Data-Driven Continuous-Time Optimal Control: A Unified Framework using Orthogonal Functions

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Abstract

We study data-driven optimal control of continuous-time linear systems over finite- and infinite-time horizons. Our approach builds on our continuous-time version of Willems et al.'s fundamental lemma and on the use of orthogonal basis functions to approximate system trajectories. We show that the solution to an optimal control problem can be approximated by a finite linear combination of basis functions and we establish error bounds for such approximations. Moreover, we approximately solve the algebraic Riccati equation and the associated optimal controller gain directly from data, opening up the possibility of optimal controller design directly from data analogue devices.

Keywords: Data-driven methods, Optimal Control, Willems' fundamental lemma, Orthogonal bases, approximation error

1 Introduction

Finite-dimensional approximations of the solution of optimal control problems can be computed using orthogonal bases decompositions of the system signals. Such techniques and the closely related spectral methods are applied especially in the nonlinear case, where it is difficult to compute analytic solutions, see for example [1–7]. The starting point in such applications is a *mathematical model* of the plant as a system of (usually first-order) differential equations. An emerging area where orthogonal bases of signal spaces are being successfully used is continuous-time *data-driven* control of linear systems, see [8–12]. In such applications the use of orthogonal bases allows to design continuous-time controllers *directly* and *exclusively* from data, with few assumptions on the system dynamics.

In this paper we build on our work [13], where Legendre bases were used to solve data-driven finite-horizon optimal control problems for continuous-time linear systems from input-output data. We extend the results to the *infinite-horizon* setting by using Laguerre-type bases. We show that the finite-dimensional approximation obtained by truncating the polynomial series representation of system trajectories converge to the optimal trajectory as the number of terms increases. A further contribution is the development of algorithms that, starting from informative data, compute increasingly accurate approximations of the maximal solution to the algebraic Riccati equation and of the corresponding optimal controller. Thus we achieve a data-driven *continuous-time controller design*; such controllers can be implemented using analogue devices. Such data-driven methods are directly applicable to the continuous-time dynamics of the to-be-controlled system. This is a desirable feature for example in power electronics, where some control schemes are implemented through analog circuitry to achieve rapid control action on inner loops and to lower implementation costs.

Among sample-based data-driven approaches to control continuous-time systems, the one closest to ours is the preprint [14], where besides (infinite-horizon) optimal control, other control problems (reference-trajectory tracking, robust pole-placement) are investigated. The main differences with [14] are the following: Firstly, in [14] it is assumed that the state derivative is directly measured. Admittedly, in this paper we do make the same assumption; however, such an assumption is not essential since the state derivative can be computed directly from samples of the state trajectory itself using "differentiation relations" (see Remark 9). Secondly, our results are stated in terms of general, not necessarily piecewise-constant inputs (see [14, Remark 2]); consequently we can use existing (sufficiently informative) measurements from previous experiments, or design new experiments using a wider class of input signals than piecewise constant ones. Finally, in [14] the role of $t \in [0, T]$ in the solution of the optimal control problem (see e.g. formula (42), the statement of Lemma 16, and footnote 5 on the same page *ibid.*) is not discussed, while we analyze thoroughly the accuracy issues arising from the truncation of polynomial-based series expansions, see Sections 3.2 and 3.3.

The paper is structured as follows. We illustrate some background material on weighted Sobolev spaces and on Laguerre polynomials respectively in Appendix A. In Section 2.1 we recall the concept of "sufficient informativity" and the continuous-time "fundamental lemma" from [13], extending it to the infinite-horizon case using Laguerre basis representations. In Section 2.2 we derive a data-driven formulation of Pontryagin's minimum principle. In Section 3.1 we summarize some results from approximation theory using polynomials. In Section 3.2 we discuss the asymptotic behavior of such approximations. We use such results in Section 3.3 to prove that the solution of the optimal control problem on finite-dimensional approximations of the system trajectories converges to the optimal solution. We show that the optimal solutions satisfy an orthogonality property in Section 3.4 and we use such property to compute approximate solutions to the optimal control problem in Section 3.5. In the last part of the paper, consisting of Section 4 and its subsections, we illustrate a data-driven approach to some classical issues related to optimal control problems. We illustrate data-driven methods to compute approximations to the algebraic Riccati equation in Section 4.1 and the computation of optimal feedback gains in Section 4.2. In Section 5 we illustrate our results with a numerical example.

Notation. Let $d, k \in \mathbb{N}$, $d \geq 1$. The space of square-summable semi-infinite sequences in $\mathbb{R}^d$ is denoted by $\ell^2(\mathbb{N}, \mathbb{R}^d)$. Given a finite sequence of vectors $x_0, \ldots, x_k \in \mathbb{R}^d$, we define $\mathrm{col}(x_0, \ldots, x_k) := \begin{bmatrix} x_0^\top & \cdots & x_k^\top \end{bmatrix}^\top$. This notation extends to matrices and to vector-valued functions. Given a vector space X, the identity operator in X is denoted as I_X; when there is no possibility of confusion, we simply write I. In the case of $X = \mathbb{R}^d$ we also write I_d. The Euclidean norm and the Frobenius norm in $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$, respectively, are denoted by $\|\cdot\|_2$ and $\|\cdot\|_\mathrm{F}$. The Moore-Penrose inverse of a matrix A is denoted by $A^\dagger$. The Kronecker product of two matrices A and B is denoted by $A \otimes B$.

Given an interval $\mathcal{I} \subset \mathbb{R}$, we denote by $\mathcal{C}^k(\mathcal{I}, \mathbb{R}^d)$ the space of k-times continuously differentiable functions from $\mathcal{I}$ to $\mathbb{R}^d$. Moreover, $L^2(\mathcal{I}, \mathbb{R}^d)$ is the space of (equivalence classes of) square-integrable functions from $\mathcal{I}$ to $\mathbb{R}^d$, equipped with the usual inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. Given $f \in L^2(\mathcal{I}, \mathbb{R}^d)$, we use blackletter typeface to represent the sequence of coefficients $\mathfrak{f} = (\mathfrak{f}_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{R}^d)$ appearing in the series expansion of f with respect to an orthonormal basis. In this context the isometric isomorphism between $L^2(\mathcal{I}, \mathbb{R}^d)$ and $\ell^2(\mathbb{N}, \mathbb{R}^d)$ is denoted by Π, i.e. $\Pi f = \mathfrak{f}$. The kth order Sobolev space associated with $L^2(\mathcal{I}, \mathbb{R}^d)$ is denoted by $H^k(\mathcal{I}, \mathbb{R}^d)$, and is equipped with the usual norm $\|\cdot\|_{H^k}$. Let $k \in \mathbb{N}$, $k \geq 1$; the first derivative of $f \in H^{k-1}(\mathcal{I}, \mathbb{R}^d)$ is denoted interchangeably by $\frac{\mathrm{d}f}{\mathrm{d}t}$, $\dot{f}$ or $f^{(1)}$ and higher derivatives are written as $\frac{\mathrm{d}^j f}{\mathrm{d}t^j} = f^{(j)}$ for $j = 0, \ldots, k-1$. Moreover, we define

$$\Lambda_k(f) = \mathrm{col}(f, \ldots, f^{(k-1)}) \in L^2(\mathcal{I}, \mathbb{R}^{kd}). \tag{1}$$

It is straightforward to verify that $\|\Lambda_k(f)\| = \|f\|_{H^{k-1}}$.

2 The continuous-time fundamental lemma and problem formulation

We consider continuous-time linear time-invariant systems governed by an input–state equation of the form

$$\dot{x} = Ax + Bu \tag{2}$$

with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. We consider the solution over either the bounded interval $(0,1)$ or on the half-axis $(0,\infty)$; such interval will be denoted by $\mathcal{I}$ whenever it is clear from the context which one we are referring to. The *behavior* of the system (2) is defined by

$$\mathcal{B} = \{\mathrm{col}(x,u) \in H^1(\mathcal{I},\mathbb{R}^n) \times L^2(\mathcal{I},\mathbb{R}^m) \,|\, \dot{x} = Ax + Bu\}\ . \tag{3}$$

Such definition is consistent with that introduced in Section 3.3 of [15].

The following result is Lemma 2 in [13].

Lemma 1 *$\mathcal{B}$ defined in (3) is a closed subset of $L^2(\mathcal{I},\mathbb{R}^{n+m})$. Moreover, $\mathcal{B}$ equipped with the standard inner product in $L^2(\mathcal{I},\mathbb{R}^{n+m})$ is a Hilbert space.*

A behavior (3) is called *controllable* if the pair (A,B) is controllable (see Definition 5.2.2 in [16] and Section 3.4.1 of [15]). We define the space of trajectories with their state variable initialized to $x(0) = x^0 \in \mathbb{R}^n$ by

$$\mathcal{B}(x^0) := \{\mathrm{col}(x,u) \in \mathcal{B} \,|\, x(0) = x^0\}. \tag{4}$$

It is straightforward to verify that the set $\mathcal{B}(x^0)$ is an affine subspace of $\mathcal{B}$; moreover, for every $w \in \mathcal{B}(x^0)$ it holds that $\mathcal{B}(x^0) = \{w\} + \mathcal{B}(0)$.

The present work focuses on the linear-quadratic optimal control problem (abbreviated OCP in the rest of the paper) described by

$$\underset{\mathrm{col}(x,u)\in\mathcal{B}(x^0)}{\mathrm{minimize}} \ \|x\|^2 + \|u\|^2. \tag{5}$$

In the following discussion, we develop a data-driven framework to solve such OCP.

Remark 2 The optimal control problem we consider in this paper is formulated as in (5) only for simplicity of exposition; the more general case of stage costs

$$\int_{\mathcal{I}} x^\top Q x + u^\top R u \,\mathrm{d}t \tag{6}$$

induced by symmetric positive-definite matrices $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ can be treated analogously. We will not enter into such details here.

2.1 A fundamental lemma

In this section we recall the notion of *persistence of excitation* and the continuous-time "fundamental lemma", see [13]. Although the results in [13] are formulated for the finite interval case, the techniques for their proofs carry over directly to the case $\mathcal{I} = (0,\infty)$.

Definition 1 ([13, Definition 16, Lemma 17][1]) We call a function $u : \mathcal{I} \to \mathbb{R}^m$ *persistently exciting of order* $k \in \mathbb{N} \setminus \{0\}$, if $u \in H^{k-1}(\mathcal{I}, \mathbb{R}^m)$ and the components of $\Lambda_k(u)$ are linearly independent in $L^2(\mathcal{I}, \mathbb{R})$, where $\Lambda_k(u)$ is defined by (1).

Definition 2 Let $\operatorname{col}(\hat{x}, \hat{u}) \in \mathcal{B}$; the *Gramian* associated with $\operatorname{col}(\hat{x}, \hat{u})$ is defined by

$$\Gamma(\hat{x}, \hat{u}) := \int_{\mathcal{I}} \operatorname{col}(\hat{x}, \hat{u}) \operatorname{col}(\hat{x}, \hat{u})^\top \, \mathrm{d}t \in \mathbb{R}^{(n+m)\times(n+m)} \,. \tag{7}$$

The following lemma links the concept of persistency of excitation to positive definiteness of the Gramian Γ.

Lemma 3 ([13, Proposition 21]) *Suppose $\mathcal{B}$ is controllable and let $\operatorname{col}(\hat{x}, \hat{u}) \in \mathcal{B}$ with $\hat{u}$ persistently exciting of order $n + 1$. Then $\Gamma(\hat{x}, \hat{u})$ is positive definite.*

Next we state a version of the "fundamental lemma", which slightly deviates from that given in [13, Corollary 24], see below in Remark 5 for a discussion.

Theorem 4 *Suppose $\mathcal{B}$ is controllable and let $\operatorname{col}(\hat{x}, \hat{u}) \in \mathcal{B}$ such that $\hat{u}$ is persistently exciting of order $n + 1$. Define*

$$\tilde{\Gamma}(\hat{x}, \hat{u}) := \int_{\mathcal{I}} \operatorname{col}(\hat{x}, \dot{\hat{x}}, \hat{u}) \operatorname{col}(\hat{x}, \hat{u})^\top \, \mathrm{d}t \in \mathbb{R}^{(2n+m)\times(n+m)}. \tag{8}$$

Then $\operatorname{col}(x, u) \in H^1(\mathcal{I}, \mathbb{R}^n) \times L^2(\mathcal{I}, \mathbb{R}^m)$ is an element of $\mathcal{B}$ if and only if there exists $g \in L^2(\mathcal{I}, \mathbb{R}^{n+m})$ such that

$$\operatorname{col}(x, \dot{x}, u) = \tilde{\Gamma}(\hat{x}, \hat{u}) g \,. \tag{9}$$

Moreover, $\operatorname{rank} \tilde{\Gamma}(\hat{x}, \hat{u}) = n + m$*.*

Proof Observe that

$$\begin{bmatrix} A & -I & B \end{bmatrix} \tilde{\Gamma}(\hat{x}, \hat{u}) = 0, \tag{10}$$

as $\operatorname{col}(\hat{x}, \hat{u}) \in \mathcal{B}$. This shows the sufficiency part of the claim. We prove the necessity part. From $\frac{d}{dt}\hat{x} = A\hat{x} + B\hat{u}$ it follows that

$$\tilde{\Gamma}(\hat{x}, \hat{u}) = \begin{bmatrix} I_n & 0 \\ A & B \\ 0 & I_m \end{bmatrix} \Gamma(\hat{x}, \hat{u}) \tag{11}$$

with $\Gamma(\hat{x}, \hat{u})$ being the Gramian defined in (7). Since by assumption $\hat{u}$ is persistently exciting of order $n+1$, $\Gamma(\hat{x}, \hat{u})$ is positive definite. Consequently, given $\operatorname{col}(x, u) \in \mathcal{B}$ the function $g = \Gamma(\hat{x}, \hat{u})^{-1} \operatorname{col}(x, u)$ satisfies (9). The rank of $\tilde{\Gamma}(\hat{x}, \hat{u})$ can be determined directly from (11). □

Whenever the trajectory $\operatorname{col}(\hat{x}, \hat{u})$ is clear from the context, we write Γ and $\tilde{\Gamma}$ instead of $\Gamma(\hat{x}, \hat{u})$ and $\tilde{\Gamma}(\hat{x}, \hat{u})$.

[1] Property (iv) of Lemma 17 in [13] should be revised to "the set of all components of $f, \dots, f^{(L-1)}$ is linearly independent in $L^2(\mathcal{I}, \mathbb{R})$".

Both the persistency of excitation condition and the construction of the data matrix $\tilde{\Gamma}$ require (higher-order) derivatives of the input and the state. For the sake of simplicity, we assume that these derivatives are measured directly. Several approaches exist to relax this assumption. The data matrix $\tilde{\Gamma}$ can be built directly from the original signals, possibly corrupted by noise, using algebraic differentiators [17, 18], as described in [19]. See also [20, 21] for alternative techniques based on filtering of system signals. An alternative method based on polynomial series expansions is discussed in Section 3.4, where state derivatives can be reconstructed without direct measurement; see Remark 9.

Remark 5 (Avoiding data redundancy) Unlike [13, Corollary 24], which uses the Gramian $\int_{\mathcal{I}} \mathrm{col}(\hat{x}, \dot{\hat{x}}, \hat{u})\,\mathrm{col}(\hat{x}, \dot{\hat{x}}, \hat{u})^\top\,\mathrm{d}t$ in the data-driven description of the behavior and thereby adds columns which are linearly dependent on those of $\tilde{\Gamma}$ in (9), the previous theorem relies on the reduced data matrix $\tilde{\Gamma}$ of full column rank, avoiding unnecessary redundancy.

Remark 6 (Hidden differential algebraic structure) An admissible g in (9) satisfies a differential-algebraic equation. Partitioning the Gramian $\tilde{\Gamma}$ as

$$\mathrm{col}(\tilde{\Gamma}_x, \tilde{\Gamma}_{\dot{x}}, \tilde{\Gamma}_u) := \tilde{\Gamma}\ , \tag{12}$$

with $\tilde{\Gamma}_x, \tilde{\Gamma}_{\dot{x}} \in \mathbb{R}^{n\times(n+m)}$ and $\tilde{\Gamma}_u \in \mathbb{R}^{m\times(n+m)}$, equation (9) becomes

$$\mathrm{col}(x, u) = \begin{bmatrix}\tilde{\Gamma}_x \\ \tilde{\Gamma}_u\end{bmatrix} g\ , \quad \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\Gamma}_x g) = \tilde{\Gamma}_{\dot{x}} g\ . \tag{13}$$

Since $x \in H^1(\mathcal{I}, \mathbb{R}^n)$, it is required that $\tilde{\Gamma}_x g \in H^1(\mathcal{I}, \mathbb{R}^n)$.

Remark 7 (Approximating the Gramian $\tilde{\Gamma}$) From a practical point of view, in the case $\mathcal{I} = (0, \infty)$ it may seem difficult to obtain a sufficiently informative trajectory $(\hat{x}, \hat{u}) \in \mathcal{B}$ to compute the Gramian matrix $\tilde{\Gamma}(\hat{x}, \hat{u})$. However, we now show that the matrix $\tilde{\Gamma}$ in (9) can be replaced with any matrix with the same image, and moreover, that a candidate matrix can be built from a trajectory of the same system, but on a finite horizon, as we presently show. Consider for example the finite-horizon case $\mathcal{I} = (0, 1)$, assume that $\hat{u}$ is persistently exciting of order $n + 1$ on $\mathcal{I}$, and denote the corresponding Gramian (8) by $\tilde{\Gamma}_0$. Analogously, assume that $\tilde{u}$ is persistently exciting of order $n + 1$ on $(0, \infty)$ and denote by $\tilde{\Gamma}_1$ the corresponding Gramian defined in (8), as in Theorem 4. Given the persistency of excitation of both input signals, it holds that $\mathrm{rank}\,\tilde{\Gamma}_0 = \mathrm{rank}\,\tilde{\Gamma}_1 = n + m$ and moreover

$$\ker \tilde{\Gamma}_0 = \ker \tilde{\Gamma}_1 = \mathrm{im}\begin{bmatrix} A & -I_n & B\end{bmatrix}.$$

Therefore, $\mathrm{im}\,\tilde{\Gamma}_0 = (\ker \tilde{\Gamma}_0^\top)^\perp = (\ker \tilde{\Gamma}_1^\top)^\perp = \mathrm{im}\,\tilde{\Gamma}_1$. Consequently, Γ_0 and Γ_1 contain the same information about the system dynamics.

2.2 Data-based minimum principle

As a complementary remark, we derive a data-driven formulation of Pontryagin's minimum principle. While this result is not essential for the subsequent analysis, it provides an alternative perspective through the introduction of a dual variable, which may be of interest for computing lower bounds for the primal problem and for assessing the distance to optimality.

We recall Pontryagin's minimum principle (see e.g. Chapter 2 of [22]). If $\operatorname{col}(x,u)$ solves OCP (5) then there exists $\lambda \in H^1(\mathcal{I},\mathbb{R}^n)$ such that

$$\begin{aligned} -\dot{\lambda} &= x + A^\top \lambda && \text{(14a)}\\ \dot{x} &= Ax + Bu && \text{(14b)}\\ 0 &= u + B^\top \lambda && \text{(14c)}\\ \lambda(1) &= 0 && \text{(14d)}\\ x(0) &= x^0\ , && \text{(14e)} \end{aligned}$$

where if $\mathcal{I} = (0,\infty)$ then the transversality condition (14d) is not applicable and is neglected. We note that it is well-known in the optimal control literature that condition (14d) does not necessarily hold for infinite horizon problems.

Using the "fundamental lemma" (Theorem 4), we obtain a data-driven version of Pontryagin's principle.

Lemma 8 (The data-driven minimum principle) *Let* $\operatorname{col}(x,u)$ *be a solution to OCP* (5). *Suppose the assumptions of Theorem 4 hold. Define* Γ *by* (7) *and* $\tilde{\Gamma}_x$, $\tilde{\Gamma}_{\dot{x}}$ *by* (12). *Let* $g \in L^2(\mathcal{I},\mathbb{R}^{n+m})$ *such that* (9) *holds; then* $\tilde{\Gamma}_x g \in H^1(\mathcal{I},\mathbb{R}^{n+m})$. *Moreover, there exists* $\lambda \in H^1(\mathcal{I},\mathbb{R}^n)$ *such that*

1. if $\mathcal{I} = (0,1)$ *then*

$$\begin{aligned} \tilde{\Gamma}_x^\top \dot{\lambda} &= \Gamma^\top \Gamma g - \tilde{\Gamma}_{\dot{x}}^\top \lambda && \text{(15a)}\\ \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\Gamma}_x g) &= \tilde{\Gamma}_{\dot{x}} g && \text{(15b)}\\ \lambda(1) &= 0 && \text{(15c)}\\ \tilde{\Gamma}_x g(0) &= x^0; && \text{(15d)} \end{aligned}$$

2. if $\mathcal{I} = (0,\infty)$ *then the equations in* (15) *hold, with the exception of* (15c).

Proof The statement $\tilde{\Gamma}_x g \in H^1(\mathcal{I},\mathbb{R}^{n+m})$ follows from the discussion in Remark 7. Consequently, only (15a) needs to be shown. From (14a) and (14c) together with $\operatorname{col}(x,u) = \Gamma g$ one obtains

$$-\begin{bmatrix} I \\ 0 \end{bmatrix} \dot{\lambda} = \Gamma g + \begin{bmatrix} A^\top \\ B^\top \end{bmatrix} \lambda. \tag{16}$$

Moreover, $\Gamma = \operatorname{col}(\tilde{\Gamma}_x, \tilde{\Gamma}_u)$ and $\tilde{\Gamma}_{\dot{x}} = \begin{bmatrix} A & B \end{bmatrix} \Gamma$. Therefore, multiplying (16) from the left with $\Gamma^\top = \Gamma$ yields (15a). $\square$

The same idea may be exploited in the analysis of optimal control problems for linear port-Hamiltonian systems, with the goal of achieving state transition under minimal energy supply, see [23]. In similar way one can derive a data-based formulation for Pontryagin's maximum principle for this setting.

3 Polynomial-based trajectories

In this section, we analyze finite-dimensional subspaces of a behavior $\mathcal{B}$ consisting of linear combinations of orthogonal Legendre and Laguerre polynomials, whose properties are recalled in Appendix A.

3.1 Orthonormal series expansion

We present the necessary notation for orthonormal series expansions and introduce specific orthonormal bases constructed from Legendre and Laguerre polynomials.

Let $\langle \cdot, \cdot \rangle$ denote the usual inner product in $L^2(\mathcal{I}, \mathbb{R}^d)$, $d \in \mathbb{N} \setminus \{0\}$, defined by $\langle f, g \rangle := \int_{\mathcal{I}} f^\top g \, \mathrm{d}t$. We consider the orthonormal basis $(\pi_i)_{i \in \mathbb{N}}$ of scalar functions in $L^2(\mathcal{I}, \mathbb{R})$ given by

$$\pi_i(t) := \begin{cases} \pi^{\mathrm{Leg}}(2t-1) & \text{if } t \in \mathcal{I} \text{ and } \mathcal{I} = (0,1), \\ \exp(-\frac{t}{2}) \pi_i^{\mathrm{Lag}}(t) & \text{if } t \in \mathcal{I} \text{ and } \mathcal{I} = (0, \infty), \end{cases} \tag{17}$$

where π_i^{Leg} and π_i^{Lag} are the i-th normalized Legendre and Laguerre polynomials, see Section A.1 and Section A.2.2. Every function $f \in L^2(\mathcal{I}, \mathbb{R}^d)$ admits a unique series expansion

$$f = \sum_{i \in \mathbb{N}} \mathfrak{f}_i \pi_i \ . \tag{18}$$

The coefficients $\mathfrak{f}_i$, $i \in \mathbb{N}$, are uniquely given by

$$\mathfrak{f}_i = \sum_{j=0}^{d-1} \langle f^\top e_j, \pi_i \rangle e_j \ , \tag{19}$$

where $\{e_0, \dots, e_{d-1}\}$ is the canonical basis in $\mathbb{R}^d$. We denote $\mathfrak{f} := (\mathfrak{f}_i)_{i \in \mathbb{N}}$; note that $\mathfrak{f} \in \ell^2(\mathbb{N}, \mathbb{R}^d)$. Let Π the isometric isomorphism defined[2] by

$$\Pi : L^2(\mathcal{I}, \mathbb{R}^d) \to \ell^2(\mathbb{N}, \mathbb{R}^d), \quad f \mapsto \mathfrak{f}. \tag{20}$$

In particular, for $f, g \in L^2(\mathcal{I}, \mathbb{R}^d)$ and $\mathfrak{f} = \Pi f$, $\mathfrak{g} = \Pi g$ the inner product satisfies

$$\langle f, g \rangle = \sum_{i \in \mathbb{N}} \mathfrak{f}_i^\top \mathfrak{g}_i. \tag{21}$$

Similarly, the outer product of $f \in L^2(\mathcal{I}, \mathbb{R}^{d_1})$ and $g \in L^2(\mathcal{I}, \mathbb{R}^{d_2})$, $d_1, d_2 \in \mathbb{N} \setminus \{0\}$, satisfies the equality

$$\int_{\mathcal{I}} f g^\top \, \mathrm{d}t = \sum_{i \in \mathbb{N}} \mathfrak{f}_i \mathfrak{g}_i^\top. \tag{22}$$

[2]In our notation for Π we neglect the dependency on the space dimension d.

We denote by P_N the orthogonal projection of $L^2(\mathcal{I}, \mathbb{R}^d)$ onto the finite-dimensional space spanned by $\{\pi_i e_j \,|\, i = 0, \dots, N \;,\;\; j = 0, \dots, d-1\}$:

$$P_N f := \sum_{i=0}^{N-1} \mathfrak{f}_i \pi_i \; . \tag{23}$$

If $f \in \operatorname{im} P_N$ then $\Pi f = \mathfrak{f} = (\mathfrak{f}_0, \dots, \mathfrak{f}_{N-1}, 0, \dots)$; in such case we identify $\Pi(\operatorname{im} P_N) \subset \ell^2(\mathbb{N}, \mathbb{R}^d)$ with $\mathbb{R}^{dN}$ and through the identification of $\mathfrak{f}$ with $\operatorname{col}(\mathfrak{f}_0, \dots, \mathfrak{f}_{N-1}) \in \mathbb{R}^{dN}$. Moreover, if $\mathfrak{f} = \operatorname{col}(\mathfrak{f}_0, \dots, \mathfrak{f}_{N-1}) \in \mathbb{R}^{dN}$ with $\mathfrak{f}_i \in \mathbb{R}^d$ for $i = 0, \dots, N-1$, then we denote with slight abuse of notation by $\Pi^{-1}\mathfrak{f}$ the function defined by $\Pi^{-1}\mathfrak{f} := \sum_{i=0}^{N-1} \mathfrak{f}_i \pi_i$.

Note that $\operatorname{im} P_N \subset H^1(\mathcal{I}, \mathbb{R}^d)$. The differentiation operator $\frac{d}{dt}$ restricted to $\operatorname{im} P_N$ admits a matrix representation in the coefficient space $\Pi(\operatorname{im} P_N) = \mathbb{R}^{dN}$. Specifically, there is a matrix $\mathcal{D}_N \in \mathbb{R}^{N \times N}$ such that

$$\mathfrak{f}^{(1)} = (\mathcal{D}_N \otimes I_d)\mathfrak{f} \tag{24}$$

for all $f \in \operatorname{im} P_N$, where $\mathfrak{f} = \Pi f \in \mathbb{R}^{dN}$ and $\mathfrak{f}^{(1)} = \Pi f^{(1)} \in \mathbb{R}^{dN}$. With respect to the orthogonal basis on $\mathcal{I} = (0, 1)$ induced by the Legendre polynomials the matrix $\mathcal{D}_N$ is given by

$$\mathcal{D}_N = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & \cdots \\ & 0 & 3 & 0 & 3 & \cdots \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix} , \tag{25}$$

whereas for the Laguerre-based orthogonal basis on $\mathcal{I} = (0, \infty)$ it is given by

$$\mathcal{D}_N = - \begin{bmatrix} \frac{1}{2} & 1 & 1 & \cdots \\ & \frac{1}{2} & 1 & \cdots \\ & & \ddots & \ddots \end{bmatrix} , \tag{26}$$

see Section A.

Since $(\pi_i)_{i \in \mathbb{N}}$ forms an orthonormal basis, $P_N f$ converges to f in $L^2(\mathcal{I}, \mathbb{R}^d)$ as $N \to \infty$. Further details on convergence in Sobolev spaces and additional material can be found in Section A.

Remark 9 (Avoiding knowledge of $\dot{x}$) For the sake of simplicity of exposition, in this paper we have assumed that the state derivative is directly measured. However, an approximation of such signal can be computed from the orthogonal basis coefficients of the state trajectory through the "differentiation relations", see (24), (25) and (26).

3.2 Approximate trajectories and their asymptotics

Provided sufficient smoothness, any trajectory in $\mathcal{B}$ can be approximated arbitrarily accurately by linear combinations of orthogonal polynomial trajectories, i.e. elements in $\mathcal{B} \cap \operatorname{im} P_N$ with large enough N.

Lemma 10 *Suppose that $\mathcal{B}$ is controllable and let $s, k \in \mathbb{N}$. If either*

1. $w \in \mathcal{B} \cap H^{k+s+n}(\mathcal{I}, \mathbb{R}^{n+m})$ *in the case* $\mathcal{I} = (0,1)$;
   or
2. $w \in \mathcal{B} \cap H^{2(k+s+n)+1}(\mathcal{I}, \mathbb{R}^{n+m})$ *and* $pw \in H^{2(k+s+n)+1}(\mathcal{I}, \mathbb{R}^{n+m})$ *for every polynomial p, in the case* $\mathcal{I} = (0, \infty)$,

then for every $N \in \mathbb{N}$ there exists $w^N \in \mathcal{B} \cap \operatorname{im} P_N$ such that $\Lambda_s(w - w^N)(0) = 0$ and

$$\|w - w^N\|_{H^s} = \mathcal{O}(N^{-k}) \quad \text{as } N \to \infty. \tag{27}$$

Proof Controllability of $\mathcal{B}$ implies the existence of an image representation, meaning that there is a polynomial matrix $M \in \mathbb{R}^{(n+m)\times m}[s]$ such that trajectories $w \in \mathcal{B}$ can be described through a latent variable ℓ as

$$w = M(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell\,, \tag{28}$$

see Theorem 6.6.1 in [16].

We consider a particular image representation, where $\deg(M) \le n+1$ and the latent variable ℓ is a flat system's output, i.e. there is $C \in \mathbb{R}^{m\times n}$ such that

$$\ell = Cx = \begin{bmatrix} C & 0 \end{bmatrix} w \tag{29}$$

and (29) for every $w = \operatorname{col}(x,u) \in \mathcal{B}$, cf. [13, Lemma 6]. Note that given $w = \operatorname{col}(x,u) \in \mathcal{B} \cap H^j(\mathcal{I}, \mathbb{R}^{n+m})$ it follows from

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\ell = C\tfrac{\mathrm{d}}{\mathrm{d}t}x = C(Ax + Bu) = C\begin{bmatrix} A & B \end{bmatrix} w \tag{30}$$

that $\ell = Cx \in H^{j+1}(\mathcal{I}, \mathbb{R}^m)$. Moreover, if w is a linear combination of orthonormal functions $\{\pi_0, \ldots, \pi_{N-1}\}$ as in (17), then also ℓ is.

Now define $\eta \in \mathbb{N}$ by $\eta := k+s+n$ if $\mathcal{I} = (0,1)$, and by $\eta := 2(k+s+n)+1$ if $\mathcal{I} = (0,\infty)$. Suppose w is a trajectory satisfying assumption 1 or 2, respectively; then $w \in H^\eta(\mathcal{I}, \mathbb{R}^{n+m})$ and the corresponding trajectory ℓ is such that $\ell \in H^{\eta+1}(\mathcal{I}, \mathbb{R}^{n+m})$. Moreover, in the case $\mathcal{I} = (0,\infty)$ it follows from (29) and (30) together with assumption 2 that $p^{k+s+n+1}\ell \in H^{2(k+s+n+1)}(\mathcal{I}, \mathbb{R}^m)$.

For every $N \in \mathbb{N}$ there is $\ell^N \in \operatorname{im} P_N$ such that

$$\|\ell - \ell^N\|_{H^{s+n+1}} = \mathcal{O}(N^{-k}) \quad \text{as } N \to \infty, \quad \Lambda_{s+n+1}(\ell - \ell^N)(0) = 0, \tag{31}$$

cf. Proposition 23 and Proposition 27. Define $w^N := M(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell^N$. In particular, $w^N \in \mathcal{B} \cap \operatorname{im} P_N$. Combining (31) with $w - w^N = M(\tfrac{\mathrm{d}}{\mathrm{d}t})(\ell - \ell^N)$ and $\deg(M) \le n+1$ yields $\Lambda_s(w - w^N)(0) = 0$ and (27). □

The previous result is notable not only for bounding the distance of a system trajectory to its polynomial-based approximation, but also because it shows that the approximating function is itself a system trajectory satisfying the same initial conditions as the approximated one. This is crucial for the analysis of optimal control problems that we perform in the next section.

3.3 Approximate optimal solutions and their asymptotics

In order to approximate the solution of the OCP (5) we restrict the optimization problem to polynomial-based system trajectories, namely:

$$\underset{\operatorname{col}(x,u)\in \operatorname{im} P_N \cap \mathcal{B}(x^0)}{\text{minimize}} \quad \|x\|^2 + \|u\|^2\,. \tag{32}$$

It follows from [22, Theorem 4.2.4] that in the finite-horizon case the solution of (5) is unique; in the infinite-horizon case, stabilizability of the system is required, see [22, Theorem 4.5.6]. Since we have assumed that $\mathcal{B}$ is controllable (see Theorem 4), each of the optimal control problems (5) and (32) admits a unique solution, denoted by $w^\star = \mathrm{col}(x^\star, u^\star)$ and $w^N = \mathrm{col}(x^N, u^N)$ in the following. The next result shows that w^N converges exponentially to $w^\star$ as $N \to \infty$.

Proposition 11 *Suppose $\mathcal{B}$ is controllable. Denote by $w^\star = \mathrm{col}(x^\star, u^\star)$ and $w^N = \mathrm{col}(x^N, u^N)$ the solutions to (5) and (32), respectively. For arbitrary $k \in \mathbb{N}$, as $N \to \infty$ it holds that*

$$0 \le \|w^N\|^2 - \|w^\star\|^2 = \mathcal{O}(N^{-k}) \,, \tag{33}$$

and moreover

$$\|w^\star - w^N\| = \mathcal{O}(N^{-k}) \,. \tag{34}$$

Proof Recall (see e.g. Section 10.2 of [24]) that the input trajectory $u^\star$ of the unique solution $w^\star = \mathrm{col}(x^\star, u^\star)$ of the OCP (5) is a feedback of the optimal state trajectory $x^\star$: $u^\star = -Kx^\star$. The feedback gain of the form $K = B^\top P$ is defined in terms of the solution P to a matrix differential equation (in the finite-horizon case) or a matrix algebraic equation (in the infinite horizon case). Namely, if $\mathcal{I} = (0,1)$ then the function $P(\cdot) : \mathcal{I} \to \mathbb{R}^{n\times n}$ is the point-wise smooth symmetric solution of the differential Riccati equation

$$\begin{aligned} \frac{d}{dt}P(\cdot) &= -P(\cdot)A - A^\top P(\cdot) + P(\cdot)BB^\top P(\cdot) - I_n \\ P(1) &= 0 \,; \end{aligned}$$

and if $\mathcal{I} = (0,\infty)$ then $P \in \mathbb{R}^{n\times n}$ is the symmetric, positive semi-definite solution of the algebraic Riccati equation

$$PA + A^\top P - PBB^\top P + I_n = 0 \,.$$

In both cases, the optimal state trajectory is defined by

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x^\star = (A - BB^\top P)x^\star, \quad u^\star = -B^\top P x^\star \,. \tag{35}$$

Moreover, it can be shown that if $\mathcal{I} = (0,1)$ then $w^\star \in \mathcal{C}^\infty(\overline{\mathcal{I}}, \mathbb{R}^{n+m})$; and if $\mathcal{I} = (0,\infty)$, $w^\star$ and all its derivatives decay exponentially, since $(A - BB^\top P)$ is Hurwitz.

As consequence of Lemma 10 for arbitrary $k \in \mathbb{N}$ and every $N \in \mathbb{N}$ there exists $v^N = \mathrm{col}(x^N, u^N) \in \mathrm{im}\, P_N$ with $x^N(0) = x^\star(0) = x^0$ such that

$$\|v^N - w^\star\| = \mathcal{O}(N^{-k}) \tag{36}$$

as $N \to \infty$. Moreover, the solution w^N of OCP (32) satisfies

$$\|w^\star\|^2 \le \|w^N\|^2 \le \|v^N\|^2 \tag{37}$$

for all $N \in \mathbb{N}$. Therefore

$$0 \le \|w^N\|^2 - \|w^\star\|^2 \le \|v^N\|^2 - \|w^\star\|^2 = (\|v^N\| + \|w^\star\|)(\|v^N\| - \|w^\star\|). \tag{38}$$

The first factor on the right hand side of (38) is bounded for all $N \in \mathbb{N}$ because of the convergence $v^N \to w^\star$ in (36). The second term can be estimated by means of the reverse triangular inequality,

$$|\|v^N\| - \|w^\star\|| \le \|v^N - w^\star\|. \tag{39}$$

Therefore, (38) together with (36) shows (33).

We now prove the second assertion (34). By the parallelogram identity

$$\|\tfrac{1}{2}(w^\star + w^N)\|^2 + \|\tfrac{1}{2}(w^\star - w^N)\|^2 = \tfrac{1}{2}\|w^\star\|^2 + \tfrac{1}{2}\|w^N\|^2. \tag{40}$$

Since the feasibility set $\mathcal{B}(x^0)$ is affine, it is also convex; this implies that

$$\|w^\star\|^2 \leq \|\tfrac{1}{2}(w^\star + w^N)\|^2. \tag{41}$$

This together with (40) and (36) implies

$$\|w^\star - w^N\|^2 \leq 2\big(\|w^N\|^2 - \|w^\star\|^2\big) = \mathcal{O}(N^{-k}) \quad \text{as } N \to \infty. \qquad \square$$

3.4 An orthogonality property

The OCP (5) is equivalent to the problem of finding the closest point to the zero vector in the affine space $\mathcal{B}(x^0)$; the restricted OCP (32) can be interpreted analogously. Given that $\mathcal{B}$ and $\operatorname{im} P_N \cap \mathcal{B}$ are Hilbert spaces with inner product defined by

$$\langle w_1, w_2 \rangle := \int_{\mathcal{I}} w_1(t)^\top w_2(t) \, \mathrm{d}t \, ,$$

the following orthogonality characterization of the solutions of the OCPs (5) and (32) is immanent.

Lemma 12 *Suppose $\mathcal{B}$ is controllable. The following statements hold:*

1. *$w^\star \in \mathcal{B}(x^0)$ solves the OCP (5) if and only if*

$$\langle w^\star, w \rangle = 0 \quad \text{for all } w \in \mathcal{B}(0) \, . \tag{42}$$

2. *$w^N \in \operatorname{im} P_N \cap \mathcal{B}(x^0)$ solves the OCP (32) if and only if*

$$\langle w^N, w \rangle = 0 \text{ for all } w \in \operatorname{im} P_N \cap \mathcal{B}(0) \, . \tag{43}$$

Proof We prove only the first assertion as the argument used to prove the second one is completely analogous. We first show the necessity part. Suppose that $w^\star \in \mathcal{B}(x^0)$ solves (5) and assume that there exists $w \in \mathcal{B}(0)$ with $\langle w^\star, w \rangle > 0$. Define $\alpha := -\langle w^\star, w \rangle \|w\|^{-2}$; note that $\alpha < 0$. Then

$$\|w^\star + \alpha w\|^2 = \|w^\star\|^2 + \alpha^2 \|w\|^2 + 2\alpha \langle w^\star, w \rangle = \|w^\star\|^2 - \frac{\langle w^\star, w \rangle^2}{\|w\|^2} < \|w^\star\|^2 \, ,$$

which contradicts the fact that $w^\star \in \mathcal{B}(x^0)$ is the solution of (5).

We prove the sufficiency part of the claim. Recall that $\mathcal{B}(x^0) = \{w^\star\} + \mathcal{B}(0)$. Using $\langle w^\star, w \rangle = 0$ for all $w \in \mathcal{B}(0)$ we obtain

$$\|w^\star + w\|^2 = \|w^\star\|^2 + \|w\|^2 \geq \|w^\star\|^2 \, , \tag{44}$$

showing that $w^\star$ solves (5). $\square$

Remark 13 In the more general case of a stage cost incorporating symmetric positive-definite weight matrices, see Remark 2, a similar orthogonality condition holds for the corresponding inner product $\langle w, \tilde{w} \rangle = \int_{\mathcal{I}} w^\top \mathcal{Q} \tilde{w} \, \mathrm{d}t$, where $\mathcal{Q} := \operatorname{diag}(Q, R)$.

3.5 Data-based solution scheme

Employing the orthogonality property we propose a data-driven framework to approximately solve the OCP (5) (see also [25] for an application of orthogonality ideas to the solution of discrete-time optimal control problems). Define Γ, $\tilde{\Gamma}$ as in (7), (8) with $\tilde{\Gamma}$ partitioned as in (12), where $\tilde{\Gamma}_x, \tilde{\Gamma}_{\dot{x}} \in \mathbb{R}^{n\times(n+m)}$ and $\tilde{\Gamma}_u \in \mathbb{R}^{m\times(n+m)}$. Given $N \in \mathbb{N}$ we define the matrices

$$
\begin{aligned}
\Psi_N &:= \mathcal{D}_N \otimes \tilde{\Gamma}_x - I_N \otimes \tilde{\Gamma}_{\dot{x}} \in \mathbb{R}^{nN\times(n+m)N} \\
\mathcal{M}_N &:= I_N \otimes \Gamma \in \mathbb{R}^{(n+m)N\times(n+m)N} \\
\mathcal{G}_N &:= I_{(n+m)N} - \Psi_N^\dagger \Psi_N \ ,
\end{aligned}
\tag{45}
$$

where $\mathcal{D}_N \in \mathbb{R}^{N\times N}$ is the differentiation matrix defined in (24).

The next lemma is a restatement of the "fundamental lemma" (Theorem 4) using series expansion coefficients.

Lemma 14 *Suppose the assumptions of Theorem 4 hold. Let $w \in L^2(\mathcal{I}, \mathbb{R}^{n+m}) \cap \operatorname{im} P_N$ and denote by $\mathfrak{w}$ the coefficient vector of w, i.e. $\mathfrak{w} = \Pi w \in \mathbb{R}^{N(n+m)}$. Define $\mathcal{M}_N$, Ψ_N and $\mathcal{G}_N$ as in (45). Then the following statements are equivalent:*

1. $w \in \operatorname{im} P_N \cap \mathcal{B}$;
2. $\mathfrak{w} = \mathcal{M}_N \mathfrak{g}$ *with* $\mathfrak{g} \in \ker \Psi_N$;
3. $\mathfrak{w} \in \operatorname{im}(\mathcal{M}_N \mathcal{G}_N)$.

Proof We show the equivalence between the first two statements. From Theorem 4 it follows that $\operatorname{col}(x,u) \in \operatorname{im} P_N \cap \mathcal{B}$ if and only there is $g \in \operatorname{im} P_N \cap L^2(\mathcal{I}, \mathbb{R}^{n+m})$ such that (9) holds, i.e.

$$
\tilde{\Gamma}_x \dot{g} - \tilde{\Gamma}_{\dot{x}} g = 0, \quad w = \operatorname{col}(x,u) = \Gamma g, \tag{46}
$$

see Remark 6. The assertion follows with $\mathfrak{g} = \Pi g$ and $\mathcal{D}_N \mathfrak{g} = \Pi \dot{g}$.

The equivalence of the last two statements follows directly from the fact that $\mathcal{G}_N$ is the projection onto $\ker \Psi_N$. □

In the next step we incorporate the initial condition $x(0)$. To this end we define the matrices

$$
\Phi_N := \begin{cases} \begin{bmatrix} 1 & -\sqrt{3} & \sqrt{5} & \cdots \end{bmatrix} \otimes \tilde{\Gamma}_x & \text{if } \mathcal{I} = (0,1), \\ \begin{bmatrix} 1 & 1 & 1 & \cdots \end{bmatrix} \otimes \tilde{\Gamma}_x & \text{if } \mathcal{I} = (0,\infty), \end{cases} \tag{47}
$$

$$
\mathcal{G}_N^0 := I_{(n+m)N} - \begin{bmatrix} \Psi_N \\ \Phi_N \end{bmatrix}^\dagger \begin{bmatrix} \Psi_N \\ \Phi_N \end{bmatrix}. \tag{48}
$$

From the theory of Legendre and Laguerre polynomials it follows that given $x \in L^2(\mathcal{I}, \mathbb{R}^n) \cap \operatorname{im} P_N$ and denoting the coefficient vector of x by $\mathfrak{x} = \Pi x \in \mathbb{R}^{Nn}$, the value of x at 0 can be computed as $x(0) = \Phi_N \mathfrak{x}$. The next result follows in a straightforward way from this fact and from Lemma 14.

Lemma 15 *Suppose the assumptions of Theorem 4 hold. Let $w \in L^2(\mathcal{I}, \mathbb{R}^{n+m}) \cap \operatorname{im} P_N$ and $\mathfrak{w} = \Pi w$. Define $\mathcal{M}_N$, Ψ_N, Φ_N and $\mathcal{G}_N^0$ as in (45), (47) and (48). Then the following statements are equivalent:*

1. $w \in \operatorname{im} P_N \cap \mathcal{B}(0)$;
2. $\mathfrak{w} = \mathcal{M}_N \mathfrak{g}$ *with* $\mathfrak{g} \in \ker \Psi_N \cap \ker \Phi_N$;
3. $\mathfrak{w} \in \operatorname{im}(\mathcal{M}_N \mathcal{G}_N^0)$.

Similarly to what we did in Lemma 14 for the "fundamental lemma", we now rewrite the orthogonality property in terms of series expansion coefficients. This reformulation leads to a data-driven characterization of the solutions of OCP (32).

Theorem 16 *Suppose the assumptions of Theorem 4 hold. Let $w^N \in L^2(\mathcal{I}, \mathbb{R}^{n+m}) \cap \operatorname{im} P_N$ and $\mathfrak{w}^N = \Pi w^N$. Define $\mathcal{M}_N$, Φ_N and $\mathcal{G}_N^0$ as in (45), (47) and (48). Then w^N solves OCP (32) if and only if $\mathfrak{w}^N = \mathcal{M}_N \mathcal{G}_N \mathfrak{h}^N$, where $\mathfrak{h}^N$ solves*

$$\begin{bmatrix} {\mathcal{G}_N^0}^\top \mathcal{M}_N^\top \mathcal{M}_N \mathcal{G}_N \\ \Phi_N \mathcal{G}_N \end{bmatrix} \mathfrak{h}^N = \begin{bmatrix} 0 \\ x^0 \end{bmatrix}. \tag{49}$$

Proof Let $w \in \operatorname{im} P_N \cap \mathcal{B}(0)$ with $\mathfrak{w} = \Pi w$. Then there exist vectors $\mathfrak{h}$ and $\mathfrak{h}^N$ in $\mathbb{R}^{(n+m)N}$ such that $\mathfrak{w} = \mathcal{M}_N \mathcal{G}_N^0 \mathfrak{h}$ and $\mathfrak{w}^N = \mathcal{M}_N \mathcal{G}_N \mathfrak{h}^N$ with $x^0 = \Phi_N \mathcal{G}_N \hat{h}^N$. Therefore,

$$\langle w^N, w \rangle = \langle w, w^N \rangle = \hat{w}^\top \hat{w}^N = \hat{h}^\top {\mathcal{G}_N^0}^\top \mathcal{M}_N^\top \mathcal{M}_N \mathcal{G}_N \hat{h}^N. \tag{50}$$

Such equality and Lemma 12 yields the assertion. □

Remark 17 When considering more general cost functions in the OCP (5), see Remarks 2 and 13, the term $\mathcal{M}_N^\top \mathcal{M}_N \mathcal{G}_N$ in data-driven formulation (49) of Theorem 16 must be replaced with $\mathcal{M}_N^\top \mathcal{Q}_N \mathcal{M}_N \mathcal{G}_N$, where

$$\mathcal{Q}_N = I_N \otimes \begin{bmatrix} Q & \\ & R \end{bmatrix}. \tag{51}$$

A direct consequence of Theorem 16 and Proposition 11 is that the results (33) and (34) on the exponential asymptotic convergence of the solutions of the OCPs (32) to those of (5) carry over also to the data-driven framework.

Proposition 18 *Suppose the assumptions of Theorem 4 hold. Define $\mathcal{M}_N$, Φ_N and $\mathcal{G}_N^0$ as in (45), (47) and (48). Let $w^\star$ be the solution of OCP (5) and consider a solution $\hat{h}^N$ of (49). Then for arbitrary $k \in \mathbb{N}$ as $N \to \infty$ it holds that*

$$0 \leq \|\mathcal{M}_N \mathcal{G}_N \hat{h}^N\|_2^2 - \|w^\star\|^2 = \mathcal{O}(N^{-k}) \tag{52}$$

and

$$\|w^\star - \Pi^{-1}(\mathcal{M}_N \mathcal{G}_N \hat{h}^N)\|^2 = \mathcal{O}(N^{-k}). \tag{53}$$

In the next section we consider the infinite-horizon case $\mathcal{I} = (0, \infty)$ and we show how approximations of the solution P to the algebraic Riccati equation and the optimal state feedback gain can be computed.

4 Data-based approximation of optimal controllers

4.1 Approximate solution of algebraic Riccati equations

We show how the result of Proposition 18 can be used to approximate the solution P of the algebraic Riccati equation

$$PA + A^\top P - PBB^\top P + I_n = 0 \tag{54}$$

associated with OCP (5) in the infinite-horizon case $\mathcal{I} = (0, \infty)$. Recall that for every initial value $x^0 \in \mathbb{R}^n$ one has

$$\left(x^0\right)^\top P x^0 = \|w^\star\|^2, \tag{55}$$

where $w^\star$ is the solution to (5). By the polarization identity

$$P_{i,j} = e_i^\top P e_j = \frac{1}{4}\left((e_i + e_j)^\top P(e_i + e_j) - (e_i - e_j)^\top P(e_i - e_j)\right). \tag{56}$$

For two fixed but otherwise arbitrary indices $1 \le i, j \le n$, we define the two initial values $x_0^0 := e_i + e_j$ and $x_1^0 := e_i - e_j$. Denote by $\hat{h}_0^N$ and $\hat{h}_1^N$ the solutions of (49) subject to $x^0 = x_0^0$ and $x^0 = x_1^0$, respectively. Then by (52) in Proposition 18 one has

$$\|\mathcal{M}_N \mathcal{G}_N \hat{h}_s^N\|_2^2 - (x_s^0)^\top P x_s^0 = \mathcal{O}(N^{-k}), \quad s = 0, 1\,; \tag{57}$$

applying (56) we conclude that

$$\frac{1}{4}\|\mathcal{M}_N \mathcal{G}_N \hat{h}_0^N\|_2^2 - \frac{1}{4}\|\mathcal{M}_N \mathcal{G}_N \hat{h}_1^N\|_2^2 - P_{i,j} = \mathcal{O}(N^{-k})\,,$$

for arbitrary $k \in \mathbb{N}$ as $N \to \infty$. The following statement summarizes this argument applied to each pair i, j of indices of the entries of P.

Proposition 19 *Suppose the assumptions of Theorem 4 hold. Let $1 \le i, j \le n$; define $x_0^0 := e_i + e_j$ and $x_1^0 := e_i - e_j$. Define $\mathcal{M}_N$, Φ_N and $\mathcal{G}_N^0$ as in (45), (47) and (48).*

Denote by $\hat{h}_0^N$ and $\hat{h}_1^N$ the solutions of (49) subject to $x^0 = x_0^0$ and $x^0 = x_1^0$, respectively. Then for arbitrary $k \in \mathbb{N}$ it holds that the (i, j)-th element of the solution P to the algebraic Riccati equation (54) satisfies

$$P_{i,j} - \left(\frac{1}{4}\|\mathcal{M}_N \mathcal{G}_N \hat{h}_0^N\|_2^2 - \frac{1}{4}\|\mathcal{M}_N \mathcal{G}_N \hat{h}_1^N\|_2^2\right) = \mathcal{O}(N^{-k}) \quad \text{as } N \to \infty\,.$$

4.2 Approximate optimal feedback gain

For the infinite time-horizon $\mathcal{I} = (0, \infty)$ the solution P of the algebraic Riccati equation (54) defines the optimal feedback gain matrix

$$K = -B^\top P. \tag{58}$$

We now discuss how to approximate K from data.

Denote by $w_j^\star = \mathrm{col}(x_j^\star, u_j^\star)$, $j = 0, \ldots, n-1$, the solution of the infinite-horizon (i.e. $\mathcal{I} = (0, \infty)$) OCP (5) with respect to the initial condition $x^0 = e_j$, $j = 0, \ldots, n-1$. Define the two time-dependent matrices

$$\begin{aligned} X^\star &= \begin{bmatrix} x_0^\star & \cdots & x_{n-1}^\star \end{bmatrix} : \mathcal{I} \to \mathbb{R}^{n \times n} \\ U^\star &= \begin{bmatrix} u_0^\star & \cdots & u_{n-1}^\star \end{bmatrix} : \mathcal{I} \to \mathbb{R}^{m \times n} . \end{aligned}$$

Since $u_j^* = K x_j^*$, $j = 0, \ldots, n-1$, it follows that

$$\int_{\mathcal{I}} U^\star (X^\star)^\top \, \mathrm{d}t = K \int_{\mathcal{I}} X^\star (X^\star)^\top \, \mathrm{d}t. \tag{59}$$

Note that the integrals on both sides are well defined, since $\mathrm{col}(x_j^\star, u_j^\star) \in L^2(\mathcal{I}, \mathbb{R}^{n+m})$. Moreover, the matrix $\int_{\mathcal{I}} X^\star (X^\star)^\top \, \mathrm{d}t$ is invertible, since $X^\star$ is continuous and $X^\star(0) = I_n$. Therefore,

$$K = \left(\int_{\mathcal{I}} U^\star (X^\star)^\top \, \mathrm{d}t \right) \left(\int_{\mathcal{I}} X^\star (X^\star)^\top \, \mathrm{d}t \right)^{-1}. \tag{60}$$

In the following we approximate the gain matrix K with solutions of the restricted OCP (32). Consider for each $w_j^\star$ its N-th-order approximation $w_j^N = \mathrm{col}(x_j^N, u_j^N)$ according to Proposition 11 and denote

$$X^N = \begin{bmatrix} x_0^N & \cdots & x_{n-1}^N \end{bmatrix}, \quad U^N = \begin{bmatrix} u_0^N & \cdots & u_{n-1}^N \end{bmatrix} \tag{61}$$

Lemma 20 *Let $\mathcal{B}$ be controllable and define $X^\star, U^\star, X^N, U^N$ and K as above. Then for every sufficiently large $N \in \mathbb{N}$ the matrix $\int_{\mathcal{I}} X^N (X^N)^\top \, \mathrm{d}t$ is invertible and for arbitrary $k \in \mathbb{N}$*

$$\left(\int_{\mathcal{I}} U^N (X^N)^\top \, \mathrm{d}t \right) \left(\int_{\mathcal{I}} X^N (X^N)^\top \, \mathrm{d}t \right)^{-1} - K = \mathcal{O}(N^{-k}) \quad \text{as } N \to \infty. \tag{62}$$

Proof First we show for

$$W^\star = \mathrm{col}(X^\star, U^\star) = \begin{bmatrix} w_0^\star & \cdots & w_{n-1}^\star \end{bmatrix} \tag{63}$$

$$W^N = \mathrm{col}(X^N, U^N) = \begin{bmatrix} w_0^N & \cdots & w_{n-1}^N \end{bmatrix}. \tag{64}$$

that

$$\int_{\mathcal{I}} \left(W^N (W^N)^\top - W^\star (W^\star)^\top \right) \mathrm{d}t = \mathcal{O}(N^{-k}) \quad \text{as } N \to \infty. \tag{65}$$

With the triangular equation and $(a+b)^2 \leq 2a^2 + 2b^2$ for all $a, b \in \mathbb{R}$ it follows that

$$\left\| \int_{\mathcal{I}} \left(W^N (W^N)^\top - W^\star (W^\star)^\top \right) \mathrm{d}t \right\|_{\mathrm{F}} \tag{66}$$

$$= \left\| \int_{\mathcal{I}} \left((W^N - W^\star)(W^N)^\top + W^\star (W^N - W^\star)^\top \right) \mathrm{d}t \right\|_{\mathrm{F}} \tag{67}$$

$$\leq \left\| \int_{\mathcal{I}} (W^N - W^\star)(W^N)^\top \mathrm{d}t \right\|_{\mathrm{F}} + \left\| \int_{\mathcal{I}} W^\star (W^N - W^\star)^\top \mathrm{d}t \right\|_{\mathrm{F}}, \tag{68}$$

where $\|\cdot\|_{\mathrm{F}}$ is the Frobenius norm. We consider the first term in (68). With the triangular inequality, the submultiplicativity of the Frobenius norm and the Cauchy–Schwarz theorem imply that

$$\left\| \int_{\mathcal{I}} (W^N - W^\star)(W^N)^\top \mathrm{d}t \right\|_{\mathrm{F}} \leq \left(\int_{\mathcal{I}} \|(W^N - W^\star)(W^N)^\top\|_{\mathrm{F}} \, \mathrm{d}t \right) \tag{69}$$

$$\leq \left(\int_{\mathcal{I}} \|W^N - W^\star\|_{\mathrm{F}} \|W^N\|_{\mathrm{F}} \, \mathrm{d}t \right)^2 \tag{70}$$

$$\leq \left(\int_{\mathcal{I}} \|W^N - W^\star\|_{\mathrm{F}}^2 \, \mathrm{d}t \int_{\mathcal{I}} \|W^N\|_{\mathrm{F}}^2 \, \mathrm{d}t \right)^{\frac{1}{2}}. \tag{71}$$

The first factor in (71) satisfies

$$\left(\int_{\mathcal{I}} \|W^N - W^\star\|_{\mathrm{F}}^2 \, \mathrm{d}t \right)^{\frac{1}{2}} = \sqrt{\sum_{j=0}^{n-1} \|w_j^N - w_j^\star\|^2} = \mathcal{O}(N^{-k}) \quad \text{as } N \to \infty \tag{72}$$

for arbitrary $k \in \mathbb{N}$ by Proposition 11, and the second factor in (71) satisfies

$$\left(\int_{\mathcal{I}} \|W^N\|_{\mathrm{F}}^2 \, \mathrm{d}t \right)^{\frac{1}{2}} = \sqrt{\sum_{j=1}^{n-1} \|w_j^N\|^2} = \mathcal{O}(1) \tag{73}$$

as $N \to \infty$. This implies that for arbitrary $k \in \mathbb{N}$ the first summand in (68) is $\mathcal{O}(N^{-k})$ as $N \to \infty$. The same asymptotic follows in a similar way for the second term in (68). This shows (65).

Consequently,

$$\int_{\mathcal{I}} X^N (X^N)^\top \mathrm{d}t - \int_{\mathcal{I}} X^\star (X^\star)^\top \mathrm{d}t = \mathcal{O}(N^{-k}) \tag{74}$$

$$\int_{\mathcal{I}} U^N (X^N)^\top \mathrm{d}t - \int_{\mathcal{I}} U^\star (X^\star)^\top \mathrm{d}t = \mathcal{O}(N^{-k}) \tag{75}$$

as $N \to \infty$.

In particular, the invertibility of the matrix $M := \int_{\mathcal{I}} X^\star (X^\star)^\top \mathrm{d}t$ implies that $M^N := \int_{\mathcal{I}} X^N (X^N)^\top \mathrm{d}t$ is invertible for every sufficiently large $N \in \mathbb{N}$. Indeed, for $N \in \mathbb{N}$ such that $\|(M - M_N)M\|_{\mathrm{F}} < 1$ the inverse of the matrix $M_N = \left(I - (M - M_N)M^{-1}\right)M$ can be represented be means of a convergent Neumann series,

$$M_N^{-1} = M^{-1} \sum_{j=0}^{\infty} \left((M - M_N)M^{-1}\right)^j = M^{-1} + \mathcal{O}(N^{-k}) \quad \text{as } N \to \infty. \tag{76}$$

This implies

$$\left(\int_{\mathcal{I}} X^N (X^N)^\top \mathrm{d}t \right)^{-1} - \left(\int_{\mathcal{I}} X^\star (X^\star)^\top \mathrm{d}t \right)^{-1} = \mathcal{O}(N^{-k}) \quad \text{as } N \to \infty. \tag{77}$$

Combining (75) and (77) finishes the proof. □

The next proposition is a consequence of the last observation and Theorem 16. It is an interpretation of Lemma 20 from a data-driven perspective.

Proposition 21 *Suppose the assumptions of Theorem 4 hold. Define $\mathcal{M}_N$, Φ_N and $\mathcal{G}_N^0$ as in (45), (47) and (48). Let $\mathfrak{H}^N$ be a solution of*

$$\begin{bmatrix} {\mathcal{G}_N^0}^\top \mathcal{M}_N^\top \mathcal{M}_N \mathcal{G}_N \\ \Phi_N \mathcal{G}_N \end{bmatrix} \mathfrak{H}^N = \begin{bmatrix} 0 \\ I_n \end{bmatrix} . \tag{78}$$

Partition

$$\begin{bmatrix} \mathfrak{X}_0^N \\ \mathfrak{U}_0^N \\ \vdots \\ \mathfrak{X}_{N-1}^N \\ \mathfrak{U}_{N-1}^N \end{bmatrix} := \mathcal{M}_N \mathcal{G}_N \mathfrak{H}^N , \quad \text{where } \mathfrak{X}_i^N \in \mathbb{R}^{n\times n}, \quad \mathfrak{U}_i^N \in \mathbb{R}^{m\times n} . \tag{79}$$

Then, for arbitrary $k \in \mathbb{N}$ and every sufficiently large $N \in \mathbb{N}$ the matrix $\sum_{i=0}^{N-1} \mathfrak{X}_i^N (\mathfrak{X}_i^N)^\top$ is invertible and as $N \to \infty$

$$\left(\sum_{i=0}^{N-1} \mathfrak{U}_i^N (\mathfrak{X}_i^N)^\top \right) \left(\sum_{i=0}^{N-1} \mathfrak{X}_i^N (\mathfrak{X}_i^N)^\top \right)^{-1} - K = \mathcal{O}(N^{-k}) . \tag{80}$$

Proof Let $\mathfrak{H}^N = \begin{bmatrix} \mathfrak{h}_0^N & \cdots & \mathfrak{h}_{n-1}^N \end{bmatrix}$ with $\mathfrak{h}_j^N \in \mathbb{R}^{(n+m)N}$. Then by Theorem 16 the solution w_j^N of OCP 32 subject the initial condition $x(0) = e_j$ satisfies $\Pi w_j^N = \mathfrak{w}_j^N = \mathcal{M}_N \mathcal{G}_N \mathfrak{h}_j^N$ for $j = 0, \ldots, n-1$. Observe that

$$\begin{bmatrix} \mathfrak{X}_i^N \\ \mathfrak{U}_i^N \end{bmatrix} = \mathfrak{W}_i^N = \begin{bmatrix} (\mathfrak{w}_0^N)_i & \cdots & (\mathfrak{w}_{n-1}^N)_i \end{bmatrix} , \quad \text{where} \quad \mathfrak{w}_j^N = \begin{bmatrix} (\mathfrak{w}_j^N)_0 \\ \vdots \\ (\mathfrak{w}_j^N)_{N-1} \end{bmatrix} . \tag{81}$$

Define X^N, U^N and W^N as in (61) and (64). Then

$$\int_{\mathcal{I}} W^N (W^N)^\top \,\mathrm{d}t = \sum_{j=0}^{n-1} \int_{\mathcal{I}} w_j^N (w_j^N)^\top \,\mathrm{d}t = \sum_{j=0}^{n-1} \sum_{i=0}^{N-1} (\mathfrak{w}_j^N)_i (\mathfrak{w}_j^N)_i^\top = \sum_{i=0}^{N-1} \mathfrak{W}_i^N (\mathfrak{W}_i^N) \tag{82}$$

and, therefore,

$$\int_{\mathcal{I}} U^N (X^N)^\top \,\mathrm{d}t = \sum_{i=0}^{N-1} \mathfrak{U}_i^N (\mathfrak{X}_i^N)^\top , \quad \int_{\mathcal{I}} X^N (X^N)^\top \,\mathrm{d}t = \sum_{i=0}^{N-1} \mathfrak{X}_i^N (\mathfrak{X}_i^N)^\top . \tag{83}$$

The last equality and Lemma 20 prove the assertion. □

We now show that the approximation of the optimal feedback gain K leads to a stabilizing feedback.

Lemma 22 *Suppose the assumptions of Theorem 4 hold. Denote by K^N and P^N, $N \in \mathbb{N}$ the data-based approximations of the optimal feedback gain K and of the solution of the algebraic Riccati equation as in Proposition 21 (equation (80)) and Proposition 19. For sufficiently large $N \in \mathbb{N}$ the closed-loop system*

$$\dot{x} = (A + BK^N)x \tag{84}$$

is stable and $x \mapsto x^\top P^N x$ is a Lyapunov function.

Proof By Proposition 21 and Proposition 19 one has

$$K^N - K = \mathcal{O}(N^{-k}), \quad P^N - P = \mathcal{O}(N^{-k}) \tag{85}$$

for arbitrary $k \in \mathbb{N}$ as $N \to \infty$. Let $M^N := A + BK^N$ and $M := A + BK$. Since the optimal feedback gain K is stabilizing, the spectrum $\sigma(M)$ of M is contained in the left open half-plane of the complex plane. By Theorem VIII.1.1 in [26]

$$\max_{\lambda \in \sigma(M)} \min_{\mu \in \sigma(M^N)} |\lambda - \mu| \leq (\|M\| + \|M^N\|)^{1-1/n} \|M - M^N\|^{1/n}, \tag{86}$$

where n is the state dimension. The term $\|M^N\|$ is bounded and $\|M - M^N\| \leq \|B\| \|K - K^N\| = \mathcal{O}(N^{-k})$ as $N \to \infty$. Therefore $\sigma(M^N) = \sigma(A + BK^N)$ is contained in the left open half-plane of the complex plane as well for all sufficiently large $N \in \mathbb{N}$, the system (84) is stable.

We proceed in showing that $x \mapsto x^\top P^N x$ is a Lyapunov function for (84). With a similar argument as before it follows from (85) that P^N is positive definite for every sufficiently large $N \in \mathbb{N}$. Moreover,

$$(M^N)^\top P^N + P^N M^N = M^\top P + PM + \mathcal{O}(N^{-k}) \quad \text{as } N \to \infty. \tag{87}$$

As $x \mapsto x^\top P x$ is a Lyapunov function for $\dot{x} = (A + BK)x$, the matrix $(M^\top P + PM)$ is negative definite, and so is $((M^N)^\top P^N + P^N M^N)$, provided that $N \in \mathbb{N}$ is sufficiently large. This shows the claim. □

5 Numerical example

We consider a controllable system of the form (2) on the interval $\mathcal{I} = (0, \infty)$, such that the uncontrolled system $\frac{\mathrm{d}}{\mathrm{d}t} x = Ax$ is unstable. For the sake of exposition and simplicity, we select a system with state dimension $n = 20$ and input dimension $m = 6$, where the system matrices A and B are drawn from a single randomly generated sample, with entries uniformly distributed in the range $[-1, 1]$. Such a choice is purely illustrative and meant to help demonstrate the underlying principles.

Instead of using the data matrix Γ corresponding to a trajectory driven by an input that is persistently exciting of order $n+1$, we use a matrix with the same image, cf. Remark 7. Such simplification avoids unnecessary complications, ensuring that the focus remains on the core concepts; numerical issues, while important, are not the primary focus of the present paper.

In Figure 1 approximate closed-loop state and input trajectories are depicted. All state components in the closed-loop system approach zero for $t \to \infty$ thus illustrating the stabilization property of the data-driven controller. The input obtained by state-feedback also approaches zero in the limit.

Figure 2 presents the pointwise deviation of the approximate to the optimal trajectory for different approximation orders N. Evidently the accuracy increases with N.

In Figure 3 the convergence of the approximate Riccati solution and the approximate optimal gain discussed in Section 4 can be observed. The convergence of the spectrum of $A - BK^N$ to that of $A - BK$ is illustrated in Table 1. In this context the distance between two non-empty compact sets is measured via the Hausdorff metric with respect to the Euclidean norm, cf. e.g. [27], which captures the largest distance

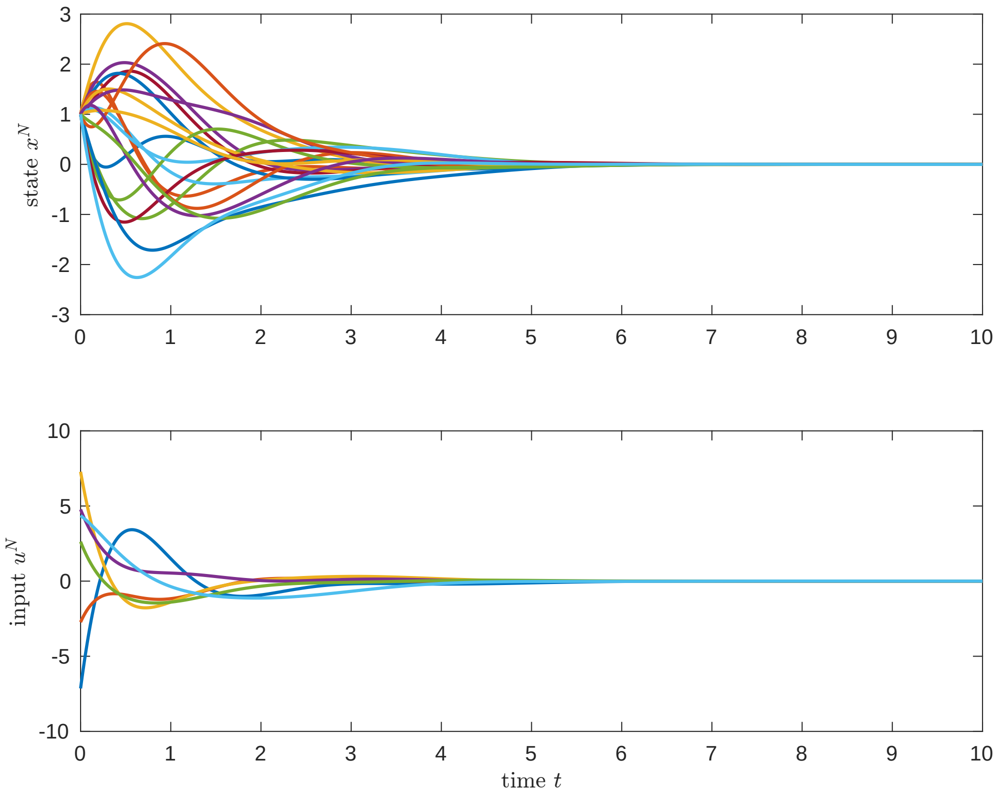

Fig. 1 Approximate optimal trajectory $w^N = \mathrm{col}(x^N, u^N)$ for the initial condition $x(0) = \mathrm{col}(1, \ldots, 1)$ with approximation order $N = 50$. On the top, the components of the state x^N are illustrated, matching the initial condition, while on the bottom the components of the input u^N are shown. It is evident that the approximate optimal controller effectively stabilizes the system to zero.

between points in one set and their closest points in the other set. This spectral convergence further leads to the stabilization of the system by the approximate optimal feedback gain K^N, as shown in Table 1.

6 Conclusions

We illustrated a data-based framework to treat in a unified way finite- and infinite-horizon optimal control problems based on the approximation of system trajectories by finite linear combinations of orthogonal basis functions. We considered optimal control problems from two different perspectives. The first one centers around the computation of *optimal trajectories*: we have shown how finite-dimensional suboptimal approximations computed from a sufficiently informative trajectory converge to the solution of an optimal control problem. The second perspective adopted in this paper centers around the data-driven computation of approximations of an optimal *controller*.

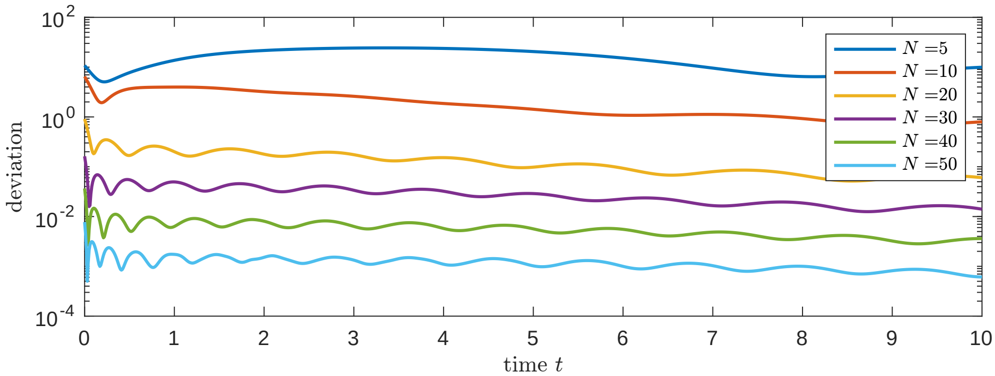

Fig. 2 Pointwise deviation between the optimal and approximate optimal trajectory in the Euclidean norm, $\|w^\star(t) - w^N(t)\|_2$, for different N. The results show that the approximate optimal trajectory converges to the optimal one. Additionally, the approximation error is oscillatory during the transient phase but smooths out in the long-term behavior.

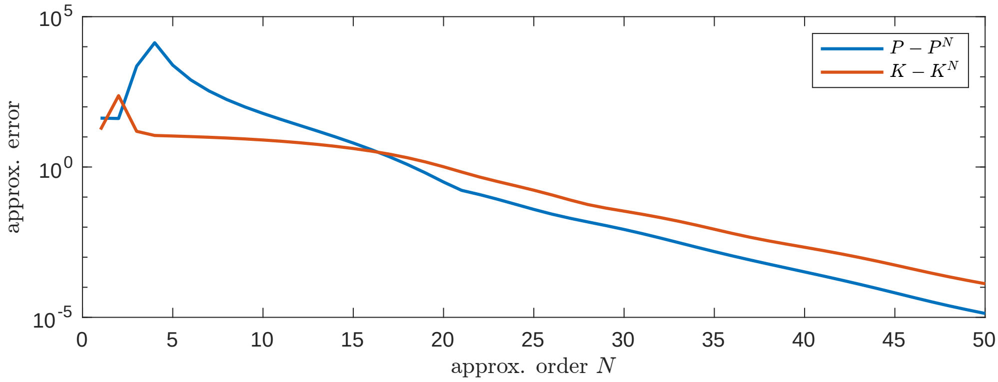

Fig. 3 The approximation error of the approximate Riccati solution P^N and the approximate optimal gain K^N with respect to the Frobenius norm, illustrating the convergence results from Section 4.

Table 1 Spectral abscissa (largest real part of a matrix's eigenvalues) of $A - BK^N$ and the Hausdorff distance between the spectra of $A - BK^N$ and $A - BK$ for different approximation orders N. The results show that the approximate optimal gain stabilizes the system, with the eigenvalues of $A - BK^N$ converging to those of the optimized closed-loop system. Consequently, the spectral abscissa approaches that of $A - BK$.

approx. order N	5	10	20	30	40	50
spectral abscissa	-0.1778	-0.4267	-0.8371	-0.8164	-0.8158	-0.8158
spectral distance	0.7254	0.7799	0.3527	0.0375	0.0028	0.0002

The results presented in this paper confirm the potential of approximation theory concepts and techniques to contribute to the solution of data-driven optimal control problems. Much work remains to be done to fully realize such potential; the following two issues are the most pressing ones. First, it is necessary to extend our results to the case of input-output data: assuming that the state variable is available for direct measurement implies a level of insight into the system dynamics that is at odds with a fully data-driven point of view on control. Secondly, computationally efficient and numerically accurate procedures must be developed to implement our methods so as to validate them on realistic applications, also on data affected by noise (see [28] on some numerical experiments in the use of orthogonal bases, and [29] for a computationally efficient and numerically stable algorithm for univariate approximation from noisy data).

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Appendix A Polynomial approximation

This section reviews fundamental results from the approximation theory of orthonormal polynomials, with a specific focus on Legendre and Laguerre polynomials. The material presented is primarily based on [30, 31].

A.1 Legendre polynomials

Consider the finite interval $\mathcal{I} = (0, 1)$ and set

$$\pi_i(t) = \sqrt{2i+1}\pi_i^{\mathrm{Leg}}(2t-1), \quad t \in \mathcal{I} = (0,1), \tag{A1}$$

where π_i^{Leg} is the i-th Legendre polynomial on $(-1, 1)$ with normalization $\pi_i^{\mathrm{Leg}}(-1) = (-1)^i$ and $\pi_i^{\mathrm{Leg}}(1) = 1$. Then, $(\pi_i)_{i\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathcal{I}, \mathbb{R})$ and

$$\pi_i(0) = \sqrt{2i+1}(-1)^i. \tag{A2}$$

We recall from Section 8.4 of [31] the formula describing the expansion coefficients of derivatives. Let $f \in H^1(\mathcal{I}, \mathbb{R})$ with $\mathfrak{f} = \Pi f$ and denote by $\mathfrak{f}^{(1)}$ the sequence of coefficients of $\frac{\mathrm{d}}{\mathrm{d}t} f$ in the polynomial representation: $\mathfrak{f}^{(1)} := \Pi(\frac{\mathrm{d}}{\mathrm{d}t} f)$. Then

$$\mathfrak{f}_i^{(1)} = \frac{(2i+1)}{2} \sum_{\substack{j=i+1 \\ i+j \text{ odd}}}^{\infty} \mathfrak{f}_i, \tag{A3}$$

i.e. $\mathfrak{f}^{(1)} = \mathcal{D}\mathfrak{f}$ with the operator $\mathcal{D} : \ell^2(\mathbb{N},\mathbb{R}) \to \ell^2(\mathbb{N},\mathbb{R})$ represented by

$$\mathcal{D} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & \cdots \\ & 0 & 3 & 0 & 3 & \cdots \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{A4}$$

Given $N \in \mathbb{N}$, we denote by $\mathcal{D}_N$ the upper left $N \times N$ submatrix of $\mathcal{D}$; note that this finite matrix represents the restriction of $\mathcal{D}$ to $\operatorname{im} P_N$ where P_N is defined by (23).

With increasing degree N the best possible approximation of a function $f \in L^2(\mathcal{I},\mathbb{R})$ in the subspace $\operatorname{im} P_N$ improves, as $P_N f \to f$ for $N \to \infty$. The next proposition quantifies the speed of convergence; recall from (1) the definition of Λ_s.

Proposition 23 *[30, Theorem 6.3] Let $k, s \in \mathbb{N}$ and $f \in H^{k+s}(\mathcal{I},\mathbb{R})$. Then for every $N \in \mathbb{N}$ there is $g^N \in \operatorname{im} P_N$ such that $\Lambda_s(f - g^N)(0) = 0$ and*

$$\|f - g^N\|_{H^s} = \mathcal{O}(N^{-k}) \quad \text{as } N \to \infty. \tag{A5}$$

A.2 Laguerre polynomials

Unlike Legendre polynomials, Laguerre polynomials are associated with an inner product that incorporates a specific weight function. Therefore, before presenting approximation results, we first introduce weighted L^2-spaces along with their corresponding Sobolev spaces. For the rest of this section we focus on the open half-axis $\mathcal{I} = (0,\infty)$.

A.2.1 Weighted Sobolev spaces

Consider the weight function r defined by

$$r(t) = \exp(-t)\,, \quad t \in \mathcal{I}. \tag{A6}$$

The weighted L^2-space associated with r and its corresponding norm are defined as

$$L^2_r(\mathcal{I},\mathbb{R}) = \{\phi : \mathcal{I} \to \mathbb{R} \text{ meas.} \mid \|\phi\|_{L^2_r} < \infty\}\,, \quad \|\phi\|_{L^2_r} := \left(\int_0^\infty r(t)\phi(t)^2\,\mathrm{d}t\right)^{\frac{1}{2}}.$$

The space $L^2_r(\mathcal{I})$ induces a family of r-weighted Sobolev spaces,

$$H^k_r(\mathcal{I},\mathbb{R}) = \{\phi \in L^2_r(\mathcal{I},\mathbb{R}) \mid \|\phi\|_{H^k_r} < \infty\}\,, \quad \|\phi\|_{H^k_r} := \left(\sum_{j=0}^k \|\phi^{(j)}\|^2_{L^2_r}\right)^{\frac{1}{2}}.$$

It turns out that the weighted spaces are isomorphic to their unweighted counterparts, which associated with the standard L^2 inner product ($r \equiv 1$).

Lemma 24 ([30]) *Consider the isometric isomorphism* $\iota : L^2_r(\mathcal{I}, \mathbb{R}) \to L^2(\mathcal{I}, \mathbb{R})$ *defined by* $\iota\phi := r^{\frac{1}{2}} f$. *Then* $\|\iota \cdot\|_{H^k}$ *defines an equivalent norm on* $H^k_r(\mathcal{I}, \mathbb{R})$, *i.e. there exists constants* $m > 0$ *and* $M > 0$ *such that for all* $\phi \in H^k_r(\mathcal{I}, \mathbb{R})$

$$m\|\phi\|_{H^k_r} \leq \|\iota\phi\|_{H^k} \leq M\|\phi\|_{H^k_r}. \tag{A7}$$

In particular, ι *restricted to* $H^k_r(\mathcal{I}, \mathbb{R})$ *is an isomorphism onto* $H^k(\mathcal{I}, \mathbb{R})$.

In order to formulate approximation results with respect to higher-order derivatives an additional family of weighted Sobolev spaces is crucial:

$$H^k_{r;\alpha}(\mathcal{I}, \mathbb{R}) = \{\phi \in H^k_r(\mathcal{I}, \mathbb{R}) \mid \|\phi\|_{H^k_{r;\alpha}} < \infty\}, \quad \|\phi\|_{H^k_{r;\alpha}} := \|p^{\frac{\alpha}{2}}\phi(\cdot)\|_{H^k_r}, \tag{A8}$$

where $\alpha \in [0, \infty)$ and $p(t) = 1 + t$.

A.2.2 Approximation by Laguerre polynomials

We consider the sequence $(\pi^{\mathrm{Lag}}_i)_{i\in\mathbb{N}}$ of Laguerre polynomials with normalization $\|\pi^{\mathrm{Lag}}_i\|_{L^2_r} = 1$ and $\pi^{\mathrm{Lag}}_i(0) = 1$ which forms an orthonormal basis in $L^2_r(\mathcal{I}, \mathbb{R})$. The projector in $L^2_r(\mathcal{I}, \mathbb{R})$ onto the space spanned by the first N, $N \in \mathbb{N}$, Laguerre polynomials is denoted by $\tilde{P}_N$

$$\tilde{P}_N\phi = \sum_{i=0}^{N-1} \langle \phi, \pi^{\mathrm{Lag}}_i \rangle_{L^2_r} \pi^{\mathrm{Lag}}_i \tag{A9}$$

The completeness of the orthogonal system $(\pi^{\mathrm{Lag}}_i)_{i\in\mathbb{N}}$ implies the strong convergence of $\tilde{P}_N$ to the identity, i.e. for all $\phi \in L^2_r(\mathcal{I}, \mathbb{R})$ one has $\tilde{P}_n\phi \to \phi$ in $L^2_r(\mathcal{I}, \mathbb{R})$ as $n \to \infty$. The next proposition provides stronger convergence results and bounds on the rate of convergence.

Proposition 25 *[30, Thm. 12.3] Let* $s, k \in \mathbb{N}$, $\alpha = 2(k+s)$. *Then there is a positive constant* c *such that for every* $\phi \in H^\alpha_{r;\alpha}(\mathcal{I}, \mathbb{R})$ *and all* $N \in \mathbb{N}$ *the following estimate holds*

$$\|(I - \tilde{P}_N)\phi\|_{H^s_r} \leq cN^{-k}\|\phi\|_{H^\alpha_{r;\alpha}}. \tag{A10}$$

In the following we extend the above approximation result with respect to an initial condition at zero.

Corollary 26 *Let* $s, k \in \mathbb{N}$, $\alpha = 2(k+s)$. *Then there exists a positive constant* c *such that for all and every* $\phi \in H^\alpha_{r;\alpha}(\mathcal{I}, \mathbb{R})$ *and all* $N \in \mathbb{N}$ *there exists* $g^N \in \operatorname{im}\tilde{P}_N$ *with* $\Lambda_s(\phi - g^N)(0) = 0$ *such that the following estimate holds*

$$\|\phi - g^N\|_{H^s_r} \leq cN^{-k}\|\phi\|_{H^\alpha_{r;\alpha}}. \tag{A11}$$

Proof Note that the point evaluation $H^s_r(\mathcal{I}, \mathbb{R}) \to \mathbb{R}^s$, $g \mapsto \Lambda_s(g)(0)$ is continuous and, therefore, there is by Proposition 25 a positive constant c_0 such that

$$\left\|\Lambda_s\big((I - \tilde{P}_N)\phi\big)(0)\right\|_2 \leq c_0N^{-k}\|\phi\|_{H^\alpha_{r;\alpha}}. \tag{A12}$$

Note that the linear map $S : \operatorname{im} P_s \to \mathbb{R}^s,\ f \mapsto \Lambda_s(f)(0)$ is a bijection. Moreover, its inverse S^{-1} considered as a map into $H_r^s(\mathcal{I}, \mathbb{R})$ is continuous, i.e. there is $c_1 > 0$ such that for every $\xi \in \mathbb{R}^s$

$$\|S^{-1}\xi\|_{H_r^s} \leq c_1 \|\xi\|_2. \tag{A13}$$

Let $N \geq s$. Choose $f^N \in \operatorname{im} P_s \subset \operatorname{im} \tilde{P}_N$ such that $\Lambda_s(f)(0) = Sf = \xi^N := \Lambda_s\big((I - P_N)\phi\big)(0)$ and define $g^N := P_N\phi - f^N \in \operatorname{im} \tilde{P}_N$. Then by construction $\Lambda_s(\phi - g^N)(0) = 0$ and

$$\|\phi - g^N\|_{H_r^s} \leq \|(I - \tilde{P}_N)\phi\|_{H_r^s} + \|f^N\|_{H_r^s} \leq (c_0 c_1 + c) N^{-k} \|\phi\|_{H^\alpha_{r;\alpha}}. \qquad \square$$

Laguerre-based orthonormal system

Following the analysis for Legendre polynomials, we now investigate analogous properties using Laguerre polynomials. To translate results from their natural weighted orthogonality setting to standard unweighted L^2 and Sobolev spaces, we employ a rescaled Laguerre basis incorporating an inverse weight factor. Let

$$\pi_i(t) = \exp(-\tfrac{t}{2})\pi_i^{\mathrm{Lag}}(t), \quad t \in \mathcal{I}, \tag{A14}$$

i.e. $\pi_i = \iota^{-1}\pi_i^{\mathrm{Lag}}$. As a consequence of Lemma 24, $(\pi_i)_{i\in\mathbb{N}}$ is an orthonormal basis in $L^2(\mathcal{I}, \mathbb{R})$. Further, $\pi_i(0) = 1$. With the recursion $\frac{\mathrm{d}}{\mathrm{d}t}\pi_{i+1}^{\mathrm{Lag}} = \frac{\mathrm{d}}{\mathrm{d}t}\pi_i^{\mathrm{Lag}} - \pi_i^{\mathrm{Lag}}$, see e.g. [30], we find

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\pi_i^{\mathrm{Lag}} = -\sum_{j=0}^{i-1}\pi_j^{\mathrm{Lag}}, \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\pi_i = -\frac{1}{2}\pi_i - \sum_{j=0}^{i-1}\pi_j. \tag{A15}$$

Therefore, given $f \in H^1(\mathcal{I}, \mathbb{R})$ with $\mathfrak{f} = \Pi f$ and $\mathfrak{f}^{(1)} = \Pi(\frac{\mathrm{d}}{\mathrm{d}t}f)$ one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i\in\mathbb{N}}\mathfrak{f}_i\pi_i\right) = \sum_{i\in\mathbb{N}}\mathfrak{f}_i\left(-\frac{1}{2}\pi_i - \sum_{j=0}^{i-1}\pi_j\right) = -\frac{1}{2}\sum_{i\in\mathbb{N}}\mathfrak{f}_i\pi_i - \sum_{j\in\mathbb{N}}\left(\sum_{i=j+1}^{\infty}\mathfrak{f}_i\right)\pi_j$$

that is

$$\mathfrak{f}_i^{(1)} = -\frac{1}{2}\mathfrak{f}_i - \sum_{j=i+1}^{\infty}\mathfrak{f}_j. \tag{A16}$$

The corresponding operator $\mathcal{D} : \ell^2(\mathcal{I}, \mathbb{R}) \to \ell^2(\mathcal{I}, \mathbb{R})$ defined by $\mathcal{D}\mathfrak{f} := \mathfrak{f}^{(1)}$ is represented by

$$\mathcal{D} = -\begin{bmatrix} \frac{1}{2} & 1 & 1 & \cdots \\ & \frac{1}{2} & 1 & \cdots \\ & & \ddots & \ddots \end{bmatrix}. \tag{A17}$$

Given $N \in \mathbb{N}$, we denote the upper left $N \times N$ submatrix of $\mathcal{D}$ by $\mathcal{D}_N$.

Similarly to the approximation with Laguerre polynomials, cf. Proposition 23, the quality of the approximation increases with the polynomial degree; the speed of convergence is characterized in the following result (see Proposition 25 and Corollary 26 in Appendix A.2.2 for the proof).

Proposition 27 *Let $k, s \in \mathbb{N}$ and $f \in H^{2(k+s)}(\mathcal{I}, \mathbb{R})$ such that $p^{k+s} f \in H^{2(k+s)}(\mathcal{I}, \mathbb{R})$, where $p(t) = (1+t)$. Then for every $N \in \mathbb{N}$ there is $g^N \in \operatorname{im} P_N$ such that $\Lambda_s(f - g^N)(0) = 0$ and*

$$\|f - g^N\|_{H^s} = \mathcal{O}(N^{-k}) \quad \text{as } N \to \infty. \tag{A18}$$

Proof Define $\alpha = 2(k+s)$ and suppose $f \in H^\alpha(\mathcal{I}, \mathbb{R})$ satisfying $p^{\frac{\alpha}{2}} f \in H^\alpha(\mathcal{I}, \mathbb{R})$ with $p(t) := 1 + t$. Consider the isomorphism ι in Lemma 24. Then $\phi := \iota^{-1} f \in H^\alpha_r(\mathcal{I}, \mathbb{R})$. Moreover,

$$p^{\frac{\alpha}{2}} \phi = p^{\frac{\alpha}{2}} \iota^{-1} f = \iota^{-1}\left(p^{\frac{\alpha}{2}} f\right) \in H^\alpha_r, \tag{A19}$$

that is $p^{\frac{\alpha}{2}} \phi \in H^\alpha_{r;\alpha}$. By Corollary 26 there exists a sequence $(\tilde{g}^N)_{N \in \mathbb{N}}$, $\tilde{g}^N \in \operatorname{im} \tilde{P}_N$, such that $\Lambda_s(\phi - \tilde{g}^N)(0) = 0$ and

$$\|\phi - \tilde{g}^N\|_{H^s_r} = \mathcal{O}(N^{-k}) \quad \text{as } N \to \infty. \tag{A20}$$

Then the assertion holds with $g^N := \iota \tilde{g}^N$. □

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