 0$.

Finally, we note that $\|\alpha_4\| - \|\alpha_2\| = (\|\alpha_1\| - \|\alpha_2\|) - (\|\alpha_1\| - \|\alpha_4\|)$, and similarly, $\|\alpha_5'\| - \|\alpha_2\| = (\|\alpha_3\| - \|\alpha_2\|) - (\|\alpha_3\| - \|\alpha_5'\|)$.

As in the proof of Lemma 6.2.15, we now deduce from this information that $\min(\|\alpha_4\| - \|\alpha_2\|, \|\alpha_5'\| - \|\alpha_2\|) < 0$, as required. ☐

Proposition 6.2.26. *Let $H_1 * \cdots * H_n$ be an $\mathfrak{S}$ free factor splitting for G. Let $(\mathbf{A}, \mathbf{x})$ and $(\mathbf{B}, \mathbf{y})$ be relative multiple Whitehead automorphisms with $\mathbf{x} \subset H_i$, $\mathbf{y} \subset H_j$, $\hat{A} \subset \{H_1, \ldots, H_n\} - \{H_i, H_j\}$, $\hat{B} \subset \{H_1, \ldots, H_n\} - \{H_j\}$, $H_i \in B_q$ for some Q, and $\hat{A} \not\subset B_q$. Let α_2 be the domain whose α-graph has $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$, and let $\alpha_1 = \alpha_2(\mathbf{A}, \mathbf{x})$ and $\alpha_3 = \alpha_2(\mathbf{B}, \mathbf{y})$. If $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is a peak, then it is reducible.*

Proof. By Lemmas 6.2.23 and 6.2.24, the path $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is homotopic in the Space of Domains to each of the paths

$$\alpha_1 \xleftarrow{(\mathbf{A}-B_q,\mathbf{x})} \alpha_2(\mathbf{A},\mathbf{x})(\mathbf{A} - B_q, \mathbf{x})^{-1} \xrightarrow{(\mathbf{B}',\mathbf{y})} \alpha_2(\mathbf{A} \cap B_q, \mathbf{x})(\mathbf{B}', \mathbf{y}) \xleftarrow{((\mathbf{A} \cap B_q)^{y_q}, \mathbf{x}^{y_q})} \alpha_3$$

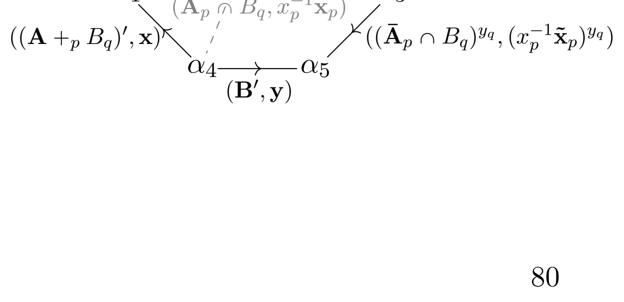
and $$\alpha_1 \xrightarrow{((\mathbf{B}+_q\hat{A})',\mathbf{y})} \alpha_2(\mathbf{A},\mathbf{x})((\mathbf{B} +_q \hat{A})', \mathbf{y}) \xleftarrow{(\mathbf{A}^{y_q},\mathbf{x}^{y_q})} \alpha_2(\mathbf{B} +_q \hat{A}, \mathbf{y}) \xrightarrow{((\bar{\mathbf{B}}_q \cap \hat{A})^{y_q}, \tilde{\mathbf{y}}_q y_q^{-1})} \alpha_3 \quad .$$

By Lemma 6.2.25, either $||\alpha_2(\mathbf{A}, \mathbf{x})(\mathbf{A} - B_q, \mathbf{x})^{-1}|| < ||\alpha_2||$ and $||\alpha_2(\mathbf{A} \cap B_q, \mathbf{x})(\mathbf{B}', \mathbf{y})|| < ||\alpha_2||$, or $||\alpha_2(\mathbf{A}, \mathbf{x})((\mathbf{B} +_q \hat{A})', \mathbf{y})|| < ||\alpha_2||$ and $||\alpha_2(\mathbf{B} +_q \hat{A}, \mathbf{y})|| < ||\alpha_2||$. Thus by Definition 6.2.2, one of the above paths is a reduction for the peak $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$. ☐

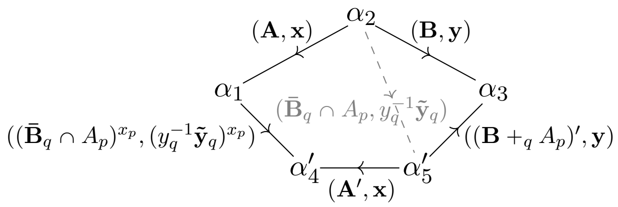
Case 4: $H_i \in \hat{B}$ and $H_j \in \hat{A}$ (say $H_i \in B_q$ and $H_j \in A_p$)

Lemma 6.2.27. *If $H_i \in B_q$ and $H_j \in A_p$, then*
$(\mathbf{A}, \mathbf{x})^{-1}(\mathbf{B}, \mathbf{y}) = ((\mathbf{A} +_p B_q)', \mathbf{x})^{-1}(\mathbf{B}', \mathbf{y})((\bar{\mathbf{A}}_p \cap B_q)^{y_q}, (x_p^{-1}\tilde{\mathbf{x}}_p)^{y_q})^{-1}$ and
$(\mathbf{A}, \mathbf{x})^{-1}(\mathbf{B}, \mathbf{y}) = ((\bar{\mathbf{B}}_q \cap A_p)^{x_p}, (y_q^{-1}\tilde{\mathbf{y}}_q)^{x_p})(\mathbf{A}', \mathbf{x})^{-1}((\mathbf{B} +_q A_p)', \mathbf{y})$, where $(\mathbf{A} +_p B_q)'$ (and $(\mathbf{B} +_q A_p)'$) are as defined in Proposition 5.5.8 (4), and $\mathbf{A}'$ and $\mathbf{B}'$ are defined similarly.

That is, there exist vertices α_4, α_5, α_4', and α_5' in our Graph of Domains such that



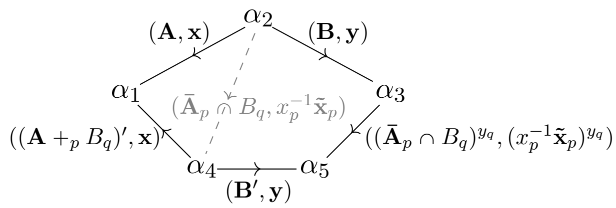
and



are both loops.

Proof. By Proposition 5.5.8 (4), $(\mathbf{A},\mathbf{x}) = (\bar{\mathbf{A}}_p \cap B_q, x_p^{-1}\tilde{\mathbf{x}}_p)((\mathbf{A} +_p B_q)', \mathbf{x})$. Now $x_p^{-1}\tilde{\mathbf{x}}_p \in H_i \in B_q$ still, and as $H_j \notin \mathbf{B}$ then $H_j \notin \bar{\mathbf{A}}_p \cap B_q$. Also $\bar{\mathbf{A}}_p \cap B_q \subseteq B_q$, so by Case 2a (Lemma 6.2.17), $(\bar{\mathbf{A}}_p \cap B_q, x_p^{-1}\tilde{\mathbf{x}}_p)(\mathbf{B}', \mathbf{y}) = (\mathbf{B},\mathbf{y})((\bar{\mathbf{A}}_p \cap B_q)^{y_q}, (x_p^{-1}\tilde{\mathbf{x}}_p)^{y_q})$. The second loop is achieved by renaming A to B and x to y, and vice versa. □

Lemma 6.2.28. *The loops*



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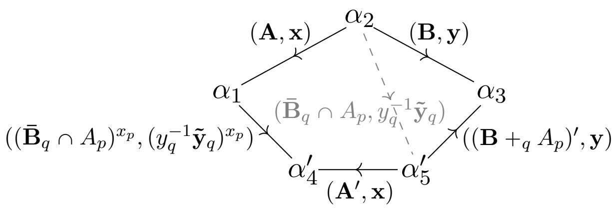

are both contractible in our Space of Domains.

Proof. As in Lemma 6.2.13, the α_1–α_2–α_4 triangle can be 'filled' with an A_i graph, and the α_2–α_3–α_5' triangle can be 'filled' with an A_j graph. Now $\bigcup(\bar{\mathbf{A}}_p \cap B_q) \subseteq B_q$ and $\bigcup(\bar{\mathbf{B}} \cap A_p) \subseteq A_p$, so (after relabelling) by Lemma 6.2.18 (Case 2a), the squares $\alpha_4 — \alpha_2 — \alpha_3 — \alpha_5 — \alpha_4$ and $\alpha_1 — \alpha_2 — \alpha_5' — \alpha_4' — \alpha_1$ are both contractible. □

Lemma 6.2.29. *If* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak with* $H_i \in \hat{B}$ *and* $H_j \in \hat{A}$*, then (up to relabelling)* $||\alpha_2(\mathbf{A},\mathbf{x})((\mathbf{A} +_p B_q)', \mathbf{x})^{-1}|| < ||\alpha_2||$ *and* $||\alpha_2(\bar{\mathbf{A}}_p \cap B_q, x_p^{-1}\tilde{\mathbf{x}}_p)(\mathbf{B}', \mathbf{y})|| < ||\alpha_2||$.

Proof. First, note that inner automorphisms stabilise each point of $\mathcal{C}_n$, and hence each domain in the Space of Domains. Writing $\gamma(z)$ for the inner automorphism which conjugates everything by z, we then see that $\alpha_1 = \alpha_2(\mathbf{A},\mathbf{x}) = \alpha_2(\mathbf{A},\mathbf{x})\gamma(x_p^{-1}) = \alpha_2(\bar{\mathbf{A}}_p, x_p^{-1}\tilde{\mathbf{x}}_p)$. Similarly, $\alpha_3 = \alpha_2(\bar{\mathbf{B}}_q, y_q^{-1}\tilde{\mathbf{y}}_q)$. Note that $\hat{\bar{\mathbf{A}}}_p = \bar{A}_p$ and $\hat{\bar{\mathbf{B}}}_q = \bar{B}_q$. Since $\mathbf{x} \subset H_i \in B_q$ and $\mathbf{y} \subset H_j \in A_p$, then $x_p^{-1}\tilde{\mathbf{x}}_p \subset H_i \notin \hat{\bar{\mathbf{A}}}_p$ and $y_q^{-1}\tilde{\mathbf{y}}_q \subset H_j \notin \hat{\bar{\mathbf{B}}}_q$.
If $A_p \cup B_q \neq \hat{H}$ then by Lemma 6.2.15, either $\|\alpha_2(\bar{\mathbf{A}}_p - \bar{B}_q, x_p^{-1}\tilde{\mathbf{x}}_p)\| < \|\alpha_2\|$ or $\|\alpha_2(\bar{\mathbf{B}}_q - \bar{A}_p, y_q^{-1}\tilde{\mathbf{y}}_q)\| < \|\alpha_2\|$. But given arbitrary sets C and D, $C - \bar{D} = C \cap D$. Hence either $\|\alpha_4\| < \|\alpha_2\|$ or $\|\alpha_5'\| < \|\alpha_2\|$.
If $A_p \cup B_q = \hat{H}$ then $\alpha_4 = \alpha_1(\mathbf{A} +_p B_q, \mathbf{x})^{-1} = \alpha_1\gamma(x_p^{-1}) = \alpha_1$. Similarly, $\alpha_5' = \alpha_3$, and since $\alpha_1 — \alpha_2 — \alpha_3$ is a peak, then $\|\alpha_2\| > \min(\|\alpha_4\|, \|\alpha_5'\|)$.
Now one of $\alpha_4 — \alpha_2 — \alpha_3$ or $\alpha_1 — \alpha_2 — \alpha_5'$ is a peak satisfying Case 2a, and the result follows from Lemma 6.2.21. □

Proposition 6.2.30. *Let* $H_1 * \cdots * H_n$ *be an* $\mathfrak{S}$ *free factor splitting for* G*. Let* $(\mathbf{A},\mathbf{x})$ *and* $(\mathbf{B},\mathbf{y})$ *be relative multiple Whitehead automorphisms with* $\mathbf{x} \subset H_i$ *and* $\mathbf{y} \subset H_j$

and $\hat{A} \subset \{H_1, \ldots, H_n\} - \{H_i\}$, $\hat{B} \subset \{H_1, \ldots, H_n\} - \{H_j\}$ *such that for some* p *and* q *we have* $H_i \in B_q$ *and* $H_j \in A_p$. *Let* α_2 *be the domain whose* α*-graph has* $\mathfrak{S}$*-labelling* $(H_1, \ldots, H_n)$, *and let* $\alpha_1 = \alpha_2(\mathbf{A}, \mathbf{x})$ *and* $\alpha_3 = \alpha_2(\mathbf{B}, \mathbf{y})$. *If* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak, then it is reducible.*

Proof. By Lemmas 6.2.27 and 6.2.28, the path $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is homotopic in the Space of Domains to each of the paths

$$\alpha_1 \xleftarrow{((\mathbf{A}+_p B_q)',\mathbf{x})} \alpha_2(\bar{\mathbf{A}}_p \cap B_q, x_p^{-1}\tilde{\mathbf{x}}_p) \xrightarrow{(\mathbf{B}',\mathbf{y})} \alpha_2(\bar{\mathbf{A}}_p \cap B_q, x_p^{-1}\tilde{\mathbf{x}}_p)(\mathbf{B}',\mathbf{y}) \xleftarrow{((\bar{\mathbf{A}}_p \cap B_q)^{y_q},(x_p^{-1}\tilde{\mathbf{x}}_p)^{y_q})} \alpha_3$$

and

$$\alpha_1 \xrightarrow{((\bar{\mathbf{B}}_q \cap A_p)^{x_p},(y_q^{-1}\tilde{\mathbf{y}}_q)^{x_p})} \alpha_2(\bar{\mathbf{B}}_q \cap A_p, y_q^{-1}\tilde{\mathbf{y}}_q)(\mathbf{A}',\mathbf{x}) \xleftarrow{(\mathbf{A}',\mathbf{x})} \alpha_2(\bar{\mathbf{B}}_q \cap A_p, y_q^{-1}\tilde{\mathbf{y}}_q) \xrightarrow{((\mathbf{B}+_q A_p)',\mathbf{y})} \alpha_3 \ .$$

By Lemma 6.2.29, either $||\alpha_2(\mathbf{A},\mathbf{x})((\mathbf{A}+_p B_q)',\mathbf{x})^{-1}|| < ||\alpha_2||$ and $||\alpha_2(\bar{\mathbf{A}}_p \cap B_q, x_p^{-1}\tilde{\mathbf{x}}_p)(\mathbf{B}',\mathbf{y})|| < ||\alpha_2||$, or $||\alpha_2(\bar{\mathbf{B}}_q \cap A_p, y_q^{-1}\tilde{\mathbf{y}}_q)(\mathbf{A}',\mathbf{x})|| < ||\alpha_2||$ and $||\alpha_2(\bar{\mathbf{B}}_q \cap A_p, y_q^{-1}\tilde{\mathbf{y}}_q)|| < ||\alpha_2||$. Thus by Definition 6.2.2, one of the above paths is a reduction for the peak $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$. □

6.3 Simple Connectivity

We have now done all the required work to conclude:

Proposition 6.3.1. *Every peak* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *(whose edges are both of Type A) in the Space of Domains is reducible (to a path of length* 2 *or* 3 *whose edges are all of Type A).*

Proof. Let $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ be a peak in the Space of Domains whose edges are both of Type A. Suppose α_2 has $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$. Then for some i and j we have $\mathbf{x} \subset H_i$ and $\mathbf{y} \subset H_j$. By assumption, $H_i \notin \hat{A}$ and $H_j \notin \hat{B}$. If $i = j$ then by Proposition 6.2.3, the peak is reducible. Otherwise, our peak falls into one of the Cases 1–4 as described in Observation 6.2.4, and by Propositions 6.2.11, 6.2.16, 6.2.22, 6.2.26, and 6.2.30, we are done (after renaming, if we fell under Case 3). □

We can now use this Peak Reduction Proposition to argue that the Space of Domains, and hence the complex $\mathcal{C}_n$, is simply connected.

Theorem 6.3.2. *Our Space of Domains is simply connected.*

Proof. Let λ be a loop in the Space of Domains. Note that any loop is homotopic to a based loop, so without loss of generality, we assume λ contains the basepoint α_0. By Corollary 5.4.4 and Proposition 6.3.1, λ is homotopic (in the Space of Domains) to a peak reduced loop λ'. But any peak reduced loop must have constant height (else it would contain some 'heighest' point, i.e. a peak). Since the basepoint has height 0 (Lemma 6.1.3) then every point in λ' must have height 0. But again by Lemma 6.1.3, the only point with height 0 is the basepoint. Hence λ' is actually the constant 'loop' at the basepoint, α_0. Thus any loop λ is homotopic to a constant loop, hence the Space of Domains is simply connected. □

Corollary 6.3.3. *The space* $\mathcal{C}_n$ *(for* $n \geqslant 5$*) is simply connected.*

Proof. This follows directly from Theorem 6.3.2 and Proposition 5.3.5. □

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