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Quantum Gravitational Information and Flat Holography

by

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To my beloved

Dad Shekar R and Mom Jeyabharathi T

I wish you the same elation I felt every time I realised something new.

Abstract

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A semiclassical analysis shows that black holes evaporate by emitting thermal Hawking radiation, leading to a violation of unitarity. Holography, which itself was inspired by the area scaling of black hole entropy, recently motivated quantum corrections to the black hole entropy through the Quantum Extremal Surface (QES) prescription, leading to the *island proposal*. Islands provide the necessary additional degrees of freedom that purify the outgoing radiation and preserve unitarity during evaporation. Islands were originally analytically studied in two-dimensional gravity. In higher dimensions, progress is limited due to the lack of explicit results for entanglement entropy (EE) of QFT states in a subregion on a higher-dimensional black hole background, which is an input to the QES prescription. We analyse entanglement entropy in particular curved backgrounds using the holographic Ryu-Takayanagi prescription. We then develop a more general systematic analysis of the properties of entanglement entropy in curved backgrounds using the replica approach. We explore the analytic $(q-1)$ expansion of Rényi entropy S_q and its variations; our setup applies to generic variations, from symmetry transformations to variations of the background metric or entangling region. Our methodology elegantly reproduces and generalises results from the literature on entanglement entropy in different dimensions, backgrounds, and states. We use our analytic expansions to explore the behaviour of entanglement entropy in static black hole backgrounds under specific scaling transformations. Using these results, we find particular conditions on the outgoing black hole radiation spectrum for the presence of islands that restore unitarity in black hole evaporation.

We then discuss holography in asymptotically flat spacetimes using the *Carrollian approach* and *flat limit of AdS/CFT*. Theories on degenerate null backgrounds are Carrollian (speed of light $c \to 0$ limit of QFTs), hence these capture massless scattering between null infinities. Carrollian conformal field theories (CCFTs) are being explored as putative duals to massless scattering in asymptotically flat spacetimes. We consider the momentum space Ward identities of Carrollian conformal field theories involving scalar operators and explicitly solve to obtain the 2 and 3-point correlators. We also analyze the Carrollian limit (vanishing speed of light $c \to 0$) of CFT 2 and 3-point functions of scalar operators in momentum space and obtain these correlators. We further discuss a generating functional for these Carrollian CFT correlators analogous to AdS/CFT. Starting with scalar fields in flat space written in coordinates adapted to null infinities, we compute the classical on-shell action. With suitable boundary conditions and source at null infinities, we show that this generates the CCFT correlators derived earlier.

A complementary approach to flat holography is to take a flat limit of AdS/CFT correlators to reproduce flat amplitudes. We also intend to systematically study massive higher-spin fields required in a consistent gravitational theory. To this end, we analyze a bulk effective field theory in AdS containing a U(1)-charged massive spin-2 field coupled to a gauge field. After holographic renormalization, we compute the one, two, and three-point functions involving two massive spin-2 fields and one gauge field. Matching with the CFT 3-point correlator of two non-conserved spin-2 operators and a conserved current, we obtain explicit mappings between the bulk minimal and gyromagnetic couplings and the boundary OPE data. Finally, we take the flat-space limit of the momentum space CFT correlator and verify that the resulting amplitude matches the expected flat-space structure.

Declaration of Authorship

I, Arvind Shekar, declare that this thesis, titled *Quantum Gravitational Information and Flat Holography*, including the work presented in it, is my own and has been generated by me as the result of my own original research. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University;
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- Where I have consulted the published work of others, this is always clearly attributed;
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- Parts of this work have appeared in the following publications and arXiv preprints:
    1. A. Shekar and M. Taylor, *Replica analysis of entanglement properties*, Phys. Rev. D 111 (2025) 066018 [2410.07312].
    2. F. Landgren and A. Shekar, *Islands and entanglement entropy in d-dimensional curved backgrounds*, arXiv:2401.01653 (2024).
    3. R. Marotta, A. Shekar, and M. Verma, *Carrollian Conformal Theories in Momentum Space*, arXiv:2512.06881 (2025).
    4. V. Nenmeli, A. Shekar and M. Verma, *From AdS to Flat Space: Massive Spin-2 Fields*, arXiv:2602.02662 (2026).

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    1. A. Shekar, *Conditions for Islands*, to appear (2026).
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Signed: .... Date: ....

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Chapter 1

Introduction

1.1 The Big Picture

An innate curiosity to understand the Universe around us is an amazing quality we are all born with. Over centuries, we have honed this process. The process is self-correcting: we interact with our environment and observe, trying to gather as much information as possible. We then use the language of mathematics to guess a pattern that fits the behavior. If our guess is correct and broad, it should hold in arenas we have not interacted with yet. We keep experimenting in new arenas until we are unlucky (or lucky) to venture upon one where our guess breaks down. By repeating this process, we have distilled most of the workings of our Universe into a minimal number of statements.

But our pursuit towards a complete understanding is far from over (which makes it all the more exciting). Through the framework called Quantum Field Theory, which incorporates quantum mechanics with the behavior of coordinate systems in inertial frames, i.e., special relativity, we understand a huge chunk of our Universe. This includes electromagnetism, the strong nuclear force, and the weak nuclear force in a combined formalism called the Standard Model. The latest experiment at CERN probing the Standard model verified it up to 13.6 TeV (length scale $\approx 10^{-20}\,m$). Another force that we have known the longest but have not completely understood is Gravity. Gravity describes the behavior of the stage, i.e., spacetime, on which the dynamics of all the other forces unfold. Gravity acts such that locally the laws in an inertial frame of reference hold. Hence, it has been described geometrically through Einstein's General Relativity. General Relativity predicts that highly massive, dense objects exert enough gravitational force to deform the trajectory of objects, trapping them in certain regions of spacetime demarcated by an event horizon. These objects are termed black holes. Shrouded by the event horizon are certain regions of spacetime where the path of certain freely falling objects abruptly end after a finite time. Geometrically the curvature of the spacetime becomes infinite and this region is termed a singularity. The presence of this infinity signals our description of gravity is incomplete.

The coupling of gravity with other forces suggests that gravity is quantised too. The usual machinery of quantum field theory on which the Standard Model is based breaks down and gives infinities for observables, when we try to quantise gravity. Also, this coupling is extremely small, and quantum gravitational effects become apparent at Planck scales ($\approx 10^{-35}\, m$). We do not yet have access to controlled experiments at such high energy scales, which gives us little to no input data to base our theories on.

We could always try to observe uncontrolled high-energy phenomena naturally occurring in our Universe and gather as much data as possible. We could also try to build self-consistent theories that incorporate all known phenomena while also quantising gravity. We have been exploring this over the past half a century. A key focus in our exploration is high energy phenomena where quantum gravitational effects become important. Black holes and their strong gravitational effects have served as a thought laboratory for our theoretical motivations and models.

Jacob Bekenstein argued in [1, 2] that black holes must also have an entropy and temperature, otherwise it would be possible to throw anything into a black hole and lose information/ entropy in the Universe, going against the second law of thermodynamics. Hawking showed in [3] that quantising fields on a black hole background while keeping the background spacetime itself unquantised (i.e., semiclassical gravity) implies that a black hole can indeed slowly release trapped objects over time and evaporate by emitting "Hawking radiation". They have a very small temperature and a very large entropy (which scales as the area of the event horizon rather than the volume scaling that one usually expects). However, the objects coming out seem to be thermally scrambled, irrespective of what formed the black hole. They have less information content than what went in. If this is true in reality, it seems that the Universe as a whole can lose information. This does not happen in any other known process so far.

A key defining feature of doing physics is predictability i.e., given certain initial conditions, we predict what happens over time. We believe the reverse is true too i.e., given the present conditions, we should be able to rewind time and guess the past conditions. This is called postdictability. If information is lost in a process then postdictability is no more a possible feature. These are however sociological expectations and it is very much possible that our Universe does indeed lose information. But, given that this stems from our lack of understanding of quantum gravity, it is very much possible that the Universe in reality preserves information and our current understanding needs a major upgrade. The latter viewpoint is what most of the physics community including me take.

By demanding **information preservation/ unitarity** as a guiding principle, we can try to understand the high energy behavior of gravity. Our goal is to build/ understand models of black hole evaporation in which all the information is encoded in the outgoing

Hawking radiation while remaining consistent with low energy behavior understood so far. We hope that this will shed light on the quantum effects of gravity. This is very much along the lines of how we discovered atoms. We first understood thermodynamic features and then put forth a microscopic realization of them. Similarly, our goal is to understand the microscopic constituents that build spacetime consistent with the thermodynamics of black holes and certain fundamental guiding principles that we expect to hold at high energies.

We quantify the information content by using a quantum information measure associated with the ignorance of the knowledge of the quantum state. When applied to subsystems these also quantify a unique feature of quantum states termed entanglement, which capture non local acausal correlations between subsystems. This measure termed "entanglement entropy" has been resourceful beyond the black hole context in studying many body quantum systems, and has been richly explored by condensed matter theorists. To preserve information during the black hole evaporation process, while being consistent with general relativity, the nature of the entanglement between subsystems across spacetime regions, must follow a particular pattern over time, and this has been our guiding principle towards adding high energy corrections to gravity.

Another peculiar feature of gravity has been the observation that the black hole entropy scales as the area of its event horizon (as opposed to the usual volume scaling). Since entropy is a measure of the information content or ignorance of the system, this implies that all the information about the black hole is present on its event horizon surface. It was suggested that this **"Holography"** is a universal feature of spacetime in the presence of gravity [4, 5], i.e., gravity has a dual description in terms of a lower-dimensional field theory living at the asymptotic boundaries of the spacetime. A concrete realisation of this from a particular high energy realisation of gravity called string theory was obtained in spacetimes with a negative cosmological constant (asymptotically Anti de Sitter spacetimes) [6–8]. Although experiments suggest that we live in a spacetime with a positive cosmological constant (i.e., asymptotically de Sitter spacetimes), this is still a good playground to learn about (quantum) gravity from the dual field theory without gravity. We have explored this rich feature called holography in asymptotically AdS spacetimes over the past couple of decades, and learnt a great deal.

*We will assume **information preservation/ unitarity** and **Holography** are features of gravity at all energy scales and use them as guiding principles towards understanding it.*

Since particular field theories without gravity are dual to gravity in one higher dimension, we expect that field theory/ quantum information measures in these theories, such as entanglement entropy and complexity of preparation of a state, must have geometric gravity duals too. Such geometric duals have been explicitly realised over the past decade, for gravity in asymptotically AdS spacetimes. For example, entanglement entropy in the dual field theory is proportional to the area of the minimal area surface in the bulk that shares the boundary with the entangling subsystem [9, 10]. This looks very similar to the area behavior of the entropy of black holes. The study of these quantum information measures have been fruitful towards understanding aspects of the dual gravity better. For example, specific entangled states in the dual field theory have been associated to wormhole geometries in the bulk gravity, suggesting that these quantum information measures and constraints on them encode gravity in the bulk.

Part I

Recently, quantum corrections to the minimal area gravity dual of entanglement entropy in the dual field theory, in asymptotically AdS spacetimes [11], suggested similar quantum corrections to the area behavior of black hole entropy. These quantum corrections seem to provide the exact additional degrees of freedom termed "*islands*", required to reproduce the behavior of the outgoing radiation entanglement entropy over time, that preserves information during a black hole evaporation process [12, 13]. However, these computations were originally analytically done for asymptotically AdS black holes in two spacetime dimensions. Studies in higher dimensions involve numerics or simplifying assumptions that do not completely capture the required behavior. This is because computing the entanglement entropy of a state on a black hole background, which is required for these quantum corrections, is hard in higher dimensions. This is, however, a crucial input required to test if the proposed quantum corrections also preserve information in evaporating black holes in dimensions greater than two. In the first part of this work, I aim to address this bottleneck. We will compute entanglement entropy in static black hole backgrounds and discuss some of its properties, using both holography and quantum field theory techniques. We will find that the presence of islands impose certain conditions on the outgoing radiation spectrum.

Part II

In the second part of this work, I will be exploring holography in spacetimes with a zero cosmological constant (asymptotically flat spacetimes). This is a good stepping stone towards understanding holography in spacetimes with a positive cosmological constant (asymptotically de Sitter spacetimes) relevant for our Universe. The value of the cosmological constant inferred from cosmological observables measured from experiments is extremely small $\approx 10^{-52}\, m^{-2}$ [14]. For the purposes of collider experiments that measure various Standard Model parameters, for modelling gravitational waveforms, and for most astrophysical observations, asymptotically flat spacetime is a perfectly good approximation. Hence, understanding holography in these spacetimes is extremely useful.

There have been several approaches towards flat holography, namely dS and AdS slicing of flat spacetime combined with our knowledge of holography in asymptotically AdS spacetimes, resulting in a codimension-2 holographic dual [15], codimension-2 celestial holography duals [16–25], codimension-1 Carrollian holography duals [26, 27], and flat limits of asymptotically AdS holography. We will be focusing on the latter two approaches.

The preceding discussion was deliberately heuristic and aimed at conveying the qualitative picture. We will now motivate these ideas using a rigorous mathematical framework.

1.2 Black hole information, holography, and entanglement entropy

In this section we will discuss how certain systems with causal horizons exhibit thermodynamic properties such as entropy and temperature.

Consider a free scalar field $\phi(x)$ in d-dimensional Minkowski spacetime with coordinate labels $x^\mu = \{x^0, x^i\}$, $i \in \{1, 2, \cdots, d-1\}$, satisfying the Klein-Gordon equation

$$\left(\Box + m^2\right) \phi(x) = 0, \quad \Box = \eta_{\mu\nu}\partial^\mu\partial^\nu, \quad \eta_{\mu\nu} = \{-, +, +, \cdots, +\}. \tag{1.2.1}$$

The indices are raised and lowered using the Minkowski metric.

Let $\{f_\mathbf{k}(x)\}$ denote a complete basis of solutions to (1.2.1), characterized by the $(d-1)$-dimensional mode vector $\mathbf{k} = \{k^1, k^2, .., k^{d-1}\}$, with the on-shell condition $(k^0)^2 = |\mathbf{k}|^2 + m^2$, imposed by (1.2.1). We can expand a general solution as

$$\phi(x) = \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}} \left(a_\mathbf{k} f_\mathbf{k} + a^\dagger_\mathbf{k} f^*_\mathbf{k}\right). \tag{1.2.2}$$

We can quantize the field by imposing the commutation relations $[a_\mathbf{k}, a^\dagger_{\mathbf{k}'}] = (2\pi)^{d-1}\delta^{d-1}(\mathbf{k} - \mathbf{k}')$, and define the vacuum by $a_\mathbf{k}|0\rangle_f = 0$.

We could alternatively consider a different complete basis of mode solutions to (1.2.1), $g_\mathbf{k}(x)$, and expand a general solution as

$$\phi(x) = \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}} \left(b_\mathbf{k} g_\mathbf{k} + b^\dagger_\mathbf{k} g^*_\mathbf{k}\right). \tag{1.2.3}$$

We can quantize the field using the commutation relation $[b_\mathbf{k}, b^\dagger_{\mathbf{k}'}] = (2\pi)^{d-1}\delta^{d-1}(\mathbf{k} - \mathbf{k}')$ and define a different vacuum $b_\mathbf{k}|0\rangle_g = 0$.

In flat spacetime, we are able to pick out a natural set of modes by demanding that they be positive frequency with respect to the time coordinate. Although the time coordinate has a Lorentz transformation freedom, the vacuum and number operator are invariant under this transformation. Therefore, every inertial observer agrees on the vacuum state and the number of particles.

The definition of vacuum and particles depends on the choice of modes. In stationary spacetimes, these modes are chosen to have positive frequency with respect to a timelike Killing vector. In the absence of causal horizons, observers define positive frequency using a globally accessible notion of time evolution, which fixes the global vacuum state. However, in the presence of causal horizons, observers generally have access only to a restricted region of spacetime and naturally associate positive frequency with the notion of time evolution appropriate to that accessible region. This, in turn, fixes the associated creation and annihilation operators and consequently the notions of vacuum and particles for those observers. These operators do not, in general, annihilate the global vacuum state. This mismatch lies at the heart of why observers with restricted causal access may perceive particles even when the field is in a global vacuum state. In special cases, such as the Unruh effect for accelerated observers in flat spacetime [28] and the Hawking effect for black holes [29], the resulting spectrum is thermal, as we will see.

In curved spacetimes, if the spacetime is stationary, we can choose a timelike Killing vector and have a preferred notion of frequency, modes, and vacuum.

1.2.1 Rindler horizons and black hole thermodynamics

We will now consider certain explicit examples and slowly build up to the black hole temperature. References for this section include [28–31]. Consider the flat $(1+1)$ dimensional (for simplicity) Minkowski space

$$ds^2 = -dt^2 + dx^2 = -dudv; \qquad -\infty < t, x < \infty, \tag{1.2.4}$$

where null coordinates $u \equiv t - x$ and $v \equiv t + x$. The equation of motion of a free Hermitian massless scalar field is

$$(-\partial_t^2 + \partial_x^2)\phi = \partial_u \partial_v \phi = 0, \tag{1.2.5}$$

The solution is $e^{i(kx-\omega t)}$ with $\omega = |k|$. These modes are positive frequency with respect to the timelike Killing vector ∂_t i.e.,

$$\mathcal{L}_{\partial_t} e^{i(kx-\omega t)} = -i\omega e^{i(kx-\omega t)}. \tag{1.2.6}$$

We have

$$\begin{aligned} &e^{-i\omega u} - \text{right moving modes} \\ &e^{-i\omega v} - \text{left moving modes} \end{aligned} \tag{1.2.7}$$

For our purposes it is enough to consider only the right moving solution, which we expand in modes as

$$\phi_R = \int_0^\infty \frac{d\omega}{2\pi\sqrt{2\omega}}(a_\omega e^{-i\omega u} + a_\omega^\dagger e^{i\omega u}) \tag{1.2.8}$$

The non zero canonical commutators are

$$[a_\omega, a_{\omega'}^\dagger] = 2\pi\delta(\omega - \omega'), \tag{1.2.9}$$

and the Minkowski vacuum $|0\rangle_M$ is defined by

$$a_\omega|0\rangle_M = 0. \tag{1.2.10}$$

Rindler observer

The coordinate system well adapted to uniformly accelerated observers in flat spacetime is the Rindler coordinate system defined by

$$t = \rho\sinh(a\eta), \qquad x = \rho\cosh(a\eta), \qquad \rho > 0 \tag{1.2.11}$$

with the metric

$$ds^2 = -(a\rho)^2 d\eta^2 + d\rho^2 = e^{2a\xi}(-d\eta^2 + d\xi^2); \qquad \rho = \frac{e^{a\xi}}{a}. \tag{1.2.12}$$

This covers the right Rindler wedge defined by $x > |t|$. We will refer to this as region A. $\rho = constant$ are hyperbolae

$$x^2 - t^2 = \rho^2. \tag{1.2.13}$$

These are worldlines of uniformly accelerated observers.

We define the Rindler null coordinates $U \equiv \eta - \xi$ and $V \equiv \eta + \xi$, in which the metric takes the form

$$ds^2 = -(a\rho)^2 dU dV. \tag{1.2.14}$$

We note

$$u = -\frac{1}{a}e^{-aU}, \qquad v = \frac{1}{a}e^{aV}. \tag{1.2.15}$$

We label coordinates in region A with the subscript A henceforth. Similarly, coordinates spanning the left Rindler wedge $x < -|t|$ can be defined by flipping the sign of the right

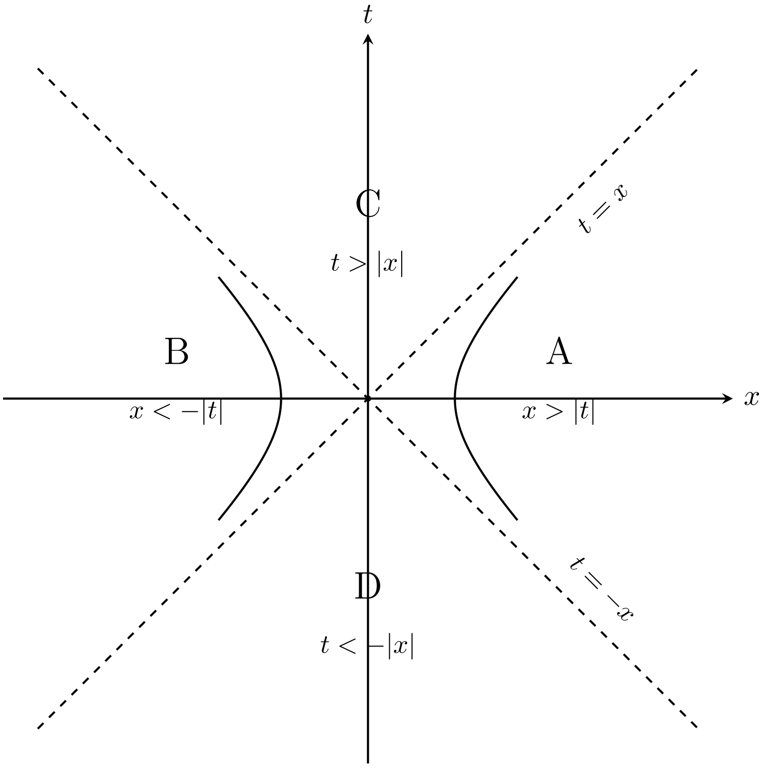


FIGURE 1.1: Minkowski spacetime divided by the null lines $t = \pm x$ into four regions. Region A is the right Rindler wedge, region B is the left Rindler wedge, while regions C and D are the future and past wedges, respectively. The dashed lines are the Rindler horizons.

hand side of (1.2.11) and (1.2.15). We will refer to this as region B, and denote the corresponding coordinates with the subscript B. We will also use Q as the placeholder for A or B. The null rays act as event horizons for Rindler observers in regions A and B.

∂_η is a Killing vector. We have

$$\partial_\eta = a(x\partial_t + t\partial_x) \tag{1.2.16}$$

This is the boost generator. It is timelike only in the wedge A and B. It is null on the null lines and spacelike in the regions C and D defined by $t > |x|$ and $t < -|x|$, respectively.

The equation of motion of a free Hermitian massless scalar field

$$(\partial^2_{\xi_Q} - \partial^2_{\eta_Q})\phi = \partial_{U_Q}\partial_{V_Q}\phi = 0, \tag{1.2.17}$$

is solved by $e^{i(k\xi_A - \Omega\eta_A)}$ in region A and $e^{i(k\xi_B - \Omega\eta_B)}$ in region B, with $\Omega = |k|$ i.e.,

$$\begin{aligned} &e^{-i\Omega U_Q} - \text{right moving mode in region Q i.e., A or B} \\ &e^{-i\Omega V_Q} - \text{left moving mode in region Q i.e., A or B} \end{aligned} \tag{1.2.18}$$

This is positive frequency with respect to the Killing vector ∂_η in region A and $-\partial_\eta$ in region B. These can be analytically continued to regions C and D labelled by $t > |x|$ and $t < -|x|$ respectively. Regions A through D span all of Minkowski space, hence these modes are as good as the Minkowski basis (1.2.7). We have

$$\phi_R = \int_0^\infty \frac{d\Omega}{2\pi\sqrt{2\Omega}} \left(b_\Omega^A e^{-i\Omega U_A} + b_\Omega^{A\dagger} e^{i\Omega U_A} + A \leftrightarrow B \right) \tag{1.2.19}$$

We have

$$b_\Omega^Q |0\rangle_R = 0 \tag{1.2.20}$$

and

$$[b_\Omega^A, b_{\Omega'}^{A\dagger}] = 2\pi\delta(\Omega - \Omega'), \qquad [b_\Omega^B, b_{\Omega'}^{B\dagger}] = 2\pi\delta(\Omega - \Omega') \tag{1.2.21}$$

and all other commutators are zero.

Notice that the Rindler vacuum $|0\rangle_R$ is different from the Minkowski vacuum $|0\rangle_M$ i.e., $a_\omega |0\rangle_R \neq 0$.

We can relate the Minkowski operators $a_\omega^{(\dagger)}$ to the Rindler operators $b_\Omega^{(\dagger)}$ using the mode expansion of the solution, given in (1.2.8), (1.2.19), and the relations between the Minkowski and Rindler null coordinates. We have,

$$b_\Omega^Q = \int_0^\infty \frac{d\omega}{2\pi} \left(\alpha_{\Omega\omega}^Q a_\omega + \beta_{\Omega\omega}^Q a_\omega^\dagger \right) \tag{1.2.22}$$

where

$$\begin{aligned}
\alpha_{\Omega\omega}^A =& \frac{1}{a} \left(\frac{\Omega}{\omega}\right)^{1/2} \left(\frac{\omega}{a}\right)^{i\Omega/a} e^{\pi\Omega/2a} \Gamma\left(-\frac{i\Omega}{a}\right), \\
\beta_{\Omega\omega}^A =& \frac{1}{a} \left(\frac{\Omega}{\omega}\right)^{1/2} \left(\frac{\omega}{a}\right)^{i\Omega/a} e^{-\pi\Omega/2a} \Gamma\left(-\frac{i\Omega}{a}\right), \\
\alpha_{\Omega\omega}^B =& e^{-i\Omega\pi/a} \alpha_{\Omega\omega}^A, \qquad \beta_{\Omega\omega}^B = e^{-i\Omega\pi/a} \beta_{\Omega\omega}^A.
\end{aligned} \tag{1.2.23}$$

Notice that,

$$\frac{|\beta_{\Omega\omega}^Q|^2}{|\alpha_{\Omega\omega}^Q|^2} = e^{-2\pi\Omega/a}. \tag{1.2.24}$$

We have

$$
\begin{aligned}
{}_M\langle 0|b^{Q\dagger}_{\Omega} b^{Q}_{\Omega'}|0\rangle_M &= \int_0^\infty \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\beta^{Q*}_{\Omega\omega}\beta^{Q}_{\Omega'\omega'}\,{}_M\langle 0|a_\omega a^\dagger_{\omega'}|0\rangle_M, \\
&= \int_0^\infty \frac{d\omega}{2\pi}\beta^{Q*}_{\Omega\omega}\beta^{Q}_{\Omega'\omega}, \\
&= \frac{2\pi\delta(\Omega-\Omega')}{e^{2\pi\Omega/a}-1}
\end{aligned}
\tag{1.2.25}
$$

This is a thermal two point function with temperature

$$
T_R = \frac{a}{2\pi}. \tag{1.2.26}
$$

Using the relation between the Rindler and Minkowski operators (1.2.22), we can write the Minkowski vacuum in terms of the Rindler vacuum. We have

$$
|0\rangle_M = \mathcal{N}\ \exp\left[\int_0^\infty d\Omega\ e^{-\pi\Omega/a}\ b^{A\dagger}_\Omega\, b^{B\dagger}_\Omega\right]\ |0\rangle_R \tag{1.2.27}
$$

where $\mathcal{N}$ is a normalization factor that can be fixed by imposing the normalization

$$
{}_M\langle 0|0\rangle_M = 1. \tag{1.2.28}
$$

Since different frequency modes are independent (1.2.21), the full vacuum can be expressed as a tensor product over frequencies i.e.,

$$
|0\rangle_M = \bigotimes_\Omega |0_\Omega\rangle_M = \bigotimes_\Omega \mathcal{N}_\Omega\ \exp\left[e^{-\pi\Omega/a}\ b^{A\dagger}_\Omega\, b^{B\dagger}_\Omega\right]\ |0\rangle_R \tag{1.2.29}
$$

with

$$
\mathcal{N} = \prod_\Omega \mathcal{N}_\Omega. \tag{1.2.30}
$$

By expanding the exponential in (1.2.29), we have

$$
|0_\Omega\rangle_M = \mathcal{N}_\Omega \sum_{n=0}^\infty e^{-\pi\Omega n/a}\ |n_\Omega\rangle_A \otimes |n_\Omega\rangle_B. \tag{1.2.31}
$$

where the number basis are defined as follows

$$
|n_\Omega\rangle_A \otimes |n_\Omega\rangle_B \equiv \frac{\left(b^{A\dagger}_\Omega\, b^{B\dagger}_\Omega\right)^n}{n!}\ |0\rangle_R \tag{1.2.32}
$$

Imposing the normalization, we have

$$
1 = {}_M\langle 0_\Omega|0_\Omega\rangle_M = \mathcal{N}_\Omega^2 \sum_{n=0}^\infty e^{-2\pi\Omega n/a} = \mathcal{N}_\Omega^2\, \frac{1}{1-e^{2\pi\Omega/a}} \tag{1.2.33}
$$

which implies,

$$\mathcal{N}_\Omega = \left(1 - e^{2\pi\Omega/a}\right)^{1/2}, \tag{1.2.34}$$

and,

$$|0\rangle_M = \prod_\Omega \left(1 - e^{-2\pi\Omega/a}\right)^{1/2} \sum_{n=0}^{\infty} e^{-\pi\Omega n/a}\, |n_\Omega\rangle_A \otimes |n_\Omega\rangle_B. \tag{1.2.35}$$

We define $q = e^{-2\pi\Omega/a}$ and the density matrix

$$\rho^{(\Omega)} = |0_\Omega\rangle_M\, {}_M\langle 0_\Omega|\,. \tag{1.2.36}$$

Tracing with respect to the region B and using ${}_B\langle m|n\rangle_B = \delta_{mn}$, gives

$$\rho_A^{(\Omega)} = \mathrm{Tr}_B\, \rho^{(\Omega)} = (1-q)\sum_{n=0}^{\infty} q^n |n\rangle_{AA}\langle n|. \tag{1.2.37}$$

In region A, we define the number operator $N_A^{(\Omega)}$ and Hamiltonian $H_A^{(\Omega)}$ as

$$N_A^{(\Omega)} = b_\Omega^{A\,\dagger}\, b_\Omega^A, \qquad H_A^{(\Omega)} = \Omega\, N_A^{(\Omega)}. \tag{1.2.38}$$

We can now express,

$$\rho_A^{(\Omega)} = \frac{1}{Z_\Omega} e^{-\beta H_A^\Omega}; \qquad \beta = \frac{2\pi}{a},\; Z_\Omega = \frac{1}{1-q}. \tag{1.2.39}$$

Since number operators for different Ω commute, we can express the full Minkowski vacuum as

$$\rho_A = |0\rangle_M\, {}_M\langle 0| = \frac{1}{Z} e^{-\beta K_A}, \tag{1.2.40}$$

where K_A is the boost generator (corresponding to the Killing vector (1.2.16)), restricted to the wedge A, given by

$$K_A = \int_{x>0} dx\, x\, T_{tt}(t=0,x). \tag{1.2.41}$$

In terms of modes (up to normal ordering), we have

$$K_A = \int_0^\infty d\Omega\, \Omega\, b_\Omega^{A\,\dagger}\, b_\Omega^A. \tag{1.2.42}$$

The key takeaways from this are the following:

- The Minkowski vacuum is not the same as the Rindler vacuum, since the timelike Killing vectors and choice of modes are different for the two observers.

The Minkowski vacuum is an entangled state formed by states in the regions A and B. The two wedges are causally disconnected from each other. Tracing out the degrees of freedom in wedge B gives a thermal density matrix in wedge A.

The presence of a horizon was crucial in getting a mixed state in one of the wedges.

- The Rindler observer experiences a temperature $a/(2\pi)$.

Black hole temperature

The key assumption in general relativity is the equivalence principle. In any curved spacetime which is a solution to the Einstein Field equations, in any local patch of spacetime one can find an inertial frame of reference by locally accelerating with a particular acceleration (freely falling frames). In other words being at rest with respect to a global coordinate system in a curved spacetime is the same as accelerating. Moreover black holes have an event horizon. These are the necessary conditions to get a mixed state (which couild be thermal) as seen from the Rindler setup.

Consider the Schwarzschild black hole solution to the Einstein field equation, given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_{d-1}^2. \tag{1.2.43}$$

where $d\Omega_{d-1}^2$ is the metric on the unit $(d-1)$-sphere, with $r \in [0, \infty)$ and $t \in (-\infty, \infty)$. In d spacetime dimensions, the blackening metric function $f(r)$ is given by

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^{d-3}. \tag{1.2.44}$$

Near the horizon $r = r_h$, we have

$$f(r) = f'(r_h)(r - r_h) + O((r - r_h)^2). \tag{1.2.45}$$

Let us define $d\rho = dr/\sqrt{f(r)}$. Near the horizon we have

$$\rho(r) \approx \int_{r_h}^{r} \frac{dr'}{\sqrt{f'(r_h)(r' - r_h)}} \approx \frac{2}{\sqrt{f'(r_h)}}\sqrt{r - r_h} \tag{1.2.46}$$

The metric near the horizon takes the form

$$ds^2 \approx -(\kappa\rho)^2 dt^2 + d\rho^2 + r_h^2 d\Omega_{d-1}^2; \qquad \kappa = \frac{f'(r_h)}{2}. \tag{1.2.47}$$

As expected, the metric near the horizon looks like a Rindler metric (1.2.12), with $a = \kappa$. From the Rindler temperature (1.2.26), we can infer that an asymptotic Schwarzschild observer would see a temperature

$$T_{BH} = \frac{a}{2\pi} = \frac{\kappa}{2\pi} = \frac{f'(r_h)}{4\pi} \tag{1.2.48}$$

In $d = 4$, $f(r) = (1 - \frac{r_h}{r})$ with $r_h = 2M$, and we have $T_{BH} = 1/(4\pi r_h) = 1/(8\pi M)$. Using $dS = dM/T$, the associated entropy of the black hole is given by

$$S = \int_0^M dM' \, 8\pi M' = 4\pi M^2 = \frac{A(r_h)}{4} \tag{1.2.49}$$

where we have fixed the integration constant by imposing $S(M = 0) = 0$.

In classical systems, changing the notion of time changes the notion of frequency. However, different classical observers, even with different notions of time measurement, still agree on the particle content of a system. When observers probe systems at scales where quantum mechanical effects become important, this need not be true. The quantum mechanical uncertainty relation $\Delta E \, \Delta t \geq 1/2$, together with the fact that the Hamiltonian H generates time evolution, $U(t) = e^{-iHt}$, reflects the deep connection between the notions of energy E and time t in quantum theory. However, this alone is not sufficient for two observers with different notions of time to disagree on the notion of vacuum. A quantum field, on the other hand, possesses an infinite number of oscillator degrees of freedom, one at every point in space. Consequently, as we have seen in this section, different mode decompositions of the field with respect to different notions of time evolution can lead to different definitions of creation and annihilation operators, and hence different vacuum states. Observers associated with different notions of time may therefore disagree on the particle content of a state. Thus, while one observer may perceive a state as vacuum, another observer with a different notion of time may perceive non-zero energy or particles.

The thermodynamic behavior of certain systems with causal horizons suggests that they possess a microscopic notion of entropy and information. To discuss the implications of this more precisely, we briefly turn to the question of how information is quantified in quantum systems.

1.2.2 Information and (entanglement) entropy

In everyday language, information refers to processed, organised, and structured data. However, in order to be more precise, we need to have a way to quantify the information content of a system, which led to Claude Shannon's definition of *self-information* [32], which satisfies the following axioms:
If p is the probability associated with an event and $I(p)$ is a measure of information/ ignorance, then

- $I(p) \geq 0$ - Information is non-negative
- $I(1) = 0$ - events that definitely occur carry no information

- $I(p_1.p_2) = I(p_1) + I(p_2)$ - The information associated with two independent events is the sum of the information content in the individual events.

- $I(p)$ is monotonically decreasing and continuous in p.

These axioms lead to the solution $I(p) = -\log_b(p)$ for $b > 0$. Hence, information is a measure of the amount of ignorance. The expectation of the information content of a random variable X with probability distribution $p_X(x)$ is,

$$S := \sum_x p_X(x) I_X(p_X(x)) = \sum_x -p_X(x)\log(p_X(x)) \tag{1.2.50}$$

This quantity is called the *Shannon entropy.* Shannon noiseless coding theorem states that the state of N identically distributed independent random variables each with entropy S can be encoded using $NS/\log 2$ bits as $N \to \infty$. Thus, entropy also captures the average number of bits required to encode the data.

Quantum systems: In quantum mechanical systems, we usually do not have the knowledge of the exact state. So, from an experimental perspective, it is natural to consider an ensemble of states $|\psi_i\rangle$, each with a classical probability p_i (we will also see later that subsystems of pure states can be interpreted similarly). We associate a matrix with this ensemble called the 'density matrix' (ρ)

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{1.2.51}$$

with the following properties: $\rho^\dagger = \rho$, $\rho \geq 0$, $\mathrm{Tr}(\rho) = 1$. The expectation value corresponding to a measurable operator $'O'$ can be shown to be $\mathrm{Tr}(\rho O)$. The definite state $|\psi\rangle\langle\psi|$ is called **pure** and the rest are **mixed**.

We can assign an information measure (the Shannon entropy) to the quantum system based on the classical probability associated with the ambiguity of the quantum state

$$S(\rho) = -\sum p_i \log(p_i) = -\mathrm{Tr}(\rho\log(\rho)) \tag{1.2.52}$$

This is the **von Neumann entropy** [33] of ρ. It is a measure of the mixedness of the system. A useful property to note is that the entropy is invariant under a unitary transformation U,

$$S(U\rho U^{-1}) = S(\rho) \tag{1.2.53}$$

Entanglement: A unique feature of a quantum mechanical system is entanglement or non local correlations. For a joint system AB made up of individual independent Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the Hilbert space is the tensor product i.e., $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Given a state ρ_{AB} in $\mathcal{H}_{AB}$, the effective state that reproduces the expectation value of operator $O_A \otimes I_B$ is the marginal state $\rho_A = \mathrm{Tr}_B \rho_{AB}$. $|\psi\rangle_{AB}$ in $\mathcal{H}_{AB}$ cannot always be factorised as $|\psi\rangle_A \otimes |\psi\rangle_B$ in $\mathcal{H}_A \otimes \mathcal{H}_B$. Also, $\mathcal{H}_{AB}$ does not always factorise (for example, in gauge theories, subsystems might not be independent due to constraints). A pure state that cannot be factorised is called **entangled**.

Given a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, there exist orthonormal sets $\{|a\rangle_A\} \subset \mathcal{H}_A$ and $\{|a\rangle_B\} \subset \mathcal{H}_B$ such that

$$|\psi\rangle = \sum_a \sqrt{p_a}\, |a\rangle_A \otimes |a\rangle_B, \qquad p_a \geq 0, \qquad \sum_a p_a = 1. \tag{1.2.54}$$

and the corresponding marginals are mixed

$$\rho_A = \sum_a p_a |a\rangle_A \langle a|, \quad \rho_B = \sum_a p_a |a\rangle_B \langle a|, \tag{1.2.55}$$

therefore, the subsystem has a non zero entropy (unlike the classical case where subsystems have zero entropy when the joint system has zero entropy). On the other hand, given an unentangled pure state, the marginals are also pure, and hence the subsystem entropy is zero. Therefore, for pure joint states, the von Neumann entropy of the subsystems is a measure of the amount of entanglement in the joint state. Hence, for pure joint states, the subsystem entropy is usually also called **Entanglement entropy**. However, if one considers a mixed joint state, the marginals are also mixed and have an entropy. So it is not necessarily true that a subsystem with entropy implies that the joint state is entangled (it is true if the joint state is pure).

We also notice from (1.2.55), that if we start with a pure entangled state, then the marginals have the same entropy, i.e.,

$$S(A) = S(A^c = B) \tag{1.2.56}$$

Purification: A mixed state on a system A is described by a density matrix ρ_A on $\mathcal{H}_A$. A *purification* of ρ_A is a pure state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ for some auxiliary system B such that

$$\rho_A = \mathrm{Tr}_B\big(|\psi\rangle\langle\psi|_{AB}\big)\,. \tag{1.2.57}$$

To construct one, diagonalize ρ_A as

$$\rho_A = \sum_a p_a\, |a\rangle_A \langle a|_A, \qquad p_a \geq 0, \qquad \sum_a p_a = 1, \tag{1.2.58}$$

and introduce an ancilla Hilbert space $\mathcal{H}_B$ with orthonormal basis $\{|a\rangle_B\}$. Then

$$|\psi\rangle_{AB} = \sum_a \sqrt{p_a}\, |a\rangle_A \otimes |a\rangle_B \tag{1.2.59}$$

is a purification, since

$$\mathrm{Tr}_B(|\psi\rangle\langle\psi|_{AB}) = \sum_a p_a \, |a\rangle_A\langle a|_A = \rho_A. \tag{1.2.60}$$

If $\mathrm{rank}(\rho_A) = r$, one may choose $\dim \mathcal{H}_B = r$ (minimal purification). Purifications are not unique: if $|\psi\rangle_{AB}$ purifies ρ_A, then so does

$$|\psi'\rangle_{AB} = (\mathbb{I}_A \otimes U_B)\, |\psi\rangle_{AB} \tag{1.2.61}$$

for any unitary U_B acting on $\mathcal{H}_B$.

Entanglement entropy of a quantum field: A regulated relativistic quantum field has a degree of freedom at every point in spacetime. The notion of a subsystem is naturally tied to a spatial subregion A of a Cauchy slice Σ (see Fig (1.2)). More precisely, to A one associates the *local algebra of observables* $\mathcal{A}(A)$ generated by field operators smeared with test functions supported in A. By causality and unitary time evolution, the physics in A is determined by its *causal domain of dependence* $D(A)$, and one may equivalently work with the algebra $\mathcal{A}(D(A))$. Given a global state ρ, the reduced state on A is defined as the restriction of the state to $\mathcal{A}(D(A))$. The entanglement entropy between A and its complement is then the von Neumann entropy of this reduced state. In continuum QFT the von Neumann entropy is UV-divergent and, strictly speaking, the Hilbert space need not factorize as $\mathcal{H} \neq \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$; nevertheless with a UV regulator (or using the split property) one can define a reduced density matrix ρ_A and

$$S(A) \equiv -\mathrm{Tr}(\rho_A \log \rho_A). \tag{1.2.62}$$

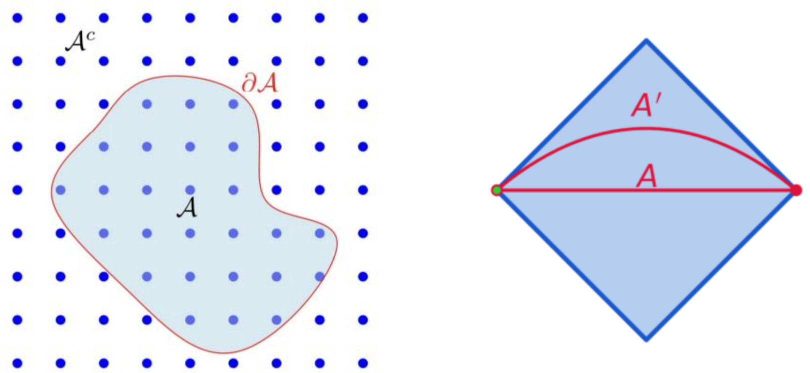


FIGURE 1.2: Entanglement Entropy in a QFT is associated with a subregion and spatial subregions with the same domain of dependence have the same entanglement entropy.

A key consequence of causal evolution is that if two spatial regions $A \subset \Sigma$ and $A' \subset \Sigma'$ have the same domain of dependence,

$$D(A) = D(A'), \tag{1.2.63}$$

then the associated local algebras coincide, $\mathcal{A}(D(A)) = \mathcal{A}(D(A'))$, and the restricted state is the same (up to the natural unitary identification by time evolution). Hence any regulator defined entanglement entropy assigned to the causal domain agrees (see Fig (1.2)):

$$\begin{gathered} S(A) = S(A'), \\ \text{whenever } D(A) = D(A'). \end{gathered} \tag{1.2.64}$$

In a local relativistic QFT in D spacetime dimensions, the leading UV divergence of the entanglement entropy for a smooth region A is an area law,

$$S(A) = \alpha \, \frac{\text{Area}(\partial A)}{\epsilon^{\,D-2}} + \cdots, \tag{1.2.65}$$

reflecting the dominance of short-distance entanglement across the entangling surface [34–36].

Rényi entropy and the replica trick: It is difficult to calculate the von Neumann entropy since it involves the logarithm of the density matrix. There is a neat trick to calculate the entanglement entropy by calculating polynomials involving the density matrix

$$S_q = \frac{1}{1-q} \log(\text{Tr}_A \rho_A^q). \tag{1.2.66}$$

The von Neumann entropy is then given by

$$\begin{aligned} S_{\text{QFT}} &= \lim_{q\to 1} S_q = \lim_{q\to 1} -\frac{\partial}{\partial q} \log(\text{Tr}_A \rho_A^q) = \lim_{q\to 1} -\frac{\text{Tr}_A(\rho_A^q \log \rho_A)}{\text{Tr}\rho_A^q} \\ &= -\text{Tr}(\rho_A \log \rho_A), \end{aligned} \tag{1.2.67}$$

i.e., we first compute $\text{Tr}_A \rho_A^q$ for positive integers and then we analytically continue q to general complex values in order to take the limit $q \to 1$. The existence and uniqueness of a proper analytic continuation is hence necessary for the replica trick to work (which is not well understood in certain systems, e.g. spin glass). S_q is called the q^{th} Rényi entropy and the trick is called the Replica trick [36–38]. We will discuss this in more detail later.

Fine-grained vs. coarse-grained entropy. The *fine-grained* entropy of a quantum system in state ρ is its von Neumann entropy. Given a set of *coarse-grained observables* $\{A_i\}$ and observed expectation values $\langle A_i \rangle$, one defines the *coarse-grained* entropy by maximizing the von Neumann entropy over all density matrices consistent with these data:

$$S_{\text{coarse}}(\{\langle A_i\rangle\}) \;\equiv\; \max_{\tilde\rho}\{-\mathrm{Tr}(\tilde\rho\log\tilde\rho)\} \quad \text{s.t.} \quad \mathrm{Tr}(\tilde\rho)=1,\; \mathrm{Tr}(\tilde\rho\, A_i)=\langle A_i\rangle \;\;\forall i. \tag{1.2.68}$$

The maximizer $\tilde\rho_*$ is the unique maximum-entropy state compatible with the constraints (a generalized Gibbs state),

$$\tilde\rho_* = \frac{1}{Z}\exp\left(-\sum_i \lambda_i A_i\right), \qquad Z = \mathrm{Tr}\exp\left(-\sum_i \lambda_i A_i\right), \tag{1.2.69}$$

with Lagrange multipliers λ_i fixed by $\mathrm{Tr}(\tilde\rho_* A_i) = \langle A_i\rangle$ [39]. By construction,

$$S_{\text{coarse}} \geq S_{\text{fine}}, \tag{1.2.70}$$

because the true state ρ is itself an allowed candidate in the maximization whenever it reproduces the same coarse-grained data $\{\langle A_i\rangle\}$.

Having established the key concepts on how to quantify information, we will now proceed back to the physics of black holes and gravity.

1.2.3 Black hole information paradox

Even before quantising fields on a curved background, there were strong hints that black holes should admit a thermodynamic interpretation. Classical general relativity endows stationary black-hole solutions with a set of laws of black-hole mechanics that closely parallel the laws of thermodynamics. In equilibrium, the surface gravity κ is constant over the event horizon (the zeroth law), suggesting an analogy with temperature.[1] Moreover, for Kerr-Newman black holes, one finds a variational identity (the first law)

$$dM = \frac{\kappa}{8\pi}\, dA + \Omega_H\, dJ + \Phi_H\, dQ. \tag{1.2.71}$$

The angular velocity of the horizon Ω_H is defined by requiring that the Killing vector field

$$\chi^\mu = (\partial_t)^\mu + \Omega_H\,(\partial_\phi)^\mu \tag{1.2.72}$$

becomes null at the event horizon r_+,

$$\chi^\mu\chi_\mu\Big|_{r=r_+} = 0. \tag{1.2.73}$$

[1] Extremal black holes provide an important caveat: they can have $\kappa = 0$.

This vector field χ^μ is the *horizon-generating Killing vector.*

The electric potential at the horizon is defined as

$$\Phi_H = -\chi^\mu A_\mu\Big|_{r=r_+}, \tag{1.2.74}$$

where A_μ is the electromagnetic gauge potential. This quantity measures the electrostatic potential difference between the horizon and infinity.

Clearly, the quantities Ω_H and Φ_H are thermodynamic potentials associated with angular momentum and charge, respectively. (1.2.71) mirrors the thermodynamic relation, $dE = T\,dS + \Omega\,dJ + \Phi\,dQ$, upon identifying the temperature $T \sim \kappa$, and the entropy $S \sim A$. Furthermore, by assuming suitable energy and cosmic-censorship conditions, Hawking proved the area theorem [40]: the horizon area is non-decreasing in any classical process,

$$\delta A \geq 0, \tag{1.2.75}$$

reminiscent of the second law of entropy.

A further motivation comes from the second law of thermodynamics: if black holes neither radiate nor carry entropy, then dropping ordinary matter into a black hole would remove its entropy from the exterior world, apparently allowing the total entropy accessible to an external observer to decrease. Bekenstein resolved this tension [1, 2] by proposing that black holes possess an entropy proportional to the horizon area, leading to the *generalised second law* in which

$$S_{\text{gen}} = S_{\text{outside}} + S_{\text{BH}} \tag{1.2.76}$$

never decreases. Hawking later placed this picture on firm grounds by quantising quantum fields on a black-hole background and showing that black holes emit thermal radiation with temperature (see section (1.2.1)).

We also assume the *Central Dogma*: [41] As seen from outside the event horizon, a black hole can be described as a quantum system with $\text{Area}(r_h)/(4)$ degrees of freedom, which evolves unitarily under time evolution.

These observations lead to the following paradox. If we start with a black hole formed by the collapse of a pure state with zero von Neumann entropy (for simplicity), then Hawking's semiclassical calculation shows that the black hole emits thermal radiation (a mixed state). At every point in time in the evaporation process, the fine-grained entropy of the Hawking radiation is equal to the entropy of the black hole (1.2.56). We also know that the thermodynamical entropy of the black hole is given by $\text{Area}(r_h)/4$. Initially, during the evaporation process, the fine-grained entropy of the Hawking radiation (which equals the entropy of the black hole) will be less than the thermodynamic entropy of the black hole. This is fine since the fine-grained entropy is always less than the coarse-grained entropy by definition. If the evolution is Unitary then the whole system must remain in a pure state (1.2.53). It is possible that the mixed radiation state is purified by the leftover black hole state, such that the whole system remains pure. However, as the black hole continues to radiate thermal Hawking radiation, at a particular time called the *Page time* $t = t_{page}$, the entropy of the radiation exceeds the thermodynamic entropy of the black hole. After this time, there are not enough degrees of freedom in the black hole to purify the radiation. The whole system is henceforth in a mixed state with non-zero von Neumann entropy. This violates unitarity, since the entanglement entropy must remain invariant under unitary evolution (1.2.53). In the extreme case when the entire black hole evaporates, we are left with only mixed thermal radiation.

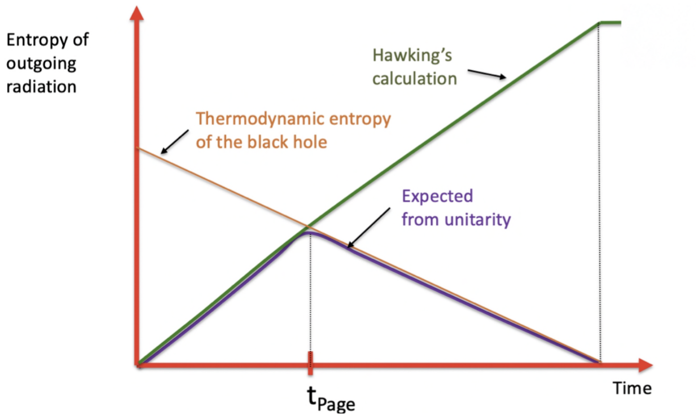


FIGURE 1.3: The orange curve captures the thermodynamic area entropy of the black hole, the green curve captures the entropy of the outgoing thermal Hawking radiation, which here is also equal to the fine-grained entropy of the black hole, and the purple curve captures the expected behavior of the black hole entropy if unitarity is preserved.

1.2.4 Attempts at a resolution

The paradox also hints at its resolution. If the fine grained entropy of the outgoing radiation (also the entropy of the black hole) is always less than the thermodynamic entropy of the black hole as shown in the purple curve in figure (1.3), then the black hole state can purify the radiation state, leading to a pure state in the whole system. This curve is called the *Page curve* [42].

The semiclassical picture does not quantise the gravitational degrees of freedom. It is possible that when we quantise it, correlations between these degrees of freedom and the outgoing radiation can purify it and reproduce the Page curve. We will see more on this resolution later on.

I briefly discuss below some developments over the years towards a resolution to the black hole information paradox

- *Modify quantum mechanics*: One possible resolution is to accept that the black hole evaporation is a non unitary process. However, this would imply that we cannot reconstruct the initial state that leads to a particular configuration.

- *Stretched Horizon and black hole complementarity*: If one tries to keep both (i) a smooth horizon for an infalling observer and (ii) recovery of information in the Hawking radiation for an outside observer, one is led to an apparent cloning tension: the same quantum information seems to be both inside (accessible to the infaller) and outside (encoded in the radiation).

  [43] proposes a phenomenological resolution of the black hole information problem based on *black hole complementarity*, implemented via a *stretched-horizon* (membrane) description for a distant observer. They formulate three postulates: (i) black hole formation and evaporation admit a *unitary* S-matrix mapping infalling matter to outgoing Hawking-like radiation; (ii) semiclassical field equations are valid outside the stretched horizon to good approximation; and (iii) a black hole of mass M behaves as a quantum system with a finite-dimensional Hilbert space of size

  $$\dim \mathcal{H}(M) \sim e^{S(M)}\,, \tag{1.2.77}$$

  providing a statistical basis for black-hole thermodynamics. The apparent conflict with a smooth horizon arises only if one insists on a single global state on a Cauchy slice that factorizes as

  $$\mathcal{H} = \mathcal{H}_{\text{bh}} \otimes \mathcal{H}_{\text{out}}\,, \tag{1.2.78}$$

  thereby treating interior and exterior correlations as simultaneously accessible. They argue this "superobserver" assumption is operationally unphysical: no single observer can both recover the information from Hawking radiation at infinity and also verify an independent interior copy. In their picture, the stretched horizon carries the requisite microscopic degrees of freedom whose coarse-grained dynamics reproduces the membrane's dissipative thermodynamic behavior (absorption, thermalization, and re-emission of information) for the outside observer, while an infalling observer experiences no special local signal at horizon crossing; the two descriptions are complementary rather than simultaneously valid.

  Complementarity becomes sharper when combined with quantum information theory: it's not just that information comes out, but when and how fast it becomes accessible.

  Patrick Hayden and John Preskill analyzed information recovery assuming unitary, rapidly mixing dynamics [44]. They found that once a black hole is past the Page time, newly thrown-in quantum information can be returned in the Hawking radiation after only a scrambling time (up to model-dependent overhead), i.e.

surprisingly quickly. This is often presented as a consistency check: complementarity can remain logically viable if the timescales/energetics prevent an observer from both collecting enough radiation to decode and then jumping in to witness duplication.

Yasuhiro Sekino and Leonard Susskind argued that black holes are among nature's fastest scramblers [45]: they can mix information over their microscopic degrees of freedom in a time scaling like $\beta \log S$ (parametrically logarithmic in the entropy). This supports the picture that information dropped in at one point on the horizon can rapidly delocalize over the stretched horizon, so later Hawking quanta can carry it out without requiring naive local propagation from deep inside.

- *Firewalls:* In [46], Almheri, Marolf, Polchinski, Sully (AMPS), propose that the following three statements cannot all be true: (i) Hawking radiation is in a pure state, (ii) the information carried by the radiation is emitted from the region near the horizon, with low energy effective field theory valid beyond some microscopic distance from the horizon, and (iii) the infalling observer encounters nothing unusual at the horizon. They show that assuming all three statements leads to an entanglement monogamy contradiction for old black holes. Their conservative resolution is to give up statement (iii) and instead posit that the infalling observer burns up at the horizon.

- *ER=EPR:* [47] discusses solutions in which two distant black holes are connected through the interior via a wormhole, or Einstein-Rosen bridge. These solutions can be interpreted as maximally entangled states of two black holes that form a complex EPR pair. They suggest that similar bridges might be present for more general entangled states. They discuss that in the case of entangled black holes one can formulate versions of the AMPS paradoxes and resolve them.

- *Nonviolent nonlocality:* In [48], Steven B. Giddings starts from the premise that if black hole evaporation is unitary, then when described in the usual semiclassical geometric language, some nonlocal mechanism must transfer quantum information from the black hole interior into the outgoing radiation. The central proposal is that this required nonlocality need not be violent to generic infalling observers: instead, information could leak out through additional, effectively nonlocal interactions that couple the interior state to relatively soft modes in the near-horizon atmosphere, producing only modest departures from Hawking's leading order thermality while still restoring purity of the full outgoing state. The paper sketches simplified, parameterized models for such transfer, argues that consistency requirements constrain the allowed form of the couplings, and notes phenomenological implications e.g., sufficiently sensitive infalling measurements could detect deviations from Hawking's predictions, and generic models may imply slightly faster decay than strictly semiclassical expectations.

- *Microstates and fuzzballs [49]:* The fuzzball program [49], developed within string theory, posits that individual black hole microstates are not well described by the semiclassical geometry with a smooth horizon and an "empty" interior. Instead, typical microstates are expected to be horizon-sized, structureful configurations, so the would-be interior is replaced by a complicated but regular stringy bound state whose structure extends all the way to where the horizon would have formed. In this picture, the thermodynamic entropy arises from counting these distinct microstate geometries, while Hawking emission is modified at the horizon scale: radiation originates from (and can interact with) the microstate's structure at the fuzzball surface, allowing information about the state to be encoded in the outgoing quanta. The semiclassical black hole with a smooth horizon is then interpreted as an effective, coarse-grained description obtained by averaging over many such microstates.

- *Soft hair*: [50] argues that asymptotically flat space BMS symmetries and their conservation laws require that black holes carry a large amount of soft supertranslation hair. They give an explicit description of soft hair in terms of soft gravitons or photons on the black hole horizon, and show that complete information about their quantum state is stored on a holographic plate at the future boundary of the horizon.

  How far this mechanism alone goes toward a full unitary evaporation is debated, but it remains a prominent observation in the broader landscape.

- *Remnants*: Remnant scenarios posit that evaporation halts or leaves a long-lived Planck-scale object that stores the information. The standard concern is that accommodating arbitrarily large information would require an enormous (often effectively infinite) internal degeneracy. [51] discusses one such proposal.

- *Singularity boundary condition:* [52] enforces a special boundary condition at the singularity to produce an effectively unitary S-matrix for evaporation.

- *Holography:* For AdS black holes, the AdS/CFT correspondence [6] (will be discussed in more detail later) provides a nonperturbative definition in which the dual boundary CFT evolves unitarily, so any apparent information loss must come from an incomplete bulk semiclassical treatment. We will discuss in great detail the proposals to complete the semiclassical picture consistent with the Page curve, using holography.

- [53] argues that a key premise behind many Page curve narratives: the idea that the gravitational Hilbert space factorizes cleanly into an interior and exterior along the radial direction, is not correct in quantum gravity because the algebra of asymptotic observables is complete (in both AdS and asymptotically flat settings). They contend that, in a standard theory of gravity, information therefore does not literally emerge from the black hole at late times in the way the Page curve story suggests, rather, it is already encoded in the asymptotic algebra, and (in principle) the entire interior can be reconstructed from exterior data. The paper explains that one can still obtain Page curves by effectively removing the Hamiltonian from the accessible exterior algebra: interpreted operationally as a detector with a "blind spot" that cannot measure part of the asymptotic region, or (in special discussions of null infinity) by discarding the Hamiltonian despite its physical accessibility, so that the resulting Page curve tracks a redistribution of information between measured and unmeasured degrees of freedom rather than fundamental information recovery.

1.2.5 Holography and entanglement

The observation that the black hole entropy scales as its horizon area (as opposed to the usual volume scaling of entropy), suggests that the maximum number of independent degrees of freedom in a gravitating region is bounded by its boundary area. [4] sharpened this into a dimensional reduction/entropy bound viewpoint, and [5] elevated it to the **holographic principle**:

the idea that quantum gravity in $(d+1)$ dimensions should be equivalent to a non-gravitational theory living on a $d-$dimensional boundary.

A concrete realisation of holography in asymptotically AdS spacetimes (AAdS) was first realised in [6]. The AdS/CFT correspondence proposed by them states the following: in the large-N limit, certain conformal field theories are exactly equivalent to string (or supergravity) theory on an AdS $\times$ compact space background, so bulk gravitational physics is fully encoded in a lower-dimensional non-gravitational QFT.

[7, 8] made it computationally concrete. For every bulk field in AdS, there is a dual gauge invariant boundary operator. Bulk gauge fields correspond to boundary symmetry currents. Asymptotically Anti de Sitter spacetimes have a boundary at spatial infinity where one needs to specify boundary conditions for the bulk field. The on-shell bulk partition functional with the boundary value of the bulk field as source acts as the generating functional for boundary QFT correlation functions

$$\mathcal{Z}_{SUGRA}[\phi_{(0)}] = \int_{\Phi\sim\phi_{(0)}} D\Phi \,\exp(-S[\Phi]) = \langle \exp\left(-\int_{\partial AAdS} \phi_{(0)}\mathcal{O}\right)\rangle_{QFT} \tag{1.2.79}$$

where the expectation value on the right hand side is over the QFT path integral. In the leading saddle point approximation, this relation becomes

$$S^{on\ shell}[\phi_{(0)}] = -W_{QFT}[\phi_{(0)}] \tag{1.2.80}$$

i.e., the on shell supergravity action is the generating function of QFT connected graphs.

In AdS/CFT, the boundary CFT is unitary and hence there is no paradox.

Ryu–Takayanagi (RT) and Hubeny Rangamani Takayanagi (HRT) formulas [9, 10]: In AdS/CFT, the RT and HRT prescriptions give a geometric computation of the entanglement entropy of a boundary subregion A in states admitting a semiclassical bulk dual. In a *time-reflection symmetric* (static) bulk geometry, one may choose the corresponding time symmetric bulk Cauchy slice, and the entanglement entropy is given by the Ryu Takayanagi (RT) formula

$$S(A) = \min_{R\sim A} \frac{\text{Area}(R)}{4G\hbar}, \tag{1.2.81}$$

where R is a codimension-2 bulk *minimal* surface anchored on the entangling surface ∂A and *homologous* to A. The homology condition $R \sim A$ means that there exists a bulk region Σ on the chosen slice such that

$$\partial\Sigma \;=\; R \cup A. \tag{1.2.82}$$

The RT formula was derived using the replica trick (1.2.67) by Lewkowycz and Maldacena [54].

For *time-dependent* states and Lorentzian bulk geometries, the minimal surface on a slice notion is not covariant. Hubeny, Rangamani and Takayanagi proposed the covariant generalization (HRT) in which R is replaced by a codimension-2 bulk surface with *vanishing null expansions* (of the two orthogonal null congruences), still anchored on ∂A and homologous to A. In static spacetimes this reduces to RT, while in dynamical settings the HRT surface captures the real-time evolution of entanglement.

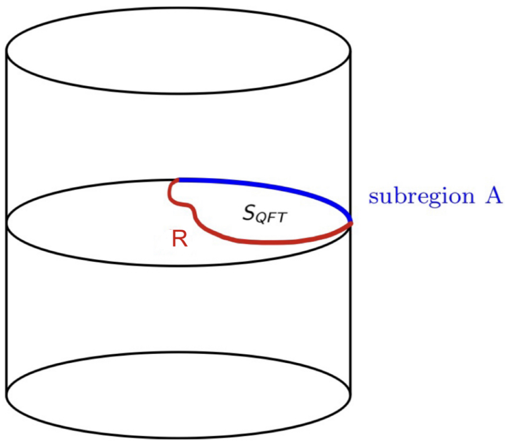


FIGURE 1.4: The entanglement entropy of a boundary subregion A is obtained by extremizing the *generalized entropy* over bulk codimension-2 surfaces R that are anchored on ∂A and homologous to A.

Example - Entanglement entropy in a $\mathbf{CFT_2}$ vacuum: Let's say we want to calculate the entanglement entropy of a CFT_2 vacuum state on an entangling region specified by $-\frac{L}{2} \leq x \leq \frac{L}{2}$, on a 2-dimensional flat background

$$ds_2^2 = -dt^2 + dx^2. \tag{1.2.83}$$

We can calculate this by using the RT prescription on AdS_3

$$ds_3^2 = \frac{L^2}{z^2}(dz^2 - dt^2 + dx^2), \tag{1.2.84}$$

with a 2-dimensional flat conformal boundary at $z = 0$. We take a constant t slice and consider the entangling region $-\frac{L}{2} \leq x \leq \frac{L}{2}$ on the conformal boundary. The codimension-2 RT surface is spanned by the worldvolume coordinate $\{z\}$, with the AdS_3 embedding $\{t, z, x(z)\}$. The induced metric on the RT surface is

$$h_{zz} = \frac{L^2}{z^2}\left(1 + x'(z)^2\right) \tag{1.2.85}$$

The area functional to be minimised is

$$S_{reg} = \frac{L}{4G_3}\int dz\, \frac{\sqrt{1 + x'(z)^2}}{z}, \tag{1.2.86}$$

with the boundary conditions $x(0) = \pm L/2$. This gives two branches, say $x_\pm$. At a turning point $z = z_0$, $x_+(z_0) = x_-(z_0)$. Minimising the area functional, we get

$$\frac{x'}{z\sqrt{1 + x'^2}} = \text{constant} = k \tag{1.2.87}$$

Solving for $x'(z)$, we have

$$x'(z)^2 = \frac{k^2 z^2}{1 - k^2 z^2}. \tag{1.2.88}$$

At the turning point z_0, we have $x'(z_0) = \infty$, which sets $k = 1/z_0$. Solving further for $x(z)$, we get

$$x_\pm(z) = c_\pm \pm \sqrt{z_0^2 - z^2}. \tag{1.2.89}$$

Imposing $x_\pm(0) = \pm L/2$, and $x_+(z_0) = x_-(z_0)$, we get

$$x^2 + z^2 = \frac{L^2}{4} \tag{1.2.90}$$

i.e., the RT surface is a semicircle with radius $L/2$ centred around $x = z = 0$. Substituting this in the area functional we get

$$\begin{aligned} S_{reg} &= 2\frac{L}{4G_3}\int_{\epsilon}^{a} \frac{a}{z\sqrt{a^2-z^2}}, \quad a = \frac{L}{2}, \\ &= \frac{L}{2G_3}\log\left(\frac{a+\sqrt{a^2-\epsilon^2}}{\epsilon}\right) \end{aligned} \tag{1.2.91}$$

ϵ is a regulator for the divergences that come from the neighborhood of the conformal boundary. The factor of 2 is because each branch captures only a quarter of the circle, and the RT surface is symmetric around $x = 0$, hence both branches contribute equally. We have

$$S = \frac{c}{3}\log\left(\frac{L}{\epsilon}\right); \quad c = \frac{3L}{2G_3}. \tag{1.2.92}$$

This matches the independent CFT_2 result by Calabrese and Cardy [55], for the vacuum entanglement entropy of an interval, where c is the Brown–Henneaux central charge.

Quantum extremal surface (QES) prescription: The quantum corrections to the holographic entanglement entropy was first considered by Faulkner, Lewkowycz and Maldacena (FLM) [56], which was then refined by Wall and Engelhardt [11]. They proposed that one has to consider a homologous surface R to the entangling subregion A and instead extremize the generalised entropy i.e.,

$$S(A) = \min \underset{R\sim A}{\mathrm{ext}}\left(\frac{Area(R)}{4G\hbar} + S_{\text{QFT enclosed by } R \text{ and } A}\right) \tag{1.2.93}$$

The resulting codimension-2 surface is called the **Quantum extremal surface**.

Subregion duality and entanglement wedge: [57] Consider a state ρ of a holographic CFT that is dual to a semiclassical, asymptotically AdS spacetime. Consider a boundary Cauchy slice and a subregion A in it. Decompose the Hilbert space as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$ and form the reduced density matrix $\rho_A = \mathrm{Tr}_{A^c}\rho$. The entanglement wedge of A, denoted $\mathrm{EW}(A)$, is defined as the bulk domain of dependence

$$\mathrm{EW}(A) = D(\Sigma_A), \tag{1.2.94}$$

where Σ_A is a bulk region on a Cauchy slice such that $\partial\Sigma_A = A \cup R_A$, with R_A the (quantum) extremal surface anchored on ∂A and homologous to A. Subregion duality (entanglement wedge reconstruction) states that bulk operators supported in $\mathrm{EW}(A)$ can be reconstructed from boundary data in A, i.e. ρ_A encodes the bulk physics in $\mathrm{EW}(A)$.

1.2.6 Islands and the Page curve

The quantum extremal surface (QES) prescription suggests similar quantum corrections to the black hole entropy. In [12, 13], the QES prescription was used to calculate the entropy of an AdS_2 black hole in JT (Jackiw–Teitelboim) gravity [58, 59] coupled to a non-gravitating flat space bath. This setup could be an approximation to a black hole in asymptotically flat space, where we neglect the effects of quantum gravity in the flat space region far away. Because AdS black holes with reflecting boundary conditions do not allow complete evaporation (the system can equilibrate with its Hawking radiation), one attaches a bath with transparent boundary conditions so that Hawking radiation can escape, enabling an evaporating setup. Moreover, pure 2-dimensional Einstein gravity has no local bulk degrees of freedom (and the Einstein–Hilbert term is essentially topological), so to get a useful AdS_2 black hole geometry with nontrivial thermodynamics, one typically uses JT dilaton gravity. JT still has no propagating bulk graviton degrees of freedom, but it admits a nontrivial dilaton profile. We consider the JT action

$$I_{JT} = \frac{1}{16\pi G_N}\int d^2x\sqrt{-g}\phi(R+2) + \frac{1}{8\pi G_N}\int dt\sqrt{-\gamma}\frac{\phi_r}{\epsilon}(K-1) \tag{1.2.95}$$

where ϵ is a UV cutoff, ϕ is the dilaton field. K is the extrinsic curvature of the asymptotic boundary i.e.,

$$K = \nabla_\mu n^\mu, \tag{1.2.96}$$

where ∇_μ are the components of the covariant derivative with respect to the bulk AdS_2 metric, and n^μ is an outward pointing unit normal to the asymptotic boundary. ϕ_r is the renormalized boundary value of the dilaton, which has the dimensions of length. It is defined via the near boundary behavior

$$\phi|_{x=-\epsilon} = \frac{\phi_r}{\epsilon}. \tag{1.2.97}$$

A simple solution for $\phi, g_{\mu\nu}$ that satisfies the usual JT boundary conditions

$$\gamma_{tt} = \frac{1}{\epsilon^2}, \quad \phi|_{x=-\epsilon} = \frac{\phi_r}{\epsilon}, \tag{1.2.98}$$

is given by

$$ds^2 = \frac{-(dx^0)^2 + (dx^1)^2}{(x^1)^2}; \quad x^1 < 0, \tag{1.2.99}$$

$$\phi(t,x) = \frac{\phi_r}{-x^1}. \tag{1.2.100}$$

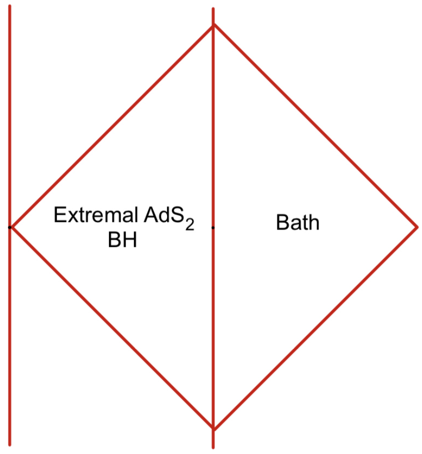


FIGURE 1.5: AdS_2 coupled to a flat bath

The conformal boundary is at $x^1 = 0$, where we will impose a transparent matching condition between the AdS and flat regions. The bulk state is taken to be in the Poincaré vacuum. Since the black hole and the bath are taken to be in equilibrium, an extremal black hole requires a bath with zero temperature. The Entanglement entropy of a CFT$_2$ on an interval AB (on a Cauchy slice), in a curved background with metric

$$ds^2 = -\frac{1}{\Omega(x^+, x^-)^2} dx^+ dx^-; \quad x^\pm = x^0 \pm x^1, \tag{1.2.101}$$

is given by

$$S(AB) = \frac{c}{6} \log \left(\frac{-\Delta x^+ \Delta x^-}{\Omega(A)\Omega(B)} \right) \tag{1.2.102}$$

Since the QES is a codimension-2 point, the area term at A is given by the dilaton profile at that point $\phi(A)/(4G_N)$. Henceforth, we will set $4G_N = 1$, i.e., absorb it into ϕ_r.

We now consider a Cauchy slice with fixed x^0, with entangling region AB, with the endpoints A at $x^1 = -a$ and B at $x^1 = b$ (see Fig (1.6)). At the AdS-bath interface, we allow matching conditions that implement transparent propagation for the CFT fields (so energy/information can flow into the bath). With this condition, the contributions from the left and the right movers are independent. The QES is given by extremizing

$$S_{gen} = \frac{\phi_r}{a} + 2\frac{c}{6} \log(a+b) - \frac{c}{6} \log(a) + \text{constant}. \tag{1.2.103}$$

For a fixed b, this is extremized at a^* given by

$$\frac{\phi_r}{a^*} = \frac{c}{6} \left(\frac{a^* - b}{a^* + b} \right). \tag{1.2.104}$$

a^* is the location of the QES. The condition above implies that $a^* > b$. For $b = 0$, we have $a^* = (6\phi_r/c)$. The entanglement wedge is given in green in the figure (1.6).

Since the whole state is pure $S(A^*B) = S((A^*B)^c)$. So the complement region has the entanglement wedge in blue, given below in Fig (1.6), which is disconnected. The region between H and A^*, which is disconnected from the asymptotic flat boundary, is called the *"Island"*. Minimising just the area contribution without the quantum corrections from the QES, will reproduce the entire AdS wedge HB, whose complement is only the blue region connected to the asymptotic flat boundary, i.e., there wouldn't be an island.

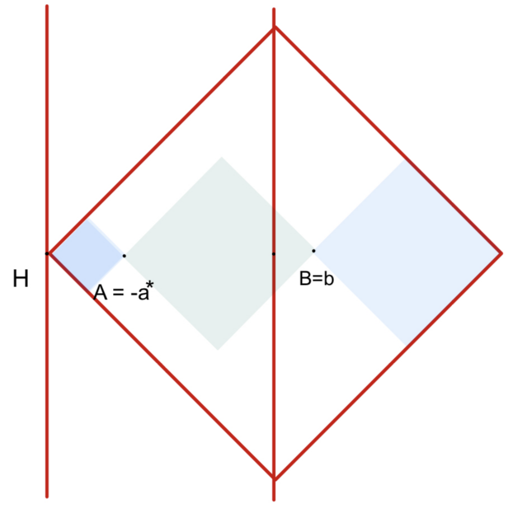


FIGURE 1.6: QES and island in the zero temperature case. Shaded in blue we see the interval $[b, \infty]$ and its entanglement wedge which also includes the island $[-\infty, -a^*]$, which is outside the horizon

Nonzero-temperature black hole coupled to a bath:

We now consider black holes at nonzero temperature coupled to a bath. More precisely, the setup can be thought of as having two copies of a black hole interacting with a bath, in the thermofield double state (see Fig (1.7)). This means that each black hole+bath system is in a thermal state. The state of the quantum fields on this background is the Hartle-Hawking state. The geometry corresponds to an AdS eternal black hole (or AdS wormhole) that is connected to two half-spaces, one on each side.

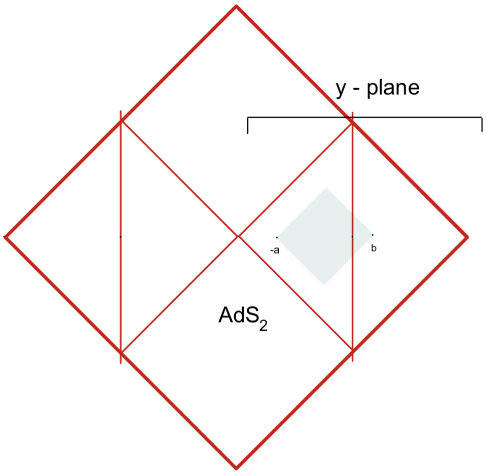


FIGURE 1.7: A black hole in thermal equilibrium with a bath. We have two $y-$planes, one on the right and one on the left. Half of each $y-$plane belongs to the bath and the other half to the black hole exterior. The two $y-$planes fit into a w-plane where the state is the Minkowski vacuum for the quantum fields of the CFT. The $y-$planes are like Rindler wedges and the state looks thermal in the $y-$coordinates

The stress tensor in the bath half-space regions is the usual thermal one with constant energy density. The stress tensor in the eternal black hole or wormhole region is that of the usual AdS_2 vacuum. We will use Poincaré coordinate $x^{\pm} = x^0 \pm x^1$, $x^1 < 0$, in the black hole region and $y^{\pm} = t \pm \sigma$ in the bath. The transparent boundary condition implies that the bulk and bath stress tensors are related as

$$\left(\frac{\partial x^+}{\partial y^+}\right)^2 T^{(x)}_{++} = T^{(y)}_{++} + \frac{c}{24\pi}\{x^+, y^+\} \tag{1.2.105}$$

If both the black hole and the bath are in vacuum as in the previous zero temperature case, i.e., $T^{(y)}_{\pm\pm} = 0 = T^{(x)}_{\pm\pm}$, the above relation, along with the requirement of having a pure state of the bath plus black hole fixes, $x^{\pm} = y^{\pm}$, which is consistent with our previous assumption.

However, for the non-zero temperature case, the transformation between the stress tensors imply

$$x^{\pm} = \tanh\left(\frac{\pi y^{\pm}}{\beta}\right), \tag{1.2.106}$$

where β is the inverse temperature. In these (y^+, y^-) coordinates, the right exterior geometry and dilaton profile are

$$ds^2 = -\frac{4\pi^2}{\beta^2} \frac{dy^+ dy^-}{\sinh^2\left[\frac{\pi}{\beta}(y^- - y^+)\right]}, \tag{1.2.107}$$

$$\phi = \frac{2\pi\phi_r}{\beta} \frac{1}{\tanh\left[\frac{\pi}{\beta}(y^- - y^+)\right]}. \tag{1.2.108}$$

The horizons are at $y^- = +\infty$ and $y^+ = -\infty$. The whole $y-$plane describes one black hole exterior together with the bath. A second $y-$plane describes the other side.

The generalized entropy of the interval $[-a, b]$ takes the form

$$S_{\text{gen}}(a) = \underbrace{\frac{\phi(A)}{4G_N}}_{\text{“area” term}} + \underbrace{S_{\text{bulk}}([-a, b])}_{\text{CFT entropy}} = \frac{2\pi\phi_r}{\beta} \frac{1}{\tanh\left(\frac{2\pi a}{\beta}\right)} + \frac{c}{6}\log\left[\frac{\sinh^2\left(\frac{\pi(a+b)}{\beta}\right)}{\sinh\left(\frac{2\pi a}{\beta}\right)}\right] + \text{constant}. \tag{1.2.109}$$

Here, we have used the standard CFT_2 interval entropy together with the appropriate warp-factor transformation implied by the map $x^\pm = \tanh(\pi y^\pm/\beta)$. In the large β limit, this reproduces (1.2.103). Extremizing with respect to a gives the QES condition

$$\frac{\sinh\left(\frac{\pi(a-b)}{\beta}\right)}{\sinh\left(\frac{\pi(a+b)}{\beta}\right)} = \frac{12\pi\phi_r}{c\beta} \frac{1}{\sinh\left(\frac{2\pi a}{\beta}\right)}. \tag{1.2.110}$$

The resulting QES lies at a point *outside* the horizon. Moreover, one finds $a \to \infty$ as $b \to \infty$, so that the quantum extremal surface approaches the black hole horizon. Note again that $a > b$.

In the regime $\phi_r/(c\beta) \gg 1$, the near-extremal black hole entropy is large compared to c and the effect of Hawking radiation is a small perturbation. In this limit, we get

$$a \approx b + \frac{\beta}{2\pi}\log\left(\frac{24\pi\phi_r}{c\beta}\right). \tag{1.2.111}$$

The region in the gravitating spacetime that is included in the entanglement wedge but is disconnected from the asymptotic bath boundary is the *island*. As in the zero-temperature case, minimizing only the area entropy contribution from the dilaton term would not reproduce this structure: the nontrivial island arises from extremizing the *generalized* entropy, i.e. including the bulk CFT contribution.

Although in the cases we have discussed, islands occur *outside* the horizon, islands *behind* the horizon were also discussed in [13, 60, 61].

It was shown in [12, 13] that the contribution to the von Neumann entropy of Hawking radiation coming from the island recovers the Page curve for an evaporating black hole.

Hence, the QES prescription that seems to provide the necessary quantum corrections consistent with a unitary black hole evaporation can also be formulated as the following island rule

$$S[\mathbf{Rad}] = \min\left\{\text{ext}\left(\frac{\text{Area}(\partial I)}{4G\hbar} + S(\text{Rad} \cup I)\right)\right\} \tag{1.2.112}$$

Rad is the radiation in the full quantum description, while Rad is the radiation in the semiclassical description. We extremize the generalized entropy over all possible islands I, and the minimum value gives the entropy of Hawking radiation. We note that the island rule in (1.2.112) is equivalent to and follows from the QES prescription.

Intuitive reason for why the QES prescription yields islands and how they recover the Page curve: Islands capture the entangled paired modes and purify the semiclassical radiation, hence reducing S_{QFT}. However, an island also brings with it an additional area cost. Before the Page time there are no islands. After the Page time, the decrease in S_{QFT} dominates the additional area term, leading to a net decrease in the generalised entropy S_{gen}. S_{gen} achieves a minimum with island(s) and hence they emerge after the Page time.

The location of the QES and the island is fixed by the QES prescription. A large island implies a big area cost. Whereas by shrinking the island we lose entangled paired modes and hence S_{QFT} increases. The balance between these competing effects determines the position of the QES.

Interpreting the semiclassical answer as a coarse-grained entropy, the QES prescription selects the fine-grained entropy by minimising the generalized entropy.

Replica wormholes: In [62, 63], it was shown that "Replica wormholes" provide a semiclassical gravitational derivation of the island/quantum extremal surface (QES) prescription for the fine-grained entanglement entropy of the black hole radiation. To compute the radiation entropy one evaluates $\text{Tr}\,\rho_R^q$ via the replica trick (1.2.67), which in gravity becomes a Euclidean path integral over q replicated geometries cyclically glued along the radiation region. Besides the “disconnected” saddle (which reproduces Hawking’s monotonically growing entropy), the gravitational path integral can admit additional saddles in which different replicas are connected through the bulk by wormholes (“replica wormholes”). Upon analytic continuation $q \to 1$, the contribution of these connected saddles yields the island rule. In evaporating black hole models, the dominance of replica-wormhole saddles after the Page time leads to a Page curve consistent with unitarity.

1.2.7 Other uses of entanglement entropy

Beyond its use in studying black hole physics, entanglement entropy has been increasingly studied as a useful measure over the last twenty years [55, 64–68]. On the one hand, entanglement entropy provides a robust measure of quantum entanglement in the context of quantum information and quantum computation[55, 69–72]: for example, it is preserved under local unitary transformations i.e. quantum gates. Entanglement entropy can also be used to characterise several features of quantum systems such as thermalization, many-body localization [73], different phases of a system [74–77], confinement [78], quantum chaos [79]. On the other hand, entanglement entropy plays a key role in gravity/gauge theory dualities via the famous proposal of Ryu-Takayanagi (1.2.81) [80]. Entanglement entropy in holographic theories thus leads to deep insights into the holographic reconstruction of dual spacetimes [81–89].

Entanglement entropy also quantifies correlations. For any bipartite quantum state ρ_{AB}, the von Neumann entropy satisfies *subadditivity*

$$S(AB) \leq S(A) + S(B), \tag{1.2.113}$$

with equality if and only if the state factorizes,

$$\rho_{AB} = \rho_A \otimes \rho_B. \tag{1.2.114}$$

The difference defines the *mutual information*

$$I(A\!:\!B) \equiv S(A) + S(B) - S(AB) \;\geq\; 0, \tag{1.2.115}$$

which quantifies the *total* correlations (classical + quantum) between A and B. A prescription to construct Rényi and von Neumann entropy from a knowledge of its correlation functions was discussed in [90].

Further, for *classical* random variable A with Shannon entropy $H(A)$, one always has

$$H(AB) \geq H(A), \qquad H(AB) \geq H(B), \tag{1.2.116}$$

In contrast, for *quantum* systems, the von Neumann conditional entropy

$$S(A|B) \equiv S(AB) - S(B) \tag{1.2.117}$$

can be negative. Equivalently, it is possible to have $S(AB) < S(B)$ (or $S(AB) < S(A)$). This cannot occur classically and is only possible in the presence of entanglement between A and B (indeed, $S(A|B) < 0$ implies that ρ_{AB} is entangled). For example, in the case where the joint state ρ_{AB} is pure and entangled, one has

$$S(AB) = 0, \qquad S(A) = S(B) > 0. \tag{1.2.118}$$

Negative conditional entropy is a hallmark of genuine quantum behavior.

Hence, entanglement entropy is a valuable quantum information measure useful to fundamentally characterise and study quantum systems. It is hence useful to compute and analyse the properties of entanglement entropy.

1.2.8 Context for the chapters in Part I

To recap, entanglement entropy plays an important role in the formulation of the black hole information loss paradox by Don Page [42]. Page formulates information loss in terms of entanglement entropy of quantum fields in the far exterior of the black hole. In Hawking's calculation [3], these quantum fields are in a thermal state and the entanglement entropy increases monotonically throughout the evaporation. Page expresses the recovery of information during the black hole evaporation in terms of the entanglement entropy increasing to a maximum, and then decreasing to zero as the black hole evaporates, signalling recovery of all information contained in the black hole.

In recent years the Page formulation of the information loss paradox, together with quantum extremal surfaces [91], has stimulated new understanding of how information is recovered. In the islands of entanglement paradigm, one considers the entanglement entropy of quantum fields in a black hole background and then uses the quantum extremal surface approach (combining black hole and quantum field entropy) to determine the quantum extremal surface. The latter has been found to have a component behind the black hole horizon, a so-called island of entanglement[12, 13, 60], which is associated with recovery of information from the black hole during the evaporation.

An essential part of the analysis of entanglement islands is the computation of entanglement entropy of quantum fields within black hole backgrounds. Most of the literature explores two-dimensional black holes within JT gravity, with the quantum fields described by a two-dimensional conformal field theory. Accordingly, the entanglement of the quantum fields is obtained from the well-known results for finite temperature two-dimensional conformal field theory, with the details of the background captured by suitable conformal transformations.

Models of islands in higher dimensions were first discussed in [92] and later in [93–95]. These are known as double holographic models (see also [96]) where gravity in an AdS_{d+1} region bounded by a pair of end-of-the-world branes is dual to a CFT_{d-1} theory living on the interception between the branes. These are examples of Karch-Randall braneworld models [97–101]. Despite the renewed interest in these models and their recent success, there are persistent challenges that arise when doing holography with AdS with a cutoff. Explicit realization of islands in higher dimensions, without resorting to braneworld models, stems in part from the lack of analytical control over entanglement entropy of the quantum fields in $d > 2$.

A proper generalisation of the islands approach to higher dimensional black holes requires computation of the entanglement of quantum fields within such backgrounds. This is computationally challenging, even for static black holes and conformal fields. Most of the quantum field theory computations in the literature relate to quantum entanglement in vacuum states, in either non-interacting quantum theories or conformal theories, in flat backgrounds [36, 38, 55, 102–111]

In the chapters in Part I, we will explicitly compute entanglement entropy in higher dimensions and systematically explore its properties with the goal of exploring if and when islands are present in higher dimensions. These explicit expressions for entanglement entropy will also be useful in various applications of entanglement entropy beyond black hole physics, that we discussed in the previous section.

1.3 Flat holography

Incorporating *holography* (1.2.5) into the framework of quantum gravity has proven to be a challenging task. Using insights from string theory, holography has been realised in spacetimes with a negative cosmological constant, namely asymptotically Anti de Sitter spacetimes via the AdS/CFT correspondence [6–8]. This correspondence has proven to be a powerful tool for exploring various aspects of quantum gravity and dual field theories, including a non-perturbative definition of quantum gravity, black hole thermodynamics from microscopic QFT counting and its quantum corrections, geometric duals to quantum information measures, precise computation of strongly coupled observables, and so on.

Despite the above remarkable achievement, it remains to extend the holographic principle to more realistic spacetimes characterized by a vanishing or positive cosmological constant, namely the asymptotically flat or de Sitter spacetimes (for reviews, see [18, 112, 113] and references therein). Although experiments indicate that our Universe is de Sitter, asymptotically flat spacetimes (AFS) capture very important phenomena from collider physics to gravitational waves. The observables in flat space correspond to scattering amplitudes and a whole compendium of inclusive observables where the time-ordering is relaxed [114]. These include expectation values of fields as well as out-of-time-order amplitudes. All these observables correspond to free states that start at asymptotic infinity, interact in the bulk, and end as free states at asymptotic infinity (one assumes that sufficiently far in the past and far in the future, the particles are so widely separated that interactions become negligible). Particles scatter between null infinities $\mathscr{I}^{\pm}$ in the case of massless fields, and timelike infinities $i^{\pm}$ in the case of massive fields. Hence, for the massless bulk fields, the natural location for the corresponding dual theory are null infinities $\mathscr{I}^{\pm}$. The spacelike boundaries of asymptotically de Sitter spacetimes makes the realisation of holography in them much more challenging. Therefore, a good stepping stone to de Sitter holography is to use the lessons from the exploration of flat holography on null boundaries as a stepping stone between holography on timelike boundaries in asymptotically anti de Sitter and spacelike boundaries in asymptotically de Sitter spacetimes. Further, by studying flat holography, we intend to understand quantum corrections to the entropy of black holes in AFS, which to a very good approximation, match realistic black holes.

However, we must note that the disconnected and null nature of the asymptotic boundaries in AFS, unlike the timelike connected boundary in AdS spacetimes that naturally hosts a relativistic QFT, makes flat space holography a more formidable task. A comparison of the known asymptotic structure and symmetries, observables, and holographic dictionary in asymptotically AdS and flat spacetimes is given in the table (1.1).

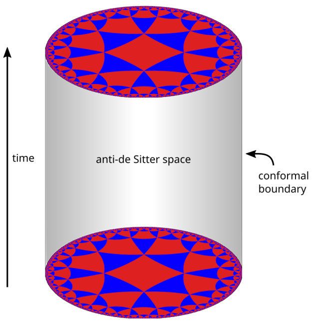
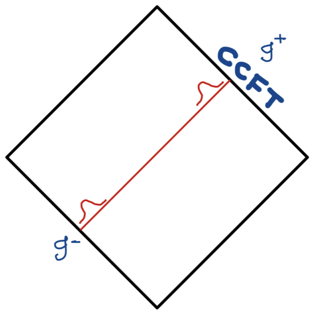
	Asymptotically AdS spacetimes 	**Asymptotically Flat Spacetimes** 
Quantum Gravity observables	AdS/CFT correlators for operators on the conformal timelike boundary.	Asymptotic flat observables between disconnected $\mathscr{I}^{\pm}$ and $i^{\pm}$.
Holographic Dictionary	$\mathcal{Z}_{\text{SUGRA}}[\phi_{(0)}] = \left\langle \exp\left(-\int_{\partial\text{AdS}} \phi_{(0)}\,\mathcal{O}\right)\right\rangle_{\text{QFT}}$	???
Source	Boundary value of the classical solution is the source	???
Divergences	Divergences around the conformal boundary	???
Asymptotic Symmetry	$SO(d,2)$ (Euclidean AdS), $SO(d+1,1)$ (Lorentzian AdS)	BMS_{d+1}

TABLE 1.1: Comparison between asymptotically AdS and asymptotically flat spacetimes.

Efforts to realise holography in AFS have taken various approaches. Early efforts include slicing flat space into AdS and dS slices and applying holography on each slice, which led to codimension 2 duals on the boundary of the light cone [15].

Another approach to flat holography that is being extensively explored is the *celestial approach* [16–25]. This is motivated by the observation that, in four-dimensional asymptotically flat spacetimes, the universal infrared behaviour of scattering amplitudes is governed by the asymptotic symmetry structure at null infinity, namely the Bondi–van der Burg–Metzner–Sachs (BMS) group [115, 116] and its extensions. In particular, soft graviton theorems, together with the associated memory effects, may be understood as consequences of these asymptotic symmetries [117]. Since the BMS group acts on the S^2 of angular directions parametrizing null infinity, the corresponding Ward identities naturally admit a two-dimensional formulation on the celestial sphere. This, in turn, suggests that flat-space scattering may be recast in a form adapted to boundary data at null infinity.

More concretely, this is achieved by Mellin transforming four-dimensional momentum-space scattering amplitudes with respect to the external particle energies, thereby expressing the S-matrix in a basis of boost eigenstates. The resulting celestial amplitudes transform covariantly as two-dimensional conformal correlators on the celestial sphere, with the four-dimensional Lorentz group acting as the global conformal group. In this representation, the asymptotic symmetry constraints of the bulk become manifest as Ward identities in the celestial description. It should be emphasized, however, that although this construction exhibits a pronounced two-dimensional conformal structure, the putative dual theory is not presently known to be an ordinary local, unitary two-dimensional conformal field theory in the conventional sense. Furthermore, while the celestial framework is intended to reorganize the full scattering problem, its most precise and well-developed results currently pertain to the infrared sector, where soft theorems and asymptotic symmetries provide the strongest evidence for such a reformulation.

More recently the *Carrollian approach* to flat holography has gained momentum [26, 27]. Historically, the Carrollian algebra was introduced as the vanishing speed of light $c \to 0$ limit of the Poincaré algebra. In this limit, the metric becomes degenerate with a nowhere vanishing Killing vector. Carrollian theories naturally arise on null hypersurfaces, on which the induced metric is degenerate, making these a natural candidate for a dual theory corresponding to massless scattering in the bulk of AFS. This is further supported by the observation that the null infinity limit of the Poincaré group gives the Carrollian conformal group. We will systematically study these Carrollian theories and use them to better understand holography in AFS.

We will also explore flat limits of AdS/CFT to understand flat holography. In the vanishing limit of the cosmological constant $\Lambda \to 0$, AdS geometry reduces to flat space geometry, and the same should also be true for the physics on these spacetimes. This flat limit is subtle and has been analysed in various setups: position space [118–121], Mellin space [122–124], partial wave expansion [120, 123], momentum space [125–130]; see also [131] for a comparison of the different formulations, and [132–143] for further work. We will analyse the flat limit of holographic CFT correlators in momentum space to yield flat amplitudes and, in the process, learn more about flat holography. The flat limit on the dual CFT is closely related to the vanishing speed of light $c \to 0$ Carroll limit of the CFT to yield Carrollian CFT (CCFT), which are dual to flat space scattering in the bulk. We also intend to explore this connection between the two approaches.

Further, massive higher spin fields are expected to play an important role in the classification of consistent gravitational theories (see [144, 145] and references therein for review and literature on the higher spin fields in general backgrounds, including AdS). In particular, it can be argued that an infinite tower of massive higher spin fields are needed for the gravitational theories to be consistent with causality at high energies [146]. However, the requirement of causality only forces the inclusion of a higher spin field above a certain energy scale. In an effective field theory (EFT), an interacting theory of a single massive higher spin field can be consistent below some energy scale when the particle masses are large compared to the exchanged momenta. In spite of the utility of massive higher spin fields, it is very tricky to construct a consistent EFT of single massive higher spin fields.

We will analyze a bulk effective field theory in AdS containing a $U(1)$-charged massive spin-2 field coupled to a gauge field, by performing the required holographic renormalization, and computing the one and two-point functions. We will also compute the renormalized bulk three-point function involving two massive spin-2 fields and one gauge field, and by comparing this with the CFT 3-point correlator of two non-conserved spin-2 operators and a conserved current, we will obtain explicit mappings between the bulk minimal and gyromagnetic couplings and the boundary OPE data. Finally, we will take the flat-space limit of the momentum space CFT correlator and verify that the resulting amplitude matches the expected flat-space structure.

1.3.1 Carrollian theories

Carrollian theories may be understood as arising from the ultra-relativistic limit of Lorentzian theories, obtained by sending the speed of light to zero, $c \to 0$. Starting from the Minkowski line element

$$ds^2 = -c^2 dt^2 + \delta_{ab} dx^a dx^b \quad \xrightarrow{c\to 0} \quad 0 \cdot dt^2 + \delta_{ab} dx^a dx^b, \tag{1.3.119}$$

one sees that the temporal part degenerates. Equivalently, the Lorentzian metric becomes degenerate in this limit, which is the basic geometric hallmark of Carrollian structure. The light cones collapse onto the time axis, so causal motion is restricted to be purely temporal: particles cannot propagate through space and the dynamics becomes ultra-local in space.

A *Carrollian structure* on a $(d+1)$-dimensional manifold M is given by a pair $(h_{\mu\nu}, v^\mu)$, where

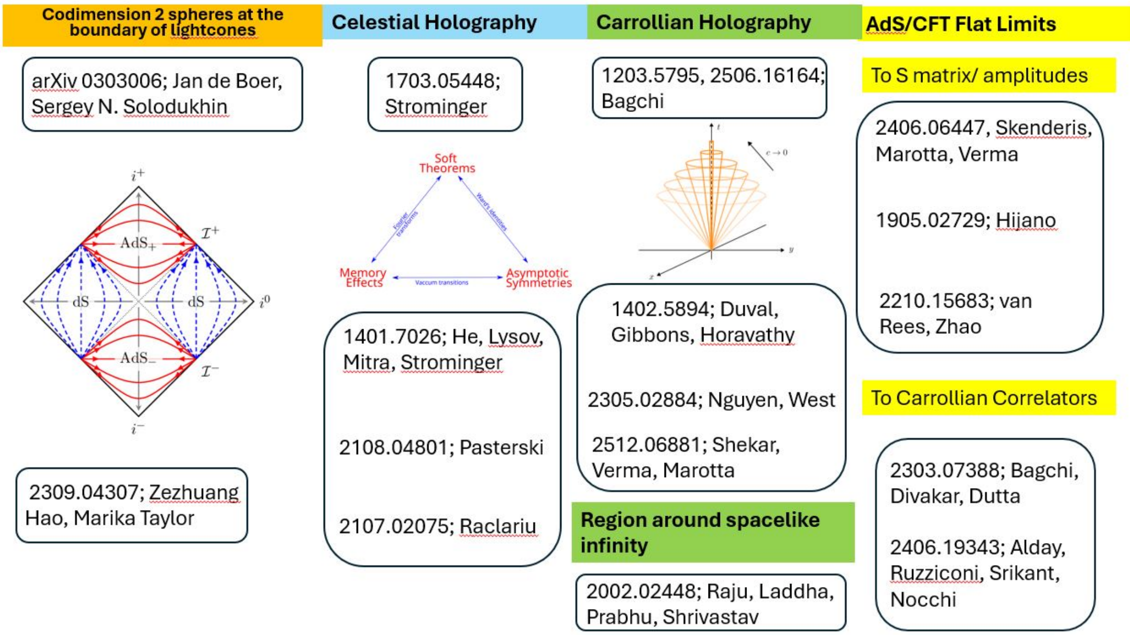


FIGURE 1.8: Various approaches to flat holography and a non-exhaustive list of references (arXiv).

- $h_{\mu\nu}$ is a symmetric rank-d covariant tensor field (a degenerate metric),
- v^μ is a nowhere-vanishing vector field spanning its kernel,

such that

$$h_{\mu\nu}v^\nu = 0. \tag{1.3.120}$$

Thus, $h_{\mu\nu}$ defines a positive-definite metric on the spatial directions transverse to v^μ, while v^μ selects the distinguished degenerate (Carrollian time) direction.

Let Σ be a null hypersurface. The pullback of the ambient spacetime metric to Σ defines a degenerate symmetric tensor $h_{\mu\nu}$ on Σ. Its kernel is generated by the null vector field tangent to the null generators of Σ. Therefore, Σ carries a natural Carrollian structure $(h_{\mu\nu}, v^\mu)$ with

$$h_{\mu\nu}v^\nu = 0. \tag{1.3.121}$$

Hence, null hypersurfaces provide canonical examples of Carrollian manifolds. This applies in particular to future and past null infinity, $\mathscr{I}^\pm$, which are null hypersurfaces. These are precisely the asymptotic regions through which massless particles enter and exit in scattering processes.

This can be seen explicitly for flat spacetime. In spherical polar coordinates

$$ds_d^2 = -c^2dt^2 + dr^2 + r^2\gamma_{ab}dx^a dx^b, \tag{1.3.122}$$

where $t \in (-\infty, \infty)$, $r \in [0, \infty)$, and x^a, with $a \in \{1, \cdots, d-2\}$, are the angular coordinates on S^{d-2}.

In retarded null coordinates $u = t - \frac{r}{c}$, we have

$$ds_d^2 = -c^2 du^2 - 2c\, du\, dr + r^2 \gamma_{ab} dx^a dx^b. \tag{1.3.123}$$

The hypersurfaces $u =$ constant are outgoing null hypersurfaces. Future null infinity $\mathscr{I}^+$ is reached by taking $r \to \infty$ with u fixed. Rescaling the metric by the conformal factor $1/r^2$ and taking this limit, we get the degenerate induced metric on future null infinity,

$$ds^2_{\mathscr{I}^+} = 0 \cdot du^2 + \gamma_{ab} dx^a dx^b. \tag{1.3.124}$$

Similarly, in advanced null coordinates $v = t + \frac{r}{c}$, the flat metric becomes

$$ds_d^2 = -c^2 dv^2 + 2c\, dv\, dr + r^2 \gamma_{ab} dx^a dx^b. \tag{1.3.125}$$

The hypersurfaces $v =$ constant are incoming null hypersurfaces. Past null infinity $\mathscr{I}^-$ is reached by taking $r \to \infty$ with v fixed. Rescaling the metric by $1/r^2$ and taking this limit, we get

$$ds^2_{\mathscr{I}^-} = 0 \cdot dv^2 + \gamma_{ab} dx^a dx^b. \tag{1.3.126}$$

We also note that

$$\begin{aligned} \partial_u &= \partial_t - c\partial_r \xrightarrow{c\to 0} \partial_t, \\ \partial_v &= \partial_t + c\partial_r \xrightarrow{c\to 0} \partial_t, \end{aligned} \tag{1.3.127}$$

i.e., in the $c \to 0$ limit, the null directions and the timelike directions degenerate.

1.3.1.1 Carrollian conformal field theory (CCFT)

The Carrollian conformal generators and algebra can be obtained from the vanishing speed of light $c \to 0$ limit of the conformal generators and algebra. Consider the d-dimensional Minkowski spacetime $\mathbb{M}^d$ parametrised by Cartesian coordinates x^μ for $\mu \in \{0, 1, ..., d-1\}$. The flat metric

$$\begin{aligned} &ds_d^2 = -(dx^0)^2 + \delta_{ij} dx^i dx^j, \quad i, j \in \{1, \cdots, d-1\}, \\ &x^0 = c x^{0\prime}, \end{aligned} \tag{1.3.128}$$

in the $c \to 0$ limit becomes

$$ds_d^2 = -0 \cdot (dx^{0\prime})^2 + \delta_{ij} dx^i dx^j. \tag{1.3.129}$$

Now consider the CFT generators

$$\begin{aligned}
P_\mu &= -i\partial_\mu \quad \text{(translations)},\\
J_{\mu\nu} &= i(x_\mu \partial_\nu - x_\nu \partial_\mu) \quad \text{(Lorentz transformations)},\\
D &= -ix^\mu \partial_\mu \quad \text{(dilations)},\\
K_\mu &= i(-x^2\partial_\mu + 2x_\mu x^\nu \partial_\nu) \quad \text{(special conformal transformations)}.
\end{aligned} \tag{1.3.130}$$

In the Carroll limit $c \to 0$, the only generators that are modified are

$$\begin{aligned}
cP_0 &\to -i\frac{\partial}{\partial x^{0\prime}} \equiv H,\\
cJ_{0i} &\to -ix_i\frac{\partial}{\partial x^{0\prime}} \equiv B_i,\\
cK_0 &\to -ix^2\frac{\partial}{\partial x^{0\prime}} \equiv K
\end{aligned} \tag{1.3.131}$$

and all other generators remain unmodified. $\{H, P_i, B_i, J_{ij}, K, K_i, D\}$ are the Carrollian Conformal generators. The commutators $[B_i, H]$ and $[B_i, K]$ become zero. All other commutators remain same as in the CFT algebra except with the generators replaced by the corresponding Carroll generators above. This is the Carrollian Conformal algebra.

It is isomorphic to the Poincaré algebra $ISO(1, d)$. We can get the Poincaré generators from the CCFT generators by redefining

$$\begin{aligned}
&\tilde{J}_{ij} = J_{ij}\,, \qquad \tilde{J}_{i0} = -\frac{1}{2}(P_i + K_i)\,, \qquad \tilde{J}_{id} = \frac{1}{2}(P_i - K_i)\,, \qquad \tilde{J}_{0d} = -D\,,\\
&\tilde{P}_0 = \frac{1}{\sqrt{2}}(H + K)\,, \qquad \tilde{P}_i = -\sqrt{2}\,B_i\,, \qquad \tilde{P}_d = \frac{1}{\sqrt{2}}(K - H)\,,
\end{aligned} \tag{1.3.132}$$

In this representation of the Poincaré algebra, the quadratic Casimir vanishes, *i.e.*,

$$\mathcal{C}_2 = \tilde{P}^\mu \tilde{P}_\mu = -\tilde{P}_0^2 + \tilde{P}^i \tilde{P}_i + \tilde{P}_d^2 = -(HK + KH) + 2B_i B^i = 0. \tag{1.3.133}$$

Therefore, the Carrollian CFT algebra captures massless scattering.

CCFT from null infinity limit of Poincaré: We can also obtain the CCFT algebra from the null infinity limit of the Poincaré algebra. Consider the $(d + 1)$-dimensional Minkowski spacetime $\mathbb{M}^{d+1}$ in Cartesian coordinates (X^0, X^a, X^d)

$$ds_{d+1}^2 = -(dX^0)^2 + \delta_{ab} dX^a dX^b + (dX^d)^2, \qquad a, b \in \{1, 2, \cdots, d-1\}. \tag{1.3.134}$$

To approach null infinity, it is useful to consider the flat null coordinates (u, r, x^a) with $a \in \{1, 2, \cdots, d-1\}$, which are related to the Cartesian coordinates as follows,

$$X^0 = \frac{1}{\sqrt{2}}[u + r(1 + x^2)], \qquad X^a = \sqrt{2}\, r\, x^a, \qquad X^d = \frac{1}{\sqrt{2}}[-u + r(1 - x^2)] \tag{1.3.135}$$

The inverse transformation is,

$$u = \frac{(X^0)^2 - X^a X_a - (X^d)^2}{\sqrt{2}\,(X^0 + X^d)}, \qquad x^a = \frac{X^a}{X^0 + X^d}, \qquad r = \frac{X^0 + X^d}{\sqrt{2}}. \tag{1.3.136}$$

The coordinates are ill-defined on the hypersurface $X^0 + X^d = 0$ where $r = 0$. Therefore

$$u \in \mathbb{R}, \qquad x^a \in \mathbb{R}, \qquad r \in \mathbb{R}\backslash\{0\} \tag{1.3.137}$$

This is a planar cut version of the Bondi coordinates. The flat metric in these coordinates takes the form,

$$ds^2_{d+1} = -2dudr + 2r^2 \delta_{ab} dx^a dx^b. \tag{1.3.138}$$

Since $g_{uu} = 0$, ∂_u is a null vector and $u =$constant are null hypersurfaces. The induced metric on the future and past null infinities, $\mathscr{I}^+$ and $\mathscr{I}^-$ is obtained by rescaling the flat metric in (1.3.138) by $1/2r^2$ and taking the limits $r \to +\infty$ and $r \to -\infty$, respectively. This gives

$$ds^2_{\mathscr{I}^\pm} = 0 \cdot du^2 + \delta_{ab} dx^a dx^b. \tag{1.3.139}$$

Consider the Poincaré generators in Cartesian coordinates

$$J_{\mu\nu} = i\left(X_\mu \frac{\partial}{\partial X^\nu} - X_\nu \frac{\partial}{\partial X^\mu}\right), \quad P_\mu = -i\,\frac{\partial}{\partial X^\mu}. \tag{1.3.140}$$

with $\mu, \nu \in \{0, 1, \cdots, d\}$.

In the flat null coordinates they take the form,

$$
\begin{aligned}
P_0 &= -\frac{i}{\sqrt{2}}\Big(\partial_r + (1+x^2)\partial_u - r^{-1}x^i\partial_i\Big),\\
P_i &= -\frac{i}{\sqrt{2}}\Big(-2x_i\partial_u + r^{-1}\partial_i\Big),\\
P_d &= -\frac{i}{\sqrt{2}}\Big(\partial_r - (1-x^2)\partial_u - r^{-1}x^i\partial_i\Big),\\
J_{0d} &= -i\Big(r\partial_r - u\partial_u - x^i\partial_i\Big),\\
J_{i0} &= \frac{i}{2}\Big(\Big(r^{-1}u + 1 + x^2\Big)\partial_i + 2x_i\Big(r\partial_r - u\partial_u - x^j\partial_j\Big)\Big),\\
J_{id} &= \frac{i}{2}\Big(\Big(r^{-1}u - 1 + x^2\Big)\partial_i + 2x_i\Big(r\partial_r - u\partial_u - x^j\partial_j\Big)\Big),\\
J_{ij} &= i\Big(x_i\partial_j - x_j\partial_i\Big).
\end{aligned}
\tag{1.3.141}
$$

In the $r \to \infty$ limit, where the generators project onto null infinity, we have a different group which is still isomorphic to $ISO(1,d)$. In this limit, we recover the generators $\tilde{P}_\mu$, $\tilde{J}_{\mu\nu}$ in (1.3.1.1), with $x^{0\prime} \to u$, and an additional $r\partial_r$ term. The operator $r\partial_r$ is the bulk realisation of the conformal dimension Δ in the CCFT. The realisation of the CCFT algebra from the null infinity limit of the Poincaré algebra provides additional motivation that CCFT captures the kinematics of massless scattering between null infinities.

By comparing the CCFT algebras obtained via the $c \to 0$ limit of the CFT algebra on $\mathbb{M}^d$ in Cartesian coordinates, and the null infinity limit of the Poincaré algebra on $\mathbb{M}^{d+1}$ in flat null coordinates, we notice that they are the same as long as we identify the t coordinate in the CCFT after the $c \to 0$ limit with the u coordinate in the CCFT on $\mathscr{I}^\pm$. This can also be seen at the level of the metric. The resulting Carrollian metrics after the limits, (1.3.129) and (1.3.139), are equivalent once we identify the coordinate $x^{0\prime}$ with u.

1.3.2 Flat limit of AdS/CFT

A complementary approach to flat holography is to consider the vanishing cosmological constant $\Lambda \to 0$ limit (flat limit) of AdS/CFT. Consider the $(d+1)$ dimensional Euclidean AdS metric in Poincaré coordinates

$$
ds^2_{AdS_{d+1}} = \frac{L^2}{z^2}(dz^2 + \delta_{ab}dx^a dx^b);\quad a,\, b \in \{1, 2, ..., d\}
$$

With $z = L\, e^{\tau/L}$, and flat limit $L \to \infty$ ($\Lambda \to 0$), we recover the Euclidean flat metric,

$$
ds^2_{AdS_{d+1}} = d\tau^2 + e^{-2\tau/L}\delta_{ab}dx^a dx^b \xrightarrow[L\to\infty]{} ds^2_{\text{flat}_{d+1}} = d\tau^2 + \delta_{ab}dx^a dx^b.
$$

We can Wick rotate this to the $(d+1)$ dimensional flat Minkowski metric by defining $\tau = it$, where now t parametrizes time. The flat limit is manifest geometrically at the level of the background metric, and we expect the associated physics on these spacetimes to admit a similar flat limit.

The observables in flat space correspond to scattering amplitudes and a whole compendium of inclusive observables where the time-ordering is relaxed [114]. These include expectation values of fields as well as out-of-time-order amplitudes. All these observables correspond to free states that start at asymptotic infinity, interact in the bulk, and end at asymptotic infinity. In asymptotically AdS spacetimes, the natural gauge-invariant observables are boundary correlators, given by insertions of local operators on the conformal timelike boundary. We saw in the previous sections that observables involving massless fields can be captured using Carrollian CFTs that naturally occur on null infinities. In this flat limit, we will show that the AdS correlators naturally give the flat amplitudes. Through the flat limit, we can reproduce any flat observable, including out-of-time-ordered amplitudes, by starting with the corresponding AdS/CFT correlator. We will, however, only discuss time-ordered observables. Unlike the Carrollian approach that has so far been explored for massless scattering, the flat limit of AdS/CFT correlators works for both massive and massless fields.

In this section, as an illustrative example we shall consider the simple case of massive scalar ϕ^3 theory in AdS and show how the flat limit of AdS, using the momentum space CFT techniques, works for the massive fields. In particular, we shall show how the tree level AdS 3-point function reduces to the 3-point vertex factor in the flat spacetime. The approach follows the momentum space techniques developed in [147]. Consider a scalar field of mass m in Euclidean AdS$_{d+1}$ (of radius L) with a cubic interaction described by the action

$$S = \int d^{d+1}x\sqrt{G}\left(\frac{1}{2}\partial_M\phi\,\partial^M\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{3!}\lambda\,\phi^3\right),$$

where the index M runs from 0 to d. We work in the Poincaré patch with coordinates $\{z, x^\mu\}$ where μ runs from 1 through d. The corresponding AdS line element is listed in (E.1.2).

By Fourier transforming with respect to the coordinates x^μ, we can extract the regular solution of the free field equation of motion ($(\Box - m^2)\phi = 0$) as

$$\phi(z, \mathbf{k}) = \mathcal{K}(z, k)\,\hat{\phi}^{(0)}(\mathbf{k}), \tag{1.3.142}$$

where $\hat{\phi}^{(0)}(\mathbf{k})$ is related to the boundary value of the field as $\hat{\phi}^{(0)}(\mathbf{k}) = \lim_{z\to 0} z^\Delta\phi(z, \mathbf{k})$, and $\mathcal{K}$ is the scalar bulk-to-boundary (Btb) propagator given in the Poincaré coordinate by

$$\mathcal{K}(z,k) = \frac{2^{\frac{d}{2}-\Delta+1}}{L^{d-\Delta-2}\Gamma[\Delta-\frac{d}{2}]}\, k^{\Delta-\frac{d}{2}} z^{d/2}\, K_{\Delta-\frac{d}{2}}(k\,z). \tag{1.3.143}$$

Here, the mass of the field ϕ is related to the conformal dimension Δ of the dual boundary operator as $\Delta(\Delta-d) = m^2L^2$. The k denotes the magnitude of the boundary momenta defined by $k^2 = \delta^{\mu\nu}k_\mu k_\nu$ (where $\delta^{\mu\nu}$ denotes the induced (inverse) Euclidean metric on the conformal boundary). The $K_\beta(z)$ is the modified Bessel function of the second kind.

The tree level CFT 3-point function in the momentum space computed using the AdS/CFT correspondence is given by [148]

$$\langle O(\mathbf{k}_1)O(\mathbf{k}_2)O(\mathbf{k}_3)\rangle = (2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\lambda L^{d+1}\left(\frac{2^{\frac{d}{2}-\Delta+1}}{\Gamma[\Delta-\frac{d}{2}]}\right)^3 I_{\frac{d}{2}-1\{\Delta-\frac{d}{2},\Delta-\frac{d}{2},\Delta-\frac{d}{2}\}}, \tag{1.3.144}$$

where $I_{\alpha\{\beta_1,\beta_2,\beta_3\}}$ is the triple K integral defined by

$$I_{\alpha\{\beta_1,\beta_2,\beta_3\}}(k_1,k_2,k_3) \equiv \int_0^\infty dz\, z^\alpha \prod_{i=1}^{3} k^{\beta_i}K_{\beta_i}(k_i\,z)\,, \tag{1.3.145}$$

whenever this integral converges. To take the flat limit, it is useful to consider the variable τ defined by $z = L\,e^{\tau/L}$. Further, if we want to keep the mass m of the particle fixed while taking the flat limit, we need to send both Δ and L to infinity keeping their ratio fixed to m. In this limit, the modified Bessel function behaves as [147]

$$K_{\Delta-\frac{d}{2}+s}(kz) = \left(\frac{\pi}{2EL}\right)^{\frac{1}{2}}\left(\frac{k}{m+E}\right)^{-mL-s} e^{-EL-E\tau}\left(1+O\left(\frac{1}{L}\right)\right), \tag{1.3.146}$$

with $E = \sqrt{k^2+m^2}$. The uplifted Euclidean $(d+1)$ dimensional momentum is defined to be $q^a = (\pm iE, k^\mu)$ with $q^2 = \delta^{ab}q_aq_b = -m^2$. The choice of signs will corresponds to in and out going states when we subsequently Wick rotate. Since we work only at tree level, we shall take all particles to be incoming and hence shall exclusively choose the + sign.

Using the identity (1.3.146), we find that in the large L limit, the bulk-to-boundary propagator takes the form,

$$\mathcal{K}(\tau,k) = \frac{e^{-E(\tau+L)}}{Z^{1/2}}\left(1+O\left(\frac{1}{L^2}\right)\right) \quad ; \qquad \frac{1}{Z^{1/2}} \equiv \left(\frac{2^{-mL}\,e^{mL}\,(L(m+E))^{mL}}{(EL)^{1/2}\,(mL)^{mL-\frac{1}{2}}}\right).$$

The factor $\frac{1}{Z^{1/2}}$ diverges as we send L to infinity . Naively, due to this, the field ϕ in (1.3.142) also diverges in this limit. To resolve this issue, we note that the source $\hat{\phi}^{(0)}(\mathbf{k})$

is arbitrary. Therefore, we can define a renormalised source as

$$\hat{\phi}^{(0)}_R(\mathbf{k}) = \frac{\hat{\phi}^{(0)}(\mathbf{k})}{Z^{1/2}}, \tag{1.3.147}$$

and scale $\hat{\phi}^{(0)}(\mathbf{k}) \to 0$ so as to keep $\hat{\phi}^{(0)}_R(\mathbf{k})$ finite. This can be done since $\hat{\phi}^{(0)}(\mathbf{k})$ is a free parameter. The conservation of spatial components of momenta is manifest from (1.3.144) due to the presence of the d dimensional delta function $\delta^d(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)$. The energy conserving delta function emerges by writing $E_i = -iq^0_i$ and integrating over τ variable as

$$\int_{-\infty}^{\infty} d\tau\, e^{-\tau(E_1+E_2+E_3)} = \int_{-\infty}^{\infty} d\tau\, e^{i\tau(q^0_1+q^0_2+q^0_3)} = (2\pi)\delta(q^0_1 + q^0_2 + q^0_3). \tag{1.3.148}$$

To make contact with Lorentzian scattering amplitudes, we Wick rotate. The $d+1$ dimensional flat space Lorentzian momenta are hence defined as $q^a = (E, k^\mu)$. With this, in the flat limit, the AdS 3-point function reduces to

$$\sqrt{Z_1 Z_2 Z_3}\Big\langle O(\mathbf{k}_1)O(\mathbf{k}_2)O(\mathbf{k}_3)\Big\rangle\Big|_{L\to\infty} = (2\pi)^{d+1}\delta^d(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)\,\delta(E_1 + E_2 + E_3)i\lambda,$$

which is the expected tree level 3-point vertex in the ϕ^3 theory in the $d+1$ dimensional flat space.

1.3.3 Relation between the Carroll limit $c \to 0$ and flat limit $\Lambda \to 0$

The Carroll limit, speed of light $c \to 0$ of the CFT on the conformal boundary of AdS, is directly related to the flat limit cosmological constant $\Lambda \to 0$ in AdS. To illustrate this it is useful to consider AdS in AdS-Bondi coordinates. Consider $\mathcal{R}^{(3,2)}$

$$ds_5^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 - (dX^4)^2 \tag{1.3.149}$$

AdS_4 can be embedded as a hypersurface in this space satisfying,

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 = -L^2 \tag{1.3.150}$$

or

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 - X^+X^- = -L^2 \tag{1.3.151}$$

where $X^\pm = X^4 \pm X^3$. We can choose a parametrization satisfying the hypersurface constraint

$$X = (X^+, X^-, X^0, X^1, X^2) = r\left(1, \frac{2u}{r} - \frac{u^2}{L^2} + 2z\bar{z}, \frac{u}{L} - \frac{L}{r}, \frac{z+\bar{z}}{\sqrt{2}}, \frac{z-\bar{z}}{\sqrt{2}i}\right) \tag{1.3.152}$$

which gives

$$ds^2_{4,AdS} = \frac{-r^2}{L^2}du^2 - 2dudr + 2r^2 dzd\bar{z} \tag{1.3.153}$$

which is the analogue of Bondi coordinates in global AdS. Here $z = e^{i\phi}\cot(\theta/2)$ maps $\{z, \bar{z}\}$ to the more familiar coordinates $\{\theta, \phi\}$ that parametrizes the unit sphere. Rescaling the metric by $1/r^2$ and taking $r \to \pm\infty$ yields the boundary metric

$$ds^2_{\partial AdS} = \frac{-1}{L^2}du^2 + 2dzd\bar{z} \tag{1.3.154}$$

In the flat limit $\Lambda \to 0$ which corresponds to $L \to \infty$, the AdS metric (1.3.153) reduces to the Minkowski line element

$$ds^2_{4,flat} = -2dudr + 2r^2 dzd\bar{z} \tag{1.3.155}$$

and the AdS boundary metric (1.3.154) reduces to the degenerate metric

$$ds^2_{\partial flat} = 0 \cdot du^2 + 2r^2 dzd\bar{z} \tag{1.3.156}$$

which is part of the Carrollian structure at $\mathscr{I}^{\pm}$. The Carroll limit $c \to 0$ of the CFT on the conformal boundary of AdS achieves a similar Carrollian theory.

1.4 Guide to the upcoming chapters

In chapter (2), following [111], we compute entanglement entropy on certain curved backgrounds in holographic field theories using the geometric gravity dual given by the Ryu-Takayanagi prescription (1.2.81).

Since entanglement entropy is a field theory measure quantifying the degree of entanglement between subsystems (regions), it depends on the field theory spectrum. The leading area piece of the black hole entropy depends on geometry. We will show that the quantum corrections to the black hole entropy given by the QES/ island prescription (1.2.93), that preserve information during black hole evaporation, impose certain conditions on the outgoing radiation spectrum. In chapter (3), following [149], we will establish these conditions.

In chapter (4), following [150], and using the replica trick, we will discuss the analytic computation of the scaling behavior of entanglement entropy in static black hole backgrounds that is necessary for deriving these conditions. Our setup applies to generic variations, from symmetry transformations to variations of the background metric or entangling region. Our methodology elegantly reproduces and generalises results from the literature on entanglement entropy in different dimensions, backgrounds, and states.

We use these analytic expansions to explore the behaviour of entanglement entropy in static black hole backgrounds under specific scaling transformations. Using these results, we can find explicit conditions on the outgoing black hole radiation spectrum for the presence of islands that reproduce the Page curve and restore unitarity in black hole evaporation.

In the second part, we discuss in detail *Flat Carrollian holography* and flat limits of AdS/CFT. In chapter (5), following [151], we consider the Carrollian conformal field theories involving scalar operators in the momentum representation. The momentum space Ward identities are explicitly solved to obtain the different branches of 2 and 3 point Carrollian conformal correlators in the momentum space. The different branches are characterized by different analytic structures in the Carrollian energies. For specific values of the conformal dimensions, the three-point functions in momentum space exhibit logarithmic behaviour. This has no analogue in position space and instead originates from singularities in the Fourier transform relating position and momentum space correlators. We also analyze the Carrollian limit of CFT 2 and 3 point functions of scalar operators in momentum space. By taking different scalings of CFT correlators with respect to the speed of light, we obtain different branches of the Carrollian conformal correlators in the momentum space.

In chapter (6), following [152], we discuss a generating functional for these Carrollian CFT correlators directly analogous to AdS/CFT. Starting with scalar fields in flat space written in flat null coordinates, we compute the on-shell action. With the right fall-off conditions for the classical solutions at null infinity and a suitable choice of source, we show that this on-shell action generates the Carrollian CFT correlators derived earlier.

A complementary approach to flat holography is to take suitable flat limits of AdS/CFT correlators to reproduce flat amplitudes. Simultaneously, we intend to systematically study massive higher spin fields that are required in a consistent gravitational theory. In chapter (7), following [153], we analyze a bulk effective field theory in AdS containing a U(1)-charged massive spin-2 field coupled to a gauge field, by performing the required holographic renormalization, and computing the one and two-point functions. We then compute the renormalized bulk three-point function involving two massive spin-2 fields and one gauge field. Matching with the CFT 3-point correlator of two non-conserved spin-2 operators and a conserved current, we obtain explicit mappings between the bulk minimal and gyromagnetic couplings and the boundary OPE data. Finally, we take the flat-space limit of the momentum space CFT correlator and verify that the resulting amplitude matches the expected flat-space structure.

Part I

Black hole Information and Entanglement Entropy

Chapter 2

Holographic analysis of entanglement entropy and islands

2.1 Introduction

Entanglement entropy has been well studied in two dimensions [55, 67], but is still much less understood in higher dimensions. In a quantum field theory it is typically studied using the replica trick (1.2.67), and heat kernel techniques, for vacuum states at zero temperature. It is subtle at finite temperatures since quantum and thermal effects mix [154–156]. Computation of entanglement entropy for generic excited states using standard quantum field theoretic methods such as heat kernels is challenging. For free massive quantum fields one can explore entanglement entropy in curved backgrounds using heat kernel techniques [157] but such methods are not applicable in the massless limit.

Our motivations for computing entanglement entropy in more general setups (curved backgrounds related to black holes, higher dimensions, and generic states), derive from the recent discussions of islands in the context of the black hole information loss paradox, that we discussed in detail in the previous chapter (1.2.6). Many of the papers discussing quantum extremal islands have explored the specific setup of two-dimensional JT gravity with the quantum fields being described by a two-dimensional conformal field theory. The use of a two-dimensional conformal field theory leads to a number of conceptual and technical simplifications, which we summarize in table 2.1. Technically, the required computations of the quantum field theory entanglement entropies make use of well-studied and well-known results for two dimensional CFTs, for single and multiple intervals. Entanglement entropy for a conformal theory in two dimensions also has a number of conceptual simplifications. Firstly, a two-dimensional metric can always be parameterised in terms of a single conformal factor; this means that we can characterise entanglement in a background with a generic metric in terms of the conformal factor.

Secondly, in two dimensions a black hole horizon is of dimension zero and one can use coordinate transformations to relate spacetimes with and without horizons. Accordingly, one can obtain the finite temperature entanglement from the zero temperature results using such coordinate transformations.

Another important conceptual and computational aspect of the two dimensional computations relates to the regularisation of the quantum field theory contribution. In the condensed matter literature it is standard to work with regulated entanglement entropy, rather than renormalise; the UV regulator is related physically to the lattice scale of the system of interest. Much of the literature computing entanglement entropy therefore focuses on regulated expressions. In a two-dimensional conformal field theory the regulated divergences are logarithmic, related to the conformal anomaly; the coefficients are proportional to the central charge and these contributions are therefore often termed universal.

The expression (1.2.93) is however a semi-classical gravity expression and in semi-classical gravity one usually works with renormalised quantum field theory contributions of regulated expressions. It was noted in [158] that finite contributions to the entanglement entropy can be obtained through differentiation with respect to a scale of the entangling region (see also [159] for a review). For example, for a region of scale l in a two-dimensional conformal field theory, the expression

$$l\frac{\partial S_{\mathrm{CFT}}}{\partial l} \tag{2.1.1}$$

is automatically finite because the (local) regulated UV divergences are independent of the (non-local) entangling region scale. There are analogous expressions for higher-dimensional CFTs involving higher numbers of derivatives. Separately, systematic renormalisation of quantum field entanglement entropy has been developed in [160]; in this approach the renormalisation of the entanglement entropy is inherited from the renormalisation scheme of the partition function.

When one applies (1.2.93) in the context of JT gravity and two-dimensional conformal field theory, one determines the quantum extremal surface by extremising with respect to the scale of the region B. This extremisation is independent of the regulator because of (2.1.1) (although the value of the actual extremised entropy in (1.2.93) does still depend on the regulator). This property is however specific to two-dimensional CFT: the first derivative of the entanglement entropy in a conformal field theory in $d > 2$ is in general not finite.

Models of islands in higher dimensions were first discussed in [92] and later in [93–95]. These are known as double holographic models (see also [96]) where gravity in an AdS_{d+1} region bounded by a pair of end-of-the-world branes is dual to a CFT_{d-1} theory living on the interception between the branes. These are examples of Karch-Randall braneworld models [97–101]. Despite the renewed interest in these models and their recent success, there are persistent challenges that arise when doing holography with AdS with a cutoff. In this chapter, we will not explore these models further.

Explicit realization of islands in higher dimensions, without resorting to braneworld models, stems in part from the lack of analytical control over entanglement entropy of the quantum fields in $d > 2$.

In this chapter, following [111], we will explore entanglement entropy in specific types of curved backgrounds, namely those related to black holes, mostly using the holographic Ryu-Takayanagi prescription (1.2.81).

In our construction, we will consider a nearly AdS_2 spacetime (NAdS_2), *i.e.*, an AdS_2 spacetime with a nontrivial dilaton profile, and perform a circular uplift, à la Kaluza-Klein, which results in AdS_3 with a compact direction. The two-dimensional dilaton becomes a dimensionful parameter in the AdS_3 metric. We then consider an appropriate field theory on the AdS_3 background. For an entangling region in this background, we compute the entanglement entropy by placing the AdS_3 spacetime on the boundary of AdS_4 and using holography. Using the Ryu-Takayanagi (RT) formula [80], the quantum field theory entanglement entropy S_{QFT} can be computed holographically by computing the area of a co-dimension two surface homologous to the entangling region. Since the fields inherited from the parent action in this model have a scale, they respect the symmetries of the background. In the setup where JT gravity with a bath is coupled to a $2d$ CFT, the explicit and spontaneous breaking of the symmetries in JT gravity renders the background incompatible with the symmetry of the $2d$ conformal fields.

We consider an annulus entangling region that naturally captures the circular extra dimension and due to the isometry in the circular direction, the entanglement entropy will only depend on one coordinate (the width of the annulus).

We generalize this construction to d-dimensions and explicitly evaluate the renormalised entanglement entropy for zero temperature states in a black hole background. This quantity has interesting applications on its own to condensed matter physics and cold atom systems in particular (see [161] for a review). In the context of islands, we are interested in the thermal entanglement entropy[1]. Near the UV boundary, the regulated entanglement entropy is dominated by the divergent piece whose structure is the same in the zero and finite temperature case, so we may use the covariant counterterm of the zero temperature renormalised entropy as the thermal regulated entropy. We place the entangling region near the UV boundary and demonstrate the non-generic existence of islands in higher dimensions.

[1]As Hawking originally pointed out [162], an observer would need to collect measurements over an infinite time scale at spatial infinity to differentiate between a mixed or pure initial state of the black hole. In the quasi-stationary regime (on timescales small in comparison with the evaporation scale, or when the entangling region is far away from the horizon) the Hawking radiation is described by an out-of-equilibrium process and we may work with the near thermal state approximation.

Islands in the $2d$ JT gravity setting	**Islands in $\mathrm{AdS}_{d\geq 3}$**
The matter fields are added to an already reduced gravitational theory. It is assumed that they do not couple to the dilaton: $\mathcal{I}_{\mathrm{JT}}[\Phi, g_{\mu\nu}] + \mathcal{I}_{\mathrm{CFT}_2}[g_{\mu\nu}]$.	The matter fields are inherited from the pure $\mathrm{AdS}_{d\geq 3}$ action and do couple to the dilaton: $\mathcal{I}[g_{\mu\nu}, \Phi]$.
The matter fields are scale-invariant and do not respect the symmetries of the background.	The matter fields have generalized conformal structure and do respect the symmetries of the background.
The structure of the divergence obscures if entropy is renormalised or regulated as it coincides with the logarithmic Weyl anomaly term.	Clear distinction between regulated and renormalised quantities. For $d > 2$ the divergence follows a power law: $S \sim \epsilon_{UV}^{d-2} A_{d-2}$ where A_{d-2} is the area of the boundary of the entangling region.
Black holes with horizons can be mapped to black holes without horizons. Entropy at finite and zero temperature is thus related via a coordinate transformation.	The background has an inherent temperature in quasi-thermal equilibrium with the temperature of the thermal fields, over small timescales relative to the evaporation process.
Heat baths collecting the Hawking radiation are required, in which an entangling region can be consistently defined due to the enhanced diffeomorphism invariance and topological nature of $2d$ gravity.	An entangling region can be consistently defined in the AdS region. We impose transparent boundary conditions at the conformal boundary but remain agnostic about what is on the other side. As elaborated on in [150], all information should be retrievable within the AdS region.
The first derivative of the entanglement entropy with respect to the scale of the entangling region, l, say: $l\frac{\partial S_{\mathrm{CFT}}}{\partial l}$ is always finite as the regulated UV divergences are independent of the (non-local) entangling region scale.	The first derivative of the entanglement entropy with respect to the scale of the entangling region is in general not finite.
Parent theory is a $4d$ near extremal black hole, with an $\mathrm{AdS}_2 \times S^2$ near horizon region.	Parent theory is pure AdS.

TABLE 2.1: Summary of conceptual and computational differences between working with a $2d$ CFT placed on JT gravity and circularly uplifted $\mathrm{AdS}_{d\geq 3}$.

2.2 Uplifting NAdS_2 to AdS_3 and beyond

Holography for AdS_2 has gone through investigation in recent years but remains less understood than its higher dimensional siblings. The main reason is that pure AdS_2

is over-constrained by its symmetries and there is no consistent notion of energy excitations above the vacuum [163–166]. This problem was partly resolved by Almheiri and Polchinski [167] by considering the leading order correction away from pure AdS$_2$ which gives 'nearly' AdS$_2$ or just NAdS$_2$. This leading order correction away from AdS$_2$ has a universal form in the sense that the gravitational backreaction can be described by a universal AdS$_2$ dilaton gravity. This two-dimensional dilaton gravity matches the leading non-conformal effects in the low energy limit of the SYK model [168, 169].

The two-dimensional dilaton gravity with a Maxwell field describes the very near horizon effective theory of five dimensional nearly extremal black holes [166], and can be obtained by a consistent circle reduction from pure AdS$_3$. Letting the Maxwell field consistently vanish gives us Jackiw–Teitelboim (JT) gravity [170, 171] (see [172] for a review) which locally has AdS$_2$ geometry with a running dilaton.

An important aspect of the holography of JT gravity is the breaking of conformal symmetry. The asymptotic symmetries of AdS$_2$ are time reparametrizations of the boundary and are also explicitly broken in JT gravity when we consider small deformation away from pure AdS$_2$[2].

There is an inherent relationship between NAdS$_2$ with a dilaton and pure AdS$_3$ in which the dilaton becomes a parameter in the metric.

The realization of a holographic description of AdS$_2 \times S^2$ or AdS$_2 \times S^3$ requires a consistent Kaluza-Klein reduction of an uplifted theory over the compact manifold. This was shown for supergravity solutions of Dp-branes and fundamental strings, which in the near-horizon limit are conformal to AdS$_{p+2} \times S^{8-p}$ [173]. These solutions can be best analysed in the dual frame defined and reviewed in [173]. The Weyl transformation to the dual frame in ten dimensions results in the following action

$$\mathcal{I}_{10} = -\frac{N^2}{(2\pi)^7\alpha'^4}\int d^{10}x\sqrt{G}N^{\gamma}e^{\gamma\phi}\left(R(G) + \beta(\partial\phi)^2 - \frac{1}{2(8-p)!N^2}|F_{8-p}|^2\right). \tag{2.2.2}$$

Here α' is the inverse string tension, G_{MN} is the 10-dimensional metric, R is the Ricci scalar, ϕ is the dilaton, F_{8-p} is an $(8-p)$-form Ramond-Ramond field strength, N denotes the integral quantized charge, and (β, γ) are constants given by

$$\begin{aligned}&\text{Dp-branes:}\quad \beta = \frac{4(p-1)(p-4)}{(7-p)^2}, \quad \gamma = \frac{2(p-3)}{7-p},\\ &\text{Fundamental strings:}\quad \beta = 0, \quad \gamma = \frac{2}{3}\end{aligned} \tag{2.2.3}$$

One can reduce the field equations of the above action over the sphere, truncating to the $(p+2)$-dimensional graviton $g_{\mu\nu}$ and scalar ϕ. The equations of motion for the

[2] JT gravity is topological and has no propagating degrees of freedom so the gravitational backreaction is governed by its symmetries.

lower-dimensional fields for both the Dp-branes and fundamental strings follow from an action of the form:

$$\mathcal{I}_{d+1} = -\mathcal{N}\int d^{d+1}\sqrt{g}e^{\gamma\phi}\Big(R+\beta(\partial\phi)^2+C\Big), \quad d=p+1. \tag{2.2.4}$$

For Dp-branes,

$$\begin{aligned} C &= \frac{(9-p)(7-p)\,\mathcal{R}^2}{2}, \quad \mathcal{R}=\frac{2}{5-p}, \\ \mathcal{N} &= \frac{(d_pN)^{(7-p)/(5-p)}g_d^{2(p-3)/(5-p)}\mathcal{R}^{(9-p)/(5-p)}}{64\pi^{(5+p)/2}(2\pi)^{(p-3)(p-2)/(5-p)}\Gamma\left(\frac{9-p}{2}\right)}, \quad d_p=(2\sqrt{\pi})^{5-p}\gamma\left(\frac{7-p}{2}\right). \end{aligned} \tag{2.2.5}$$

where g_d^2 is the Yang-Mills coupling related to the string coupling g_s, by

$$g_d^2 = g_s(2\pi)^{p-2}(\alpha')^{(p-3)/2}. \tag{2.2.6}$$

For fundamental strings,

$$C=6, \quad \mathcal{N}=\frac{g_sN^{3/2}(\alpha')^{1/2}}{6\sqrt{2}}. \tag{2.2.7}$$

For arbitrary β, d and γ, the equations of motion for the scalar field and metric are [173]

$$\begin{aligned} &-R_{\mu\nu}+(\gamma^2-\beta)\partial_\mu\phi\partial_\nu\phi+\gamma\nabla_\mu\partial_\nu\phi-\frac{\gamma^2+d(\gamma^2-\beta)}{\gamma^2-\beta}g_{\mu\nu}=,0 \\ &\nabla^2\phi+\gamma(\partial\phi)^2-\frac{\gamma\left(d(\gamma^2-\beta)+\gamma^2\right)}{(\gamma^2-\beta)^2}=0, \\ &R+\beta(\partial\phi)^2+\frac{\left(d(\gamma^2-\beta)+\gamma^2\right)\left(d(\gamma^2-\beta)-\beta\right)}{(\gamma^2-\beta)^2}=0. \end{aligned} \tag{2.2.8}$$

We express the solutions in the $(d+1)$-dimensional coordinates $x^\mu \equiv \{\rho, x^i\}$ with $i \in \{1,2,\cdots,d\}$. The equations of motion admit AdS_{d+1} solutions with a running scalar.

$$ds^2_{d+1}=\frac{1}{\rho^2}\Big(d\rho^2+\delta_{ij}dx^idx^j\Big), \quad e^\phi=\rho^{2\alpha} \tag{2.2.9}$$

provided that [174]

$$\alpha=-\frac{\gamma}{2(\gamma^2-\beta)}, \quad C=\frac{\Big(d(\gamma^2-\beta)+\gamma^2\Big)\Big(d(\gamma^2-\beta)+\beta\Big)}{(\gamma^2-\beta)^2}. \tag{2.2.10}$$

We can thus treat both Dp-brane and fundamental string cases simultaneously, by processing the field equations for arbitrary (d,β,γ) and writing (α, C) in terms of these parameters. With a unit AdS radius, the radial coordinate ρ has dimensions of $(\text{length})^2$ while the dilaton e^ϕ has dimensions $(\text{length})^{2\alpha}$.

The action (2.2.4) can always be interpreted as the reduction of an AdS gravity theory in $(2\sigma+1)$ dimensions [174, 175] where

$$2\sigma = (d - 2\alpha\gamma). \tag{2.2.11}$$

σ can take non-integer values. This reduction ansatz is over a $(2\sigma - d)$-dimensional torus,

$$ds^2_{2\sigma+1} = ds^2_{d+1} + e^{2\gamma\phi}(dz.dz)_{2\sigma-d} \tag{2.2.12}$$

where the first term on the RHS is (2.2.9) and the second term is the torus metric.

A particular case of interest in the context of nearly AdS$_2$ (NAdS$_2$) is $\beta = 0$, where (2.2.10) implies

$$2\alpha\gamma = -1, \quad C = d(d+1). \tag{2.2.13}$$

In the NAdS$_2$ case, the reduction is over a circle from AdS$_3$. Since the scalar kinetic term vanishes, it can always be rescaled. It is convenient to choose this rescaling such that $\gamma = 1$. There is an immediate generalization to a reduction including a Kaluza-Klein gauge field A_μ:

$$ds^2 = ds^2_{d+1} + e^{2\phi}\Big(dy + A_\mu dx^\mu\Big)^2 \tag{2.2.14}$$

The scalar field ϕ controls the radius of the circle and from here on we will denote the dilaton e^ϕ as Φ. The resulting reduced action is

$$\mathcal{I} = -\mathcal{N}\int d^{d+1}x\sqrt{-g_{d+1}}\,\Phi\left(R^{(d+1)} + d(d+1) - \frac{1}{4}\Phi^2 F_{\mu\nu}F^{\mu\nu}\right) + \mathcal{I}^{(d+1)}_{\text{ct}}. \tag{2.2.15}$$

Here, $\mathcal{N}$ is a normalization constant, g_{d+1} denotes the determinant of the $(d+1)$-dimensional metric, $R^{(d+1)}$ is the corresponding Ricci scalar, $\mathcal{I}^{(d+1)}_{\text{ct}}$ are the counterterms including the Gibbons-Hawking-York boundary terms, and $F_{\mu\nu}$ is the Maxwell term from the Kaluza-Klein gauge field. We will work in the $A_\mu = 0$ limit so that the metric does not have any off-diagonal components on the circle. We thus get a simple diagonal uplift$\leftrightarrow$reduction. With the AdS$_3$ metric normalized to have radius one, we get

$$ds^2_3 = ds^2_2 + \Phi^2 dy^2 \tag{2.2.16}$$

where

$$ds^2_2 = \frac{1}{x^2}\left(-dt^2 + dx^2\right); \quad \Phi = -\frac{\phi_c}{x}. \tag{2.2.17}$$

When reducing AdS_3 over a circle, we take the periodicity of the y-direction to be 2π. We start from the AdS_3 action

$$\mathcal{I}_3 = -\mathcal{N}\int d^3x\sqrt{-g_3}\left(R^{(3)}+2\right), \quad \mathcal{N} = \frac{1}{16\pi G_3}. \tag{2.2.18}$$

where $R^{(3)}$ is the Ricci scalar corresponding to the metric (2.2.16), and g_3 is its determinant. Integrating over the circular direction gives the reduced action of the form

$$\mathcal{I}_2 = -\hat{\mathcal{N}}\int d^2x\sqrt{-g_2}\,\Phi\left(R^{(2)}+2\right) + S_{\mathrm{ct}}, \quad \hat{\mathcal{N}} = \frac{1}{16\pi G_2}, \tag{2.2.19}$$

where $R^{(2)}$ is the Ricci scalar corresponding to the metric (2.2.17), g_2 is its determinant, S_{ct} is a boundary term, and $\hat{\mathcal{N}} = 2\pi\mathcal{N}$, *i.e.*, $\frac{1}{G_2} = \frac{2\pi}{G_3}$. As a boundary condition, we fix the value of the dilaton Φ close to the conformal boundary. At the conformal boundary $x = -\epsilon$, with $\epsilon \to 0$ we have

$$\Phi|_{x=-\epsilon} = \frac{\phi_c}{\epsilon}. \tag{2.2.20}$$

Making the change of coordinates $x^+ = t + x$ and $x^- = t - x$, we have

$$ds_2^2 = \frac{-4dx^+dx^-}{(x^- - x^+)^2}; \quad \Phi = \frac{2\phi_c}{x^- - x^+} \tag{2.2.21}$$

with a unit curvature length. Now letting $x^\pm = \tanh\frac{\pi y^\pm}{\beta}$ we get

$$ds_2^2 = -\left(\frac{4\pi^2}{\beta^2}\frac{dy^+dy^-}{\sinh^2\left[\frac{\pi}{\beta}(y^- - y^+)\right]}\right); \quad \Phi = \frac{2\pi\phi_c}{\beta}\frac{1}{\tanh\frac{\pi}{\beta}(y^- - y^+)}. \tag{2.2.22}$$

This is the familiar two-dimensional metric considered in [12]. Now, let

$$y^\pm = t \pm \sigma, \tag{2.2.23}$$

with the Lorenzian time $t = -i\tau$. Taken together, we can write the uplifted three-dimensional metric as

$$ds_3^2 = \left(\frac{2\pi}{\beta}\right)^2\left[\frac{d\sigma^2 + d\tau^2}{\sinh^2\left(\frac{2\pi\sigma}{\beta}\right)} + \frac{\phi_c^2 dy^2}{\tanh^2\left(\frac{2\pi\sigma}{\beta}\right)}\right], \quad \Phi = -\frac{2\pi}{\beta}\phi_c\frac{1}{\tanh\left(\frac{2\pi\sigma}{\beta}\right)}. \tag{2.2.24}$$

We can also move to the more familiar Schwarzschild coordinates via

$$\sigma = \left(\frac{\beta}{2\pi}\log\frac{r}{\sqrt{r\left(r+\frac{4\pi}{\beta}\right)}}\right) \tag{2.2.25}$$

in which the metric becomes

$$ds_3^2 = r\left(r + \frac{4\pi}{\beta}\right) d\tau^2 + \frac{dr^2}{r\left(r + \frac{4\pi}{\beta}\right)} + \phi_c^2 \left(r + \frac{2\pi}{\beta}\right)^2 dy^2. \tag{2.2.26}$$

This is the metric for an eternal black hole. The horizon is at $r = 0$, where the radius of the y circle remains finite. In two dimensions, going between Poincaré and global Schwarzschild coordinates corresponds to going from zero to finite temperature. However, in higher dimensions, this is not the case.

We will now proceed to compute the entanglement entropy of a field theory (with the entangling region specified below) on this AdS_3 background.

2.3 Dimensional analysis

On a fixed Cauchy slice, we study a cylindrical entangling region with topology $\mathbb{R}_x \times S_y^1$, where $\mathbb{R}_x$ is restricted to the interval $[x_1, x_2]$ and the circumference of S_y^1 is L_y. This region is topologically equivalent to an annulus.

Flat Background: Consider first a CFT_3 on the flat background

$$ds_3^2 = -dt^2 + dx^2 + \phi_c^2 dy^2. \tag{2.3.27}$$

At zero temperature, we have translational invariance in the x-direction *i.e.*, the entanglement entropy only depends on $\Delta x = (x_2 - x_1)$ and not on x_1 and x_2 individually. We also have isometry along the y-direction, so the entanglement entropy must be proportional to the circumference L_y. This constrains the zero temperature renormalised entanglement entropy to be of the form

$$S_{\text{ren}} \sim c\frac{L_y}{\Delta x} \tag{2.3.28}$$

where the dimensionless constant c measures the number of degrees of freedom, although it is not exactly a central charge.

At a finite temperature T, the renormalised entanglement entropy should take the form

$$S_{\text{ren}} \sim c\frac{L_y}{\Delta x} f\left(\frac{\Delta x}{T}\right) \tag{2.3.29}$$

where $f\left(\frac{\Delta x}{T}\right)$ is a function of dimensionless ratios which will reduce to (2.3.28) as $T \to 0$. For large T, the expression will asymptote to some power law:

$$S_{\text{ren}} \sim cL_y \frac{T^\alpha}{(\Delta x)^{\alpha+1}}. \tag{2.3.30}$$

AdS$_3$ Background: Now consider the case of a CFT$_3$ on an AdS$_3$ background (with AdS radius ℓ_3), which in Poincaré coordinates can be written as

$$ds_3^2 = \frac{\ell_3^2}{x^2}\left(-dt^2 + dx^2 + \phi_c^2 dy^2\right) \tag{2.3.31}$$

with the conformal factor $\Omega(x) = (\ell_3/x)$. This is a local conformal transformation as it depends on the position x and hence we have now lost the inherent translational invariance so the entanglement entropy generally depends explicitly on x_1 and x_2. The entangling region geometry has parameters (L_y, x_1, x_2), as well as the implicit parameter ℓ_3. We still have isometry along the y-direction, so the entanglement entropy is proportional to L_y. Hence, at zero temperature, we have

$$S_{\text{ren}} \sim c\frac{L_y}{x_{1,2}} g\left(\frac{\ell_3}{x_2}, \frac{x_1}{x_2}\right) \tag{2.3.32}$$

where $g\left(\frac{\ell_3}{x_2}, \frac{x_1}{x_2}\right)$ is a function of dimensionless ratios.

2.4 Entanglement entropy on a flat background

The renormalised entanglement entropy for a CFT$_3$ living on a flat background was studied in general dimensions in [160] for a disk entangling region. We will consider an annulus entangling region and compute the entanglement entropy of a CFT$_3$ on a flat background. To do this, we consider the entangling region on the conformal 3-dimensional flat boundary of AdS$_4$, and use the holographic Ryu-Takayanagi (RT) prescription [9]. Our metric is parametrized by ϕ_c, which controls the radius of the circular direction. The AdS$_4$ metric is given by

$$ds_4^2 = \ell^2 \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}(-dt^2 + dx^2 + \phi_c^2 dy^2) \tag{2.4.33}$$

where ρ is an auxiliary bulk direction into which the RT surface propagates. On a constant time Cauchy slice consider the entangling region A of an annulus with width L along the x direction, spanning all values of y: $0 \leq x \leq L$, $0 \leq y \leq 2\pi$, as illustrated in figure 2.1.

We work with a static gauge in which the time coordinate is constant, and the candidate RT surface is spanned by the worldvolume coordinates $x^\alpha = \{\rho, y\}$, with the AdS$_4$ embedding $x^m = \{t, \rho, x(\rho), y\}$. The candidate RT surface is symmetric along the y direction due to the isometry along y. It is also symmetric about the inflection point $x(\rho_0) = L/2$. When computing the minimal surface, we can thus consider only the area corresponding to $0 \leq x \leq L/2$ and then double it. The induced metric on the candidate

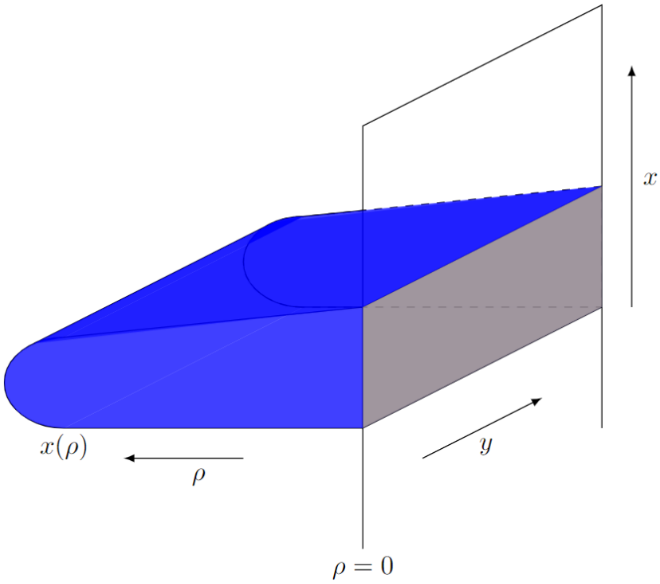


FIGURE 2.1: A slice of the annulus entangling region (brown), in the circular y-direction, at the constant time. The RT surface (blue) propagates into the bulk direction. The equation of motion $x(\rho)$ is the minimal surface minimizing the area functional.

RT surface is

$$h_{\rho\rho} = \frac{\ell^2}{4\rho^2} + \frac{1}{\rho}(\partial_\rho x)^2 , \quad h_{yy} = \frac{\phi_c^2}{\rho}. \tag{2.4.34}$$

The area functional to be minimized is

$$S = \frac{1}{4G_4}\int_0^{2\pi} dy \times 2\int_\epsilon^{\rho_0} d\rho\sqrt{h} = \frac{1}{2G_4}\int_0^{2\pi} dy \int_\epsilon^{\rho_0} d\rho\sqrt{\frac{\phi_c^2 x'(\rho)^2}{\rho^2} + \frac{\ell^2\phi_c^2}{4\rho^3}} \tag{2.4.35}$$

where ϵ is a UV-cutoff, and the factor of 2 in front of the ρ-integral is due to the symmetry around the inflection point. Now let us define

$$\mathcal{L} = \sqrt{\frac{\phi_c^2 x'(\rho)^2}{\rho^2} + \frac{\ell^2\phi_c^2}{4\rho^3}}. \tag{2.4.36}$$

Since there is no explicit $x(\rho)$ dependence the equation of motion simply takes the form

$$\frac{d}{d\rho}\left(\frac{\partial\mathcal{L}}{\partial x'(\rho)}\right) = 0 \tag{2.4.37}$$

which gives us

$$\frac{\phi_c x'(\rho)}{\rho^2\sqrt{\frac{\ell^2}{4\rho^3} + \frac{x'(\rho)^2}{\rho^2}}} = k \tag{2.4.38}$$

where k is the integration constant. Solving for $x'(\rho)^2$ gives

$$x'(\rho)^2 = \frac{k^2\ell^2\rho}{4\left(\phi_c^2 - k^2\rho^2\right)}. \tag{2.4.39}$$

At the turning point, ρ_0, we have the boundary condition $x'(\rho_0) \to \infty$ which fixes k as

$$k = \frac{\phi_c}{\rho_0}. \tag{2.4.40}$$

Solving further for $x(\rho)$ we get

$$x_\pm(\rho) = c_\pm \pm \frac{\ell\rho^{3/2}\,{}_2F_1\left(\frac{1}{2},\frac{3}{4};\frac{7}{4};\frac{\rho^2}{\rho_0^2}\right)}{3\rho_0} \tag{2.4.41}$$

There are two branches with $x_+(0) = 0$ which implies $c_+ = 0$, and $x_-(0) = L$ which implies $c_- = L$. Further using $x_+(\rho_0) = x_-(\rho_0)$, we get

$$\rho_0 = \frac{4L^2\Gamma\left(\frac{5}{4}\right)^2}{\ell^2\pi\Gamma\left(\frac{3}{4}\right)^2}. \tag{2.4.42}$$

Now, substituting (2.4.39) along with (2.4.40) into (2.4.35), we get the regulated entanglement entropy

$$S_{\text{reg}} = \frac{\pi\phi_c\ell}{2G_4}\int_\epsilon^{\rho_0} \frac{1}{\sqrt{\rho^3\left(1-\left(\frac{\rho}{\rho_0}\right)^2\right)}}d\rho. \tag{2.4.43}$$

By letting $x = \frac{\rho}{\rho_0}$ the above integral can be rewritten as

$$S_{\text{reg}} = \frac{\pi\phi_c\ell}{2G_4}\int_{\tilde{\epsilon}=(\epsilon/\rho_0)}^{1} \frac{1}{\sqrt{\rho_0}}\frac{x^{-\frac{3}{2}}}{\sqrt{1-x^2}}dx \tag{2.4.44}$$

which evaluates to

$$S_{\text{reg}} = -\frac{\pi\phi_c\ell\,{}_2F_1\left(-\frac{1}{4},\frac{1}{2};\frac{3}{4};x^2\right)}{G_4\sqrt{\rho_0 x}}\Bigg|_{\tilde{\epsilon}}^{1} \tag{2.4.45}$$

$$= \frac{\pi^{3/2}\phi_c\ell\Gamma\left(-\frac{1}{4}\right)}{4G_4\sqrt{\rho_0}\Gamma\left(\frac{1}{4}\right)} + \frac{\pi\phi_c\ell\,{}_2F_1\left(-\frac{1}{4},\frac{1}{2};\frac{3}{4};\ \tilde{\epsilon}^2\right)}{G_4\sqrt{\epsilon}}. \tag{2.4.46}$$

We could also obtain the upper limit by employing the formula

$$\int_0^1 dx x^{\mu-1}(1-x^\lambda)^{\nu-1} = \frac{B(\frac{\mu}{\lambda},\nu)}{\lambda}, \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.4.47}$$

When $\epsilon \to 0$, S_{reg} has a divergent piece which can be removed by adding the local covariant counterterm

$$S_{\text{ct}} = -\frac{\pi \phi_c \ell}{G_4 \sqrt{\epsilon}}. \tag{2.4.48}$$

The renormalised entanglement entropy thus takes the form

$$S_{\text{ren}} = -\frac{\pi^2 \Gamma\left(\frac{3}{4}\right)^2}{G_4 \Gamma\left(\frac{1}{4}\right)^2} \frac{2\phi_c \ell^2}{L}. \tag{2.4.49}$$

2.5 Entanglement entropy on a curved background

To compute the entanglement entropy of a field theory living on an AdS_3 or BTZ background, with a specified entangling region, using the holographic Ryu-Takayanagi prescription, we consider bulk spacetimes whose conformal boundary is AdS_3 or BTZ. A geometry satisfying this requirement is provided by the AdS_4 C-metric, belonging to the Plebanski-Demianski family of type-D metrics. This is a class of exact solutions to the Einstein field equations with many applications, first studied in [176]. It describes two black holes of opposite charge and equal mass, connected by a non-traversable wormhole. A cosmic string passes through the wormhole, which pulls the black holes, causing them to accelerate from each other at a constant rate (see also [177] for a further interpretation of the C-metric). We can write down the C-metric as

$$ds^2 = \frac{\ell^2}{(\ell + \hat{r}\rho)^2} \left(-H(\hat{r}) dt^2 + \frac{d\hat{r}^2}{H(\hat{r})} + \hat{r}^2 \left(\frac{d\rho^2}{G(\rho)} + G(\rho) dy'^2 \right) \right) \tag{2.5.50}$$

where

$$H(\hat{r}) = \frac{\hat{r}^2}{\ell_3^2} + \kappa - \frac{\mu \ell}{\hat{r}}, \tag{2.5.51}$$

$$G(\rho) = 1 - \kappa\rho^2 - \mu\rho^3. \tag{2.5.52}$$

$\kappa = \pm 1, 0$ gives different background geometries and the AdS_4 radius is given by

$$\ell_4 = \left(\frac{1}{\ell^2} + \frac{1}{\ell_3^2} \right)^{-1/2}. \tag{2.5.53}$$

We set $\mu = 0$ and note that with the change of coordinates $(\rho, \hat{r}, y') \to (\sigma, r, \tilde{y})$

$$\cosh\sigma = \frac{\ell_3}{\ell_4} \frac{\sqrt{1 + \frac{\hat{r}^2\rho^2}{\ell_3^2}}}{\left|1 + \frac{\hat{r}\rho}{\ell}\right|}, \quad r = \hat{r}\sqrt{\frac{1 - \kappa\rho^2}{1 + \frac{\hat{r}^2\rho^2}{\ell_3^2}}}, \quad y' = \phi_c \tilde{y}, \tag{2.5.54}$$

the geometry can be written as

$$ds_4^2 = \ell_4^2 d\sigma^2 + \frac{\ell_4^2}{\ell_3^2}\cosh^2\sigma\left(\frac{dr^2}{\frac{r^2}{\ell_3^2}+\kappa} - \left(\frac{r^2}{\ell_3^2}+\kappa\right)dt^2 + r^2\phi_c^2 d\tilde{y}^2\right). \tag{2.5.55}$$

The conformal boundary is at $\sigma \to \infty$. Factoring out the divergent Weyl factor, the boundary metric is

$$ds_3^2 = \frac{\ell_4^2}{\ell_3^2}\left(\frac{dr^2}{\frac{r^2}{\ell_3^2}+\kappa} - \left(\frac{r^2}{\ell_3^2}+\kappa\right)dt^2 + r^2\phi_c^2 d\tilde{y}^2\right). \tag{2.5.56}$$

$\kappa = +1$ corresponds to global AdS_3, $\kappa = -1$ corresponds to the non-rotating BTZ black hole, and $\kappa = 0$ corresponds to the massless BTZ geometry [178, 179].

For $\kappa = +1$, transforming the AdS_3 slices from global to Poincaré coordinates (see appendix A.2), the metric becomes

$$ds_4^2 = d\sigma^2\ell_4^2 + \ell_4^2\cosh^2\sigma\left(\frac{dx^2 - d\tilde{t}^2}{x^2} + \frac{\phi_c^2 dy^2}{x^2}\right). \tag{2.5.57}$$

The boundary metric (at $\sigma \to \infty$) is the uplifted AdS_2 metric we have been considering:

$$ds_3^2 = \ell_4^2\left(\frac{dx^2 - d\tilde{t}^2}{x^2} + \frac{\phi_c^2 dy^2}{x^2}\right). \tag{2.5.58}$$

On a $\tilde{t}$ constant slice, we consider the annulus entangling region $x \in [x_1, x_2]$, $y \in [0, 2\pi]$. We parametrize the candidate RT surface of the annulus by the worldvolume coordinates $x^\alpha = \{\sigma, y\}$, and the embedding coordinates are $x^m = \{t, \sigma, x(\sigma), y\}$. The area functional for the regulated entropy is

$$S_{\text{reg}} = \frac{1}{4G_4}\int_0^{2\pi} dy\left(\int_{\frac{1}{\epsilon}}^{\sigma_0} d\sigma\mathcal{L}\left((x_b(\sigma), x_b'(\sigma), \sigma\right) + \int_{\sigma_0}^{\frac{1}{\epsilon}} d\sigma\mathcal{L}\left((x_a(\sigma), x_a'(\sigma), \sigma\right)\right) \tag{2.5.59}$$

where

$$\mathcal{L}\left((x(\sigma), x'(\sigma), \sigma\right) = \frac{\ell_4^2\phi_c\cosh\sigma}{x(\sigma)}\sqrt{\frac{\cosh\sigma x'(\sigma)^2}{x(\sigma)^2}+1}. \tag{2.5.60}$$

Here, the RT surface does not enjoy a reflection symmetry about the inflection point, and we thus get two branches of the solution for the equation of motions: $x_a(\sigma)$, $x_b(\sigma)$. However, solving the differential equation to obtain the equation of motion will be a daunting task. In appendix A.1, we analyze the RT surface analytically and present an implicit solution. It can be seen that the turning point of the RT surface is close to the conformal boundary. We also perform asymptotic analysis near the conformal boundary and obtain the divergent pieces of the regulated entropy, which exactly coincide with the explicit computation of the covariant counterterms in section 2.5.1, as expected. To bypass the difficulty of obtaining explicit equations of motion from the RT surface, we will, in section 2.5.2, present a method to recover the entanglement entropy of a CFT on an AdS_3 background from the flat limit.

2.5.1 Explicit computation of covariant counterterms

In this section, we explicitly compute the covariant counterterms of the entanglement entropy, using holographic renormalization. Similar to the disk entangling region considered in [160], the divergence for an annulus entangling region is manifestly removed by the covariant counterterm

$$S_{\mathrm{ct}} = -\frac{1}{4G_4}\int_{\partial A} dx\sqrt{\tilde{h}} \tag{2.5.61}$$

where ∂A is the boundary of the entangling region and $\tilde{h}$ is the determinant of the induced metric on ∂A. We have two disconnected circles at the boundary of the entangling region, one at x_1 and the other one at x_2. The embedding coordinates are $x^m = \{t, y, x\}$ and ∂A simply has one coordinate: $x^\alpha = \{y\}$. Thus, the induced metric takes the form

$$\tilde{h}_{yy} = (\partial_y(y))^2\, g_{yy} = \ell_4^2 \cosh^2\sigma \frac{\phi_c^2}{x^2}. \tag{2.5.62}$$

The covariant counterterm thus becomes

$$S_{\mathrm{ct}} = -\frac{1}{4G_4}\int_{\partial A} \ell_4 \cosh\sigma \frac{\phi_c}{x} dy = -\frac{\ell_4\pi}{4G_4}\frac{\phi_c}{\sqrt{\epsilon}}\left(\frac{1}{x_2} - \frac{1}{x_1}\right) \tag{2.5.63}$$

where in the last line we have imposed $\lim_{\sigma\to\infty}\cosh\sigma \to \frac{1}{2\sqrt{\epsilon}}$[3]. This counterterm exactly agrees with the divergent piece in the asymptotic analysis (A.1.27). The minus sign in the parentheses on the RHS comes because the circle intersecting x_1 (x_2) is integrated counterclockwise (clockwise). In appendix A.1 we obtain this result from the asymptotic analysis of the area functional (2.5.59).

2.5.2 Entanglement entropy from the flat limit

Here, we will recover the full explicit solution of the entanglement entropy on a curved background by studying the flat limit. We have from (2.5.59) that the area functional takes the form

$$S_{\mathrm{reg}} = \frac{\ell_4^2}{4G_4}(2\pi\phi_c)\left[\int_{\frac{1}{\epsilon}}^{\sigma_0} d\sigma\mathcal{L}\left((x_b(\sigma), x_b'(\sigma), \sigma\right) + \int_{\sigma_0}^{\frac{1}{\epsilon}} d\sigma\mathcal{L}\left((x_a(\sigma), x_a'(\sigma), \sigma\right)\right] \tag{2.5.64}$$

[3] it might seem peculiar that the cut-off here is dimensionless but this is because we have used $\epsilon = u = e^{-2\sigma}$ so for physical purposes dimensionality for the regulator can easily be reinstated.

where

$$\mathcal{L}(x(\sigma), x'(\sigma), \sigma) = \sqrt{\frac{\cosh^2\sigma\left(\frac{\cosh^2\sigma x'(\sigma)^2}{x(\sigma)^2}+1\right)}{x(\sigma)^2}}. \tag{2.5.65}$$

By adding the counterterm (2.5.63) the renormalised entropy becomes

$$S_{\text{ren}} = \frac{\ell_4^2}{4G_4}(2\pi\phi_c)\lim_{\epsilon\to 0}\left[\left(\int_{\frac{1}{\epsilon}}^{\sigma_0} d\sigma\mathcal{L}|_{x_b(\sigma)} + \int_{\sigma_0}^{\frac{1}{\epsilon}} d\sigma\mathcal{L}|_{x_a(\sigma)}\right) - \frac{1}{2\sqrt{\epsilon}}\left(\frac{1}{x_1}-\frac{1}{x_2}\right)\right]. \tag{2.5.66}$$

From dimensional analysis, we derived that the renormalised entanglement entropy should take the form

$$S_{\text{ren}} \sim c\frac{L_y}{x_1}g\left(\frac{\ell_3}{x_2},\frac{x_1}{x_2}\right). \tag{2.5.67}$$

Comparing this with (2.5.66) we make the identifications

$$\begin{aligned} c &= \frac{\ell_4^2}{4G_4} \\ L_y &= 2\pi\phi_c \\ \frac{1}{x_1}g\left(\frac{\ell_3}{x_2},\frac{x_1}{x_2}\right) &= \lim_{\epsilon\to 0}\left[S_{\text{reg}} - \frac{1}{2\sqrt{\epsilon}}\left(\frac{1}{x_1}-\frac{1}{x_2}\right)\right]. \end{aligned} \tag{2.5.68}$$

From AdS$_4$ in global coordinates (2.5.55) and its conformal boundary metric (2.5.56), it is clear that in the limit $\ell_3 \to \infty$, the conformal AdS$_3$ boundary ($\kappa = 1$) reduces to the flat metric. The Poincaré coordinates reduce to flat coordinates in the flat limit $\ell_3 \to \infty$: $x \to x_f, t \to t_f, y \to y_f$ and $\ell_4 \to \ell$. We must hence recover the renormalised entanglement entropy of a CFT$_3$ on a flat background from the flat limit $\ell_3 \to \infty$ of the renormalised entanglement entropy of a CFT$_3$ on an AdS$_3$ background.

We notice from (2.5.66), and (2.5.68) that the function g is independent of ℓ_3 in Poincaré coordinates, *i.e.*, $g(x_1/x_2)$. Therefore, we can recover the renormalised flat entanglement entropy (2.4.49) in the flat limit iff

$$\frac{1}{x_1}g\left(\frac{x_1}{x_2}\right) = \frac{N}{x_1}\left(\frac{\frac{x_1}{x_2}}{1-\frac{x_1}{x_2}}\right) = \frac{N}{(x_2-x_1)}, \quad N = -4\pi\frac{\Gamma\left(\frac{3}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2}. \tag{2.5.69}$$

Consequently, the renormalised entanglement entropy on a curved AdS$_3$ background must have the same functional form as the renormalised flat entanglement entropy, such that in the flat limit it reduces to the flat result

$$\frac{\pi\phi_c\ell_4^2}{2G_4}\frac{N}{(x_2-x_1)} \xrightarrow{\ell_3\to\infty} \frac{\pi\phi_c\ell^2}{2G_4}\frac{N}{(x_{f2}-x_{f1})} \tag{2.5.70}$$

The renormalised entanglement entropy on an AdS_3 background then takes the form

$$S_{\text{ren}}^{\mathrm{AdS}_3} = -4\pi \frac{\Gamma\left(\frac{3}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2} c L_y \frac{1}{(x_2 - x_1)}. \tag{2.5.71}$$

Remarks on the curved boundary of $\mathbf{AdS}_4$: When placing a non-compact space on a conformal boundary, one has to be careful. The AdS_4 C-metric is plagued with conical singularities in the deep exterior, owed to the string threading through the wormhole connecting the two black holes. An entangling region can always be sufficiently small so that the resulting RT surface does not intersect these conical singularities from the cosmic string. We are only interested in the parametric behavior of the minimal RT surface, and only the coordinates of the entangling region on the AdS_3 boundary dictate if it is in contact with the singularity. In the context of the island rule, when extremizing the entangling region, it could in principle be that its minimal RT surface intersects the singularities of the bulk manifold. It is not well understood what the implications of this would be, and we leave this for future work. In either case, such a discrepancy would be expected to manifest itself in the calculation.

When going to Poincaré coordinates in (2.5.57) we have done hyperbolic slicing in such a way that only the Poincaré patch of the conformal boundary is considered. All the zero-temperature states live in this patch and since the parent theory is inherently conformal[4] the bulk curvature radius would only appear in an overall factor to the entropy; and so, we may use this conformal-like transformation in the case at hand.

2.5.3 Area terms

So far we have studied the contribution from the conformal fields to the generalized entropy, on a flat and curved background. However, we also have geometrical area contributions to the generalized entropy which we will compute for the respective cases for our annulus entangling region.

Flat case: Consider again the flat metric

$$ds_3^2 = -dt_f^2 + dx_f^2 + \phi_c^2 dy^2 \tag{2.5.72}$$

where as usual $0 < y \leq 2\pi$. If we now consider a surface at point x taking all values of y, the induced metric on the boundary of the entangling region, $\partial\Sigma$, is $ds_1^2 = \phi_c^2 dy^2$. The area of this surface is

$$S_{\text{area}} = \frac{1}{4G_3} \int_0^{2\pi} dy \sqrt{\phi_c^2} = \frac{\pi \phi_c}{2G_3}. \tag{2.5.73}$$

[4]not to be confused with the $3d$ fields that enjoy generalized conformal structure.

AdS$_3$ case: With the metric in Poincaré coordinates

$$ds_3^2 = \frac{\ell_3^2}{x^2}(-dt^2 + dx^2 + \phi_c^2 dy^2) \tag{2.5.74}$$

the induced metric on the boundary of the entangling region is $ds_1^2 = \left(\frac{\phi_c \ell_3}{x}\right)^2 dy^2$. The area term in this case becomes

$$S_{\text{area}} = \frac{1}{4G_3}\int_0^{2\pi} dy \sqrt{\frac{\phi_c^2 \ell_3^2}{x^2}} = \frac{\pi \phi_c \ell_3}{2G_3 x}. \tag{2.5.75}$$

2.6 Generalization to higher dimensions

2.6.1 d-dimensional static uncharged black holes in AdS

We can generalize our analysis to higher dimensions by considering a class of static black hole solutions to Einstein's equations in d dimensions with a negative cosmological constant and a $(d-2)$-dimensional horizon topology of positive, zero, or negative curvature [179–181]

$$ds_d^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 h_{ij}(y)dy^i dy^j \tag{2.6.76}$$

$$f(r) = k - \frac{\omega_d m}{r^{d-3}} + \frac{r^2}{l^2} \tag{2.6.77}$$

with the coordinates labelled as $x^\mu = \{t, r, y^i\}$, $(i = 1, ..., (d-2))$ and $h_{ij}(y^i)$ being the horizon metric. The horizon is taken to be a compact orientable manifold M^{d-2} with $Vol(M^{d-2}) = \int d^{d-2}x\sqrt{h}$, and

$$\omega_d = \frac{16\pi G}{(d-2)Vol(M^{d-2})}. \tag{2.6.78}$$

l is a parameter with dimensions of length, and m has dimensions of inverse length. With this form of f, one can check that the metric in (2.6.76) satisfies

$$R_{\mu\nu} = -\frac{(d-1)}{l^2} g_{\mu\nu} \tag{2.6.79}$$

with the horizon metric satisfying

$$R_{ij}(h) = (d-3)kh_{ij}. \tag{2.6.80}$$

As pointed out in [179], the metric (2.6.76) solves Einstein's field equations with a negative cosmological constant for any value of k, provided the horizon is itself Einstein. The horizon can be of positive, zero, or negative curvature depending on the sign of k, opening up the possibility for black hole solutions with non-spherical horizon topology.

Black hole solutions restricted to horizons with constant curvature are asymptotically locally AdS for all values of m.

2.6.2 Entangling region in higher dimensions

We will consider a higher-dimensional version of an annulus spanning all values of the $(d-2)$ isometric directions, instead of just one. At a constant time Cauchy slice, our entangling region is a $(d-1)$-dimensional manifold, $A = M^{d-2} \times [x_1, x_2]$, as depicted in figure 2.2. All directions in the submanifold M^{d-2} are isometric. The entanglement entropy will still only functionally depend on one coordinate. If we did not have complete isometry along the directions of M^{d-2}, we would have to consider our entangling region to also depend on the finite length in the corresponding y^i directions.

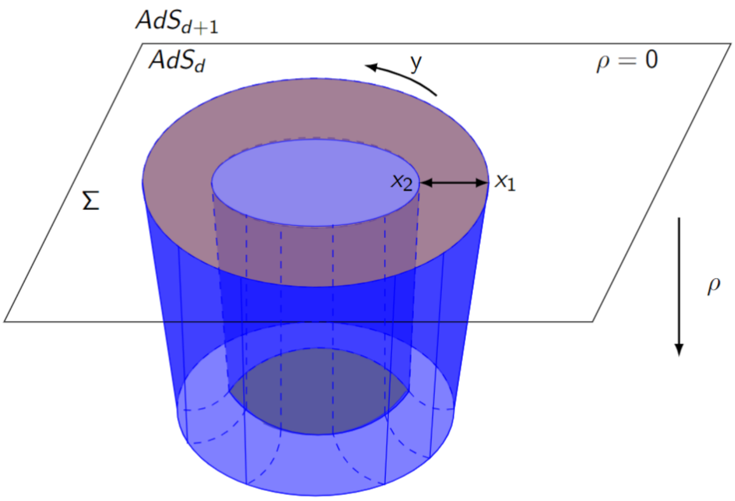


FIGURE 2.2: Annulus entangling region (brown) at the constant time Cauchy slice Σ. The RT surface (blue) propagates into the bulk direction. Here y captures the $(d-2)$ isometric direction.

2.6.3 Structure of the area term in general dimensions

The geometrical contribution to the generalized entropy is given by the area of the QES *i.e.*, the boundary of the entangling region, ∂A:

$$S_{\text{area}} = \frac{1}{4G_d} \int_{\partial A} d^{d-2}x \sqrt{\tilde{h}} \tag{2.6.81}$$

where $\tilde{h}_{ij}$ is the induced metric on ∂A. We consider the black hole solutions in the previous section given by the metric ansatz (2.6.76) so

$$\tilde{h}_{ij} = r^2 h_{ij}. \tag{2.6.82}$$

Evaluating the area term (3.3.8) gives

$$S_{\text{area}} = \frac{1}{4G_d}\int_{\partial A} d^{d-2}x\sqrt{r^{2(d-2)}h} = \frac{\Omega_{d-2}}{4G_d}r^{d-2}; \quad d > 2 \tag{2.6.83}$$

where Ω_{d-2} is the volume of the compact orientable manifold capturing the horizon *i.e.* an Einstein space with an arbitrary constant curvature. We note that the area entropy term is monotonic in r.

In two dimensions, the boundary of the entangling region is a point, and the entangling region is a one-dimensional interval. In three dimensions, the entangling region is circularly uplifted to an annulus, the topology is that of a circle times an interval: $S^1 \times I$, with the boundary being the circle. In this case, the extra dimension is an angular coordinate running between $[0, 2\pi)$, so here we would have $\Omega = 2\pi$. For $d > 3$, we could have spherical, toroidal (of genus ≥ 1), or hyperbolic manifolds as boundaries of the entangling region.

2.6.4 Flat background

We start with a d-dimensional flat metric on the conformal boundary of AdS$_{d+1}$, given by

$$ds^2_{d+1} = \ell^2_{d+1}\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(-dt^2 + dr^2 + \phi_c^2 d\Omega^2_{d-2}\right). \tag{2.6.84}$$

On the d dimensional flat conformal boundary at $\rho = 0$, we choose a t constant slice, and consider the entangling region A: $r \in [0, L]$, spanning the submanifold M^{d-2} with metric $d\Omega^2_{d-2}$. Using the holographic RT prescription, the regulated entanglement entropy is given by minimising the area functional

$$S = \frac{\Omega_{d-2}}{4G_{d+1}}\int d\rho\sqrt{\frac{l^2_{d+1}}{4\rho^2} + \frac{r'(\rho)^2}{\rho}}\left(\frac{\phi_c^2}{\rho}\right)^{\frac{d-2}{2}}, \tag{2.6.85}$$

for candidate surfaces $r(\rho)$ spanning M^{d-2}, that are homologous to the boundary entangling region A, and anchored on ∂A. The equation of motion for the minimal surface is given by

$$r'(\rho)^2 = \frac{\ell^2_{d+1}}{4\rho\left(\frac{\rho^{1-d}\phi_c^{2d-4}}{k^2} - 1\right)} \tag{2.6.86}$$

where k is an integration constant. Using that $r'(\rho) \to \infty$ as $\rho \to \rho_0$ fixes k:

$$k^2 = \phi_c^{2(d-2)}\rho_0^{1-d}. \tag{2.6.87}$$

The solution for $r(\rho)$ is

$$r_\pm(\rho) = c_\pm \pm \ell_{d+1} \left(\frac{\rho^{d/2}\rho_0^{(1-d)/2}}{d} \right) {}_2F_1\left[\frac{1}{2}, \frac{d}{2(d-1)}; 1 + \frac{d}{2(d-1)}; \left(\frac{\rho_0}{\rho}\right)^{1-d} \right] \tag{2.6.88}$$

There are two branches with $r_+(0) = 0$ which implies $c_+ = 0$, and $r_-(0) = L$ which implies $c_- = L$. Further using $r_+(\rho_0) = r_-(\rho_0)$, we get

$$\rho_0 = \frac{L^2}{4\pi\, \ell_{d+1}^2} \frac{\Gamma\left[\frac{1}{2(d-1)}\right]^2}{\Gamma\left[\frac{d}{2(d-1)}\right]^2}. \tag{2.6.89}$$

Substituting (2.6.86) and (2.6.87) back into the integrand of the area functional (2.6.85) gives

$$S_{\text{reg}} = \frac{2\Omega_{d-2}}{4G_{d+1}} \frac{1}{2} \ell_{d+1} \phi_c^{d-2} \int_\epsilon^{\rho_0} d\rho\, \rho^{-d/2} \sqrt{\frac{1}{1 - \left(\frac{\rho}{\rho_0}\right)^{d-1}}}. \tag{2.6.90}$$

The factor of 2 indicates that the RT surface is symmetric about $r = L/2$. Using $x = (\rho/\rho_0)^{d-1}$, we have

$$S_{\text{reg}} = \frac{\Omega_{d-2}\, \ell_{d+1} \phi_c^{d-2}}{4G_{d+1}} \frac{\rho_0^{1-d/2}}{d-1} \int_{(\epsilon/\rho_0)^{d-1}}^{1} dx\, x^{a-1} (1-x)^{-1/2}; \qquad a = \frac{2-d}{2(d-1)}. \tag{2.6.91}$$

Evaluating the incomplete beta integral above, we get

$$S_{\text{reg}} = \frac{\Omega_{d-2}\, \ell_{d+1} \phi_c^{d-2}}{4G_{d+1}} \frac{\rho_0^{1-d/2}}{d-1} \left[\frac{x^a}{a} {}_2F_1\left(a, \frac{1}{2}; a+1; x\right) \right] \Bigg|_{(\epsilon/\rho_0)^{d-1}}^{1}, \qquad d \neq 2. \tag{2.6.92}$$

For $d = 2$, we have

$$S_{reg} = \frac{\ell_3}{4G_3} \log\left(\frac{1 + \sqrt{1 - \epsilon/\rho_0}}{1 - \sqrt{1 - \epsilon/\rho_0}} \right) \xrightarrow{\epsilon \to 0} \frac{\ell_3}{2G_3} \log\left(\frac{L}{\ell_3\, \epsilon} \right) \tag{2.6.93}$$

For $d \neq 2$, when $\epsilon \to 0$, S_{reg} has a divergent piece which can be removed by adding the local covariant counterterm

$$S_{\text{ct}} = \frac{\Omega_{d-2}\, \ell_{d+1} \phi_c^{d-2}}{4G_{d+1}} \frac{\epsilon^{1-\frac{d}{2}}}{1 - \frac{d}{2}} \tag{2.6.94}$$

We have,

$$
\begin{aligned}
S_{\text{ren}} &= \frac{\Omega_{d-2}\,\ell_{d+1}\phi_c^{d-2}}{4G_{d+1}}\,\frac{\rho_0^{1-d/2}}{d-1}\,\frac{\sqrt{\pi}\,\Gamma(a)}{\Gamma\left(a+\frac{1}{2}\right)} \\
&= \frac{\Omega_{d-2}\,\ell_{d+1}^{d-1}\phi_c^{d-2}}{G_{d+1}}\,\frac{2^{\,d-3}\,\pi^{\frac{d-1}{2}}}{2-d}\left[\frac{\Gamma\left(\frac{d}{2(d-1)}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)}\right]^{d-1} L^{2-d}, \qquad d \neq 2. \qquad (2.6.95)
\end{aligned}
$$

In the last line, we substituted the turning point (2.6.89). The counterterm can also be evaluated using

$$
S_{\text{ct}} = \frac{1}{4G_{d+1}}\int_{\partial A} d^{d-2}x\sqrt{\tilde{h}}. \qquad (2.6.96)
$$

2.6.5 Curved background

The AdS$_{d+1}$ metric with a conformal boundary covering the Poincaré patch is given by

$$
\begin{aligned}
ds^2 &= d\sigma^2\ell_{d+1}^2 + \ell_{d+1}^2\cosh^2\sigma\,\frac{1}{x^2}(-dt^2+dx^2+\phi_c^2 d\Omega_{d-2}), \\
\ell_{d+1} &= \left(\frac{1}{\ell^2}+\frac{1}{\ell_d^2}\right)^{-1/2}. \qquad (2.6.97)
\end{aligned}
$$

On the AdS$_d$ conformal boundary at $\sigma \to \infty$, we choose a t constant slice, and consider the entangling region A: $x \in [x_1, x_2]$, spanning the submanifold M^{d-2} with metric $d\Omega_{d-2}^2$. The area functional for the regulated entanglement entropy is given by

$$
S_{\text{reg}} = \frac{\Omega_{d-2}}{4G_{d+1}}\left(\int_{\frac{1}{\epsilon}}^{\sigma_0} d\sigma\mathcal{L}\left((x_b(\sigma), x_b'(\sigma), \sigma\right) + \int_{\sigma_0}^{\frac{1}{\epsilon}} d\sigma\mathcal{L}\left((x_a(\sigma), x_a'(\sigma), \sigma\right)\right) \qquad (2.6.98)
$$

with

$$
\mathcal{L} = \ell_{d+1}\sqrt{1+x'(\sigma)^2\frac{\cosh^2\sigma}{x(\sigma)^2}}\left(\frac{\ell_{d+1}^2\cosh^2\sigma\phi_c^2}{x^2}\right)^{\frac{d-2}{2}}, \qquad (2.6.99)
$$

for candidate surfaces $x(\sigma)$ spanning M^{d-2}, that are homologous to the boundary entangling region A, and anchored on ∂A.

By letting $u = e^{-2\sigma}$, so that the conformal boundary is at $u = 0$, we can write the above integrand as

$$
\mathcal{L} = \ell_{d+1}^2\frac{\phi_c}{x(u)}\frac{1}{4u^{3/2}}\sqrt{(u^2+1)\left(1+u(u^2+1)\right)\frac{x'(u)^2}{x(u)^2}}\left(\ell_{d+1}^2\frac{(u^2+1)}{4u}\frac{\phi_c^2}{x(u)^2}\right)^{\frac{d-3}{2}}. \qquad (2.6.100)
$$

Just as in the three-dimensional example in section 2.5.2, the ℓ_d dependence is only through ℓ_{d+1}, which is an overall factor in the area functional. Hence, to be consistent with the flat result in the flat limit $\ell_d \to \infty$, the functional dependence of the AdS_d entanglement entropy on the non-isometric direction, here the Poincaré coordinate x, must be the same as the entanglement entropy on the flat background (2.6.95). The counterterm from (2.6.96), is given by

$$S_{\rm ct} = -\frac{\Omega_{d-2}}{4G_{d+1}} \left(\frac{\ell_{d+1}^{d-2}\,\phi_c^{d-2}}{2^{d-2}\,\epsilon^{\frac{d}{2}-1}} \left(\frac{1}{x_2^{d-2}} - \frac{1}{x_1^{d-2}} \right) \right). \tag{2.6.101}$$

We could also consider the global metric

$$ds^2 = \ell_{d+1}^2 d\sigma^2 + \frac{\ell_{d+1}^2}{\ell_d^2}\cosh^2\sigma \left(-\left(\frac{r^2}{\ell_d^2}+\kappa\right)dt^2 + \frac{dr^2}{\left(\frac{r^2}{\ell_d^2}+\kappa\right)} + r^2 \tilde{h}_{ij}d\tilde{y}^i d\tilde{y}^j \right). \tag{2.6.102}$$

On the d dimensional conformal boundary at $\sigma = \infty$, we choose a t constant slice, and consider the entangling region $r \in [r_1, r_2]$, spanning $\tilde{y}^i$'s. The regulated entanglement entropy is given by

$$S_{\rm reg} = \frac{\tilde{\Omega}_{d-2}}{4G_4}\left(\int_{1/\epsilon}^{\sigma_0} \mathcal{L}\Big|_{r_1(\sigma)} d\sigma + \int_{\sigma_0}^{1/\epsilon} \mathcal{L}\Big|_{r_2(\sigma)} d\sigma \right) \tag{2.6.103}$$

where

$$\mathcal{L} = \ell_{d+1}\sqrt{\left(1+\cosh^2\sigma \frac{r'(\sigma)^2}{(r(\sigma)^2+\kappa\ell_d^2)}\right)\left(\frac{\ell_{d+1}^2}{\ell_d^2}\cosh^2\sigma\, r(\sigma)^2\right)^{\frac{d-2}{2}}}\,. \tag{2.6.104}$$

The counterterm here becomes

$$S_{\rm ct} = -\frac{\tilde{\Omega}_{d-2}}{4G_{d+1}}\left(\frac{\ell_{d+1}}{2\sqrt{\epsilon}\,\ell_d}\right)^{d-2}\left(r_2^{d-2} - r_1^{d-2}\right). \tag{2.6.105}$$

The Poincaré class of solutions corresponds to $\kappa = 0$. With $\kappa = 0$, $r = \ell_d^2/x$, $\phi_c\, y^i = \ell_d\, \tilde{y}^i$, the global metric (2.6.102) reduces to the Poincaré patch (2.6.97). For $\kappa \neq 0$ we have

$$S_{\rm reg} = \frac{\Omega_{d-2}}{4G_{d+1}}\left(\int_{\frac{1}{\epsilon}}^{\sigma_0} d\sigma \mathcal{L}\left((x_b(\sigma), x_b'(\sigma), \sigma\right) + \int_{\sigma_0}^{\frac{1}{\epsilon}} d\sigma \mathcal{L}\left((x_a(\sigma), x_a'(\sigma), \sigma\right) \right) \tag{2.6.106}$$

with

$$\mathcal{L} = \ell_{d+1}\sqrt{1 + x'(\sigma)^2 \frac{\ell_d^2\cosh^2\sigma}{x(\sigma)^2(\ell_d^2+\kappa\, x(\sigma)^2)}\left(\frac{\ell_{d+1}^2\cosh^2\sigma\phi_c^2}{x^2}\right)^{\frac{d-2}{2}}}\,. \tag{2.6.107}$$

It is clear that this reduces to (2.6.99) when $\kappa = 0$. When $\kappa = 0$, the ℓ_d dependence is only through ℓ_{d+1}, which is an overall factor in the area functional, hence to be consistent with the flat result in the flat limit $\ell_d \to \infty$, the entanglement entropy corresponding to the Poincaré patch has to have the same functional form as the flat result:

$$S_{\text{ren}} = \frac{\Omega_{d-2}\,\ell_{d+1}^{d-1}\phi_c^{d-2}}{G_{d+1}}\,\frac{2^{\,d-3}\,\pi^{\frac{d-1}{2}}}{2-d}\left[\frac{\Gamma\left(\frac{d}{2(d-1)}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)}\right]^{d-1} L^{2-d}, \qquad d \neq 2. \tag{2.6.108}$$

2.7 Islands in higher dimensions

The island rule (1.2.93) can be written as

$$S_{\text{EE}} = \min\left[\operatorname*{ext}_{r_2}\left(S_{\text{gen}} = S_{\text{CFT}}(r_1, r_2) + S_{\text{area}}(r_2)\right)\right] \tag{2.7.109}$$

where r_1 is being fixed and we are extremizing over r_2 such that S_{gen} is minimized.

We have shown that the entanglement entropy of conformal fields can be evaluated explicitly in any dimension in the $\kappa = 0$ class of solutions. This is the entanglement entropy at zero temperature. However, beyond two dimensions, the thermal behavior sits differently in the entanglement entropy *i.e.*, one cannot move between zero and finite temperature via a coordinate transformation. In the context of islands, we will thus work with the thermal regulated entropy.

For an entangling region near the UV boundary, the regulated entanglement entropy is dominated by the divergent piece whose structure is the same in the zero and finite temperature case. Working with the divergent piece could allow us to see if there is a bound on the regulator that would admit islands, *i.e.*, we can probe the regime of validity of the UV cutoff that admits islands, and beyond which new physics may have to be considered.

For an island to exist, the QES must lie at a finite point, *i.e.*, it should not share a boundary with asymptotic infinity. Let r_2^* denote such a finite QES candidate. A necessary condition is

$$\boxed{r_2^* < \infty, \qquad S'_{\text{QFT}}(r_2^*) = -\,S'_{\text{area}}(r_2^*)} \tag{2.7.110}$$

with $S'_{\text{area}}(r_2^*) > 0$ for $r_2^* \geq 0$, due to the monotonicity of S_{area}. If there are several such finite QES candidates, the physical QES is chosen to be the one that minimizes the generalized entropy, *i.e.* the one with the smallest value of $S_{\text{gen}}(r_2^*)$.

Poincaré coordinates: In this case, the generalized entropy $S_{\text{gen}} = -S_{\text{ct}} + S_{\text{area}}$ becomes

$$S_{\text{gen}} = \frac{\Omega_{d-2}}{4G_{d+1}} \left(\frac{\ell_{d+1}\phi_c}{\sqrt{\epsilon}}\right)^{d-2} \left(\frac{1}{x_2^{d-2}} - \frac{1}{x_1^{d-2}}\right) + \frac{\Omega_{d-2}}{4G_d} \left(\frac{\phi_c \ell_d}{x_2}\right)^{d-2}. \tag{2.7.111}$$

For a fixed x_1 it is clear that (2.7.111) does not have any extremal points in x_2 in any parameter space. Thus, the boundary of the island x_2^* can never be at a finite location.

Global coordinates: In this case, we have

$$S_{\text{gen}} = \frac{\Omega_{d-2}}{4G_{d+1}} \left(\frac{\ell_{d+1}}{\sqrt{\epsilon}\ell_d}\right)^{d-2} \left(r_2^{d-2} - r_1^{d-2}\right) + \frac{\Omega_{d-2}}{4G_d} r_2^{d-2} \tag{2.7.112}$$

Similarly, $S_{\text{gen}}(r_2)$ has no extremal points and is minimized at $r_2 = 0$. Thus, no non-trivial islands can form in this case either.

We remain agnostic about the existence of an explicit heat bath beyond the transparent boundary conditions.

2.8 Discussion

In this chapter, we study islands by placing fields on an $\text{AdS}_{d\geq 3}$ black hole background to model an evaporating black hole. The action of this system is obtained by circularly uplifting NAdS_2 gravity. In the island literature, it is often the case that the action of the conformal fields (modeling the Hawking radiation) is supplemented to the gravitational action. Our construction is more general in the sense that the fields are inherited from the parent theory as supposed to coupled to an already dimensionally reduced gravitational theory. The fields in our construction have a scale (they enjoy generalized conformal structure as opposed to being just conformal) and their symmetries are thus compatible with those of curved backgrounds. If we reduce our higher dimensional construction to $d = 2$ we do not reproduce the familiar JT gravity setting coupled to a CFT_2 [12]; our fields will instead be a CFT_3 reduced over a circle.

The presence of islands in [12] is owed to the boundary of the spatial entangling region in $2d$ being a zero-dimensional surface, and thus the leading UV divergence coincides with the logarithmic term arising from the Weyl anomaly. It is this logarithmic piece that drives the island formation in the sense that it provides the generalized entropy with a local minimum, specifying the finite location of the quantum extremal surface. At the same time, it is also this logarithmic piece that obscures if the entropy is regulated or renormalised. We point out that the non-generic characteristics of the entanglement entropy of a $2d$ CFT obscures how and if the JT gravity setting should generalize to higher dimensions.

As a first step in the direction of exemplifying a realization of islands in higher dimensions, we considered the thermal entanglement entropy of a $\mathrm{CFT}_{d\geq 3}$ on an $\mathrm{AdS}_{d\geq 3}$ black hole background and take the entangling region to be a (higher-dimensional) annulus. The consequence of working with this entangling region is that when we circularly uplift NAdS_2 to higher dimensions, the entanglement entropy will still only depend on one scale: the width of the annulus, as the angular directions are isometric. If we place a part of the entangling region near the UV boundary it will be dominated by the divergent piece given by the covariant counterterms we obtained. Using this as the thermal entropy of the fields, we observe that the generalized entropy will never have any extremal points which means that there cannot exist any non-trivial islands.

It is not necessarily the case that the regulated entropy will expose the same features for islands as a renormalised quantity. As we noticed in this chapter, computing the entanglement entropy on an annulus entangling region, even in slightly non-trivially curved backgrounds beyond flat space, such as CFT_{d+1} on AdS_{d+1} is already challenging. The differential equation governing the RT surface (obtained by minimising the area functional) is non-linear and cannot be solved analytically in this case. Although we obtained the entanglement entropy from symmetry arguments and flat space limit in this case, more relevant backgrounds, such as a thermal state on a static black hole background, is clearly much more challenging. The RT surface is again governed by a non-linear differential equation. Symmetry and flat limits do not fix S_{QFT}, unlike the AdS case. However, to probe islands, we need the renormalised entanglement entropy for an annulus entangling region on a black hole background. In chapter (4), we will use the replica trick (1.2.67) to directly compute entanglement entropy and its properties for field theory on a static black hole background. We will see that this computation makes no reference to holography and hence is valid for all quantum field theories.

The exploration of islands in this chapter naturally poses the following questions: how generic are islands in the case of a higher-dimensional black hole evaporation? Should the field theories and their spectrum on these backgrounds satisfy certain conditions for the presence of islands? In chapter (3), following [149], we will explore these questions.

Chapter 3

Conditions for Islands

In this chapter, using the quantum extremal surface/ island prescription (1.2.93), we will discuss specific conditions on the quantum field theory or the spectrum of the outgoing black hole radiation for the presence of islands.

3.1 Static black holes

We consider static black hole solutions to Einstein's equations in d dimensions, remaining agnostic about the spacetime asymptotics *i.e.* the value of the cosmological constant. We will, however, note some features specific to the case of a negative cosmological constant. The ones with spherical horizon topology were constructed and studied in [182–184], four dimensional planar and toroidal cases in [185–187], four dimensional solutions in which the topology of the horizon is an arbitrary genus Riemann surface were constructed in [188–193]. Higher-dimensional constant curvature black holes with negative cosmological constant were obtained in [180, 181].The generalization of the metric ansatz in [188–190] to d dimensional black hole solutions with horizon topology, a $(d-2)-$ dimensional Einstein manifold of positive, zero or negative curvature was considered in [179]. The following d dimensional metric ansatz is considered

$$ds_d^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 h_{ij}(x)dx^i dx^j \tag{3.1.1}$$

$$f(r) = k - \frac{\omega_d m}{r^{d-3}} + \frac{r^2}{l^2} \tag{3.1.2}$$

with the coordinates labelled as $x^\mu = \{t, r, x^i\}$, $(i = 1, ..., (d-2))$ and $h_{ij}(x)$ the horizon metric with dependence on coordinates x^i only. The horizon is taken to be a compact orientable manifold M^{d-2} with $Vol(M^{d-2}) = \int d^{d-2}x\sqrt{h}$, and

$$\omega_d = \frac{16\pi G}{(d-2)Vol(M^{d-2})} \tag{3.1.3}$$

l is the AdS radius with dimensions of length ($l \to \infty$ for asymptotically flat black holes), and m has dimensions of inverse length. With this form of f, one can check that the metric in (2.6.76) satisfies

$$R_{\mu\nu} = -\frac{(d-1)}{l^2} g_{\mu\nu} \tag{3.1.4}$$

with the horizon metric satisfying

$$R_{ij}(h) = (d-3)kh_{ij}. \tag{3.1.5}$$

We note that the metric $g_{\mu\nu}$ is a solution to Einstein's equations with a negative cosmological constant for any value of k, provided the horizon is itself Einstein. The horizon can be of positive, zero, or negative curvature depending on the sign of k, opening up the possibility for black hole solutions with non-spherical horizon topology.

Computing the Riemann curvature, it can be seen that for $m = 0$, the global metric is locally isometric to AdS space, provided that the horizon is itself a constant curvature space, *i.e.*,

$$R_{\mu\nu\rho\sigma} = \frac{1}{l^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \quad \text{if} \tag{3.1.6}$$

$$R_{ijkl}(h) = k(h_{ik}h_{jl} - h_{il}h_{jk}). \tag{3.1.7}$$

Hence, we further impose that the horizon be a constant curvature space for $m = 0$. We have three possible horizon topologies depending on the value of k - elliptic horizons ($k = 1$), flat horizons ($k = 0$), hyperbolic horizons ($k = -1$) [180]. From the form of f in (2.6.77), we notice that the metric at infinity is determined by the cosmological constant term for any value of m. Since $m = 0$ is locally isometric to AdS space, the black hole solutions restricted to horizons with constant curvature are asymptotically locally AdS for all values of m.

Restricting the metric such that the polynomial $r^{d-3}l^2 f$ has a simple positive root r_+ with $f(r) > 0$ for all $r > r_+$, describes the exterior of a black hole with a non-degenerate horizon. For $k = 1$, the authors of [182–184] have considered solutions with spherical topology. $k = 1$ only requires the horizon to be Einstein and does not restrict to spherical topology. For $k = 0$ and $m > 0$ the polynomial $r^{d-3}l^2 f$ has a simple positive root r_+ with $f(r) > 0$ for $r > r_+$. Thus, $k = 0$ corresponds to black hole solutions with toroidal topology. For $k = -1$, it was shown that we have a black hole interpretation in four dimensions [188–193], with the horizon topology being a Riemann surface of genus greater than one, and in five dimensions [179] it was shown that there are non-degenerate horizons with hyperbolic geometry.

3.1.1 Thermodynamics of black holes in AdS

Since we are looking at black hole evaporation in AdS, let us quickly review the qualitative aspects of how it differs from the asymptotically flat case. The thermodynamics of Schwarzchild-AdS ($k = 1$) black holes were discussed in [183, 194]. For $r_+ << l$, the temperature decreases with increasing r_+, just like in the asymptotically flat Schwarzschild black hole. However, there is an intermediate value $r_+ \approx l$ at which the temperature reaches a global minimum T_{min} after which it becomes an increasing function of r_+. Hence, the heat capacity of large AdS black holes is positive. This implies that the black hole can reach a stable equilibrium with its radiation. For $T > T_{min}$, there are two possible solutions with opposing signs of specific heat.

3.2 Topology of the background, and entangling region

Static black holes: We will consider static uncharged black holes with an arbitrary cosmological constant in d dimensions, discussed in the previous section. These black hole solutions are Riemannian manifolds $\mathbb{R} \times \mathbb{R}^+ \times M^{d-2}$, where M^{d-2} is a compact orientable manifold capturing the topology of the horizon.

Entangling region: We pick a constant time Cauchy slice and consider the entangling region $r_1 \leq r \leq r_2$ spanning the range of M^{d-2} as a suitable choice to capture the dynamics of evaporation of the black hole owing to the spherical or toroidal symmetries in M^{d-2}. The consequence of this is that the QES only depends on r. So our entangling region is a $(d-1)$ dimensional manifold, $A = M^{d-2} \times [r_1, r_2]$.

If we did not have complete isometry along the directions of M^{d-2} (for example, stationary black holes), we would have to consider our entangling region to also depend on the finite length in the corresponding x^i direction(s) with no isometry.

3.3 Structure of the area term in general dimensions

The geometrical contribution to the generalized entropy is given by the area of the QES *i.e.* the boundary of the entangling region, ∂A:

$$S_{\text{area}} = \frac{1}{4G_d} \int_{\partial A} d^{d-2}x\sqrt{\tilde{h}} \tag{3.3.8}$$

where $\tilde{h}_{ij}$ is the induced metric on $\partial\Sigma$.

We consider the black hole solutions in the previous section given by the metric ansatz (2.6.76)

$$\tilde{h}_{ij} = \partial_i x^\mu \partial_j x^\nu g_{\mu\nu} = r^2 h_{ij}. \tag{3.3.9}$$

Evaluating the area term (3.3.8) gives

$$S_{\text{area}} = \frac{1}{4G_d} \int_{\partial A} d^{d-2}x \sqrt{r^{2(d-2)}h} = \frac{Vol(M^{d-2})}{4G_d} r^{d-2}; \quad d > 2 \tag{3.3.10}$$

where we recall $Vol(M^{d-2}) = \int d^{d-2}x\sqrt{h}$ is the volume of the compact orientable manifold capturing the horizon. This is a dimensionless constant.

We must note that the area term will receive suitable corrections if we introduce other fields, such as a dilaton, in the action which give corrections to the metric. We also notice that the area entropy term is monotonic in r.

Examples: For $d = 2$ the the QES is a point and the entangling region is a one dimensional interval. For $d = 3$ the entangling region is circularly uplifted to an annulus, the topology is that of a circle times an interval: $S^1 \times I$, with the QES being the circle. In this case the extra dimension is an angular coordinate running between $[0, 2\pi)$ so here we would have $Vol(M^1) = 2\pi$. Since the circle is intrinsically flat we will only get one type of horizon in this case. However, for $d > 3$ we could have spherical, toroidal (of genus ≥ 1), hyperbolic horizons.

3.4 Requirements to get islands

3.4.1 Constraints on S_{QFT}

The premise of islands follows from the semi-classical prescription for entropy, i.e. the quantum extremal surface approach. The quantum extremal surface (QES) prescription [91] states that the generalised entropy associated with a subregion A is obtained from extremising the contributions of the geometric area and the entanglement entropy of the quantum fields in subregion A:

$$S_{\hbar^0}(A) = \min \left[\underset{A}{\text{ext}} \left(S_{\text{gen}} = S_{\text{QFT}}(A) + S_{\text{area}}(\partial A) \right) \right]. \tag{3.4.11}$$

The quantum extremal surface is used in the island prescription [12, 13] as a way to recover the Page curve and restore unitarity, by including quantum $O(\hbar^0)$ corrections to the Hawking radiation from an evaporating black hole.

Specifically, if $S[\mathbf{Rad}]$ is the total entropy and $S_{\text{QFT}}[A]$ is the renormalised entropy of the quantum fields (here radiation) associated with a subregion A, we have

$$S[\mathbf{Rad}] = \min\left[\operatorname*{ext}_{A}\left(S_{\text{gen}} = S_{\text{QFT}}(A) + S_{\text{area}}(\partial A)\right)\right]. \tag{3.4.12}$$

We vary the subregion A until the generalised entropy is extremized and then choose the minimum among them. This minimum generalised entropy associated with a particular subregion gives the entropy of the radiation in the quantum description (to first order in the $\hbar$ expansion). It has been shown to recover the Page curve in certain setups.

If the subregion A_{min} is finite i.e., it does not share a boundary with asymptotic infinity, then the complementary region (which has the same entropy, provided we started with a pure state) will be disconnected and has a region whose boundaries are disconnected from the asymptotic infinity. The region which is disconnected is termed an island, and the presence of islands has been shown to be key in recovering the Page curve.

Now let us consider the extremisation of the generalised entropy in a static black hole background. Given the underlying symmetry of such a background, the extremisation corresponds to considering quantum fields in a slab region bounded by inner and outer radii, and varying the radius. The area term in the generalised entropy is purely geometric and typically behaves monotonically in the radius. From (2.6.83) we have

$$S'_{\text{area}}(r) = \frac{Vol(M^{d-2})(d-2)}{4G_d} r^{d-3}; \quad d > 2. \tag{3.4.13}$$

Thus $S'_{\text{area}}(r) > 0$ for $r \geq 0$. The existence of islands, i.e. the quantum extremal surface having a component that is disconnected from asymptotic infinity, can thus be translated to conditions on the scaling of S_{QFT} with the radius. Let us fix one of the radii to $r = r_1$ and vary the other radius $r = r_*$. The existence of a finite extremum of the generalized entropy guarantees the existence of a finite QES candidate not anchored at asymptotic infinity, i.e., a candidate island. A necessary condition for r_* to be such a QES candidate is

$$\boxed{r_* < \infty, \qquad S'_{\text{QFT}}(r_*) = -\,S'_{\text{area}}(r_*)} \tag{3.4.14}$$

The existence of such an r_* therefore guarantees the existence of a candidate finite QES, and hence a corresponding candidate island. The QES is then obtained by choosing the extremum r_* such that

$$S_{\text{gen}}(r_*) \leq S_{\text{gen}}(r_i) \quad \forall\, r_i \text{ such that } S'_{\text{gen}}(r_i) = 0 \tag{3.4.15}$$

3.5 Discussion

S_{area} is purely geometric *i.e.* it depends only on the metric. S_{QFT} also depends on the quantum field theory and the entangling region. Therefore, the above condition on S_{QFT} for the existence of islands (3.4.14) imposes conditions on the quantum field theory parameters. From (3.4.14), the scaling of S_{QFT} with the entangling region is the relevant field theory quantity to probe for the existence of islands.

In chapter (4), following [150], we will use the replica trick (1.2.67) to directly compute entanglement entropy and its properties (including its scaling with the entangling region) for field theory on a static black hole background. We will explore the analytic $(q-1)$ expansion of Rényi entropy S_q (1.2.66) and its variations; our setup applies to generic variations, from symmetry transformations to variations of the background metric or entangling region. We will see that our methodology elegantly reproduces and generalises results from the literature on entanglement entropy in different dimensions, backgrounds and states. We will use these analytic expansions to explore the behaviour of entanglement entropy in static black hole backgrounds under scaling transformations relevant for probing islands discussed above. We also note that these analytic computations make no reference to holography and the asymptotics of the spacetime and hence is valid for all quantum field theories and static backgrounds.

Chapter 4

Replica analysis of entanglement properties

4.1 Introduction

Generalisation of the islands approach to higher dimensional black holes requires computation of the entanglement of quantum fields within such backgrounds. As we have seen in the previous chapters, this is computationally challenging, even for static black holes and conformal fields. Most of the quantum field theory computations in the literature relate to quantum entanglement in vacuum states, in either non-interacting quantum theories or conformal theories, in flat backgrounds [36, 38, 55, 102–111]. Holographic approaches to entanglement entropy allow access to a wider range of states in strongly interacting quantum field theories, but these computations still primarily cover quantum theories in conformally flat backgrounds i.e. the boundary of the holographic geometry is conformally flat. Moreover, as seen in chapter (2), the equations governing the Ryu-Takayanagi (RT) surface cannot be solved analytically even for simple setups such as a field theory on an AdS background. The equations governing the RT surface for a field theory on a black hole background will be more involved and cannot be solved analytically.

The relevant setup for exploring islands of entanglement is quantum field theory within a black hole background. Here, it is important that the background is viewed as representing a typical state in the thermal ensemble, rather than the thermal state itself. In practice, this means that the geometry is well represented by the black hole geometry outside the horizon, but that we are agnostic about the details of the black hole interior, which capture the specifics of the microstate. The entanglement entropy region is a slab outside the horizon i.e., the region between specified inner and outer radii, see Figure 4.1. The quantum fields are implicitly assumed to be in quasi-equilibrium with the background, and their state is thus approximated to leading order by the thermal state. Again, however, one should keep in mind that the actual state of this radiation is pure, rather than thermal.

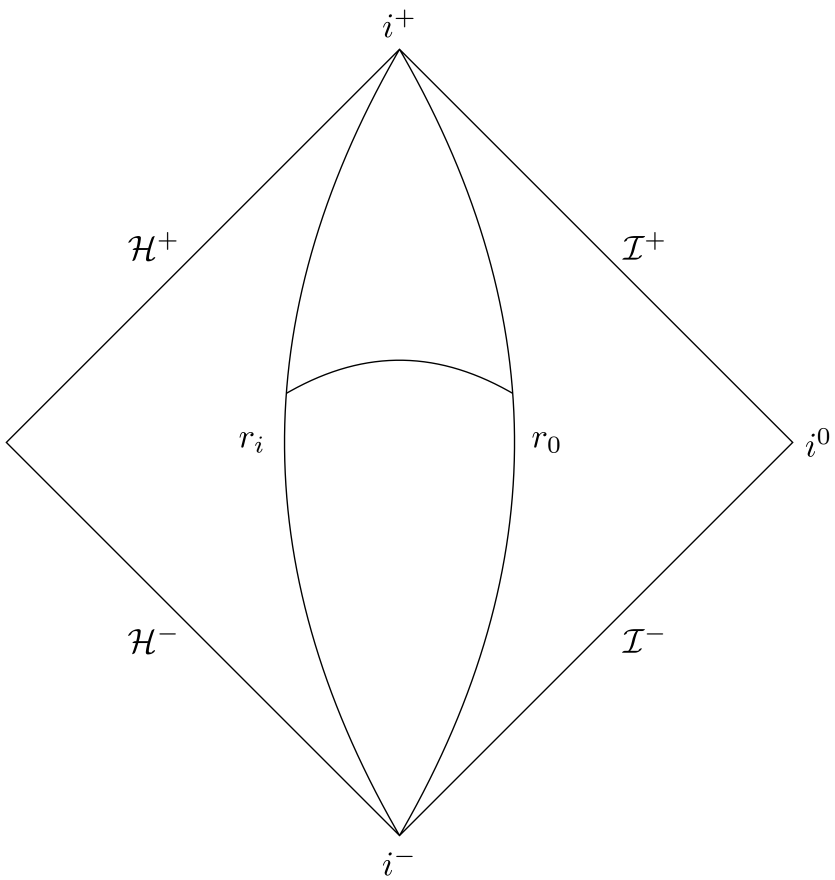


FIGURE 4.1: Entangling region on asymptotically flat black hole background bounded by inner radius r_i and outer radius r_o

Thus, entanglement islands require analysis of entanglement entropy at finite temperature in black hole backgrounds. Interesting examples of backgrounds include black strings: near extremal black strings have AdS_3/BTZ near horizon geometries, and thus one is led to consider entanglement entropy within slab regions outside the horizon in such geometries. Given the underlying symmetry of such backgrounds, the most natural representation of the quantum fields is as conformal field theories i.e., there are no characteristic scales in the quantum field theory. Analysis of this system would allow interpretation of islands within the holographic duals to the asymptotically AdS_3 near horizon regions [111].

More generally, to determine the existence and features of islands, one would need to analyse entanglement entropy in generic black hole backgrounds - including the asymptotic regions, not just near horizon regions. In this chapter, we initiate a systematic analysis of the properties of entanglement in such backgrounds, using the replica trick [36–38].

To recap, the entanglement entropy corresponding to a reduced density matrix ρ_A, associated with a region A takes the form:

$$S_{\rm QFT} = -{\rm Tr}(\rho_A \log \rho_A). \tag{4.1.1}$$

The premise of the replica approach is to rewrite this as a limit of the Rényi entropy S_q:

$$S_{\rm QFT} = \lim_{q\to 1}\left(S_q\right) \equiv \lim_{q\to 1}\left(-\frac{\partial}{\partial q}(\log({\rm Tr}\rho_A^q))\right). \tag{4.1.2}$$

This implicitly assumes that the Rényi entropy S_q can be analytically continued in q to $q = 1$. Such an analytic continuation is not guaranteed and can be subtle in systems with a nontrivial replica structure. In particular, in spin glass models, the naive replica-symmetric analytic continuation fails [195–200]. However, known replica anomalies are in discrete systems, rather than in continuum quantum field theories. Moreover, we are agnostic about the details of the quantum field theory describing the fields outside the black hole; our entanglement analysis would be valid for any quantum field theory in which there is no replica anomaly.

In this chapter we explore the analytic $(q-1)$ expansion of the Rényi entropy S_q and its variations δS_q, and use these expansions to compute entanglement and its variations. Our setup applies to generic variations, from symmetry transformations through to variations of the background metric or the shape of the entangling region. Here we focus on the example of variation under a specific scaling transformation: as we discuss in the conclusions, this quantity is key to determining whether there are islands of entanglement.

Analytic $(q-1)$ expansions have been used in previous literature on entanglement entropy, particularly for computing the universal (conformal anomaly driven) contributions to entanglement entropy in vacua of conformal field theories. In this context, systematic studies of $(q-1)$ expansions of curvature invariants were carried out in [108–110].

In this chapter we extend such analysis to generic quantum states on static backgrounds, exemplifying our approach with conformal field theories. Let $\langle T_{\mu\nu}\rangle^{[0]}$ represent the stress tensor of the quantum state on the base manifold. The stress tensor on the replica manifold $\langle T_{\mu\nu}\rangle_q$ is then expressed analytically as

$$\langle T_{\mu\nu}\rangle_q = \langle T_{\mu\nu}\rangle^{[0]} + (q-1)\langle T_{\mu\nu}\rangle^{[1]} + \cdots \tag{4.1.3}$$

The leading correction $\langle T_{\mu\nu}\rangle^{[1]}$ is determined from the stress tensor on the base manifold. We derive explicitly the equations for this correction term for states in a conformal field theory, from the conservation and trace equations of the stress tensor. We then use the analytic expansion of the stress tensor to compute properties of entanglement entropy in such states, focussing on variations of entanglement entropy.

In the case of vacuum states of conformal field theories, the stress tensor on the base manifold is expressed in terms of curvatures, with the trace of the stress tensor corresponding to the conformal anomaly. Our methodology reproduces results from the literature on entanglement in such states, and its variations under scaling transformations.

Entanglement in quasithermal states has not been studied extensively in the literature. In general quantum entanglement is primarily studied for vacuum and near vacuum states; in finite temperature states, quantum effects are entwined with thermal behaviour. As discussed above, entanglement in quasithermal states is however relevant to the study of islands of entanglement in black holes. Accordingly, we use our methodology to analyse the replica stress tensor for quasithermal states, and the corresponding properties of entanglement in such states. Our results make contact with earlier literature on stress tensors of finite temperature states in backgrounds with conical deficits, obtained in the context of cosmology and cosmic strings.

The plan of this chapter is as follows. In section 4.2 we review the replica trick for entanglement entropy and in section 4.3 we discuss the geometric properties of replica manifolds. In section 4.4 we consider the parametrisations of variations of the entanglement entropy. In section 4.5 we set up the analytic expansion in $(q-1)$ of the entanglement entropy and its variations. These quantities involve the $(q-1)$ expansion of the stress tensor and in section 4.6 we show how the replica stress tensor can be determined from the stress tensor on the base manifold, illustrating with conformal field theory examples. In section 4.7 we solve the equations for the replica stress tensor explicitly in the context of vacuum states of conformal field theories on static backgrounds. Finally, in sections 4.8 and 4.9 we give results for the entanglement entropy variations in vacuum and finite temperature states in such backgrounds. We conclude in section 4.10, discussing how our results relate to entanglement islands.

4.2 Entanglement entropy of a QFT and the replica trick

Let us consider a quantum field theory living on a $d-$dimensional manifold M_1 with a general non-dynamical metric. Consider a state $|\psi\rangle$ on the manifold M_1 with the corresponding density matrix $\rho = |\psi\rangle\langle\psi|$. If the Hilbert space $\mathcal{H}$ on M_1 can be factorised as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ where $\mathcal{H}_A$ is the Hilbert space corresponding to a subregion A on M_1, then we can define a reduced density matrix $\rho_A = \mathrm{Tr}_{\bar{A}}\rho$ corresponding to the subregion A. A discussion on how to define states and density matrices in the path integral formalism [201] is in appendix B.2.

The entanglement entropy over a region A on the manifold M_1 is given by the von Neumann entropy corresponding to the density matrix ρ_A associated with the region A

$$S_{\text{QFT}}(A) = -\text{Tr}(\rho_A \log \rho_A). \tag{4.2.4}$$

When ρ corresponds to a pure quantum state, we have $S(A) = S(\overline{A})$. S_{QFT} is difficult to calculate owing to the requirement to calculate the logarithm of the density matrix. We can instead work with polynomials of the density matrix. The q^{th} Rényi entropy S_q is given by,

$$S_q = \frac{1}{1-q} \log(\text{Tr}_A \rho_A^q). \tag{4.2.5}$$

The von Neumann entropy is then given by

$$\begin{aligned} S_{\text{QFT}} &= \lim_{q\to 1} S_q = \lim_{q\to 1} -\frac{\partial}{\partial q} \log(\text{Tr}_A \rho_A^q) \\ &= \lim_{q\to 1} -\frac{\text{Tr}_A(\rho_A^q \log \rho_A)}{\text{Tr}\rho_A^q} = -\text{Tr}(\rho_A \log \rho_A). \end{aligned} \tag{4.2.6}$$

This is called the replica trick [36–38]. We first compute $\text{Tr}_A \rho_A^q$ for positive integers, and then we analytically continue q to general complex values in order to take the limit $q \to 1$. The existence of a well-defined analytic continuation is therefore necessary for the replica trick to work, which can be subtle in systems with nontrivial replica structure, such as spin glasses [195–200].

In a QFT computing $\text{Tr}_A \rho_A^q$ amounts to calculating the Euclidean path integral over a $q-$fold replicated manifold M_q (henceforth referred to as replica manifold) obtained by taking q copies of the original manifold M_1 and gluing them with specific boundary conditions along the couple of identical open cuts at the $\tau = \tau_0$ slice where the entangling region $x \in A$ is defined on each manifold. If $\phi_i(x,\tau)$ is the field on the i^{th} copy of the manifold M_1,

$$\text{Tr}_A \rho_A^q = \frac{1}{Z_1^q} \int_{BC} D\phi_1 D\phi_2 ... D\phi_q \exp\left(-\int_{\mathbb{R}^d} d\tau d^{d-1}x \sum_{i=1}^{q} \mathcal{L}_E[\phi_i(x,\tau)] \right) \equiv \frac{Z_q}{Z_1^q}. \tag{4.2.7}$$

Along the couple of identical open cuts in each manifold, the above path integral is restricted by the boundary conditions(BC) given by

$$\begin{aligned} &\phi_i(x,\tau_0^+) = \phi_i(x,\tau_0^-) = \phi_i(x \notin A);\ \text{if } x \notin A\ , \\ &\phi_i(x,\tau_0^+) = \phi_{i+1}(x,\tau_0^-);\ \text{if } x \in A\ , \end{aligned} \tag{4.2.8}$$

where $i \in \{1,2,...,q\}$ and $q+i \equiv i$. In the replica path integral above, on each one-manifold M_1, the fields along the points $x \notin A$ are identified and summed over all such possible field configurations $\phi_i(x \notin A)$, and the points $x \in A$ are cyclically identified between one-manifolds by the above boundary conditions.

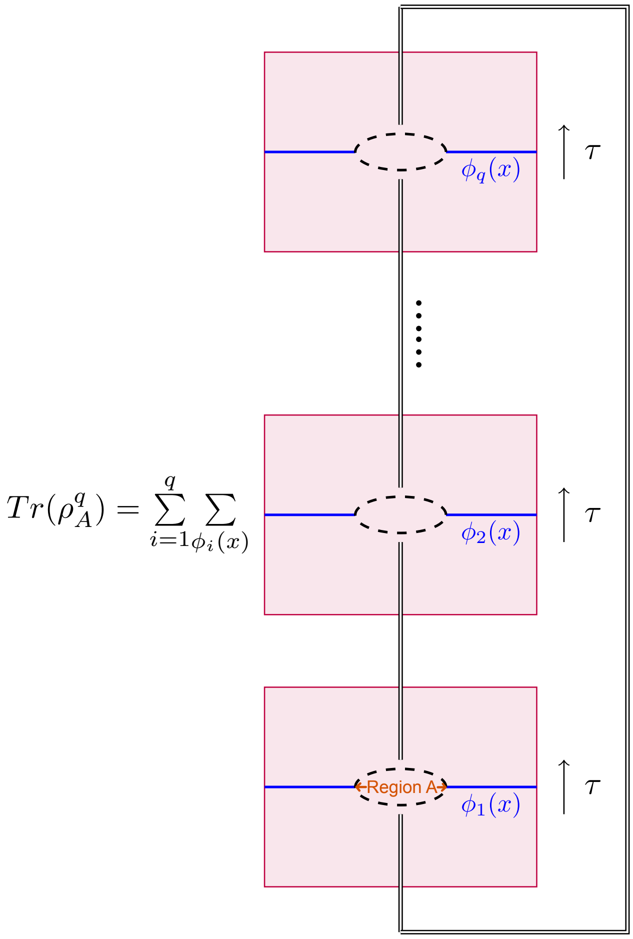


FIGURE 4.2: q copies of manifold M_1 with replica boundary conditions

It now follows that

$$S_{\mathrm{QFT}} = \lim_{q \to 1} \frac{1}{1-q} \left(W_q - q W_1 \right), \tag{4.2.9}$$

where $W_q = \log Z_q$ is the connected generating functional on the replica manifold M_q.

From now on we will refer to terms that depend on every point in spacetime as densities and terms that take a value over the entire manifold (integrated value of densities over the entire manifold) as global terms. Expanding W_q, we have

$$W_q = qW_1 + (q-1)W^{[1]} + \mathcal{O}((q-1)^2). \tag{4.2.10}$$

If we ignore the subtleties due to the boundary condition (4.2.8) (which would contribute in powers of $(q-1)$ for any quantity calculated on M_q), we simply have q copies of M_1 contributing one W_1 on each copy, giving us the $\mathcal{O}((q-1)^0)$ term in the expression above. Such an expansion is true for any global quantity (as opposed to densities) calculated on M_q. Densities on M_q (say F_q) have an expansion of the form,

$$F_q = F_1 + (q-1)F^{[1]} + \mathcal{O}((q-1)^2). \tag{4.2.11}$$

In other words, any global quantity on M_q (say G_q) can be written as

$$\begin{aligned} G_q = \int_{M_q} F_q &= \int_{M_q} (F_1 + (q-1)F^{[1]} + \mathcal{O}((q-1)^2)) \\ &= q\int_{M_1} F_1 + (q-1)\lim_{q\to 1}\int_{M_q} F^{[1]} + \mathcal{O}((q-1)^2) \\ &= qG_1 + (q-1)G^{[1]} + \mathcal{O}((q-1)^2). \end{aligned} \tag{4.2.12}$$

Using (4.2.10) in (4.2.9) we have the following expressions for the Rényi and entanglement entropy,

$$S_q = -(W^{[1]} + (q-1)W^{[2]} + ...), \tag{4.2.13}$$

$$S_{\text{QFT}} = -W^{[1]}. \tag{4.2.14}$$

One can think of the entanglement structure between two subregions as being encoded in the replica corrections to W_q arising due to the replica boundary conditions (4.2.8). S_{QFT} is clearly dependent on only the correction $W^{[1]}$, but S_q is dependent on all the replica corrections (terms with $(q-1)^{i>0}$) to W_q.

4.3 Geometry of the replica manifold

The metric and entangling region on M_1 along with the replica boundary conditions (4.2.8) completely determines the metric on the replica manifold M_q. We can impose the cyclic boundary condition (4.2.8) on the fields on the replica manifold by constructing a manifold respecting the boundary conditions and then defining a field on it. Hence, the replica manifold M_q is made up of $q-$copies of the base manifold M_1 glued by the replica boundary conditions (4.2.8). Therefore, the metric on the replica manifold depends on, the metric of M_1 as well as the shape of the entangling region since that determines how the $q-$copies of M_1 are glued respecting (4.2.8). In the section that follows we will see how the replica metric $g^{(q)}_{\mu\nu}$ depends on both these details. We will keep $g_{\mu\nu}$ on M_1 general and keep the entangling region A to be of an arbitrary shape.

4.3.1 Replica manifold M_q and Replica metric $g^{(q)}_{\mu\nu}$

For the sake of highlighting each aspect of the replica metric $g^{(q)}_{\mu\nu}$, we will restrict $g_{\mu\nu}$ and the shape of the entangling region A at varying levels and later put all aspects together for arbitrary metric and entangling region shape.

4.3.1.1 Entangling regions with no extrinsic curvature

Let us first consider metric $g_{\mu\nu}$ and entangling region A on M_1 such that the entangling boundary Σ has no extrinsic curvature [108]. Any entangling region in 2-dimensions trivially satisfies this. In $d > 2$ an example would be planar entangling region on a flat metric

$$ds^2_{M_1} = d\tau^2 + dx^2 + \delta_{ij} dz^i dz^j, \tag{4.3.15}$$

with entangling region $x \in [x_1, x_2]$ on a τ =constant slice, spanning $z^i{}'s$.

For sake of simplicity let us suppress the boundary directions Σ in M_1 and consider only the resulting manifold $\mathbb{R}^2$. We do this since along these directions the metric remains unaffected going from M_1 to M_q. Away from the boundary of the entangling region, the replica boundary conditions imply that the replica manifold is simply q copies of the base manifold M_1, see figure (4.3). Therefore in the bulk of the entangling region (i.e., sufficiently away from endpoints/ entangling boundaries) the replica manifold inherits the same topology and metric as the base manifold M_1.

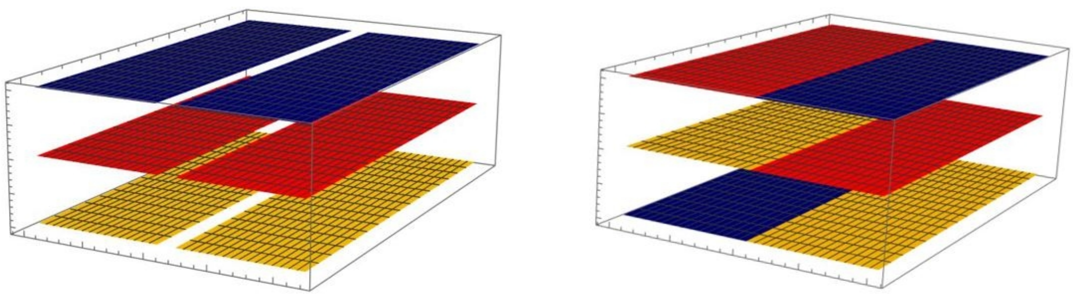


FIGURE 4.3: $q = 3$ copies of manifold after replica boundary conditions - in the bulk of the entangling region

Locally around the boundary of the entangling region the metric on M_1 is flat

$$ds^2_{M_1} \text{ around boundary} = d\tau^2 + dx^2. \tag{4.3.16}$$

Around the entangling boundary, the replica boundary conditions imply that the replica manifold locally takes the topology of a $q-$fold spiral sheet with two ends attached as shown in figure (4.4). One can choose a chart $\{\rho, \psi\}$ where ρ is the distance from the boundary with $\rho \in [0, \infty)$ and ψ is the angle from the cut in the first manifold with $\psi \in [0, 2\pi q)$. If $\{x_0, \tau_0\}$ is the endpoint, we have,

$$\begin{aligned} x - x_0 &= \rho \cos(\psi), \\ \tau - \tau_0 &= \rho \sin(\psi). \end{aligned} \tag{4.3.17}$$

Every 2π range of ψ starting from 0 with $\rho \in [0, \infty)$ on the replica manifold covers one complete range of $\{x, \tau\}$ i.e., one copy of M_1. There are q copies of M_1 in M_q.

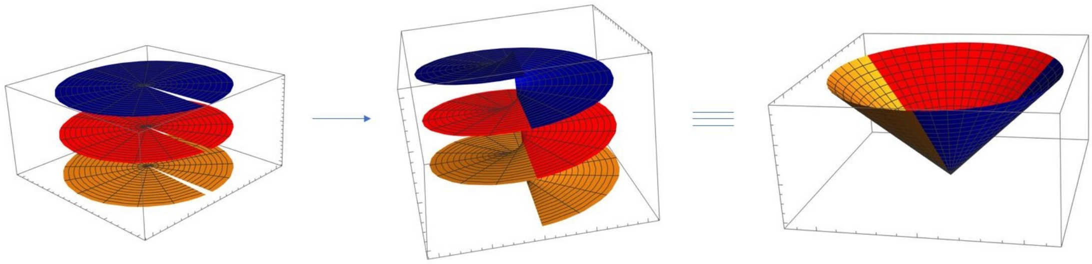


FIGURE 4.4: $q = 3$ copies of manifold after replica boundary conditions - locally around the boundary of the entangling region (the free cut in the first manifold is identified with the free cut in the last manifold although it isn't shown in the figure). This is diffeomorphic to segments of a cone

One can consider a smooth diffeomorphism that maps the replica space around the boundary, to a cone as shown in figure (4.4) i.e., every copy of M_1 in M_q around the boundary is mapped to $1/q^{th}$ segment of a cone. Therefore the replica manifold around the boundary has the topology $C_q \times \Sigma$ where C_q is the conical space with the corresponding range of the angle $[0, 2\pi q)$. When $q \to 1$, $C_q \to \mathbb{R}^2$ and we recover the base manifold M_1 around the boundary of the entangling region i.e., $\mathbb{R}^2 \times \Sigma$

The cone can be seen as embedded in a 3-dimensional (pseudo) Euclidean manifold,

$$\begin{aligned} X &= q\rho' \cos(\psi'/q), \\ Y &= q\rho' \sin(\psi'/q), \\ Z &= \sqrt{|1-q^2|}\rho', \\ i.e., Z^2 &= \frac{|1-q^2|}{q^2}(X^2 + Y^2), \end{aligned} \tag{4.3.18}$$

where $\rho' \in [0, \infty)$ is the distance from the Z-axis and $\psi' \in [0, 2\pi q)$ is the angle from X-axis. We have the metric on the cone,

$$ds^2_{\text{cone}} = dX^2 + dY^2 + dZ^2 = d\rho'^2 + \rho'^2 d\psi'^2. \tag{4.3.19}$$

The conical singularity can be regulated by redefining $Z = \sqrt{|1-q^2|} f(\rho', a)$ with $\partial_{\rho'} f(\rho', a)|_{\rho'=0} = 0$ and $f(\rho', a)|_{a \to 0} = \rho'$, where a is a regulating parameter. We are essentially introducing a smooth transition in derivative around the tip in order to regulate the singularity. After regulating the metric on the cone goes to,

$$ds^2_{\text{reg cone}} = U(\rho', a) d\rho'^2 + \rho'^2 d\psi'^2, \tag{4.3.20}$$

where $U(\rho', a) = q^2 + (1 - q^2)\partial_{\rho'} f(\rho', a)$.

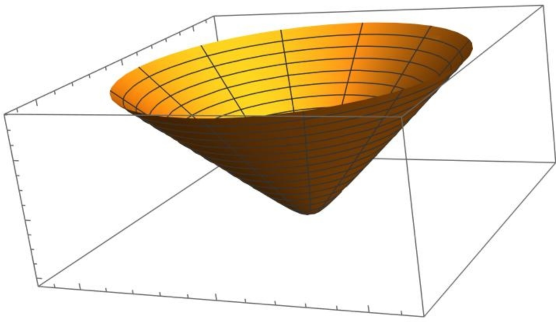


FIGURE 4.5: Regulated cone

Using the properties of $f(\rho, a)$, it can be shown that $U(\rho, a)$ satisfies

$$U(\rho, a)\Big|_{\rho=0} \to q^2; \quad U(\rho, a)\Big|_{\rho>>a} \to 1, \tag{4.3.21}$$

i.e., in the $\rho \to 0$ limit, the metric becomes $q^2 d\rho^2 + \rho^2 d\psi^2 = d\tilde{\rho}^2 + \tilde{\rho}^2 d\tilde{\psi}^2$ where $\{\tilde{\rho} = q\rho, \tilde{\psi} = \psi/q\}$ in which there is no conical singularity as desired. Outside the local neighborhood around $\rho = 0$, the metric expansion simply yields the flat metric as expected.

A particular $U(\rho, a)$ satisfying the limits (4.3.21) is

$$U(\rho, a) = \frac{\rho^2 + a^2 q^2}{\rho^2 + a^2}; \quad f(\rho, a) = \rho - a \tan^{-1}(\rho/a). \tag{4.3.22}$$

Every such choice of the function $U(\rho, a)$ satisfying the limits in (4.3.21) corresponds to choosing different surfaces to roll off the tip. Our calculations must be insensitive to these choices and must only depend on the defining properties of $U(\rho, a)$ given by the limits (4.3.21). Hence, we will keep $U(\rho, a)$ general in our discussions along with the conditions given by (4.3.21). The first condition in (4.3.21) fixes

$$U(\rho, a) = 1 + (q - 1)(2 + O(\rho)) + \mathcal{O}((q - 1)^2), \tag{4.3.23}$$

i.e.,

$$U|_{q=1} = 1; \quad \partial_q U|_{q=1,\rho=0} = 2. \tag{4.3.24}$$

We could also express these limits in the coordinate $x = \rho/a$ in which U has no a dependence

$$U(x)\Big|_{x=0} \to q^2; \quad U(x)\Big|_{x>>1} \to 1. \tag{4.3.25}$$

This metric on the (regulated) cone is also the same metric on the $q-$fold spiral sheeted replica manifold around the entangling boundary in $\{\rho, \psi\}$ coordinates with the diffeomorphism $\rho' \to \rho, \psi' \to \psi$. We will also reinstate the z^i directions. The metric along the z^i directions on M_q remains same as that on M_1[1]. Locally around the boundary, we have,

$$ds^2_{q,reg} = g^{(q)}_{\mu\nu} dx^\mu dx^\nu|_{bdy} = U(\rho, a) d\rho^2 + \rho^2 d\psi^2 + \delta_{ij} dz^i dz^j, \tag{4.3.26}$$

where metric on Σ has no extrinsic curvature and is same as the metric on M_1. We will refer to (4.3.26) as the unsquashed conical metric (to differentiate it from the squashed conical metric that follows).

In the bulk of the entangling region i.e., sufficiently away from boundary,

$$g^{(q)}_{\mu\nu} = g_{\mu\nu}, \tag{4.3.27}$$

as discussed before.

Therefore, the metric on the replica manifold $g^{(q)}_{\mu\nu}$ (replica metric) varies from the metric on M_1 i.e., $g_{\mu\nu}$ only in a small region around the end-points/ boundary of the entangling region (characterised by $r = r_1$ or $r = r_2$, $\tau = \tau_0$ and spanning M^{d-2}), where it has the topology $C_q \times \Sigma$, where C_q is the conical space with the corresponding range of the angle $[0, 2\pi q)$.

4.3.1.2 Entangling regions with extrinsic curvature(s): squashed Cone

Let us now consider entangling regions and metrics where the entangling boundary Σ has an extrinsic curvature [110]. An example would be, non planar entangling regions (spherical, cylindrical etc...) in flat space:

$$ds^2_{M_1} = d\tau^2 + dr^2 + d\Omega^2_{d-2}. \tag{4.3.28}$$

[1] If we however did not have symmetries along the y^i directions (Ex: stationary case) and chose a finite entangling region along z^i, then $g^{(q)}_{ij}$ on M_q would have conical singularities as well, and would be different from g_{ij} on M_1.

Consider the entangling region with boundary Σ given by $r = r_0$, $\tau =$constant, spanning all other coordinates. In suitable polar coordinates defined by

$$\begin{aligned} r - r_0 &= \rho\cos(\psi), \\ \tau - \tau_0 &= \rho\sin(\psi), \end{aligned} \tag{4.3.29}$$

the flat space metric can be written in the form,

$$ds^2_{M_1} = d\rho^2 + \rho^2 d\psi^2 + (r_0 + \rho\cos\psi)^2 h_{ij}(y)dy^i dy^j, \tag{4.3.30}$$

for a spherical entangling region with $h_{ij}(y)dy^i dy^j = (d\theta^2 + \sin^2\theta d\phi^2)$ in $d = 4$, and

$$ds^2_{M_1} = d\rho^2 + \rho^2 d\psi^2 + (r_0 + \rho\cos\psi)^2 d\phi^2 + dz^2, \tag{4.3.31}$$

for a cylindrical entangling region.

Under replica boundary conditions the replica manifold M_q and the replica metric $g^{(q)}_{\mu\nu}$ has similar features as discussed before - remains same as M_1 and $g_{\mu\nu}$ in the bulk of the entangling region, has the topology $C_q \times \Sigma$ locally around the boundary of the entangling region (C_q is the conical space with the corresponding range of the angle $[0, 2\pi q)$). We could hence try a similar regularization procedure, modifying $g^{(q)}_{\rho\rho}|_{bdy}$ as before to smoothen the conical tip. Locally around an entangling boundary, we have

$$\begin{aligned} ds^2_q =& g^{(q)}_{\mu\nu} dx^\mu dx^\nu|_{bdy} \\ =& U(\rho,a)d\rho^2 + \rho^2 d\psi^2 + (r_0 + \rho\cos(\psi))^2 h_{ij}(y)dy^i dy^j; \text{ spherical,} \\ =& U(\rho,a)d\rho^2 + \rho^2 d\psi^2 + (r_0 + \rho\cos(\psi))^2 d\phi^2 + dz^2; \text{ cylindrical.} \end{aligned} \tag{4.3.32}$$

However, the resulting metric in this case has a curvature singularity (for example in the Ricci scalar) at $\rho = 0$. Unlike the previous case (i.e., the case where Σ has no extrinsic curvature), in the metric on M_q above, ∂_ψ is not a Killing vector (due to explicit ψ dependence in $g^{(q)}_{ij}$). Hence the $O(2)$ symmetry in the $\{\rho, \psi\}$ directions present in the previous case is now absent. Instead, we now have a discrete symmetry $\psi \to \psi + 2\pi k$ for $k \in \mathbb{Z}$, hence this is termed a squashed conical singularity. The geometry near Σ is a warped product of two spaces. This is regularised by introducing a new regularization parameter p, modifying the warp factor $(r_0 + \rho\cos(\psi)) \to (r_0 + \rho^p c^{1-p}\cos(\psi))$. Locally around an entangling boundary, we have

$$\begin{aligned} ds^2_q =& g^{(q)}_{\mu\nu} dx^\mu dx^\nu|_{bdy} \\ =& U(\rho,a)d\rho^2 + \rho^2 d\psi^2 + (r_0 + \rho^p c^{1-p}\cos(\psi))^2 h_{ij}(y)dy^i dy^j; \text{ spherical,} \qquad (4.3.33) \\ =& U(\rho,a)d\rho^2 + \rho^2 d\psi^2 + (r_0 + \rho^p c^{1-p}\cos(\psi))^2 d\phi^2 + dz^2; \text{ cylindrical.} \qquad (4.3.34) \end{aligned}$$

The curvature singularity vanishes for $p \geq 2$. Since we will be dealing with integrals quadratic in curvature, therefore it is enough if $p \geq 1$. We choose $p(q) = q$. We will refer to (4.3.33), (4.3.34) as squashed conical metrics. Different regularisation schemes than the ones described here have been considered in [202–204].

In $\{r, \tau, y^i\}$ coordinates the replica metric locally around an entangling boundary takes the form,

$$\begin{aligned}
ds_q^2 &= g^{(q)}_{\mu\nu} dx^\mu dx^\nu|_{bdy} \\
&= \frac{(r-r_0)^2 + (\tau-\tau_0)^2 U(\rho,a)}{(r-r_0)^2 + (\tau-\tau_0)^2} d\tau^2 + \frac{(r-r_0)^2 U(\rho,a) + (\tau-\tau_0)^2}{(r-r_0)^2 + (\tau-\tau_0)^2} dr^2 \\
&+ \frac{(r-r_0)(\tau-\tau_0)(U(\rho,a)-1)}{(r-r_0)^2 + (\tau-\tau_0)^2} dr d\tau \\
&+ \Bigg((r_0 + (r-r_0)[(r-r_0)^2 + (\tau-\tau_0)^2]^{\frac{p-1}{2}} c^{1-p})^2 h_{ij}(y) dy^i dy^j; \text{ spherical} \\
&\text{or } (r_0 + (r-r_0)[(r-r_0)^2 + (\tau-\tau_0)^2]^{\frac{p-1}{2}} c^{1-p})^2 d\phi^2 + dz^2; \text{ cylindrical} \Bigg).
\end{aligned} \tag{4.3.35}$$

Notice that one requires q copies of $\{r, \tau, y^i\}$ to cover the entire replica manifold.

Static metric: More generally let us consider static metric in coordinates $\{\tau, r, \tilde{y}^i\}$,

$$ds^2_{M_1} = g_{\tau\tau}(r) d\tau^2 + g_{rr}(r) dr^2 + r^2 h_{ij}(y) d\tilde{y}^i d\tilde{y}^j. \tag{4.3.36}$$

Consider the entangling region with boundary Σ characterised by τ =constant and $f(r, \tilde{y}^i)$ =constant. Around the codimension-2 entangling boundary, we expand the metric in suitable coordinates $\{x^a\}_{a=1}^2$, $\{y^i\}_{i=1}^{d-2}$ as follows [110],

$$\begin{aligned}
ds^2_{M_1} =& dx_1^2 + dx_2^2 + (\gamma_{ij}(y) + 2x_2 K^{x_1}_{ij} + \dots) dy^i dy^j + \dots \\
=& d\rho^2 + \rho^2 d\psi^2 + (\gamma_{ij}(y) + 2\rho\cos(\psi) K^{x_1}_{ij} + \dots) dy^i dy^j + \dots
\end{aligned} \tag{4.3.37}$$

where we have used

$$x_1 = \rho\cos(\psi), \quad x_2 = \rho\sin(\psi). \tag{4.3.38}$$

x_1 and x_2 span the transverse space to Σ. The entangling boundary Σ is at $x_1 = 0 = x_2$, spanning $y^{i\prime}s$. γ_{ij} is the induced metric. In the static case, the extrinsic curvature corresponding to the normal vector directed along ∂_τ is zero. Therefore we have only one non-zero extrinsic curvature $K^{x_1}_{ij}$ for the normal vector along ∂_{x_1}

If the entangling region corresponds to $r = r_0$, $\tau = \tau_0$ spanning $y^{i\prime}s$, we then have

$$x_1 = r - r_0; \quad x_2 = \tau - \tau_0, \tag{4.3.39}$$

and the extrinsic curvature

$$K_{ij}^{r} = r_0 g^{rr}(r_0) h_{ij}; \quad K_{ij}^{\tau} = 0. \tag{4.3.40}$$

The corresponding metric on the replica manifold has a conical singularity at Σ and a curvature singularity as discussed before. We have the regulated replica metric given by,

$$\begin{aligned} ds_q^2 \text{ expanded around endpoints } &= g_{\mu\nu}^{(q)} dx^\mu dx^\nu|_{bdy} \\ = U(\rho, a) d\rho^2 + \rho^2 d\psi^2 + (\gamma_{ij}(y) &+ 2\rho^p c^{1-p} \cos(\psi) K_{ij}^{x_1} + \dots) dy^i dy^j + \dots \end{aligned} \tag{4.3.41}$$

4.3.1.3 General metric and arbitrary entangling region shape

Since the replica metric $g_{\mu\nu}^{(q)}$ differs from the metric on M_1 only at Σ where it picks up a conical singularity and an additional curvature singularity (in the case of a non zero extrinsic curvature at Σ), it is useful to consider the expansion of a general metric around the boundary Σ of an entangling region of an arbitrary shape. Such a foliation perturbatively in distance from the entangling boundary was considered in [105]

$$\begin{aligned} ds_{M_1}^2 = &\left(\delta_{ab} - \frac{1}{3}\mathcal{R}_{acbd}\Big|_{\Sigma} x^c x^d\right) dx^a dx^b + \left(A_i + \frac{1}{3} x^b \varepsilon^{de} \mathcal{R}_{ibde}\Big|_{\Sigma}\right) \varepsilon_{ac} x^a dx^c dy^i \\ &+ \left(\gamma_{ij} + 2K_{aij} x^a + x^a x^c \left(\delta_{ac} A_i A_j + \mathcal{R}_{iacj}|_{\Sigma} + K_{cil} K_{aj}{}^{l}\right)\right) dy^i dy^j + \mathcal{O}\left(x^3\right), \end{aligned} \tag{4.3.42}$$

where $\{y^i\}_{i=1}^{d-2}$ parametrize Σ and $\{x^a\}_{a=1}^{2}$ the 2-dimensional transverse space. The entangling surface Σ is located at $x^a = 0$, γ_{ij} is the corresponding induced metric, ϵ_{ac} is the volume form of the transverse space, $R_{\mu\nu\alpha\beta}$ and K_{ij}^{a} are the background and extrinsic curvatures, respectively, and A_i is the analog of the Kaluza-Klein vector field associated with dimensional reduction over the transverse space, that comes from g_{ic}. Note that by construction the structure of Σ is built into the above ansatz. Upto linear order in distance from the entangling boundary, the metric reduces to

$$\begin{aligned} ds_{M_1}^2 = &\delta_{ab} dx^a dx^b + A_i \varepsilon_{ac} x^a dx^c dy^i + (\gamma_{ij} + 2K_{aij} x^a) dy^i dy^j + \mathcal{O}\left(x^2\right) \\ = &d\rho^2 + \rho^2 d\psi^2 + A_i \varepsilon_{ac} x^a dx^c dy^i + (\gamma_{ij} \\ &+ 2\rho \cos(\psi) K_{1ij} + 2\rho \sin(\psi) K_{2ij}) dy^i dy^j + \mathcal{O}\left(x^2\right), \end{aligned} \tag{4.3.43}$$

which is a generalization of (4.3.37) to non-static metrics. We have used (4.3.38) to express in $\{\rho, \psi, y^i\}$ coordinates.

On M_q we can regularise the conical and curvature singularity in $g^{(q)}_{\mu\nu}$ as before,

$$\begin{aligned}
& ds^2_{M_q} \text{ around boundary} = g^q_{\mu\nu} dx^\mu dx^\nu|_{\text{around } \Sigma} \\
=& U(\rho, a) d\rho^2 + \rho^2 d\psi^2 + (\gamma_{ij} + 2\rho^p c^{1-p} \cos(\psi) K_{1ij} \\
& + 2\rho^p c^{1-p} \sin(\psi) K_{2ij}) dy^i dy^j + A_i \varepsilon_{ac} x^a dx^c dy^i \\
& + \mathcal{O}\left(x^2\right).
\end{aligned} \tag{4.3.44}$$

Notice that sufficiently far away from the entangling boundary caustics might be encountered and the coordinate system will break down. Hence, the metric expansions (4.3.42) and (4.3.44) are only valid locally around Σ. This is sufficient since the replica metric differs from the metric on M_1 only around Σ, which (4.3.44) captures.

In general, the replica metric is simply a local expansion around the entangling boundary. This can be noticed by taking the $q \to 1$ limit in the replica metric and noticing that we only recover a flat metric. Therefore, replica metric and coordinates are only well defined in a small region around the entangling boundary specified by the conical regulator $'a'$. Away from the entangling boundary the metric is same as that on M_1. Specifically, in the case of flat metric on M_1, the corresponding replica metric and coordinates are well defined throughout M_q.

4.4 QFT entanglement entropy (EE) and its variations

We are ultimately interested in understanding the conditions on the spectrum of a QFT modeling the Hawking radiation on a black hole background in the semiclassical regime (i.e., metric is not quantised) if unitarity has to be preserved. We follow the recent island rule [12, 13] which uses the Quantum extremal surface prescription [91]. See the discussion in section 4.10 following (3.4.11) for a more detailed discussion. For this purpose, it is enough to understand how the EE of the QFT i.e., S_{QFT} varies under the scaling of the entangling region (3.4.12). Since energy is inversely related to length, the scaling of the entangling region, including the scaling of all other length parameters in the theory, is related to the RG flow up to a negative sign.

4.4.1 General QFTs and RG flows

For a d-dimensional QFT on a Riemannian manifold M_1, we consider the behavior of the action under the local renormalization group [205, 206] where the cut-off and the energy scale depends on the spatial coordinates say x. Since we need well-defined local operator equations, it is necessary to be able to define general finite local operators $\mathcal{O}_i(x)$ with corresponding sets of renormalizable couplings λ^i which form a closed set under operator mixing. This is ensured by promoting the coupling to be a function of

x, $\lambda^i(x)$, so that they act as sources for the corresponding local operators. We have the QFT action I_{QFT} given by

$$I = \int_M d^d x \sqrt{g} \lambda^i(x) \mathcal{O}_i(x) \subset I_{\text{QFT}}. \tag{4.4.45}$$

I remains invariant under the following local classical dilatation

$$\begin{aligned} &g_{\mu\nu}(x) \to e^{2\sigma(x)} g_{\mu\nu}(x), \quad \lambda^i(x) \to e^{\beta^i(\lambda)\sigma(x)} \lambda^i(x), \\ &\mathcal{O}_i(x) \to e^{-(d+\beta^i(\lambda))\sigma(x)} \mathcal{O}_i(x), \end{aligned} \tag{4.4.46}$$

where the Weyl transformation, capturing local rescaling of the metric by an arbitrary continuous function $\sigma(x)$, is an extension of the usual renormalization group to a local renormalization group. $\beta^i(\lambda) = \Lambda(x) \frac{d\lambda^i(x)}{d\Lambda(x)}$ is the beta function, a vector field on the space of couplings and $\Lambda(x)$ is the x dependent energy scale.

For a strictly renormalizable theory the operators are marginal (*i.e.* $\mathcal{O}_i$ has dimension d and in $\beta^i(\lambda) = k_i \lambda^i + O(\lambda^2)$ we have $k_i = 0$). For now, we will restrict our attention to marginal operators.

We have the connected generating functional,

$$e^{W[\lambda^i(x)]} = \int D[\phi] e^{-S_0[\phi] - \int_M d^d x \sqrt{g} \lambda^i(x) \mathcal{O}_i(x)}, \tag{4.4.47}$$

with

$$\langle T_{\mu\nu}(x) \rangle = \frac{-2}{\sqrt{g}} \frac{\delta W}{\delta g^{\mu\nu}(x)}, \tag{4.4.48}$$

$$\langle \mathcal{O}_i(x) \rangle = \frac{-1}{\sqrt{g}} \frac{\delta W}{\delta \lambda^i(x)}. \tag{4.4.49}$$

We have the following operator that captures the variation w.r.t. the energy scale $\Lambda(x)$

$$\begin{aligned} \Delta_\Lambda \equiv \Lambda(x) \frac{\partial}{\partial \Lambda(x)} = -\delta L^i \frac{\delta}{\delta L^i} &= \int d^d x \left(\delta_\Lambda g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} + \delta \lambda^i(x) \frac{\delta}{\delta \lambda^i(x)} \right) \\ &= \int d^d x \left(\delta g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} + \sigma(x) \beta^i(\lambda) \frac{\delta}{\delta \lambda^i(x)} \right), \end{aligned} \tag{4.4.50}$$

where in the last line we have used $\sigma(x)\beta^i(\lambda) = \Lambda(x) \frac{d\lambda^i(x)}{d\Lambda(x)}$. The variation with respect to the energy scale is inversely related to varying the length scales L^i. We will have a corresponding local renormalization group operator on the replica manifold with the metric and theory now on the replica manifold M_q, and the integral now promoted to be on the replica manifold.

Dilatation Ward Identity and anomaly: Considering the operation of Δ_Λ on the generating functional $W[\lambda^i(x)]$ from (4.4.47), the classical dilatation symmetry (4.4.46) manifests as the trace Ward identity. To keep it general, we will consider the operation of the replica local renormalization group operator on the replica connected generating functional W_q

$$\int_{M_q} d^d x \left(2\sigma(x) g^{(q)}_{\mu\nu}(x) \frac{\delta}{\delta g^{(q)}_{\mu\nu}(x)} + \sigma(x)\beta^i(\lambda)\frac{\delta}{\delta\lambda^i(x)} \right) W_q[\lambda^i(x)] = \int_{M_q} d^d x \sqrt{g_q}(-\mathcal{A}_{M_q}),$$
$$\mathcal{A}_{M_q} = \sigma(x) \left\langle T^\mu_\mu(x) \right\rangle_{M_q} + \sigma(x)\beta^i(\lambda) \left\langle \mathcal{O}_i \right\rangle_{M_q}. \tag{4.4.51}$$

In the expression above we have used that the dilatation symmetry is under $\delta g^{(q)}_{\mu\nu} = 2\sigma(x) g^{(q)}_{\mu\nu}(x)$ and $\lambda^i(x) \to e^{\beta^i(\lambda)\sigma(x)}\lambda^i(x)$ and expressions (4.4.48), (4.4.49). Δ_Λ with this transformation is also the local renormalization group operator, under which the generating functional is dilatation invariant upto a QFT dilatation anomaly $\mathcal{A}$. If we consider a CFT, $\mathcal{A}$ would simply be the well-known Weyl anomaly. The expression (4.4.51) is essentially the local renormalization group equation.

4.4.2 Scale transformations of entanglement

Let us now look at how the Rényi $q-$entropy, S_q and the EE, S_{QFT} varies under the scaling of the coordinates $x^\mu \to x^{\mu\prime} = x^\mu + \xi^\mu(x^\mu)$ (this could include scaling of the entangling region boundary, symmetry transformations, and/or deformations of the background metric). With the definition $\delta_{\xi^i} F = F(x^i + \xi^i) - F(x^i)$, we have

$$\begin{aligned}
\delta_L S_q &= \Big(\sum_i \delta_{\xi^i} + \sum_j \delta_{l^j} + \delta_g \Big) S_q = -\Delta_\Lambda S_q \\
&= -\int_M d^d x \left(\delta_\Lambda g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} + \sigma(x)\beta^i(\lambda)\frac{\delta}{\delta\lambda^i(x)} \right) \left(\frac{1}{1-q}(W_q - qW_1) \right) \\
&= \int_M d^d x \Bigg\{ \left(\left[\delta_L g_{\mu\nu} = \Big(\sum_i \delta_{\xi^i} + \sum_j \delta_{l^j} + \delta_g \Big) g_{\mu\nu} \right] \frac{\delta}{\delta g_{\mu\nu}} - \sigma(x)\beta^i(\lambda)\frac{\delta}{\delta\lambda^i(x)} \right) \\
&\qquad \left(\frac{1}{1-q}(W_q - qW_1) \right) \Bigg\}
\end{aligned} \tag{4.4.52}$$

$$\begin{aligned}\delta_L S_q =& \frac{1}{q-1}\Bigg\{\frac{1}{2}\int_{M_q} d^d x\sqrt{g_q}\Big(\sum_i(\mathcal{L}_{\xi^i}+B_i)+\sum_j \delta_{l^j}+\delta_g\Big)g^{(q)}_{\mu\nu}\left\langle T^{\mu\nu}\right\rangle_q \\ &-\frac{1}{2}q\int_{M_1} d^d x\sqrt{g}\Big(\sum_i(\mathcal{L}_{\xi^i}+B_i)+\sum_j \delta_{l^j}+\delta_g\Big)g_{\mu\nu}\left\langle T^{\mu\nu}\right\rangle \\ &-\int_{M_q} d^d x\sqrt{g_q}\sigma(x)\beta^i(\lambda)\left\langle \mathcal{O}_i(x)\right\rangle_q + q\int_{M_1} d^d x\sqrt{g}\sigma(x)\beta^i(\lambda)\left\langle \mathcal{O}_i(x)\right\rangle_{q=1}\Bigg\};\end{aligned} \tag{4.4.53}$$

and

$$\delta_L S_{\rm QFT} = \lim_{q\to 1}\delta_L S_q. \tag{4.4.54}$$

here l^i are other variable length scales. We use δ_{l^j} to denote the variation of these length scales and δ_g the deformations of the metric. We have used,

$$\delta_{\xi^i} g^{(q)}_{\mu\nu} = \mathcal{L}_{\xi^i} g^{(q)}_{\mu\nu} + B_i g^{(q)}_{\mu\nu}. \tag{4.4.55}$$

$\mathcal{L}_{\xi^i} g^{(q)}_{\mu\nu}$ is the Lie derivative along ξ^i, $B_i g^{(q)}_{\mu\nu}$ captures the contribution from $dx^\mu dx^\nu$ in the line element i.e., under one of the coordinates $x^a \to x^{a\prime} = x^a + \xi^a(x^\mu)$ we have

$$\begin{aligned}&g^{(q)}_{\mu\nu}(\mathbf{x})dx^\mu dx^\nu \to g^{(q)}_{\mu\nu}(x^a+\xi^a(\mathbf{x}))d(x^\mu+\delta^\mu_a\xi^a(\mathbf{x}))d(x^\nu+\delta^\nu_a\xi^a(\mathbf{x})) = g^{(q)\prime}_{\mu\nu}(\mathbf{x})dx^\mu dx^\nu;\\ &\delta_{\xi^a} g^{(q)}_{\mu\nu} = g^{(q)\prime}_{\mu\nu} - g^{(q)}_{\mu\nu}.\end{aligned} \tag{4.4.56}$$

The variations of S_q and $S_{\rm QFT}$ (4.4.52) captures S_q and $S_{\rm QFT}$ up to an overall constant. When one of the directions x^i corresponds to an isometry on M_1 i.e., $\mathcal{L}_{\xi^i} g_{\mu\nu} = 0$, and the entangling region does not have a boundary along this direction then $\mathcal{L}_{\xi^i} g^{(q)}_{\mu\nu} = 0$. In the presence of entangling boundaries along x^i, M_q and $g^{(q)}_{\mu\nu}$ develops a conical singularity along the boundary which breaks the isometry along x^i.

Notice that (4.4.52) reduces to expressions in [105], where perturbation in the reduced density matrix ρ_A in M_1 under deformations of $g_{\mu\nu}$ and entangling surface Σ is calculated, to calculate the perturbation of $S_{\rm QFT}$ given by (4.2.4).

Calculation of variation of entropy: For the most general metric and entangling region with boundary Σ characterised by $\{x^a\}_{a=1}^2 = 0$, the replica metric $g^{(q)}_{\mu\nu}$ locally around Σ is given by (4.3.44), and sufficiently away from the boundary we have $g^{(q)}_{\mu\nu} = g_{\mu\nu}$. The replica stress tensor $\langle T_{\mu\nu}\rangle_q$ can now be obtained by solving the replica conservation equations on M_q,

$$\nabla^{(q)}_\mu \langle T_{\mu\nu}\rangle_q = 0. \tag{4.4.57}$$

In the case of a CFT ($\beta^i(\lambda) = 0$) we additionally have

$$\langle T^{\mu}{}_{\mu} \rangle_q = \mathcal{A}[g^{(q)}_{\mu\nu}]. \tag{4.4.58}$$

These are enough to evaluate the scaling of the CFT Rényi and entanglement entropy given by (4.4.52) with $\beta^i(\lambda) = 0$.

For calculating the scaling of the entanglement entropy of a QFT we can further simplify the problem by Taylor expanding $g^{(q)}_{\mu\nu}$ and $\langle T_{\mu\nu} \rangle_q$ around $q = 1$,

$$\begin{aligned} g^{(q)}_{\mu\nu} &= g_{\mu\nu} + (q-1)\left(g^{[1]}_{\mu\nu} = (\partial_q g^{(q)}_{\mu\nu})|_{q=1} \right) + \mathcal{O}\left((q-1)^2\right), \\ \sqrt{g_q} &= \sqrt{g} + \frac{q-1}{2}\sqrt{g} g^{\sigma\kappa} (\partial_q g^{(q)}_{\sigma\kappa})|_{q=1} + \mathcal{O}\left((q-1)^2\right), \\ \langle T^{\mu\nu} \rangle_q &= \langle T^{\mu\nu} \rangle + (q-1) \langle T^{\mu\nu} \rangle^{[1]} + \mathcal{O}\left((q-1)^2\right). \end{aligned} \tag{4.4.59}$$

Using this in (4.4.52) and keeping only $\mathcal{O}((q-1)^0)$ terms in the scaling of the q^{th} Rényi entropy (since we take the limit $q \to 1$ of this to get the scaling of S_{QFT} with the scaling of the entangling region), we get

$$\begin{aligned} \delta_L S_{\text{QFT}} = \lim_{q\to 1} \Bigg\{ & \Bigg(I_1 \equiv \frac{1}{2} \int_{M_q} d^d x \sqrt{g} \delta_L ((\partial_q g^{(q)}_{\mu\nu})|_{q=1}) \langle T^{\mu\nu} \rangle \\ & + \frac{1}{4} \int_{M_q} d^d x \sqrt{g} g^{\sigma\kappa} (\partial_q g^{(q)}_{\sigma\kappa})|_{q=1} \delta_L g_{\mu\nu} \langle T^{\mu\nu} \rangle \Bigg) \\ & + \left(I_2 \equiv \frac{1}{2} \int_{M_q} d^d x \sqrt{g} \left(\delta_L g_{\mu\nu} \right) \langle T^{\mu\nu} \rangle^{[1]} \right) \\ - \frac{1}{q-1} \Bigg(& \int_{M_q} d^d x \sqrt{g_q} \sigma(x) \beta^i(\lambda) \langle \mathcal{O}_i(x) \rangle_q - q \int_{M_1} d^d x \sqrt{g} \sigma(x) \beta^i(\lambda) \langle \mathcal{O}_i(x) \rangle_{q=1} \Bigg) \Bigg\}, \end{aligned} \tag{4.4.60}$$

where $\delta_L = \sum_i \delta_{\xi^i} + \sum_j \delta_{l^j} + \delta_g$ and $\delta_{\xi^i} g^{(q)}_{\mu\nu}$ is given by (4.4.55). We have also used,

$$\int_{M_q} d^d x \sqrt{g} \left(\delta_L g_{\mu\nu} \langle T^{\mu\nu} \rangle \right) = q \int_{M_1} d^d x \sqrt{g} \left(\delta_L g_{\mu\nu} \langle T^{\mu\nu} \rangle \right). \tag{4.4.61}$$

Note that there are implicit q dependencies coming from the M_q integration limits which we have not series expanded in the above expression (4.4.60). Specifically, the q dependence comes from the ψ integration on M_q with integration limits $[0, 2\pi q)$. We expect these to give terms with factors of $q^{i\geq 0}$ since we are integrating over q copies of M_1 (if the ψ integration had resulted in factors of $(q-1)^{i<0}$ then we had to retain $O((q-1)^2)$ terms in the series expansion of replica metric and replica stress tensor). We hence retain the limit $q \to 1$ for such unaccounted factors of $q^{i>0}$.

If

$$\int_{M_q} \sqrt{g}\partial_q g^{(q)}_{\mu\nu}|_{q=1} F^{\mu\nu} = 0, \text{ and}$$
$$\int_{M_q} \sqrt{g}\delta_L \left(\partial_q g^{(q)}_{\mu\nu}|_{q=1}\right) F^{\mu\nu} = 0, \tag{4.4.62}$$

where $F^{\mu\nu}$ is on M_1, then

$$\delta_L S_{\text{QFT}} = \lim_{q\to 1}\left\{\left(I_2 \equiv \frac{1}{2}\int_{M_q} d^d x\sqrt{g}\left(\delta_L g_{\mu\nu}\right)\langle T^{\mu\nu}\rangle^{[1]}\right)\right.$$
$$\left. - \frac{1}{q-1}\left(\int_{M_q} d^d x\sqrt{g_q}\sigma(x)\beta^i(\lambda)\left\langle\mathcal{O}_i(x)\right\rangle_q - q\int_{M_1} d^d x\sqrt{g}\sigma(x)\beta^i(\lambda)\left\langle\mathcal{O}_i(x)\right\rangle_{q=1}\right)\right\}. \tag{4.4.63}$$

For the conformal vacuum of CFT, $\beta^i(\lambda) = 0$ and the second line in the above expression is trivial.

$$\delta_L S_{\text{QFT}} = \lim_{q\to 1}\frac{1}{2}\int_{M_q} d^d x\sqrt{g}\left(\delta_L g_{\mu\nu}\right)\langle T^{\mu\nu}\rangle^{[1]}. \tag{4.4.64}$$

The $\mathcal{O}(q-1)$ correction to the replica stress tensor is the only quantity on M_q required to compute the variation of S_{CFT} with the scaling of the entangling region. When there are enough symmetries, $\langle T^{\mu\nu}\rangle^{[1]}$ can be determined in terms of $\langle T^{\mu\nu}\rangle$ on M_1 using the replica conservation equations (4.4.57). We will show later that this is possible in the static case. (4.4.62) is reasonable, since $\partial_q g^{(q)}_{\mu\nu}|_{q=1}$ is non zero only at the entangling boundary Σ where there is a conical singularity contribution. We will also explicitly show this in the general case and specifically in the static case in later sections.

For a general QFT, given the action and connected generating functional on M_1, one can obtain the corresponding action and connected generating functional on M_q by replacing $g_{\mu\nu}$ by $g^{(q)}_{\mu\nu}$. In addition to solving $\langle T_{\mu\nu}\rangle_q$ explicitly using (4.4.48), one can then also solve for the one-point functions on M_q defined by (4.4.49) which can then be used to evaluate (4.4.52).

For the specific case, when $g_{\mu\nu}$ is flat and the entangling surface is planar and we consider a vacuum state,

- [105], [207] gives the N point vacuum correlation function on M_q in terms of specific $N+1$ point vacuum correlation functions on M_1 given by,

$$-\lim_{q\to 1}\partial_q\langle\mathcal{O}_1(x_1)\ldots\mathcal{O}_N(x_N)\rangle_q = \langle\mathcal{O}_1(x_1)\ldots\mathcal{O}_N(x_N)\,K_0\rangle_c, \tag{4.4.65}$$

where $K_0 = 2\pi H_R$, and H_R is the Rindler (modular) Hamiltonian.

$$K_0 = -2\pi \int d^{d-2}y \int_0^{\infty} dx_1 x_1 T_{22}\left(x_1, x_2 = 0, y\right), \tag{4.4.66}$$

here $\{x_1, x_2\}$ span the transverse space to Σ (located at $x_1 = x_2 = 0$).

- Replica stress tensor one-point functions on corresponding replica space given by (4.3.26) (unsquashed cone) has also been considered in the literature on cosmic string backgrounds which have a similar conical singularity structure - massless scalar [208–210], massive scalar [211], fermion [212], photon [213, 214], graviton [214]. We will discuss more on this in later sections.

4.5 Entanglement entropy on static backgrounds

We will now restrict to static backgrounds on M_1 where we will carry out completely explicit computations.

4.5.1 Metric on M_1

We restrict to static metrics on M_1. We can always choose coordinates/ gauge $\{\tau, r, y^i\}$ in which the metric can be written as,

$$ds^2_{M_1} = g_{\tau\tau}(r)d\tau^2 + g_{rr}(r)dr^2 + r^2 h_{ij}(y)dy^i dy^j, \tag{4.5.67}$$

here $\tau \in \mathbb{R}$ is the Euclidean time coordinate, $r \in [0, \infty)$ is a spatial radial coordinate, y^i's are $(d-2)$ spatial coordinates that span a compact orientable submanifold M^{d-2} with $Vol(M^{d-2}) = \int d^{d-2}x\sqrt{h}$.

Our analysis is valid for any asymptotics. It is however worth noting that in AdS we can have non-trivial codimension-2 h_{ij} topologies,

$$R_{ijkl}(h) = k(h_{ik}h_{jl} - h_{il}h_{jk}), \tag{4.5.68}$$

i.e., h_{ij} is a constant curvature space. We have three possible codimension-2 topologies depending on the value of k - elliptic horizons $(k = 1)$, flat horizons $(k = 0)$, hyperbolic horizons $(k = -1)$ [180].

Global flat space: We will also consider flat space with metric

$$ds^2_{M_1} = d\tau^2 + dx^2 + \phi_0^2\, h_{ij}(y)dy^i dy^j, \tag{4.5.69}$$

with constant ϕ_0 and y^i spans a compact submanifold. Here

$$\int d^{d-2}x\sqrt{det(g_{ij})} = \phi_0^{d-2}\int d^{d-2}x\sqrt{h}, \tag{4.5.70}$$

is regulated.

We will also consider the flat spacetime in planar slicing, in which case the metric takes the form,

$$ds^2_{M_1} = d\tau^2 + dx^2 + \delta_{ij}dz^i dz^j, \tag{4.5.71}$$

where now x and z^i are spatial cartesian coordinates with $x, z^i \in (-\infty, \infty)$.

4.5.2 Entangling region

We are interested in computing the entanglement entropy of a region given by: τ constant, $r_1 \leq r \leq r_2$ (or $x_1 \leq x \leq x_2$) spanning the range of y^i's (or z^i's) i.e., M^{d-2}. Due to symmetries along the y^i (or z^i) directions (spherical or toroidal or translation symmetries), we think such an entangling region is suitable to capture all the interesting dynamics. So our entangling region is a $(d-1)$ dimensional manifold, $A = M^{d-2} \times I$ ($I \in [r_1, r_2]$ or $[x_1, x_2]$). Notice that the entangling region has two disconnected boundaries at $r = r_1$ and $r = r_2$.

If we do not have complete isometry along the directions of M^{d-2} (for example, perpendicular to the axis of rotation in the stationary metric), we would have to consider our entangling region to also span a finite length in the corresponding y^i direction(s) with no isometry, in order to capture all interesting dynamics.

4.5.3 Scaling of entanglement entropy

As mentioned before we are ultimately interested in calculating how the EE of a black hole starting in a pure state varies during evaporation, using the Quantum extremal surface prescription. We assume that the metric remains static throughout evaporation. In doing so we have ignored backreaction or are considering that this evaporation is a quasi-equilibrium process. For this purpose, it is enough to understand how the EE of the QFT modeling the Hawking radiation i.e., S_{QFT} varies under the scaling of the entangling region (4.5.2) in the r direction (3.4.11). On static spacetimes, this is closely related to the action of the renormalization group operator on S_{QFT}. The variation with respect to the energy scale is inversely related to varying the length Δr (the only length scale in our setup).[2]

$$\Delta_\Lambda \equiv \Lambda(x)\frac{\partial}{\partial\Lambda(x)} = -\delta(\Delta r)\frac{\delta}{\delta(\Delta r)}. \tag{4.5.72}$$

Let us now look at how the EE $S_{\rm QFT}$ varies under the scaling of the entangling region in the r direction i.e., under the action of $\delta\Delta r\frac{\delta}{\delta\Delta r}$. Using (4.4.50) on the expression for $S_{\rm QFT}$ given by (4.2.9), we have

$$\begin{aligned}
&\delta(\Delta r)\frac{\delta}{\delta(\Delta r)}S_{QFT} = -\Delta_\Lambda S_{QFT} \\
&= -\int d^dx\left(\delta_\Lambda g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} + \sigma(x)\beta^i(\lambda)\frac{\delta}{\delta\lambda^i(x)}\right)\left(\lim_{q\to1}\frac{1}{(1-q)}(W_q - qW_1)\right) \\
&= \int d^dx\left(\delta_{\Delta r} g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} - \sigma(x)\beta^i(\lambda)\frac{\delta}{\delta\lambda^i(x)}\right)\left(\lim_{q\to1}\frac{1}{(1-q)}(W_q - qW_1)\right).
\end{aligned} \tag{4.5.73}$$

Using (4.4.48) and (4.4.49), we have

$$\begin{aligned}
&\delta(\Delta r)\frac{\delta}{\delta(\Delta r)}S_{\rm QFT} \\
&=\lim_{q\to1}\frac{1}{(q-1)}\left(\frac{1}{2}\int_{M_q}d^dx\sqrt{g_q}\left(\delta_{\Delta r}g^{(q)}_{\mu\nu}\left\langle T^{\mu\nu}\right\rangle_q\right) - \frac{1}{2}q\int_{M_1}d^dx\sqrt{g}\left(\delta_{\Delta r}g_{\mu\nu}\left\langle T^{\mu\nu}\right\rangle\right)\right. \\
&\quad\left. -\int_{M_q}d^dx\sqrt{g_q}\sigma(x)\beta^i(\lambda)\left\langle\mathcal{O}_i(x)\right\rangle_q + q\int_{M_1}d^dx\sqrt{g}\sigma(x)\beta^i(\lambda)\left\langle\mathcal{O}_i(x)\right\rangle_{q=1}\right).
\end{aligned} \tag{4.5.74}$$

Changing the length Δr of the entangling region can also be achieved by scaling the coordinates (measuring scale) in the opposite sense which gets reflected in the change of the metric above ($\delta_{\Delta r}g^{(q)}_{\mu\nu}$).

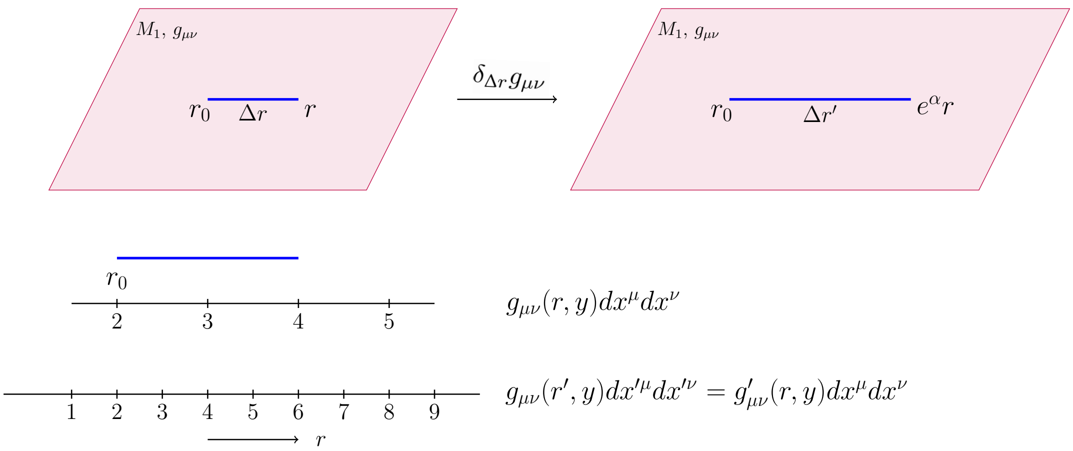


FIGURE 4.6: Scaling coordinates and metric

[2]In non-static cases where we have to take finite lengths in other directions for the entangling region, their variation would also add to the variation of the energy scale.

S_{QFT} only receives entangling region and its localised endpoint contributions: From (4.5.74) we can see that under r-scaling, S_{QFT} depends on the difference between integrals on the replica manifold and the base manifold. Specifically, the (replica) integrals involve the (replica) stress tensor expectation value with respect to the quantum state under consideration and the $r-$scaling of the (replica) metric on the (replica) manifold. As we have already noticed, the metric on the replica manifold differs from the base metric only in a localised region around the boundary of the entangling region, where it has a conical singularity. The corresponding replica stress tensor expectation value on the replica manifold $\langle T^{\mu\nu}\rangle_q$ differs from the base manifold stress tensor expectation value $\langle T^{\mu\nu}\rangle$ via purely metric dependent terms which are localised around the boundary of the entangling region, and non metric dependent (regulator independent) terms which are non trivial over the entire entangling region (we will discuss the regulator independent terms in detail in the following sections). Hence, calculating the $r-$scaling of S_{QFT} amounts to calculating the difference between integrals on the replica and base manifold, or in other words, extracting these localised entangling boundary and entangling region contributions (here, coming from the stress tensor and the metric).

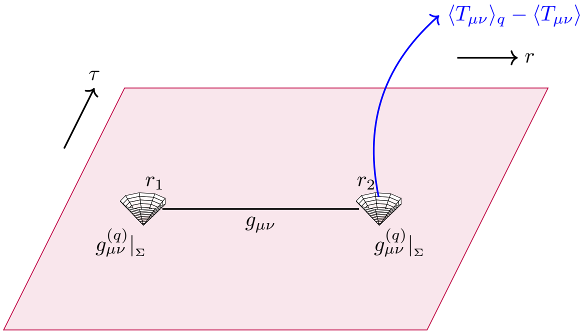


FIGURE 4.7: Localised endpoint contribution of metric and stress tensor expectation on replica manifold

On M_q, we will expand the metric and the stress tensor expectation value around $q = 1$ up to $\mathcal{O}(q-1)$ as before using (4.4.59). Using this in (4.5.74), we get

$$\begin{aligned}
\delta(\Delta r)\frac{\delta S_{\text{QFT}}}{\delta(\Delta r)} = \lim_{q\to 1}\Bigg\{ &\Bigg(I_1 \equiv \frac{1}{2}\int_{M_q} d^dx\sqrt{g}\delta_{\Delta r}((\partial_q g^{(q)}_{\mu\nu})|_{q=1})\,\langle T^{\mu\nu}\rangle \\
&+ \frac{1}{4}\int_{M_q} d^dx\sqrt{g}g^{\sigma\kappa}(\partial_q g^{(q)}_{\sigma\kappa})|_{q=1}\delta_{(\Delta r)}g_{\mu\nu}\,\langle T^{\mu\nu}\rangle\Bigg) \\
&+ \Bigg(I_2 \equiv \frac{1}{2}\int_{M_q} d^dx\sqrt{g}\,(\delta_{\Delta r}g_{\mu\nu})\,\langle T^{\mu\nu}\rangle^{[1]}\Bigg) \\
-\frac{1}{q-1}\Bigg(\int_{M_q} d^dx\sqrt{g_q}\sigma(x)\beta^i(\lambda)\,&\langle \mathcal{O}_i(x)\rangle_q - q\int_{M_1} d^dx\sqrt{g}\sigma(x)\beta^i(\lambda)\,\langle \mathcal{O}_i(x)\rangle_{q=1}\Bigg)\Bigg\}.
\end{aligned} \tag{4.5.75}$$

Alternatively, we can choose not to expand the determinant of the metric g_q on the replica manifold M_q, since we will be evaluating these integrals on the replica manifold.

$$\delta(\Delta r)\frac{\delta S_{\mathrm{QFT}}}{\delta(\Delta r)} = \lim_{q\to 1}\left\{\left(I_1' = \frac{1}{2}\int_{M_q} d^d x\sqrt{g_q}\delta_{\Delta r}((\partial_q g_{\mu\nu}^{(q)})|_{q=1})\,\langle T^{\mu\nu}\rangle\right)\right.$$
$$+\left(I_2' = \frac{1}{2}\int_{M_q} d^d x\sqrt{g_q}\,(\delta_{\Delta r} g_{\mu\nu})\,\langle T^{\mu\nu}\rangle^{[1]}\right)$$
$$\left.-\frac{1}{q-1}\left(\int_{M_q} d^d x\sqrt{g_q}\sigma(x)\beta^i(\lambda)\,\langle\mathcal{O}_i(x)\rangle_q - q\int_{M_1} d^d x\sqrt{g}\sigma(x)\beta^i(\lambda)\,\langle\mathcal{O}_i(x)\rangle_{q=1}\right)\right\}. \tag{4.5.76}$$

It however does not make a difference to the final result.

In (4.5.75) and (4.5.76) the integral I_1 or I_1' captures the replica contribution coming from the difference between replica and base metric, and I_2 or I_2' captures the replica contribution coming from the difference between replica and base stress tensor expectation value. One must note that it is the difference between terms on M_q and M_1 that matters for S_{QFT} and not the actual terms on the replica manifold M_q.

4.5.3.1 Local representation of regulator dependent replica terms

In this section we will briefly review how to deal with integrals over the conical singular manifold M_q of quantities defined on M_q, following [108]. In expression (4.5.75) and (4.5.76), we have integrals of the kind

$$I = \lim_{q\to 1}\frac{1}{(q-1)}\int_{M_q} d^d x\sqrt{g_q}(F_{\mu\nu}^{(q)} - F_{\mu\nu}^{(1)})G^{\mu\nu(1)}, \tag{4.5.77}$$

where $F_{\mu\nu}^{(q)}$, $G_{\mu\nu}^{(q)}$ are tensors on M_q, and $F_{\mu\nu}^{(1)}$, $G^{\mu\nu(1)}$ are the corresponding $O((q-1)^0)$ terms whose values reduce to corresponding tensors on M_1. In the integral above, we can expand $\sqrt{g_q}$ in powers of $(q-1)$ and keep terms up to order $(q-1)$. After this expansion, we also have integrals with $\sqrt{g}$ in the volume element. If $F_{\mu\nu}^{(q)}$ depends only on the replica metric then $(F_{\mu\nu}^{(q)} - F_{\mu\nu}^{(1)})$ is localised around the conical singular boundary $r = r_0$ of the entangling region. For the purpose of evaluating these integrals we could always write a local representation of $F_{\mu\nu}^{(q)}$ on M_q

$$F_{\mu\nu}^{(q)} = F_{\mu\nu}^{(1)} + N_{\mu\nu}(q-1)\delta_\Sigma(\rho) + \mathcal{O}\left((q-1)^2\right), \tag{4.5.78}$$

where $N_{\mu\nu}$ is given by,

$$N_{\mu\nu} = \lim_{q\to 1}\frac{1}{(q-1)Vol(M_{d-2})}\int_{M_q} d^d x\sqrt{g_q}(F_{\mu\nu}^{(q)} - F_{\mu\nu}^{(1)}), \tag{4.5.79}$$

where we have defined (normalised) the delta function $\delta_\Sigma(\rho)$ as,

$$\int_{M_q} d^d x \sqrt{g_q} f \delta_\Sigma(\rho) = \int_\Sigma d^{d-2} x \sqrt{h} f, \tag{4.5.80}$$

and $Vol(M_{d-2}) = \int_\Sigma d^{d-2} x \sqrt{h}$. With this we have,

$$I = N_{\mu\nu} \int_\Sigma d^{d-2} x \sqrt{h} G^{\mu\nu(1)}. \tag{4.5.81}$$

When working with integrals of the kind I (i.e., involving the regulating function $U(\rho, a)$), it is convenient to work in $x = \rho/a$ coordinate since the regulating function $U(x)$ is independent of concial singularity regulator (a) in these coordinates (see (4.3.25)) which makes it easy to read off the a dependence in the integral.

Volume element: The volume element in $x = \rho/a$ coordinate is given by

$$\sqrt{g_q} d^d x = a^2 x \sqrt{U(x)} \sqrt{det(g_{ij})} dx d\psi d^{d-2} y^i; \tag{4.5.82}$$

where

$$g_{ij} = \phi_0^2 h_{ij},\, \delta_{ij}; \tag{4.5.83}$$

for $g^{(q)}_{\mu\nu}$ given by (4.3.26), and

$$g_{ij} = \gamma_{ij}(y) + 2\rho^p c^{1-p} \cos(\psi) K^{x_1}_{ij} + \dots \tag{4.5.84}$$

for the static case (4.3.41). For $g^{(q)}_{\mu\nu}$ given by (4.3.33), we have

$$\sqrt{g_q} d^d x = a^2 x \sqrt{U(x)} \left(r_0 + (ax)^p c^{1-p} \cos(\psi) \right)^{(d-2)} \sqrt{h} dx d\psi d^{d-2} y^i. \tag{4.5.85}$$

We have not written the explicit form for a general non-static replica metric (4.3.44) since it also involves non-diagonal components and is easier to write for the specific case required.

We expand the volume element corresponding to (4.3.33) in powers of a,

$$\begin{aligned}
&\sqrt{g_q}d^dx \\
=&a^2x\sqrt{U(x)}\sqrt{h}dxd\psi d^{d-2}y^i \\
&\Bigg(\frac{1}{24}a^{4p}x^{4p}c^{4(1-p)}(d-5)(d-4)(d-3)(d-2)r_0^{d-6}\cos^4(\psi) \\
&+\frac{1}{6}a^{3p}x^{3p}c^{3(1-p)}(d-4)(d-3)(d-2)r_0^{d-5}\cos^3(\psi) \\
&+\frac{1}{2}a^{2p}x^{2p}c^{2(1-p)}(d-3)(d-2)r_0^{d-4}\cos^2(\psi) \\
&+a^px^pc^{1-p}(d-2)r_0^{d-3}\cos(\psi)+r_0^{d-2}+O(a^{5p})\Bigg).
\end{aligned} \tag{4.5.86}$$

We need the limit in which the regulator $a \to 0$ (since this amounts to setting $U \to 1$). In $d-$dimensions, the most subleading term in a in the volume element has a dependence $a^{i=2+(d-2)p}$(flat squashed cones) and $a^{i=2}$(flat, zero extrinsic curvature). Therefore, in a $d-$dimensional spacetime, we need to consider all terms in $(F^{(q)}_{\mu\nu} - F^{(1)}_{\mu\nu})$ with coefficient $1/a^{r\geq i}$.

A similar expression and result for $\sqrt{g}d^dx$ (appearing in replica integrals) can be obtained by simply setting $q \to 1$ ($U(x) = 1$) in (4.5.82) and (4.5.86).

Example: Consider the following regulated conical singular metric

$$\begin{aligned}
&ds_q^2 = e^{\sigma}(U(\rho,a)d\rho^2 + d\psi^2) = e^{\sigma}(ds^2_{\tilde{C}}); \\
&\sigma = \sigma_1\rho^2 + \sigma_2\rho^4 + ...
\end{aligned} \tag{4.5.87}$$

We have the Ricci scalar given by,

$$R = e^{-\sigma}R_{\tilde{C}} - e^{-\sigma}\Box_{\tilde{C}}\sigma. \tag{4.5.88}$$

$R_{\tilde{C}}$ is the Ricci on $ds^2_{\tilde{C}}$ given by $\frac{U'(\rho)}{\rho U^2}$. We have

$$\begin{aligned}
&\int_{M_{q(reg)}}\sqrt{g_q}R \\
&= 2\pi q\int_{\rho=0}^{\rho=\infty}d\rho\frac{U'(\rho)}{U^{3/2}} - \int_{\rho=0,\psi=0}^{\rho=\infty,\psi=2\pi q}\sqrt{U}\rho d\rho d\psi\Box_{\tilde{C}}\sigma \\
&= 4\pi(1-q) - \int_{\rho=0,\psi=0}^{\rho=\infty,\psi=2\pi q}\sqrt{U}\rho d\rho d\psi\Box_{\tilde{C}}\sigma.
\end{aligned}$$

The differences between the replica and base space tensors that depend on the regulator (a) are zero for $\rho >> a$. Therefore, for integrals involving the difference it is enough to do the integrals between $\rho \in [0, \rho_0]$ for $\rho_0 >> a$. This must give the same result as taking the limit ρ_0 to ∞. For example:

$$\begin{aligned}\int_{M_{q(reg)}} \sqrt{g_q} R_q - \int_M \sqrt{g} R &= \left(2\pi q \int_{\rho=0}^{\rho_0} d\rho \frac{U'(\rho)}{U^{3/2}}\right)\Bigg|_{\rho_0 >> a} \\ &= 4\pi q \left(\frac{1}{q} - \frac{1}{U(\rho_0, a)}\right)\Bigg|_{\rho_0 >> a} = 4\pi(1-q).\end{aligned} \tag{4.5.89}$$

Let Σ be the singular set, i.e., $\rho = 0$. The first term is independent of regulator (a), the second term depends on a but in the $a \to 0$ it reduces to the integral curvature on the smooth domain $M_q \backslash \Sigma$ i.e.,

$$\lim_{a \to 0} \int_{M_{q(reg)}} \sqrt{g_q} R = 4\pi(1-q) + \int_{M_q \backslash \Sigma} R. \tag{4.5.90}$$

We can write the local representation for R,

$$R^{(q)} = R + 2\frac{(1-q)}{q}\delta(\rho). \tag{4.5.91}$$

In higher dimensions, for the metric with no extrinsic curvature for boundary Σ embedded in M_1, the replica metric is the unsquashed cone metric given by

$$ds_q^2 = e^\sigma (U(\rho, a) d\rho^2 + d\psi^2 + (\gamma_{ij}(x^i) + h_{ij}(x^i)\rho^2) dx^i dx^j + ...) = e^\sigma (d\tilde{s}^2), \tag{4.5.92}$$

where $\sigma = \sigma_1 \rho^2 + \sigma_2 \rho^4 + ...$, we can similarly show, following [108],

$$\begin{aligned} R^{(q)} &= R + 4\pi(1-q)\delta_\Sigma, \\ {R^{\mu}{}_{\nu}}^{(q)} &= R^{\mu}{}_{\nu} + 2\pi(1-q)(n^\mu n_\nu)\delta_\Sigma, \\ {R^{\mu\nu}{}_{\alpha\beta}}^{(q)} &= R^{\mu\nu}{}_{\alpha\beta} + 2\pi(1-q)((n^\mu n_\alpha)(n^\nu n_\beta) - (n^\mu n_\beta)(n^\nu n_\alpha))\delta_\Sigma, \end{aligned} \tag{4.5.93}$$

where $\int_M \delta_\Sigma f = \int_\Sigma f$ and $n^k = n^k_\mu dx^\mu$ are two orthonormal vectors orthogonal to the boundary Σ with $n_\mu n_\nu = \sum_{k=1}^2 n^k_\mu n^k_\nu$. Notice that the replica metric has a non zero curvature even if the metric on M_1 has zero curvature (for example flat metric on M_1).

The conical singular replica metric expanded around endpoints, i.e., (4.3.26), does not include the e^σ conformal factor. Therefore, we will only get the $\mathcal{O}(q-1)$ terms when computing any geometric quantity. However, since we are computing differences between replica and base manifold quantities, this difference automatically cancels the $\mathcal{O}((q-1)^0)$ piece, giving us the $\mathcal{O}(q-1)$ terms. Therefore, the replica metric without the conformal e^σ factor is enough for calculating r-scaling of S_{QFT} or such integral differences between replica and base manifold. In any case, the $O((q-1)^0)$ term is simply the quantity evaluated on the smooth domain $M_q\backslash\Sigma$ or on M_1.

If $A^{(q)}$ and $B^{(q)}$ are two quantities on M_q, that satisfy

$$\begin{aligned} A^{(q)} &= A + (q-1)\, N_1\, \delta_\Sigma + O((q-1)^2) \\ B^{(q)} &= B + (q-1)\, N_2\, \delta_\Sigma + O((q-1)^2), \end{aligned} \tag{4.5.94}$$

where $\int_M \delta_\Sigma\, f = \int_\Sigma f$, then,

$$\int_{M_q} A^{(q)} B^{(q)} = q \int_{M_1} AB + (q-1) \int_\Sigma (N_1 B + N_2 A) + O((q-1)^2). \tag{4.5.95}$$

Using this and (4.5.93) we have

$$\begin{aligned} \int_{M_q} R^{(q)2}\Big|_{K^i_{\mu\nu}=0} &= q \int_{M_1} R^2 - 8\pi(q-1) \int_\Sigma R + O((q-1)^2), \\ \int_{M_q} R^{(q)}_{\mu\nu} R^{\mu\nu\,(q)}\Big|_{K^i_{\mu\nu}=0} &= q \int_{M_1} R_{\mu\nu}R^{\mu\nu} - 4\pi(q-1) \int_\Sigma R_{\mu\nu} n^\mu_i n^\nu_i + O((q-1)^2), \\ \int_{M_q} R^{(q)}_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta\,(q)}\Big|_{K^i_{\mu\nu}=0} &= \\ q \int_{M_1} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} &- 8\pi(q-1) \int_\Sigma R_{\mu\nu\alpha\beta} n^\mu_i n^\alpha_i n^\nu_i n^\beta_i + O((q-1)^2). \end{aligned} \tag{4.5.96}$$

In the case of a general entangling region (general background and foliation) with two non zero extrinsic curvatures of Σ, $K^{(i)}_{\mu\nu}$ with $i = 1, 2$, the replica metric takes the form (4.3.44). The integrals of Riemann contractions given above are modified in the case of such squashed cones due to additional contributions from the two extrinsic curvature terms (see [110]). We have,

$$\begin{aligned} \int_{M_q} R^{(q)2} &= q \int_{M_1} R^2 - 8\pi(q-1) \int_\Sigma R + O((q-1)^2), \\ \int_{M_q} R^{(q)}_{\mu\nu} R^{\mu\nu\,(q)} &= q \int_{M_1} R_{\mu\nu}R^{\mu\nu} - 4\pi(q-1) \int_\Sigma \left(R_{\mu\nu} n^\mu_i n^\nu_i - \frac{1}{2} K^2 \right) + O((q-1)^2), \\ \int_{M_q} R^{(q)}_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta\,(q)} &= \\ q \int_{M_1} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} &- 8\pi(q-1) \int_\Sigma \left(R_{\mu\nu\alpha\beta} n^\mu_i n^\alpha_i n^\nu_i n^\beta_i - \mathrm{Tr}\ K^2 \right) + O((q-1)^2), \end{aligned} \tag{4.5.97}$$

where $K^2 = \sum_{i=1}^2 K^{(i)}_{\mu\nu} K^{(i)\mu\nu}$ and $\mathrm{Tr}\ K^2 = \sum_{i=1}^2 (\mathrm{Tr}\ K^{(i)})^2$. We have also suppressed the volume element in the integrals above i.e., $\int_{M_q} \equiv \int_{M_q} d^d x \sqrt{g_q}$ and $\int_\Sigma \equiv \int_\Sigma d^{d-2} x \sqrt{h}$.

Specifically for spherical entangling boundaries in 4 dimensions where replica metric is given by (4.3.33),

$$\int_\Sigma K^2 = 16\pi, \quad \int_\Sigma \text{Tr } K^2 = 8\pi, \tag{4.5.98}$$

and for cylindrical entangling boundaries in 4 diimensions where replica metric is given by (4.3.34),

$$\int_\Sigma K^2 = \int_\Sigma \text{Tr } K^2 = 2\pi \frac{L}{r_0}, \tag{4.5.99}$$

where L is the length of the cylinder in the z direction.

Using (4.5.97), we also note down integrals of certain useful combination of Riemann contractions that show up in trace anomalies in 4 dimensions,

$$\begin{aligned}
&\int_{M_q} d^d x\sqrt{g_q}\left(E_4^{(q)} = R^{(q)}_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu(q)} - 4R^{(q)}_{\mu\nu}R^{\mu\nu(q)} + R^{(q)2}\right)\\
&= \int_{M_1} d^d x\sqrt{g}E_4 + 16\pi^2(1-q)I_a^\Sigma,\\
&\int_{M_q} d^d x\sqrt{g_q}\left(I_4^{(q)} = R^{(q)}_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu(q)} - 2R^{(q)}_{\mu\nu}R^{\mu\nu(q)} + \frac{1}{3}R^{(q)2}\right)\\
&= \int_{M_1} d^d x\sqrt{g}I_4 + 16\pi^2(1-q)I_c^\Sigma,
\end{aligned} \tag{4.5.100}$$

where

$$\begin{aligned}
I_a^\Sigma &= \frac{1}{2\pi}\int_\Sigma (R_{abab} - 2R_{aa} + R + K^2 - \text{Tr } K^2),\\
I_c^\Sigma &= \frac{1}{2\pi}\int_\Sigma \left(R_{abab} - R_{aa} + \frac{1}{3}R + \frac{1}{2}K^2 - \text{Tr } K^2\right).
\end{aligned} \tag{4.5.101}$$

R is the Ricci scalar of metric $g_{\mu\nu}$, n_a^μ, $a = 1, 2$ is a pair of unit vectors orthogonal to the surface Σ, $R_{aa} = R_{\mu\nu}n_a^\mu n_a^\nu$, $R_{abab} = R_{\alpha\beta\mu\nu}n_a^\alpha n_b^\beta n_a^\mu n_b^\nu$, is projection of the Ricci and Riemann tensor onto the subspace orthogonal to surface Σ.

Specific values of integrals of Riemmann contraction and K^2, Tr K^2 in 5 and 6 dimensions for different choices of hypersurface Σ have also been calculated in [110].

4.5.3.2 Scaling of entangling region

Let $r = r_1$ and $r = r_2$ be the two boundaries of the entangling region with $r_2 > r_1$. On a Minkowski background, we have translation symmetry along the $r-$direction, and hence $S_{\text{QFT,flat}}$ depends only on the difference $(r_2 - r_1)$. However, there is no $r-$translation symmetry on non-Minkowski static metric solutions in (4.5.67). Hence S_{QFT} on non-Minkowski static spacetimes depends independently on r_1 and r_2 and not just on the difference $(r_2 - r_1)$. Hence, we fix one end of the entangling region and scale the other to capture this dependence. This scaling behavior is also useful in the discussion of Quantum extremal surfaces [91]. If r_0 is the fixed end, we have

$$|r-r_0| \to e^{\alpha}|r-r_0| \implies r \to r' = r_0 + e^{\alpha}(r-r_0). \tag{4.5.102}$$

This works for both $r > r_0$ and $r < r_0$.

We will keep the farther end r_2 fixed, and scale r_1 i.e., $r_2 \to r_2$ and $r_1 \to r_2 + e^{\alpha}(r_1 - r_2)$. Under this scaling $S_{\text{QFT}}[r_1, r_2] \to S_{\text{QFT}}[r_2 + e^{\alpha}(r_1 - r_2), r_2]$. The LHS of (4.5.75) is now the difference between both of these upto $\mathcal{O}(\alpha)$

$$S_{\text{QFT}}[r_2 + e^{\alpha}(r_1 - r_2), r_2] - S_{\text{QFT}}[r_1, r_2] = -\alpha(\Delta r)\frac{\partial S_{\text{QFT}}}{\partial r_1} + \mathcal{O}(\alpha^2), \tag{4.5.103}$$

where $\Delta r = (r_2 - r_1)$.

If we however keep r_1 fixed and scale r_2, i.e., $r_1 \to r_1$ and $r_2 \to r_1 + e^{\alpha}(r_2 - r_1)$, we have

$$S_{\text{QFT}}[r_1, r_1 + e^{\alpha}(r_2 - r_1)] - S_{\text{QFT}}[r_1, r_2] = \alpha(\Delta r)\frac{\partial S_{\text{QFT}}}{\partial r_2} + \mathcal{O}(\alpha^2). \tag{4.5.104}$$

On the RHS of (4.5.75) we implement the same scaling as follows. On the RHS we have $\delta_{\Delta r} g_{\mu\nu}$, $\delta_{\Delta r} g^{(q)}_{\mu\nu}$ and $\delta_{\Delta r}(\partial_q g^{(q)}_{\mu\nu})|_{q=1}$. To demonstrate the scaling we will use $f_{\mu\nu}$ as a proxy for $g_{\mu\nu}$, $g^{(q)}_{\mu\nu}$ and $\partial_q g^{(q)}_{\mu\nu}$. Under scaling $f_{\mu\nu}(x)dx^{\mu}dx^{\nu} \to f_{\mu\nu}(x')dx'^{\mu}dx'^{\nu} = f'_{\mu\nu}(x,\alpha)dx^{\mu}dx^{\nu}$ (here x'^{μ} denotes scaled coordinates and in our case we are only scaling the r direction keeping r_0 fixed). Here we keep one endpoint r_0 (r_1 or r_2) fixed and scale all other points along the r direction as in equation (4.5.102). Then $\delta_{\Delta r} f_{\mu\nu}(x) = f'_{\mu\nu}(x,\alpha) - f_{\mu\nu}(x)$ upto $\mathcal{O}(\alpha)$

$$\begin{aligned}
\delta_{\Delta r} f_{rr}(\tau, r, x^i) &= \alpha\left((r-r_0)\partial_r + 2\right) f_{rr}(\tau, r, x^i), \\
\delta_{\Delta r} f_{r\tau}(\tau, r, x^i) &= \alpha\left((r-r_0)\partial_r + 1\right) f_{r\tau}(\tau, r, x^i), \\
\delta_{\Delta r} f_{\tau\tau}(\tau, r, x^i) &= \alpha(r-r_0)\partial_r f_{\tau\tau}(\tau, r, x^i), \\
\delta_{\Delta r} f_{ij}(\tau, r, x^i) &= \alpha(r-r_0)\partial_r f_{ij}(\tau, r, x^i).
\end{aligned} \tag{4.5.105}$$

4.5.4 Replica metric contribution to S_{QFT}

The replica contribution in integral I_1 in (4.5.75) is from the difference between the replica and the base metric. This contribution is entirely from the regulated conical singularity and is localised around the endpoints of the entangling region. Hence, using the replica metric expanded around the boundary, given by (4.3.26) and (4.3.32), is enough to capture this. To calculate I_1 we first calculate the local representation of $g^{(q)}_{\mu\nu}$ and $\delta_{\Delta r} g^{(q)}_{\mu\nu}$.

Noting $p(q=1)=1$, in $\{\rho,\psi,y^i\}$ basis, we have,

$$(\partial_q g^{(q)}_{\mu\nu})|_{q=1}=\begin{cases}\partial_q U(\rho)|_{q=1}\equiv U^{[1]}(\rho);\quad \mu=\nu=\rho,\\ 2\rho\cos(\psi)(r_0+\rho\cos(\psi))\log\left(\frac{\rho}{c}\right)p'(q)|_{q=1}h_{ij}(y);\\ \mu=i,\nu=j\ \text{(flat metric (4.3.33))},\\ 2\rho\left(\cos(\psi)K^1_{ij}+\sin(\psi)K^2_{ij}\right)\log\left(\frac{\rho}{c}\right)p'(q)|_{q-1};\\ \mu=i,\nu=j\ \text{(general metric (4.3.44))},\\ 0;\ \text{otherwise.}\end{cases} \tag{4.5.106}$$

In $\{r,\tau,y^i\}$ basis (using (4.3.17)), we have,

$$(\partial_q g^{(q)}_{\mu\nu})|_{q=1}=\begin{cases}\frac{(r-r_0)^2U^{[1]}(\rho)}{(r-r_0)^2+(\tau-\tau_0)^2}=\cos(\psi)^2U^{[1]}(\rho);\quad \mu=\nu=r,\\ \frac{(\tau-\tau_0)^2U^{[1]}(\rho)}{(r-r_0)^2+(\tau-\tau_0)^2}=\sin(\psi)^2U^{[1]}(\rho);\quad \mu=\nu=\tau,\\ \frac{(r-r_0)(\tau-\tau_0)U^{[1]}(\rho)}{(r-r_0)^2+(\tau-\tau_0)^2}=\sin(\psi)\cos(\psi)U^{[1]}(\rho);\\ \mu=r,\nu=\tau\ \text{or vice versa.}\end{cases}$$

All other components are the same as those in (4.5.106). We also have,

$$\alpha(r-r_0)\partial_r f(\rho,\psi)=\alpha\cos(\psi)(\cos(\psi)\rho\partial_\rho f-\sin(\psi)\partial_\psi f). \tag{4.5.107}$$

Using this and (4.5.105) we have,

$$\begin{aligned}&\delta_{\Delta r}((\partial_q g^{(q)}_{\mu\nu})|_{q=1})\\ =&\alpha\left(U^{[1]}(x)\cos(\psi)^2[3-\cos(2\psi)]+x\partial_x U^{[1]}(x)\cos(\psi)^4\right);\quad \mu=\nu=r,\\ =&\alpha\frac{\sin(2\psi)^2}{4}\left(-2U^{[1]}(x)+x\partial_x U^{[1]}(x)\right);\quad \mu=\nu=\tau,\\ =&\alpha\cos(\psi)^4\left(2\tan(\psi)^3U^{[1]}(x)+\tan(\psi)x\partial_x U^{[1]}(x)\right);\ \mu=r,\nu=\tau\ \text{or vice versa,}\\ =&2\alpha\rho\cos(\psi)p'(q)|_{q=1}\left(\cos^2(\psi)(r_0+\rho\cos(\psi))+\log\left(\frac{\rho}{c}\right)(r_0+2\rho\cos(\psi))\right)h_{ij};\\ &\mu=i,\nu=j\ \text{for}\ g^{(q)}_{\mu\nu}-(4.3.33),\\ =&2\alpha\rho\cos(\psi)p'(q)|_{q=1}\left\{\frac{\sin(2\psi)}{2}K_{2ij}+K_{1ij}\left(\cos^2(\psi)+\log\left(\frac{\rho}{c}\right)\right)\right.\\ &+\log\left(\frac{\rho}{c}\right)\Big(\cos(\psi)\left(\sin(\psi)\rho\partial_\rho K_{2ij}+\cos(\psi)\rho\partial_\rho K_{1ij}\right)\\ &\left.+\sin(\psi)(\sin(\psi)\partial_\psi K_{2ij}-\cos(\psi)\partial_\psi K_{1ij})\Big)\right\};\quad \mu=i,\nu=j\ \text{for}\ g^{(q)}_{\mu\nu}-(4.3.44),\\ =&0;\ \text{otherwise,}\end{aligned} \tag{4.5.108}$$

where $x = \rho/a$. We have kept $U(\rho, a)$ general in the expressions above.

We notice that $\partial_q g^{(q)}_{\mu\nu}|_{q=1}$ and $\delta_{\Delta r}((\partial_q g^{(q)}_{\mu\nu})|_{q=1})$ have only an $\mathcal{O}(a^{i>0})$ term. Since the volume element (4.5.82) is at least $O(a^2)$, therefore

$$\int_{\rho=0,\psi=0;\forall y^i}^{\rho=\rho_0,\psi=2\pi q} d^d x \sqrt{g_q} \partial_q g^{(q)}_{\mu\nu}|_{q=1} = 0 \text{ as } \{a \to 0, \rho_0 \to \infty\}, \tag{4.5.109}$$

$$\int_{\rho=0,\psi=0;\forall y^i}^{\rho=\rho_0,\psi=2\pi q} d^d x \sqrt{g_q} \delta_{\Delta r}(\partial_q g^{(q)}_{\mu\nu})|_{q=1} = 0 \text{ as } \{a \to 0, \rho_0 \to \infty\}. \tag{4.5.110}$$

Although the integral vanishes in the $a \to 0$ limit (due to positive powers of a in the integrand), the integral itself diverges before taking the $a \to 0$ limit. This divergence is due to the upper limit being ∞ in the ρ integral which can be made finite by integrating upto a finite upper limit ρ_0, and then taking the limit $\{a \to 0, \rho_0 \to \infty\}$ such that the terms involving the combination of both tend to zero i.e., $a \to 0$ faster than the leading power of ρ_0 (or terms involving ρ_0) tends to ∞. Since one can always choose how fast $a \to 0$, the integral (4.5.109) and (4.5.110) holds.

We notice that (4.5.109) is also true for a general replica metric (4.3.44) (for entangling surface $x^a = 0$; $a = 1, 2$), as can be seen from (4.5.106) having only $\mathcal{O}(a^{i>0})$ terms. Now, even if we scale $\partial_q g^{(q)}_{\mu\nu}|_{q=1}$corresponding to general replica metric (4.3.44), by an general length scaling δ_L, since δ_L is dimensionless, therefore $\delta_L(\partial_q g^{(q)}_{\mu\nu}|_{q=1})$ will also have only $\mathcal{O}(a^{i>0})$ terms, therefore we have

$$\int_{\rho=0,\psi=0;\forall y^i}^{\rho=\infty,\psi=2\pi q} d^d x \sqrt{g_q} \delta_L(\partial_q g^{(q)}_{\mu\nu})|_{q=1} = 0 \text{ as } \{a \to 0, \rho_0 \to \infty\}. \tag{4.5.111}$$

Therefore, for the replica integrals, we can use the local representation

$$\begin{aligned} g^{(q)}_{\mu\nu} &= g_{\mu\nu} \text{ up to } \mathcal{O}((q-1)), & (4.5.112)\\ \delta_{\Delta r}(g^{(q)}_{\mu\nu}) &= \delta_{\Delta r}(g_{\mu\nu}) \text{ up to } \mathcal{O}((q-1)), \\ \delta_L(g^{(q)}_{\mu\nu}) &= \delta_L(g_{\mu\nu}) \text{ up to } \mathcal{O}((q-1)). & (4.5.113)\end{aligned}$$

i.e., for the purpose of a replica integral we have $\partial_q g^{(q)}_{\mu\nu}|_{q=1} = 0$ and $\delta_{\Delta r}(\partial_q g^{(q)}_{\mu\nu})|_{q=1} = 0$. Therefore

$$\begin{aligned} I_1 \equiv& \frac{1}{2}\int_{M_q} d^d x \sqrt{g} \delta_{\Delta r}(\partial_q g^{(q)}_{\mu\nu})|_{q=1} \langle T^{\mu\nu}\rangle \\ &+ \frac{1}{4}\int_{M_q} d^d x \sqrt{g} g^{\sigma\kappa}(\partial_q g^{(q)}_{\sigma\kappa})|_{q=1} \delta_{\Delta r} g_{\mu\nu} \langle T^{\mu\nu}\rangle = 0, \\ I'_1 \equiv& \frac{1}{2}\int_{M_q} d^d x \sqrt{g_q} \delta_{\Delta r}(\partial_q g^{(q)}_{\mu\nu})|_{q=1} \langle T^{\mu\nu}\rangle = 0. \end{aligned} \tag{4.5.114}$$

4.5.5 Replica stress tensor contribution to $S_{\rm QFT}$

We now have,

$$\begin{aligned}\{-\alpha(r_2-r_1)\}\frac{\partial S_{\rm QFT}}{\partial r_1}&=\lim_{q\to 1}\left\{\left(I_2\equiv\frac{1}{2}\int_{M_q}d^dx\sqrt{g}\,(\delta_{\Delta r}g_{\mu\nu})\,\langle T^{\mu\nu}\rangle^{[1]}\right)\right.\\&\left.-\frac{1}{q-1}\left(\int_{M_q}d^dx\sqrt{g_q}\sigma(x)\beta^i(\lambda)\,\langle\mathcal{O}_i(x)\rangle_q-q\int_{M_1}d^dx\sqrt{g}\sigma(x)\beta^i(\lambda)\,\langle\mathcal{O}_i(x)\rangle_{q=1}\right)\right\}.\end{aligned} \tag{4.5.115}$$

I_2 captures the contribution coming from the difference between the replica and base stress tensor expectation value. Using (4.5.105) we have,

$$\begin{aligned}&-\alpha(r_2-r_1)\frac{\partial S_{\rm QFT}}{\partial r_1}\\&=\alpha\lim_{q\to 1}\frac{1}{(q-1)}\left\{\int_{M_q}d^dx\sqrt{g}\left[\frac{(r-r_2)}{2}(\partial_r g_{rr})\left\langle T^{rr}\right\rangle^{[1]}+\left(\frac{(r-r_2)}{2}(\partial_r g_{\tau\tau})-g_{\tau\tau}\right)\right.\right.\\&\qquad\left.\left\langle T^{\tau\tau}\right\rangle^{[1]}-rr_2h_{ij}\left\langle T^{ij}\right\rangle^{[1]}+\left(g_{\mu\nu,\,q=1}\left\langle T^{\mu\nu}\right\rangle_q-\left\langle T^{\mu}_{\mu}\right\rangle_{q=1}\right)\right]\\&\qquad\left.-\left(\int_{M_q}d^dx\sqrt{g_q}\beta^i(\lambda)\,\langle\mathcal{O}_i(x)\rangle_q-q\int_{M_1}d^dx\sqrt{g}\beta^i(\lambda)\,\langle\mathcal{O}_i(x)\rangle_{q=1}\right)\right\}.\end{aligned} \tag{4.5.116}$$

where the transformation is such that $\sigma(x)=\alpha$.

We could add $\alpha\lim_{q\to 1}\int_{M_q}d^dx\sqrt{g}\left\langle T^{\mu\nu}\right\rangle_q(\partial_q g^{(q)}_{\mu\nu})|_{q=1}=0$ (using (4.5.109) or (4.5.112)) in the last line to get the trace of stress tensor on M_q.

Here we have kept the farther end r_2 fixed and scaled all other points along the r direction. Alternatively, we could fix r_1 and scale all other points. In that case we would get the expression above i.e., (4.5.116) with r_1 interchanged with r_2.

Scaling τ: Since $S_{\rm QFT}$ on a static spacetime is independent of τ, therefore we could also scale the τ direction suitably with no effect other than writing our expressions more compactly as we will see.

We fix r_2 and scale all other points in the r direction as described in section (4.5.3.2) and in addition scale $\tau\to\tau'=e^{\alpha}\tau$. Under this we have $S_{\rm QFT}[r_1,r_2,\tau]\to S_{\rm QFT}[r_2+e^{\alpha}(r_1-r_2),r_2,e^{\alpha}\tau]$. On the LHS we have

$$
\begin{aligned}
&S_{\mathrm{QFT}}[r_2 + e^{\alpha}(r_1 - r_2), r_2, e^{\alpha}\tau] - S_{\mathrm{QFT}}[r_1, r_2, \tau] \\
&= -\alpha(\Delta r)\frac{\partial S_{\mathrm{QFT}}}{\partial r_1} + \tau\frac{\partial S_{\mathrm{QFT}}}{\partial \tau} = -\alpha(\Delta r)\frac{\partial S_{\mathrm{QFT}}}{\partial r_1}.
\end{aligned}
\tag{4.5.117}
$$

since $\frac{\partial S_{\mathrm{QFT}}}{\partial \tau} = 0$ on a static background.

On the RHS of (4.5.115) we have to replace $\delta_{\Delta r} g_{\mu\nu}$ with $\delta_{\Delta\tau}\delta_{\Delta r} g_{\mu\nu}$. Where the latter stands for now scaling the r and τ coordinates together in both the metric and the measure in the line element, and taking the difference between the resulting metric and the unscaled metric upto $\mathcal{O}(\alpha)$. In the static case none of the metric components depend on τ and there are no cross terms such as $drd\tau$ in the line element. Therefore scaling τ along with r in the line element only introduces one non-trivial term for the $\delta_{\Delta\tau}\delta_{\Delta r} g_{\tau\tau}$ component, where there is an additional term $2\alpha g_{\tau\tau}$ coming from scaling $d\tau^2$. We have

$$
\delta_{\Delta\tau}\delta_{\Delta r} g_{\tau\tau} = \alpha(r - r_2)\partial_r g_{\tau\tau} + 2\alpha g_{\tau\tau}. \tag{4.5.118}
$$

Introducing this in (4.5.115) we get

$$
\begin{aligned}
&\{-\alpha(r_2 - r_1)\}\frac{\partial S_{\mathrm{QFT}}}{\partial r_1} \\
&= \alpha \lim_{q\to 1}\frac{1}{(q-1)}\Bigg\{\int_{M_q} d^d x\sqrt{g_q}\Bigg[\frac{(r-r_2)}{2}(\partial_r g_{rr})\Big\langle T^{rr}\Big\rangle^{[1]} + \frac{(r-r_2)}{2}(\partial_r g_{\tau\tau})\Big\langle T^{\tau\tau}\Big\rangle^{[1]} \\
&\qquad - r r_2 h_{ij}\Big\langle T^{ij}\Big\rangle^{[1]} + \left(\Big\langle T^{\mu}_{\mu}\Big\rangle_q - \Big\langle T^{\mu}_{\mu}\Big\rangle_{q=1}\right)\Bigg] \\
&\qquad - \left(\int_{M_q} d^d x\sqrt{g_q}\beta^i(\lambda)\left\langle \mathcal{O}_i(x)\right\rangle_q - q\int_{M_1} d^d x\sqrt{g}\beta^i(\lambda)\left\langle \mathcal{O}_i(x)\right\rangle_{q=1}\right)\Bigg\}.
\end{aligned}
\tag{4.5.119}
$$

Instead if we keep only r_1 fixed, we get the same expression as above with $r_1 \leftrightarrow r_2$.

Using the dilatation Ward identity, where $\mathcal{A}_q$ is the QFT dilatation anomaly,

$$
\Big\langle T^{\mu}_{\mu}\Big\rangle_q = \mathcal{A}_q - \beta^i(\lambda)\left\langle \mathcal{O}_i\right\rangle_q. \tag{4.5.120}
$$

Using (4.5.120) in (4.5.119), we have,

$$
\begin{aligned}
&\{-\alpha(r_2-r_1)\}\frac{\partial S_{\text{QFT}}}{\partial r_1} \\
&= \alpha \lim_{q\to 1}\frac{1}{(q-1)}\Bigg\{\int_{M_q} d^dx\sqrt{g_q}\Bigg[\frac{(r-r_2)}{2}(\partial_r g_{rr})\Big\langle T^{rr}\Big\rangle^{[1]} + \frac{(r-r_2)}{2}(\partial_r g_{\tau\tau})\Big\langle T^{\tau\tau}\Big\rangle^{[1]} \\
&\qquad\qquad - rr_2 h_{ij}\Big\langle T^{ij}\Big\rangle^{[1]} + \Big(\mathcal{A}_q - \mathcal{A}\Big)\Bigg] \\
&\qquad - 2\Bigg(\int_{M_q} d^dx\sqrt{g_q}\beta^i(\lambda)\left\langle \mathcal{O}_i(x)\right\rangle_q - q\int_{M_1} d^dx\sqrt{g}\beta^i(\lambda)\left\langle \mathcal{O}_i(x)\right\rangle_{q=1}\Bigg)\Bigg\}.
\end{aligned} \tag{4.5.121}
$$

4.5.6 Comments on S_{QFT} - Local vs global contributions

Let us try to understand the expression for the entanglement entropy (4.2.14). Given the specific state and the entangling region, W_1 is simply a constant independent of the shape of the entangling region. This is because $e^{W_1} = Z_1$ is the trace of the density matrix obtained by sewing the entangling region.

The information about the entangling region is in $W^{[1]}$ (or in general also in $W_q - qW_1$) via the replica boundary conditions involved in Z_q.

We will refer to terms that depend on every point in spacetime as densities and terms that take a value over the entire manifold (integrated value of densities over the entire manifold) as global terms.

Metric dependent terms localise at the boundary of the entangling region: The replica boundary conditions lead to conical singularities in the replica metric only at the boundaries of the entangling region. The conical singularity contributes non trivially, locally in a radius a (the concial singularity regulator), and $g^{(q)}_{\mu\nu} - g_{\mu\nu}$ is localised at the boundaries of the entangling region. Terms in $W_q - qW_1$ density that depends only on $g^{(q)}_{\mu\nu} - g_{\mu\nu}$ (metric dependent terms) will also be localised at the boundary (of entangling region) and these give boundary (of entangling region) dependent terms in $W_q - qW_1$ (in the limit the regulator $a \to 0$). Specifically, these boundary terms contribute to $W^{[1]}$, leading to dependence on boundary coordinates and hence details of the entangling region in S_{QFT}. More generally replica corrections (i.e., terms with $(q-1)^{i>0}$ coefficient) in density terms on M_q (such as the replica corrections to replica stress tensor $\langle T^{\mu\nu}\rangle_q - \langle T^{\mu\nu}\rangle$) that depend on only replica corrections to replica metric $g^{(q)}_{\mu\nu} - g_{\mu\nu}$ (metric dependent terms) are localised at the boundary (of entangling region). These contribute to boundary dependence in global terms on M_q. Notice that these metric-dependent density terms depend on the conical singularity regulator (a) and hence we will sometimes refer to it as regulator dependent terms.

State dependent terms are non local and receive contributions over the entire entangling region: There are also non metric dependent, i.e., regulator independent terms in replica corrections to density terms (including $\langle T^{\mu\nu}\rangle_q - \langle T^{\mu\nu}\rangle$). We will refer to them as state-dependent or regulator independent contributions. These provide non-trivial contributions over the entire manifold and not just the entangling boundaries. Therefore global terms also receive contributions from the entire manifold. Although the replica boundary conditions affects the metric only in a small neighborhood around the boundary of the entangling region, quantities on the replica manifold (replica states, replica correlation functions) respecting these boundary conditions can differ from the corresponding quantities on M_1 throughout the manifold (boundary conditions can alter solutions throughout the manifold). For example: If we start with flat space on M_1 which has translational symmetry along one of the spatial directions in the entangling region, and if the entangling region is finite along that spatial direction then the corresponding replica manifold and metric will now have a conical singularity at the boundary of this entangling region. This breaks the translation symmetry along this spatial direction on the replica manifold and the corresponding quantities (density terms) on the replica manifold will now have terms that reflect this broken symmetry which would be trivial on M_1. Outside the entangling region, since the fields are not identified between different copies of M_1, we do not expect densities to have any replica corrections in this region. Therefore, in general the replica corrections (i.e., terms with $(q-1)^{i>0}$ coefficient) in densities (such as $W_q - qW_1$ density, $\langle T^{\mu\nu}\rangle_q - \langle T^{\mu\nu}\rangle$), sourced by the replica state is expected to have non-trivial contributions throughout the entangling region only, and not just the conical singular endpoints. These contribute terms that depend on the full entangling region in replica corrections to global terms. Since regulator independent density terms receive contributions throughout the entangling region only, non-local dependencies such as dependence on the dimensions (example: length, volume) of the entangling region in S_{QFT}, for example, come from such terms.

The metric/ regulator dependent contributions localised at the boundary completely determine the leading UV/ divergent behavior in S_{QFT}, while the regulator independent (state dependent) terms contribute all the finite non local terms in S_{QFT}.

4.6 Replica stress tensor

In expression (4.5.115), that captures how S_{QFT} varies with the scaling of the entangling region, given the metric and the entangling region, the only quantity in I_2 that we have not explicitly evaluated in terms of quantities on M_1 is the replica corrections to the replica stress tensor, specifically $\langle T^{\mu\nu}\rangle^{[1]}$. In this section, we discuss the stress tensor expectation value on the replica manifold and express them in terms of stress tensor expectation on M_1. Results in section (4.5) apply to QFT on a static background[3]. For the discussion on Replica stress tensor, we will restrict our attention to conformal field theories on a $d-$dimensional general background on manifold M_1 unless specified otherwise later. The theory on the resulting replica manifold M_q will also be a CFT as can be seen from the replica partition function in (4.2.7) with boundary conditions (4.2.8).

In general, for a state in a CFT on M_q (including M_1) one can express the (replica)stress tensor as the sum of a traceless part ($t^{\mu\nu(q)}$) and a non zero trace part ($\chi^{\mu\nu(q)}[g^{(q)}_{\mu\nu}]$)

$$\langle T^{\mu\nu}\rangle_q = t^{\mu\nu(q)} + \chi^{\mu\nu(q)}[g^{(q)}_{\mu\nu}]. \tag{4.6.122}$$

$t^{\mu\nu(q)}$ characterises the specific state, referred to as state part of the replica stress tensor expectation (not to be confused with state dependent part or regulator independent part since we can have regulator dependent terms in $t^{\mu\nu(q)}$ which together with other terms are traceless), while $\chi^{\mu\nu(q)}[g^{(q)}_{\mu\nu}]$ captures the anomaly and the (replica) Casimir or zero-point energy. It depends only on the (replica) metric $g^{(q)}_{\mu\nu}$, and we will refer to it as the geometric part of the replica stress tensor expectation (not necessarily same as the metric dependent or regulator dependent part).

4.6.1 Replica stress tensor expectation - Trace and Conservation equations

For this discussion, we will be inherently working with the renormalised stress tensor. Since the theory on M_q is a CFT (with replica boundary conditions) given we have a CFT on M_1, we have the following constraints on the replica stress tensor expectation,

Trace equations -

$$t^{\mu}{}_{\mu}{}^{(q)} = 0 \text{ (Tracelessness of state part of replica stress tensor)}, \tag{4.6.123}$$

$$\chi^{\mu(q)}_{\mu}[g^{(q)}_{\mu\nu}] = \mathcal{A}[g^{(q)}_{\mu\nu}] \quad \text{(Trace anomaly for a CFT on } M_q). \tag{4.6.124}$$

These two equations imply,

$$\langle T^{\mu}{}_{\mu}\rangle_q = \mathcal{A}[g^{(q)}_{\mu\nu}]. \tag{4.6.125}$$

The replica trace equations above hold only for states in CFTs.

[3] In non-static cases the expression remains same as (4.5.75) or (4.5.76), except one has to include variations of length scale along the other finite y^i directions in the entangling region.

Conservation equations -

$$\nabla_{\mu}{}^{(q)}\langle T^{\mu\nu}\rangle_q = \nabla_{\mu}{}^{(q)}(t^{\mu\nu(q)} + \chi^{\mu\nu(q)}[g_{\mu\nu}^{(q)}]) = 0, \tag{4.6.126}$$

where $\nabla_{\mu}{}^{(q)}$ is the covariant derivative with respect to $g_{\mu\nu}^{(q)}$, and we have conservation of the full replica stress tensor expectation.

The replica conservation equation above holds for any QFT. The state and geometric parts are not separately conserved. Also, notice that the decomposition into state and geometric parts is not unique but the split is convenient in some of the analysis below. One can always add traceless terms ${t'_{\mu\nu}}^{(q)}$ to $t_{\mu\nu}^{(q)}$ and ${\chi'_{\mu\nu}}^{(q)}$ to $\chi_{\mu\nu}^{(q)}$ satisfying

$$\nabla_{\mu}^{(q)}({t'_{\mu\nu}}^{(q)} + {\chi'_{\mu\nu}}^{(q)}) = 0. \tag{4.6.127}$$

The state part can also have regulator dependent terms which are traceless.

The replica stress tensor expectation $\langle T_{\mu\nu}\rangle_q$ differs from base stress tensor expectation $\langle T_{\mu\nu}\rangle$ only in a localised region around the endpoints of the entangling region for purely metric dependent terms and over the entire entangling region for state-dependent (regulator independent) terms. The replica metric, regulating function $U(\rho, a)$, Christoffels, and the state and geometric parts of the replica stress tensor can be expressed as a series in $(q-1)$

$$\begin{aligned}
g_{\mu\nu}^{(q)} &= {g_{\mu\nu}}^{[0]} + (q-1){g_{\mu\nu}}^{[1]} + \mathcal{O}((q-1)^2),\\
U(\rho, a) &= 1 + (q-1)U^{[1]}(\rho, a) + \mathcal{O}((q-1)^2),\\
{\Gamma^{\mu}_{\ \nu\sigma}}^{(q)} &= {\Gamma^{\mu}_{\ \nu\sigma}}^{[0]} + (q-1){\Gamma^{\mu}_{\ \nu\sigma}}^{[1]} + \mathcal{O}((q-1)^2),\\
{\chi_{\mu\nu}}^{(q)} &= {\chi_{\mu\nu}}^{[0]} + (q-1){\chi_{\mu\nu}}^{[1]} + \mathcal{O}((q-1)^2),\\
{t_{\mu\nu}}^{(q)} &= {t_{\mu\nu}}^{[0]} + (q-1){t_{\mu\nu}}^{[1]} + \mathcal{O}((q-1)^2).
\end{aligned} \tag{4.6.128}$$

Conservation of replica stress tensor expectation gives

$$\nabla_{\mu}{}^{(q)}\langle T^{\mu}_{\ \sigma}\rangle_q = \partial_\mu\langle T^{\mu}_{\ \sigma}\rangle_q + {\Gamma^{\mu}_{\ \mu\lambda}}^{(q)}\langle T^{\lambda}_{\ \sigma}\rangle_q - {\Gamma^{\lambda}_{\ \mu\sigma}}^{(q)}\langle T^{\mu}_{\ \lambda}\rangle_q = 0, \tag{4.6.129}$$

i.e.,

$$\begin{aligned}
\nabla_{\mu}{}^{(q)}\langle T^{\mu}_{\ \sigma}\rangle_q = {}&\partial_\mu\left(g^{\mu a(q)}\langle T_{a\sigma}\rangle_q\right) + \left({\Gamma^{\mu}_{\ \mu\lambda}}^{[0]} + (q-1){\Gamma^{\mu}_{\ \mu\lambda}}^{[1]}\right)\left(g^{\lambda a(q)}\langle T_{a\sigma}\rangle_q\right)\\
&- \left({\Gamma^{\lambda}_{\ \mu\sigma}}^{[0]} + (q-1){\Gamma^{\lambda}_{\ \mu\sigma}}^{[1]}\right)\left(g^{\mu a(q)}\langle T_{a\lambda}\rangle_q\right) = 0.
\end{aligned} \tag{4.6.130}$$

At $\mathcal{O}(q-1)$

$$\begin{aligned}
&\partial_\mu\left(g^{\mu a[0]}\langle T_{a\sigma}\rangle^{[1]}\right) + {\Gamma^{\mu}_{\ \mu\lambda}}^{[0]}g^{\lambda a[0]}\langle T_{a\sigma}\rangle^{[1]} - {\Gamma^{\lambda}_{\ \mu\sigma}}^{[0]}g^{\mu a[0]}\langle T_{a\lambda}\rangle^{[1]}\\
&= -\partial_\mu\left(g^{\mu a[1]}\langle T_{a\sigma}\rangle^{[0]}\right) - {\Gamma^{\mu}_{\ \mu\lambda}}^{[0]}g^{\lambda a[1]}\langle T_{a\sigma}\rangle^{[0]} - {\Gamma^{\mu}_{\ \mu\lambda}}^{[1]}g^{\lambda a[0]}\langle T_{a\sigma}\rangle^{[0]}\\
&\quad + {\Gamma^{\lambda}_{\ \mu\sigma}}^{[0]}g^{\mu a[1]}\langle T_{a\lambda}\rangle^{[0]} + {\Gamma^{\lambda}_{\ \mu\sigma}}^{[1]}g^{\mu a[0]}\langle T_{a\lambda}\rangle^{[0]}.
\end{aligned} \tag{4.6.131}$$

Metric compatibility $\nabla^{[0]}_{\mu} g^{\mu a[0]} = 0$ on M_1 implies

$$\partial_\mu g^{\mu a[0]} + \Gamma^{\mu}{}_{\mu\lambda}{}^{[0]} g^{\lambda a[0]} + \Gamma^{a}{}_{\mu\lambda}{}^{[0]} g^{\mu\lambda[0]} = 0. \tag{4.6.132}$$

Using this, LHS in (4.6.131) reduces to

$$\begin{aligned} LHS =& g^{\mu a[0]} \partial_\mu \langle T_{a\sigma}\rangle^{[1]} + \langle T_{a\sigma}\rangle^{[1]} \left(\partial_\mu g^{\mu a[0]} + \Gamma^{\mu}{}_{\mu\lambda}{}^{[0]} g^{\lambda a[0]} \right) - \Gamma^{\lambda}{}_{\mu\sigma}{}^{[0]} g^{\mu a[0]} \langle T_{a\lambda}\rangle^{[1]} \\ =& g^{\mu a[0]} \partial_\mu \langle T_{a\sigma}\rangle^{[1]} - \Gamma^{a}{}_{\mu\lambda}{}^{[0]} g^{\mu\lambda[0]} \langle T_{a\sigma}\rangle^{[1]} - \Gamma^{\lambda}{}_{\mu\sigma}{}^{[0]} g^{\mu a[0]} \langle T_{a\lambda}\rangle^{[1]}. \end{aligned} \tag{4.6.133}$$

Metric compatibility $\nabla^{(q)}_{\mu} g^{\mu a(q)} = 0$ on M_q implies

$$\partial_\mu g^{\mu a(q)} + \Gamma^{\mu}{}_{\mu\lambda}{}^{(q)} g^{\lambda a(q)} + \Gamma^{a}{}_{\mu\lambda}{}^{(q)} g^{\mu\lambda(q)} = 0. \tag{4.6.134}$$

At $\mathcal{O}(q-1)$

$$\partial_\mu g^{\mu a[1]} + \Gamma^{\mu}{}_{\mu\lambda}{}^{[0]} g^{\lambda a[1]} + \Gamma^{a}{}_{\mu\lambda}{}^{[0]} g^{\mu\lambda[1]} + \Gamma^{\mu}{}_{\mu\lambda}{}^{[1]} g^{\lambda a[0]} + \Gamma^{a}{}_{\mu\lambda}{}^{[1]} g^{\mu\lambda[0]} = 0. \tag{4.6.135}$$

Using this, RHS in (4.6.131) reduces to

$$\begin{aligned} RHS =& - g^{\mu a[1]} \partial_\mu \langle T_{a\sigma}\rangle^{[0]} - \left(\partial_\mu g^{\mu a[1]} + \Gamma^{\mu}{}_{\mu\lambda}{}^{[0]} g^{\lambda a[1]} \right) \langle T_{a\sigma}\rangle^{[0]} - \Gamma^{\mu}{}_{\mu\lambda}{}^{[1]} g^{\lambda a[0]} \langle T_{a\sigma}\rangle^{[0]} \\ & + \Gamma^{\lambda}{}_{\mu\sigma}{}^{[0]} g^{\mu a[1]} \langle T_{a\lambda}\rangle^{[0]} + \Gamma^{\lambda}{}_{\mu\sigma}{}^{[1]} g^{\mu a[0]} \langle T_{a\lambda}\rangle^{[0]} \\ =& - g^{\mu a[1]} \partial_\mu \langle T_{a\sigma}\rangle^{[0]} + \left(\Gamma^{a}{}_{\mu\lambda}{}^{[0]} g^{\mu\lambda[1]} + \Gamma^{\mu}{}_{\mu\lambda}{}^{[1]} g^{\lambda a[0]} + \Gamma^{a}{}_{\mu\lambda}{}^{[1]} g^{\mu\lambda[0]} \right) \langle T_{a\sigma}\rangle^{[0]} \\ & - \Gamma^{\mu}{}_{\mu\lambda}{}^{[1]} g^{\lambda a[0]} \langle T_{a\sigma}\rangle^{[0]} + \Gamma^{\lambda}{}_{\mu\sigma}{}^{[0]} g^{\mu a[1]} \langle T_{a\lambda}\rangle^{[0]} + \Gamma^{\lambda}{}_{\mu\sigma}{}^{[1]} g^{\mu a[0]} \langle T_{a\lambda}\rangle^{[0]} \\ =& \left(-g^{\mu a[1]} \partial_\mu + \Gamma^{a}{}_{\mu\lambda}{}^{[0]} g^{\mu\lambda[1]} + \Gamma^{a}{}_{\mu\lambda}{}^{[1]} g^{\mu\lambda[0]} \right) \langle T_{a\sigma}\rangle^{[0]} \\ & + \left(\Gamma^{\lambda}{}_{\mu\sigma}{}^{[0]} g^{\mu a[1]} + \Gamma^{\lambda}{}_{\mu\sigma}{}^{[1]} g^{\mu a[0]} \right) \langle T_{a\lambda}\rangle^{[0]}. \end{aligned} \tag{4.6.136}$$

Combining both sides of equality in (4.6.131), given by (4.6.133) and (4.6.136),

$$\boxed{\begin{aligned} &\left(g^{\mu a[0]} \partial_\mu - \Gamma^{a}{}_{\mu\lambda}{}^{[0]} g^{\mu\lambda[0]} \right) \langle T_{a\sigma}\rangle^{[1]} - \Gamma^{\lambda}{}_{\mu\sigma}{}^{[0]} g^{\mu a[0]} \langle T_{a\lambda}\rangle^{[1]} \\ &= - \left(g^{\mu a[1]} \partial_\mu - \Gamma^{a}{}_{\mu\lambda}{}^{[0]} g^{\mu\lambda[1]} - \Gamma^{a}{}_{\mu\lambda}{}^{[1]} g^{\mu\lambda[0]} \right) \langle T_{a\sigma}\rangle^{[0]} \\ &\quad + \left(\Gamma^{\lambda}{}_{\mu\sigma}{}^{[0]} g^{\mu a[1]} + \Gamma^{\lambda}{}_{\mu\sigma}{}^{[1]} g^{\mu a[0]} \right) \langle T_{a\lambda}\rangle^{[0]} \end{aligned}} \tag{4.6.137}$$

This is of the form

$$L^{a[0]} \langle T_{a\sigma}\rangle^{[1]} - M^{\lambda a}{}_{\sigma}{}^{[0]} \langle T_{a\lambda}\rangle^{[1]} = -L^{a[1]} \langle T_{a\sigma}\rangle^{[0]} + M^{\lambda a}{}_{\sigma}{}^{[1]} \langle T_{a\lambda}\rangle^{[0]}. \tag{4.6.138}$$

where

$$L^{a(q)} = g^{\mu a(q)} \partial_\mu - \Gamma^{a}{}_{\mu\lambda}{}^{(q)} g^{\mu\lambda(q)}, \tag{4.6.139}$$

$$M^{\lambda a}{}_{\sigma}{}^{(q)} = \Gamma^{\lambda}{}_{\mu\sigma}{}^{(q)} g^{\mu a(q)}. \tag{4.6.140}$$

Equation (4.6.137) are d coupled first-order linear differential equation involving $\mathcal{O}(q-1)$ replica stress tensor corrections $\langle T_{\mu\nu}\rangle^{[1]}$ on the LHS, and on RHS we have the sources: stress tensor on M_1 i.e., $\langle T_{\mu\nu}\rangle^{[0]}$, $\mathcal{O}(q-1)$ replica metric corrections $g^{[1]}_{\mu\nu}$, metric on M_1. We can also write (4.6.137) in an integral form using Stokes theorem. However to solve for $\langle T_{\mu\nu}\rangle^{[1]}$, we need the explicit form of the sources along with more linear equations involving $\langle T_{\mu\nu}\rangle^{[1]}$ (since we have $d(d+1)/2$ unknowns and $(d+1)$ linear equations). In the presence of certain symmetries, we can reduce the number of unknowns involved. We will see in the following sections that in the static case, the replica conservation and trace equations are enough.

Note on homogeneous solution to $\langle T_{\mu\nu}\rangle^{[1]}$: We also notice that the conservation equations on M_1 i.e.,

$$\nabla^{[0]}_{\mu}\langle T^{\mu}{}_{\sigma}\rangle^{[0]} = 0, \tag{4.6.141}$$

imply

$$L^{a\,[0]}\langle T_{a\sigma}\rangle^{[0]} - M^{\lambda a}{}_{\sigma}{}^{[0]}\langle T_{a\lambda}\rangle^{[0]} = 0, \tag{4.6.142}$$

i.e., it is the same as the replica conservation equation (4.6.138) with no sources and with $\langle T_{\mu\nu}\rangle^{[1]}$ replaced by $\langle T_{\mu\nu}\rangle^{[0]}$. However, notice that the homogeneous solution to $\langle T^{\mu}{}_{\sigma}\rangle^{[1]}$ is not only $\langle T^{\mu}{}_{\sigma}\rangle^{[0]}$ since the presence of a conical singularity in M_q at the boundary breaks any translation symmetry along ρ in M_1. Therefore terms that we usually set to zero in $\langle T^{\mu}{}_{\sigma}\rangle^{[0]}$ due to translation symmetry on M_1 can be non trivial in $\langle T^{\mu}{}_{\sigma}\rangle^{[1]}$ on M_q.

In the case of squashed cone the translation symmetry along ψ is also broken and we only have the discrete symmetry i.e., $\psi \to \psi + 2\pi k$ where $k \in \mathbb{Z}$ which might introduce ψ dependence and more terms on M_q. Even if there were no translation symmetry on M_1, to begin with, the homogeneous solution is still modified due to the presence of conical and curvature singularities (in squashed cone) on M_q. In other words we need to be careful when fixing the integration constants in the general homogeneous solution to $\langle T^{\mu}{}_{\sigma}\rangle^{[1]}$(they are not the same as those in $\langle T^{\mu}{}_{\sigma}\rangle^{[0]}$ on M_1). The homogeneous solution to $\langle T^{\mu}{}_{\sigma}\rangle^{[1]}$ has some more terms in addition to $\langle T^{\mu}{}_{\sigma}\rangle^{[0]}$.

4.6.2 Local and non local contributions

As discussed before in section (4.5.6) the replica corrections to the replica stress tensor i.e., $\langle T_{\mu\nu}\rangle_q - \langle T_{\mu\nu}\rangle$ are localised around the boundary of the entangling region for purely metric dependent terms; and around the full entangling region for regulator independent (state dependent) terms.

Notice that the regulated replica metric $g^{(q)}_{\mu\nu}$ i.e., (4.3.44) is in general valid (is a local expansion) in only a small neighborhood around the boundary of the entangling region. Hence, once we make use of replica metric (4.3.44) in the replica conservation equation (4.6.126), the $\mathcal{O}(q-1)$ corrections to the replica stress tensor $\langle T_{\mu\nu}\rangle^{[1]}$ calculated from the corresponding replica conservation equations will also be valid only in a small neighborhood of the boundary of the entangling region.

This is enough to extract the $\mathcal{O}(q-1)$ corrections to the purely metric dependent parts of the replica stress tensor since this by definition depends only on $(g^{(q)}_{\mu\nu} - g_{\mu\nu})$ which is non zero only in a small neighborhood (determined by the distance from the conical singularity set by the regulator (a)) around the conical singular endpoints (since the replica metric differs from the metric on M_1 only in a small neighborhood around the conical singular endpoints). The localised contributions around $\rho = 0$, of the $\mathcal{O}(q-1)$ corrections to the purely metric dependent parts of the replica stress tensor, can be explicitly calculated using (4.5.78). For the purely metric dependent parts of the solution $\langle T_{\mu\nu}\rangle^{[1]}$ we also have

$$\langle T_{\mu\nu}\rangle^{[1]}|_{\rho>>a} \to 0, \tag{4.6.143}$$

where a is the regulator. For integrals involving such terms it is enough to perform the ρ integral between $[0, \rho_0 >> a]$ since the integrand is zero for $\rho >> a$.

The regulator independent (state-dependent) replica stress tensor (at all orders in $(q-1)$) are non trivial due to broken symmetries on M_q that were originally present in M_1 background metric (because of conical singularity or states that break the symmetry on background metric on M_1). They are non trivial over the entire entangling region and do not have a regulating radius (a) dependence. Since these terms must go to zero as $q \to 1$ (since symmetries are restored as $q \to 1$), they cannot have a non trivial $O((q-1)^0)$ term i.e., they do not have a non trivial counterpart on M_1. This is unlike the regulator dependent case where we also had a non trivial $O(1)$ term.

In the case of a disconnected boundary (at $r = r_1$ and $r = r_2$) we have two conical singularities, one at each boundary. The net contribution to the $\mathcal{O}(q-1)$ replica stress tensor is the sum of the contributions from each conical singularity. We perform the integral restricted around the entangling region. As mentioned before, the replica boundary conditions do not affect the different copies of M_1 outside the entangling region. For the regulator independent parts of the replica stress tensor, we consider the contribution around r_1 in the semi disc of radius $(r_2 - r_1)/2$, with $r \geq r_1$ centered around r_1, and contribution around r_2 in semi disc of radius $(r_2 - r_1)/2$ with $r \leq r_2$ centered around r_2. We perform the integrals for the regulator-independent parts of the replica stress tensor restricted to these semi discs. On a flat background on M_1, due to translational invariance, the $\mathcal{O}(q-1)$ replica stress tensor has identical contributions from each conical singularity and the net contribution is the sum of individual conical contribution, therefore in this case the integral is the same as integrating over a disc of radius $(r_2 - r_1)/2$ around one of the conical singularities.

4.6.3 Replica stress tensor - Static metric

So far in the replica conservation equations we have not made a choice of the metric on M_1. Although the $(d+1)$ - replica conservation and trace equations hold true on any general background, on static backgrounds we can exploit certain symmetries to reduce the number of unknown replica stress tensor components to match the number of equations.

The replica conservation and trace equations can be explicitly expanded for the most general metric (4.3.44). However, since all components of the replica metric are non-trivial in the general case, it is rather easy to expand it for the particular case of interest as and when necessary. On static backgrounds we can work in a gauge in which - $g^{(q)}_{\rho i} = g^{(q)}_{\psi i} = g^{(q)}_{\rho\psi} = 0$, see (4.3.41), we will now explicitly write down the replica conservation and trace equations in this case.

Note that $\nabla_\mu{}^{(q)}$ is non-trivial, and on replica manifold one needs Christoffels $\Gamma^\mu{}_{\nu\sigma}{}^{(q)}$ to explicitly write down the conservation equations on M_q, for instance: metric with zero $\Gamma^\mu{}_{\nu\sigma}$ on M_1 have non zero $\Gamma^\mu{}_{\nu\sigma}{}^{(q)}$. For the replica metric $g^{(q)}_{\mu\nu}$ corresponding to the static background on M_1, the non-trivial Christoffel symbols are:

$$
\begin{aligned}
\Gamma^\rho{}_{\rho\rho}{}^{(q)} &= \frac{1}{2}U^{-1}\partial_\rho U, & \Gamma^i{}_{\psi k}{}^{(q)} &= \frac{1}{2}g^{ij\,(q)}\partial_\psi g^{(q)}_{jk}, \\
\Gamma^\rho{}_{\psi\psi}{}^{(q)} &= -U^{-1}\rho, & \Gamma^\rho{}_{ij}{}^{(q)} &= -\frac{1}{2}U^{-1}\partial_\rho g^{(q)}_{ij}, \\
\Gamma^\psi{}_{\rho\psi}{}^{(q)} &= \frac{1}{\rho}, & \Gamma^\psi{}_{ij}{}^{(q)} &= -\frac{1}{2\rho^2}\partial_\psi g^{(q)}_{ij}, \\
\Gamma^i{}_{\rho k}{}^{(q)} &= \frac{1}{2}g^{ij\,(q)}\partial_\rho g^{(q)}_{jk}, & \Gamma^i{}_{kl}{}^{(q)} &= \Gamma^i{}_{kl}{}^{[0]},
\end{aligned}
$$

with

$$
\begin{aligned}
g^{(q)}_{ij} =&\delta_{ij}, \text{ or} \\
=&(r_0 + \rho^p c^{1-p}\cos(\psi))^2 h_{ij}(y), \text{ or} \\
=&(r_0 + \rho^p c^{1-p}\cos(\psi))^2; \quad i = j = \phi, \text{ and } \delta_{ij}; \text{ otherwise, or} \\
=&(\gamma_{ij} + 2\rho^p c^{1-p}\cos(\psi)K^1_{ij} + \dots).
\end{aligned} \tag{4.6.144}
$$

The first three equalities correspond to planar, spherical, and cylindrical entangling regions in flat space respectively, and the last equality corresponds to a static background.

We have,

$$
\begin{aligned}
&L^{a[0]}\langle T_{a\sigma}\rangle^{[1]} \\
&= \left(\partial_\rho - \frac{\Gamma^\rho{}_{\psi\psi}}{\rho^2} - \Gamma^\rho{}_{ij}g^{ij}\right)\langle T_{\rho\sigma}\rangle^{[1]} + \left(\frac{1}{\rho^2}\partial_\psi - \Gamma^\psi{}_{ij}g^{ij}\right)\langle T_{\psi\sigma}\rangle^{[1]} + \left(g^{ik}\partial_k - \Gamma^i{}_{jk}g^{jk}\right)\langle T_{i\sigma}\rangle^{[1]} \\
&= \left(\partial_\rho + \frac{1}{\rho} + \Gamma^i{}_{i\rho}\right)\langle T_{\rho\sigma}\rangle^{[1]} + \frac{1}{\rho^2}\left(\partial_\psi + \Gamma^i{}_{i\psi}\right)\langle T_{\psi\sigma}\rangle^{[1]} + \left(\partial_k + \Gamma^i{}_{ik}\right)\langle T^k_\sigma\rangle^{[1]},
\end{aligned}
$$

here we have used $-\Gamma^\rho{}_{ij}g^{ij} = \Gamma^i{}_{i\rho}$ and $-\rho^2\Gamma^\psi{}_{ij}g^{ij} = \Gamma^i{}_{i\psi}$.

In general, explicit coordinate ($\{\rho,\psi\}$) dependence of $L^{a(q)}\langle T_{a\lambda}\rangle^{(q)}$ can be written but $M^{\lambda a(q)}_{\sigma}\langle T_{a\lambda}\rangle^{(q)}$ depends on the choice of σ. We will write down the full explicit replica stress tensor conservation equations in subsequent sections.

4.6.3.1 Replica stress tensor: Geometric part

The geometric part of stress tensor on M_1 i.e., $\chi_{\mu\nu}[g_{\mu\nu}]$ and its trace i.e., the trace anomaly $\chi^\mu{}_\mu[g_{\mu\nu}] = \mathcal{A}[g_{\mu\nu}]$ are well known. They are non-trivial for an even-dimensional CFT on a curved background and are given by

$$
\left\langle \chi^\mu{}_\mu \right\rangle = \frac{1}{(4\pi)^{d/2}}\left(\sum_j c_{dj} I_j^{(d)} - (-)^{\frac{d}{2}} a_d E^{(d)} + \sum_j d_{dj} D^i J_i^{(d)}\right), \tag{4.6.145}
$$

where $E^{(d)}$ is the Euler density in d dimensions (type A anomaly), $I_j^{(d)}$ are independent Weyl invariants (type B anomalies), and the last term denotes the type D anomalies which are total derivatives that could be canceled by the Weyl variation of local covariant counterterms (scheme dependent). We can express $E^{(d)}$ in the form,

$$
E^{(d)} = \frac{1}{2^{d/2}}\delta^{\nu_1\cdots\nu_d}_{\mu_1\cdots\mu_d} R^{\mu_1\mu_2}_{\nu_1\nu_2}\cdots R^{\mu_{d-1}\mu_d}_{\nu_{d-1}\nu_d}. \tag{4.6.146}
$$

For example in $d=2$

$$
\chi_{\mu\nu}[g_{\mu\nu}]^{d=2} = -\frac{c}{48\pi} R g_{\mu\nu}. \tag{4.6.147}
$$

In $d=4$ there is one Weyl invariant and in $d=6$ there are three Weyl invariants,

$$
\chi^\mu{}_\mu[g_{\mu\nu}]^{d=4} = -\frac{a}{(4\pi)^2}E^{(4)} + \frac{c}{(4\pi)^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}. \tag{4.6.148}
$$

When d is odd

$$
\chi_{\mu\nu}[g_{\mu\nu}]^{d\ \text{odd}} = 0. \tag{4.6.149}
$$

In general, for a conformally flat background $\overline{g}_{\mu\nu}(x) = e^{2\sigma(x)}\eta_{\mu\nu} = \Omega^2\eta_{\mu\nu}$ (with stress tensor vacuum expectation value normalized to 0 in flat space) the following relation between the vacuum stress tensor expectation (the only contribution is the geometric part) and the trace anomaly holds [215, 216],

$$\frac{\delta\sqrt{-\bar{g}}\,\langle\bar{T}^{\mu\nu}(x)\rangle}{\delta\sigma\,(x')} = 2\frac{\delta\sqrt{-\bar{g}}\left\langle\bar{T}^{\lambda}_{\lambda}\,(x')\right\rangle}{\delta\bar{g}_{\mu\nu}(x)}. \tag{4.6.150}$$

The solution of which is given by

$$\begin{aligned}\langle\bar{T}^{\mu}_{\nu}\rangle = \Omega^{-4}\,\langle T^{\mu}_{\nu}\rangle - \frac{a_4}{(4\pi)^2}\left[\left(4\bar{R}^{\lambda}_{\rho}\bar{W}^{\rho\mu}{}_{\lambda\nu} - 2\bar{H}^{\mu}_{\nu}\right) - \Omega^{-4}\left(4R^{\lambda}_{\rho}W^{\rho\mu}{}_{\lambda\nu} - 2H^{\mu}_{\nu}\right)\right]\\ - \frac{\gamma}{6(4\pi)^2}\left[I^{\mu}_{\nu} - \Omega^{-4}I^{\mu}_{\nu}\right] - 8\frac{c_4}{(4\pi)^2}\left[\bar{D}^{\rho}\bar{D}_{\lambda}\left(\bar{W}^{\rho\mu}{}_{\lambda\nu}\ln\Omega\right) + \frac{1}{2}\bar{R}^{\lambda}_{\rho}\bar{W}^{\rho\mu}{}_{\lambda\nu}\ln\Omega\right],\end{aligned} \tag{4.6.151}$$

where,

$$\begin{aligned}H_{\mu\nu} &\equiv -\frac{1}{2}\left[g_{\mu\nu}\left(\frac{R^2}{2} - R^2_{\lambda\rho}\right) + 2R^{\lambda}_{\mu}R_{\nu\lambda} - \frac{4}{3}RR_{\mu\nu}\right],\\ I_{\mu\nu} &\equiv 2D_{\mu}D_{\nu}R - 2g_{\mu\nu}D^2R - 2RR_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R^2.\end{aligned} \tag{4.6.152}$$

Replica geometric part (solution): We can get the geometric part of the replica stress tensor by taking

$$\chi^{(q)}_{\mu\nu}[g^{(q)}_{\mu\nu}] = \chi_{\mu\nu}[g_{\mu\nu} \to g^{(q)}_{\mu\nu}], \tag{4.6.153}$$

since it immediately solves equation (4.6.124).

Therefore, we have

$$\chi^{(q)}_{\mu\nu}[g^{(q)}_{\mu\nu}]^{d=2} = -\frac{c}{48\pi}R^{(q)}g^{(q)}_{\mu\nu}. \tag{4.6.154}$$

We have,

$$\chi^{[1]}_{\mu\nu} = \frac{\partial\chi_{\mu\nu}}{\partial g_{\sigma\eta}}g^{[1]}_{\sigma\eta}. \tag{4.6.155}$$

4.6.3.2 Replica stress tensor: State part

We now determine the traceless replica state part and the full replica stress tensor from that on the base. We assume that the states respect certain symmetries of the background. For example, on static backgrounds, we assume that the states satisfy $\langle T_{ij}\rangle = 0\ \forall\ i \neq j$ where $y^i \in \Sigma$, since we have spherical or planar or hyperbolic symmetry along $y^{i\prime}s$ and hence $g_{ij} = 0\ \forall\ i \neq j$. On a static metric we also have $g_{\rho i} = 0 = g_{\psi i}$[4]. We consider states that respect these symmetries as well i.e., states that

[4] This is not true in non static spacetimes.

satisfy $\langle T_{\rho i}\rangle = 0 = \langle T_{\psi i}\rangle$. These states are of interest in the context of the black hole problem since we expect the average behavior of the hawking radiation to also respect this symmetry. While calculating S_{QFT} in our examples, we consider states that are common to all theories, such as the vacuum and thermal states.

We will work with the base manifold stress tensor expectation for a general state on M_1 (i.e., we consider a general, state part of the base stress tensor $t_{\mu\nu}$ denoted as $t^{[0]}_{\mu\nu}$) unless specified.

Tracelessness of $t^{(q)}_{\mu\nu}$ i.e., (4.6.123) gives

$$t^{\mu}{}_{\mu}{}^{(q)} = g^{\mu\nu\,(q)} t_{\mu\nu}{}^{(q)} = 0. \tag{4.6.156}$$

Using (4.6.128), and considering only the $\mathcal{O}(q-1)$ term, we have

$$g^{\mu\nu\,[0]} t_{\mu\nu}{}^{[1]} + g^{\mu\nu\,[1]} t_{\mu\nu}{}^{[0]} = 0. \tag{4.6.157}$$

This gives

$$t^{[1]}_{\rho\rho} + \frac{1}{\rho^2} t^{[1]}_{\psi\psi} + g^{ij} t^{[1]}_{ij} = U^{[1]}(\rho, a) t^{[0]}_{\rho\rho} - g^{ij\,[1]} t^{[0]}_{ij}. \tag{4.6.158}$$

The trace equation for the full replica stress tensor i.e., (4.6.125) gives,

$$\langle T^{\rho}{}_{\rho}\rangle_q + \frac{1}{\rho^2}\langle T_{\psi\psi}\rangle_q + \langle T^{i}{}_{i}\rangle_q = \mathcal{A}[g^{(q)}_{\mu\nu}]. \tag{4.6.159}$$

Conservation of replica stress tensor expectation i.e., (4.6.126) gives

$$\nabla_{\mu}{}^{(q)}\langle T^{\mu}{}_{\sigma}\rangle_q = \partial_{\mu}\langle T^{\mu}{}_{\sigma}\rangle_q + \Gamma^{\mu}{}_{\mu\lambda}{}^{(q)}\langle T^{\lambda}{}_{\sigma}\rangle_q - \Gamma^{\lambda}{}_{\mu\sigma}{}^{(q)}\langle T^{\mu}{}_{\lambda}\rangle_q = 0. \tag{4.6.160}$$

1. For $\sigma = \rho$

$$\begin{aligned}\nabla_{\mu}{}^{(q)}\langle T^{\mu}{}_{\rho}\rangle_q = 0 = (\partial_\rho + \Gamma^{\psi}{}_{\psi\rho}{}^{(q)} + \Gamma^{i}{}_{i\rho}{}^{(q)})\langle T^{\rho}{}_{\rho}\rangle_q + (\partial_\psi + \Gamma^{i}{}_{i\psi}{}^{(q)})\langle T^{\psi}{}_{\rho}\rangle_q \\ + (\partial_k + \Gamma^{i}{}_{ik}{}^{(q)})\langle T^{k}{}_{\rho}\rangle_q - \Gamma^{\psi}{}_{\psi\rho}{}^{(q)}\langle T^{\psi}{}_{\psi}\rangle_q - \Gamma^{i}{}_{k\rho}{}^{(q)}\langle T^{k}{}_{i}\rangle_q.\end{aligned} \tag{4.6.161}$$

Using the replica Christoffels (4.6.144) and series expanding in $(q-1)$ using (4.6.128) and considering only the $\mathcal{O}(q-1)$ term, we have

$$\begin{aligned}&\left(\partial_\rho + \frac{1}{\rho} + \Gamma^{i}{}_{i\rho}\right)\langle T_{\rho\rho}\rangle^{[1]} + \frac{1}{\rho^2}(\partial_\psi + \Gamma^{i}{}_{i\psi})\langle T_{\psi\rho}\rangle^{[1]} - \frac{1}{\rho^3}\langle T_{\psi\psi}\rangle^{[1]} + (\partial_k + \\ &\Gamma^{i}{}_{ik})\langle T^{k}{}_{\rho}\rangle^{[1]} - \Gamma^{i}{}_{k\rho}\langle T^{k}_{i}\rangle^{[1]} = \left(\partial_\rho + \frac{1}{\rho} + \Gamma^{i}{}_{i\rho}\right)(U^{[1]}\langle T_{\rho\rho}\rangle^{[0]}) - \Gamma^{i}{}_{i\rho}{}^{[1]}\langle T_{\rho\rho}\rangle^{[0]} \\ &- \frac{1}{\rho^2}\Gamma^{i}{}_{i\psi}{}^{[1]}\langle T_{\psi\rho}\rangle^{[0]} + \Gamma^{i}{}_{k\rho}{}^{[1]}\langle T^{k}_{i}\rangle^{[0]}.\end{aligned} \tag{4.6.162}$$

If we use the trace equation for the full replica stress tensor i.e., (4.6.159) to eliminate $\langle T^\rho_\rho\rangle_q$, we get

$$\frac{1}{\rho^2}(\partial_\psi+\Gamma^i{}_{i\psi})\langle T_{\rho\psi}\rangle^{[1]}-\frac{1}{\rho^2}(\partial_\rho+\Gamma^i{}_{i\rho})\langle T_{\psi\psi}\rangle^{[1]}-\left(\partial_\rho+\frac{1}{\rho}+\Gamma^i{}_{i\rho}\right)\Big(\langle T^k_k\rangle^{[1]}$$
$$-\mathcal{A}^{[1]}[g^{(q)}_{\mu\nu}]\Big)+(\partial_k+\Gamma^i{}_{ik})\langle T^k_\rho\rangle^{[1]}-\Gamma^i{}_{k\rho}\langle T^k_i\rangle^{[1]}=\frac{1}{\rho^2}(\Gamma^i{}_{i\rho}{}^{[1]}\langle T_{\psi\psi}\rangle^{[0]}$$
$$-\Gamma^i{}_{i\psi}{}^{[1]}\langle T_{\rho\psi}\rangle^{[0]})+\Gamma^i{}_{i\rho}{}^{[1]}\left(\langle T^k_k\rangle^{[0]}-\mathcal{A}[g_{\mu\nu}]\right)+\Gamma^i{}_{k\rho}{}^{[1]}\langle T^k_i\rangle^{[0]}. \qquad (4.6.163)$$

Using (4.6.122) implies,

$$\frac{1}{\rho^2}(\partial_\psi+\Gamma^i{}_{i\psi})t^{[1]}_{\rho\psi}-\frac{1}{\rho^2}(\partial_\rho+\Gamma^i{}_{i\rho})t^{[1]}_{\psi\psi}-\left(\partial_\rho+\frac{1}{\rho}+\Gamma^i{}_{i\rho}\right)t^k{}_k{}^{[1]}+(\partial_k$$
$$+\Gamma^i{}_{ik})t^k{}_\rho{}^{[1]}-\Gamma^i{}_{k\rho}t^k{}_i{}^{[1]}=-\frac{1}{\rho^2}(\partial_\psi+\Gamma^i{}_{i\psi})\chi^{[1]}_{\rho\psi}+\frac{1}{\rho^2}(\partial_\rho+\Gamma^i{}_{i\rho})\chi^{[1]}_{\psi\psi} \qquad (4.6.164)$$
$$+\left(\partial_\rho+\frac{1}{\rho}+\Gamma^i{}_{i\rho}\right)\left(\chi^k{}_k{}^{[1]}-\mathcal{A}^{[1]}[g^{(q)}_{\mu\nu}]\right)-(\partial_k+\Gamma^i{}_{ik})\chi^k{}_\rho{}^{[1]}+\Gamma^i{}_{k\rho}\chi^k{}_i{}^{[1]}$$
$$+\frac{1}{\rho^2}(\Gamma^i{}_{i\rho}{}^{[1]}\langle T_{\psi\psi}\rangle^{[0]}-\Gamma^i{}_{i\psi}{}^{[1]}\langle T_{\rho\psi}\rangle^{[0]})+\Gamma^i{}_{i\rho}{}^{[1]}\left(\langle T^k_k\rangle^{[0]}-\mathcal{A}[g_{\mu\nu}]\right)+\Gamma^i{}_{k\rho}{}^{[1]}\langle T^k_i\rangle^{[0]},$$

where we have collected the source terms (i.e., those expressed in terms of the stress tensor on the base manifold) on the RHS.

2. For $\sigma=\psi$

$$\nabla_\mu{}^{(q)}\langle T^\mu_\psi\rangle_q=0=(\partial_\rho+\Gamma^\rho{}_{\rho\rho}{}^{(q)}+\Gamma^i{}_{i\rho}{}^{(q)}-\Gamma^\rho{}_{\psi\psi}{}^{(q)}g_{\rho\rho}{}^{(q)}g^{\psi\psi(q)})\langle T^\rho_\psi\rangle_q$$
$$+(\partial_\psi+\Gamma^i{}_{i\psi}{}^{(q)})\langle T^\psi_\psi\rangle_q+(\partial_k+\Gamma^i{}_{ik}{}^{(q)})\langle T^k_\psi\rangle_q-\Gamma^i{}_{k\psi}{}^{(q)}\langle T^k_i\rangle_q. \qquad (4.6.165)$$

Similarly using (4.6.144) and (4.6.128) and retaining only the $\mathcal{O}(q-1)$ terms, we get

$$\left(\partial_\rho+\frac{1}{\rho}+\Gamma^i{}_{i\rho}\right)\langle T_{\rho\psi}\rangle^{[1]}+\frac{1}{\rho^2}(\partial_\psi+\Gamma^i{}_{i\psi})\langle T_{\psi\psi}\rangle^{[1]}+(\partial_k+\Gamma^i{}_{ik})\langle T^k_\psi\rangle^{[1]}$$
$$-\Gamma^i{}_{k\psi}\langle T^k_i\rangle^{[1]}=\left(\frac{1}{2}\partial_\rho U^{[1]}+U^{[1]}\partial_\rho+\frac{U^{[1]}}{\rho}+\Gamma^i{}_{i\rho}U^{[1]}\right)\langle T_{\rho\psi}\rangle^{[0]}$$
$$-\Gamma^i{}_{i\rho}{}^{[1]}\langle T_{\rho\psi}\rangle^{[0]}-\frac{1}{\rho^2}\Gamma^i{}_{i\psi}{}^{[1]}\langle T_{\psi\psi}\rangle^{[0]}+\Gamma^i{}_{k\psi}{}^{[1]}\langle T^k_i\rangle^{[0]}. \qquad (4.6.166)$$

Using (4.6.122) implies,

$$\left(\partial_\rho + \frac{1}{\rho} + \Gamma^i{}_{i\rho}\right) t^{[1]}_{\rho\psi} + \frac{1}{\rho^2}(\partial_\psi + \Gamma^i{}_{i\psi}) t^{[1]}_{\psi\psi} + (\partial_k + \Gamma^i{}_{ik}) t^k{}_\psi{}^{[1]} - \Gamma^i{}_{k\psi} t^k{}_i{}^{[1]} =$$
$$- \left(\partial_\rho + \frac{1}{\rho} + \Gamma^i{}_{i\rho}\right) \chi^{[1]}_{\rho\psi} - \frac{1}{\rho^2}(\partial_\psi + \Gamma^i{}_{i\psi}) \chi^{[1]}_{\psi\psi} - (\partial_k + \Gamma^i{}_{ik}) \chi^k{}_\psi{}^{[1]} + \Gamma^i{}_{k\psi} \chi^k{}_i{}^{[1]}$$
$$+ \left(\frac{1}{2}\partial_\rho U^{[1]} + U^{[1]}\partial_\rho + \frac{U^{[1]}}{\rho} + \Gamma^i{}_{i\rho} U^{[1]}\right) \left(\langle T_{\rho\psi}\rangle^{[0]} = t^{[0]}_{\rho\psi} + \chi^{[0]}_{\rho\psi}\right)$$
$$-\Gamma^i{}_{i\rho}{}^{[1]} \langle T_{\rho\psi}\rangle^{[0]} - \frac{1}{\rho^2}\Gamma^i{}_{i\psi}{}^{[1]} \langle T_{\psi\psi}\rangle^{[0]} + \Gamma^i{}_{k\psi}{}^{[1]} \langle T^k_i\rangle^{[0]}. \tag{4.6.167}$$

3. For $\sigma = i$

$$\nabla_\mu{}^{(q)} \langle T^\mu{}_i\rangle_q = 0 = (\partial_\rho + \Gamma^\rho{}_{\rho\rho}{}^{(q)} + \Gamma^\psi{}_{\psi\rho}{}^{(q)} + \Gamma^k{}_{k\rho}{}^{(q)}) \langle T^\rho{}_i\rangle_q - \Gamma^k{}_{\rho i}{}^{(q)} \langle T^\rho{}_k\rangle_q$$
$$+ (\partial_\psi + \Gamma^k{}_{k\psi}{}^{(q)}) \langle T^\psi{}_i\rangle_q - \Gamma^k{}_{\psi i}{}^{(q)} \langle T^\psi{}_k\rangle_q + (\partial_k + \Gamma^l{}_{lk}{}^{(q)}) \langle T^k{}_i\rangle_q - \Gamma^k{}_{li}{}^{(q)} \langle T^l{}_k\rangle_q$$
$$- \Gamma^\rho{}_{ki}{}^{(q)} \langle T^k{}_\rho\rangle_q - \Gamma^\psi{}_{ki}{}^{(q)} \langle T^k{}_\psi\rangle_q. \tag{4.6.168}$$

Similarly using (4.6.144) and (4.6.128) and retaining only the $\mathcal{O}(q-1)$ terms, we get

$$\left(\partial_\rho + \frac{1}{\rho} + \Gamma^k{}_{k\rho}\right) \langle T_{\rho i}\rangle^{[1]} + \frac{1}{\rho^2}(\partial_\psi + \Gamma^k{}_{k\psi}) \langle T_{\psi i}\rangle^{[1]} + (\partial_k + \Gamma^l{}_{lk}) \langle T^k_i\rangle^{[1]}$$
$$- \Gamma^k{}_{\rho i} \langle T_{\rho k}\rangle^{[1]} - \Gamma^k{}_{\psi i} \frac{1}{\rho^2} \langle T_{\psi k}\rangle^{[1]} - \Gamma^k{}_{li} \langle T^l{}_k\rangle^{[1]} + f^{[0]}_{ki}(\rho,\psi,x^i) \langle T^k_\rho\rangle^{[1]}$$
$$- \Gamma^\psi{}_{ki} \langle T^k_\psi\rangle^{[1]} = \left(\frac{\partial_\rho U^{[1]}}{2} + \frac{U^{[1]}}{\rho} + \Gamma^k{}_{k\rho} U^{[1]} + U^{[1]}\partial_\rho\right) \langle T_{\rho i}\rangle^{[0]} \tag{4.6.169}$$
$$- \Gamma^k{}_{\rho i} U^{[1]} \langle T_{\rho k}\rangle^{[0]} + U^{[1]} f^{[0]}_{ki}(\rho,\psi,x^i) \langle T^k_\rho\rangle^{[0]} - \Gamma^k{}_{k\rho}{}^{[1]} \langle T_{\rho i}\rangle^{[0]} + \Gamma^k{}_{\rho i}{}^{[1]} \langle T_{\rho k}\rangle^{[0]}$$
$$- \frac{1}{\rho^2}\Gamma^k{}_{k\psi}{}^{[1]} \langle T_{\psi i}\rangle^{[0]} + \frac{1}{\rho^2}\Gamma^k{}_{\psi i}{}^{[1]} \langle T_{\psi k}\rangle^{[0]} - f^{[1]}_{ki}(\rho,\psi,x^i) \langle T^k_\rho\rangle^{[0]} + \Gamma^\psi{}_{ki}{}^{[1]} \langle T^k_\psi\rangle^{[0]},$$

where $f^{(q)}_{ki}(\rho,\psi,x^i) = (1/2)\partial_\rho g^{(q)}_{ki} = f^{[0]}_{ki}(\rho,\psi,x^i) + (q-1) f^{[1]}_{ki}(\rho,\psi,x^i) + O((q-1)^2)$. Notice that there are no $\langle T_{ij}\rangle^{[0]}$ sources in the $\sigma = i$ equation. This is because $\Gamma^i{}_{jk}{}^{(q)} = \Gamma^i{}_{jk}{}^{[0]}$. We also notice that in $d = 2,3,4$, $\chi^{[0]}_{\mu i} = 0$ for $\mu \in \{\rho,\psi\}$ and hence $\chi^{[1]}_{\mu i} = 0$ for $\mu \in \{\rho,\psi\}$ (see section (4.6.3.1)).

Tracelessness and conservation gives us $(d+1)$ equations to solve for $\langle T_{\mu\nu}\rangle^{[1]}$ (and hence $t_{\mu\nu}{}^{[1]}$) in terms of $\langle T_{\mu\nu}\rangle$ (and $t_{\mu\nu}{}^{[0]}$) for any state satisfying the symmetry requirements. Although, there are $d(d+1)/2$ independent components of $\langle T_{\mu\nu}\rangle^{[1]}$ (or $t_{\mu\nu}{}^{[1]}$) and only $(d+1)$ equations to solve for them, under the presence of certain symmetries we can reduce the number of independent components to match the number of constraint equations. This is true for a static metric on M_1. Static metrics on M_1 that are solutions to Einstein field equations with $\Lambda > 0$(dS), $\Lambda < 0$(AdS), $\Lambda = 0$(flat), have spherical or planar or hyperbolic symmetry along $y^{i\prime}s \in \Sigma$ hence $g_{ij} = 0$ for $i \neq j$. On a static metric we also have $g_{\rho i} = 0 = g_{\psi i}$[5]. We consider states that respect these symmetries as well i.e., states that satisfy

$$\langle T_{\rho i}\rangle = 0 = \langle T_{\psi i}\rangle. \tag{4.6.170}$$

These states are of interest in the context of a black hole problem since we expect the average behavior of the Hawking radiation to also respect this symmetry.

Therefore, all the source terms in the i-replica conservation equation i.e., (4.6.169) vanish, giving us only the homogeneous solution. The replica metric also has the symmetries discussed above i.e., it has spherical or planar or hyperbolic symmetry along $y^{i\prime}s \in \Sigma$ (there are no non trivial replica corrections along $y^{i\prime}s$ [6]) hence $g^{(q)}_{ij} = 0$ for $i \neq j$ and $g^{(q)}_{\rho i} = 0 = g^{(q)}_{\psi i}$. Therefore, for the states considered on M_1, we have

$$\langle T_{\rho i}\rangle^{[1]} = 0 = \langle T_{\psi i}\rangle^{[1]}, \tag{4.6.171}$$
$$\langle T_{ij}\rangle^{[1]} = \langle T_{ij}\rangle^{[0]} \quad \forall\, i \neq j. \tag{4.6.172}$$

With this the $\sigma = i$ conservation equation on M_q (4.6.168) gives,

$$(\partial_k + \Gamma^l{}_{lk})\langle T^k{}_i\rangle^{[1]} - \Gamma^k{}_{li}\langle T^l{}_k\rangle^{[1]} = 0. \tag{4.6.173}$$

(4.6.171) further implies $t^{[1]}_{\mu i} = -\chi^{[1]}_{\mu i}$ for $\mu \in \{\rho, \psi\}$. Since in $d = 2, 3, 4$, $\chi^{[0]}_{\mu i} = 0 = \chi^{[1]}_{\mu i}$ for $\mu \in \{\rho, \psi\}$, therefore in $d = 2, 3, 4$

$$t^{[1]}_{\rho i} = t^{[1]}_{\psi i} = 0, \tag{4.6.174}$$
$$t^{[1]}_{ij} = \langle T_{ij}\rangle^{[0]} - \chi^{[1]}_{ij}. \tag{4.6.175}$$

On static spacetime since $g_{ij} = 0$ for $i \neq j$ (due to symmetries in Σ) we have $\langle T_{ij}\rangle^{[0]} = 0$ for $i \neq j$ (since the only tensors in Σ are g_{ij} and tensors formed using it, therefore $\langle T_{ij}\rangle^{[0]}$ must be proportional to g_{ij} or tensors formed using it). We do not consider states that violate this. For such states (4.6.168) further reduces to,

$$\boxed{\text{For a fixed } i: \quad (\partial_i + \sum_l \Gamma^l{}_{li})\langle T^i{}_i\rangle^{[1]} - \sum_l \Gamma^l{}_{li}\langle T^l{}_l\rangle^{[1]} = 0.} \tag{4.6.176}$$

We also note $\Gamma^l{}_{li} = \frac{1}{2} g^{ll}\partial_i g_{ll}$ (no sum over l).

We consider certain specific cases below. For $g_{ij} = \delta_{ij}$ or $d \leq 3$ we have $\Gamma^i{}_{jk} = 0$.

[5] This is not true in non static spacetimes.

[6] In non static case if we take a finite entangling region along $y^{i\prime}s$, this won't be true.

In this case, we simply have,

$$\text{For a fixed } i\text{: } \partial_i \langle T^i{}_i \rangle^{[1]} = 0. \tag{4.6.177}$$

In $d = 4$ static black holes in AdS, dS and flat we can choose coordinates such that $g_{ij}dy^i dy^j = r^2(d\theta^2 + h_{\phi\phi}(\theta)d\phi^2)$, where

$$h_{\phi\phi}(\theta) = \sin^2(\theta); \quad k = 1 \text{ (spherical)}, \tag{4.6.178}$$
$$= \sinh^2(\theta) \text{ or } \cosh^2(\theta) \text{ or } e^{-2\theta}; \quad k = -1 \text{ (hyperbolic)}, \tag{4.6.179}$$
$$g_{ij}dy^i dy^j = \frac{r^2}{l^2}(d\theta^2 + d\phi^2); \quad k = 0 \text{ (planar)}, \tag{4.6.180}$$

here k is the parameter introduced in (4.5.68). We have

$$(\partial_\theta + \frac{1}{2h_{\phi\phi}}\partial_\theta h_{\phi\phi})\langle T^\theta{}_\theta \rangle^{[1]} - \frac{1}{2h_{\phi\phi}}\partial_\theta h_{\phi\phi}\langle T^\phi{}_\phi \rangle^{[1]} = 0, \tag{4.6.181}$$
$$\partial_\phi \langle T^\phi{}_\phi \rangle^{[1]} = 0. \tag{4.6.182}$$

Therefore in the static case the only non-trivial replica stress tensor expectation corrections are $\{\langle T_{\rho\rho}\rangle^{[1]}, \langle T_{\rho\psi}\rangle^{[1]}, \langle T_{\psi\psi}\rangle^{[1]}, \langle T\,_{ii}\rangle^{[1]}\}$ - a total of $(d+1)$ components determined by the $(d+1)$ linear equations - the trace anomaly condition i.e., (4.6.159) and the d conservation equations (4.6.163), (4.6.166), (4.6.176). We have,

$\mathcal{O}(q-1)$ **Trace condition:**

$$\boxed{\langle T_{\rho\rho}\rangle^{[1]} + \frac{1}{\rho^2}\langle T_{\psi\psi}\rangle^{[1]} + \langle T^i{}_i\rangle^{[1]} = U^{[1]}(\rho, a)\langle T_{\rho\rho}\rangle^{[0]} + \mathcal{A}^{[1]}[g^{(q)}_{\mu\nu}].} \tag{4.6.183}$$

$\mathcal{O}(q-1)$ **ρ-component of Conservation:**

$$\boxed{\begin{aligned} &\left(\partial_\rho + \frac{1}{\rho} + \Gamma^i{}_{i\rho}\right)\langle T_{\rho\rho}\rangle^{[1]} + \frac{1}{\rho^2}(\partial_\psi + \Gamma^i{}_{i\psi})\langle T_{\psi\rho}\rangle^{[1]} - \frac{1}{\rho^3}\langle T_{\psi\psi}\rangle^{[1]} \\ &-\sum_i \Gamma^i{}_{i\rho}\langle T^i{}_i\rangle^{[1]} = \left(\partial_\rho + \frac{1}{\rho} + \Gamma^i{}_{i\rho}\right)(U^{[1]}\langle T_{\rho\rho}\rangle^{[0]}) - \Gamma^i{}_{i\rho}{}^{[1]}\langle T_{\rho\rho}\rangle^{[0]} \\ &\qquad -\frac{1}{\rho^2}\Gamma^i{}_{i\psi}{}^{[1]}\langle T_{\psi\rho}\rangle^{[0]} + \sum_i \Gamma^i{}_{i\rho}{}^{[1]}\langle T^i{}_i\rangle^{[0]}, \end{aligned}} \tag{4.6.184}$$

or using trace equation to eliminate $\langle T_{\rho\rho}\rangle_q$,

$$\boxed{\begin{aligned} &\frac{1}{\rho^2}(\partial_\psi + \Gamma^i{}_{i\psi})\langle T_{\rho\psi}\rangle^{[1]} - \frac{1}{\rho^2}(\partial_\rho + \Gamma^i{}_{i\rho})\langle T_{\psi\psi}\rangle^{[1]} \\ &- \left(\partial_\rho + \frac{1}{\rho} + \Gamma^i{}_{i\rho}\right)\left(\langle T^k{}_k\rangle^{[1]} - \mathcal{A}^{[1]}[g^{(q)}_{\mu\nu}]\right) - \sum_i \Gamma^i{}_{i\rho}\langle T^i{}_i\rangle^{[1]} = \frac{1}{\rho^2}(\Gamma^i{}_{i\rho}{}^{[1]}\langle T_{\psi\psi}\rangle^{[0]} \\ &-\Gamma^i{}_{i\psi}{}^{[1]}\langle T_{\rho\psi}\rangle^{[0]}) + \Gamma^i{}_{i\rho}{}^{[1]}\left(\langle T^k{}_k\rangle^{[0]} - \mathcal{A}[g_{\mu\nu}]\right) + \sum_i \Gamma^i{}_{i\rho}{}^{[1]}\langle T^i{}_i\rangle^{[0]}. \end{aligned}} \tag{4.6.185}$$

$\mathcal{O}(q-1)$ ψ-component of Conservation:

$$\boxed{\begin{aligned} &\left(\partial_\rho + \frac{1}{\rho} + \Gamma^i{}_{i\rho}\right)\langle T_{\rho\psi}\rangle^{[1]} + \frac{1}{\rho^2}(\partial_\psi + \Gamma^i{}_{i\psi})\langle T_{\psi\psi}\rangle^{[1]} - \sum_i \Gamma^i{}_{i\psi}\langle T^i{}_i\rangle^{[1]} \\ &= \left(\frac{1}{2}\partial_\rho U^{[1]} + U^{[1]}\partial_\rho + \frac{U^{[1]}}{\rho} + \Gamma^i{}_{i\rho}U^{[1]}\right)\langle T_{\rho\psi}\rangle^{[0]} \\ &-\Gamma^i{}_{i\rho}{}^{[1]}\langle T_{\rho\psi}\rangle^{[0]} - \frac{1}{\rho^2}\Gamma^i{}_{i\psi}{}^{[1]}\langle T_{\psi\psi}\rangle^{[0]} + \sum_i \Gamma^i{}_{i\psi}{}^{[1]}\langle T^i{}_i\rangle^{[0]}. \end{aligned}} \tag{4.6.186}$$

We can now use (4.6.176) to solve for $\langle T_{ii}\rangle^{[1]}$, then (4.6.185) and (4.6.186) to solve for $\langle T_{\rho\psi}\rangle^{[1]}$ and $\langle T_{\psi\psi}\rangle^{[1]}$ in terms of $\langle T_{\rho\psi}\rangle^{[0]}$, $\langle T_{\psi\psi}\rangle^{[0]}$, $\langle T_{ii}\rangle^{[0]}$ and $U^{[1]}(\rho, a)$, $p(q)$. (4.6.183) can then be used to solve for $\langle T_{\rho\rho}\rangle^{[1]}$ in terms of $\langle T_{\mu\nu}\rangle^{[0]}$, $U^{[1]}(\rho, a)$, $p(q)$. The scaling of the Rényi entropies which depends on $\langle T_{\mu\nu}\rangle_q$, and EE (4.5.115) which depends only on $\langle T_{\mu\nu}\rangle^{[1]}$, can be expressed completely in terms of $\langle T_{\mu\nu}\rangle^{[0]}$ for a general state (i.e., stress tensor on M_1 only).

Since $\chi^{[1]}_{\mu\nu}$ is already known (4.6.3.1), we could also directly calculate $t^{[1]}_{\mu\nu}$ using the conservation, trace equations above and (4.6.122). Terms with $\chi^{[1]}_{\mu\nu}$ now act as additional source terms.

4.6.4 Sources - Stress Tensor on M_1

The sources in the trace and conservation equations (4.6.183)-(4.6.186) are $\langle T_{\rho\psi}\rangle^{[0]}$, $\langle T_{\psi\psi}\rangle^{[0]}$, $\langle T_{\rho\rho}\rangle^{[0]}$, $\langle T_{ii}\rangle^{[0]}$ (i.e., stress tensor on base manifold M_1) and $\mathcal{A}^{[1]}[g^{(q)}_{\mu\nu}]$.

Flat space, vacuum:

$$\langle T_{\mu\nu}\rangle^{[0]} = 0 = t^{[0]}_{\mu\nu}. \tag{4.6.187}$$

We can choose renormalisation scheme such that $\chi_{\mu\nu}[\eta_{\mu\nu}] = 0$.

Curved space, vacuum:

$$\langle T_{\mu\nu}\rangle^{[0]} = \chi^{[0]}_{\mu\nu}. \tag{4.6.188}$$

Thermal/equilibrium fluid stress tensor: The stress tensor at temperature T_{QFT} in a CFT is of particular interest. Some of the static backgrounds are also a black hole with inherent temperature T_{BH}, which will be equivalent to T_{QFT} in (quasi) thermal equilibrium.

The finite temperature stress tensor is expressed more elegantly as an equilibrium relativistic fluid tensor, with a conformal equation of state.

In general, the relativistic fluid tensor can be expressed as [217]

$$t^{[0]}_{\mu\nu} = \epsilon u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu), \tag{4.6.189}$$

where ϵ is the energy density, p the pressure, $u_\mu u^\mu = -1$. In a static gauge ($x^\mu = \{\tau, r, x^i\}$), one can choose $\hat{u} = \partial/\partial\tau$ or $u_\mu = \delta^\tau_\mu$. We have,

$$t^{[0]}_{\mu\nu} = (\epsilon + p)\delta^\tau_\mu \delta^\tau_\nu + p g_{\mu\nu}, \tag{4.6.190}$$

i.e.,

$$\begin{aligned} t^{[0]}_{rr} &= p g_{rr}, \\ t^{[0]}_{\tau\tau} &= \epsilon + p(1 + g_{\tau\tau}), \\ t^{[0]}_{r\tau} &= p g_{r\tau}, \\ t^{[0]}_{ij} &= p g_{ij}, \end{aligned} \tag{4.6.191}$$

or using (4.3.17)

$$\begin{aligned} t^{[0]}_{\rho\rho} &= p g_{rr}\cos(\psi)^2 + (\epsilon + p(1+g_{\tau\tau}))\sin(\psi)^2 + p g_{r\tau}\sin(2\psi), \\ t^{[0]}_{\psi\psi} &= p g_{rr}\rho^2\sin(\psi)^2 + (\epsilon + p(1+g_{\tau\tau}))\rho^2\cos(\psi)^2 - p g_{r\tau}\rho^2\sin(2\psi), \\ t^{[0]}_{\rho\psi} &= -p g_{rr}\rho\sin(\psi)\cos(\psi) + (\epsilon + p(1+g_{\tau\tau}))\rho\sin(\psi)\cos(\psi) + p g_{r\tau}\rho\cos(2\psi), \\ t^{[0]}_{ij} &= p g_{ij}. \end{aligned} \tag{4.6.192}$$

In $(-1, +1, +1, ..., +1)$ signature where $u_\mu u^\mu = -1$, tracelessness ${t^\mu}_\mu^{[0]} = 0$ implies,

$$\epsilon = (d-1)p. \tag{4.6.193}$$

This is the well-known equation of state for a conformal theory. Therefore in a CFT we can rewrite the state part of the thermal stress tensor as,

$$t^{[0]}_{\mu\nu} = p(g_{\mu\nu} + d u_\mu u_\nu) = p(g_{\mu\nu} + d\delta^\tau_\mu \delta^\tau_\nu). \tag{4.6.194}$$

In the Euclidean $(+1, +1, +1, ..., +1)$ signature we have $u_\mu u^\mu = 1$, and $\epsilon = -(d+1)p$, which implies,

$$t^{[0]\,E}_{\mu\nu} = p(g_{\mu\nu} - d u_\mu u_\nu) = p(g_{\mu\nu} - d\delta^\tau_\mu \delta^\tau_\nu), \tag{4.6.195}$$

where in the second equality we have chosen a static gauge ($x^\mu = \{\tau, r, x^i\}$), with $\hat{u} = \partial/\partial\tau$ or $u_\mu = \delta^\tau_\mu$.

In flat space we can choose a gauge in which $g_{\mu\nu} = \delta_{\mu\nu}$, with $p = CT^d$.

The conservation equation gives [217],

$$\nabla^\mu t^{[0]}_{\mu\nu} = 0 = \partial_\nu p + d D_u p\, u_\nu + dp\, a_\nu, \tag{4.6.196}$$

where

$$D_u p = u_\mu \nabla^\mu p, \tag{4.6.197}$$

$$a_\nu = u_\mu \nabla^\mu u_\nu. \tag{4.6.198}$$

This is often projected along ($u^\nu \nabla^\mu t^{[0]}_{\mu\nu}$) and transverse ($\Delta^{\alpha\nu} \nabla^\mu t^{[0]}_{\mu\nu}$ ($\Delta^{\alpha\nu} = g^{\alpha\nu} + u^\alpha u^\nu$)) to the fluid. For a general relativistic fluid, these projections give

$$-D_u \epsilon - (\epsilon + p)(\nabla^\nu u_\nu) = 0, \tag{4.6.199}$$

$$(\epsilon + p) a_\nu + \nabla^\perp_\nu p = 0; \quad \nabla^\perp_\nu = \Delta^{\mu\nu} \nabla_\nu. \tag{4.6.200}$$

In our context $\epsilon + p = dp$ and thus,

$$\nabla^\nu u_\nu = -\frac{d-1}{d} D_u(\ln p), \tag{4.6.201}$$

$$a_\nu = -\frac{1}{d} \nabla^\perp_\nu (\ln p). \tag{4.6.202}$$

The theory of relativistic hydrodynamics i.e., derivative expansion around equilibrium is well established; these conservation relations are corrected order by order.

We will now explicitly solve the conservation and trace equations on M_q in certain examples.

4.7 Replica stress tensor - vacuum states

In this section, we consider a CFT vacuum state on a $d-$dimensional metric. Let us first consider unsquashed cones in M_q. These correspond to static metrics on M_1 that have zero extrinsic curvature for boundary Σ embedded in M_1. Among solutions to Einstein field equations, this is trivially true for all metrics in 2 dimensions, 3 dimensional metrics of the form

$$g_{\tau\tau}(r)d\tau^2 + g_{rr}(r)dr^2 + \phi_0^2 dy^2, \tag{4.7.203}$$

with the entangling region τ constant, $r_1 \leq r \leq r_2$ spanning the range of y^i's, and flat backgrounds in cartesian coordinates in $d \geq 2$ dimensions i.e., metrics of form (4.5.71) with planar entangling region - τ constant, $x_1 \leq x \leq x_2$ spanning the range of z^i's i.e., M^{d-2}. On these backgrounds all christoffels with y^i or z^i component(s) vanish. The replica metric is given by the unsquashed conical metric (4.3.26). Along the Σ components the replica metric is the same as the metric on M_1. The Killing vectors on $g^{(q)}_{\mu\nu}$ are $\frac{1}{\rho}\partial_\psi$, ∂_i.

4.7.1 Connected entangling boundary with no extrinsic curvature

In this case the full replica trace and conservation equations given by (4.6.159), (4.6.161), (4.6.165), (4.6.168) reduce to:

Trace condition:

$$\langle T^\rho{}_\rho\rangle_q + \langle T^\psi{}_\psi\rangle_q + \langle T^i{}_i\rangle_q = \mathcal{A}[g^{(q)}_{\mu\nu}]. \tag{4.7.204}$$

ρ-component of Conservation:

$$\left(\partial_\rho + \frac{1}{\rho}\right)\langle T^\rho{}_\rho\rangle_q + \frac{1}{\rho^2}\partial_\psi\langle T_{\rho\psi}\rangle_q - \frac{1}{\rho}\langle T^\psi{}_\psi\rangle_q = 0, \tag{4.7.205}$$

or using trace equation to eliminate $\langle T^\rho_\rho\rangle_q$,

$$\frac{1}{\rho^2}\partial_\psi\langle T_{\rho\psi}\rangle_q - \left(\partial_\rho + \frac{2}{\rho}\right)\langle T^\psi{}_\psi\rangle_q - \left(\partial_\rho + \frac{1}{\rho}\right)\left(\langle T^k_k\rangle_q - \mathcal{A}[g^{(q)}_{\mu\nu}]\right) = 0. \tag{4.7.206}$$

ψ-component of Conservation:

$$\left(U^{-1}\partial_\rho - \frac{1}{2}U^{-2}\partial_\rho U + \frac{U^{-1}}{\rho}\right)\langle T_{\rho\psi}\rangle_q + \partial_\psi\langle T^\psi{}_\psi\rangle_q = 0. \tag{4.7.207}$$

i-component of Conservation:

$$\text{For a fixed } i\text{: } \partial_i \langle T^i{}_i\rangle_q = 0. \tag{4.7.208}$$

Let us consider the vacuum state on M_1, for which $\langle T_{\mu\nu}\rangle^{[0]} = 0$. The solutions must be independent of ψ and y^i since $\frac{1}{\rho}\partial_\psi$, ∂_i are Killing vectors on M_q in the unsquashed case and the vacuum state respects this symmetry [7]. Since $\mathcal{A}[g^{(q)}_{\mu\nu}]$ depends on only $g^{(q)}_{\mu\nu}$ it is independent of ψ in the unsquashed case. Considering that the vacuum state itself does not introduce any preferred tensor or dimensionful parameter, and $g^{(q)}_{\rho\psi} = 0$ and that the only invariant vectors on M_q are $\frac{1}{\rho}\partial_\psi$, ∂_i, we can conclude $\langle T^\rho{}_\psi\rangle_q = 0$. We also note that $\langle T^\mu{}_\nu\rangle_q$ has mass dimension d in $d-$dimensions.

Since all replica solutions are independent of ψ, (4.7.204)-(4.7.208) gives,

$$\begin{aligned}
\text{Trace: } & \langle T^\rho{}_\rho\rangle_q + \langle T^\psi{}_\psi\rangle_q + \langle T^i{}_i\rangle_q = \mathcal{A}[g^{(q)}_{\mu\nu}], \\
\rho \text{ conservation: } & \left(\partial_\rho + \frac{1}{\rho}\right)\langle T^\rho{}_\rho\rangle_q - \frac{1}{\rho}\langle T^\psi{}_\psi\rangle_q = 0, \\
\psi \text{ conservation: } & \left(\partial_\rho + \frac{1}{2}U^{-1}\partial_\rho U + \frac{1}{\rho}\right)\langle T^\rho{}_\psi\rangle_q = 0, \\
i \text{ conservation: } & \partial_i\langle T^i{}_i\rangle_q = 0;\ \text{for a fixed } i,
\end{aligned} \tag{4.7.209}$$

using trace equation to eliminate $\langle T^\rho{}_\rho\rangle_q$ in ρ conservation equation,

$$\begin{aligned}
&\rho \text{ conservation and trace:} \\
&\left(\partial_\rho + \frac{2}{\rho}\right)\langle T^\psi{}_\psi\rangle_q + \left(\partial_\rho + \frac{1}{\rho}\right)\left(\langle T^k{}_k\rangle_q - \mathcal{A}[g^{(q)}_{\mu\nu}]\right) = 0.
\end{aligned} \tag{4.7.210}$$

Solving the ψ conservation equation above, we get,

$$\langle T^\rho{}_\psi\rangle_q = \frac{a_1}{\rho\sqrt{U}}. \tag{4.7.211}$$

For this to be dimensionally correct, the constant a_1 must have mass dimension $(d-1)$. Since there are no dimensionful constants when we consider the vacuum on M_1, the only solution to the above case is $a_1 = 0$.

We see that in the absence of anomaly (d odd), the PDEs *i.e.*, (4.7.209) and hence their solutions are independent of the regulator (a). The only dimensionful parameter is ρ. Therefore, in the absence of anomaly the solution $\langle T^\mu{}_\nu\rangle_q \propto \frac{1}{\rho^d}$ in $d-$dimensions, with the dimensionless proportionality constant depending on d and q. This constant has to be of $\mathcal{O}(q-1)$ since $\langle T^\mu{}_\nu\rangle_{q=1} = 0$.

Alternatively, conformal invariance under $g^{(q)}_{\mu\nu} \to \omega^2 g^{(q)}_{\mu\nu}$ for constant ω implies that under this transformation $T^\mu{}_{\nu(q)} \to \omega^{-d} T^\mu{}_{\nu(q)}$. Under the absence of anomaly the vev

[7] In general we consider the theory on M_q and the states in the spectrum that respect global symmetries.

$\langle T^\mu{}_\nu \rangle_{q,ren}$ obeys the same transformation law. We also have

$$ds_q^2 = \omega^2 \left(d\left(\frac{\rho}{\omega}\right)^2 + \left(\frac{\rho}{\omega}\right) d\phi^2 + \delta_{ij} d\left(\frac{y^i}{\omega}\right) d\left(\frac{y^j}{\omega}\right)\right). \tag{4.7.212}$$

Therefore, $\langle T^\mu{}_\nu \rangle_{q,ren} = \omega^{-d} \langle T^\mu{}_\nu \left(\frac{\rho}{\omega}\right)\rangle_{q,ren}$ which implies $\langle T^\mu{}_\nu \rangle^{[1]} \propto \frac{1}{\rho^d}$.

In the presence of anomaly there is now an additional dimensionful parameter a (conical singularity regulator) and hence $\langle T^\mu{}_\nu \rangle_q$ now has an additional term which depends on dimensionful parameters ρ and a and dimensionless constants d and q. This is not necessarily only the geometric part of the replica stress tensor ($\chi^{\mu(q)}_\nu$) since the state part can also have a dependence due to reasons specified around (4.6.127) (i.e., non-uniqueness in state and geometric decomposition).

Equation (4.7.208) implies

$$\langle T^i{}_j \rangle_q = g_1(\rho, a)\delta^i{}_j = \left(g(\rho, a) + \frac{C_d}{\rho^d}\right) \delta^i{}_j. \tag{4.7.213}$$

We have assumed that the functional form is same in every direction in Σ. This is true since the replica q and a dependence is same along all directions in Σ in $g^{(q)}_{\mu\nu}$ (it is trivial for the unsquashed case) and the vacuum state on M_1 has no preference in directions in Σ. We also have,

$$\langle T^\psi{}_\psi \rangle_q = g_2(\rho, a). \tag{4.7.214}$$

We define $f(\rho, a) = \mathcal{A}[g^{(q)}_{\mu\nu}] - (d-2)g(\rho, a)$. The ρ conservation and trace equation (4.7.210) implies,

$$\begin{aligned} \frac{1}{\rho^2}\partial_\rho(\rho^2 \langle T^\psi{}_\psi \rangle_q) &= \left(\partial_\rho + \frac{1}{\rho}\right)\left(\mathcal{A}[g^{(q)}_{\mu\nu}] - \langle T^k{}_k \rangle_q\right) \\ &= \left(\partial_\rho + \frac{1}{\rho}\right)\left(f(\rho, a) - (d-2)\frac{C_d}{\rho^d}\right), \end{aligned} \tag{4.7.215}$$

which implies

$$\langle T^\psi{}_\psi \rangle_q = (1-d)\frac{C_d(1-\delta_{d,2})}{\rho^d} + f - \frac{1}{\rho^2}\int d\rho\rho f + \frac{c_1}{\rho^2}. \tag{4.7.216}$$

The trace equation along with (4.7.213), (4.7.216) implies,

$$\langle T^\rho{}_\rho \rangle_q = \frac{C_d(1-\delta_{d,2})}{\rho^d} + \frac{1}{\rho^2}\int d\rho\rho f - \frac{c_1}{\rho^2}. \tag{4.7.217}$$

In $d \neq 2$ we have the constant $c_1 = 0$ (using dimensional analysis for renormalised stress tensor). Therefore in $d \geq 2$ we have,

$$\begin{aligned} \langle T^{\psi}{}_{\psi}\rangle_q &= (1-d)\frac{C_d}{\rho^d} + f - \frac{1}{\rho^2}\int d\rho \rho f, \\ \langle T^{\rho}{}_{\rho}\rangle_q &= \frac{C_d}{\rho^d} + \frac{1}{\rho^2}\int d\rho\rho f, \end{aligned} \tag{4.7.218}$$

where we have identified C_2 with $-c_1$ in $d = 2$.

In the solutions above there is an arbitrary function $g(\rho, a)$ which encodes the regulator a dependence in $\langle T_{ii}\rangle_q$. The non trivial terms involving the coefficient $C_d(q)$ in the replica corrections to stress tensor vev show up because the presence of a conical singularity in M_q at the boundary breaks any translation symmetry along (r, τ) that is present on M_1.

In the above case we have considered a CFT (i.e., it satisfies trace condition (4.7.204)). If we however consider a QFT, the stress tensor on the replica and base manifold satisfy only the conservation equations i.e., in this case we have (4.7.205), (4.7.207), (4.7.208) which reduce to (4.7.209). If the QFT has no other dimensionful parameters, in this case the non trivial solutions are given by:

$$\begin{aligned} \langle T^{\psi}{}_{\psi}\rangle_q &= \frac{Q_d(q)}{\rho^d} + F(\rho, a, q), \\ \langle T^{\rho}{}_{\rho}\rangle_q &= \frac{Q_d(q)}{(1-d)\rho^d} + \frac{1}{\rho}\int d\rho F(\rho, a, q), \\ \langle T^{i}{}_{j}\rangle_q &= \left(\frac{R_d(q)}{\rho^d} + G(\rho, a, q)\right)\delta^{i}{}_{j}. \end{aligned} \tag{4.7.219}$$

In the presence of dimensionful parameters such as mass the dependence on ρ in the a independent term changes, and hence the coefficients depending on q and d change respecting the conservation equations (4.7.205), (4.7.207), (4.7.208).

For example, consider the case of a free QFT with mass m (with an m^2 dependent term in Lagrangian density). In such cases the m dependence that shows up in the vev of the stress tensor is an overall m^2 factor. Using the conservation equations i.e., in this case (4.7.205), (4.7.207), (4.7.208) which reduce to (4.7.209), we have,

$$\begin{aligned} \langle T^{\psi}{}_{\psi}\rangle_q &= \frac{m^2 M_d(q)}{\rho^{d-2}} + F_1(\rho, a, q), \\ \langle T^{\rho}{}_{\rho}\rangle_q &= \frac{m^2 M_d(q)}{(3-d)\rho^{d-2}} + \frac{1}{\rho}\int d\rho F_1(\rho, a, q), \\ \langle T^{i}{}_{j}\rangle_q &= \left(\frac{m^2 S_d(q)}{\rho^{d-2}} + G_1(\rho, a, q)\right)\delta^{i}{}_{j}. \end{aligned} \tag{4.7.220}$$

For the traceless part of $\langle T^\mu{}_\nu\rangle_q$ we have,

$$S_d(q) = \frac{(d-4)}{(3-d)(d-2)} M_d(q). \tag{4.7.221}$$

The terms with the regulator a dependence (purely replica metric-dependent terms) are localised around the endpoints of the entangling region. The terms without a dependence (state-dependent terms) describe the state and are non-local i.e., it has contribution over the entire entangling region. We treat both terms appropriately as described in section (4.6.2).

In general the regulator a independent/ state-dependent part of $\langle T^\mu{}_\nu\rangle_q$ in $d-$dimensions is of form,

$$\begin{aligned}\langle T^\mu{}_\nu\rangle_{q,\,a=0} &= \frac{1}{\rho^d}\Bigg[C_d(q)\,\mathrm{diag}((1-d),1,1,1,...,1)\\
&+\mathrm{diag}\left(Q_d(q),\frac{Q_d(q)}{1-d},R_d(q),R_d(q),...,R_d(q)\right)\\
&+(m\rho)^2 M_d(q)\,\mathrm{diag}\left(1,\frac{1}{3-d},\frac{d-4}{(3-d)(d-2)},...,\frac{d-4}{(3-d)(d-2)}\right)\\
&+(m\rho)^2\,\mathrm{diag}\left(N_d(q),\frac{N_d(q)}{3-d},S_d(q),S_d(q),...,S_d(q)\right)\Bigg],\end{aligned} \tag{4.7.222}$$

where the basis is in order (ψ,ρ,y^i). We will denote the coefficients in the $\mathcal{O}(q-1)$ contribution $\langle T^\mu{}_\nu\rangle^{[1]}_{a=0}$ using $A^{[1]}$ where A is the corresponding q-dependent coefficient in $\langle T^\mu{}_\nu\rangle_{q,a=0}$.

We can also directly solve the replica trace and conservation equations (4.7.204)-(4.7.208) at $\mathcal{O}(q-1)$ to get $\langle T^\mu{}_\nu\rangle^{[1]}_{a=0}$ directly in terms of $\langle T^\mu{}_\nu\rangle^{[0]}_{a=0}$. It is worth noting these as well,

$\mathcal{O}(q-1)$ **Trace condition:**

$$\langle T^\rho_{\ \rho}\rangle^{[1]} + \langle T^\psi_{\ \psi}\rangle^{[1]} + \langle T^i_{\ i}\rangle^{[1]} = \mathcal{A}^{[1]}[g^{(q)}_{\mu\nu}]. \tag{4.7.223}$$

$\mathcal{O}(q-1)$ **ρ-component of Conservation:**

$$\left(\partial_\rho + \frac{1}{\rho}\right)\langle T^\rho_{\ \rho}\rangle^{[1]} + \frac{1}{\rho^2}\partial_\psi\langle T_{\psi\rho}\rangle^{[1]} - \frac{1}{\rho}\langle T^\psi_{\ \psi}\rangle^{[1]} = 0, \tag{4.7.224}$$

or using trace equation to eliminate $\langle T_{\rho\rho}\rangle_q$,

$$\frac{1}{\rho^2}\partial_\psi\langle T_{\rho\psi}\rangle^{[1]} - \frac{1}{\rho^2}\partial_\rho\left(\rho^2\langle T^\psi_{\ \psi}\rangle^{[1]}\right) - \left(\partial_\rho+\frac{1}{\rho}\right)\left(\langle T^k_{\ k}\rangle^{[1]} - \mathcal{A}^{[1]}[g^{(q)}_{\mu\nu}]\right) = 0. \tag{4.7.225}$$

$\mathcal{O}(q-1)$ ψ**-component of Conservation:**

$$\left(\partial_\rho + \frac{1}{\rho}\right)\langle T_{\rho\psi}\rangle^{[1]} + \partial_\psi \langle T^{\psi}{}_{\psi}\rangle^{[1]} = \left(\frac{1}{2}\partial_\rho U^{[1]} + U^{[1]}\partial_\rho + \frac{U^{[1]}}{\rho}\right)\langle T_{\rho\psi}\rangle^{[0]}. \tag{4.7.226}$$

$\mathcal{O}(q-1)$ i**-component of Conservation:**

$$\text{For a fixed } i\text{: } \partial_i \langle T^{i}{}_{i}\rangle^{[1]} = 0. \tag{4.7.227}$$

Written in this form we notice that the only non zero stress tensor sources on M_1 is $\langle T_{\rho\psi}\rangle^{[0]}$. This is because in the full replica trace and conservation equations for unsquashed conical case, i.e., (4.7.204)-(4.7.208), the replica q-dependence via $U(\rho, a)$ is only associated with coefficient of $\langle T_{\rho\psi}\rangle_q$ and $\mathcal{A}[g^{[q]}_{\mu\nu}]$. We also notice that

$$\begin{aligned}
\langle T^{\rho}{}_{\mu}\rangle^{[1]} &= -U^{[1]}\langle T_{\rho\mu}\rangle^{[0]} + \langle T_{\rho\mu}\rangle^{[1]}; \quad \mu \in \{\rho, \psi\},\\
\langle T^{\psi}{}_{\mu}\rangle^{[1]} &= \frac{1}{\rho^2}\langle T_{\psi\mu}\rangle^{[1]},\\
\langle T^{i}{}_{j}\rangle^{[1]} &= g^{ik}\langle T_{kj}\rangle^{[1]},
\end{aligned} \tag{4.7.228}$$

therefore when both indices are lowered we also have an additional stress tensor source on $M_1 : \langle T_{\rho\rho}\rangle^{[0]}$.

Cosmic string literature: The study of the vacuum polarisation is also important in condensed matter systems with conical defects. For example, in graphitic cones one has $q = 5/6, 2/3, 1/2, 1/3, 1/6$. All these types of cones have been experimentally observed [218].

The exact form of the proportionality constant (depending on q) depends on the specific theory. A similar conical singular structure as on an unsquashed M_q arises in the presence of a cosmic string. The non zero vev on M_q (including that of the energy momentum tensor) induced by this conical singularity has been well studied in the cosmic string literature.

In $d = 4$ the vev of the renormalised energy momentum tensor on M_q for the massless scalar case has been considered in [208, 209], the massive scalar case in [211], the massless and massive fermionic case in [212]

$$
\begin{aligned}
&\langle T^\mu_\nu\rangle^{scalar}_q \\
&= \frac{1}{1440\pi^2\rho^4}\Bigg[\left(\frac{1}{q^4}-1\right)\mathrm{diag}(-3,1,1,1)+20(6\xi-1)\left(\frac{1}{q^2}-1\right)\mathrm{diag}\left(\frac{3}{2},\frac{-1}{2},1,1\right) \\
&\qquad + 15(m\rho)^2\left(\frac{1}{q}-1\right)\left(12\xi-1-\frac{1}{q}\right)\mathrm{diag}(-1,1,0,0) \\
&\qquad - 15(m\rho)^2\left(\frac{1}{q^2}-1\right)\mathrm{diag}(0,0,1,1)\Bigg],
\end{aligned}
\tag{4.7.229}
$$

where the basis is in order (ψ,ρ,y^1,y^2) and ξ is a parameter which fixes the coupling of the scalar field to gravity by means of the curvature R. We can map this result to solutions we have obtained via conservation and trace equations. In the above expression $\mathrm{diag}(-3,1,1,1)$ term corresponds to the massless CFT solutions (4.7.213), (4.7.216), (4.7.217). Note that for $\xi = 1/6$ with $m = 0$ we recover the CFT result; $\mathrm{diag}\left(\frac{3}{2},\frac{-1}{2},1,1\right)$ term corresponds to the massless scalar QFT solution (4.7.219); $\mathrm{diag}(0,0,1,1)$ term to the massive scalar solution (4.7.220); and $\mathrm{diag}(-1,1,0,0)$ term to the massive traceless scalar solution (4.7.221). Comparing (4.7.222) with the expression above we can read off the exact factors and q dependence in $C_d(q), Q_d(q), M_d(q), N_d(q), R_d(q), S_d(q)$.

A more general expression for a CFT in $d = 4$ is given by

$$
\langle T^\mu_\nu\rangle_q = \frac{f(q)}{1440\pi^2\rho^4}\mathrm{diag}(-3,1,1,1),
\tag{4.7.230}
$$

where

$$
f(q) = \frac{1}{16q^4}\left(1-q^2\right)\left[2\left(1+\frac{q^2}{3}\right)a+\left(1-q^2\right)(2b-c)\right],
\tag{4.7.231}
$$

with

$$
\begin{aligned}
a &= 12\left(n_s+6n_f+12n_v\right), \\
b &= -4\left(n_s+\frac{11}{2}n_f+62n_v\right), \\
c &= -240n_v.
\end{aligned}
\tag{4.7.232}
$$

for a free field multiple containing n_s conformal scalars, n_f Dirac fermions and n_v gauge vector fields.

Comparing (4.7.222) with this we can read off

$$
C_d(q) = \frac{f(q)}{1440\pi^2}.
\tag{4.7.233}
$$

Expressions in $d-$dimensions for casimir effect on M_q for free massless scalar field is considered in [210]. In $d = 4$ the vev of the renormalised energy momentum tensor on M_q for the electromagnetic field (photon) is considered in [213, 214] and linearised gravitational field (graviton) in [214]

$$\langle T^{\mu}_{\ \nu}\rangle^{\,photon}_{q} = \frac{\left(\frac{1}{q^2}-1\right)\left(\frac{1}{q^2}+11\right)}{720\pi^2\rho^4}\,\mathrm{diag}(-3,1,1,1), \tag{4.7.234}$$

$$\langle T^{\mu}_{\ \nu}\rangle^{\,graviton}_{q} = \frac{\left(\frac{1}{q}-1\right)}{720\pi^2\rho^4}\,\mathrm{diag}[-3g(q),g(q),f(q),f(q)], \tag{4.7.235}$$

where

$$\begin{aligned} f(q) &= \frac{121}{q^3}+\frac{121}{q^2}+\frac{1421}{q}-1459, \\ g(q) &= \frac{121}{q^3}+\frac{121}{q^2}-\frac{829}{q}+251. \end{aligned} \tag{4.7.236}$$

The photon case corresponds to the massless CFT solutions (4.7.213), (4.7.216), (4.7.217), and the graviton case corresponds to the massless QFT solution (4.7.219).

Conical N point correlations in terms of flat space $(N+1)$ point correlations: When $g_{\mu\nu}$ is flat and the entangling surface is planar, [105, 207] gives the $N-$point vacuum correlation function on M_q in terms of specific $(N+1)$ point vacuum correlation functions on M_1 involving an additional Rindler (modular) Hamiltonian insertion (see (4.4.65)). For a CFT the $\mathcal{O}(q-1)$ result in [105], [207] matches with the results we got in (4.7.222) expanded upto $\mathcal{O}(q-1)$ with $(Q_d^{[1]} = M_d^{[1]} = N_d^{[1]} = R_d^{[1]} = S_d^{[1]} = 0)$. [105, 207] make use of the CFT stress tensor 2-point function $\langle T_{\mu\nu}T_{22}\rangle$ fixed by conformal symmetry [219, 220]. The non zero coefficient $C_d^{[1]}$ for $d = 4$ can be read off from comparing with stress tensor one point function directly calculated on conical backgrounds given by (4.7.230),

$$C^{[1]}_{d=4} = -\pi^2\frac{C_T}{120}, \tag{4.7.237}$$

where C_T is the charge related to the $B-$type conformal anomaly. For a free field multiple containing n_s conformal scalars, n_f Dirac fermions and n_v gauge vector fields we have

$$C_T = \frac{1}{3\pi^4}(n_s+6n_f+12n_v). \tag{4.7.238}$$

The modular Hamiltonian is known only for certain symmetric geometries, such as spherical [221] and planar regions in flat space, and is not known for generic entangling boundaries and backgrounds which limits its use in determining correlation functions on

M_q. However, the replica conservation and trace equations on static backgrounds that we have established, determines $\langle T_{\mu\nu}\rangle_q$ for any state and entangling surface on static backgrounds. We have $(d+1)$ equations for $(d+1)$ unknowns - $\langle T_{\rho\rho}\rangle_q, \langle T_{\psi\psi}\rangle_q, \langle T_{\rho\psi}\rangle_q, \langle T_{ii}\rangle_q$.

4.7.2 Disconnected entangling boundaries with no extrinsic curvature

Multiple Conical Singularities literature: The results so far in this section hold for spaces with one conical singularity. But when we have an entangling region with a disconnected boundary such as the one we have (i.e., on static metric (4.5.67), $r = r_1$ and $r = r_2$ spanning y^i on a τ constant slice are two disconnected boundaries), M_q has two conical singularities, one at each boundary. If we have an entangling region with multiple disconnected boundaries then we have the manifold M_q with multiple conical singularities, one at each boundary. We can always have a local replica metric and replica stress tensor around each conical singularity but we would like to have a global replica metric and replica stress tensor, since the terms that do not depend on the conical regulator (a) can contribute in the entire entangling region and hence non locally. Finding a single chart that captures the global replica manifold and replica metric with multiple conical singularities might be difficult or not possible for a general curved background in M_1, but it is possible when the metric on M_1 is flat, which is the case we will be focussing on.

Flat space with multiple conical singularities and correlators on them have been discussed in literature on spaces with multiple cosmic strings where there is a conical singularity at the location of each cosmic string [222, 223].

Consider a planar entangling region $x = x_1$ spanning z^i on a $\tau = \tau_0$ slice on flat metric (4.3.15) on M_1. The replica manifold M_q has one conical singularity at $x = x_1$, $\tau = \tau_0$ and the corresponding replica metric around the conical singularity is given by,

$$ds^2_{q\text{around bdy}} = d\rho^2 + \rho^2 d\psi^2 + \delta_{ij}dz^i dz^j; \quad \rho \in [0, \infty), \psi \in [0, 2\pi q), \tag{4.7.239}$$

where $x - x_1 = \rho\cos(\psi)$ and $\tau - \tau_0 = \rho\sin(\psi)$. With a change of coordinates $\rho = qr^q$ and $\psi = q\theta$ we can write the metric in the form

$$ds^2_{q\text{around bdy}} = e^{-4V(r)}(dr^2 + r^2 d\theta^2) + \delta_{ij}dz^i dz^j; \quad r \in [0, \infty), \psi \in [0, 2\pi), \tag{4.7.240}$$

where $V(r) = 2\lambda\log(r)$ and $\lambda = (q-1)/4$. The advantage of doing this is that the q dependence is now only in λ and not in the range of any of the coordinates.

The generalization to replica metric with multiple (say N) conical singularities (one at each $\{x_i, \tau_i\}$, $i \in \{1, 2, ..., N\}$), corresponding to M_1 with flat metric (4.3.15) is given by,

$$\begin{aligned}
&ds^2_{q\text{around bdy}} = e^{-4\sum_{i=1}^{N} V_i(r_i)}(dr^2 + r^2 d\theta^2) + \delta_{ij}dz^i dz^j; \\
&r \in [0, \infty), \psi \in [0, 2\pi),
\end{aligned} \tag{4.7.241}$$

where $V_i = 2\lambda_i \log(r_i)$, $r_i = \sqrt{(x - x_i)^2 + (\tau - \tau_i)^2}$ is the distance from each conical sigularity, $\lambda_i = (q_i - 1)/4$. In our case $q_i = q$ and $\tau_i = \tau_0 \ \forall\, i$.

For a CFT on M_1 we can now use the replica conservation and trace equations $\nabla^{(q)}_\mu \langle T^\mu{}_\nu\rangle_q = 0$, $T^\mu{}_\mu = \mathcal{A}^{(q)}[g^{(q)}_{\mu\nu}]$. The homogeneous solution in $\{r, \tau, y^i\}$ basis is given by

$$\begin{aligned}
\langle T^\mu{}_\nu\rangle_q &= C\sum_{i=1}^{N}\frac{\lambda_i(q)}{r_i^4}\begin{pmatrix} 1 - \frac{d(\tau-\tau_i)^2}{r_i^2} & \frac{d(x-x_i)(\tau-\tau_i)}{r_i^2} \\ \frac{d(x-x_i)(\tau-\tau_i)}{r_i^2} & (1-d) + \frac{d(\tau-\tau_i)^2}{r_i^2}\end{pmatrix}; \quad \{\mu,\nu\} \in \{r,\tau\}, \\
&= C\sum_{i=1}^{N}\frac{\lambda_i(q)}{r_i^4}\delta^\mu{}_\nu; \quad \{\mu,\nu\} \in \{i,j\},
\end{aligned} \tag{4.7.242}$$

where C is a constant. In the case of N conical singularities the net contribution is the sum of contribution from each conical singularity.

Replica stress tensor vev: We will now consider the replica stress tensor vev in the case of two disconnected planar entangling boundaries on flat space. In the case of an entangling region with disconnected boundary $x = x_1$ and $x = x_2$ spanning $z^{i\prime}s$ on a constant time slice, we have two conical singularities at the two disconnected boundaries. We also have an additional dimensionful parameter $L = x_2 - x_1$ on which the $\langle T^\mu{}_\nu\rangle_q$ can depend on. Since the boundaries Σ has no extrinsic curvature, we still have ∂_ψ and ∂_i as Killing vectors on M_1 and M_q. We will now solve (4.7.209) taking these into account. The $i-$conservation equation implies

$$\langle T^i{}_i\rangle_q = \tilde{g}_1(\rho, a, L) = (d-2)\left(\frac{C_d(q)}{\rho^d}K(\rho, L) + \tilde{g}(\rho, a, L)\right), \tag{4.7.243}$$

where $C_d(q)$ is a $\mathcal{O}(q-1)$ function of q, and $K(\rho, L)$ is a dimensionless function capturing the L dependence in the regulator independent part. Using (4.7.210),we get

$$\begin{aligned}
&\langle T^\psi{}_\psi\rangle_q = \frac{1}{\rho^2}\int d\rho\rho\partial_\rho(\rho(\mathcal{A}^{(q)} - \langle T^i{}_i\rangle_q)) \\
=&\frac{-1}{\rho^2}\left(\frac{(d-2)C_d(q)}{\rho^{d-2}}K(\rho, L) - \int d\rho(d-2)\frac{C_d(q)}{\rho^{d-1}}K(\rho, L)\right) \\
&+ \frac{1}{\rho^2}\int d\rho\rho\partial_\rho(\rho\tilde{f}(\rho, a)) \\
=&\frac{(1-d)C_d(q)}{\rho^d}K(\rho, L) + \frac{1}{\rho^2}\int d\rho\partial_\rho K(\rho, L)\frac{C_d(q)}{\rho^{d-2}} + \tilde{f} - \frac{1}{\rho^2}\int d\rho\rho\tilde{f},
\end{aligned} \tag{4.7.244}$$

where $\tilde{f}(\rho, a) = \mathcal{A}^{(q)} - (d-2)\tilde{g}(\rho, a)$. The trace equation implies,

$$\begin{aligned}\langle T^{\rho}{}_{\rho}\rangle_q &= \mathcal{A}^{(q)} - \langle T^{\psi}{}_{\psi}\rangle_q - \langle T^{i}{}_{i}\rangle_q \\ &= \frac{C_d(q)}{\rho^d} K(\rho, L) - \frac{1}{\rho^2}\int d\rho \partial_\rho K(\rho, L)\frac{C_d(q)}{\rho^{d-2}} + \frac{1}{\rho^2}\int d\rho\rho\tilde{f}. \end{aligned} \tag{4.7.245}$$

The function $K(\rho, L)$ must satisfy the following properties,

1. It is dimensionless since $\langle T^{\mu}{}_{\nu}\rangle_q$ has mass dimension d which implies $K(\rho, L)$ can only depend on the ratio ρ/L.
2. For $\rho >> L$, $K(\rho/L) \to 1$, since far away from the entangling region it must be indistinguishable from a single conical singularity.

Both these conditions imply $K(\rho, L)$ must be of the form,

$$K(\rho, L) = \sum_{i\geq 0} A_i \left(\frac{L}{\rho}\right)^i; \quad i \in \mathbb{Z}. \tag{4.7.246}$$

We have,

$$\begin{aligned}\langle T^{r}{}_{r}\rangle_q|_{a=0} &= \langle T^{\rho}{}_{\rho}\rangle_q \cos^2(\psi) + \langle T^{\psi}{}_{\psi}\rangle_q \sin^2(\psi) \\ &= \frac{C_d(q)}{\rho^d} K(\rho, L)[1 - d\sin^2(\psi)] - \frac{\cos(2\psi)}{\rho^2}\int d\rho\partial_\rho K(\rho, L)\frac{C_d(q)}{\rho^{d-2}}. \end{aligned} \tag{4.7.247}$$

4.8 Entanglement entropy - vacuum states

We will now calculate the scaling of the Rényi and EE for a vacuum state on a d dimensional background, and an entangling boundary with no extrinsic curvature. We make use of the results for the replica stress tensor vev from the previous section. Using (4.5.115), the scaling of S_{QFT} in this case is given by,

$$-\alpha\Delta r\frac{\partial S_{d\text{ dim}}}{\partial r_1} = \lim_{q\to 1}\frac{1}{2}\int_{M_q} d^d x\sqrt{g}\left(\delta_{\Delta r} g_{rr}\langle T^{rr}\rangle^{[1]} + \delta_{\Delta r} g_{\tau\tau}\langle T^{\tau\tau}\rangle^{[1]}\right), \tag{4.8.248}$$

where we have kept r_2 fixed and scaled r_1. We can instead scale r_2 and keep r_1 fixed simply by interchanging r_1 and r_2 in the expression above. We recover the scaling of the EE in the limit $q \to 1$. Notice also that $\delta_{\Delta r} g_{r\tau} = \delta_{\Delta r} g_{ij} = 0$ in static backgrounds with zero extrinsic curvature for Σ.

On a flat background metric, we have translation symmetry along r or x, therefore S_{QFT} only depends on Δr or Δx. Therefore, it is enough to consider the scaling of S_{QFT} with respect to scaling the entire interval $\Delta r \to e^{\alpha}\Delta r$ or $\Delta x \to e^{\alpha}\Delta x$, as opposed to keeping one of the boundaries fixed and scaling the other. On scaling the metric, we have

$$\delta_{\Delta r} g_{\mu\nu} = 2\alpha g_{rr}; \quad \mu = \nu = r, \tag{4.8.249}$$

$$= 0; \quad \text{otherwise.} \tag{4.8.250}$$

In flat space vacuum CFT state we have $\langle T_{\mu\nu}\rangle^{[0]} = 0$, therefore

$$\langle T^{\mu}{}_{\nu}\rangle^{[1]} = g^{\mu\alpha[1]}\langle T_{\alpha\nu}\rangle^{[0]} + g^{\mu\alpha[0]}\langle T_{\alpha\nu}\rangle^{[1]} = g^{\mu\alpha[0]}\langle T_{\alpha\nu}\rangle^{[1]}, \tag{4.8.251}$$

i.e., for the vacuum state we can raise and lower indices in $\langle T^{\mu}{}_{\nu}\rangle^{[1]}$ simply by using the metric on the base manifold M_1, therefore,

$$\Delta r \frac{\partial S_{d\ \mathrm{dim}}}{\partial \Delta r} = \lim_{q\to 1} \int_{M_q} d^d x \sqrt{g} \langle T^r_r\rangle^{[1]}. \tag{4.8.252}$$

Notice that the scaling is now with respect to Δr. This is because in flat space $\delta_{\Delta r} g^{\mu\nu}$ is independent of whether we scale with respect to r_1 or r_2 or $r_2 - r_1 = \Delta r$. This reflects the translational symmetry along r in flat space.

As seen in the previous section, the replica stress tensor has regulator (a) dependent terms and regulator-independent terms, which needs to be treated appropriately when performing replica integrals (4.6.2). We will separately calculate the contributions from both kind of terms. The complete result for scaling of Rényi and EE is the sum of both contributions.

4.8.1 Regulator independent contribution

Let us now consider only the regulator (a) independent contribution to the replica stress tensor vev. These are non local and contribute over the entire entangling region.

Approach 1 - As the sum of individual conical stress tensor: We first consider the regulator (a) independent contribution to the replica stress tensor vev given by (4.7.222). In $\{r, \tau, y^i\}$ basis we have

$$
\begin{aligned}
\langle T^r_{\ r}\rangle^{[1]}_{a=0} =& \langle T^\rho_{\ \rho}\rangle^{[1]}_{a=0}\cos^2(\psi) + \langle T^\psi_{\ \psi}\rangle^{[1]}_{a=0}\sin^2(\psi)\\
=& \frac{1}{\rho^d}\left[C^{[1]}_d + \frac{Q^{[1]}_d}{1-d}\right][1-d\sin^2(\psi)] + \frac{m^2}{\rho^{(d-2)}}(M^{[1]}_d + N^{[1]}_d)\left[\frac{1+(2-d)\sin^2(\psi)}{(3-d)}\right],\\
\langle T^\tau_{\ \tau}\rangle^{[1]}_{a=0} =& \langle T^\rho_{\ \rho}\rangle^{[1]}_{a=0}\sin^2(\psi) + \langle T^\psi_{\ \psi}\rangle^{[1]}_{a=0}\cos^2(\psi)\\
=& \frac{1}{\rho^d}\left[C^{[1]}_d + \frac{Q^{[1]}_d}{1-d}\right][(1-d)+d\sin^2(\psi)] + \frac{m^2}{\rho^{(d-2)}}(M^{[1]}_d + N^{[1]}_d)\left[1+\frac{(d-2)}{(3-d)}\sin^2(\psi)\right],\\
\langle T^r_{\ \tau}\rangle^{[1]}_{a=0} =& \langle T^\tau_{\ r}\rangle^{[1]}_{a=0} = \left(\langle T^\rho_{\ \rho}\rangle^{[1]}_{a=0} - \langle T^\psi_{\ \psi}\rangle^{[1]}_{a=0}\right)\cos(\psi)\sin(\psi)\\
=& \left(\frac{d}{\rho^d}\left[C^{[1]}_d + \frac{Q^{[1]}_d}{1-d}\right] + \frac{(d-2)m^2}{(3-d)\rho^{(d-2)}}\left(M^{[1]}_d + N^{[1]}_d\right)\right)\cos(\psi)\sin(\psi),\\
\langle T^i_{\ j}\rangle^{[1]}_{a=0} =& \frac{\delta^i_j}{\rho^d}\left(C^{[1]}_d + R^{[1]}_d + (m\rho)^2\left[\frac{(d-4)M^{[1]}_d}{(3-d)(d-2)} + S^{[1]}_d\right]\right).
\end{aligned}
\tag{4.8.253}
$$

We have $C_d(q)$ given by (4.7.233). From this we see that $C^{[1]}_d = \frac{-(n_s+6n_f+12n_v)}{360\pi^2}$. We can read off the other undetermined coefficients depending on q by comparing our result (4.7.222) with the result in literature obtained by directly computing on conical manifolds for scalar case, given by (4.7.229).

In the case of a disconnected boundary (at $r = r_1$ and $r = r_2$) we have two conical singularity, one at each boundary. As we have just seen in the case of multiple conical singularities, the net contribution to the $\mathcal{O}(q-1)$ replica stress tensor is the sum of contribution from each conical singularity. We perform the integral restricted around the entangling region. In $\{r, \tau\}$ plane these are semi discs of radius $(r_2 - r_1)/2$, with $r \geq r_1$ around r_1 and $r \leq r_2$ around r_2. Since the $\mathcal{O}(q-1)$ replica stress tensor has identical contributions from each conical singularity and the net contribution is the sum of individual conical contribution, this integral is the same as integrating over a disc of radius $(r_2 - r_1)/2$ around one of the conical singularities.

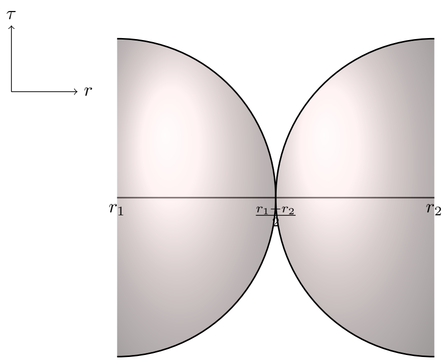


Figure 4.8: Region of integration

Using (4.8.252), the non local contribution to the entanglement entropy $S_{d\text{ dim, bulk}}$ (i.e, contribution depending on the entire entangling region and not just the entangling boundary) in $d-$dimensional flat space with an entangling region of size Δr with a zero extrinsic curvature boundary is

$$\begin{aligned}
&\Delta r \frac{\partial S_{d\text{ dim, bulk}}}{\partial \Delta r}\Bigg|_{\text{reg indep}} \\
=&\lim_{q\to 1}\int_\Sigma d^{d-2}x\sqrt{h}\int_0^{2\pi q} d\psi \int_\epsilon^{\frac{r_2-r_1}{2}} d\rho\rho\langle T^r_r\rangle^{[1]}_{a=0} \\
=&2\pi Vol(M_{d-2})\int_\epsilon^{\frac{r_2-r_1}{2}} d\rho\left(\frac{A_d}{\rho^{d-1}}+\frac{m^2B_d}{\rho^{d-3}}\right) \\
=&2\pi A_2\log\left(\frac{\Delta r}{2\epsilon}\right)+\frac{\pi}{4}m^2B_2(\Delta r)^2; \quad d=2, \\
=&2\pi Vol(M_{d-2})\left\{A_4\left(-\frac{2}{(\Delta r)^2}+\frac{1}{2\epsilon^2}\right)+m^2B_4\log\left(\frac{\Delta r}{2\epsilon}\right)\right\}; \quad d=4, \\
=&2\pi Vol(M_{d-2})\left\{\frac{A_d}{2-d}\left(\frac{2^{d-2}}{(\Delta r)^{d-2}}-\frac{1}{\epsilon^{d-2}}\right)+\frac{m^2B_d}{4-d}\left(\frac{2^{d-4}}{(\Delta r)^{d-4}}-\frac{1}{\epsilon^{d-4}}\right)\right\}; \\
&\text{otherwise,}
\end{aligned} \tag{4.8.254}$$

where $\Delta r = r_2 - r_1$ and $Vol(M_{d-2}) = \int_\Sigma d^{d-2}x\sqrt{h}$. In 3 dimensional flat space with Σ spanning a finite length $2\pi\phi_0$ as in (4.7.203), $Vol(M_{d-2}) = 2\pi\phi_0$.

We have,

$$\begin{aligned}
A_d &= \frac{(2-d)}{2}\left(C^{[1]}_d+\frac{Q^{[1]}_d}{(1-d)}\right), \\
B_d &= \frac{(d-4)(M^{[1]}_d+N^{[1]}_d)}{2(d-3)}.
\end{aligned} \tag{4.8.255}$$

In $d=4$, we can read off the specific values of the coefficients from the cosmic string literature (4.7.1). For the scalar case from (4.7.229) we have

$$\begin{aligned}
C^{[1]}_4 &= \frac{-1}{360\pi^2}, \quad Q^{[1]}_4 = \frac{1}{24\pi^2}(1-6\xi), \\
M^{[1]}_4 &= \frac{-1}{48\pi^2}(1-6\xi), \quad N^{[1]}_4 = 0.
\end{aligned} \tag{4.8.256}$$

For photon from (4.7.234)

$$C^{[1]}_4 = \frac{-1}{30\pi^2}, \quad Q^{[1]}_4 = M^{[1]}_4 = N^{[1]}_4 = 0. \tag{4.8.257}$$

For graviton from (4.7.235)

$$Q^{[1]}_4 = \frac{-7}{5\pi^2}, \quad C^{[1]}_4 = M^{[1]}_4 = N^{[1]}_4 = 0. \tag{4.8.258}$$

Approach 2 - Using replica stress tensor with explicit L dependence: Using the result for the replica stress tensor vev in the case of disconnected planar boundaries, given by (4.7.247), the regulator dependent contribution to the scaling of the Entanglement entropy is given by,

$$\begin{aligned}
\Delta r \frac{\partial S_{d \text{ dim, bulk}}}{\partial \Delta r}\bigg|_{\text{reg indep}} &= \lim_{q\to 1} \int_{M_q} d^d x \sqrt{g} \langle T^r_r \rangle^{[1]} \\
= \lim_{q\to 1} \int_\Sigma d^{d-2}x\sqrt{h} \int_0^{2\pi q} d\psi \int_\epsilon^{\alpha L} d\rho\rho & \left(\frac{C_d^{[1]}}{\rho^d} K(\rho, L)[1 - d\sin^2(\psi)] \right. \\
& \left. - \frac{\cos(2\psi)}{\rho^2} \int d\rho \partial_\rho K(\rho, L) \frac{C_d^{[1]}}{\rho^{d-2}} \right) \\
= (2-d)\pi Vol(M_{d-2}) \int_\epsilon^{\alpha L} d\rho\rho & \left(\frac{C_d^{[1]}}{\rho^d} \left[K(\rho, L) = \sum_i A_i \left(\frac{L}{\rho}\right)^i \right] \right) \\
= (2-d)\pi Vol(M_{d-2}) C_d^{[1]} \sum_{i\geq 0} & \left(\frac{1}{L^{d-2}} \frac{A_i}{\alpha^{d-2+i}(2-d-i)} - \frac{A_i L^i}{\epsilon^{d-2+i}(2-d-i)} \right); \; i \in \mathbb{Z} \\
= (2-d)\pi Vol(M_{d-2}) C_d^{[1]} \sum_{i\geq 0} & \left(\frac{1}{L^{d-2}} \frac{B_i}{\alpha^{d-2+i}} - \frac{B_i}{\epsilon^{d-2}} \frac{L^i}{\epsilon^i} \right); \quad i \in \mathbb{Z},
\end{aligned} \tag{4.8.259}$$

where $B_i = A_i/(2-d-i)$. Notice that the scaling is now with respect to Δr. This is because in flat space $\delta_{\Delta r} g^{\mu\nu}$ is independent of whether we scale with respect to r_1 or r_2 or $r_2 - r_1 = \Delta r$. This reflects the translational symmetry along r in flat space. We have the finite part of EE given by,

$$S_{d \text{ dim, bulk}}\bigg|_{\text{reg } a \text{ indep, finite}} = \frac{\pi Vol(M_{d-2}) C_d^{[1]}}{L^{d-2}} \sum_{i\geq 0} \left(\frac{B_i}{\alpha^{d-2+i}} \right). \tag{4.8.260}$$

By comparing the coefficient of the finite part with the expression we have in [111], we find that

$$\pi C_d^{[1]} \sum_{i\geq 0} \frac{B_i}{\alpha^{d-2+i}} = 2^{d-3} \pi^{\frac{d}{2}-1} \frac{l^{d-1}}{(2-d)G} {}_2F_1\left(\frac{1}{2}, -\frac{d-2}{2(d-1)}; \frac{d}{2(d-1)}; 1\right) \left(\frac{\Gamma\left(\frac{1}{2(d-1)}\right)}{\Gamma\left(\frac{d}{2(d-1)}\right)} \right)^{2-d}, \tag{4.8.261}$$

where the ratio l^{d-1}/G is related to the central charge c in the case of even d and to the number of degrees of freedom in the case of odd d.

Approach 3: When there are two disconnected boundaries, the net contribution to the replica stress tensor is the sum of contribution from each boundary with a conical singularity. In general the coordinates are well defined in only a small local patch around each conical singularity and we need multiple charts to cover the entire manifold in the case of two conical singularities. If $\{\rho_1, \psi_1, y^i\}$ and $\{\rho_2, \psi_2, y^i\}$ are the two local coordinates around each boundary conical singularity, we have

$$\langle T^r_r \rangle_{a=0} = C_d(q) \left(\frac{1}{\rho_1^d}(1 - d\sin^2\psi_1) + \frac{1}{\rho_2^d}(1 - d\sin^2\psi_2) \right). \tag{4.8.262}$$

We will now consider a new set of coordinates $\{\rho, \psi\}$ with $\rho = 0$ at $r = (r_1 + r_2)/2$ and $\psi = 0$ along the axis running from r_1 to r_2, we have

$$\begin{aligned}
\rho_1 &= \sqrt{\frac{L^2}{4} + L\rho\cos(\psi) + \rho^2}, \\
\rho_2 &= \sqrt{\frac{L^2}{4} - L\rho\cos(\psi) + \rho^2}, \\
\psi_1 &= \pm\cos^{-1}\left(\frac{L + 2\rho\cos(\psi)}{2\sqrt{\frac{L^2}{4} + L\rho\cos(\psi) + \rho^2}} \right) + 2\pi q, \\
\psi_2 &= \pm\cos^{-1}\left(\frac{2\rho\cos(\psi) - L}{2\sqrt{\frac{L^2}{4} - L\rho\cos(\psi) + \rho^2}} \right) + 2\pi q; \quad q \in \mathbb{Z}.
\end{aligned} \tag{4.8.263}$$

We can now calculate the EE with the integral restricted to the entangling region $\rho \in [0, \frac{L=(r_2-r_1)}{2}]$ and $\psi \in [0, 2\pi q)$. This again gives

$$S_{d\text{ dim, bulk}}\Big|_{\text{reg } a \text{ indep, finite}} \propto \frac{1}{L^{d-2}}. \tag{4.8.264}$$

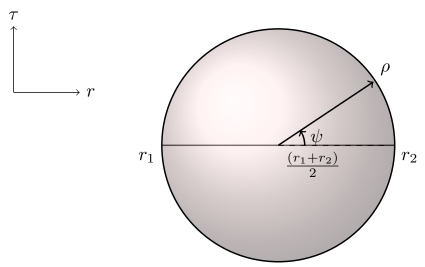


FIGURE 4.9: Region of integration

For the regulator-independent replica stress tensor, when there are two disconnected boundaries, it is not very clear what the right region of integration is, although all the approaches we have considered give the same Δr or L dependence. However, for the regulator-dependent replica stress tensor parts, the replica corrections vanish beyond the radius set by the regulator and their integral only picks up the local contribution around each boundary.

4.8.2 Regulator dependent contribution

Let us now consider only the regulator (a) dependent contribution to the replica stress tensor vev. The $\mathcal{O}(q-1)$ terms in these, contribute locally around each conical singularity to the replica integrals. These are given by the regulator a dependent parts of the solutions (4.7.213), (4.7.216), (4.7.217),

$$\begin{aligned}
\langle T^i{}_j\rangle - \langle T^i{}_j\rangle|_{a=0} &= g(\rho,a)\delta^i{}_j, \\
\langle T^\psi{}_\psi\rangle - \langle T^\psi{}_\psi\rangle|_{a=0} &= f - \frac{1}{\rho^2}\int d\rho\rho f, \\
\langle T^\rho{}_\rho\rangle - \langle T^\rho{}_\rho\rangle|_{a=0} &= \frac{1}{\rho^2}\int d\rho\rho f, \\
\langle T_{\rho\psi}\rangle &= 0 = t_{\rho\psi},
\end{aligned} \tag{4.8.265}$$

where $f(\rho,a) = \mathcal{A}^{(q)} - g(\rho,a)$. In 2 dimensions we have $g(\rho,a) = 0$. In $d > 2$ for a planar entangling boundary, we can always choose $g(\rho,a) = 0$ (since the manifold is $C_2 \times \Sigma$ and $g^{(q)}_{ij}$ has no a dependence). Using the coordinate transformation (4.3.17) we have,

$$\langle T^r_r\rangle^{[1]} - \langle T^r_r\rangle^{[1]}|_{a=0} = \frac{\cos(2\psi)}{\rho^2}\int d\rho\rho\mathcal{A}^{[1]} + \mathcal{A}^{[1]}\sin^2(\psi). \tag{4.8.266}$$

4.8.2.1 2 dimensional CFT

Let us consider a CFT on a 2-dimensional static metric.

$$ds^2_{M_1} = g_{\tau\tau}(r)d\tau^2 + g_{rr}(r)dr^2, \tag{4.8.267}$$

where $r \in [0,\infty)$, $\tau \in (-\infty,\infty)$. We consider the entangling region $r \in [r_1, r_2]$ at $\tau = \tau_0$.

The replica metric in a local region around the endpoint $r = r_0$ (r_1 or r_2) is given by

$$ds^2_{M_q} \text{ around endpoint} = U(\rho,a)d\rho^2 + \rho^2 d\psi^2, \tag{4.8.268}$$

where $\rho \in [0,\infty)$ and $\psi \in [0, 2\pi q)$. We have $(r - r_0) = \rho\cos\psi$, $(\tau - \tau_0) = \rho\sin\psi$, and the conical regulating function $U(\rho,a)|_{\rho\to 0} = q^2$, $U(\rho,a)|_{\rho>>a} = 1$. The non trivial contribution in the difference of replica and base quantities is from locally around the conical singularity.

In 2-dimensions the geometric part of the stress tensor is given by

$$\chi_{\mu\nu}[g_{\mu\nu}] = \left(n = -\frac{c}{48\pi}\right) R g_{\mu\nu}, \tag{4.8.269}$$

where c is the central charge.

Therefore we have (see section (4.6.3.1)),

$$\chi^{(q)}_{\mu\nu}[g^{(q)}_{\mu\nu}] = -\frac{c}{48\pi}R^{(q)}g^{(q)}_{\mu\nu}. \tag{4.8.270}$$

From the replica metric around the entangling boundary given by (4.8.268), we have

$$\begin{aligned}
R^{(q)} - R &= \frac{\partial_\rho U}{\rho U^2},\\
\mathcal{A}^{(q)} - \mathcal{A} &= 2nR^{(q)} = 2n\frac{\partial_\rho U}{\rho U^2} = \left(\mathcal{A}^{[1]} = 2n\frac{\partial_\rho U^{[1]}}{\rho}\right)(q-1) + O((q-1)^2),\\
\chi^{(q)}_{\rho\rho} - \chi_{\rho\rho} &= n\frac{\partial_\rho U}{\rho U} = \left(\chi^{[1]}_{\rho\rho} = n\frac{\partial_\rho U^{[1]}}{\rho}\right)(q-1) + O((q-1)^2),\\
\chi^{(q)}_{\psi\psi} - \chi_{\psi\psi} &= n\frac{\rho\partial_\rho U}{U^2} = \left(\chi^{[1]}_{\psi\psi} = n\rho\partial_\rho U^{[1]}\right)(q-1) + O((q-1)^2),\\
\chi^{(q)}_{\rho\psi} - \chi_{\rho\psi} &= 0.
\end{aligned} \tag{4.8.271}$$

We have skipped the conformal factor e^σ in the replica metric which captures $\mathcal{O}(q-1)^0$ terms from M_1 as well. This is fine since we are only interested in the $\mathcal{O}(q-1)$ terms. Notice $\chi^{\rho[1]}_{\rho} = \chi^{\psi[1]}_{\psi}$.

2 dimensional CFT vacuum state on a flat background: We will focus on the vacuum state on a flat background. On a flat background we have $\chi^{[0]}_{\mu\nu} = 0$ but $\chi^{(q)}_{\mu\nu} \neq 0$. In a $d = 2$ CFT, we now consider the regulator dependent part of the $\mathcal{O}(q-1)$ replica stress tensor sourced by the trace anomaly $\mathcal{A}^{(q)}[g^{(q)}_{\mu\nu}]$ on M_q. We have

$$\begin{aligned}
\langle T^\psi{}_\psi\rangle^{[1]} - \langle T^\psi{}_\psi\rangle^{[1]}|_{a=0} &= \mathcal{A}^{[1]} - \frac{1}{\rho^2}\int d\rho\rho\mathcal{A}^{[1]} = \frac{2n}{\rho^2}\left(\rho\partial_\rho U^{[1]} - U^{[1]}\right),\\
\langle T^\rho{}_\rho\rangle^{[1]} - \langle T^\rho{}_\rho\rangle^{[1]}|_{a=0} &= \frac{1}{\rho^2}\int d\rho\rho\mathcal{A}^{[1]} = \frac{2nU^{[1]}}{\rho^2},\\
\langle T_{\rho\psi}\rangle &= 0 = t_{\rho\psi},
\end{aligned} \tag{4.8.272}$$

and

$$\begin{aligned}
\langle T^r_r\rangle^{[1]} - \langle T^r_r\rangle^{[1]}|_{a=0} &= \frac{\cos(2\psi)}{\rho^2}\int d\rho\rho\mathcal{A}^{[1]} + \mathcal{A}^{[1]}\sin^2(\psi)\\
&= \frac{2n}{\rho^2}(U^{[1]}\cos(2\psi) + \rho\partial_\rho U^{[1]}\sin^2(\psi)).
\end{aligned} \tag{4.8.273}$$

Around each boundary/endpoint of the entangling region, the local representation of $F^{[1]}_{\mu\nu}$ (the superscript is used to indicate that it is an $\mathcal{O}(q-1)$ coefficient) is given by (see (4.5.78))

$$\lim_{q\to 1}\frac{\delta(\rho)}{2\pi q}\int_{M_q}(\sqrt{U}\text{ or }1)\rho\, d\rho\, d\psi\, F^{[1]}_{\mu\nu}. \tag{4.8.274}$$

The choice of $\sqrt{g_q}$ or $\sqrt{g}$ in the volume element gives same result in the limit $q \to 1$.

The local representation of $\langle T^r_r\rangle^{[1]} - \langle T^r_r\rangle^{[1]}|_{a=0}$ around each boundary of the entangling region is,

$$\langle T^r_r\rangle^{[1]} - \langle T^r_r\rangle^{[1]}|_{a=0} = -2n\delta(\rho). \tag{4.8.275}$$

We have,

$$\begin{aligned}
\Delta r\frac{\partial S_{\mathrm{CFT}_2}}{\partial \Delta r}\bigg|_{\text{reg dep}} =& \lim_{q\to 1}\left(\int_{M_q} d^2x\sqrt{g}\langle T^r_r\rangle^{[1]} - \langle T^r_r\rangle^{[1]}|_{a=0}\right)\\
=& \lim_{q\to 1}\left[\int_{\rho=0,\psi=0}^{\rho=\infty,\psi=2\pi q}\rho d\rho d\psi\Big(-2n(\delta(\rho)|_{\text{around } r_1}+\delta(\rho)|_{\text{around } r_2})\Big)\right]\\
=& -8\pi n = \frac{c}{6}.
\end{aligned} \tag{4.8.276}$$

Putting together this and the regulator independent replica stress tensor contribution to EE in 2 dim CFT given by (4.8.254), we have

$$S_{\mathrm{CFT}_2} = \frac{c}{6}\log\left(\frac{\Delta r}{\epsilon_{\mathrm{UV}}}\right). \tag{4.8.277}$$

This is the well known result for EE of vacuum on a 2-dimensional flat background.

Alternatively, we can also look at the decomposition of the $\mathcal{O}(q-1)$ replica corrections into state and geometric parts. We also have the state part of the replica stress tensor corrections $t^{[1]}_{\mu\nu}$ given by,

$$t^{\rho}{}_{\rho}{}^{[1]} = \langle T^{\rho}{}_{\rho}\rangle^{[1]} - \chi^{\rho}{}_{\rho}{}^{[1]} = -t^{\psi}{}_{\psi}{}^{[1]} = \frac{C_2^{[1]}}{\rho^2} + n\left(\frac{2U^{[1]}}{\rho^2} - \frac{\partial_\rho U^{[1]}}{\rho}\right). \tag{4.8.278}$$

Notice that the state part of the replica stress tensor correction $t^{[1]}_{\mu\nu}$ also has metric dependent terms.

Using the coordinate transformation (4.3.17) we have,

$$\chi^{[1]}_{rr} = \chi^{[1]}_{\tau\tau} = n\frac{\partial_\rho U^{[1]}}{\rho}, \tag{4.8.279}$$

$$\chi^{[1]}_{r\tau} = 0, \tag{4.8.280}$$

$$\mathcal{A}^{[1]} = 2n\frac{\partial_\rho U^{[1]}}{\rho}. \tag{4.8.281}$$

Similarly,

$$\begin{aligned}
t^{r}_{r}{}^{[1]} &= -t^{\tau}_{\tau}{}^{[1]} = t^{\rho}{}_{\rho}{}^{[1]}\cos(2\psi),\\
t^{r}_{\tau}{}^{[1]} &= t^{\tau}_{r}{}^{[1]} = t^{\rho}{}_{\rho}{}^{[1]}\sin(2\psi).
\end{aligned} \tag{4.8.282}$$

We have $\langle T^{\mu}_{\nu}\rangle^{[1]} = t^{\mu\,[1]}_{\nu} + \chi^{\mu\,[1]}_{\nu}$.

The local representation of $\chi^{[1]}_{\mu\nu}$ around each boundary of the entangling region is,

$$\chi^{[1]}_{rr} = \chi^{[1]}_{\tau\tau} = -2n\delta(\rho), \tag{4.8.283}$$
$$\chi^{[1]}_{r\tau} = 0, \tag{4.8.284}$$
$$\mathcal{A}^{[1]} = -4n\delta(\rho). \tag{4.8.285}$$

Similarly the local representation of $t^{[1]}_{\mu\nu}$ around each boundary of the entangling region is,

$$t^{r\,[1]}_{r} = t^{\tau\,[1]}_{\tau} = t^{r\,[1]}_{\tau} = 0. \tag{4.8.286}$$

In flat space vacuum CFT state we have $\chi^{[0]}_{\mu\nu} = t^{[0]}_{\mu\nu} = 0$, therefore

$$\begin{aligned} \chi^{\mu\,[1]}_{\nu} &= g^{\mu\alpha\,[1]}\chi^{[0]}_{\alpha\nu} + g^{\mu\alpha\,[0]}\chi^{[1]}_{\alpha\nu} = g^{\mu\alpha\,[0]}\chi^{[1]}_{\alpha\nu}, \\ t^{\mu\,[1]}_{\nu} &= g^{\mu\alpha\,[1]}t^{[0]}_{\alpha\nu} + g^{\mu\alpha\,[0]}t^{[1]}_{\alpha\nu} = g^{\mu\alpha\,[0]}t^{[1]}_{\alpha\nu}. \end{aligned} \tag{4.8.287}$$

We have,

$$\begin{aligned} \Delta r\frac{\partial S_{\mathrm{CFT}_2}}{\partial \Delta r} =& \lim_{q\to 1}\left(\int_{M_q} d^2x\sqrt{g}\chi^{r\,[1]}_{r}\right) \\ =& \lim_{q\to 1}\left[\int_{\rho=0,\psi=0}^{\rho=\infty,\psi=2\pi q}\rho d\rho d\psi\Big(-2n(\delta(\rho)|_{\text{around }r_1} + \delta(\rho)|_{\text{around }r_2})\Big)\right] \\ =& -8\pi n = \frac{c}{6}. \end{aligned} \tag{4.8.288}$$

which implies

$$S_{\mathrm{CFT}_2} = \frac{c}{6}\log\left(\frac{\Delta r}{\epsilon_{\mathrm{UV}}}\right). \tag{4.8.289}$$

Massive scalar field on a flat background: In 2 dimensional QFTs on flat space, in the presence of mass m the regulator-independent part of the replica stress tensor (4.7.222) has an additional m dependent term which additionally contributes to the EE as can be seen in (4.8.254),

$$\Delta r\frac{\partial S_{\mathrm{CFT}_2}}{\partial \Delta r} = \frac{c}{6} + \frac{\pi}{4}m^2(M_2^{[1]} + N_2^{[1]})(\Delta r)^2, \tag{4.8.290}$$

which implies

$$S_{\mathrm{CFT}_2} = \frac{c}{6}\log\left(\frac{\Delta r}{\epsilon_{\mathrm{UV}}}\right) + \frac{\pi}{8}m^2(M_2^{[1]} + N_2^{[1]})(\Delta r)^2. \tag{4.8.291}$$

Similarly for the fermionic field one has to use the additional contributions in the regulator-independent part of the replica stress tensor given in [212].

2 dimensional CFT on a general background: In case of a general metric, we have

$$\begin{aligned}
-\alpha\Delta r\frac{\partial S_{\mathrm{CFT}_2}}{\partial r_1} &= \lim_{q\to 1}\frac{1}{2}\int_{M_q} d^d x\sqrt{g}\left(\delta_{\Delta r}g^{rr}\langle T_{rr}\rangle^{[1]}+\delta_{\Delta r}g^{\tau\tau}\langle T_{\tau\tau}\rangle^{[1]}\right)\\
=&\lim_{q\to 1}\frac{1}{2}\int_{M_q} d^d x\sqrt{g}\left(\left(\delta_{\Delta r}g^{rr}\chi^{[1]}_{rr}+\delta_{\Delta r}g^{\tau\tau}\chi^{[1]}_{\tau\tau}\right)+\left(\delta_{\Delta r}g^{rr}t^{[1]}_{rr}+\delta_{\Delta r}g^{\tau\tau}t^{[1]}_{\tau\tau}\right)\right)\\
=&\lim_{q\to 1}\frac{1}{2}\int_{M_q}\rho d\rho d\psi\alpha\Big(2g^{rr}+(r-r_2)\partial_r g^{rr}+(r-r_2)\partial_r g^{\tau\tau}\Big)\Big(-2n(\delta(\rho)|_{r_1}+\delta(\rho)|_{r_2})\Big)\\
&+\lim_{q\to 1}\frac{1}{2}\int_{M_q} d^d x\sqrt{g}\left(\delta_{\Delta r}g^{rr}t^{[1]}_{rr}+\delta_{\Delta r}g^{\tau\tau}t^{[1]}_{\tau\tau}\right)\\
=&-2\pi n\left\{2(g^{rr}(r_1)+g^{rr}(r_2))+(r_1-r_2)\partial_r g^{rr}|_{r_1}+(r_1-r_2)\partial_r g^{\tau\tau}|_{r_1}\right\}\\
&+\lim_{q\to 1}\frac{1}{2}\int_{M_q} d^d x\sqrt{g}\left(\delta_{\Delta r}g^{rr}t^{[1]}_{rr}+\delta_{\Delta r}g^{\tau\tau}t^{[1]}_{\tau\tau}\right). \qquad (4.8.292)
\end{aligned}$$

For the state dependent part one has to solve (4.7.204)-(4.7.208), where we have a non zero source $\chi^{[0]}_{\mu\nu}$ and in the case of general state $t^{[0]}_{\mu\nu}$. Notice that the scaling of EE above is not with respect to Δr, but with respect to the boundary coordinate r_1. This reflects the lack of translation symmetry along r, on a general metric.

4.8.2.2 Even dimensional CFT

In odd dimensional CFTs since the trace anomaly $\mathcal{A}[g_{\mu\nu}]=0=\mathcal{A}^{(q)}[g^{(q)}_{\mu\nu}]$ therefore there are no regulator dependent contributions to EE. The scaling of the EE is completely given by (4.8.254).

In even dimensional CFTs the trace anomaly on M_1 is non zero and is given by (4.6.145). The anomaly on the replica manifold $\mathcal{A}^{(q)}[g^{(q)}_{\mu\nu}]$ is given by the same expression but with $g_{\mu\nu}$ replaced by $g^{(q)}_{\mu\nu}$,

$$\mathcal{A}^{(d,q)}[g^{(q)}_{\mu\nu}]=\left\langle\chi^{\mu}{}_{\mu}\right\rangle_{d,q}=\frac{1}{(4\pi)^{d/2}}\left(\sum_j c_{dj}I^{(d,q)}_j-(-1)^{\frac{d}{2}}a_d E^{(d,q)}+\sum_j d_{dj}D^i J^{(d,q)}_i\right). \qquad (4.8.293)$$

In general, the replica metric in this case locally around the entangling boundary is given by (4.3.26),

$$\begin{aligned}
&ds^2_q \text{ expanded around boundary}, reg = g^{(q)}_{\mu\nu}dx^\mu dx^\nu|_{bdy}\\
&=U(\rho,a)d\rho^2+\rho^2 d\psi^2+\delta_{ij}dz^i dz^j. \qquad (4.8.294)
\end{aligned}$$

We notice that the replica metric has no ψ dependence and hence the replica anomaly in any even dimensional CFT, for this replica metric, is also independent of ψ. From (4.8.266), we have

$$\langle T^r_r\rangle^{[1]} - \langle T^r_r\rangle^{[1]}|_{a=0} = \frac{\cos(2\psi)}{\rho^2}\int d\rho\rho\mathcal{A}^{[1]} + \mathcal{A}^{[1]}\sin^2(\psi). \tag{4.8.295}$$

Using (4.5.78), the local representation of $\langle T^r_r\rangle^{[1]} - \langle T^r_r\rangle^{[1]}|_{a=0}$ around each boundary of the entangling region is,

$$\langle T^r_r\rangle^{[1]} - \langle T^r_r\rangle^{[1]}|_{a=0} = \frac{\delta_\Sigma(\rho)\pi q}{Vol(M_{d-2})}\int_{M_q} d\rho d^{d-2}x\sqrt{h}\rho\mathcal{A}^{[1]} = N\delta_\Sigma(\rho). \tag{4.8.296}$$

We have,

$$\begin{aligned}
\Delta r\frac{\partial S_{\mathrm{CFT}_d}}{\partial\Delta r}\bigg|_{\text{reg dep}} =& \lim_{q\to1}\left(\int_{M_q} d^dx\sqrt{g}\left(\langle T^r_r\rangle^{[1]} - \langle T^r_r\rangle^{[1]}|_{a=0}\right)\right)\\
=& \lim_{q\to1}\left[N\int_{M_q}\rho d\rho d\psi d^{d-2}\sqrt{h}\Big(\delta_\Sigma(\rho)|_{\text{around } r_1} + \delta_\Sigma(\rho)|_{\text{around } r_2}\Big)\right]\\
=& 2\pi\int_{M_q} d\rho d^{d-2}x\sqrt{h}\rho\mathcal{A}^{[1]} = 2\pi Vol(M_{d-2})\int_0^\infty d\rho\rho\mathcal{A}^{[1]},
\end{aligned} \tag{4.8.297}$$

where in the last equality we have made use of the fact that we are dealing with a planar entangling region and hence $g^{[1]}_{\mu\nu}$ and $\mathcal{A}^{[1]}$ are independent of $y^{i\prime}s$.

2 dimensions: See (4.8.276)

$$\mathcal{A}^{[1]} = -\frac{c}{24}R^{[1]} \implies \Delta r\frac{\partial S_{\mathrm{CFT}_2}}{\partial\Delta r}\bigg|_{\text{reg dep}} = \frac{c}{6}. \tag{4.8.298}$$

4 dimensions:

$$\mathcal{A}^{(q)}[g^{(q)}_{\mu\nu}] = \chi^\mu{}_\mu[g_{\mu\nu}]^{d=4} = -\frac{a}{(4\pi)^2}E^{(q)}_4 + \frac{c}{(4\pi)^2}I^{(q)}_4, \tag{4.8.299}$$

where $I^{(q)}_4$ is the term involving weyl contraction $C^{(q)}_{\mu\nu\rho\sigma}C^{(q)\mu\nu\rho\sigma}$.

$$\begin{aligned}
E^{(q)}_4 &= R^{(q)}_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu(q)} - 4R^{(q)}_{\mu\nu}R^{\mu\nu(q)} + R^{(q)2},\\
I^{(q)}_4 &= R^{(q)}_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu(q)} - 2R^{(q)}_{\mu\nu}R^{\mu\nu(q)} + \frac{1}{3}R^{(q)2}.
\end{aligned} \tag{4.8.300}$$

We make use of known integrals of Riemann contractions on replica metric i.e., (4.5.97), (4.5.100) to get,

$$\begin{aligned}\Delta r \frac{\partial S_{\mathrm{CFT}_d}}{\partial \Delta r}\bigg|_{\text{reg dep}} =& 2\pi \int_{M_q} d\rho d^{d-2}x\sqrt{h}\rho\mathcal{A}^{[1]} \\ =& (aI_a^{\Sigma} - cI_c^{\Sigma}).\end{aligned} \tag{4.8.301}$$

This is the universal logarithmic contribution to S_{QFT}. For a planar entangling region on flat space the Ricci scalar of the background, the extrinsic curvature of Σ and projection of Ricci and Riemann tensors on Σ is zero, therefore (4.8.301) is zero.

5 and 6 dimensions: We can similarly compute the scaling of regulator-dependent contribution to EE in higher dimensions by using (4.8.297). The replica trace anomaly is given by (4.6.145) with $g_{\mu\nu}$ replaced by $g^{(q)}_{\mu\nu}$. The relevant replica integrals in 5 and 6 dimensions are listed in [110].

The replica anomaly is independent of the choice of the CFT state. Therefore, the results in this section hold for any CFT state.

General deformation: So far we have considered a particular kind of scaling i.e., we scale the entangling region along its finite length direction. Consider now (4.4.63) where we consider a general deformation of the background metric and the entangling surface geometry captured by $\delta_L g_{\mu\nu}$. For the CFT replica stress tensor given by (4.7.222) (i.e., all other coefficients except $C_d(q)$ is zero), it was shown in [105] that two terms in (4.3.42) contribute to logarithmic terms in $\delta_L S_{\mathrm{QFT}}$ in 4-dimensions,

$$\delta_L g_{ij} = x^a x^c R_{iacj}; \quad \delta_L g_{ab} = \frac{-1}{3} x^c x^d R_{acbd}. \tag{4.8.302}$$

We have,

$$\delta_L S_{\mathrm{QFT}} = \frac{c}{6\pi} \int d^2y (\delta^{ac}\delta^{bd} R_{abcd} + \delta^{ij}\delta^{ac} R_{iacj}) \log(l/\epsilon). \tag{4.8.303}$$

here l is the characteristic length scale of perturbations and ϵ is a UV cutoff. We have made use of (4.7.237). (4.7.238) can be written as $C_T = 40c/\pi^4$ where c is the CFT central charge corresponding to the Weyl tensor contractions in the 4-dimensional trace anomaly. This should be compared with Solodukhin's formula [109] for the universal part of entanglement entropy in the case of a 4-dimensional CFT.

4.8.3 General QFT

The replica stress tensor vev $\langle T_{\mu\nu}\rangle^{[1]}$ for a planar entangling boundary in a general unitary d dimensional field theory (free or interacting) was given in [207]

$$\langle T_{ij}\left(x_1,x_2,y\right)\rangle^{[1]} = -\left\langle T_{ij}\left(x_1,x_2,y\right)K_0\right\rangle \\ = \frac{A_d\delta_{ij}}{(d-1)^2\Gamma(d)}\int_0^\infty d\mu\left(c^{(0)}(\mu)-(d-1)c^{(2)}(\mu)\right)\mu^2K_0(\mu\rho), \tag{4.8.304}$$

$$\langle T_{ab}\left(x_1,x_2,y\right)\rangle^{[1]} = -\left\langle T_{ab}\left(x_1,x_2,y\right)K_0\right\rangle \\ = \frac{A_d}{(d-1)^2\Gamma(d)}\int_0^\infty d\mu\left(c^{(0)}(\mu)+(d-1)(d-2)c^{(2)}(\mu)\right) \\ \left(\mu^2K_0(\mu\rho)\left(\delta_{ab}-\frac{x_ax_b}{\rho^2}\right)+\frac{\mu K_1(\mu\rho)}{\rho}\left(\delta_{ab}-\frac{2x_ax_b}{\rho^2}\right)\right). \tag{4.8.305}$$

where x_a with $a=1,2$ are transverse to the entangling boundary and $y^{i\prime}s$ are along the entangling boundary. μ is the spectral parameter and $c^{(0)}(\mu)$ and $c^{(2)}(\mu)$ are spectral functions that encode the information of the exact QFT, and $A_d=\frac{\pi^{d/2}}{(d+1)2^{d-2}\Gamma(d/2)}$.

One could easily repeat the calculation for EE scaling that we just did for a CFT, for a $d-$dimensional field theory using the expressions above. We have,

$$\Delta r\frac{\partial S_{d\text{ dim, bulk}}}{\partial\Delta r}\Bigg|_{\text{reg indep}} \\ = \lim_{q\to1}\int_\Sigma d^{d-2}x\sqrt{h}\int_0^{2\pi q}d\psi\int_\epsilon^{\frac{r_2-r_1}{2}}d\rho\rho\langle T^r_r\rangle^{[1]}_{a=0} \\ = \lim_{q\to1}Vol(M_{d-2})\int_0^{2\pi q}d\psi\int_\epsilon^{\frac{r_2-r_1}{2}}d\rho\Bigg(\frac{A_d}{(d-1)^2\Gamma(d)}\int_0^\infty d\mu C(\mu) \\ \left(\mu^2\rho K_0(\mu\rho)\sin^2(\psi)-\mu K_1(\mu\rho)\cos(2\psi)\right)\Bigg) \\ = \frac{\pi A_dVol(M_{d-2})}{(d-1)^2\Gamma(d)}\int_\epsilon^{\frac{r_2-r_1}{2}}d\rho\left(\int_0^\infty d\mu C(\mu)\mu^2\rho K_0(\mu\rho)\right) \\ = \frac{-\pi A_dVol(M_{d-2})}{(d-1)^2\Gamma(d)}\left(\int_0^\infty d\mu C(\mu)(F(\mu,(r_2-r_1)/2)-F(\mu,\epsilon))\right), \tag{4.8.306}$$

with $F(\mu,\rho)=\mu\rho K_1(\mu\rho)$, $C(\mu)=\left(c^{(0)}(\mu)+(d-1)(d-2)c^{(2)}(\mu)\right)$. In the CFT limit $c^{(0)}(\mu)\propto\mu^{d-2}\delta(\mu)$ and $c^{(2)}(\mu)=C_T\frac{d-1}{d}\mu^{d-3}$, this reproduces the CFT result we have in (4.7.222), and (4.8.254) with

$$C_d^{[1]}=-\frac{C_T\pi^{d/2}\Gamma(d/2)}{\Gamma(d+2)}, \tag{4.8.307}$$

where C_T is the charge that appears in the 2 point energy momentum correlation function in a $d-$dimensional CFT.

4.8.4 Entangling boundary with extrinsic curvature

Consider a general CFT state on flat space on M_1 with a spherical or cylindrical entangling boundary i.e.,

$$ds^2_{M_1,sph} = -dt^2 + dr^2 + r^2 h_{ij}(y) dy^i dy^j, \tag{4.8.308}$$

with entangling region $r_1 \leq r \leq r_2$ spanning the sphere directions y^i on a constant t slice.

or

$$ds^2_{M_1,cyl} = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2, \tag{4.8.309}$$

with entangling region $r_1 \leq r \leq r_2$ spanning ϕ and $\frac{-L}{2} \leq z \leq \frac{L}{2}$ on a constant t slice.

The replica metric locally around the boundary of entangling region $r = r_0 \in \{r_1, r_2\}$ is given by (4.3.33) and (4.3.34) i.e.,

$$\begin{aligned} ds^2_{q \text{ around } \Sigma, sph} &= U(\rho) d\rho^2 + \rho^2 d\psi^2 + (r_0 + \rho^p c^{1-p} \cos(\psi))^2 h_{ij}(y) dy^i dy^j, \\ ds^2_{q \text{ around } \Sigma, cyl} &= U(\rho) d\rho^2 + \rho^2 d\psi^2 + (r_0 + \rho^p c^{1-p} \cos(\psi))^2 d\phi^2 + dz^2. \end{aligned} \tag{4.8.310}$$

Using (4.5.115), the scaling of S_{QFT} in this case is given by,

$$\alpha \Delta r \frac{\partial S_{d \text{ dim}}}{\partial \Delta r} = \lim_{q \to 1} \frac{1}{2} \int_{M_q} d^d x \sqrt{g}\, \delta_{\Delta r} g_{\mu\nu} \langle T^{\mu\nu} \rangle^{[1]} \tag{4.8.311}$$

Since S_{QFT} depends only on Δr due to translation symmetry along r, we consider the scaling of S_{QFT} with respect to $\Delta r \to e^\alpha \Delta r$ or $r \to e^\alpha r$. On scaling the metric (4.8.308) and (4.8.309), we get

$$\begin{aligned} \delta_{\Delta r} g_{\mu\nu, sph} &= 2\alpha \eta_{\mu\nu, sph}, \quad \mu = \nu = r, \text{ and } \mu, \nu \in \{i, j\}, \\ \delta_{\Delta r} g_{\mu\nu, cyl} &= 2\alpha \eta_{\mu\nu, cyl}, \quad \mu = \nu = r \text{ or } \phi, \end{aligned} \tag{4.8.312}$$

therefore,

$$\Delta r \frac{\partial S_{d \text{ dim, sph}}}{\partial \Delta r} = \lim_{q \to 1} \int_{M_q} d^d x \sqrt{g} \left(\langle T^{rr} \rangle^{[1]} + r^2 h_{ij} \langle T^{ij} \rangle^{[1]} \right), \tag{4.8.313}$$

$$\Delta r \frac{\partial S_{d \text{ dim,cyl}}}{\partial \Delta r} = \lim_{q \to 1} \int_{M_q} d^4 x \sqrt{g} \left(\langle T^{rr} \rangle^{[1]} + r^2 \langle T^{\phi\phi} \rangle^{[1]} \right). \tag{4.8.314}$$

If we consider flat space and vacuum state, we have symmetry along τ and $-L/2 \leq z \leq L/2$, therefore S_{QFT} is also independent of τ and z. We will restrict to vacuum state and scale τ and z as well. We get

$$\begin{aligned} \delta_{\Delta r} g_{\mu\nu,sph} &= 2\alpha\eta_{\mu\nu,sph}, \\ \delta_{\Delta r} g_{\mu\nu,cyl} &= 2\alpha\eta_{\mu\nu,cyl}. \end{aligned} \tag{4.8.315}$$

Therefore, for the vacuum state, we have

$$\Delta r \frac{\partial S_{d\text{ dim}}}{\partial \Delta r} = \lim_{q\to 1} \int_{M_q} d^d x \sqrt{g}\, \eta_{\mu\nu} \langle T^{\mu\nu} \rangle^{[1]}. \tag{4.8.316}$$

Furthermore, for vacuum state we have $\langle T_{\mu\nu} \rangle^{[0]} = 0$, which implies $\eta_{\mu\nu} \langle T^{\mu\nu} \rangle^{[1]} = \langle T^{\mu}_{\mu} \rangle^{[1]}$. This implies

$$\Delta r \frac{\partial S_{d\text{ dim}}}{\partial \Delta r} = \lim_{q\to 1} \int_{M_q} d^d x \sqrt{g} \langle T^{\mu}_{\mu} \rangle^{[1]} = \lim_{q\to 1} \int_{M_q} d^d x \sqrt{g} \mathcal{A}^{[1]}. \tag{4.8.317}$$

As we see from the expressions above the only contribution to EE scaling for a spherical or cylindrical entangling region on flat background is from the regulator dependent replica anomaly $\mathcal{A}^{(q)}$ at $\mathcal{O}(q-1)$. There are no regulator independent replica stress tensor contributions, that were present in the case of planar disconnected entangling boundaries.

The replica anomaly contribution above is very similar to the regulator dependent planar contribution (4.6.145), except now the replica metric (4.8.310) and hence the replica anomaly can depend on the coordinate ψ and have non zero extrinsic curvature for boundaries $r = r_0$. These integrals were performed in [110] and we have listed them in (4.5.97), (4.5.98), (4.5.99).

2 dimensions:

$$\mathcal{A}^{[1]} = -\frac{c}{24} R^{[1]} \implies \Delta r \frac{\partial S_{\text{CFT}_2}}{\partial \Delta r}\bigg|_{\text{reg dep}} = \frac{c}{6}. \tag{4.8.318}$$

4 dimensions:

$$\mathcal{A}^{(q)}[g^{(q)}_{\mu\nu}] = \chi^{\mu}{}_{\mu}[g_{\mu\nu}]^{d=4} = -\frac{a}{(4\pi)^2} E_4^{(q)} + \frac{c}{(4\pi)^2} I_4^{(q)}. \tag{4.8.319}$$

Using known integrals of Riemann contractions on replica metric with extrinsic curvature i.e., (4.5.97), (4.5.100), we have

$$\begin{aligned} \Delta r \frac{\partial S_{\text{CFT}_4}}{\partial \Delta r} =& 2\pi \int_{M_q} d\rho d^{d-2} x \sqrt{h} \rho \mathcal{A}^{[1]} \\ =& (a I_a^{\Sigma} - c I_c^{\Sigma}). \end{aligned} \tag{4.8.320}$$

For a spherical and cylindrical Σ on flat space we only have contributions from integrals of K^2 and Tr K^2. Using the value of these integrals in both cases given by (4.5.98), (4.5.99), we have

$$\begin{aligned}\Delta r\frac{\partial S_{\mathrm{CFT}_4}}{\partial \Delta r} =&4a; \quad \text{Spherical } \Sigma,\\ =&\frac{L}{2r_0}c; \quad \text{Cylindrical } \Sigma.\end{aligned} \tag{4.8.321}$$

at each entangling boundary. Therefore, the net contribution in our case would be the sum of the contributions from the two disconnected entangling boundaries at $r_0 = \{r_1, r_2\}$. The spherical Σ isolates central charge a contribution and the cylindrical Σ isolates central charge c contribution.

5 and 6 dimensions: We can similarly compute the scaling of EE in higher dimensions by using (4.8.317). The replica trace anomaly is given by (4.6.145) with $g_{\mu\nu}$ replaced by $g^{(q)}_{\mu\nu}$. The relevant replica integrals in 5 and 6 dimensions are listed in [110].

4.9 Entanglement entropy - non vacuum states

4.9.1 Thermal state in odd dimensional CFTs

Odd dimensional CFTs do not have an anomaly, and more generally $\chi^{[0]}_{\mu\nu} = 0 = \chi^{(q)}_{\mu\nu}$. We consider this case with static metrics on M_1 satisfying $\Gamma^i_{k\mu} = 0$, $\mu \in \{\rho, \psi\}$. This is true for planar entangling region in flat space on M_1

$$ds^2_{M_1} = d\tau^2 + dr^2 + \delta_{ij}dx^i dx^j. \tag{4.9.322}$$

It is also true for a CFT on a $d-$dimensional static metric with a finite volume along the x^i directions

$$ds^2_{M_1} = g_{\tau\tau}(r)d\tau^2 + g_{rr}(r)dr^2 + \phi_0^2(d\Omega^2_{d-2} = h_{ij}dx^i dx^j), \tag{4.9.323}$$

where $r \in [0, \infty)$, $\tau \in (-\infty, \infty)$, M_{d-2} is compact.

We consider the usual entangling region $r \in [r_1, r_2]$ at $\tau = \tau_0$ spanning the x^i directions.

The replica metric in a local region around the boundary $r = r_0$ (r_1 or r_2) is given by

$$ds^2_{M_q} \text{ around endpoint} = U(\rho, a)d\rho^2 + \rho^2 d\psi^2 + \phi_0^2 h_{ij}dx^i dx^j, \tag{4.9.324}$$

where $(r - r_0) = \rho\cos\psi$, $(\tau - \tau_0) = \rho\sin\psi$, and the conical regulating function $U(\rho, a)|_{\rho\to 0} \to q^2$, $U(\rho, a)|_{\rho>>a} \to 1$. We could always use coordinate $x = \rho/a$ where $U(x)$ is independent of regulator a. We have ${\Gamma^i_{i\rho}}^{(q)} = 0 = {\Gamma^i_{i\psi}}^{(q)}$.

4.9.1.1 Replica stress tensor

The replica trace and conservation equations (4.6.183)-(4.6.186), (4.6.176) reduce to

$\mathcal{O}(q-1)$ **Traceless condition:**

$$t^{[1]}_{\rho\rho} + \frac{1}{\rho^2} t^{[1]}_{\psi\psi} + t^{i}{}_{i}{}^{[1]} = U^{[1]}(\rho,a) t^{[0]}_{\rho\rho}. \tag{4.9.325}$$

$\mathcal{O}(q-1)$ ρ**-component of Conservation:**(which includes trace condition)

$$\frac{1}{\rho^2}\partial_\psi t^{[1]}_{\rho\psi} - \frac{1}{\rho^2}\partial_\rho t^{[1]}_{\psi\psi} - \left(\partial_\rho + \frac{1}{\rho}\right) t^{i}{}_{i}{}^{[1]} = 0. \tag{4.9.326}$$

$\mathcal{O}(q-1)$ ψ**-component of Conservation:**

$$\left(\partial_\rho + \frac{1}{\rho}\right) t^{[1]}_{\rho\psi} + \frac{1}{\rho^2}\partial_\psi t^{[1]}_{\psi\psi} = \left(\frac{1}{2}\partial_\rho U^{[1]} + U^{[1]}\partial_\rho + \frac{U^{[1]}}{\rho}\right) t^{[0]}_{\rho\psi}. \tag{4.9.327}$$

$\mathcal{O}(q-1)$ i**-component of Conservation:**

$$\text{For a fixed } i\text{: } (\partial_i + \sum_l \Gamma^l{}_{li})\langle T^i{}_i\rangle^{[1]} - \sum_l \Gamma^l{}_{li}\langle T^l{}_l\rangle^{[1]} = 0. \tag{4.9.328}$$

We will now calculate the scaling of the EE in this case. When the metric on M_1 is flat, scaling the metric using (4.5.105), we have

$$\delta_{\Delta r} g_{\mu\nu} = 2\alpha; \quad \mu = \nu = r, \tag{4.9.329}$$

$$= 0; \quad \text{otherwise.} \tag{4.9.330}$$

Therefore using (4.5.115), the scaling of the EE is given by,

$$\Delta r \frac{\partial S_{\mathrm{CFT}_d}}{\partial \Delta r} = \lim_{q\to 1} \int_{M_q} d^d x \sqrt{g}\, t^{rr\,[1]}. \tag{4.9.331}$$

Using the coordinate transformation (4.3.17) we have,

$$t^{rr} = U^{-2} t^{(q)}_{\rho\rho} (\cos(\psi))^2 - \frac{2U^{-1}}{\rho} t^{(q)}_{\rho\psi} \cos(\psi)\sin(\psi) + \frac{1}{\rho^2} t^{(q)}_{\psi\psi} (\sin(\psi))^2, \tag{4.9.332}$$

$$\begin{aligned} t^{rr\,[1]}\Big|_{\text{around } r_0,\tau_0} &= t^{[1]}_{\rho\rho}(\cos(\psi))^2 - \frac{2}{\rho} t^{[1]}_{\rho\psi}\cos(\psi)\sin(\psi) + \frac{1}{\rho^2} t^{[1]}_{\psi\psi}(\sin(\psi))^2 \\ &\quad - 2U^{[1]} t^{[0]}_{\rho\rho}(\cos(\psi))^2 - \frac{2U^{[1]}}{\rho} t^{[0]}_{\rho\psi}\cos(\psi)\sin(\psi). \end{aligned} \tag{4.9.333}$$

We will consider the CFT thermal stress tensor on a flat background ((4.6.192) with $\epsilon = -(d+1)p$ and $g_{rr} = g_{\tau\tau} = 1$, $g_{r\tau} = 0$, $g_{ij} = \phi_0^2 h_{ij}$)

$$\begin{aligned} t^{[0]}_{\rho\psi} &= (-dp/2)\rho\sin(2\psi), \\ t^{[0]}_{\rho\rho} &= p[1 - d(\sin(\psi))^2], \\ t^{[0]}_{\psi\psi} &= p\rho^2[1 - d(\cos(\psi))^2], \\ t^{[0]}_{ij} &= p(g_{ij} = \phi_0^2 h_{ij} \text{ or } \delta_{ij}). \end{aligned} \tag{4.9.334}$$

Homogeneous solution: Let us first consider the homogeneous solution to the replica trace and conservation equations above. (4.9.328) has no sources and has the solution

$$t^{i[1]}_i = \frac{(d-2)C^{[1]}}{\rho^d} + t^{i[0]}_i. \tag{4.9.335}$$

The $\frac{C^{[1]}}{\rho^d}$ term reflects the lack of ρ translation symmetry due to the conical singularity on the replica manifold. Since the $i-$conservation equations on M_1 and M_q are the same with $t^{i[0]}_i \to t^{i[1]}_i$, noting (4.6.1) we additionally have the $t^{i[0]}_i$ term.

Noting that $t^{i[0]}_i$ and $t^{i[1]}_i$ are independent of ψ, (4.9.326) and (4.9.327) implies,

$$\left(1 + 3\rho\partial_\rho + \rho^2\partial_\rho^2 + \partial_\psi^2\right) t^{[1]}_{\rho\psi} = 0. \tag{4.9.336}$$

We use the ansatz $t^{[1]}_{\rho\psi} = A\rho^\Delta m(\psi)$, This gives

$$\begin{aligned} \partial_\psi^2 m(\psi) &= -k^2 m(\psi); \quad k^2 = (\Delta+1)^2 \\ \implies \Delta &= (-k-1) \text{ or } (k-1), \end{aligned} \tag{4.9.337}$$

which implies

$$m(\psi) = c_1\cos(k\psi) + c_2\sin(k\psi). \tag{4.9.338}$$

We have the following replica corrections to the replica stress tensor,

$$t^{[1]}_{\rho\psi} = \sum_k (A_k\rho^{-(k+1)} + B_k\rho^{(k-1)})(c_1\cos(k\psi) + c_2\sin(k\psi)). \tag{4.9.339}$$

Noting our observations in (4.6.1), we must have $t^{[1]}_{\rho\psi} = t^{[0]}_{\rho\psi}$ which implies $B_2 c_2 = -dp/2$, $c_1 = 0$, $A_k = 0\ \forall\, k$ and $B_{k\neq 2} = 0$. With this we have

$$t^{[1]}_{\rho\psi} = \frac{-dp}{2}\rho\sin(2\psi). \tag{4.9.340}$$

(4.9.326) implies

$$\begin{aligned}
t^{[1]}_{\psi\psi} =& \int d\rho \left[\partial_\psi t^{[1]}_{\rho\psi} - \rho^2 \left(\partial_\rho + \frac{1}{\rho}\right) t^{i}{}_{i}{}^{[1]} \right] \\
=& \sum_k (B_k \rho^k - A_k \rho^{-k})(c_2 \cos(k\psi) - c_1 \sin(k\psi)) \\
&+ (1-d)\frac{C^{[1]}_d}{\rho^{d-2}} + \frac{(2-d)}{2} p\rho^2 + c_3(\psi) \\
=& [t^{[0]}_{\psi\psi} = p\rho^2(1 - d\cos^2(\psi))] + (1-d)\frac{C^{[1]}_d}{\rho^{d-2}},
\end{aligned} \tag{4.9.341}$$

where we have chosen $c_3(\psi) = 0$ on dimensional grounds. (4.9.325) implies,

$$\begin{aligned}
t^{[1]}_{\rho\rho} &= -\frac{1}{\rho^2} t^{[1]}_{\psi\psi} - t^{i}{}_{i}{}^{[1]} \\
&= \sum_k (A\rho^{-k-2} - B\rho^{k-2})(c_2 \cos(k\psi) - c_1 \sin(k\psi)) \\
&= [t^{[0]}_{\rho\rho} = p\rho^2(1 - d\sin^2(\psi))] + \frac{C^{[1]}_d}{\rho^d}.
\end{aligned} \tag{4.9.342}$$

Notice that ∂_ψ is a Killing vector on M_1 and M_q but the thermal state breaks this symmetry. Considering this and the discussion in (4.6.1), the homogeneous solution to $t^{[1]}_{\mu\nu}$ contains the vacuum part which has ψ translation symmetry and the thermal homogeneous part which does not have this symmetry i.e.,

$$t^{[1]}_{\mu\nu}|_{homogeneous} = \frac{C^{[1]}}{\rho^d}((1-d)\rho^2, 1, 1, 1, 1, ..., 1) + t^{[0]}_{\mu\nu}(\rho,\psi)|_{thermal}, \tag{4.9.343}$$

where the basis is in order $\{\psi, \rho, x^i\}$.

Full solution: Let us consider a general CFT state. Combining (4.9.326) and (4.9.327) into a second order PDE in $t^{[1]}_{\rho\psi}$ only,

$$\begin{aligned}
(\partial_\psi^2 + \rho^2\partial_\rho^2 + 3\rho\partial_\rho + 1) t^{[1]}_{\rho\psi}(\rho,\psi) =& \Bigg(\left(1 + 2\rho\partial_\rho + \frac{1}{2}\rho^2\partial_\rho^2\right) U^{[1]}(\rho,a) \\
&+ \left(3\rho U^{[1]}(\rho,a) + \frac{3}{2}\rho^2\partial_\rho U^{[1]}(\rho,a)\right)\partial_\rho + \rho^2 U^{[1]}(\rho,a)\partial_\rho^2 \Bigg) t^{[0]}_{\rho\psi}(\rho,\psi),
\end{aligned} \tag{4.9.344}$$

We will now demand

$$t^{[0]}_{\rho\psi} = h(\rho)m(\psi), \tag{4.9.345}$$

$$\partial_\psi^2 m(\psi) = -n^2 m(\psi) \text{ i.e., } m(\psi) = c_1\cos(n\psi) + c_2\sin(n\psi), \tag{4.9.346}$$

for arbitrary constants n, c_1, c_2.

The solution to (4.9.344) is

$$t^{[1]}_{\rho\psi}(\rho,\psi) = g(\rho)m(\psi), \tag{4.9.347}$$

where $g(\rho)$ satisifies the first-order ODE

$$\begin{aligned}
&\rho\left(2\rho g''(\rho) + 6g'(\rho)\right) - 2(n^2-1)g(\rho) = 2U^{[1]}(\rho,a)\rho^2 \\
&h''(\rho) + 3\left(\rho^2 U^{[1]\prime}(\rho,a) + 2\rho U^{[1]}(\rho,a)\right)h'(\rho) + h(\rho) \\
&\left(\rho\left(\rho U^{[1]\prime\prime}(\rho,a) + 4U^{[1]\prime}(\rho,a)\right) + 2U^{[1]}(\rho,a)\right).
\end{aligned} \tag{4.9.348}$$

This can be solved for $f(\rho)$ but we will not write the solution here.

(4.9.345) is not true for a general CFT state, however,it is true for the CFT thermal stress tensor on a flat background (4.9.334). Restricting further to this state i.e., $h(\rho) = -dp\rho/2$ and $m(\psi) = \sin(2\psi)$ (i.e., a particular solution to (4.9.346) with $n = 2$), we have

$$t^{[1]}_{\rho\psi} = \left\{\frac{d_1}{\rho^3} + d_2\rho + \frac{dp}{16}af(x)\right\}\sin(2\psi), \tag{4.9.349}$$

with

$$\begin{aligned}
f(x) = x\Bigg(&2U^{[1]}(x) + xU^{[1]\prime}(x) \\
&- \int_1^x \left(\frac{8U^{[1]}(x')}{x'} + 7U^{[1]\prime}(x') + x'U^{[1]\prime\prime}(x')\right)dx'\Bigg),
\end{aligned} \tag{4.9.350}$$

where $x = \rho/a$. d_1, d_2 are integration constants. Comparing with the homogeneous solution (4.9.340) and noting (4.6.1) we get

$$d_1 = 0; \quad d_2 = \frac{-dp}{2}, \tag{4.9.351}$$

i.e.,

$$t^{[1]}_{\rho\psi} = t^{[0]}_{\rho\psi} + \frac{dp}{16}af(x)\sin(2\psi). \tag{4.9.352}$$

(4.9.328) has no sources and has the same solution as the homogeneous solution on M_q

$$t^{i[1]}_i = \frac{(d-2)C^{[1]}_d}{\rho^d} + t^{i[0]}_i. \tag{4.9.353}$$

Using (4.9.326),

$$
\begin{aligned}
t^{[1]}_{\psi\psi} &= \int d\rho \partial_\psi t^{[1]}_{\rho\psi} - \int d\rho\rho\partial_\rho(\rho t^{i}{}_{i}{}^{[1]}) \\
&= \left\{ \frac{-d_1}{\rho^2} + d_2\rho^2 + \frac{dp}{8}a^2 \int dx f(x) \right\} \cos(2\psi) + d_3(\psi) - \frac{\rho^2}{2}(t^{i}{}_{i}{}^{[1]} = t^{i}{}_{i}{}^{[0]} = (d-2)p) \\
&= (1-d)\frac{C_d^{[1]}}{\rho^{d-2}} + t^{[0]}_{\psi\psi} + \frac{dp}{8}a^2 \int dx f(x)\cos(2\psi) + d_3(\psi).
\end{aligned}
\tag{4.9.354}
$$

We again choose $d_3(\psi) = 0$ on dimensional grounds. Using (4.9.325)

$$
\begin{aligned}
t^{[1]}_{\rho\rho} &= -\frac{t^{[1]}_{\psi\psi}}{\rho^2} - t^{i[1]}_{i} + U^{[1]}t^{[0]}_{\rho\rho} \\
&= \left\{ \frac{d_1}{\rho^4} - d_2 - \frac{dp}{8x^2}\int dx f(x) \right\} \cos(2\psi) - \frac{t^{i[1]}_{i}}{2} + U^{[1]}t^{[0]}_{\rho\rho} \\
&= \frac{C_d^{[1]}}{\rho^d} + t^{[0]}_{\rho\rho} - \frac{dp}{8x^2}\int dx f(x)\cos(2\psi) + U^{[1]}t^{[0]}_{\rho\rho}.
\end{aligned}
\tag{4.9.355}
$$

We have,

$$
\begin{aligned}
t^{rr\,[1]} =& t^{[1]}_{\rho\rho}(\cos(\psi))^2 - \frac{2}{\rho}t^{[1]}_{\rho\psi}\cos(\psi)\sin(\psi) + \frac{1}{\rho^2}t^{[1]}_{\psi\psi}(\sin(\psi))^2 \\
&- 2U^{[1]}t^{[0]}_{\rho\rho}(\cos(\psi))^2 - \frac{2U^{[1]}}{\rho}t^{[0]}_{\rho\psi}\cos(\psi)\sin(\psi) \\
=& \frac{C_d^{[1]}}{\rho^d}(1 - d\sin^2(\psi)) + t^{[0]}_{rr} = \left(p\left(1 - \frac{d}{2}\right) - d_2 + d_1\frac{\cos(4\psi)}{\rho^4} \right) + \\
& I(x,\psi) = \Bigg[\left(U^{[1]}t^{[0]}_{\rho\rho} - \frac{dp}{8x^2}\int dx f(x)\cos(2\psi) \right)\cos(\psi)^2 - \left(\frac{dp}{16x}f(x)\sin^2(2\psi) \right) \\
&+ \left(\frac{dp}{8x^2}\int dx f(x)\cos(2\psi) \right)\sin(\psi)^2 - 2U^{[1]}t^{[0]}_{\rho\rho}(\cos(\psi))^2 - \frac{2U^{[1]}}{\rho}t^{[0]}_{\rho\psi}\cos(\psi)\sin(\psi) \Bigg].
\end{aligned}
\tag{4.9.356}
$$

The universal parts of $U^{[1]}(x)$ is determined by (4.3.25). We have $U^{[1]}(x)|_{x\to 0} = 2$, $U^{[1]}(x)|_{x>>1} = 0$. Notice that $t^{[1]}_{\mu\nu}$ with $\mu, \nu \in \{\rho, \psi\}$ depends on non-universal terms in the regulating function i.e., it seems to depend on the choice of the regulator. We will work with a specific regulating function (4.3.22) i.e., $U(x,a) = \frac{x^2+q^2}{x^2+1}$ with $U^{[1]}(x,a) = \frac{2}{1+x^2}$. With this choice, we have,

$$
f(x) = 8x\left(\log\left(1 + \frac{1}{x^2}\right) - \frac{1}{1+x^2} \right),
\tag{4.9.357}
$$

$$
\int f(x)dx = 8x^2 \log\left(\sqrt{1 + \frac{1}{x^2}} \right); \quad x = \frac{\rho}{a}.
\tag{4.9.358}
$$

4.9.1.2 Entanglement Entropy

The scaling of the EE in this case is given by,

$$\begin{aligned}\Delta r\frac{\partial S_{\mathrm{CFT}_d}}{\partial \Delta r} &= \lim_{q\to 1}\int_{M_q} d^dx\sqrt{g}t^{[1]}_{rr}\\ &= \lim_{q\to 1, a\to 0}\int_{M_{d-2},\psi=0}^{2\pi q}\sqrt{h}d^{d-2}y^i d\psi\left[\int_\epsilon^{\frac{r_2-r_1}{2}}d\rho\rho\left(\frac{C_d^{[1]}}{\rho^d}[1-d\sin^2(\psi)]\right.\right.\\ &\left.\left.+\left(t^{[0]}_{rr}=p\left(1-\frac{d}{2}\right)-d_2+d_1\frac{\cos(4\psi)}{\rho^4}\right)\right)+a^2\int_\epsilon^\infty dxxI(x,\psi)\right] \end{aligned} \tag{4.9.359}$$

$$\begin{aligned}\Delta r\frac{\partial S_{\mathrm{CFT}_d}}{\partial \Delta r} &= \lim_{q\to 1, a\to 0}\pi q Vol(M_{d-2})\left[\int_\epsilon^{\frac{r_2-r_1}{2}}d\rho\rho\left((2-d)\frac{C_d^{[1]}}{\rho^{d-1}}+(2-d)p-2d_2\right)\right.\\ &\left.+a^2p\int_\epsilon^\infty dxx\left(\frac{2(d-1)}{1+x^2}-d\log\left(1+\frac{1}{x^2}\right)\right)\right]\\ &= \lim_{q\to 1}\pi Vol(M_{d-2})\left\{C_d^{[1]}\left(\frac{2^{d-2}}{(\Delta r)^{d-2}}-\frac{1}{\epsilon^{d-2}}\right)+\frac{(2-d)p-2d_2}{8}(\Delta r)^2\right\}\\ &= \pi Vol(M_{d-2})\left\{C_d^{[1]}\left(\frac{2^{d-2}}{(\Delta r)^{d-2}}-\frac{1}{\epsilon^{d-2}}\right)+\frac{p}{4}(\Delta r)^2\right\}. \end{aligned} \tag{4.9.360}$$

We have kept d_1, d_2 arbitrary until the last equality where we have used (4.9.351). As mentioned in (4.6.2), we restrict the ρ integral between 0 to $\frac{r_2-r_1}{2}$ for the regulator-independent homogeneous replica stress tensor contributions, and between 0 to ∞ for the regulator dependent replica stress tensor contributions.

We have chosen a specific regulating function $U(x,a)$. But from the second equality above, we see that the a dependence in the regulator-dependent integral goes as a^2, and hence the integral goes to zero as $a\to 0$ independent of the choice of $U(x,a)$. So the contribution to the scaling of EE is purely from the regulator independent homogeneous part of the replica stress tensor.

When solving the replica trace and conservation equations (4.9.325)-(4.9.328) in the case of a disconnected boundary $\Delta r = L = (r_2 - r_1)$ distance apart, we can consider L to be one more dimensionful parameter that the replica stress tensor $\langle T^\mu_\nu\rangle_q$ can in general depend on. If we follow an approach similar to the vacuum case we find that the stress tensor and EE change as follows

$$t_i^{i[1]} = \frac{(d-2)C_d^{[1]}}{\rho^d}K(\rho/L)+t_i^{i[0]}K_1(\rho/L), \tag{4.9.361}$$

where $K(\rho/L)$ and $K_1(\rho/L)$ are dimensionless functions which tend to 1 as $\rho >> L$, which implies

$$K(\rho, L) = \sum_{i\geq 0} A_i \left(\frac{L}{\rho}\right)^i; \quad K_1(\rho, L) = \sum_{i\geq 0} A_i^{(1)} \left(\frac{L}{\rho}\right)^i; \, i \in \mathbb{Z}. \tag{4.9.362}$$

$$t^{[1]}_{\rho\psi} = \left\{\frac{d_1}{\rho^3} + d_2\rho + \frac{dp}{16} a f(x)\right\} \sin(2\psi), \tag{4.9.363}$$

$$\begin{aligned}
t^{[1]}_{\psi\psi} =& \int d\rho \partial_\psi t^{[1]}_{\rho\psi} - \int d\rho\rho\partial_\rho(\rho t^i{}_i{}^{[1]}) \\
=& \left\{\frac{-d_1}{\rho^2} + d_2\rho^2 + \frac{dp}{8}a^2 \int dx f(x)\right\} \cos(2\psi) + d_3(\psi) \\
& - \frac{\rho^2}{2}(d-2)pK_1(\rho/L) - (d-2)p \int d\rho \frac{\rho^2}{2}\partial_\rho K_1(\rho/L) \\
& + \frac{(1-d)C^{[1]}_d}{\rho^{d-2}} K(\rho, L) + \int d\rho \, \partial_\rho K(\rho, L) \frac{C^{[1]}_d}{\rho^{d-2}},
\end{aligned} \tag{4.9.364}$$

$$\begin{aligned}
t^{[1]}_{\rho\rho} =& - \frac{t^{[1]}_{\psi\psi}}{\rho^2} - t^{i[1]}_i + U^{[1]} t^{[0]}_{\rho\rho} \\
=& \left\{\frac{d_1}{\rho^4} - d_2 - \frac{dp}{8x^2} \int dx f(x)\right\} \cos(2\psi) + U^{[1]} t^{[0]}_{\rho\rho} \\
& - \frac{(d-2)}{2} pK_1(\rho/L) + \frac{(d-2)p}{\rho^2} \int d\rho \, \frac{\rho^2}{2}\partial_\rho K_1(\rho/L) \\
& + \frac{C^{[1]}_d}{\rho^d} K(\rho, L) - \frac{1}{\rho^2} \int d\rho \, \partial_\rho K(\rho, L) \frac{C^{[1]}_d}{\rho^{d-2}},
\end{aligned} \tag{4.9.365}$$

$$
\begin{aligned}
t^{[1]}_{rr} =& t^{[1]}_{\rho\rho}(\cos(\psi))^2 - \frac{2}{\rho}t^{[1]}_{\rho\psi}\cos(\psi)\sin(\psi) + \frac{1}{\rho^2}t^{[1]}_{\psi\psi}(\sin(\psi))^2 \\
&- 2U^{[1]}t^{[0]}_{\rho\rho}(\cos(\psi))^2 - \frac{2U^{[1]}}{\rho}t^{[0]}_{\rho\psi}\cos(\psi)\sin(\psi) \\
=& \frac{C^{[1]}_d}{\rho^d}K(\rho/L)(1 - d\sin^2(\psi)) + p\left(1 - \frac{d}{2}\right)K_1(\rho/L) \\
&- d_2 + d_1\frac{\cos(4\psi)}{\rho^4} + I(x,\psi) + \frac{\cos(2\psi)}{\rho^2} \\
&\int d\rho\left((d-2)p\frac{\rho^2}{2}\partial_\rho K_1(\rho/L) - \partial_\rho K(\rho,L)\frac{C^{[1]}_d}{\rho^{d-2}}\right). \qquad (4.9.366)
\end{aligned}
$$

We have,

$$
\begin{aligned}
\Delta r\frac{\partial S_{d\text{ dim, bulk}}}{\partial\Delta r}\bigg|_{\text{reg indep}} &= \lim_{q\to1}\int_{M_q}d^dx\sqrt{g}t^{[1]}_{rr} \\
=& \lim_{q\to1}\int_\Sigma d^{d-2}x\sqrt{h}\int_0^{2\pi q}d\psi\int_\epsilon^{\alpha L}d\rho\rho\left(\frac{C^{[1]}_d}{\rho^d}K\left(\frac{\rho}{L}\right)\right. \\
&\left.(1 - d\sin^2(\psi)) + p\left(1-\frac{d}{2}\right)K_1\left(\frac{\rho}{L}\right) - d_2\right) \\
=& \pi Vol(M_{d-2})\left[(2-d)\sum_{i\geq0}\left(\frac{C^{[1]}_d}{L^{d-2}}\frac{B_i}{\alpha^{d-2+i}} + p\frac{B^{(1)}_i}{\alpha^{i-2}}L^2\right.\right. \\
&\left.\left.- \frac{C^{[1]}_d B_i}{\epsilon^{d-2}}\frac{L^i}{\epsilon^i} - pB^{(1)}_i\frac{L^i}{\epsilon^{i-2}}\right) - d_2\alpha^2L^2\right], \qquad (4.9.367)
\end{aligned}
$$

where $i \in \mathbb{Z}$ and $B_i = A_i/(2-d-i)$, $B^{(1)}_i = A^{(1)}_i/(2-i)$.

4.10 Conclusions

We begin this section by summarising the main results of this chapter. We have considered the properties of entanglement entropy on a generic static background (4.4.52), (4.5.74). This analysis requires the replica stress tensor, which is determined using the replica stress tensor conservation (4.6.137) as well as other defining properties, such as the trace equation in a CFT. Note that the geometry of the entangling region is captured by the metric on the replica space. On static backgrounds we can uniquely and explicitly solve for the replica stress tensor expectation value using the replica stress tensor conservation equations (4.6.176), (4.6.185), (4.6.186) and trace equation (4.6.183) in the case of a CFT. The $\mathcal{O}(q-1)$ replica stress tensor expectation value completely determines the scaling of the entanglement entropy (see (4.5.115)).

Most computations of entanglement and Rényi entropies have been limited to vacuum states on flat space in specific entangling subregions. The approach presented in this chapter provides a way to calculate the scaling of Rényi and entanglement entropies (and the quantities themselves up to a constant) for general states and subregions on arbitrary backgrounds. It also sets up the methodology required to calculate replica stress tensor correlations in any state given the corresponding stress tensor correlations on the base manifold.

We have presented explicit computations for vacuum and quasithermal states for a subregion bounded by two planes in flat space using the replica metric corresponding to an unsquashed cone (4.3.26). We have also given explicit computations in spherical and cylindrical entangling boundaries in flat space. A natural extension would be to consider other states and subregions on static backgrounds. While our focus has been on specific scaling transformations, we could analyse in a similar way the behaviour of entanglement under generic variations, including deformations of the background metric, using (4.4.52).

Another straightforward extension would be to consider other states (non vacuum and non thermal) on flat backgrounds in higher dimensions. The replica conservation and trace equations that determine $\langle T_{\mu\nu}\rangle^{[1]}$ will now have sources $\langle T_{\mu\nu}\rangle^{[0]}$ and $\mathcal{A}^{[1]}[g^{[q]}_{\mu\nu}]$ (in even dimensions). We could also consider a more complicated entangling boundary geometry. The expansion of the metric around a general foliation is given by (4.3.42) and the corresponding replica metric is given in (4.3.44). Using these in the replica conservation and trace equations gives the replica stress tensor expectation value, and using that one can obtain the scaling of entanglement entropy and also the Rényi entropy for any entangling boundary of interest.

Further analysis of entanglement and its variations in non-flat static backgrounds would use the replica metric (4.3.41) which has explicit dependence on the extrinsic curvature of the entangling boundary. In this generic context the replica conservation and trace equations are more complicated, due to the non-vanishing connection on the base space, although still straightforwardly solvable case by case.

To extend to stationary black holes, one would need to extend the computation of entanglement to the corresponding stationary backgrounds. One way of approaching this would be to make use of the generating boost or rotation transforms that relate stationary and static black holes. For example, the defining equations for the replica stress tensor in the stationary case will be related to those in the static case by the corresponding generating transformations. A more ambitious step would be to extend our analysis to entanglement in generic backgrounds, in which there is no time translation symmetry. Interesting first steps in this direction could follow from considering de Sitter type backgrounds.

Carrollian and celestial CFTs have recently been extensively discussed in the context of holography in asymptotically flat spacetimes. Since our computation is a field theory computation using replica trick and does not rely on holography or on the nature of the asymptotics of the space, one could straightforwardly extend our methodology to compute entanglement entropy in Carrollian and celestial field theories. The trace and conservation equations will be modified, reflecting the underlying symmetry structure of these theories [224–229]. It would be interesting to explore entanglement in such theories, and relate to the proposed holographic descriptions. More generally, explicit computations of S_{QFT} in higher dimensions also facilitate exploration of the quantum focussing conjecture [230], with the possibility of refining bounds further.

Let us now turn to the connections between our analysis and islands of entanglement. The scaling of the entanglement entropy with the entangling region on a static black hole background (4.5.121), which is relevant for probing islands, depends on the correlators in the theory. For a CFT, it depends on the stress tensor correlators. Therefore the condition for the presence of an island discussed in (3.4.14), imposes conditions on these correlators or in other words the spectrum of the field theory governing the outgoing radiation.

Using the scaling of entanglement entropy in static black hole backgrounds computed in this chapter, along with the conditions for islands (3.4.14), one can compute the explicit constraints on the spectrum of the outgoing radiation.

It was shown in [95, 231–233], that in theories with long-range gravity where Gauss law holds, islands which by definition are disconnected from the asymptotic region of spacetime do not constitute consistent entanglement wedges. This is because the energy of an excitation localized to the island can be detected from outside the island. It was argued that this can be resolved if islands appear in conjunction with a massive graviton. Resolutions retaining massless graviton, have been discussed in the context of the de Sitter information paradox[234] and in the presence of black holes with baby universes[235]. It will be interesting to see how our conditions on the scaling of S_{QFT} for the existence of islands, which in turn impose conditions on the QFT spectrum, compare with these.

Part II

Flat Holography

Chapter 5

Carrollian Conformal Theories in Momentum Space

5.1 Introduction

Holography is expected to be an integral feature of any theory of quantum gravity [4, 5]. It was mainly introduced to explain the Bekenstein–Hawking entropy–area law [1, 3, 236], and it asserts that all physical information in a gravitational theory can be represented by a lower–dimensional, non-gravitational theory defined on its boundary. However, incorporating this feature into the framework of quantum gravity has proven to be a challenging task. The current efforts try to understand how holography works in certain fixed classes of spacetimes. E.g., using the insights from string theory, the holographic principle has been concretely realized in the asymptotically Anti de Sitter space times via the AdS/CFT correspondence [6–8]. This correspondence has proven to be a powerful tool for exploring various aspects of quantum gravity.

Despite the above remarkable achievement, it remains to extend the holographic principle to more realistic spacetimes characterized by a vanishing or positive cosmological constant, namely the asymptotically flat or de Sitter spacetimes (for reviews, see [18, 112, 113] and references therein). In fact, flat-space holography is expected to be especially rich [237]. In AdS/CFT, each bulk background (specified by choices of moduli, fluxes, and compactification data) corresponds to a particular boundary theory (possibly a family related by exactly marginal couplings). Flat space behaves differently: an asymptotically flat spacetime can contain very large, weakly curved interior regions where the physics matches different choices of moduli parameters. E.g., it has been argued that starting from type IIA string theory on $R^{3,1} \times T^6$, one can engineer large interior regions inside the flat spacetimes which resemble many other familiar vacua, such as type IIB on $\mathrm{AdS}_5 \times S^5$ or $\mathrm{AdS}_3 \times S^3 \times T^4$, M theory on $\mathrm{AdS}_4 \times S^7$ or $\mathrm{AdS}_7 \times S^4$, M theory on $R^{10,1}$ as well as type IIA and IIB on $R^{9,1}$. Hence, the boundary holographic description of type IIA string theory on $R^{3,1} \times T^6$ must contain information about all these possibilities [237]. This makes flat-space holography both fascinating and challenging, necessitating a dedicated study.

To make progress in our understanding of flat holography, it is fruitful to study this from multiple complementary directions. In particular, some proposals for the dual boundary descriptions have been put forward for the low-energy (supergravity) sector of the bulk. For the massless bulk fields, the expected natural "location" of the dual data is the null infinity ($\mathscr{I}^+ \cup \mathscr{I}^-$) of the asymptotically flat spacetime. The asymptotic symmetry group of asymptotically flat spacetimes is an infinite dimensional extension of the Poincaré group, namely, Bondi-Metzner-Sachs (BMS) group [115, 116]. Building on this fact, there are two closely related approaches for analyzing the physics at the null boundary of asymptotically flat spacetimes.

In the first approach, known as Celestial (CFT) approach, one makes connection between the bulk scattering amplitudes and the correlators on the celestial sphere at infinity [16–25]. For a $(d+1)$-dimensional bulk, the celestial sphere has dimension $(d-1)$ and hence the boundary theory is viewed as a $(d-1)$-dimensional conformal theory living on that sphere [18]. On the other hand, in the second approach, known as Carrollian (CFT) approach [22, 228, 238–265], one relates the $(d+1)$-dimensional bulk dynamics to a $d-$dimensional Carrollian conformal field theory living on the d dimensional null infinity $\mathscr{I}^+ \cup \mathscr{I}^-$ [256]. Both celestial and Carrollian approaches are complementary ways to analyze the theory at null infinity in asymptotically flat space times and have been shown to be equivalent [252].

In this chapter, following [151], we focus on the Carrollian approach. Historically, the Carrollian algebra was introduced as the vanishing speed of light $c \to 0$ limit of the Poincaré algebra [26, 27]. In this limit, the Lorentzian metric becomes degenerate, and the theory becomes ultra-local. Carroll structures also arise naturally on null hypersurfaces, on which the induced metric is degenerate. This is also true for null infinities in particular. A different symmetry group implies that the properties of Carrollian theories are quite different from the standard QFTs we are used to. In particular, the standard analytic properties of Poincaré invariant QFTs do not directly apply to these theories. On the other hand, the bulk theories have well defined analytic properties. Hence, for the duality to work, we need to know how the bulk analytic properties translate to boundary analytic properties.

One way to approach this problem is to first note that the Carrollian conformal theories have mostly been studied in the position representation. In this representation, the analytic properties are not manifest as is also the case for usual Poincare invariant QFTs for which the analytic properties are not transparent in the position space. However, it is well known that the momentum representation makes the analytic properties of traditional QFTs quite manifest. E.g., the physical properties such as locality, causality and unitarity can be translated into some analytic behaviour of the momentum space amplitude. The locality requires some kind of polynomial boundedness of amplitudes for theories involving finite number of derivatives. The causality is ensured by behavior under the analytic continuation of the amplitudes. The unitarity shows up in various dispersion relations and cutting rules of the amplitudes.

This hints that for the Carrollian conformal field theories (CCFT) on the null boundaries, we can hope to get some insights about the analytic properties by analyzing them in the momentum space. The branch cuts and the poles in the momentum space correlators may tell us about these properties. E.g., the analytic structure in energy space along the null direction can tell us about the causality in the boundary Carrollian theory. Momentum space correlators carry information about energy flow direction. If under reversal of energy flow directions, some specific Carrollian correlators vanish or develop singularities, it may signal a breakdown of Carrollian causality.

Another motivation for studying the Carrollian theories in momentum space comes from the fact that asymptotic symmetries of flat spacetimes have been shown to be closely related to soft theorems, memory effects and flat space S matrices (see, e.g.[117, 266]). The relations between these are most naturally expressed in terms of momenta [267–269]. Hence, a momentum space formulation of the flat space boundary dynamics can provide the most direct bridge between these aspects of the bulk gravitational physics.

With these motivations in mind, we study the momentum representation of the Carrollian conformal field theories. Formally, momentum-space correlators can be obtained by Fourier transforming the corresponding position-space expressions. However, as already emphasised in the CFT literature, position-space correlators are typically written for separated points and, in general, do not admit a well-defined Fourier transform prior to renormalisation [129, 270–295]. Our approach will be to first derive the Carrollian conformal Ward identities in the momentum space. The momentum space correlation functions are solutions of these Ward identities. We shall explicitly consider the solutions of these Ward identities for 2 and 3 point functions. Unlike the standard CFTs whose 2 and 3-point functions are completely fixed by the conformal symmetry and are characterized by single terms (for scalar operators), the similar correlation functions of the Carrollian CFTs have more than one term (even for scalar operators)[1]. Further, the 3-point correlation functions are not completely fixed by the symmetry alone. This is another feature of the Carrollian conformal theories. However, this is not unexpected.

[1]We shall refer to the different terms in the Carrollian conformal correlators as different branches. These branches are characterized by different structures of Carrollian energy conserving delta functions (see, e.g., equation (5.4.71) for 3-point case). For 2-point functions, the two branches are also known as electric and magnetic branches.

The amplitudes in the bulk theories are also not completely fixed and if the Carrollian conformal theories have to be their boundary dual, then these Carrollian conformal theories should also have some degree of non uniqueness.

The momentum space Ward identities also have logarithmic solutions for specific values of the Carrollian conformal dimensions. These solutions do not have any direct Fourier analog in the position space. Instead, it turns out that the Fourier transforms of the position space solutions are singular for specific values of the Carrollian conformal dimensions. They need regularization. The logarithmic terms can arise due to the regularization process. This is analogous to the standard CFTs in momentum space where the regularization gives rise to such logarithmic terms and the regularized correlators satisfy the anomalous Ward identities [148, 296, 297]. However, what is different in Carrollian conformal theories is that, here, the logarithmic terms we obtain in the momentum space satisfy the original Ward identities and not the anomalous ones.

We shall also study the boundary Carrollian conformal theories by taking a suitable limit of Lorentzian CFT correlators. It turns out that the Carrollian approach to flat holography can emerge if we consider the flat and ultra-relativistic limits of the bulk AdS and its boundary respectively [253, 258]. These two limits are closely related. In the limit where we take the AdS radius to be very large, the AdS resembles flat space. In this limit, the AdS correlators morph into the flat space S matrices [298, 299]. On the boundary side, the flat limit corresponds to a Carrollian limit where we send the speed of light to zero [253]. In other words, the flat limit of the AdS geometry corresponds to an ultra-relativistic contraction of the boundary conformal algebra [253]. From this perspective, Carrollian holography can be studied using the standard AdS/CFT duality. This approach has the advantage of inheriting well-understood structures from AdS/CFT. Thus, some aspects of both the bulk as well as boundary theories in the flat spacetimes can be studied by taking the flat/Carrollian limit of the AdS/CFT correspondence. In [129, 147, 153, 300], the flat limit has been analyzed for the bulk dynamics directly using the momentum space techniques. Here, we shall consider the Carrollian limit of boundary CFT theory in the momentum space.

The approach to flat holography by using the flat limit of AdS has certain limitations. One way to see this using the boundary side is to note that Carrollian conformal theories on a null hypersurface can be obtained by taking a Inönü-Wigner type contraction of the standard conformal field theories where we send the speed of light c to zero. In this limit, the conformal algebra gives rise to Carrollian conformal algebra. In this Carrollian limit, the CFT correlators become Carrollian conformal correlators. The different branches of 2 and 3-point Carrollian conformal correlators are obtained by different $c \to 0$ scalings in this limit. This approach, which is tightly connected to the ultra-relativistic limit of CFT correlators, reproduces only a restricted set of Carrollian correlators on the null boundary of flat space. Relativistic CFT correlators obey the full conformal Ward identities, which are more constraining than their Carrollian counterparts, and the corresponding space of allowed parameters is therefore smaller. These restrictions persist in the $c \to 0$ limit, since the limit affects the space–time geometry but leaves the intrinsic CFT data unchanged. As a consequence, the Carrollian correlators obtained in this manner realise only a distinguished subclass of the general solutions permitted by Carrollian conformal symmetry. The limitations to this approach to flat holography can also be seen by considering the compactification of 10-dimensional theory to 4 dimensions [301].

In Lorentzian CFTs, the Wightman and time ordered correlators have different interpretations. However, in the Carrollian limit $c \to 0$, the notion of time disappears. Hence, in this limit, both of these types of correlators are expected to give the same results (upto a normalization). We shall consider both the Wightman and time ordered correlators in the momentum space and consider their ultra-relativistic limit. We shall find that in this limit, they indeed give rise to same Carrollian conformal correlators which differ only by some overall constants.

In this chapter, we shall set up all the necessary boundary ingredients for discussing flat–space holography in momentum space. We will discuss the connection with the bulk dynamics in chapter (6).

The rest of the chapter is organized as follows. In section 5.2, we review Carrollian geometry and Carrollian conformal algebra. We also review how the Carrollian conformal algebra arises from the Inönü–Wigner contraction. In section 5.3, we shall summarize 2 and 3 point Carrollian conformal correlators in the position space in arbitrary dimensions. We shall also clarify to what extent the Ward identities fix these correlators. In particular, we clarify what freedom is available in the 3-point correlators. In section 5.4, we derive the momentum space Carrollian conformal Ward identities and solve the resulting differential equations to obtain the 2 and 3 point correlators in the momentum space. In section 5.5, we consider the Inönü-Wigner contraction of the CFT 2 and 3 point functions and show that different branches of 2 and 3-point Carrollian conformal correlators can be obtained by different $c \to 0$ scalings. We end with a discussion in section 5.6. The appendix contains some reviews and details of some derivations. In appendix C.1, we note our conventions and some useful identities. In C.2, we review some aspects of the Lorentzian CFT correlators in the momentum space. In particular, equation (C.2.15) gives a useful representation of fully time ordered 3-point function in momentum space as a convolution over two point functions. This representation will be useful for taking the Carrollian limit of time ordered 3-point function. In appendix C.3, we review some useful delta distribution identities which are used in the analysis of momentum space Ward identities. In appendix C.4, we compute the momentum space Carrollian conformal correlators directly using the Fourier transform of the position space correlators for generic values of the Carrollian conformal dimensions. Finally, in appendix C.5, we give an alternative evaluation of equation (5.5.118) which is required for taking the Carrollian limit of two point function in momentum space. In the main text, it has been derived using the contour methods.

5.2 Review of the Carrollian conformal group and representations

Carrollian geometry can be formulated in several equivalent ways. A geometrically transparent construction arises by considering the geometry induced on null planes in Minkowski spacetime $\mathbb{M}^{d+1}$, with coordinates (X^0, X^i, X^d), where $i = 1, \ldots, d-1$, and metric

$$ds^2 = \eta_{\mu\nu}\, dX^\mu dX^\nu = -\, du\, dv + \delta_{ij}\, dX^i dX^j, \qquad \mu = 0, \ldots, d\ . \tag{5.2.1}$$

Here $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$, and we have introduced the null coordinates

$$u = X^0 - X^d, \qquad v = X^0 + X^d\ . \tag{5.2.2}$$

The hypersurfaces defined by $v =$ constant form a family of null planes $\mathcal{J} \simeq \mathbb{R} \times \mathbb{R}^{d-1}$, each parametrized by the coordinates $x^\alpha = (u, X^i)$, with $\alpha = 0, 1, \ldots, d-1$, induced from the ambient coordinates (X^0, X^i, X^d) onto the hypersurface. Their normal vector $\hat{n}_\mu = \partial_\mu v = (1, 0, \ldots, 1)$ is itself null since $\eta^{\mu\nu}\hat{n}_\mu \hat{n}_\nu = 0$.

The metric on these hypersurfaces can be obtained by pulling back the $(d+1)$–dimensional Minkowski metric onto $v =$ constant. Setting $dv = 0$ in equation (5.2.1), one finds

$$ds^2_{\mathcal{J}} \;= \delta_{ij}\, dX^i dX^j \;\equiv\; q_{\alpha\beta}\, dx^\alpha dx^\beta, \qquad \alpha, \beta = 0, 1, \ldots, d-1. \tag{5.2.3}$$

The metric is degenerate being of the form $q_{\alpha\beta} = (0,\, \delta_{ij})$, and there exists a non-zero vector field n^β satisfying

$$q_{\alpha\beta}\, n^\beta = 0\ . \tag{5.2.4}$$

This vector generates the kernel of $q_{\alpha\beta}$ and defines the null direction of the Carrollian structure. In the present coordinates, it is given by

$$n \equiv \partial_u = (1,\, 0^i), \tag{5.2.5}$$

that is, the tangent vector to the null plane. The pair $(q_{\alpha\beta},\, n = \partial_u)$ thus defines the canonical flat Carrollian structure on the null hypersurface.

This geometry precisely corresponds to that obtained by taking the ultra–relativistic limit $c \to 0$, which leads to a degenerate metric structure in which the temporal direction becomes null, while the spatial sections remain endowed with a non–degenerate Euclidean metric. Mathematically, this limit can be realized through the Inönü–Wigner contraction of the conformal algebra $\mathfrak{so}(2,d)$ in the ultra–relativistic limit $c \to 0$. To see this explicitly, let us start from the d–dimensional Minkowski metric

$$ds^2_{\mathbb{M}^d} = -(dX^0)^2 + \delta_{ij}\, dX^i dX^j, \tag{5.2.6}$$

and restore the dependence on the speed of light. The contraction can be implemented by rescaling the speed of light as [302]

$$c \to \epsilon c \quad \text{with} \quad \epsilon \to 0 \ , \tag{5.2.7}$$

At the level of metric, we write $X^0 = \epsilon\, u$ and then take the limit $\epsilon \to 0$ to recover the degenerate null–plane metric

$$ds^2_{\mathbb{M}^d} \ \longrightarrow \ ds^2_{\mathcal{J}} = 0\, du^2 + \delta_{ij}\, dX^i dX^j \ , \tag{5.2.8}$$

which reproduces the Carrollian geometry induced on $\mathcal{J}$. The algebra of conformal isometries of Minkowski space $\mathbb{M}^d$, $\mathfrak{so}(2,d)$, is given by [303]

$$\begin{aligned}
[\mathcal{D}, \mathcal{P}_\mu] &= i\, \mathcal{P}_\mu, \qquad [\mathcal{D}, \mathcal{K}_\mu] = -\, i\, \mathcal{K}_\mu, \qquad [\mathcal{K}_\mu, \mathcal{P}_\nu] = -\, 2i\, (\eta_{\mu\nu}\mathcal{D} - \mathcal{L}_{\mu\nu})\,, \\
[\mathcal{K}_\rho, \mathcal{L}_{\mu\nu}] &= i\, (\eta_{\rho\mu}\mathcal{K}_\nu - \eta_{\rho\nu}\mathcal{K}_\mu)\,, \qquad [\mathcal{P}_\rho, \mathcal{L}_{\mu\nu}] = i\, (\eta_{\rho\mu}\, \mathcal{P}_\nu - \eta_{\rho\nu}\mathcal{P}_\mu)\,, \\
[\mathcal{L}_{\mu\nu}, \mathcal{L}_{\rho\sigma}] &= i\, (\eta_{\nu\rho}\mathcal{L}_{\mu\sigma} + \eta_{\mu\sigma}\mathcal{L}_{\nu\rho} - \eta_{\mu\rho}\mathcal{L}_{\nu\sigma} - \eta_{\nu\sigma}\mathcal{L}_{\mu\rho})\,.
\end{aligned} \tag{5.2.9}$$

The generators of translations, dilatations, Lorentz transformations, and special conformal transformations are represented in coordinate space, respectively, as

$$\begin{aligned}
\mathcal{P}_\mu &= -\, i\, \partial_\mu, \qquad \mathcal{D} = -\, i\, X^\mu \partial_\mu, \\
\mathcal{L}_{\mu\nu} &= i\, (X_\mu \partial_\nu - X_\nu \partial_\mu)\,, \qquad \mathcal{K}_\mu = i\left(2X_\mu\, X^\nu \partial_\nu - X^2 \partial_\mu\right).
\end{aligned} \tag{5.2.10}$$

The dilatation generator is unaffected by the Inönü–Wigner contraction given in equation (5.2.7), acting trivially on the spatial components of translations, rotations, and special conformal transformations, which remain finite in the limit [26, 27]. The algebra satisfied by these generators is therefore the same as the $\mathfrak{so}(2,d)$ algebra. For the time components, instead, one obtains

$$
\begin{aligned}
&\mathcal{P}_0 \longrightarrow \frac{1}{\epsilon}\,\mathcal{H}, \qquad \text{with} \quad \mathcal{H} = -\,i\,\partial_u, \\
&\mathcal{L}_{0i} = i\,(X_0\,\partial_i - X_i\,\partial_0) \;\Rightarrow\; i\left(\epsilon\,u\,\partial_i - \frac{1}{\epsilon}\,X_i\,\partial_u\right) \simeq \frac{1}{\epsilon}\,\mathcal{B}_i, \qquad \text{with} \quad \mathcal{B}_i = -\,i\,X_i\,\partial_u, \\
&\mathcal{K}_0 \longrightarrow i\left[2\epsilon\,u\left(u\partial_u + X^i\partial_i\right) - \left(-\,\epsilon^2 u^2 + X_i^2\right)\frac{1}{\epsilon}\,\partial_u\right] \simeq \frac{1}{\epsilon}\,\mathcal{K}, \qquad \text{with} \quad \mathcal{K} = -\,i\,X_i^2\,\partial_u\ .
\end{aligned}
$$

These generators satisfy the Carrollian conformal algebra

$$
\begin{aligned}
[\mathcal{L}_{ij},\,\mathcal{B}_k] &= i\,(\delta_{jk}\mathcal{B}_i - \delta_{ik}\mathcal{B}_j)\,, \qquad [\mathcal{P}_j,\,\mathcal{B}_i] = -\,i\,\delta_{ij}\,\mathcal{H}, \\
[\mathcal{D},\,\mathcal{K}] &= -\,i\,\mathcal{K}, \qquad [\mathcal{K},\,\mathcal{H}] = 0, \qquad [\mathcal{K},\,\mathcal{P}_i] = 2\,i\,\mathcal{B}_i, \\
[\mathcal{H},\,\mathcal{K}_i] &= 2\,i\,\mathcal{B}_i, \qquad [\mathcal{D},\,\mathcal{H}] = i\,\mathcal{H} \;\;;\;\; [\mathcal{K}_i,\,\mathcal{B}_j] = -i\delta_{ij}\,\mathcal{K}.
\end{aligned} \tag{5.2.11}
$$

Here the time translation $\mathcal{H}$, the Carrollian boosts $\mathcal{B}_i$, and the temporal conformal generator $\mathcal{K}$ replace their relativistic counterparts in the contracted algebra.

The conformal Carroll algebra in d dimensions is isomorphic to the algebra $\mathfrak{iso}(1,d)$ of the Poincaré group in d+1 dimensions. This can be seen by defining

$$
\begin{aligned}
&J_{ij} = \mathcal{L}_{ij}, \qquad J_{i0} = \tfrac{1}{2}\,(\mathcal{P}_i + \mathcal{K}_i)\,, \qquad J_{id} = \tfrac{1}{2}\,(\mathcal{P}_i - \mathcal{K}_i)\,, \qquad J_{0d} = -\,\mathcal{D}, \\
&\hat{P}_0 = \tfrac{1}{\sqrt{2}}\,(\mathcal{H} + \mathcal{K})\,, \qquad \hat{P}_i = -\,\sqrt{2}\,\mathcal{B}_i, \qquad \hat{P}_d = \tfrac{1}{\sqrt{2}}\,(\mathcal{K} - \mathcal{H})\,.
\end{aligned} \tag{5.2.12}
$$

The set $\{J_{MN},\,\hat{P}_M\}$ with $M, N = 0, \ldots, d$, then satisfies the commutation relations of the (d+1)-dimensional Poincaré algebra. In other words,

$$
\mathfrak{ccarr}(d) \;\simeq\; \mathfrak{iso}(1,d)\ . \tag{5.2.13}
$$

This correspondence provides a geometrical interpretation: the Carrollian conformal symmetries acting on a null hypersurface in d dimensions are isomorphic to the Poincaré symmetries of a (d+1)-dimensional Minkowski spacetime. Moreover, the quadratic Casimir of the (d+1)–dimensional Poincaré algebra, which is isomorphic to the Carrollian conformal algebra, vanishes identically,

$$
C_2 = \eta^{MN}\,\hat{P}_M\hat{P}_N = 0\ , \tag{5.2.14}
$$

reflecting the fact that the associated momentum vector is null in the (d+1)-dimensional Minkowski space $\mathbb{R}^{1,d}$ underlying $\mathfrak{iso}(1,d)$. The finite-dimensional conformal Carrollian algebra admits an infinite-dimensional enhancement, in which the generators $\mathcal{H}$, $\mathcal{B}_i$, and $\mathcal{K}$ are promoted to the supertranslation generator $M_f = f(x^i)\,\partial_t$ (see, for example, [304]). This algebra is isomorphic to the BMS algebra and is crucial if we consider the null hypersurface to be the null boundaries of the asymptotically flat spacetimes.

However, in this chapter we restrict our analysis to its finite-dimensional conformal Carrollian version.

The transformation properties of generic rank-s tensor fields $\mathcal{O}_{\mathbf{s}}(x^\alpha) \equiv \mathcal{O}_{i_1 \dots i_s}(x^\alpha)$, forming irreducible tensor representations of the Carrollian $SO(d-1)$ rotation group, are constructed by first analysing the stability subgroup of the origin and then extending them to arbitrary points through the action of the translation generators $\mathcal{P}_i$ and $\mathcal{H}$. The stability subgroup contains the Carrollian generators $\{\mathcal{J}_{ij},\, \mathcal{D},\, \mathcal{B}_i,\, \mathcal{K}_i,\, \mathcal{K}\}$. The generators of spatial rotations in $\mathbb{R}^{d-1}$, namely $\mathcal{J}_{ij}$, and the dilatation generator $\mathcal{D}$ commute and can therefore be simultaneously diagonalized (see for example [251, 305])

$$[\mathcal{J}_{ij},\, \mathcal{O}_{\mathbf{s}}(0)] = i\,\Sigma_{ij}\,\mathcal{O}_{\mathbf{s}}(0) \qquad , \qquad [\mathcal{D},\, \mathcal{O}_{\mathbf{s}}(0)] = i\,\Delta\,\mathcal{O}_{\mathbf{s}}(0), \tag{5.2.15}$$

where Σ_{ij} denotes the representation of the $SO(d{-}1)$ spin operator and Δ is the Carrollian weight (or scaling dimension) of $\mathcal{O}_{\mathbf{s}}$. Furthermore, since the generators $\mathcal{B}_i$ and $\mathcal{K}_i$ transform as vectors under $SO(d{-}1)$, their non-trivial action on the representation containing $\mathcal{O}_{\mathbf{s}}(0)$ would produce an infinite tower of higher-rank fields. To ensure a finite-dimensional representation, one therefore imposes

$$[\mathcal{B}_i,\, \mathcal{O}_{\mathbf{s}}(0)] = [\mathcal{K}_i,\, \mathcal{O}_{\mathbf{s}}(0)] = 0\ . \tag{5.2.16}$$

From the Carrollian algebra we also have:

$$\begin{aligned}[\mathcal{K},\, \mathcal{O}_{\mathbf{s}}(0)] &= \frac{\delta^{ij}}{d-1}\left[[\mathcal{K}_i,\, \mathcal{O}_{\mathbf{s}}(0)]\,,\, \mathcal{B}_j\right] + [\mathcal{K},\, \mathcal{O}_{\mathbf{s}}(0)] - \frac{\delta^{ij}}{d-1}\left[[\mathcal{B}_j,\, \mathcal{O}_{\mathbf{s}}(0)]\,,\, \mathcal{K}_j\right] \\ &= 0 \end{aligned} \tag{5.2.17}$$

The operators $\mathcal{K}_\alpha = (\mathcal{K},\, \mathcal{K}_i)$ and $\mathcal{P}_\alpha = (\mathcal{H},\, \mathcal{P}_i)$ satisfy the commutation relations

$$\begin{aligned} \Big[\mathcal{D},\, \Big[\mathcal{K}_\alpha,\, \mathcal{O}_{\mathbf{s}}(0)\Big]\Big] &= i\,(\Delta-1)\,\Big[\mathcal{K}_\alpha,\, \mathcal{O}_{\mathbf{s}}(0)\Big], \\ \Big[\mathcal{D},\, \Big[\mathcal{P}_\alpha,\, \mathcal{O}_{\mathbf{s}}(0)\Big]\Big] &= i\,(\Delta+1)\,\Big[\mathcal{P}_\alpha,\, \mathcal{O}_{\mathbf{s}}(0)\Big]. \end{aligned} \tag{5.2.18}$$

Hence, $\mathcal{K}_\alpha$ and $\mathcal{P}_\alpha$ act respectively as the lowering and raising operators of the Carrollian conformal dimension. Equation (5.2.16) can thus be interpreted as defining the operator associated with the lowest-weight state of conformal dimension Δ. By analogy with standard conformal field theory, this condition characterizes a *primary Carrollian field*. The field at a generic point of the null plane is obtained from its value at the origin by the action of the Carrollian translation operators $\mathcal{P}_\alpha$, namely,

$$\mathcal{O}_{\mathbf{s}}(x) = e^{\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)}\,\mathcal{O}_{\mathbf{s}}(0)\,e^{-\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)}\ . \tag{5.2.19}$$

The action of the translation operators on a generic element $\mathcal{A}$ of the Carrollian conformal algebra, at the level of commutation relations, is given by

$$[\mathcal{A},\, \mathcal{O}_{\mathbf{s}}(x)] = e^{i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)}\Big[e^{-\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)}\mathcal{A}e^{\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)},\, \mathcal{O}_{\mathbf{s}}(0)\Big]\, e^{-\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)}\;. \tag{5.2.20}$$

This relation expresses how commutators involving $\mathcal{O}_{\mathbf{s}}(0)$ are mapped to those involving $\mathcal{O}_{\mathbf{s}}(x)$ under finite Carrollian translations. By using Hadamard's lemma[2] we can write

$$[\mathcal{H},\, \mathcal{O}_{\mathbf{s}}(x)] \;=\; e^{\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)}\Big[\mathcal{H},\, \mathcal{O}_{\mathbf{s}}(0)\Big]\, e^{-\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)} = -\,i\,\partial_u\mathcal{O}_{\mathbf{s}}(x)\,. \tag{5.2.21}$$

Similarly, one finds[3]

$$[\mathcal{P}_i,\, \mathcal{O}_{\mathbf{s}}(x)] = -\,i\,\partial_i\mathcal{O}_{\mathbf{s}}(x)\,. \tag{5.2.22}$$

For the $SO(d-1)$ rotation generators and for dilatations we have

$$\begin{aligned} e^{-\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)}\mathcal{L}_{ij}e^{\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)} &= \mathcal{L}_{ij} + (x_i\mathcal{P}_j - x_j\mathcal{P}_i)\,, \\ e^{-\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)}\mathcal{D}e^{\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)} &= \mathcal{D} -\, x^\alpha\mathcal{P}_\alpha\,. \end{aligned} \tag{5.2.23}$$

These relations give

$$\begin{aligned} [\mathcal{L}_{ij},\, \mathcal{O}_{\mathbf{s}}(x)] &= e^{ix^\alpha\mathcal{P}_\alpha}\,[\mathcal{L}_{ij},\, \mathcal{O}_{\mathbf{s}}(0)]\, e^{-ix^\alpha\mathcal{P}_\alpha} - i(x_j\partial_i - x_i\partial_j)\mathcal{O}_{\mathbf{s}}(x) \\ &= -\,i\,[-\,\Sigma_{ij} + (x_j\partial_i - x_i\partial_j)]\,\mathcal{O}_{\mathbf{s}}(x)\,, \\ [\mathcal{D},\, \mathcal{O}_{\mathbf{s}}(x)] &= e^{ix^\alpha\mathcal{P}_\alpha}\,[\mathcal{D},\, \mathcal{O}_{\mathbf{s}}(0)]\, e^{-ix^\alpha\mathcal{P}_\alpha} + i\, x^\alpha\, \partial_\alpha\mathcal{O}_{\mathbf{s}}(x) \\ &= i\,\Big[\,\Delta + u\partial_u + x^i\partial_i\Big]\,\mathcal{O}_{\mathbf{s}}(x)\,. \end{aligned} \tag{5.2.24}$$

Finally, the last commutation relations that will be used below are

$$\begin{aligned} e^{-\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)}\mathcal{K}\,e^{\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)} &= \mathcal{K} - 2x^i\,\mathcal{B}_i\, + x^i\,x_i\,\mathcal{H}\,, \\ e^{-\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)}\mathcal{B}_i\,e^{\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)} &= \mathcal{B}_i - x_i\,\mathcal{H}\,, \\ e^{-\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)}\mathcal{K}_i\,e^{\,i\,(u\,\mathcal{H}+x^i\mathcal{P}_i)} &= \mathcal{K}_i + 2u\mathcal{B}_i + 2x_i\mathcal{D} - 2x^j\mathcal{L}_{ij} \\ &\quad -2ux_i\mathcal{H} + x^2\mathcal{P}_i - 2x^jx_i\mathcal{P}_j\,. \end{aligned} \tag{5.2.25}$$

These relations imply

$$\begin{aligned} [\mathcal{K}, \mathcal{O}_{\mathbf{s}}(x)] &= -\,i\,x^2\,\partial_u\mathcal{O}_{\mathbf{s}}(x)\,, \qquad [\mathcal{B}_i, \mathcal{O}_{\mathbf{s}}(x)] = i\,x_i\,\partial_u\mathcal{O}_{\mathbf{s}}(x)\,, \\ [\,\mathcal{K}_i,\, \mathcal{O}_{\mathbf{s}}(x)\,] &= i\Big(2x_i\,\Delta + 2x_i\,u\,\partial_u + 2x_i\,x^j\,\partial_j - x^2\partial_i - 2x^k\,\Sigma_{ik}\Big)\mathcal{O}_{\mathbf{s}}(x)\,. \end{aligned} \tag{5.2.26}$$

[2]The version of Hadamard's lemma used here is $e^ABe^{-A} = \sum_{n=0}^{\infty}\frac{1}{n!}[A,B]^n$, where $[A,B]^n \equiv [A,[A,\ldots,[A,B]\ldots]]$ denotes the n-fold nested commutator.

[3]In these expressions we have used the identity $\partial_\alpha\mathcal{O}_{\mathbf{s}}(x) = i\,e^{ix^\alpha\mathcal{P}_\alpha}\,[\mathcal{P}_\alpha,\, \mathcal{O}_{\mathbf{s}}(0)]\,e^{-ix^\alpha\mathcal{P}_\alpha}$.

The commutation relations obtained above determine how all Carrollian generators act on fields at a generic spacetime point x, once their action at the origin is specified. They provide the essential ingredients for realizing Carrollian conformal symmetry on both scalar and spinning fields. In the following sections, we shall extensively employ them, in particular for scalar fields, to construct the CCFT correlators.

5.3 Position space Carrollian correlators

The study of Carrollian conformal correlators has so far been mostly carried out in position space, and primarily in specific spacetime dimensions. In arbitrary dimensions, a systematic analysis was presented in [254] (also see [261]). Here, we summarize the structure of the two and three point Carrollian conformal correlators involving scalar operators in the position space.

We denote a generic Carrollian conformal correlator involving spinning operators $\mathcal{O}_{\mathbf{s}}$ of spin s and conformal dimension Δ_s as

$$\mathcal{A}(u_1, \mathbf{x}_1; \cdots ; u_n, \mathbf{x}_n) = \Big\langle \, \mathcal{O}_{\mathbf{s}_1}(x_1) \cdots \mathcal{O}_{\mathbf{s}_n}(x_n) \, \Big\rangle, \qquad x \equiv (u, \mathbf{x}) \,. \tag{5.3.27}$$

The variation of a field under the action of a generic CCFT generator, denoted by $\mathcal{T}_A$, is defined as

$$\delta \mathcal{O}_{\mathbf{s}}(x) = [\mathcal{T}_A, \, \mathcal{O}_{\mathbf{s}}(x)] = i \, T_A \, \mathcal{O}_{\mathbf{s}}(x) \,, \qquad A = 1, \cdots, \dim[\mathfrak{ccarr}(d)], \tag{5.3.28}$$

where T_A denotes the differential operator that realizes the action of $\mathcal{T}_A$ on the field $\mathcal{O}_{\mathbf{s}}(x)$, as given in the previous section.

Assuming that the vacuum is invariant under the CCFT symmetry group, i.e.,

$$\mathcal{T}_A \, |0\rangle = 0 \,, \tag{5.3.29}$$

the action of $\mathcal{T}_A$ on an n-point function gives

$$\langle 0 | \, \mathcal{T}_A \prod_{a=1}^{n} \phi(x_a) \, |0\rangle \quad = \quad \sum_{a=1}^{n} \Big\langle 0 \Big| \, \phi(x_1) \cdots [\mathcal{T}_A, \, \phi(x_a)] \cdots \phi(x_n) \Big| 0 \Big\rangle = 0 \,. \tag{5.3.30}$$

Using $[\mathcal{T}_A, \mathcal{O}_{\mathbf{s}}(x_a)] = i \, T_A^{(a)} \, \mathcal{O}_{\mathbf{s}}(x_a)$, one obtains the Ward identities

$$\sum_{a=1}^{n} T_A^{(a)} \Big\langle \phi(x_1) \cdots \phi(x_n) \Big\rangle = 0 \,. \tag{5.3.31}$$

Specialising the CCFT algebra to the case of correlation functions leads to the Ward identities obeyed by Carrollian correlators. In d dimensional position space, they take the following form

$$\begin{aligned}
\mathcal{H} \quad &: \quad \sum_{a=1}^{n} \frac{\partial}{\partial u_a}\, \mathcal{A}(u_1, \mathbf{x}_1; \cdots; u_n, \mathbf{x}_n) = 0 \\
\mathcal{P}_i \quad &: \quad \sum_{a=1}^{n} \frac{\partial}{\partial x_a^i}\, \mathcal{A}(u_1, \mathbf{x}_1; \cdots; u_n, \mathbf{x}_n) = 0 \\
\mathcal{B}_i \quad &: \quad \sum_{a=1}^{n} x_a^i \frac{\partial}{\partial u_a}\, \mathcal{A}(u_1, \mathbf{x}_1; \cdots; u_n, \mathbf{x}_n) = 0 \\
\mathcal{J}_{ij} \quad &: \quad \sum_{a=1}^{n} \Big(-\Sigma_{ij}^{(a)} - x_i^a\, \partial_j^a + x_j^a\, \partial_i^a\Big)\, \mathcal{A}(u_1, \mathbf{x}_1; \cdots; u_n, \mathbf{x}_n) = 0 \\
\mathcal{D} \quad &: \quad \sum_{a=1}^{n} \left(\Delta_a + u_a\, \partial_{u_a} + x_a^i \frac{\partial}{\partial x_a^i}\right) \mathcal{A}(u_1, \mathbf{x}_1; \cdots; u_n, \mathbf{x}_n) = 0 \\
\mathcal{K}_i \quad &: \quad \sum_{a=1}^{n} \left(2x_i^a\, \Delta_a - 2\, x_a^j\, \Sigma_{ij}^{(a)} + 2u_a\, x_i^a\, \partial_{u_a} + 2x_i^a\, x_a^j\, \partial_j^a - x_a^2\, \partial_i^a\right) \mathcal{A}(u_1, \mathbf{x}_1; \cdots; u_n, \mathbf{x}_n) = 0\,, \\
\mathcal{K} \quad &: \quad \sum_{a=1}^{n} x_a^2 \frac{\partial}{\partial u_a}\, \mathcal{A}(u_1, \mathbf{x}_1; \cdots; u_n, \mathbf{x}_n) = 0
\end{aligned} \tag{5.3.32}$$

where $i = 1, 2, \ldots, d-1$ and $x_a^2 \equiv \delta_{ij}\, x_a^i x_a^j$.

We next review how to derive the two and three point correlation functions by solving the position space Ward identities.

5.3.1 Two point function

In the two point case, the $\mathcal{H}$ and $\mathcal{P}_i$ Ward identities simply require that the correlator depends only on the differences of the coordinates,

$$\mathcal{A}(u_1, \mathbf{x}_1;\, u_2, \mathbf{x}_2) \equiv \mathcal{A}(u_{12},\, \mathbf{x}_{12})\,,$$

where $u_{12} = u_1 - u_2$ and $\mathbf{x}_{12} = \mathbf{x}_1 - \mathbf{x}_2$. The boost Ward identity associated with $\mathcal{B}_i$ imposes

$$(x_1^i \partial_{u_1} + x_2^i \partial_{u_2})\, G^{(2)}(u_{12},\, \mathbf{x}_{12}) = x_{12}^i\, \partial_{u_1} G^{(2)}(u_{12},\, \mathbf{x}_{12}) = 0\,, \tag{5.3.33}$$

The above equation implies that the most general two point function can only be of the form

$$G^{(2)}(u_{12},\, \mathbf{x}_{12}) = G(\mathbf{x}_{12}) + F(u_{12})\, \delta^{(d-1)}(\mathbf{x}_{12})\,. \tag{5.3.34}$$

This structure naturally splits into two independent branches. The first term, $G(\mathbf{x}_{12})$, is entirely supported on the spatial slice $\mathbb{R}^{d-1}$ and depends only on the spatial separation

$\mathbf{x}_{12}$. It is usually referred as the *magnetic branch.* The second term, $F(u_{12})\,\delta^{(d-1)}(\mathbf{x}_{12})$, is localized in space and depends solely on the Carrollian "time" coordinate u_{12}. It is usually referred as the *electric branch.* The action of the dilatation generator imposes the constraint

$$\Big(u_{12}\,\partial_{u_{12}} + x^i_{12}\,\partial_{x^i_{12}} + \Delta_1 + \Delta_2\Big)\Big[G(\mathbf{x}_{12}) + F(u_{12})\,\delta^{(d-1)}(\mathbf{x}_{12})\Big] = 0\,. \qquad (5.3.35)$$

Expanding this expression yields[4]

$$\begin{aligned}&\Big(\Delta_1 + \Delta_2 + x^i_{12}\,\partial_{x^i_{12}}\Big)G(\mathbf{x}_{12}) = 0;\\ &\Big(\Delta_1 + \Delta_2 - (d-1) + u_{12}\,\partial_{u_{12}}\Big)F(u_{12}) = 0\,.\end{aligned} \qquad (5.3.36)$$

The first equation controls the scaling behaviour of the spatial (magnetic) branch, while the second one constrains the temporal (electric) branch through its dependence on u_{12}. This condition ensures the full invariance of the magnetic branch under the Carrollian special conformal transformation. One can check that the 2-point function consistent with the above Ward identities is given by [254]

$$G(\mathbf{x}_{12}) = \frac{B_1}{|\mathbf{x}_{12}|^{\Delta_1+\Delta_2}}\delta_{\Delta_1,\Delta_2} \quad ; \quad F(u_{12}) = \frac{B_2}{|u_{12}|^{\Delta_1+\Delta_2-d+1}} \qquad (5.3.37)$$

Finally, note that the presence of the delta function, which enforces $x^i_1 = x^i_2$, implies that the term $F(u_{12})\,\delta^{(d-1)}(\mathbf{x}_{12})$ automatically satisfies the $\mathcal{K}_i$ Ward identity for arbitrary values of Δ_1 and Δ_2. In contrast, the spatial contribution $G(\mathbf{x}_{12})$ solves the $\mathcal{K}_i$ Ward identity only when the two operators have equal conformal dimensions. Thus, we finally arrive at following expression of two point function

$$\Big\langle\, \mathcal{O}_1(u_1,\mathbf{x}_1)\mathcal{O}_2(u_2,\mathbf{x}_2)\Big\rangle = \frac{B_1}{|\mathbf{x}_{12}|^{\Delta_1+\Delta_2}}\delta_{\Delta_1,\Delta_2} + \frac{B_2}{|u_{12}|^{\Delta_1+\Delta_2-d+1}}\delta^{d-1}(\mathbf{x}_{12}) \qquad (5.3.38)$$

5.3.2 Three point function

For the 3-point function also, the invariance under the $\mathcal{H}$ and $\mathcal{P}_i$ generators imply, as in the two-point case, that the correlator depends only on the coordinate differences. However, only two independent coordinate differences can be formed using three points since the third one is linearly dependent on the other two. E.g.,

$$u_{13} = u_{12} - u_{23}\,, \qquad x^i_{13} = x^i_{12} - x^i_{23}\,.$$

[4]In deriving the second identity we used the standard homogeneity property of the Dirac delta distribution, $x^i\,\partial_{x^i}\,\delta^{(d-1)}(\mathbf{x}) = -(d-1)\,\delta^{(d-1)}(\mathbf{x})$.

Therefore, it is convenient to express the three-point correlator as a function of two independent variables. One possible choice is, for instance,

$$\Big\langle \mathcal{O}_1(u_1, \mathbf{x}_1)\mathcal{O}_2(u_2, \mathbf{x}_2)\mathcal{O}_3(u_3, \mathbf{x}_3)\Big\rangle \equiv \mathcal{A}_3(u_{12}, \mathbf{x}_{12};\, u_{23}, \mathbf{x}_{23}) \tag{5.3.39}$$

The Ward identity associated with the Carrollian boost generator $\mathcal{B}_i$, with the choice made in equation (5.3.39) can be expressed as

$$\left(x^i_{12}\partial_{u_{12}} + x^i_{23}\partial_{u_{23}}\right)\mathcal{A}_3(u_{12}, \mathbf{x}_{12};\, u_{23}, \mathbf{x}_{23}) = 0 \tag{5.3.40}$$

Equation (5.3.40) implies that the three-point correlator is either completely independent of the Carrollian temporal variables u_{ab}, or any dependence on u_{ab} can only appear multiplied by the corresponding spatial delta functions $\delta^{(d-1)}(\mathbf{x}_{ab})$. In other words, Carrollian boost invariance implies that the correlator can be decomposed into five distinct sectors: a *purely magnetic branch*, depending only on the spatial separations $\mathbf{x}_{ab}$; a *purely electric branch*, fully localized in space through the products of $\delta^{(d-1)}(\mathbf{x}_{ab})$ and depending solely on the Carrollian "time" variables u_{ab}; and three *mixed branches*, which involve the temporal differences u_{ab} and are multiplied by the corresponding spatial delta functions $\delta^{(d-1)}(\mathbf{x}_{ab})$ [254][5]

$$\begin{aligned}\Big\langle \mathcal{O}_1(u_1, \mathbf{x}_1)\mathcal{O}_2(u_2, \mathbf{x}_2)\mathcal{O}_3(u_3, \mathbf{x}_3)\Big\rangle \\ = \; & G_1(u_1, u_2, u_3)\delta^{d-1}(\mathbf{x}_{12})\delta^{d-1}(\mathbf{x}_{23}) + G_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) + G_3(u_{23}, \mathbf{x}_{12})\delta^{d-1}(\mathbf{x}_{23}) \\ & + G_4(u_{31}, \mathbf{x}_{23})\delta^{d-1}(\mathbf{x}_{31}) + G_5(u_{12}, \mathbf{x}_{31})\delta^{d-1}(\mathbf{x}_{12})\end{aligned} \tag{5.3.41}$$

The remaining Ward identities fix the precise form of the functions G_i. We now describe these functions one by one.

The function G_1 is not completely fixed by Carrollian conformal invariance. In particular, the Carrollian special conformal Ward identity does not impose any additional constraint not implied by other Ward identities. The Carrollian dilatation Ward identity implies that G_1 must be a homogeneous function of degree $2(d-1) - \Delta_t$ (where $\Delta_t = \Delta_1 + \Delta_2 + \Delta_3$), i.e.,

$$G_1(\lambda u_1, \lambda u_2, \lambda u_3) = \lambda^{2d-2-\Delta_t} G_1(u_1, u_2, u_3) \tag{5.3.42}$$

[5] Ref. [254] shows that the most general solution to the $\mathcal{B}_i$ Ward identity also includes collinear contributions. However, as demonstrated in the same reference, once the $\mathcal{K}_i$ Ward identity is imposed, these terms can be consistently reabsorbed into the definition of the magnetic branch.

Any function having the form[6]

$$G_1(u_1,u_2,u_3) = (u_{23})^{2d-2-\Delta_t} f\left(\frac{u_{12}}{u_{23}}\right), \tag{5.3.43}$$

where f is an arbitrary function, will satisfy all the Ward identities. In particular, the dilatation Ward identity will be satisfied by (5.3.43) since

$$\sum_{a=1}^{3} u_a \frac{\partial}{\partial u_a} G_1(u_1,u_2,u_3) = (2d-2-\Delta_t)\, G_1(u_1,u_2,u_3) \tag{5.3.44}$$

and

$$\sum_{a=1}^{3} x_a^i \frac{\partial}{\partial x_a^i} \delta^{d-1}(\mathbf{x}_{12})\delta^{d-1}(\mathbf{x}_{23}) = -2(d-1). \tag{5.3.45}$$

An example of a function which satisfies above equations is [254]

$$G_1(u_1,u_2,u_3) = \sum_{p+q+r=\Delta_1+\Delta_2+\Delta_3-2(d-1)} \frac{C_1}{|u_{12}|^p |u_{23}|^q |u_{13}|^r} \tag{5.3.46}$$

However, this is only one example of a function allowed by the dilatation Ward identity. Any function consistent with the form (5.3.43) is also acceptable and will satisfy all the Carrollian conformal Ward identities.

The dilatation and $\mathcal{K}_i$ Ward identities for the magnetic term G_2 coincide with those of a Euclidean $(d-1)$-dimensional CFT. Therefore, the solution is identical to the one in that case, namely

$$G_2(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = \frac{C_2}{|\mathbf{x}_{12}|^{\Delta_1+\Delta_2-\Delta_3}|\mathbf{x}_{23}|^{\Delta_2+\Delta_3-\Delta_1}|\mathbf{x}_{13}|^{\Delta_1+\Delta_3-\Delta_2}} \tag{5.3.47}$$

For the mixed branches, for example $G_3(u_{23},\mathbf{x}_{12})$, the dilatation Ward identity takes the form

$$\left(\Delta_t + u_{23}\frac{\partial}{\partial u_{23}} + x_{12}^i \frac{\partial}{\partial x_{12}^i} - (d-1)\right) G_3(u_{23},\mathbf{x}_{12}) = 0. \tag{5.3.48}$$

It is solved by the function

$$G_3(u_{23},\mathbf{x}_{12}) = \frac{C_3}{|\mathbf{x}_{12}|^a |u_{23}|^b} \quad , \qquad a+b = \Delta_t - (d-1) \tag{5.3.49}$$

[6]Naively, one can assume the function f to be a function of two ratios, namely, $f(\frac{u_{13}}{u_{23}}, \frac{u_{12}}{u_{23}})$. However, the second ratio $\frac{u_{13}}{u_{23}}$ can be written in terms of the other since

$$\frac{u_{13}}{u_{23}} = \frac{u_{12}}{u_{23}} + 1$$

The Carrollian special conformal Ward identity instead turns out to be

$$\begin{aligned}
&\Big(x_1^i\,\Delta_1 + x_2^i\,\Delta_2 + x_3^i\,\Delta_3 + (u_2 x_2^i - u_3 x_3^i)\,\frac{\partial}{\partial u_{23}} + (x_1^i x_1^j - x_2^i x_2^j)\,\frac{\partial}{\partial x_{12}^j} - \frac{1}{2}(x_1^2 - x_2^2)\,\frac{\partial}{\partial x_{12\,i}} \\
&+ (x_2^i x_2^j - x_3^i x_3^j)\,\frac{\partial}{\partial x_{23}^j} - \frac{1}{2}(x_2^2 - x_3^2)\,\frac{\partial}{\partial x_{23\,i}}\Big)\,G_3(u_{23}, \mathbf{x}_{12})\,\delta^{(d-1)}(\mathbf{x}_{23}) \\
&= \left(\Delta_1 - \frac{a}{2}\right) x_{12}^i G_3(u_{23}, \mathbf{x}_{12})\,\delta^{(d-1)}(\mathbf{x}_{23}) \\
&= 0 \qquad (5.3.50)
\end{aligned}$$

This gives

$$a = 2\Delta_1 \qquad , \qquad b = \Delta_2 + \Delta_3 - \Delta_1 - (d-1) \qquad (5.3.51)$$

The remaining functions in (5.3.41) are fixed in the same way by the Carrollian conformal Ward identities and are given by

$$\begin{aligned}
G_4(u_{31}, \mathbf{x}_{23}) &= \frac{C_4}{|\mathbf{x}_{23}|^{2\Delta_2}\,|u_{31}|^{\Delta_3+\Delta_1-\Delta_2-d+1}} \\
G_5(u_{12}, \mathbf{x}_{31}) &= \frac{C_5}{|\mathbf{x}_{31}|^{2\Delta_3}\,|u_{12}|^{\Delta_1+\Delta_2-\Delta_3-d+1}} \qquad (5.3.52)
\end{aligned}$$

The constants C_i appearing in the expressions of G_i will, in general, depend upon d and Δ_i.

5.4 Carrollian correlators from momentum space Ward identities

As mentioned earlier, most of the results involving the Carrollian conformal correlators are only available in the position space. However, motivated by the close connection between the Carrollian conformal field theories, flat holography and scattering amplitudes, it is desirable to have a systematic momentum space formulation of the Carrollian theories. In this section, we obtain the 2 and 3 point Carrollian conformal correlators in the momentum space. We shall first derive the Ward identities in the momentum space. These Ward identities will then be used to find the differential equations constraining the 2 and 3 point correlations functions. These differential equations will then be solved to determine the explicit form of the 2 and 3 point correlators in the momentum space.

5.4.1 Momentum space Ward identities

To write the Ward identities in the momentum space, we consider the Fourier transform

$$\mathcal{O}_{\mathbf{s}}(\omega,\mathbf{p}) = \int d^{d-1}x\, du\, \mathcal{O}_{\mathbf{s}}(u,\mathbf{x}) e^{i(u\,\omega + x^i p_i)} \quad ; \qquad p_\mu \equiv (\omega,\, \mathbf{p}) \tag{5.4.53}$$

We now multiply the Ward identities in (5.3.32) with $e^{i(u_a\omega_a + x^i_a p_{ia})}$ from left and integrate over the position variables. For the terms involving the derivatives, we perform an integration by parts, ignoring the boundary terms, and write the surviving coordinates u_a and x^i_a as derivatives with respect to ω_a and p^i_a, respectively. Finally, using the above definition of the transform, we get the momentum space Ward identities

$$\begin{aligned}
\mathcal{H} \quad &: \quad \sum_{a=1}^{n} \omega_a \mathcal{A}\left(\omega_1, \mathbf{p}_1, \ldots, \omega_n,\, \mathbf{p}_n\right) = 0 \\
\mathcal{P}_i \quad &: \quad \sum_{a=1}^{n} p^i_a\, \mathcal{A}\left(\omega_1, \mathbf{p}_1, \ldots, \omega_n,\, \mathbf{p}_n\right) = 0 \\
\mathcal{B}_i \quad &: \quad \sum_{a=1}^{n} \omega_a \frac{\partial}{\partial p^a_i} \mathcal{A}\left(\omega_1, \mathbf{p}_1, \ldots, \omega_n,\, \mathbf{p}_n\right) = 0 \\
\mathcal{J}_{ij} \quad &: \quad \sum_{a=1}^{n} \left[\Sigma^a_{ij} - i\left(p^a_j \frac{\partial}{\partial p^a_i} - p^a_i \frac{\partial}{\partial p^a_j}\right)\right] \mathcal{A}\left(\omega_1, \mathbf{p}_1, \ldots, \omega_n,\, \mathbf{p}_n\right) = 0 \\
\mathcal{D} \quad &: \quad \sum_{a=1}^{n} \left[\Delta_a - d - \omega_a \frac{\partial}{\partial \omega_a} - p^i_a \frac{\partial}{\partial p^i_a}\right] \mathcal{A}\left(\omega_1, \mathbf{p}_1, \ldots, \omega_n,\, \mathbf{p}_n\right) = 0 \\
\mathcal{K}_i \quad &: \quad \sum_{a=1}^{n} \left[\Delta_a \frac{\partial}{\partial p^i_a} + \Sigma^a_{ij} \frac{\partial}{\partial p_{aj}} - d \frac{\partial}{\partial p^i_a} - \omega_a \frac{\partial^2}{\partial p^i_a \partial \omega_a} - p_{aj} \frac{\partial^2}{\partial p^i_a \partial p_{aj}} + \frac{1}{2} p_{ai} \frac{\partial^2}{\partial p^j_a \partial p_{aj}}\right] \mathcal{A} = 0 \\
\mathcal{K} \quad &: \quad \sum_{a=1}^{n} \omega_a \frac{\partial^2}{\partial p^a_i\, \partial p_{ia}} \mathcal{A}\left(\omega_1, \mathbf{p}_1, \ldots, \omega_n,\, \mathbf{p}_n\right) = 0
\end{aligned} \tag{5.4.54}$$

where, we have defined the momentum space correlator $\mathcal{A}\left(\omega_1, \mathbf{p}_1, \ldots, \omega_n,\, \mathbf{p}_n\right)$ by

$$\mathcal{A}\left(\omega_1, \mathbf{p}_1, \ldots, \omega_n,\, \mathbf{p}_n\right) = \int \prod_{a=1}^{n} d^d x_a e^{i\sum_{b=1}^{n}(u_b\omega_b + x^i_b\, p^b_i)} \Big\langle\, \mathcal{O}_{\mathbf{s}_1}(x_1) \ldots \mathcal{O}_{\mathbf{s}_n}(x_n) \Big\rangle \tag{5.4.55}$$

Note that the correlators appearing in the Ward identities (5.4.54) include the energy momentum conserving delta functions. This is in contrast to the momentum space Ward identities one considers in usual CFTs where the overall energy momentum conserving delta functions are brought outside the Ward identity generators [306]. The reason for this is that in the Carrollian case, the Ward identities allow for more general solutions where we may not necessarily have the usual overall energy and momentum conserving delta functions as we shall see below.

5.4.2 Ward identity constraints

5.4.2.1 2-point function

We start by finding the consequences of different Ward identities on the 2-point correlation function of scalar fields. The $\mathcal{P}_i$ Ward identity implies that the function $\mathcal{A}$ must take the form

$$\mathcal{A}_2(\omega_1, \mathbf{p}_1, \omega_2, \mathbf{p}_2) \;=\; \delta^{d-1}(\mathbf{p}_1 + \mathbf{p}_2)\tilde{\mathcal{A}}_2(\omega_1, \mathbf{p}_1, \omega_2, \mathbf{p}_2) \tag{5.4.56}$$

On the other hand, the $\mathcal{H}$ Ward identity implies

$$\begin{aligned}\mathcal{A}_2(\omega_1, \mathbf{p}_1, \omega_2, \mathbf{p}_2) =&\delta^{d-1}(\mathbf{p}_1 + \mathbf{p}_2)\\ &\Big[\delta(\omega_1+\omega_2)F_1(\omega_1, \mathbf{p}_1, \omega_2, \mathbf{p}_2) + \delta(\omega_1)\delta(\omega_2)F_2(\omega_1, \mathbf{p}_1, \omega_2, \mathbf{p}_2)\Big]\end{aligned}$$

Note that in the second term, the product of delta functions $\delta(\omega_1)\delta(\omega_2)$ sets the ω_1 and ω_2 to zero and hence we can write

$$F_2(\omega_1, \mathbf{p}_1, \omega_2, \mathbf{p}_2) = F_2(\mathbf{p}_1, \mathbf{p}_2) \tag{5.4.57}$$

Ultra-local contact terms supported at vanishing momenta, $\delta^{d-1}(\mathbf{p}_1)\delta^{d-1}(\mathbf{p}_2)$ are, in principle, compatible with translation invariance, but are ruled out by the full set of Carrollian Ward identities.

Next, we consider the consequences of $\mathcal{B}_i$ Ward identity

$$\begin{aligned}0 =&\left(\omega_1\frac{\partial}{\partial p_i^1} + \omega_2\frac{\partial}{\partial p_i^2}\right)\\ &\left[\delta^{d-1}(\mathbf{p}_1 + \mathbf{p}_2)\Big(\delta(\omega_1+\omega_2)F_1(\omega_1, \mathbf{p}_1, \omega_2, \mathbf{p}_2) + \delta(\omega_1)\delta(\omega_2)F_2(\mathbf{p}_1, \mathbf{p}_2)\Big)\right]\\ =&\delta(\omega_1+\omega_2)\left(\omega_1\frac{\partial}{\partial p_i^1} + \omega_2\frac{\partial}{\partial p_i^2}\right)\left[\delta^{d-1}(\mathbf{p}_1 + \mathbf{p}_2)F_1(\omega_1, \mathbf{p}_1, \omega_2, \mathbf{p}_2)\right]\\ =&2\delta(\omega_1+\omega_2)\delta^{d-1}(\mathbf{p}_1 + \mathbf{p}_2)\omega_1\frac{\partial}{\partial p_i^1}F_1(\omega_1, \mathbf{p}_1, \omega_2)\end{aligned} \tag{5.4.58}$$

This can be satisfied if F_1 is chosen to be independent of the momenta along the $\mathbb{R}^{d-1}$ directions, i.e.,

$$\frac{\partial}{\partial p_i^1}F_1(\omega_1, \mathbf{p}_1, \omega_2) = 0 \qquad\Longrightarrow\qquad F_1 = F_1(\omega_1, \omega_2) \tag{5.4.59}$$

Thus, the 2-point function takes the form

$$\mathcal{A}_2(\omega_1, \mathbf{p}_1, \omega_2, \mathbf{p}_2) \; = \; \delta^{d-1}(\mathbf{p}_1 + \mathbf{p}_2)\Big[\delta(\omega_1 + \omega_2)F_1(\omega_1, \omega_2) + \delta(\omega_1)\delta(\omega_2)F_2(\mathbf{p}_1, \mathbf{p}_2)\Big] \tag{5.4.60}$$

The $\mathcal{J}_{ij}$ Ward identity can be satisfied if we assume that the correlator depends only upon the magnitude of the momenta p_i^a along the R^{d-1} directions. Let's denote the magnitude of the vector p_a^i as

$$k_a \;\; \equiv \;\; |\mathbf{p}_a| = \sqrt{\sum_{i=1}^{d-1}(p_a^i)^2} \tag{5.4.61}$$

For later applications, we note the following chain rule identities

$$\frac{\partial}{\partial p^i} \;\; = \;\; \frac{p_i}{k}\frac{\partial}{\partial k} \quad ; \qquad \frac{\partial^2}{\partial p^i \partial p^j} \;\; = \;\; \left[\frac{\delta_{ij}}{k} - \frac{p_i\, p_j}{k^3}\right]\frac{\partial}{\partial k} + \frac{p_i\, p_j}{k^2}\frac{\partial^2}{\partial k^2} \tag{5.4.62}$$

$$p^j\,\frac{\partial^2}{\partial p^i \partial p^j} \;\; = \;\; p_i\frac{\partial^2}{\partial k^2} \quad ; \qquad p_i\frac{\partial^2}{\partial p_j \partial p^j} \;\; = \;\; \frac{(d-2)p_i}{k}\frac{\partial}{\partial k} + p_i\frac{\partial^2}{\partial k^2} \tag{5.4.63}$$

Now, acting with the $\mathcal{J}_{ij}$ operator and using the identity (C.3.22), we see that $\mathcal{J}_{ij}$ trivially annihilates the first term of (5.4.60). Similarly, the action of $\mathcal{J}_{ij}$ on the second term gives

$$\begin{aligned}
&\sum_{a=1}^{2}\left[-i\left(p_j^a\frac{\partial}{\partial p_i^a} - p_i^a\frac{\partial}{\partial p_j^a}\right)\right]\delta^{d-1}(\mathbf{p}_1 + \mathbf{p}_2)F_2(k_1) \\
&\qquad = \;\; -i\left[\frac{\partial}{\partial P^i}\delta^{d-1}(\mathbf{P})\sum_{a=1}^{2}p_a^j - \frac{\partial}{\partial P^j}\delta^{d-1}(\mathbf{P})\sum_{a=1}^{2}p_a^i\right]F_2(k_1) \\
&\qquad\qquad -i\delta^{d-1}(\mathbf{p}_1 + \mathbf{p}_2)\sum_{a=1}^{2}\left[-i\left(p_j^a\, p_i^a - p_i^a\, p_j^a\right)\right]\frac{1}{k_1}\frac{\partial}{\partial k_1}F_2(k_1) \\
&\qquad = \;\; 0
\end{aligned} \tag{5.4.64}$$

with $P^i \equiv \sum_{a=1}^{n} p_a^i$ and $n = 2$.

The $\mathcal{K}$-Ward identity is trivially satisfied. The first term in (5.4.60) is annihilated due to the identity (C.3.25) while the 2nd term is annihilated since we have $\omega_a = 0$ due to the delta functions.

Next, we apply the dilation Ward identity. Acting with the dilatation generator, we get

$$\begin{aligned}
0 &= \sum_{a=1}^{2}\left[\Delta_a - d - \omega_a\frac{\partial}{\partial\omega_a} - p_a^i\frac{\partial}{\partial p_a^i}\right]\mathcal{A}_2(\omega_1,\mathbf{p}_1,\omega_2,\,\mathbf{p}_2)\\
&= (\Delta_1+\Delta_2-2d)\delta^{d-1}\left(\sum_{b=1}^{2}\mathbf{p}_b\right)\left[\delta\left(\omega_1+\omega_2\right)F_1(\omega_1,\omega_2)+\delta(\omega_1)\delta(\omega_2)\,F_2(\mathbf{p}_1,\mathbf{p}_2)\right]\\
&\quad -\delta(\omega_1)\delta(\omega_2)\delta^{d-1}(\mathbf{P})\left[-(1+d)F_2(\mathbf{p}_1,\mathbf{p}_2)+\sum_{a=1}^{2}p_a^i\frac{\partial}{\partial p_a^i}F_2(\mathbf{p}_1,\mathbf{p}_2)\right]\\
&\quad -\delta^{d-1}(\mathbf{P})\delta\left(\omega_1+\omega_2\right)\left[-dF_1(\omega_1,\omega_2)+\sum_{a=1}^{2}\omega_a\frac{\partial}{\partial\omega_a}F_1(\omega_1,\omega_2)\right] \qquad (5.4.65)
\end{aligned}$$

The above Ward identity is satisfied if the functions F_1 and F_2 satisfy

$$\left[\Delta_1+\Delta_2-d-\omega_1\frac{\partial}{\partial\omega_1}\right]F_1(\omega_1) = 0 \quad ; \qquad (5.4.66)$$

$$\left[\Delta_1+\Delta_2-(d-1)-p_1^i\frac{\partial}{\partial p_1^i}\right]F_2(\mathbf{p}_1) = 0. \qquad (5.4.67)$$

The solutions of those differential equations are given by

$$F_1(\omega_1)=\omega_1^{\Delta_1+\Delta_2-d} \quad ; \qquad F_2(\mathbf{p}_1)=|\mathbf{p}_1|^{\Delta_1+\Delta_2-d+1} \qquad (5.4.68)$$

Finally, we need to consider the special conformal Ward identity $\mathcal{K}_i$. The action of $\mathcal{K}_i$ generator on the 2-point function is given by

$$\begin{aligned}
&\sum_{a=1}^{2}\left[(\Delta_a-d)\frac{\partial}{\partial p_a^i}-\omega_a\frac{\partial^2}{\partial p_a^i\partial\omega_a}-p_{aj}\frac{\partial}{\partial p_a^i\partial p_{aj}}+\frac{1}{2}p_{ai}\frac{\partial^2}{\partial p_a^j\partial p_{ai}}\right]\mathcal{A}_2(\omega_1,\mathbf{p}_1,\omega_2,\mathbf{p}_2)\\
=\;&\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2)\delta(\omega_1+\omega_2)\\
&\sum_{a=1}^{2}\left[(\Delta_a-d)\frac{\partial}{\partial p_a^i}-p_{aj}\frac{\partial^2}{\partial p_{aj}\partial\omega_a}-p_{aj}\frac{\partial^2}{\partial p_a^i\partial p_{aj}}+\frac{1}{2}p_{ai}\frac{\partial^2}{\partial p_a^j\partial p_{a_j}}\right]F_1\\
&+\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2)\delta(\omega_1)\delta(\omega_2)\\
&\sum_{a=1}^{2}\left[(\Delta_a-(d-1))\frac{\partial}{\partial p_a^i}-p_{aj}\frac{\partial^2}{\partial p_a^i\partial p_a^j}+\frac{1}{2}p_{ai}\frac{\partial^2}{\partial p_a^j\partial p_{aj}}\right]F_2
\end{aligned}$$

where, we have again used the delta function identities to commute the delta functions across the special conformal generator. The first term involving F_1 now vanishes since it does not depend upon the momenta p_a^i. From the 2nd term, we get

$$p_{1i}\left[\left(\Delta_1-\frac{d}{2}\right)\frac{1}{k_1}\frac{\partial}{\partial k_1}-\frac{1}{2}\frac{\partial^2}{\partial k_1^2}\right]F_2(\mathbf{p}_1,\mathbf{p}_2)=0 \qquad (5.4.69)$$

This is the special conformal generator for a Euclidean CFT in $d-1$ dimensions and as is well known, it imposes the condition $\Delta_1 = \Delta_2$.

5.4.2.2 3-point function

For the 3-point correlator, the $\mathcal{P}_i$ Ward identity again imposes the overall spatial momentum conservation

$$\mathcal{A}_3(\omega_1, \mathbf{p}_1, \omega_2, \mathbf{p}_2, \omega_3, \mathbf{p}_3) = \delta^{d-1}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)\tilde{\mathcal{A}}_3(\omega_1, \mathbf{p}_1, \omega_2, \mathbf{p}_2, \omega_3, \mathbf{p}_3) \tag{5.4.70}$$

On the other hand, the $\mathcal{H}$ Ward identity implies

$$\begin{aligned}
&\mathcal{A}_3(\omega_1, \mathbf{p}_1, \omega_2, \mathbf{p}_2, \omega_3, \mathbf{p}_3) \\
&\quad = \delta^{d-1}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)\Big[\delta(\omega_1+\omega_2+\omega_3)F_1 + \delta(\omega_1)\delta(\omega_2)\delta(\omega_3)F_2 \\
&\qquad + \delta(\omega_1)\delta(\omega_2+\omega_3)F_3 + \delta(\omega_2)\delta(\omega_1+\omega_3)F_4 + \delta(\omega_3)\delta(\omega_1+\omega_2)F_5\Big]
\end{aligned} \tag{5.4.71}$$

Thus, we have a total of 5 independent functions appearing in the 3-point Carrollian correlator of scalar operators. Each of these functions depends upon the 6 variables ω_i and $\mathbf{p}_i$ (for $i = 1, 2, 3$) to begin with. For further restricting the form of the functions F_i, we need to consider the rest of the Ward identities. We consider them one by one.

$\mathcal{B}_i$ Ward identity

The functions F_i appearing in the 3-point function are all independent. Hence, each term in (5.4.71) must be annihilated by the $\mathcal{B}_i$ Ward identity individually. For the F_1 term, this means

$$\begin{aligned}
0 &= \delta(\omega_1+\omega_2+\omega_3)\left(\omega_1\frac{\partial}{\partial p_i^1} + \omega_2\frac{\partial}{\partial p_i^2} + \omega_3\frac{\partial}{\partial p_i^3}\right)\delta^{d-1}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)F_1 \\
&= \frac{\partial}{\partial P^i}\delta^{d-1}(\mathbf{P})(\omega_1+\omega_2+\omega_3)\delta(\omega_1+\omega_2+\omega_3)F_1 \\
&\quad + \delta(\omega_1+\omega_2+\omega_3)\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \\
&\quad \left[\omega_1\left(\frac{\partial}{\partial p_1^i} - \frac{\partial}{\partial p_3^i}\right) + \omega_2\left(\frac{\partial}{\partial p_2^i} - \frac{\partial}{\partial p_3^i}\right)\right] F_1(\omega_1, \mathbf{p}_1, \omega_2, \mathbf{p}_2) \\
&= 0 + \delta(\omega_1+\omega_2+\omega_3)\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\left[\omega_1\frac{\partial}{\partial p_1^i} + \omega_2\frac{\partial}{\partial p_2^i}\right] F_1(\omega_1, \mathbf{p}_1, \omega_2, \mathbf{p}_2)
\end{aligned} \tag{5.4.72}$$

In going from the first to second line, we have used the identity (C.3.19). In going from the second to third line, we have used the fact that F_1 can be made independent of $\mathbf{p}_3^i$ (due to the delta function) and hence its' derivative with respect to $\mathbf{p}_3^i$ vanishes. The above equation now implies

$$\frac{\partial}{\partial p_1^i}F_1(\omega_1,\,\mathbf{p}_1,\,\omega_2,\,\mathbf{p}_2) \;=\; 0 \;=\; \frac{\partial}{\partial p_2^i}F_1(\omega_1,\,\mathbf{p}_1,\,\omega_2,\,\mathbf{p}_2) \;\Rightarrow\; F_1 \equiv F_1(\omega_1,\omega_2) \tag{5.4.73}$$

The $\mathcal{B}_i$ Ward identity on F_2 term is trivially satisfied due to the presence of the product of delta function $\delta(\omega_1)\delta(\omega_2)\delta(\omega_3)$. On the F_3 term, we get

$$\begin{aligned} 0 &= \delta(\omega_1)\delta(\omega_2+\omega_3)\left(\omega_2\frac{\partial}{\partial p_i^2}+\omega_3\frac{\partial}{\partial p_i^3}\right)\delta^{d-1}(\mathbf{p}_1+\,\mathbf{p}_2+\,\mathbf{p}_3)F_3 \\ &= \delta(\omega_1)\,\delta(\omega_2+\omega_3)\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\omega_2\left(\frac{\partial}{\partial p_2^i}-\frac{\partial}{\partial p_3^i}\right)F_3(\mathbf{p}_1;\,\omega_2,\,\mathbf{p}_2;\,\omega_3,\,\mathbf{p}_3) \end{aligned}$$

The above equation will be satisfied provided

$$\frac{\partial}{\partial p_2^i}F_3(\mathbf{p}_1;\,\omega_2,\,\mathbf{p}_2;\,\omega_3,\,\mathbf{p}_3) = \frac{\partial}{\partial p_3^i}F_3(\mathbf{p}_1;\,\omega_2,\,\mathbf{p}_2;\,\omega_3,\,\mathbf{p}_3) \tag{5.4.74}$$

This gives

$$F_3(\mathbf{p}_1;\,\omega_2,\,\mathbf{p}_2;\,\omega_3,\,\mathbf{p}_3) = F_3(\omega_2,\,\mathbf{p}_2+\mathbf{p}_3) \equiv F_3(\omega_2,\,\mathbf{p}_1) \equiv F_3(\omega_3,\,\mathbf{p}_1) \tag{5.4.75}$$

where we have discarded constant contributions, independent of $\mathbf{p}_2$ and $\mathbf{p}_3$, that carry no dynamical information. In writing the above expression, we have implicitly made use of the delta functions $\delta(\omega_1)\,\delta(\omega_2+\omega_3)\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)$ which multiplies F_3 term. The analysis for F_4 and F_5 follows exactly as for F_3 implying

$$F_4 = F_4(\omega_1,\,\mathbf{p}_2) \quad ; \qquad F_5 = F_5(\omega_1,\,\mathbf{p}_3) \tag{5.4.76}$$

Thus, the $\mathcal{B}_i$ generator restricts the correlator to have the form

$$\begin{aligned} &\mathcal{A}_3(\omega_1,\,\mathbf{p}_1;\,\omega_2,\,\mathbf{p}_2,\,\omega_3,\,\mathbf{p_3}) \\ &\quad= \delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\Big[\delta(\omega_1+\omega_2+\omega_3)F_1(\omega_1,\,\omega_2)+\delta(\omega_1)\delta(\omega_2)\delta(\omega_3)F_2(\mathbf{p}_1,\,\mathbf{p}_2) \\ &\qquad+\delta(\omega_1)\delta(\omega_2+\omega_3)F_3(\omega_2,\,\mathbf{p}_1)+\delta(\omega_2)\delta(\omega_1+\omega_3)F_4(\omega_1,\,\mathbf{p}_2) \\ &\qquad+\delta(\omega_3)\delta(\omega_1+\omega_2)F_5(\omega_1,\mathbf{p}_3)\Big] \end{aligned}$$

$\mathcal{K}$ Ward identity

The action of $\mathcal{K}$ generator on the F_1 term is given by

$$\begin{aligned} 0 &= \delta(\omega_1+\omega_2+\omega_3)F_1(\omega_1,\omega_2)\sum_{a=1}^{3}\omega_a\frac{\partial^2}{\partial p_a^i\partial p_{ia}}\delta^{d-1}(\mathbf{P}) \\ &= \delta(\omega_1+\omega_2+\omega_3)F_1(\omega_1,\omega_2)\sum_{a=1}^{3}\omega_a\frac{\partial^2}{\partial P^i\partial P_i}\delta^{d-1}(\mathbf{P}) \end{aligned} \tag{5.4.77}$$

The last equation is trivially satisfied. The term involving F_2 is also trivially annihilated by $\mathcal{K}$ generator due to the delta function over each p_a. For F_3, we have

$$\begin{aligned} &\sum_{a=1}^{3}\omega_a\frac{\partial^2}{\partial p_a^i\partial p_{ia}}\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\delta(\omega_1)\delta(\omega_2+\omega_3)F_3(\omega_2,\,\mathbf{p}_1) \\ =\;& \delta(\omega_1)\delta(\omega_2+\omega_3)F_3(\omega_2,\,\mathbf{p}_1)\,\omega_2\left(\frac{\partial^2}{\partial P^i\partial P_i}-\frac{\partial^2}{\partial P^i\partial P_i}\right)\delta^{d-1}(\mathbf{P}) \\ &+\delta(\omega_1)\delta(\omega_2+\omega_3)\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)F_3(\omega_2,\,\mathbf{p}_1)\omega_2\left(\frac{\partial^2}{\partial p_2^i\partial p_{i2}}-\frac{\partial^2}{\partial p_3^i\partial p_{i3}}\right)F_3(\omega_2,\,\mathbf{p}_1) \end{aligned}$$

The first term in the RHS of above expression identically vanishes. The second term also vanishes if we note that we can write $F_3(\omega_2,\mathbf{p}_1)=F_3(\omega_2,-\mathbf{p}_2-\mathbf{p}_3)$ due to the presence of delta function over spatial momenta. The analysis of the action of $\mathcal{K}$ generators on the F_4 and F_5 terms is exactly identical to the F_3 case. Thus, we see that $\mathcal{K}$ Ward identity does not impose any constraint on F_i.

$\mathcal{J}_{ij}$ Ward identity

The $\mathcal{J}_{ij}$ Ward-identity imposes the $SO(d-1)$ invariance along R^{d-1}. It implies that the correlator can depend only upon the magnitude of the momenta along these directions. This can be shown explicitly by observing that the generator $\mathcal{J}_{ij}$ commutes with $\delta^{(d-1)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)$ and by using the identities given in equation (5.4.62). Thus, denoting the magnitude of the momenta p_a^i by k_a, the $\mathcal{J}_{ij}$ Ward identity implies

$$\begin{aligned} &\mathcal{A}_3(\omega_1,\,\mathbf{p}_1;\,\omega_2,\,\mathbf{p}_2,\,\omega_3,\,\mathbf{p}_3) \\ &\quad=\;\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\Big[\delta(\omega_1+\omega_2+\omega_3)F_1(\omega_1,\,\omega_2)+\delta(\omega_1)\delta(\omega_2)\delta(\omega_3)F_2(k_1,\,k_2) \\ &\qquad+\delta(\omega_1)\delta(\omega_2+\omega_3)F_3(\omega_2,\,k_1)+\delta(\omega_2)\delta(\omega_1+\omega_3)F_4(\omega_1,\,k_2) \\ &\qquad+\delta(\omega_3)\delta(\omega_1+\omega_2)F_5(\omega_1,k_3)\Big] \end{aligned}$$

Dilatation Ward identity

Again, each term in the 3-point function must be annihilated by the dilatation generator. On the first term, it gives

$$\begin{aligned} 0 &= \delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\delta(\omega_1+\omega_2+\omega_3) \\ &\quad \left[\Delta_t - 3\,d + 1 + (d-1) - \sum_{a=1}^{2}\omega_a\frac{\partial}{\partial\omega_a}\right] F_1(\omega_1,\omega_2) \end{aligned}$$

where we have defined $\Delta_t = \Delta_1+\Delta_2+\Delta_3$ and used the delta function identities (C.3.23).

The action of dilatation generator on F_2 gives

$$\begin{aligned} 0 &= (\Delta_t - 3\,d)\prod_{a=1}^{3}\delta(\omega_a)\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)F_2 \\ &\quad -\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)F_2\sum_{a=1}^{3}\omega_a\frac{\partial}{\partial\omega_a}\prod_{b=1}^{3}\delta(\omega_b) \\ &\quad -\prod_{b=1}^{3}\delta(\omega_b)F_2\sum_{a=1}^{3}p_a^i\frac{\partial}{\partial p_a^i}\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \\ &\quad -\prod_{b=1}^{3}\delta(\omega_b)\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\sum_{a=1}^{2}p_a^i\frac{\partial}{\partial p_a^i}F_2 \\ &= \delta(\omega_1)\delta(\omega_2)\delta(\omega_3)\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \\ &\quad \left[\Delta_t - 3\,d + 3 + (d-1) - \sum_{a=1}^{3}k_a\frac{\partial}{\partial k_a}\right] F_2 \end{aligned} \tag{5.4.78}$$

where we have used the identity

$$\begin{aligned} \sum_{a=1}^{3}\omega_a\frac{\partial}{\partial\omega_a}\prod_{b=1}^{3}\delta(\omega_b) = \sum_{a=1}^{3}\omega_a\frac{\partial}{\partial\omega_a}\delta(\omega_a)\prod_{b\neq a}\delta(\omega_b) &= -\sum_{a=1}^{3}\delta(\omega_a)\prod_{b\neq a}\delta(\omega_b) \\ &= -3\prod_{a=1}^{3}\delta(\omega_a) \end{aligned}$$

Next, the action of dilatation generator on F_3 gives

$$\begin{aligned} 0 = \delta(\omega_1)\delta(\omega_2+\omega_3)\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \\ \left[\Delta_t - 2d + 1 - \omega_2\frac{\partial}{\partial\omega_2} - k_1\frac{\partial}{\partial k_1}\right] F_3(\omega_2,k_1) \end{aligned} \tag{5.4.79}$$

where, we have used the identity

$$\sum_{a=1}^{3}p_a^i\frac{\partial}{\partial p_a^i}F_3(\omega_2,\,k_1) \quad = \quad k_1\frac{\partial}{\partial k_1}F_3(\omega_2,k_1) \tag{5.4.80}$$

and

$$\begin{aligned}\sum_{a=1}^{3}\omega_a\frac{\partial}{\partial\omega_a}\delta(\omega_1)\delta(\omega_2+\omega_3) &= \delta(\omega_2+\omega_3)\omega_1\frac{\partial}{\partial\omega_1}\delta(\omega_1)+\delta(\omega_1)\sum_{a=2}^{3}\omega_a\frac{\partial}{\partial\omega_a}\delta(\omega_2+\omega_3)\\ &= -\delta(\omega_2+\omega_3)\delta(\omega_1)-\delta(\omega_1)\delta(\omega_2+\omega_3)\\ &= -2\delta(\omega_1)\delta(\omega_2+\omega_3) \qquad (5.4.81)\end{aligned}$$

The analysis for F_4 and F_5 is identical to the case of F_3. Hence, we can write the resulting condition for them by changing the momentum labels in (5.4.79). Collecting all the results, the dilatation Ward identity gives following differential equations for F_i

$$\left[\Delta_t-2\,d-\sum_{a=1}^{2}\omega_a\frac{\partial}{\partial\omega_a}\right]F_1(\omega_1,\omega_2) = 0 \qquad (5.4.82)$$

$$\left[\Delta_t-2d+2-\sum_{a=1}^{3}k_a\frac{\partial}{\partial k_a}\right]F_2(k_1,k_2) = 0 \qquad (5.4.83)$$

$$\left[\Delta_t-2d+1-\omega_2\frac{\partial}{\partial\omega_2}-k_1\frac{\partial}{\partial k_1}\right]F_3(\omega_2,k_1) = 0 \qquad (5.4.84)$$

$$\left[\Delta_t-2d+1-\omega_1\frac{\partial}{\partial\omega_1}-k_2\frac{\partial}{\partial k_2}\right]F_4(\omega_1,k_2) = 0 \qquad (5.4.85)$$

$$\left[\Delta_t-2d+1-\omega_1\frac{\partial}{\partial\omega_1}-k_3\frac{\partial}{\partial k_3}\right]F_5(\omega_1,k_3) = 0 \qquad (5.4.86)$$

Carrollian special conformal Ward identity $\mathcal{K}_i$

Finally, we consider the effect of the carrollian special conformal Ward identity $\mathcal{K}_i$ on the 3-point function. As before, the corresponding generator acts on each F_i independently. For the F_1 term, we have (denoting $\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3$ as before)

$$\begin{aligned}0 &= \sum_{a=1}^{3}\left[(\Delta_a-d)\frac{\partial}{\partial p_{ai}}-\omega_a\frac{\partial^2}{\partial p_a^i\partial\omega_a}-p_{aj}\frac{\partial^2}{\partial p_a^i\partial p_{aj}}+\frac{1}{2}p_{ai}\frac{\partial^2}{\partial p_a^j\partial p_{aj}}\right]\\ &\qquad \delta(\omega_1+\omega_2+\omega_3)\,\delta^{d-1}(\mathbf{P})F_1\\ &= \delta(\omega_1+\omega_2+\omega_3)\frac{\partial}{\partial P^i}\delta^{d-1}(\mathbf{P})\left[\Delta_t-2d-\sum_{a=1}^{3}\omega_a\frac{\partial}{\partial\omega_a}\right]F_1(\omega_1,\omega_2) \quad (5.4.87)\end{aligned}$$

where we have used identities (C.3.23), (C.3.26) and (C.3.27) in going from the first to second line. The differential operator in the above expression is same as the one appearing in the dilatation Ward identity (5.4.82) on F_1. Hence, the Carrollian special conformal Ward identity does not impose any additional constraint on F_1.

Next, the action of $\mathcal{K}_i$ generator on F_2 is given by

$$\begin{aligned}
0 \;=\; & \sum_{a=1}^{3}\left[(\Delta_a-d)\frac{\partial}{\partial p_{ai}}-\omega_a\frac{\partial^2}{\partial p_a^i\partial\omega_a}-p_{aj}\frac{\partial^2}{\partial p_a^i\partial p_{aj}}+\frac{1}{2}p_{ai}\frac{\partial^2}{\partial p_a^j\partial p_{aj}}\right] \\
& \delta^{d-1}(\mathbf{P})\prod_{b=1}^{3}\delta(\omega_b)\,F_2(k_1,\,k_2,\,k_3) \\
=\; & \prod_{b=1}^{3}\delta(\omega_b)\frac{\partial}{\partial P^i}\delta^{d-1}(\mathbf{P})\left[\sum_{a=1}^{3}(\Delta_a-d)+3+d-\sum_{a=1}^{3}p_a^j\frac{\partial}{\partial p_a^j}-1\right]F_2 \\
& +\prod_{b=1}^{3}\delta(\omega_b)\frac{\partial}{\partial P^j}\delta^{d-1}(\mathbf{P})\left[-\sum_{a=1}^{3}\left(p_a^j\frac{\partial}{\partial p_a^i}-p_{ai}\frac{\partial}{\partial p_a^j}\right)\right]F_2 \\
& +\prod_{b=1}^{3}\delta(\omega_b)\delta^{d-1}(\mathbf{P})\left[\sum_{a=1}^{3}(\Delta_a-d+1)\frac{\partial}{\partial p_a^i}-p_a^j\frac{\partial^2}{\partial p_a^i\partial p_a^j}+\frac{1}{2}p_{ai}\frac{\partial^2}{\partial p_a^j\partial p_{aj}}\right]F_2
\end{aligned} \tag{5.4.88}$$

The first line of the above expression vanishes by the dilatation Ward identity (5.4.83) and the second line vanishes by $\mathcal{J}_{ij}$ Ward identity. The 3rd line is the standard special conformal Ward identity in an Euclidean CFT in $(d-1)$ dimensions (see for example [306, 307]). Expressing the operator in the last line in terms of the derivatives with respect to the magnitudes of momenta, we get the equation satisfied by F_2 to be

$$\sum_{a=1}^{3}p_a^i\left[\left(\Delta_a-\frac{d}{2}\right)\frac{1}{k_a}\frac{\partial}{\partial k_a}-\frac{1}{2}\frac{\partial^2}{\partial k_a^2}\right]F_2 \;=\; 0 \tag{5.4.89}$$

Next, we consider the action of the $\mathcal{K}_i$ generator on the F_3 term. We shall evaluate the action of the different terms in $\mathcal{K}_i$ generator one by one. The first term is given by

$$\begin{aligned}
& \sum_{a=1}^{3}(\Delta_a-d)\;\frac{\partial}{\partial p_a^i}\delta(\omega_1)\,\delta(\omega_2+\omega_3)\delta^{d-1}(\mathbf{P})\,F_3(\omega_2,k_1) \\
& =\delta(\omega_1)\delta(\omega_2+\omega_3) \\
& \left\{(\Delta_t-3\,d)\,F_3\frac{\partial}{\partial P^i}\delta^{d-1}(\mathbf{P})+(\Delta_2+\Delta_3-2d)\delta^{d-1}(\mathbf{P})\frac{p_2^i+p_3^i}{k_1}\frac{\partial}{\partial k_1}F_3(\omega_2,\,k_1)\right\}
\end{aligned}$$

The second term gives

$$\begin{aligned}
& -\sum_{a=1}^{3}\omega_a\frac{\partial^2}{\partial\omega_a\partial p_a^i}\delta(\omega_1)\delta(\omega_2+\omega_3)\,\delta^{d-1}(\mathbf{P})\,F_3(\omega_2,\,k_1) \\
& =\;\delta(\omega_1)\,\delta(\omega_2+\omega_3) \\
& \left[\frac{\partial}{\partial P^i}\delta^{d-1}(\mathbf{P})\left(2-\omega_2\frac{\partial}{\partial\omega_2}\right)F_3+\delta^{d-1}(\mathbf{P})(p_2^i+p_3^i)\left(\frac{1}{k_1}\frac{\partial}{\partial k_1}-\frac{\omega_2}{k_1}\frac{\partial^2}{\partial k_1\partial\omega_2}\right)F_3\right]
\end{aligned}$$

where we have used the first identity in (C.3.23) and used $\frac{\partial}{\partial\omega_3}F_3 = 0$.

The 3rd term of the special conformal generator acting on F_3 gives

$$\begin{aligned}
&-\sum_{a=1}^{3} p_{aj}\frac{\partial^2}{\partial p_{aj}\partial p_a^i}\delta(\omega_1)\delta(\omega_2+\omega_3)\,\delta^{d-1}(\mathbf{P})F_3(\omega_2,\,k_1)\\
=\;& -\delta(\omega_1)\delta(\omega_2+\omega_3)\Bigg[\frac{\partial}{\partial P^i}\delta^{d-1}(\mathbf{P})\Big(-d+k_1\frac{\partial}{\partial k_1}\Big)F_3\\
&+\frac{\partial}{\partial P^j}\delta^{d-1}(\mathbf{P})\frac{(p_2+p_3)^i(p_2+p_3)^j}{k_1}\frac{\partial}{\partial k_1}F_3\\
&+\delta^{d-1}(\mathbf{P})(p_2+p_3)_i\,\frac{\partial^2}{\partial k_1^2}F_3\Bigg]
\end{aligned}$$

where we have used

$$\sum_{a=1}^{3} p_a^j\frac{\partial F_3}{\partial p_a^i} = (p_2^j+p_3^j)\frac{p_2^i+p_3^i}{k_1}\frac{\partial F_3}{\partial k_1} \quad ; \quad \sum_{a=1}^{3} p_a^j\frac{\partial F_3}{\partial p_a^j} = k_1\frac{\partial F_3}{\partial k_1} \tag{5.4.90}$$

and,
$$\sum_{a=1}^{3} p_a^j\frac{\partial^2}{\partial p_a^j\partial p_a^i}F_3(p_2,\,k_1) = (p_2+p_3)_i\,\frac{\partial^2}{\partial k_1}F_3(p_2,\,k_1) \tag{5.4.91}$$

The 4th term of the $\mathcal{K}_i$ generator acting on F_3 gives

$$\begin{aligned}
&\frac{1}{2}\sum_{a=1}^{3} p_{ai}\frac{\partial^2}{\partial p_a^j\partial p_{aj}}\delta(\omega_1)\delta(\omega_2+\omega_3)\,\delta^{d-1}(\mathbf{P})F_3(\omega_2,\,k_1)\\
=\;& \frac{1}{2}\delta(\omega_1)\delta(\omega_2+\omega_3)\Bigg[-2F_3\frac{\partial}{\partial P^i}\delta^{d-1}(\mathbf{P})\\
&+2\frac{\partial}{\partial P^j}\delta^{d-1}(\mathbf{P})\frac{(p_2+p_3)_i\,(p_2+p_3)^j}{k_1}\frac{\partial}{\partial k_1}F_3\\
&+\delta^{d-1}(\mathbf{P})(p_2+p_3)_i\Big(\frac{d-2}{k_1}\frac{\partial}{\partial k_1}+\frac{\partial^2}{\partial k_1^2}\Big)F_3(\omega_2,k_1)\Bigg]
\end{aligned}$$

where we have used

$$\begin{aligned}
\sum_{a=1}^{3} p_{ai}\frac{\partial}{\partial p_{aj}}F_3(\omega_2,k_1) &= \frac{(p_2+p_3)_i\,(p_2+p_3)^j}{k_1}\frac{\partial}{\partial k_1}F_3(\omega_2,k_1)\\
\sum_{a=1}^{3} p_{ai}\frac{\partial^2}{\partial p_a^j\partial p_{aj}}F_3(\omega_2,\,k_1) &= (p_2+p_3)_i\Big[\frac{d-2}{k_1}\frac{\partial}{\partial k_1}+\frac{\partial^2}{\partial k_1^2}\Big]F_3(\omega_2,k_1)
\end{aligned} \tag{5.4.92}$$

Combining all the terms, the action of Carrollian special conformal Ward identity on F_3 gives

$$\begin{aligned}
&\delta(\omega_1)\delta(\omega_2+\omega_3)\Bigg[\frac{\partial}{\partial P^i}\delta^{d-1}(\mathbf{P})\left(\Delta_t-2d+1-\omega_2\frac{\partial}{\partial\omega_2}-k_1\frac{\partial}{\partial k_1}\right)F_3(\omega_2,\,k_1)\\
&+\delta^{d-1}(\mathbf{P})(p_2+p_3)_i\left(\left(\Delta_2+\Delta_3-\frac{3d}{2}\right)\frac{1}{k_1}\frac{\partial}{\partial k_1}-\frac{\omega_2}{k_1}\frac{\partial^2}{\partial\omega_2\partial k_1}-\frac{1}{2}\frac{\partial^2}{\partial k_1^2}\right)F_3(\omega_2,\,k_1)\Bigg]\\
&=0
\end{aligned} \tag{5.4.93}$$

The first line vanishes by the dilatation Ward identity (5.4.84). The action of $\mathcal{K}_i$ on the remaining F_4 and F_5 terms can be similarly obtained. Thus, collecting all the results, the Carrollian special conformal Ward identity $\mathcal{K}_i$ imposes the following constraints

$$\sum_{a=1}^{3}p_a^i\left[\left(\Delta_a-\frac{d}{2}\right)\frac{1}{k_a}\frac{\partial}{\partial k_a}-\frac{1}{2}\frac{\partial^2}{\partial k_a^2}\right]F_2(k_1,k_2) = 0 \tag{5.4.94}$$

$$\left[\left(\Delta_2+\Delta_3-\frac{3d}{2}\right)\frac{1}{k_1}\frac{\partial}{\partial k_1}-\frac{\omega_2}{k_1}\frac{\partial^2}{\partial\omega_2\partial k_1}-\frac{1}{2}\frac{\partial^2}{\partial k_1^2}\right]F_3(\omega_2,\,k_1) = 0 \tag{5.4.95}$$

$$\left[\left(\Delta_1+\Delta_3-\frac{3d}{2}\right)\frac{1}{k_2}\frac{\partial}{\partial k_2}-\frac{\omega_1}{k_2}\frac{\partial^2}{\partial\omega_1\partial k_2}-\frac{1}{2}\frac{\partial^2}{\partial k_2^2}\right]F_4(\omega_1,\,k_2) = 0 \tag{5.4.96}$$

$$\left[\left(\Delta_1+\Delta_2-\frac{3d}{2}\right)\frac{1}{k_3}\frac{\partial}{\partial k_3}-\frac{\omega_1}{k_3}\frac{\partial^2}{\partial\omega_1\partial k_3}-\frac{1}{2}\frac{\partial^2}{\partial k_3^2}\right]F_5(\omega_1,\,k_3) = 0 \tag{5.4.97}$$

5.4.3 Solutions of Ward identities

5.4.3.1 2-point function

The solution to the differential equations (5.4.67) and (5.4.69), which constrain the two-point function, has already been provided in equation (5.4.68). For completeness, we recall here its full expression, giving the general form of the Carrollian two-point function in momentum space

$$\begin{aligned}
&\Big\langle \mathcal{O}_{\Delta_1}(\omega_1,\mathbf{p}_2)\,\mathcal{O}_{\Delta_2}(\omega_2,\,\mathbf{p}_2)\Big\rangle\\
&=\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2)\left[\delta(\omega_1+\omega_2)\;b_1\;\omega_1^{\Delta_1+\Delta_2-d}+\delta(\omega_1)\delta(\omega_2)\;b_2\;|\mathbf{p_1}|^{2\Delta_1-d+1}\right]
\end{aligned} \tag{5.4.98}$$

where b_1 and b_2 are arbitrary constants which may depend upon d and Δ_i.

5.4.3.2 3-point function

A generic d dimensional Carrollian CFT 3-point function of scalar operators in the momentum space takes the form

$$\begin{aligned}
&\mathcal{A}_3(\omega_1,\,\mathbf{p}_1;\,\omega_2,\,\mathbf{p}_2,\,\omega_3,\,\mathbf{p_3})\\
&\quad= \delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\Big[\delta(\omega_1+\omega_2+\omega_3)F_1(\omega_1,\,\omega_2)+\delta(\omega_1)\delta(\omega_2)\delta(\omega_3)F_2(k_1,\,k_2)\\
&\qquad+\delta(\omega_1)\delta(\omega_2+\omega_3)F_3(\omega_2,\,k_1)+\delta(\omega_2)\delta(\omega_1+\omega_3)F_4(\omega_1,\,k_2)\\
&\qquad+\delta(\omega_3)\delta(\omega_1+\omega_2)F_5(\omega_1,k_3)\Big]
\end{aligned}$$

We consider the general solutions of each F_i one by one. Starting with F_1, we note that it is the solution of equation (5.4.82). Now, the equation (5.4.82) imposes the scale invariance and only tells us that F_1 is a homogeneous function of degree $\Delta_t - 2d$. Hence, the most general solution must take the form

$$F_1(\omega_1,\omega_2) \;=\; \omega_1^{\Delta_t-2d}\, f\left(\frac{\omega_2}{\omega_1}\right) \tag{5.4.99}$$

where f can be any arbitrary function[7].

Next, we consider F_2. It is given by the solutions of the differential equations (5.4.83) and (5.4.95). These two equations are recognized to be the momentum space dilatation and special conformal Ward identities of 3 point function involving scalar operators in $d-1$ dimensional Euclidean CFT [306, 307]. Hence, its solution can be immediately written down in terms of the triple K integrals

$$F_2(k_1,k_2,k_3) \;=\; c_2\int_0^\infty dy\, y^{\frac{d-3}{2}}\prod_{a=1}^{3} k_a^{\Delta_a-\frac{d-1}{2}}K_{\Delta_a-\frac{d-1}{2}}(k_a y) \tag{5.4.100}$$

For special values of Δ_a, the above expression diverges and needs regularization. The analysis for this follows exactly the case of standard CFTs. For more details on this, see, e.g., [148].

Next, F_3 is given by the solutions of (5.4.84) and (5.4.95). For generic values of Δ_a, the solution is given by

$$F_3(\omega_2,k_1) \;=\; c_3\,(k_1)^{2\Delta_1-d+1}(\omega_2)^{\Delta_2+\Delta_3-\Delta_1-d} \;+\; d_3\,(\omega_2)^{\Delta_1+\Delta_2+\Delta_3-2d+1} \tag{5.4.101}$$

[7] Due to the delta function over the "energies", $\omega_3=-\omega_2-\omega_1$, there is only one independent energy ratio. Indeed, $\frac{\omega_3}{\omega_1}=-1-\frac{\omega_2}{\omega_1}$.

In the special case $\Delta_1 = \frac{d-1}{2}$, the two solutions above coincide and an additional solution involving logarithm emerges

$$F_3(\omega_2, k_1) = e_3 \, (\omega_2)^{\Delta_1+\Delta_2+\Delta_3-2d+1} \ln\left(\frac{k_1}{\omega_2}\right) \quad ; \quad \Delta_1 = \frac{d-1}{2} \tag{5.4.102}$$

Note that the above logarithmic solution is valid for arbitrary values of Δ_2 and Δ_3. It only requires Δ_1 to be specific.

Finally, we consider the solutions of F_4 and F_5. Since the differential equations for these are similar to F_3, their solutions can be written down immediately by permuting the indices. Hence, we have

$$\begin{aligned} F_4(\omega_1, k_2) &= c_4 \, (k_2)^{2\Delta_2-d+1} (\omega_1)^{\Delta_1+\Delta_3-\Delta_2-d} + d_4 \, (\omega_1)^{\Delta_1+\Delta_2+\Delta_3-2d+1} \\ F_5(\omega_1, k_3) &= c_5 \, (k_3)^{2\Delta_3-d+1} (\omega_1)^{\Delta_1+\Delta_2-\Delta_3-d} + d_5 \, (\omega_1)^{\Delta_1+\Delta_2+\Delta_3-2d+1} \end{aligned} \tag{5.4.103}$$

Again for Δ_2 and Δ_3 equal to $\frac{d-1}{2}$, the two solutions coincide and we have additional solutions

$$\begin{aligned} F_4(\omega_1, k_2) &= e_4 \, (\omega_1)^{\Delta_1+\Delta_2+\Delta_3-2d+1} \ln\left(\frac{k_2}{\omega_1}\right) \quad ; \quad \Delta_2 = \frac{d-1}{2} \\ F_5(\omega_1, k_3) &= e_5 \, (\omega_1)^{\Delta_1+\Delta_2+\Delta_3-2d+1} \ln\left(\frac{k_3}{\omega_1}\right) \quad ; \quad \Delta_3 = \frac{d-1}{2} \end{aligned} \tag{5.4.104}$$

A final remark concerns the origin and interpretation of the logarithmic terms that appear when solving the Ward identities in momentum space. These logarithmic solutions do not correspond to additional branches of the correlator in position space. Instead, they arise at special loci in the parameter space $\{\Delta_a, d\}$ where the transform of the position-space correlators becomes singular and requires regularization. This can be seen explicitly by considering the transform of the $|\mathbf{x}_{12}|^{-2\Delta_1}$ factor appearing in equation (5.3.49)

$$\int d^{d-1}x \, \frac{e^{i\mathbf{p}_1\cdot\mathbf{x}_{12}}}{|\mathbf{x}_{12}|^{2\Delta_1}} = \frac{2^{d-1-2\Delta_1} \, \pi^{\frac{d-1}{2}}}{\Gamma(\Delta_1)} \, \Gamma\!\left(\frac{d-1-2\Delta_1}{2}\right) k_1^{2\Delta_1-(d-1)}. \tag{5.4.105}$$

The prefactor is singular when $\Delta_1 = \frac{d-1}{2}$, i.e. when the exponent of $k_1^{2\Delta_1-(d-1)}$ reaches its critical value. In this limit, we get (denoting $\epsilon = d-1-2\Delta_1$)

$$\lim_{\epsilon\to 0} \Gamma\left(\frac{\epsilon}{2}\right) k_1^{-\epsilon} = \frac{2}{\epsilon} - 2\log k_1 + O(\epsilon) \tag{5.4.106}$$

The above equation shows that the regularized transform acquires a $\log k_1$ contribution, as in equation (5.4.102).

A completely analogous mechanism affects the transform of the u_{23}-dependent factor in equation (5.3.49). A logarithmic dependence on ω_2 as in (5.4.102) emerges only when an additional constraint relating the parameters Δ_a are satisfied. These constraints do not directly emerge from solving the Ward identities in position space. This mismatch indicates that the regularization of the transform must be handled with care. These issues deserve further investigation and we leave these for a future work.

We also remark that the logarithmic term in (5.4.102) are divergent as $k_1 \to 0$ or ∞ for fixed values of ω_2. These singularities can be regulated by introducing a cutoff, and in this sense the variable ω_2 naturally plays the role of an effective IR/UV regulator as we span the spatial momentum space $\mathbf{p}_1$. Similarly, the logarithmic term also diverges as $\omega_2 \to 0$ or ∞ for fixed values of k_1. These singularities can be regulated by introducing a cutoff, and hence the variable k_1 can be thought as an effective IR/UV regulator as we span the Carrollian energy space ω_2.

5.5 Carrollian limit of CFT correlators

In section (1.3.1.1) we obtained the Carrollian CFT algebra in two different ways: from the $c \to 0$ limit of the CFT algebra on $\mathbb{M}^d$ in Cartesian coordinates, and from the null infinity limit of the Poincaré algebra on $\mathbb{M}^{d+1}$ in flat null coordinates. This suggests that CCFTs naturally occur on null infinities $\mathscr{I}^\pm$ of flat spacetime, and captures the kinematics of massless scattering between null infinities. *By comparing the CCFT algebras obtained via both these limits, we notice that they are the same as long as we identify the t coordinate in the CCFT after the $c \to 0$ limit with the u coordinate in the CCFT on $\mathscr{I}^\pm$.* This can also be seen at the level of the metric. The resulting Carrollian metrics after the limits, (1.3.129) and (1.3.139), are equivalent once we identify the coordinate $x^{0\prime}$ with u.

The Carrollian limit of CFT correlators has been considered in position space for some branches [253, 258]. However, as emphasized earlier, the momentum space provides a natural language for the flat-space amplitudes. Therefore, it is useful to have control over the Carrollian limit of CFT correlators directly in the momentum space, along with the position space.

In this section, we shall consider the Carrollian limit of the standard CFT correlators. We shall show how, in this limit, the CFT 2 and 3 point functions give rise to 2 and 3 point Carrollian conformal correlators. We will keep the conformal dimensions of CFT-operators arbitrary; under this assumption, we do not expect to reproduce, in the $c \to 0$ limit, the logarithmic behavior discussed in the previous section. The main strategy would be to explicitly keep track of the factors of speed of light c in the correlators and send it to zero. By rescaling the correlators with different powers of c, we shall obtain different branches of 2 and 3 point Carrollian conformal correlators.

In the CFT correlators, the speed of light c always appears with time coordinate, so the conventional notion of time ceases to apply when we send c to zero. As a consequence, the correlators lose their standard causal interpretation for spatially separated points. Due to this, the Carrollian limit of Wightman and time ordered correlators requires seperate treatments. We shall consider both Wightman as well as time-ordered CFT correlators in our analysis.

5.5.1 2-point function

We begin this section by reviewing the Carrollian limit of the two-point function in position space. This limit amounts to taking $(c\,u,\,\mathbf{x}) \;\to\; (0,\,\mathbf{x})$ which is equivalent to considering the ultra-relativistic limit of vanishing light velocity, $c \to 0$. In this limit, as discussed previously, the conformal group $SO(2,d)$ of Minkowski space $\mathbb{M}^{1,d-1}$ reduces to the massless Poincaré group $ISO(1,d)$, which acts as the group of conformal isometries of the null plane $\mathscr{J} = \mathbb{R}\times\mathbb{R}^{d-1}$. The natural starting point for studying the Carrollian limit are therefore the CFT correlation functions in Lorentzian signature.

We start by reviewing the Carrollian limit of time ordered 2-point function in the position space following [253, 258]. As reviewed in appendix C.2, the Lorentzian time-ordered CFT two-point function takes the form (see, e.g., [253])

$$\Big\langle T\Big(\mathcal{O}(t_1,\mathbf{x}_1)\,\mathcal{O}(t_2,\mathbf{x}_2)\Big)\Big\rangle = \frac{C_2(\Delta)}{[-c^2t_{12}^2+|\mathbf{x}_{12}|^2+i\epsilon]^{\Delta}}\,. \tag{5.5.107}$$

In the Carrollian limit $c\to 0$, two distinct behaviors of the two-point function emerge. For $\mathbf{x}_{12}\neq 0$, the two-point simply reduces to $|\mathbf{x}_{12}|^{-2\Delta}$ in the limit $c\to 0$. The case $\mathbf{x}_{12}=0$ is instead singular. Combining both situations, we can schematically write [258]

$$\Big\langle T\Big(\mathcal{O}(t_1,\mathbf{x}_1)\,\mathcal{O}(t_2,\mathbf{x}_2)\Big)\Big\rangle \xrightarrow[c\to 0]{} \frac{C_2(\Delta)}{|\mathbf{x}_{12}|^{2\Delta}} + \frac{1}{c^{\alpha}}\,g(\Delta,\,t)\,\delta^{(d-1)}(\mathbf{x}_{12})\,. \tag{5.5.108}$$

where the function $g(\Delta,t)$ and the constant α need to be fixed. The first and second terms in the RHS of above expression correspond respectively to the *magnetic* and *electric* branches of the Carrollian two-point function in position space. The function $g(\Delta,t)$ appearing in the electric branch is determined by multiplying the time-ordered two-point function by c^{α} (which in the limit $c\to 0$ suppresses the magnetic contribution) and then integrating over the spatial coordinates for removing the delta-function. The result of the integration gives [258]

$$g(\Delta,\, t_{12}) = \frac{\pi^{\frac{d-1}{2}}\,\Gamma[\Delta - \frac{d-1}{2}]}{\Gamma[\Delta]} \frac{C_2(\Delta)}{(-t_{12}^2 + i\epsilon)^{\Delta - \frac{d-1}{2}}} \quad ; \quad \alpha = 2\Delta - d + 1 \tag{5.5.109}$$

After taking the $c \to 0$ limit of the CFT correlator, the resulting CCFT correlator is same as the CCFT correlator on $\mathscr{I}^{\pm}$, once we identify the resulting t coordinate with the u coordinate in the CCFT correlator on $\mathscr{I}^{\pm}$. The rationale for this is discussed in detail in (5.5). We shall follow a similar procedure for taking the Carrollian limit of 2-point functions in the momentum space. We start with the Wightman 2-point function.

5.5.1.1 Wightman

The Wightman (Lorentzian) two–point function in momentum space for a scalar primary operator $\mathcal{O}$ of conformal dimension Δ is given by (see, e.g., [308])

$$G_2^W(p) \;\equiv\; \langle\!\langle \mathcal{O}(p)\, \mathcal{O}(-p) \rangle\!\rangle = \frac{\pi^{\frac{d}{2}+1}\,\Theta\Big(p^0 - |\mathbf{p}|\Big)}{2^{\,2\Delta-d-1}\,\Gamma(\Delta)\,\Gamma\Big(\Delta - \frac{d}{2} + 1\Big)} \;\frac{1}{c^{2\Delta-d}(\omega^2 - c^2\,|\mathbf{p}|^2)^{\frac{d}{2}-\Delta}} \tag{5.5.110}$$

where Θ is the Heaviside step function ($\Theta(x) = 1$ for $x > 0$, $\Theta(x) = 0$ for $x \le 0$), and we have written

$$-p^2 = (p^0)^2 - |\mathbf{p}|^2 \quad ; \quad p^0 = \frac{\omega}{c}. \tag{5.5.111}$$

The Heaviside factor enforces support on the future timelike (or null) cone, $p^0 = \frac{\omega}{c} \geq |\mathbf{p}| \geq 0$. In the Carrollian limit, with fixed ω,

$$\lim_{c\to 0^+} \Theta\Big(\frac{\omega}{c} - |\mathbf{p}|\Big) \;\equiv\; \Theta(\omega) = 1. \tag{5.5.112}$$

Moreover, taking the limit $\omega \to 0^+$ (from the positive side) one still preserves the condition $\Theta(\omega) \to 1$. As in position space, the Carrollian limit of the two–point function can be expressed as

$$G_2^W(p) \xrightarrow{c\to 0} \frac{\pi^{\frac{d}{2}+1}}{2^{2\Delta-d-1}\,\Gamma[\Delta - \frac{d}{2} + 1]\,\Gamma(\Delta)} \frac{1}{c^{2\Delta-d}} \left[\omega^{2\Delta-d} + \frac{1}{c^{\alpha}}\,\delta(\omega) g(|\mathbf{p}|)\right] \;\; ; \tag{5.5.113}$$

The first term captures the regular contribution for non-vanishing energy, while the second term, proportional to the delta function, accounts for the singular contribution originating from the threshold region $\omega \simeq c|\mathbf{p}| \to 0$. This contact term arises only for $\Delta < \frac{d}{2}$ where the two points develop a divergence in the Carrollian limit. At this stage, there are two possible Carrollian limits depending on the value of the scaling exponent

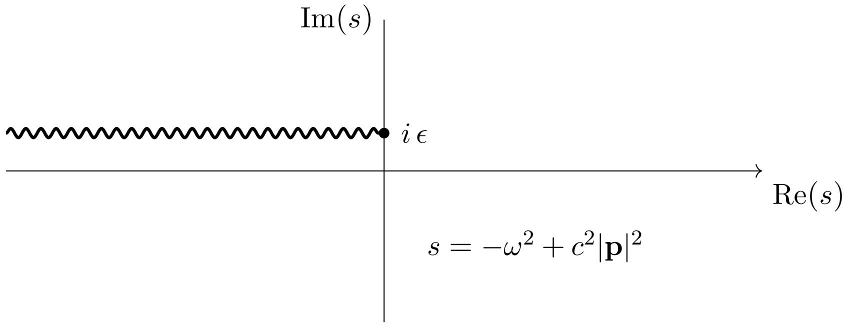


FIGURE 5.1: The branch cut in the $s \equiv p^2c^2$ plane for the time ordered 2-point function in equation (5.5.116).

α. The function $g(|\mathbf{p}|)$ is fixed by integrating[8]

$$\lim_{c\to 0} c^{2\Delta-d+\alpha} \int_{-\infty}^{\infty} d\omega\, G_2^W(p) = \frac{\pi^{\frac{d}{2}+1}\,\Gamma[d-2\Delta-1]}{4\,\Gamma[-\Delta+\frac{d}{2}+1]\,\Gamma(\Delta)}\,\frac{c^\alpha}{c^{d-2\Delta-1}}\,|\mathbf{p}|^{2\Delta-d+1} \quad (5.5.114)$$

that fixes $\alpha = d-2\Delta-1$ and gives

$$c^\beta\, G_2^W(p) \xrightarrow{c\to 0} \begin{cases} \frac{\pi^{\frac{d}{2}+1}}{2^{2\Delta-d-1}\,\Gamma[\Delta-\frac{d}{2}+1]\,\Gamma(\Delta)}\,\omega^{2\Delta-d} & \beta = 2\Delta-d \\ \frac{\pi^{\frac{d}{2}+1}\,\Gamma[d-2\Delta-1]}{4\,\Gamma[\frac{d}{2}-\Delta+1]\,\Gamma[\Delta]}\,|\mathbf{p}|^{2\Delta-d+1} & \beta = -1 \end{cases} \quad (5.5.115)$$

This expression is consistent with Eq. (5.4.98), evaluated in the electric branch, with $\Delta_1 = \Delta_2 \equiv \Delta$.

5.5.1.2 Time ordered

The time ordered two point CFT correlation function in the Lorentzian signature takes the form [308]

$$G_2^T(p) = -i\frac{\pi^{\frac{d}{2}}\,\Gamma[\frac{d}{2}-\Delta]}{2^{\Delta-d}\,\Gamma[\Delta]}\,\frac{1}{c^{2\Delta-d}(-\omega^2+c^2|\mathbf{p}|^2-i\epsilon)^{\frac{d}{2}-\Delta}} \quad ; \quad \epsilon > 0 \quad (5.5.116)$$

Compared to the Wightman two-point function (5.5.110), it differs only by an overall factor and the $i\epsilon$ prescription (see Fig.5.1).

The analysis is the same as for the Wightman 2-point function. Hence, we write

$$G_2^T(p) \xrightarrow{c\to 0} -i\frac{\pi^{\frac{d}{2}}\,\Gamma[\frac{d}{2}-\Delta]}{2^{\Delta-d}\,\Gamma[\Delta]}\,\frac{1}{c^{2\Delta-d}}\left[\omega^{2\Delta-d} + \frac{1}{c^\alpha}\,\delta(\omega)\tilde{g}(|\mathbf{p}|)\right]. \quad (5.5.117)$$

[8]Here, we have used the identity $\int_{c|\mathbf{p}|}^{\infty} d\omega\,(\omega^2-c^2|\mathbf{p}|^2)^{\Delta-\frac{d}{2}} = (2c|\mathbf{p}|)^{2\Delta-d+1}B(\Delta-\frac{d}{2}+1,\,-2\Delta+d-1)$, for $B(x,y)$ the Euler Beta function.

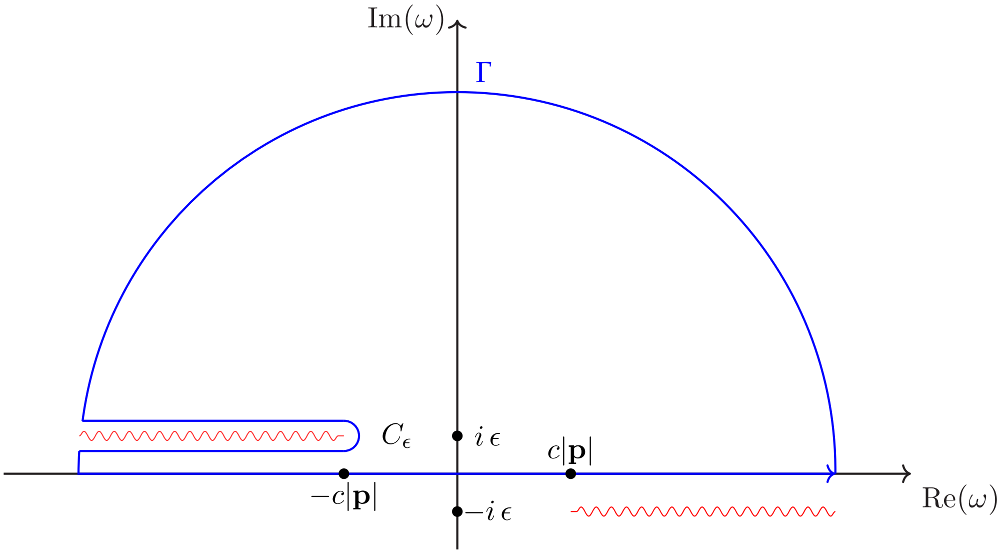


FIGURE 5.2: The contour choice (shown in blue) for performing the ω integration in equation (5.5.118). The red wiggly lines represent the two branch cuts due to the factors in denominator.

Again, to determine the function $\tilde{g}(|\mathbf{p}|)$ and the coefficient α, we can integrate both sides over ω. The integration over the left side involves the integral

$$\int_{-\infty}^{\infty} \frac{d\omega}{\left[(-\omega + c\,|\mathbf{p}| - i\epsilon)(\omega + c|\mathbf{p}| - i\epsilon)\right]^{\frac{d}{2}-\Delta}} \tag{5.5.118}$$

This can be evaluated directly in a brute force way, carefully keeping track of the phase factors in the denominators. Alternatively, we can evaluate it by a contour method. Here, we give the argument based on contour deformation while providing the details of direct evaluation in the appendix. Both approaches, of course, give the same result. In the contour method, we consider the integrand in (5.5.118) evaluated along the contour shown in figure 5.2 in the complex ω plane. For arbitrary Δ, the branch cuts start from the $\pm c|\mathbf{p}| \mp i\epsilon$ points and extend to $\pm\infty \mp i\epsilon$. Assuming $\frac{d}{2} - \Delta > 0$, the contributions from the arc at infinity vanish. In addition, the integral over the semicircle C_ϵ centered at the branch point $-c|\mathbf{p}| + i\epsilon$ also vanishes. This implies that the desired integral along the real axis is given by the discontinuity across the branch cut in the second quadrant leaving us with the identity

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{d\omega}{\left[(-\omega + c\,|\mathbf{p}| - i\epsilon)(\omega + c|\mathbf{p}| - i\epsilon)\right]^{\frac{d}{2}-\Delta}} \\
&= \; -\left[e^{-i\pi(\frac{d}{2}-\Delta)} \int_{-\infty}^{-c|\mathbf{p}|} + e^{i\pi(\frac{d}{2}-\Delta)} \int_{-c|\mathbf{p}|}^{-\infty}\right] \frac{d\omega}{\left[(-\omega + c\,|\mathbf{p}|)(\omega + c|\mathbf{p}|)\right]^{\frac{d}{2}-\Delta}}
\end{aligned} \tag{5.5.119}$$

The evaluation of the two integrals in the right hand side follows the same method as in the case of Wightman two-point function, yielding

$$\int_{-\infty}^{\infty} \frac{d\omega}{[(-\omega + c\,|\mathbf{p}| - i\epsilon)(\omega + c|\mathbf{p}| - i\epsilon)]^{\frac{d}{2}-\Delta}} \;=\; \frac{2i\pi\Gamma\left(d - 2\Delta - 1\right)}{\Gamma^2\left(\frac{d}{2} - \Delta\right)}(2c|\mathbf{p}|)^{2\Delta-d+1} \tag{5.5.120}$$

This fixes the function $\tilde{g}$ in (5.5.117), giving the Carrollian limit of the time ordered CFT 2-point function to be

$$c^\beta\, G_2^T(p) \xrightarrow{c\to 0} \begin{cases} -i\frac{\pi^{\frac{d}{2}}\,\Gamma[\frac{d}{2}-\Delta]}{2^{\Delta-d}\,\Gamma[\Delta]}\;\omega^{2\Delta-d} & \beta = 2\Delta - d \\ \frac{2^{\Delta}\pi^{\frac{d}{2}+1}\,\Gamma[d-2\Delta-1]}{4\,\Gamma[\Delta-\frac{d}{2}]\,\Gamma[\Delta]}\;|\mathbf{p}|^{2\Delta-d+1} & \beta = -1 \end{cases} \tag{5.5.121}$$

Comparing (5.5.115) and (5.5.121), we see that the Carrollian limit of Wightman and time ordered Lorentzian CFT 2-point functions differ only in the overall constants.

We conclude by noting that, although the CCFT algebra is obtained from the CFT algebra via an ultrarelativistic contraction, the same limit applied to CFT correlators generates only a subset of the full Carrollian solution space. The reason is that relativistic CFT two-point functions are already constrained by the full conformal symmetry to satisfy $\Delta_1 = \Delta_2$, a condition that follows from invariance under the algebra $\mathfrak{so}(2,d)$. Consequently, the ultra-relativistic limit inherits this restriction and cannot access the most general Carrollian structures allowed by the CCFT Ward identities.

5.5.2 3-point function

In this section, we consider the Carrollian limit of the Lorentzian CFT 3-point function. We start with the time ordered case.

5.5.2.1 Time ordered

The 3-point function of three scalar operators with conformal dimensions $\Delta_a \quad (a = 1,2,3)$ in momentum space in the Lorentzian signature can be obtained by transforming the corresponding expression in position space. The explicit transform is quite cumbersome [309]. Fortunately, it is possible to work with an integral representation which expresses the Lorentzian momentum space 3-point function as a convolution over the product of three two-point functions

$$\mathcal{A}_3(p_1,p_2,p_3) \;=\; \delta^d(p_1+p_2+p_3)\,\mathcal{N}(\Delta_1\,,\Delta_2\,,\Delta_3)\left\langle\!\!\left\langle \mathcal{O}_{\Delta_1}(p_1)\,\mathcal{O}_{\Delta_2}(p_2)\,\mathcal{O}_{\Delta_3}(p_3)\right\rangle\!\!\right\rangle \tag{5.5.122}$$

where

$$\left\langle\!\!\left\langle \mathcal{O}_{\Delta_1}(p_1)\, \mathcal{O}_{\Delta_2}(p_2)\, \mathcal{O}_{\Delta_3}(p_3) \right\rangle\!\!\right\rangle$$
$$= \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - i\epsilon)^{\frac{d}{2}-\beta_3} [(p_2+k)^2 - i\epsilon]^{\frac{d}{2}-\beta_1} [(p_1-k)^2 - i\epsilon]^{\frac{d}{2}-\beta_2}} \tag{5.5.123}$$

Here, we have introduced the quantities, $\beta_c = \Delta_{ab} = \frac{\Delta_a + \Delta_b - \Delta_c}{2}$, ($a,\, b,\, c \in \{1,2,3\}$ with a, b, c distinct) and the normalization factor is given by

$$\mathcal{N}(\Delta_1\,,\Delta_2\,,\Delta_3) = \prod_{a=1}^{3} \left[-i \frac{\pi^{\frac{d}{2}}\, \Gamma[\frac{d}{2} - \Delta_a]}{2^{2\Delta_a - d}\Gamma[\Delta_a]} \right] \tag{5.5.124}$$

The details of the derivation of this representation of the correlator are given in appendix C.2.2.

In the above representation of the correlator, the manifest cyclic symmetry under the exchange of the three scalar operators is apparently lost since (5.5.123) only depends upon the external momenta p_1 and p_2. However, the symmetry under all external momenta can be shown by a change of the integration variable k. For example, the shift $k \to k + p_1$ rewrites, after using momentum conservation, the correlator in terms of the momenta (p_1, p_3) instead of (p_1, p_2).

Since the three-point function is represented as a product of three 2-point functions, the analysis of the Carrollian limit proceeds along the same lines as in the previous section. In what follows, we consider the Carrollian limit for different branches one by one.

Branch with $\delta(\omega_1 + \omega_2 + \omega_3)$

For this branch, none of the zeroth components of the external momenta vanish. Hence, we can directly take the limit $c \to 0$. A convenient way to do this is to rescale the loop momentum k^μ by p_1, namely

$$k^\mu \to |p_1| k^\mu \quad , \qquad d^d k \to |p_1|^d\, d^d k\,.$$

where, we have introduced the Lorentzian norm [309]

$$|p| = \sqrt{|-(p^0)^2 + \mathbf{p}^2|} \tag{5.5.125}$$

With the above rescaling, the correlator factorizes as

$$\langle\!\langle\, \mathcal{O}_{\Delta_1}(p_1)\, \mathcal{O}_{\Delta_2}(p_2)\, \mathcal{O}_{\Delta_3}(p_3) \,\rangle\!\rangle \;\; = \;\; |p_1|^{\Delta_1+\Delta_2+\Delta_3-2d}\, f\left(\frac{p_2}{|p_1|},\, \Delta_i\right), \tag{5.5.126}$$

where f is a dimensionless function that depends only on the ratio $\frac{p_2}{|p_1|}$ (with $|p_1|$ being the norm of d dimensional Lorentzian momenta p_1^μ). The Carrollian limit can now be taken by replacing

$$p_1 \simeq \frac{\omega_1}{c}, \tag{5.5.127}$$

Substituting these scalings, we obtain

$$\lim_{c\to 0} c^{2d-\Delta_1-\Delta_2-\Delta_3} \langle\!\langle \mathcal{O}_{\Delta_1}(\omega_1)\, \mathcal{O}_{\Delta_2}(\omega_2)\, \mathcal{O}_{\Delta_3}(\omega_3)\rangle\!\rangle = \left(\omega_1^2\right)^{\frac{\Delta_1+\Delta_2+\Delta_3-2d}{2}} f\left(\frac{\omega_2}{\omega_1}\right). \tag{5.5.128}$$

This result is in agreement with the solution of the Ward identities for the branch F_1 given in equation (5.4.99). In contrast with (5.4.99), the function $f\left(\frac{\omega_2}{\omega_1}\right)$ in (5.5.128), arising in the small-c limit, is not arbitrary. This behaviour is analogous to the ultra-relativistic limit of the two-point function, as discussed at the end of Section 5.5.1.2.

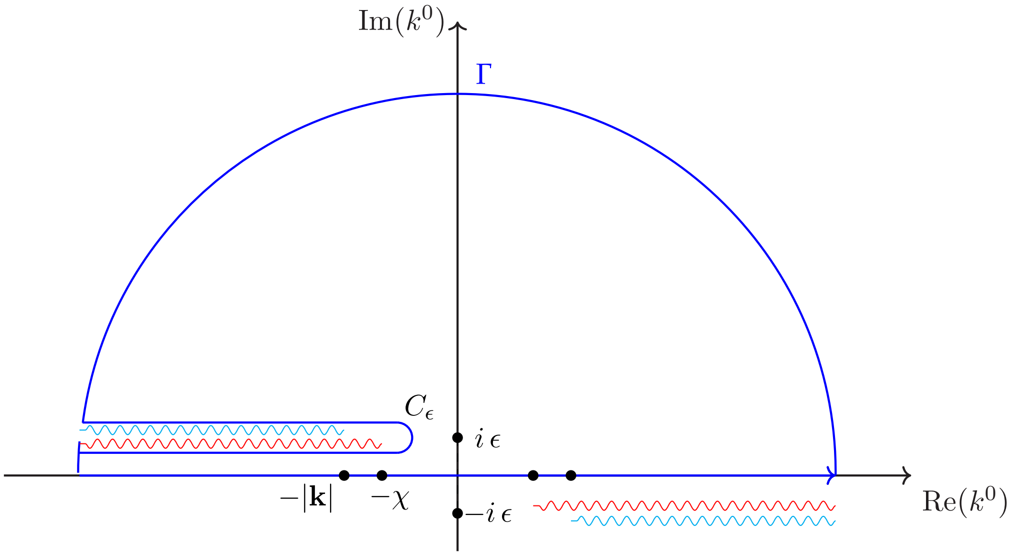
Branch with $\delta(\omega_1)\delta(\omega_2+\omega_3)$

For the 3-point branches involving one or more vanishing energy components, we need to look for additional contributions to the Carrollian limit, scaling with different powers c^α of the speed of light. These arise from specific kinematic regions. For (5.5.123), these correspond to

$$\chi_1 = ck^0 \pm c|\mathbf{k}-\mathbf{p}_1| \simeq 0\ , \quad \text{or} \qquad \chi_2 = -ck^0 \pm c|\mathbf{k}+\mathbf{p}_2| \simeq 0\ , \tag{5.5.129}$$

or from configurations in which both the above conditions are simultaneously satisfied. These contributions are proportional to $\delta(\omega_1)$ or $\delta(\omega_2)$ (and also to $\delta(\omega_3)$ if one chooses a representation of the CFT correlator in which the momentum p_3 appears explicitly), as well as to the product $\delta(\omega_1)\,\delta(\omega_2)\,\delta(\omega_3)$. The latter delta function structure originates from the overall energy-conserving delta function in the CFT correlator, which enforces $\omega_1+\omega_2+\omega_3 = 0$.

The functions multiplying the delta functions in the energies are fixed by projecting the CFT correlator onto the corresponding support. For instance, the coefficient of $\delta(\omega_1)$ is obtained by integrating equation (5.5.123) with respect to ω_1 at fixed $\omega_2 \neq 0$ (so that the $\delta(\omega_2)$ term does not contribute). We can now fix the coefficient of the $\delta(\omega_1)$ term by integrating equation (5.5.123) with respect to ω_1. A similar analysis can be extended to the other branches that emerge in the Carrollian limit. The integral over ω_1 is performed



$$\chi = p_2^0 + |\mathbf{p}_2 + \mathbf{k}| \leq 0$$
$$|\mathbf{k}| > \chi \quad , \quad p_2^0 \leq 0$$

FIGURE 5.3: The contour choice for performing the k^0 integral in equation (5.5.131). There are a total of four branch points described in equation (5.5.132).

in the same way as in equation (5.5.120) with the result

$$\begin{aligned}\int_{-\infty}^{\infty} d\omega_1 \, &\frac{1}{[(p_1-k)^2 - i\epsilon]^{\frac{d}{2}-\beta_2}} \\ &= 2i\, c^{d-2\beta_2} \, \sin\Big[\pi\Big(\tfrac{d}{2}-\beta_2\Big)\Big]\, B\Big(1-\tfrac{d}{2}+\beta_2,\, d-2\beta_2-1\Big)\, |\mathbf{p}_1-\mathbf{k}|^{\,1-d+2\beta_2}.\end{aligned} \tag{5.5.130}$$

The next step is to compute the integral over the integration momenta k

$$\int \frac{dk^0\, d^{d-1}k}{(2\pi)^d} \frac{1}{(k^2-i\epsilon)^{\frac{d}{2}-\beta_3}\,[(p_2+k)^2-i\epsilon]^{\frac{d}{2}-\beta_1}|\mathbf{p}_1-\mathbf{k}|^{d-2\beta_2-1}} \tag{5.5.131}$$

As in the two-point case, the integration over k^0 can be carried out by contour method analyzing the analytic structure of the integrand, with particular attention to the branch cuts generated by the non-integer exponents of terms in the denominators. The location of these cuts is determined by the $i\epsilon$ prescription. The branch cuts in the complex k^0–plane start at

$$k^0 = \pm|\mathbf{k}|\mp i\epsilon, \qquad k^0 = -p_2^0 \pm |\mathbf{k}+\mathbf{p}_2|\mp i\epsilon \tag{5.5.132}$$

and extend to $\pm\infty\mp i\epsilon$.[9] Their relative positions depend on the values of the external four-momentum p_2 and the spatial momentum $\mathbf{k}$. Two distinct configurations may arise

$$-|\mathbf{k}| \;<\; -\chi \;\equiv\; -p_2^0 - |\mathbf{p}_2 + \mathbf{k}| \;<\; -p_2^0 + |\mathbf{p}_2 + \mathbf{k}|\,, \tag{5.5.133}$$

$$\text{and;} \qquad -|\mathbf{k}| \;>\; -\chi \quad\Longrightarrow\quad \chi > |\mathbf{k}|\,. \tag{5.5.134}$$

Since the above relations are valid for any value of $\mathbf{p}_2$ and $\mathbf{k}$, in the Carrollian limit ($c \to 0$) these conditions become (for $c\,(|\mathbf{p}_2 + \mathbf{k}| - |\mathbf{k}|) \simeq 0$)

$$\begin{aligned} \chi \geq |\mathbf{k}| \;\;&\Longrightarrow\;\; \omega_2 + c\,|\mathbf{p}_2 + \mathbf{k}| \;\geq\; c\,|\mathbf{k}| \;\;\Longrightarrow\;\; \omega_2 \geq 0\,, \\ \chi \leq |\mathbf{k}| \;\;&\Longrightarrow\;\; \omega_2 + c\,|\mathbf{p}_2 + \mathbf{k}| \;\leq\; c\,|\mathbf{k}| \;\;\Longrightarrow\;\; \omega_2 \leq 0\,. \end{aligned} \tag{5.5.135}$$

When the second condition is satisfied, ω_2 is negative, while χ can take either sign. In the Carrollian limit this leads to

$$\omega_2 \leq 0 \;\;\Rightarrow\;\; \begin{cases} \chi \geq 0 \;\;\Longrightarrow\;\; 0 \,\geq\, \omega_2 \,\geq\, -c\,|\mathbf{p}_2 + \mathbf{k}| \;=\; -c\,|\mathbf{k}| \;\simeq\; 0, \quad \forall\,|\mathbf{k}| \in [0,\infty)\,, \\ \chi \leq 0 \;\;\Longrightarrow\;\; 0 \,\geq\, \omega_2 \,\leq\, -c\,|\mathbf{p}_2 + \mathbf{k}| \;=\; -c\,|\mathbf{k}| \;\simeq\; 0, \quad \forall\,|\mathbf{k}| \in [0,\infty)\,. \end{cases} \tag{5.5.136}$$

In the last step we have imposed $c|\mathbf{k}| \simeq 0$, as appropriate in the ultra-relativistic (small–c) regime, thereby isolating the contribution from the region singled out by equation (5.5.129). From equations (5.5.135) and (5.5.136), the case $|\mathbf{k}| \geq \chi$ is compatible with both positive and negative values of χ. However, the case $\chi \leq 0$ selects only the value $\omega_2 = 0$ in the small-c limit. We therefore restrict our analysis to the Carrollian limit with $\chi \geq 0$, which is compatible with $\omega_2 \leq 0$. The Carrollian limit for the case $|\mathbf{k}| \leq \chi$ with $\omega_2 \geq 0$ can be treated analogously.

Now, we consider the k^0 integral in (5.5.131) along the contour shown in figure 5.3. Under the assumption

$$2d - 2\beta_1 - 2\beta_3 - 1 \geq 0 \;\Longrightarrow\; \Delta_2 \leq d - \frac{1}{2}\;, \tag{5.5.137}$$

the contribution from the arc at infinity vanishes[10]. The integral along the semicircle centered around the branch point $-\chi + i\epsilon$ also vanishes. Thus, the desired integral along the real axis is

[9] If the conformal dimensions of the operators were integers, the corresponding singularity would reduce to an integer-order pole or be absent. In the present discussion we do not consider this special case.

[10] This is a strong constraint on the scaling dimension. The 3-point CCFT correlator derived using the CCFT Ward identities did not possess such a constraint. Whether this is an artefact of the integral representation of the 3-point CFT correlator, and how the $c \to 0$ limit was evaluated, is not clear. A good check would be to analyse a similar Carrollian limit in position space and see if that yields a similar constraint. Also, note that the operator $r\partial_r$ is the bulk realisation of the scaling dimension, as discussed below (1.3.141). We will see in chapter 6 that the on-shell action that generates the CCFT correlators seems to pick particular scaling dimensions. Moreover, unitarity constraints similar to those in CFTs impose constraints on the scaling dimension in CCFTs. We are yet to investigate if these conditions are related.

determined by the discontinuities across the cuts

$$
\begin{aligned}
&\int \frac{dk^0}{2\pi} \frac{1}{(k^2 - i\epsilon)^{\frac{d}{2}-\beta_3} \left[(p_2+k)^2 - i\epsilon\right]^{\frac{d}{2}-\beta_1} |\mathbf{p}_1 - \mathbf{k}|^{d-2\beta_2-1}} \\
\equiv\ & \left[2i \sin\pi(d-\beta_1-\beta_3) \int_{-|\mathbf{k}|}^{-\infty} \ +\ 2\,i\sin\pi\left(\frac{d}{2}-\beta_1\right) \int_{-\chi}^{-|\mathbf{k}|}\right] \frac{dk^0}{2\pi} \\
&\times \frac{1}{|k^0 - |\mathbf{k}||^{\frac{d}{2}-\beta_3}\, |k^0 + |\mathbf{k}||^{\frac{d}{2}-\beta_3}\, |p_2^0 + k^0 - |\mathbf{k}+\mathbf{p}_2||^{\frac{d}{2}-\beta_1} |p_2^0 + k^0 + |\mathbf{k}+\mathbf{p}_2||^{\frac{d}{2}-\beta_1}} \\
\equiv\ & \frac{i}{\pi} \sin\pi(d-\beta_1-\beta_3)\, I_1 + \frac{i}{\pi}\sin\pi\left(\frac{d}{2}-\beta_1\right) I_2
\end{aligned}
\tag{5.5.138}
$$

We now evaluate the two contributions separately. After a suitable change of variables, the first integral can be recast in the form

$$
I_1 = -(2|\mathbf{k}|)^{1-2d+2\beta_1+2\beta_3} \int_0^\infty \frac{dt}{t^{\frac{d}{2}-\beta_3}\,(t+1)^{\frac{d}{2}-\beta_3}\,(t+A_1)^{\frac{d}{2}-\beta_1}\,(t+A_2)^{\frac{d}{2}-\beta_1}}
\tag{5.5.139}
$$

where

$$
\begin{aligned}
A_1 &= \frac{-\frac{\omega_2}{c} + |\mathbf{k}| - |\mathbf{p}_2 + \mathbf{k}|}{2|\mathbf{k}|} = \frac{-\chi + |\mathbf{k}|}{2\,|\mathbf{k}|} \geq \frac{-|\mathbf{k}| + |\mathbf{k}|}{2|\mathbf{k}|} \geq 0 \\
A_2 &= \frac{-\frac{\omega_2}{c} + |\mathbf{k}| + |\mathbf{p}_2 + \mathbf{k}|}{2|\mathbf{k}|}
\end{aligned}
\tag{5.5.140}
$$

In the intermediate steps, the factors in the denominators in (5.5.139) involve the absolute values. However, as shown above, A_1 is positive definite. Similarly, since we are considering the case $\omega_2 \leq 0$, it follows that $A_2 \geq 0$. Thus, noting that the range of integration is over positive values of t, it follows that all the factors in the denominator, in the chosen kinematic region, are positive definite. Therefore, we have removed the absolute value from all the terms in the denominator. By performing a further change of variable $t = \frac{y}{1-y}$, we get

$$
\begin{aligned}
I_1 &= -\frac{(2|\mathbf{k}|)^{1-2d+2\beta_1+2\beta_3}}{(A_1\,A_2)^{\frac{d}{2}-\beta_1}} \int_0^1 dy\, y^{-\frac{d}{2}+\beta_3}\,(1-y)^{2d-2\beta_3-2\beta_1-2} \prod_{i=1}^{2}\left(1 + \frac{1-A_i}{A_i} y\right)^{\beta_1-\frac{d}{2}} \\
&= -\frac{(2|\mathbf{k}|)^{1-2d+2\beta_1+2\beta_3}}{(A_1\,A_2)^{\frac{d}{2}-\beta_1}} \frac{\Gamma[\beta_3 - \frac{d}{2}+1]\,\Gamma[2d - 2(\beta_3+\beta_1) - 1]}{\Gamma[\frac{3d}{2} - 2\beta_1 - \beta_3]} \\
&\quad F_1\left(\beta_3 - \frac{d}{2} + 1;\, \frac{d}{2} - \beta_1,\, \frac{d}{2} - \beta_1;\, \frac{3}{2}d - 2\beta_1 - \beta_3;\, \frac{A_1 - 1}{A_1},\, \frac{A_2-1}{A_2}\right)
\end{aligned}
\tag{5.5.141}
$$

where we have used the identity given in equation (5.5.180). Here $F_1(a;b,b';c;p,q)$ is the Appell hypergeometric function of the first kind.

The second integral gives

$$\begin{aligned} I_2 &= \int_{-\omega}^{-|\mathbf{k}|} \frac{dk^0}{|k^0-|\mathbf{k}||^{\frac{d}{2}-\beta_3}\,|k^0+|\mathbf{k}||^{\frac{d}{2}-\beta_3}\,|p_2^0+k^0-|\mathbf{k}+\mathbf{p}_2||^{\frac{d}{2}-\beta_1}|p_2^0+k^0+|\mathbf{k}+\mathbf{p}_2||^{\frac{d}{2}-\beta_1}} \\ &= -(2|\mathbf{k}|)^{1-2d+2\beta_3+2\beta_1}\int_0^{\frac{-\chi+|\mathbf{k}|}{2|\mathbf{k}|}} \frac{dt}{|t|^{\frac{d}{2}-\beta_3}\,|t-1|^{\frac{d}{2}-\beta_3}\,|t-A_1|^{\frac{d}{2}-\beta_1}\,|t-A_2|^{\frac{d}{2}-\beta_1}} \\ &= -(2|\mathbf{k}|)^{1-2d+2\beta_3+2\beta_1}\int_0^{A_1} \frac{dt}{t^{\frac{d}{2}-\beta_3}\,(1-t)^{\frac{d}{2}-\beta_3}\,(A_1-t)^{\frac{d}{2}-\beta_1}\,(A_2-t)^{\frac{d}{2}-\beta_1}} \end{aligned} \tag{5.5.142}$$

In going from 1st to 2nd line, we have made the coordinate transformation $k^0=(2t-1)|\mathbf{k}|$. We have also used the condition $|\mathbf{k}|\geq\chi$ which holds in the chosen Kinematic region. In going to the 3rd line, we have removed the absolute values. To see this, we note that $t\in[0,A_1]$ implies $|t-A_1|=(A_1-t)$. Furthermore $A_2\geq A_1$ and since $t\in[0,A_1]$, we have $t\leq A_2$ and therefore $|t-A_2|=(A_2-t)$. We have also $A_1=\frac{1}{2}\left(1-\frac{\chi}{|\mathbf{k}|}\right)$, with $0\leq\chi\leq|\mathbf{k}|$. It follows that $A_1\leq 1$, and therefore $|t-1|=1-t$. We now make a change of variable $t=-\frac{z\alpha}{1-z\alpha}$ with $\alpha=\frac{\chi-|\mathbf{k}|}{\chi+|\mathbf{k}|}\leq 0$, getting

$$\begin{aligned} I_2 &= \frac{(-1)^{\frac{d}{2}-\beta_3+1}}{(A_1A_2)^{\frac{d}{2}-\beta_1}}\frac{1}{\alpha^{\frac{d}{2}-\beta_3-1}}\int_0^1 dz\,\frac{(2|\mathbf{k}|)^{2\beta_1+2\beta_3-2d+1}(1-z\,\alpha)^{d-2\beta_3+d-2\beta_1-2}}{z^{\frac{d}{2}-\beta_3}\,(1+\frac{(1-A_1)}{A_1}\alpha\,z)^{\frac{d}{2}-\beta_1}\,(1+\frac{(1-A_2)}{A_2}\alpha\,z)^{\frac{d}{2}-\beta_1}} \\ &= (-1)^{\frac{d}{2}-\beta_3+1}\frac{(2|\mathbf{k}|)^{2\beta_1+2\beta_3-2d+1}}{(A_1A_2)^{\frac{d}{2}-\beta_1}}\frac{1}{\alpha^{\frac{d}{2}-\beta_3-1}(\beta_3-\frac{d}{2}+1)}\,F_D^{(3)}\Bigg[\beta_3-\frac{d}{2}+1, \\ &\quad\left\{-2(d-\beta_3-\beta_1-1),\,\frac{d}{2}-\beta_1,\,\frac{d}{2}-\beta_1\right\},\,\beta_3-\frac{d}{2}+2,\,\left\{\alpha,\,\alpha\,\frac{A_1-1}{A_1},\,\alpha\,\frac{A_2-1}{A_2}\right\}\Bigg] \end{aligned} \tag{5.5.143}$$

where $F_D^{(m)}$ denotes the Lauricella hypergeometric function of type D, defined by the integral representation

$$F_D^{(m)}(a;\,b_1,\ldots,b_m;\,c;\,x_1,\ldots,x_m)=\frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)}\int_0^1 t^{\,a-1}\,(1-t)^{\,c-a-1}\prod_{j=1}^{m}\big(1-t\,x_j\big)^{-b_j}\,dt \tag{5.5.144}$$

with $\mathrm{Re}(c)>\mathrm{Re}(a)>0$.[11]

We can now take the Carrollian limit $c\to 0$. In this limit, we first note that

$$A_1\simeq A_2\simeq-\frac{\omega_2}{2c\,|\mathbf{k}|},\qquad \frac{A_1-1}{A_1}\;\simeq\;\frac{A_2-1}{A_2}\;\simeq\;\alpha\;\simeq\;1\,. \tag{5.5.145}$$

[11]The condition $\mathrm{Re}(c)>\mathrm{Re}(a)>0$ in our notations are $\beta_3-\frac{d}{2}>-1$. The factor $(k^2-i\epsilon)^{\frac{d}{2}-\beta_3}$ goes to the numerator, when $\beta_3>\frac{d}{2}$. Hence, the identity (5.5.144) has been used in (5.5.143) in the sense of analytic continuation.

With the above results, the integrals in equations (5.5.141) and (5.5.143) simplify to give

$$I_2 \simeq (-1)^{\frac{d}{2}-\beta_3} I_1 \simeq c^{d-2\beta_1} \frac{(2|\mathbf{k}|)^{2\beta_3-d+1}}{(\omega_2^2)^{\frac{d}{2}-\beta_1}} B\Big(\beta_1 - \tfrac{d}{2} + 1,\, d - 2\beta_1 - 1\Big) \tag{5.5.146}$$

where $B(x, y)$ denotes the Euler Beta function. Finally, by using the identity

$$\int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{|\mathbf{p}_1 - \mathbf{k}|^{d-2\beta_2-1}|\mathbf{k}|^{d-2\beta_3-1}} = \frac{\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d-1}{2} - \beta_2 - \beta_3\right)}{\Gamma\left(\frac{d-1}{2} - \beta_2\right)\Gamma\left(\frac{d-1}{2} - \beta_3\right)} B(\beta_2, \beta_3)|\mathbf{p}_1|^{2\beta_2+2\beta_3-d+1} \tag{5.5.147}$$

we can now evaluate the last integral in equation (5.5.131), getting

$$\begin{aligned}
&\lim_{c\to 0} c^{2(\beta_1+\beta_2-d)} \int_{-\infty}^{\infty} d\omega_1 \langle\!\langle\, \mathcal{O}_{\Delta_1}(p_1)\, \mathcal{O}_{\Delta_2}(p_2)\, \mathcal{O}_{\Delta_3}(p_3) \,\rangle\!\rangle \\
&= -2^{2\beta_3-d+1}\pi^{\frac{d-3}{2}} \sin\pi\left(\frac{d}{2} - \beta_2\right)\left[\sin\pi\,(d - \beta_1 - \beta_3) + (-1)^{\frac{d}{2}-\beta_3}\sin\pi\left(\frac{d}{2} - \beta_1\right)\right] \\
&\quad B\left(\beta_2 - \frac{d}{2} + 1,\, d - 2\beta_2 - 1\right) B\left(\beta_1 - \frac{d}{2} + 1,\, d - 2\beta_1 - 1\right) B\left(\beta_3,\, \beta_1\right) \\
&\quad \frac{\Gamma\left[\frac{d-1}{2} - \beta_3 - \beta_1\right]}{\Gamma\left[\frac{d-1}{2} - \beta_3\right]\Gamma\left[\frac{d-1}{2} - \beta_1\right]} (\omega_2)^{\Delta_2+\Delta_3-\Delta_1-d}\,|\mathbf{p}_1|^{\Delta_1-d+1}
\end{aligned} \tag{5.5.148}$$

The above expression matches with the first term in the F_3 branch (5.4.101) with a specific value of the overall constant c_3 (which depends upon the spacetime boundary dimension d and the conformal dimension Δ_i).

The results of the Carrollian limit for the other branches containing $\delta(\omega_2)$ and $\delta(\omega_3)$ can be obtained in a similar manner and can be written down by permuting the indices in the above result.

Branch with $\delta(\omega_1)\delta(\omega_2)\delta(\omega_3)$

The Carrollian branch containing $\delta(\omega_1)\delta(\omega_2)\delta(\omega_3)$ originates from the region in which both conditions in equation (5.5.129) are satisfied simultaneously. In this region, ω_1, ω_2, and consequently ω_3 (due to the presence of overall delta function $\delta(\omega_1 + \omega_2 + \omega_3)$) vanish giving rise to the factor $\delta(\omega_1)\,\delta(\omega_2)\,\delta(\omega_3)$. In this case, the behavior of the CFT correlator in the Carrollian limit $c \to 0$ is determined by integrating over both ω_1 as well as ω_2. Using the identity (5.5.120), we find

$$\begin{aligned}
&\int_{-\infty}^{\infty} \prod_{i=1}^{2} d\omega_i \langle\langle \mathcal{O}_{\Delta_1}(p_1)\, \mathcal{O}_{\Delta_2}(p_2)\, \mathcal{O}_{\Delta_3}(p_3) \rangle\rangle \\
&= \frac{(2i)^3 c^2}{2\pi} \prod_{i=1}^{3}\left[\sin\pi\left(\frac{d}{2} - \beta_i\right) B\left(1 - \frac{d}{2} + \beta_i,\, d - 2\beta_i - 1\right)\right] \\
&\quad \int_{-\infty}^{\infty} \frac{d^{d-1}k}{(2\pi)^{d-1}} |\mathbf{k}|^{1-d+2\beta_3}|\mathbf{p}_1 - \mathbf{k}|^{1-d+2\beta_2}\,|\mathbf{p}_2 + \mathbf{k}|^{1-d+2\beta_1}
\end{aligned} \tag{5.5.149}$$

The momentum-space integral can be evaluated in terms of triple-K integrals. Writing $\delta_t = \delta_1 + \delta_2 + \delta_3$, one finds [306]

$$\int_{-\infty}^{\infty} \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{|\mathbf{k}|^{\delta_1}\,|\mathbf{p}_1 - |\mathbf{k}|^{\delta_2}\,|\mathbf{p}_2 + \mathbf{k}|^{\delta_3}} = \frac{2^{4\frac{d-1}{2}}}{(4\pi)^{\frac{d-1}{2}}} \frac{I_{\frac{d-1}{2}-1\{\frac{d-1}{2}-\delta_t+\delta_j\}}}{\prod_{a=1}^{3}\Gamma(\delta_a)\Gamma(d-1-\delta_t)} \tag{5.5.150}$$

with parameters

$$-\delta_t + \delta_a = \Delta_a - (d-1), \qquad \nu = \tfrac{d-3}{2}, \qquad \sigma_a = \Delta_a - \frac{(d-1)}{2} \tag{5.5.151}$$

and the triple-K integral defined by

$$I_{\nu\{\sigma_1,\sigma_2,\sigma_3\}}(k_1,k_2,k_3) = \int_0^\infty dy\, y^\nu \prod_{j=1}^{3} k_j^{\sigma_j} K_{\sigma_j}(k_j\, y)\,. \tag{5.5.152}$$

This result is in agreement with equation (5.4.100), which was obtained by solving the Conformal Carrollian Ward identities.

5.5.2.2 Wightman

The 3-point CFT Wightman correlator in momentum space in the Lorentzian signature can be expressed in the form [309] (see also [310, 311] for discussions of Wightman functions in CFT)

$$\mathcal{A}_3 \;=\; \tilde{a}(\{\beta_i\}) \int \frac{d^dk}{(2\pi)^d} \frac{\theta(p_2^0 + k^0 - |\mathbf{p}_2 + \mathbf{k}|)\theta(p_1^0 - k^0 - |\mathbf{p}_1 - \mathbf{k}|)\theta(k^0 - |\mathbf{k}|)}{|p_2 + k|^{d-2\beta_1}|p_1 - k|^{d-2\beta_2}|k|^{d-2\beta_3}} \tag{5.5.153}$$

where

$$\tilde{a}(\{\beta_i\}) = c_{123} \frac{2^{3(d+1)-2\beta_t}\pi^{3(d/2+1)}}{\prod_{j=1}^{3}\Gamma(\beta_j - d/2+1)\Gamma(\beta_j)} \quad ;\ \beta_j = \frac{\Delta_1+\Delta_2+\Delta_3}{2} - \Delta_j. \tag{5.5.154}$$

The zeroth component of the momentum vector p^μ is denoted as $p^0 = \frac{\omega}{c}$ and the norm $|p|$ is defined as[12]

$$|p| \equiv \sqrt{(p^0)^2 - |\mathbf{p}|^2} = \sqrt{\left(\frac{\omega}{c}\right)^2 - |\mathbf{p}|^2} \tag{5.5.155}$$

Again, the correlator in (5.5.153) is not symmetric under the exchange of the momenta since $(p_3^0,\, \mathbf{p}_3)$ do not appear. However, with a change of variable

$$k^0 \to k^0 - p_2^0 \;\;;\;\; \mathbf{k} \to \mathbf{k} - \mathbf{p}_2 \tag{5.5.156}$$

the Wightman correlator in equation (5.5.153) becomes

$$\mathcal{A}_3 \;=\; \tilde{a}(\{\beta_i\}) \int \frac{d^dk}{(2\pi)^d} \frac{\theta(k^0 - |\mathbf{k}|)\theta(-p_3^0 - k^0 - |\mathbf{p}_3 + \mathbf{k}|)\theta(k^0 - p_2^0 - |\mathbf{k} - \mathbf{p}_2|)}{|k|^{d-2\beta_1}|p_3 + k|^{d-2\beta_2}|k - p_2|^{d-2\beta_3}} \tag{5.5.157}$$

[12] In (5.5.155), there is no ambiguity in the sign inside the square root since the Wightman correlator vanishes for spacelike or null like external momenta

This depends upon the momenta p_2 and p_3. Similarly, with the change of variable

$$k^0 \to k^0 + p_1^0 \quad ; \quad \mathbf{k} \to \mathbf{k} + \mathbf{p}_1 \tag{5.5.158}$$

the 3-point function can be made to depend upon the momenta p_1 and p_3. One could again take a sum of the three equivalent expressions divided by 3 to make apparent the symmetry in the exchange of the three external momenta.

The Heaviside step functions in the expression of Wightman 3-point functions imply that $p_1^0 \geq 0, p_3^0 \leq 0$ while p_2^0 can be both positive as well as negative. This will be useful below when we consider the Carrollian limit of the above 3-point function. We shall now show how the different branches of Carrollian 3-point functions emerge when we take the limit $c \to 0$.

Branch with $\delta(\omega_1 + \omega_2 + \omega_3)$

For this branch, we can again directly take the limit $c \to 0$ since none of the external zeroth components of the momenta vanish. As in the time ordered case, if we rescale the loop momenta as

$$k^\mu \to |p_1| k^\mu \quad , \qquad d^d k \to |p_1|^d \, d^d k \, .$$

the denominator inside the integral in (5.5.153) rescales as

$$\frac{1}{|p_2 + k|^{d-2\beta_1} |p_1 - k|^{d-2\beta_2} |k|^{d-2\beta_3}} \quad \to \quad \frac{|p_1|^{\Delta_t - 3d}}{\left|\frac{p_2}{|p_1|} + k\right|^{d-2\beta_1} \left|\frac{p_1}{|p_1|} - k\right|^{d-2\beta_2} |k|^{d-2\beta_3}} \tag{5.5.159}$$

Next, under this rescaling, the first theta function becomes

$$\theta(p_2^0 + k^0 - |\mathbf{p}_2 + \mathbf{k}|) \to \theta\Bigg(|p_1| \left(\frac{p_2^0}{|p_1|} + k^0 - \left|\frac{\mathbf{p}_2}{|p_1|} + \mathbf{k}\right|\right)\Bigg) = \theta\Bigg(\frac{p_2^0}{|p_1|} + k^0 - \left|\frac{\mathbf{p}_2}{|p_1|} + \mathbf{k}\right|\Bigg) \tag{5.5.160}$$

where, we have used the fact that $|p_1| > 0$. Hence, rescaling the theta function by $|p_1|$ will not change its value. Now, in the limit $c \to 0$, we would have

$$\frac{p_2^0}{|p_1|} \simeq \frac{\omega_2}{|\omega_1|} \quad ; \qquad \frac{\mathbf{p}_2}{|p_1|} \simeq c \, \frac{\mathbf{p}_2}{|\omega_1|} \simeq 0 \tag{5.5.161}$$

This shows that the argument of the theta function becomes a function of the ratio $\frac{\omega_2}{|\omega_1|}$. Following a similar manipulation as above, it is easy to see that the other two theta functions become independent of the external momenta in the limit $c \to 0$. Hence, the Wightman correlator again factorizes and we get the expected result

$$\lim_{c \to 0} c^{2d - \Delta_1 - \Delta_2 - \Delta_3} \mathcal{A}_3 = \left(\omega_1^2\right)^{\frac{\Delta_1 + \Delta_2 + \Delta_3 - 2d}{2}} \, f\left(\frac{\omega_2}{\omega_1}\right) . \tag{5.5.162}$$

where f is some function which can be explicitly evaluated. Again, unlike (5.4.99), the function $f\left(\frac{\omega_2}{\omega_1}\right)$ in (5.5.162), arising in the Carrollian limit, is not arbitrary.

Branch with $\delta(\omega_1)\delta(\omega_2)\delta(\omega_3)$

The branch with 3 delta functions in the energy direction should take the following form in the Carrollian limit

$$\lim_{c\to 0} c^{\alpha}\mathcal{A}_3 = f_1(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\delta(\omega_1)\delta(\omega_2)\delta(\omega_3) \tag{5.5.163}$$

for some α. We can determine the function f_1 by integrating both sides over ω_1 and ω_2 as follows

$$\begin{aligned} c^{\alpha}\int_{-\infty}^{\infty} & d\omega_1 d\omega_2\ \mathcal{A}_3 \\ &= \tilde{a}(\{\beta_i\})c^{\alpha}\int_{-\infty}^{\infty} d\omega_1 d\omega_2 \int \frac{d^dk}{(2\pi)^d}\frac{\theta(p_2^0+k^0-|\mathbf{p}_2+\mathbf{k}|)\theta(p_1^0-k^0-|\mathbf{p}_1-\mathbf{k}|)\theta(k^0-|\mathbf{k}|)}{|p_2+k|^{d-2\beta_1}|p_1-k|^{d-2\beta_2}|k|^{d-2\beta_3}} \end{aligned} \tag{5.5.164}$$

The integration over ω_1 is given by

$$\begin{aligned} \int_{-\infty}^{\infty} & d\omega_1 \frac{\theta(p_1^0-k^0-|\mathbf{p}_1-\mathbf{k}|)}{((p_1^0-k^0)^2-|\mathbf{p}_1-\mathbf{k}|^2)^{\frac{d}{2}-\beta_2}} \\ &= \int_{ck^0+c|\mathbf{p}_1-\mathbf{k}|}^{\infty} d\omega_1 \frac{1}{((\frac{\omega_1}{c}-k^0)-|\mathbf{p}_1-\mathbf{k}|)^{\frac{d}{2}-\beta_2}((\frac{\omega_1}{c}-k^0)+|\mathbf{p}_1-\mathbf{k}|)^{\frac{d}{2}-\beta_2}} \\ &= \frac{2^{2\beta_2-d+1}c}{|\mathbf{p}_1-\mathbf{k}|^{d-2\beta_2-1}}\int_0^{\infty} dx \frac{1}{x^{\frac{d}{2}-\beta_2}(x+1)^{\frac{d}{2}-\beta_2}} \\ &= \frac{2^{2\beta_2-d+1}c}{|\mathbf{p}_1-\mathbf{k}|^{d-2\beta_2-1}} B\left(\beta_2-\frac{d}{2}+1, d-2\beta_2-1\right) \end{aligned} \tag{5.5.165}$$

In going from the 1st to 2nd equality, we have made a coordinate change

$$\omega_1 = (2c|\mathbf{p}_1-\mathbf{k}|x + ck^0 + c|\mathbf{p}_1-\mathbf{k}|) \quad\implies\quad d\omega_1 = 2c|\mathbf{p}_1-\mathbf{k}|\ dx \tag{5.5.166}$$

In a similar way, the integration over ω_2 is given by

$$\int_{-\infty}^{\infty} d\omega_2 \frac{\theta(p_2^0+k^0-|\mathbf{p}_2+\mathbf{k}|)}{|p_2+k|^{d-2\beta_1}} = \frac{2^{2\beta_1-d+1}c}{|\mathbf{p}_2+\mathbf{k}|^{d-2\beta_1-1}} B\left(\beta_1-\frac{d}{2}+1, d-2\beta_1-1\right) \tag{5.5.167}$$

With the above integrations, the k^0 dependence in (5.5.164) simplifies and the integration over this can be easily performed

$$\int \frac{dk^0}{(2\pi)}\frac{\theta(k^0-|\mathbf{k}|)}{|k|^{d-2\beta_3}} = \frac{1}{2\pi}\frac{2^{2\beta_3-d+1}}{|\mathbf{k}|^{d-2\beta_3-1}} B\left(\beta_3-\frac{d}{2}+1, d-2\beta_3-1\right) \tag{5.5.168}$$

Combining all the terms, the final expression is given by

$$f_1(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) = c^{2+\alpha}\hat{a}(\{\beta_i\})\int \frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{1}{|\mathbf{p}_1-\mathbf{k}|^{d-2\beta_2-1}|\mathbf{p}_2+\mathbf{k}|^{d-2\beta_1-1}|\mathbf{k}|^{d-2\beta_3-1}} \tag{5.5.169}$$

Choosing $\alpha = -2$ gives the desired result. In the above expression, we have defined

$$\hat{a}(\{\beta_i\}) = \frac{\tilde{a}(\{\beta_i\})}{2\pi} \prod_{i=1}^{3} 2^{2\beta_i - d+1} B\left(\beta_i - \frac{d}{2} + 1, d - 2\beta_i - 1\right) \tag{5.5.170}$$

The integral in (5.5.169) is a triple K integral [306] (see equation (C.1.6)) giving

$$f_1(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = c_{123}\, a(d, \Delta_i)\, I_{\frac{d-3}{2}\{\Delta_1 - \frac{d-1}{2}, \Delta_2 - \frac{d-1}{2}, \Delta_3 - \frac{d-1}{2}\}}(k_1, k_2, k_3) \tag{5.5.171}$$

where

$$a(d, \Delta_i) = \frac{\pi^{1+d} 2^{\frac{9+3d}{2} - \Delta_1 - \Delta_2 - \Delta_3}}{\Gamma\left(\frac{\Delta_2+\Delta_3-\Delta_1+d-1}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_3-\Delta_2+d-1}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3+d-1}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2+\Delta_3-d+1}{2}\right)} \tag{5.5.172}$$

Branch with $\delta(\omega_1)\delta(\omega_2 + \omega_3)$

In this case, we expect

$$\lim_{\substack{c \to 0 \\ \omega_1 \to 0}} c^{\sigma_1} \mathcal{A}_3 = f_3(\omega_2, \mathbf{p}_1)\delta(\omega_1)\delta(\omega_2 + \omega_3) \tag{5.5.173}$$

We first perform the integration over ω_1. This is done above in equation (5.5.165) using which we obtain

$$c^{\sigma_1} \int_{-\infty}^{\infty} d\omega_1 \mathcal{A}_3 = c^{\sigma_1+1} a_1 \int \frac{d^d k}{(2\pi)^d} \frac{\theta(p_2^0 + k^0 - |\mathbf{p}_2 + \mathbf{k}|)\theta(k^0 - |\mathbf{k}|)}{|p_2 + k|^{d-2\beta_1} |\mathbf{p}_1 - \mathbf{k}|^{d-2\beta_2-1} |k|^{d-2\beta_3}} \tag{5.5.174}$$

where

$$a_1 = 2^{2\beta_2 - d+1} B\left(\beta_2 - \frac{d}{2} + 1, d - 2\beta_2 - 1\right) \tilde{a}(\beta_i) \tag{5.5.175}$$

Next, we need to perform the integration over the variable k^0. To do this integral, we note that the Heaviside step functions imply

$$k^0 \geq |\mathbf{k}| \quad \text{and} \qquad k^0 \geq -p_2^0 + |\mathbf{p}_2 + \mathbf{k}| \tag{5.5.176}$$

Both the conditions must be satisfied simultaneously. There are two possibilities for which both the above conditions can be satisfied

$$|\mathbf{k}| \geq -p_2^0 + |\mathbf{p}_2 + \mathbf{k}| \quad \text{or} \qquad |\mathbf{k}| \leq -p_2^0 + |\mathbf{p}_2 + \mathbf{k}| \tag{5.5.177}$$

A simple analysis shows that the second condition in (5.5.177) is not compatible with having $\delta(\omega_1)$. Hence, we discard this option and consider the first one. Thus, the integration over k^0

becomes

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{dk^0}{(2\pi)}\frac{\theta(p_2^0+k^0-|\mathbf{p}_2+\mathbf{k}|)\theta(k^0-|\mathbf{k}|)}{|p_2+k|^{d-2\beta_1}|k|^{d-2\beta_3}}\\
&=\int_{|\mathbf{k}|}^{\infty}\frac{dk^0}{(2\pi)}\frac{1}{(\frac{\omega_2}{c}+k^0-|\mathbf{p}_2+\mathbf{k}|)^{\frac{d}{2}-\beta_1}(\frac{\omega_2}{c}+k^0+|\mathbf{p}_2+\mathbf{k}|)^{\frac{d}{2}-\beta_1}(k^0-|\mathbf{k}|)^{\frac{d}{2}-\beta_3}(k^0+|\mathbf{k}|)^{\frac{d}{2}-\beta_3}}\\
&=\frac{2^{2\beta_3+2\beta_1-2d+1}}{|\mathbf{k}|^{2d-2\beta_1-2\beta_3-1}}\int_0^{\infty}\frac{dx}{(2\pi)}\frac{1}{(x+q)^{\frac{d}{2}-\beta_1}(x+r)^{\frac{d}{2}-\beta_1}x^{\frac{d}{2}-\beta_3}(1+x)^{\frac{d}{2}-\beta_3}}\\
&=\frac{2^{2\beta_3+2\beta_1-2d+1}}{|\mathbf{k}|^{2d-2\beta_1-2\beta_3-1}}\frac{1}{(ab)^{\frac{d}{2}-\beta_1}}\int_0^1\frac{dy}{(2\pi)}\frac{y^{\beta_3-\frac{d}{2}}(1-y)^{2d-2\beta_1-2\beta_3-2}}{(1-\tilde{q}y)^{\frac{d}{2}-\beta_1}(1-\tilde{r}y)^{\frac{d}{2}-\beta_1}}\\
&=\frac{2^{2\beta_3+2\beta_1-2d+1}}{|\mathbf{k}|^{2d-2\beta_1-2\beta_3-1}}\frac{\Gamma(\beta_3-\frac{d}{2}+1)\Gamma(2d-2\beta_1-2\beta_3-1)}{2\pi(qr)^{\frac{d}{2}-\beta_1}\Gamma(\frac{3d}{2}-2\beta_1-\beta_3)}\\
&\quad F_1\left(\beta_3-\frac{d}{2}+1;\frac{d}{2}-\beta_1,\frac{d}{2}-\beta_1;\frac{3d}{2}-2\beta_1-\beta_3;\tilde{a},\tilde{b}\right)
\end{aligned}\tag{5.5.178}$$

In going from 1st to 2nd equality, we have made a coordinate change $k^0=2|\mathbf{k}|x+|\mathbf{k}|$ while in going from 2nd to 3rd equality, we have made the coordinate change $x=\frac{y}{1-y}$. We have also defined $\tilde{q}=\frac{q-1}{q}$ and $\tilde{r}=\frac{r-1}{r}$ with

$$q\equiv\frac{1}{2}+\frac{\omega_2}{2c|\mathbf{k}|}-\frac{|\mathbf{p}_2+\mathbf{k}|}{2|\mathbf{k}|}\quad;\qquad r=\frac{1}{2}+\frac{\omega_2}{2c|\mathbf{k}|}+\frac{|\mathbf{p}_2+\mathbf{k}|}{2|\mathbf{k}|}\tag{5.5.179}$$

In writing the last equality, we have used the Appell hypergeometric identity

$$\int_0^1 dy\frac{y^{\alpha-1}(1-y)^{\gamma-\alpha-1}}{(1-py)^{\beta}(1-qy)^{\beta'}}=\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}F_1(\alpha;\beta,\beta';\gamma;p,q);\quad(\alpha>0;\quad\gamma-\alpha>0)\tag{5.5.180}$$

We want to specialize the result in (5.5.178) in the limit $c\to 0$. In this limit, we have

$$q\simeq\frac{\omega_2}{2c|\mathbf{k}|}\quad;\qquad r\simeq\frac{\omega_2}{2c|\mathbf{k}|}\quad;\qquad \tilde{q}\simeq 1-\frac{2c|\mathbf{k}|}{\omega_2}\quad;\qquad \tilde{r}\simeq 1-\frac{2c|\mathbf{k}|}{\omega_2}\tag{5.5.181}$$

In this limit, we can simplify the expression using the following Appell hypergeometric identity

$$\lim_{x\to1}F_1(\alpha;\beta,\beta';\gamma;x;x)=\lim_{x\to1}{}_2F_1(\alpha,\beta+\beta';\gamma;x)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta-\beta')}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta-\beta')}\tag{5.5.182}$$

Using the above identity together with

$$\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{1}{|\mathbf{p}_1-\mathbf{k}|^{d-2\beta_2-1}|\mathbf{k}|^{d-2\beta_3-1}}=\frac{\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d-1}{2}-\beta_2-\beta_3\right)}{\Gamma\left(\frac{d-1}{2}-\beta_2\right)\Gamma\left(\frac{d-1}{2}-\beta_3\right)}B(\beta_2,\beta_3)|\mathbf{p}_1|^{2\beta_2+2\beta_3-d+1}\tag{5.5.183}$$

the expression in (5.5.178) gets simplified and we finally get

$$c^{\sigma_1}\int_{-\infty}^{\infty} d\omega_1 \mathcal{A}_3 = c^{\sigma_1+1+d-2\beta_1}\, c_{123} b_1(d,\Delta_i)\, \omega_2^{2\beta_1-d}|\mathbf{p}_1|^{2\beta_2+2\beta_3-d+1} \tag{5.5.184}$$

where

$$b_1(d,\Delta_i) = \frac{\pi^{2d+\frac{1}{2}} 2^{3d-\Delta_1-\Delta_2-\Delta_3}\Gamma\left(\frac{1}{2}(d-2\Delta_1-1)\right)}{\Gamma(\Delta_1)\Gamma\left(\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_3)\right)\Gamma\left(\frac{1}{2}(-d-\Delta_1+\Delta_2+\Delta_3+2)\right)} \tag{5.5.185}$$

By choosing $\sigma_1 = 2\beta_1 - d - 1$, we get the desired Carrollian branch

$$f_3(\omega_2,\mathbf{p}_1) = c_{123} b_1(d,\Delta_i)\, \omega_2^{\Delta_2+\Delta_3-\Delta_1-d}|\mathbf{p}_1|^{2\Delta_1-d+1} \tag{5.5.186}$$

Branch with $\delta(\omega_2)\delta(\omega_1+\omega_3)$

In this case, we expect

$$\lim_{c\to 0} c^{\sigma_2}\mathcal{A}_3 = f_4(\omega_1,|\mathbf{p}_2|)\delta(\omega_2)\delta(\omega_1+\omega_3) \tag{5.5.187}$$

To fix the form of the function f_4, we perform the integration over ω_2. The relevant integral was done in equation (5.5.167) which gives

$$c^{\sigma_2}\int_{-\infty}^{\infty} d\omega_2 \mathcal{A}_3 = c^{\sigma_2+1} a_2(\{\beta_i\})\int \frac{d^dk}{(2\pi)^d}\frac{\theta(p_1^0-k^0-|\mathbf{p}_1-\mathbf{k}|)\theta(k^0-|\mathbf{k}|)}{|p_1-k|^{d-2\beta_2}|\mathbf{p}_2+\mathbf{k}|^{d-2\beta_1-1}|k|^{d-2\beta_3}} \tag{5.5.188}$$

where

$$a_2 = 2^{2\beta_1-d+1}B\left(\beta_1-\frac{d}{2}+1, d-2\beta_1-1\right)\tilde{a} \tag{5.5.189}$$

Next, we need to perform the integration over k^0. Due to the step theta functions, the range of k^0 is restricted. The two theta functions imply following conditions

$$k^0 \geq |\mathbf{k}| \quad \text{and} \qquad k^0 \leq p_1^0 - |\mathbf{p}_1-\mathbf{k}| \tag{5.5.190}$$

Both the conditions must be satisfied simultaneously. There are two possibilities

$$|\mathbf{k}| \geq p_1^0 - |\mathbf{p}_1-\mathbf{k}| \quad \text{or} \qquad |\mathbf{k}| \leq p_1^0 - |\mathbf{p}_1-\mathbf{k}| \tag{5.5.191}$$

In the first case, the two conditions of (5.5.190) are mutually exclusive and can't be satisfied. Hence, the integration range would be null and the integration would be trivially zero. In the second case, the range of k^0 integration would be from $|\mathbf{k}|$ to $p_1^0 - |\mathbf{p}_1-\mathbf{k}|$. Hence, the integration over k^0 gives

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{dk^0}{(2\pi)}\frac{\theta(p_1^0-k^0-|\mathbf{p}_1-\mathbf{k}|)\theta(k^0-|\mathbf{k}|)}{|p_1-k|^{d-2\beta_2}|k|^{d-2\beta_3}}\\
=&\int_{|\mathbf{k}|}^{p_1^0-|\mathbf{p}_1-\mathbf{k}|}\frac{dk^0}{(2\pi)}\left(\frac{1}{(\frac{\omega_1}{c}-k^0-|\mathbf{p}_1-\mathbf{k}|)^{\frac{d}{2}-\beta_2}(\frac{\omega_1}{c}-k^0+|\mathbf{p}_1-\mathbf{k}|)^{\frac{d}{2}-\beta_2}}\right)\\
&\left(\frac{1}{(k^0-|\mathbf{k}|)^{\frac{d}{2}-\beta_3}(k^0+|\mathbf{k}|)^{\frac{d}{2}-\beta_3}}\right),\\
=&\frac{2^{2\beta_2+2\beta_3-2d+1}}{|\mathbf{k}|^{2d-2\beta_2-2\beta_3-1}}\int_0^a\frac{dx}{(2\pi)}\frac{1}{x^{\frac{d}{2}-\beta_3}(1+x)^{\frac{d}{2}-\beta_3}(s-x)^{\frac{d}{2}-\beta_2}(t-x)^{\frac{d}{2}-\beta_2}},\\
=&\frac{2^{2\beta_2+2\beta_3-2d+1}}{|\mathbf{k}|^{2d-2\beta_2-2\beta_3-1}}\frac{1}{(st)^{\frac{d}{2}-\beta_2}}\int_0^{\frac{1}{\tilde{a}}}\frac{dy}{(2\pi)}\frac{y^{-\frac{d}{2}+\beta_3}(1-y)^{2d-2\beta_2-2\beta_3}}{(1-\tilde{s}y)^{\frac{d}{2}-\beta_2}(1-\tilde{t}y)^{\frac{d}{2}-\beta_2}}.
\end{aligned}
\tag{5.5.192}
$$

In going from the first to second equality, we have made the coordinate change $k^0=(2|\mathbf{k}|x+|\mathbf{k}|)$ whereas in going from the second to third equality, we have made the coordinate transformation $x=\frac{y}{1-y}$. We have defined $\tilde{s}=\frac{1+s}{s}$ and $\tilde{t}=\frac{1+t}{t}$, where

$$
s=-\frac{1}{2}+\frac{\omega_1}{2c|\mathbf{k}|}-\frac{|\mathbf{p}_1-\mathbf{k}|}{2|\mathbf{k}|}\quad;\quad t=-\frac{1}{2}+\frac{\omega_1}{2c|\mathbf{k}|}+\frac{|\mathbf{p}_1-\mathbf{k}|}{2|\mathbf{k}|}
\tag{5.5.193}
$$

As before, we only need to know the above integral in the limit $c\to 0$. In this limit, we have

$$
s\simeq\frac{\omega_1}{2c|\mathbf{k}|}\quad;\quad t\simeq\frac{\omega_1}{2c|\mathbf{k}|}\quad;\quad \tilde{s}\simeq 1+\frac{2c|\mathbf{k}|}{\omega_1}\quad;\quad \tilde{t}\simeq\frac{2c|\mathbf{k}|}{\omega_1}
$$

At the leading order $\tilde{s}\simeq\tilde{t}\simeq 1$ and the integral can be performed as before to obtain

$$
c^{\sigma_2}\int_{-\infty}^{\infty}d\omega_2\mathcal{A}_3 = c^{\sigma_2+1+d-2\beta_2}c_{123}b_2(d,\Delta_i)\omega_1^{2\beta_2-d}|\mathbf{p}_2|^{2\beta_1+2\beta_3-d+1}
\tag{5.5.194}
$$

where,

$$
b_2(d,\Delta_i) = \frac{\pi^{2d+\frac{1}{2}}2^{3d-\Delta_1-\Delta_2-\Delta_3}\Gamma\left(\frac{1}{2}\left(d-2\Delta_2-1\right)\right)}{\Gamma\left(\Delta_2\right)\Gamma\left(\frac{1}{2}\left(\Delta_1-\Delta_2+\Delta_3\right)\right)\Gamma\left(\frac{1}{2}\left(-d+\Delta_1-\Delta_2+\Delta_3+2\right)\right)}
\tag{5.5.195}
$$

By choosing $\sigma_1=2\beta_2-d-1$, we get the desired Carrollian branch

$$
f_4(\omega_1,|\mathbf{p}_2|)=c_{123}b_2(d,\Delta_i)\,\omega_1^{\Delta_1-\Delta_2+\Delta_3-d}|\mathbf{p}_2|^{2\Delta_2-d+1}
\tag{5.5.196}
$$

Branch with $\delta(\omega_3)\delta(\omega_1+\omega_2)$

In this case, we expect

$$
\lim_{c\to 0}c^{\sigma_3}\mathcal{A}_3=f_5(\omega_2,|\mathbf{p}_3|)\delta(\omega_3)\delta(\omega_1+\omega_2)
\tag{5.5.197}
$$

For getting this Carrollian branch, we need to consider the representation (5.5.188) of the 3-point function. The integration over ω_3 is given by

$$\int_{-\infty}^{\infty} d\omega_3 \frac{\theta(-p_3^0 - k^0 - |\mathbf{p}_3 + \mathbf{k}|)}{((p_3^0 + k^0)^2 - |\mathbf{p}_3 + \mathbf{k}|^2)^{\frac{d}{2}-\beta_2}} \quad = \quad \frac{2^{2\beta_2 - d+1} c}{|\mathbf{p}_3 + \mathbf{k}|^{d-2\beta_2-1}} B\left(\beta_2 - \frac{d}{2} + 1, d - 2\beta_2 - 1\right) \tag{5.5.198}$$

Next, we need to perform the integration over k^0. Using the same procedure as before and making the similar approximation, we find

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{dk^0}{(2\pi)} \frac{\theta(k^0 - p_2^0 - |\mathbf{k} - \mathbf{p}_2|)\theta(k^0 - |\mathbf{k}|)}{|k - p_2|^{d-2\beta_3} |k|^{d-2\beta_1}} \\
&= \frac{2^{2\beta_1 - d+1}}{|\mathbf{k}|^{d-2\beta_1 - 1}} \frac{\Gamma(\beta_1 - \frac{d}{2} + 1)\Gamma(d - 2\beta_1 - 1)}{2\pi\Gamma(\frac{d}{2} - \beta_1)} \left(\frac{\omega_2}{c}\right)^{2\beta_3 - d}
\end{aligned} \tag{5.5.199}$$

Note that due to the Heaviside theta functions, the integration variable must satisfy the conditions

$$k^0 \geq |\mathbf{k}| \quad \text{and} \quad k^0 \geq p_2^0 + |\mathbf{k} - \mathbf{p}_2| \tag{5.5.200}$$

Again, there are two possibilities

$$|\mathbf{k}| \geq p_2^0 + |\mathbf{k} - \mathbf{p}_2| \quad \text{or} \quad |\mathbf{k}| \leq p_2^0 + |\mathbf{k} - \mathbf{p}_2| \tag{5.5.201}$$

The second option is not possible in this case since due to the delta functions $\delta(\omega_3)\delta(\omega_1 + \omega_2)$, we must have $\omega_2 < 0$ since ω_1 is always positive. Finally, using the identity (5.5.147), we obtain

$$c^{\sigma_3} \int_{-\infty}^{\infty} d\omega_3 \mathcal{A}_3 \quad = \quad c^{\sigma_3 + 1 + d - 2\beta_3} c_{123} b_3(d, \Delta_a) \omega_2^{2\beta_3 - d} |\mathbf{p}_3|^{2\beta_1 + 2\beta_2 - d + 1} \tag{5.5.202}$$

where

$$b_3(d, \Delta_i) \quad = \quad \frac{\pi^{2d+\frac{1}{2}} 2^{3d - \Delta_1 - \Delta_2 - \Delta_3} \Gamma\left(\frac{1}{2}\left(d - 2\Delta_3 - 1\right)\right)}{\Gamma\left(\Delta_3\right) \Gamma\left(\frac{1}{2}\left(\Delta_1 + \Delta_2 - \Delta_3\right)\right) \Gamma\left(\frac{1}{2}\left(-d + \Delta_1 + \Delta_2 - \Delta_3 + 2\right)\right)} \tag{5.5.203}$$

By choosing $\sigma_3 = 2\beta_3 - d - 1$, we get the desired Carrollian branch

$$f_5(\omega_2, |\mathbf{p}_3|) = c_{123} b_3(d, \Delta_a)\, \omega_2^{\Delta_1 + \Delta_2 - \Delta_3 - d} |\mathbf{p}_3|^{2\Delta_3 - d + 1} \tag{5.5.204}$$

5.6 Discussion

In this chapter, we have studied the Carrollian conformal theories in the momentum space. We have focused on the scalar operators and only used those aspects of the theories which are dictated by the symmetry. For this, we have used the Ward identities corresponding to the Carrollian conformal symmetry. By taking a transform of the position space Ward identities, we first obtained these Ward identities in the momentum space and then solved them for the 2 and

3 point functions. Unlike standard conformal field theories, the Ward identities of Carrollian conformal theories admit multiple structures reflecting the fact that the Carrollian manifold has a null direction. The 2-point function has two independent structures known as electric and magnetic branches. On the other hand, the 3-point function admits 5 independent branches. For the 2-point function, both branches are completely fixed up to some normalization constant. On the other hand, for the 3-point functions, four of the branches are fixed by symmetries whereas one of the branch is only fixed up to a function of ratios of the Carrollian energies.

We also considered the Carrollian limit of the standard conformal field theory correlators. In the Carrollian limit $c \to 0$, we showed how to recover the different branches of the Carrollian conformal theories in the momentum space. We considered both Wightman as well as time-ordered correlators. If we put the CFT on the AdS boundary, then their Carrollian limit $c \to 0$ can be interpreted as the flat limit of AdS from the boundary perspective. In the bulk, the flat limit corresponds to taking the large AdS radius limit $R \to \infty$. By combining (and considering all massless particles) the bulk analysis in momentum space done in [129, 147, 153, 300] with the Carrollian limit of the boundary CFT developed here, one would obtain a unified momentum–space description of bulk and boundary dynamics in flat spacetime. Such a framework could provide new insights into the emergence of flat Carrollian holography from the flat and ultra-relativistic limits of AdS/CFT. A detailed presentation of the bulk analysis, including the flat limit of AdS correlators in momentum space for massless fields, will be discussed in more detail in [152].

As mentioned in the introduction, one application of the momentum space analysis considered in this chapter is in the study of the analytic properties of Carrollian conformal theories on the flat space null boundaries. Explicit analytic structures of Carrollian conformal correlators in the momentum space would be useful for this purpose. This may also require the regularization of Carrollian conformal correlators for special values of the Carrollian dimensions and a better understanding of the logarithmic terms discussed earlier. We leave these questions for future work.

In this chapter, we have not considered the 4 and higher point correlators. However, these correlators must also be a solution of the Ward identities. In the usual CFTs, the higher point functions in the momentum space can be represented geometrically as simplex integrals [312, 313]. The structure of the higher point Carrollian conformal correlators are expected to be more complicated due to the presence of several branches as well as the fact that symmetries do not completely fix even the 3-point correlators. Hence, we expect more freedom in the higher point Carrollian conformal correlators.

We have focused only on the null boundaries in this chapter. This is relevant for the massless bulk fields. However, for the massive bulk fields, we also need to take into account the timelike and spacelike components of the flat space boundaries [229, 314, 315]. The theories living on these boundary components are still very poorly understood compared to the theories on the null boundaries. It would be interesting to see whether we can use similar momentum space techniques used in this chapter to study the theories on the time like and spacelike boundaries.

In chapter (6), following [152], we will explore a generating functional for the CCFT correlators between the null boundaries of flat spacetime.

Chapter 6

Carrollian Generating Functional

In a gauge/gravity correspondence gauge-invariant boundary operators are in one-to-one correspondence with bulk fields. Their boundary values act as sources for the dual operators and are fixed by solving the classical equations of motion with appropriate boundary conditions. Asymptotically Anti de Sitter spacetimes have a boundary at spatial infinity where one needs to specify boundary conditions for the bulk field. In the AdS/CFT correspondence, the on-shell bulk partition functional with the boundary value of the bulk field as source acts as the generating functional for boundary QFT correlation functions

$$\mathcal{Z}_{SUGRA}[\phi_{(0)}] = \int_{\Phi\sim\phi_{(0)}} D\Phi \,\exp(-S[\Phi]) = \langle \exp\left(-\int_{\partial AAdS} \phi_{(0)}\mathcal{O}\right)\rangle_{QFT} \tag{6.0.1}$$

where the expectation value on the right hand side is over the QFT path integral. In the leading saddle point approximation, this relation becomes

$$S^{on\,shell}[\phi_{(0)}] = -W_{QFT}[\phi_{(0)}] \tag{6.0.2}$$

i.e., the on-shell supergravity action is the generating function of QFT connected graphs. We apply this paradigm to Carrollian flat holography for a free complex scalar field.

Minkowski spacetime in flat null coordinates: Consider the $(d+1)$-dimensional Minkowski spacetime $\mathbb{M}^{d+1}$ in Cartesian coordinates (X^0, X^a, X^d)

$$ds^2_{d+1} = -(dX^0)^2 + \delta_{ab}dX^a dX^b + (dX^d)^2, \qquad a, b \in \{1, 2, \cdots, d-1\}. \tag{6.0.3}$$

To approach null infinity, it is useful to consider the flat null coordinates (u, r, x^a) with $a \in \{1, 2, \cdots, d-1\}$, which are related to the Cartesian coordinates as follows,

$$X^0 = \frac{1}{\sqrt{2}}[u + r(1 + x^2)], \qquad X^a = \sqrt{2}\, r\, x^a, \qquad X^d = \frac{1}{\sqrt{2}}[-u + r(1 - x^2)] \tag{6.0.4}$$

The inverse transformation is,

$$u = \frac{(X^0)^2 - X^a X_a - (X^d)^2}{\sqrt{2}\,(X^0 + X^d)}, \qquad x^a = \frac{X^a}{X^0 + X^d}, \qquad r = \frac{X^0 + X^d}{\sqrt{2}}. \tag{6.0.5}$$

The coordinates are ill-defined on the hypersurface $X^0 + X^d = 0$ where $r = 0$. Therefore

$$u \in \mathbb{R}, \qquad x^a \in \mathbb{R}, \qquad r \in \mathbb{R}\backslash\{0\} \tag{6.0.6}$$

This is a planar cut version of the Bondi coordinates. The flat metric in these coordinates takes the form,

$$ds^2_{d+1} = -2dudr + 2r^2\delta_{ab}dx^a dx^b. \tag{6.0.7}$$

Since $g_{uu} = 0$, ∂_u is a null vector and u =constant are null hypersurfaces. The induced metric on the future and past null infinities, $\mathscr{I}^+$ and $\mathscr{I}^-$ is obtained by rescaling the flat metric in (1.3.138) by $1/2r^2$ and taking the limits $r \to +\infty$ and $r \to -\infty$, respectively. This gives

$$ds^2_{\mathscr{I}^\pm} = 0 \cdot du^2 + \delta_{ab}dx^a dx^b. \tag{6.0.8}$$

6.1 Free massless scalar field

Consider the case of a complex scalar field with free action,

$$S = -\int d^{d+1}x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi\right) \tag{6.1.9}$$

The equation of motion in flat null coordinates (6.0.7) takes the form

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\big) = \left(\frac{1}{2r^2}\partial_a\partial^a - \frac{(d-1)}{r}\partial_u - 2\partial_u\partial_r\right)\Phi(u,r,x^a) = 0 \tag{6.1.10}$$

As specified in eq. (1.3.137), the chosen coordinate system is singular at $r = 0$ and future and past null infinities $\mathscr{I}^\pm$ are reached in the limits $r \to \pm\infty$ respectively. For this reason, we treat the equation of motion separately on the two disconnected coordinate patches $r > 0$ and $r < 0$ and we write the field in the piecewise form:

$$\phi(r,\,u,\,x) \equiv \begin{cases} \phi_+(u,\,r,\,x) & r > 0 \\ \phi_-(u,\,r,x) & r < 0 \end{cases} \tag{6.1.11}$$

where $\phi_\pm$ satisfy the same bulk equation on their respective domain. A relation between ϕ_+ and ϕ_- can be imposed only after specifying additional matching or regularity conditions across $r = 0$.

We want to find the solutions and asymptotic behaviour of the field ϕ for both positive and negative values of r. To analyse the solution of the equation of motion on the $r > 0$ coordinate patch, which approaches $\mathcal{I}^+$ in the limit $r \to +\infty$, we propose the following series ansatz:

$$\phi_+(r,u,x) = \frac{1}{r^\Delta}\sum_{n=0}^{\infty}\frac{\phi_n^+(u,x)}{r^n}, \qquad r > 0. \tag{6.1.12}$$

Plugging this into the equation of motion (6.1.10), we get

$$r^{-\Delta-1} \sum_{n=0}^{\infty} \left(r^{-n-1} \nabla^2 \phi_n^+ + r^{-n} \left[-2(d-1) + 4(\Delta+n) \right] \partial_u \phi_n^+ \right) = 0 \tag{6.1.13}$$

where $\nabla^2 = \partial_a \partial^a$. This implies

$$\begin{aligned} &(2\Delta - (d-1))\partial_u \phi_n^+ = 0, \\ &\nabla^2 \phi_{n-1}^+ + \left[-2(d-1) + 4(\Delta+n) \right] \partial_u \phi_n^+ = 0, \qquad n \in \{1, 2, ...\}; \ \phi_{-1} = 0. \end{aligned} \tag{6.1.14}$$

We have two possible solutions. One with

$$\Delta = \frac{(d-1)}{2}, \qquad \phi_0^+(u, x) \text{ unfixed}, \tag{6.1.15}$$

and another with

$$\partial_u \phi_0^+ = 0, \qquad \Delta \text{ unfixed}. \tag{6.1.16}$$

The analysis on the $r < 0$ patch is entirely analogous, with ϕ_- replacing ϕ_+, throughout.

6.1.1 Solution with $\Delta = (d-1)/2$

We will consider the solution with $\Delta = (d-1)/2$.

6.1.1.1 Series solution

We have

$$\phi_n(u, x) = \int_0^u du' \left(-\frac{\nabla^2}{4n} \right) \phi_{n-1}(u', x) + \psi_n(x) \tag{6.1.17}$$

Let us explicitly write down a few of the terms,

$$\begin{aligned} \phi_1(u, x) &= \int_0^u du_1 \left(-\frac{\nabla^2}{4} \right) \phi_0(u_1, x) + \psi_1(x) \\ \phi_2(u, x) &= \int_0^u du_1 \int_0^{u_1} du_2 \, \frac{1}{2} \left(-\frac{\nabla^2}{4} \right)^2 \phi_0(u_2, x) + \frac{1}{2} \left(-\frac{\nabla^2}{4} \right) u \, \psi_1(x) + \psi_2(x) \\ \phi_3(u, x) &= \int_0^u du_1 \int_0^{u_1} du_2 \int_0^{u_2} du_3 \, \frac{1}{3!} \left(-\frac{\nabla^2}{4} \right)^3 \phi_0(u_3, x) \\ &\quad + \frac{1}{3!} \left(-\frac{\nabla^2}{4} \right)^2 \frac{u^2}{2} \psi_1(x) + \frac{1}{3} \left(-\frac{\nabla^2}{4} \right) u \, \psi_2(x) + \psi_3(x) \\ &\vdots \\ \phi_n(u, x) &= \int_0^u du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{n-1}} du_n \, \frac{1}{n!} \left(-\frac{\nabla^2}{4} \right)^n \phi_0(u_n, x) \\ &\quad + \left(\psi_n(x) + \sum_{m=1}^{n-1} \frac{(n-m)! \, u^m}{n! m!} \left(\frac{-\nabla^2}{4} \right)^m \psi_{n-m}(x) \right) \end{aligned} \tag{6.1.18}$$

The full solution is

$$\phi(u,r,x) = \phi_{rad}(u,r,x) + \phi_{sub}(u,r,x) \tag{6.1.19}$$

where $\phi_{rad}(u,r,x)$ is the radiative solution corresponding to plane waves in Cartesian coordinates given by

$$\begin{aligned}
&\phi_{rad}(u,r,x)\\
=&r^{(1-d)/2}\left(\phi_0(u,x) + \sum_{n=1}^{\infty}\int_0^u du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{n-1}} du_n \, \frac{1}{n!}\left(-\frac{\nabla^2}{4r}\right)^n \phi_0(u_n,x)\right)\\
=&r^{(1-d)/2}\,\exp\left(\frac{-\nabla^2}{4r}(\partial_u)^{-1}\right)\phi_0(u,x).
\end{aligned} \tag{6.1.20}$$

with the understanding that $\partial_u^{-1} f(u) = \int_0^u du'\, f(u')$. Under a Fourier transform from $(u,r,x) \to (\omega,r,k)$ as defined in (5.4.53), we have $(\partial_u)^{-1} \to 1/(-i\omega)$ and $\partial_a \to -ik_a$. This implies

$$\phi_{rad}(\omega,r,k) = r^{(1-d)/2}\exp\left(\frac{ik^2}{4\omega r}\right)\phi_0(\omega,k) \tag{6.1.21}$$

ϕ_{sub} is the subradiative part of the solution given by,

$$\phi_{sub}(u,r,x) =r^{(1-d)/2}\left(\sum_{n=1}^{\infty}\frac{\psi_n(x)}{r^n} + \sum_{n=2}^{\infty}\frac{1}{r^n}\sum_{m=1}^{n-1}\frac{(n-m)!\,u^m}{n!m!}\left(\frac{-\nabla^2}{4}\right)^m \psi_{n-m}(x)\right) \tag{6.1.22}$$

It is subleading in r compared to the radiative solution since it is $O\left(r^{\frac{(-1-d)}{2}}\right)$. We notice from (6.1.12) and (6.1.18), that the subradiative solution can also be reorganised as

$$\phi_{sub}(u,r,x) =r^{(1-d)/2}\left[\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{n!}{m!(m+n)!}\left(-\frac{\nabla^2}{4}\frac{u}{r}\right)^m\left(\frac{\psi_n}{r^n}\right)\right]. \tag{6.1.23}$$

Using,

$$I_n(z) = \sum_{m=0}^{\infty}\frac{1}{m!(m+n)!}\left(\frac{z}{2}\right)^{2m+n}, \qquad n \in \mathbb{Z}_{\geq 0}, \tag{6.1.24}$$

we have,

$$\phi_{sub}(u,r,x) = r^{(1-d)/2}\left[\sum_{n=1}^{\infty}\left(\sqrt{-\frac{\nabla^2}{4}\frac{u}{r}}\right)^{-n} I_n\left(\sqrt{-\nabla^2\frac{u}{r}}\right)\left(\frac{n!\,\psi_n(x)}{r^n}\right)\right] \tag{6.1.25}$$

or equivalently in momentum space defined by (5.4.53)

$$\phi_{sub}(u,r,k) = r^{(1-d)/2}\left[\sum_{n=1}^{\infty} n!\,r^{-n}\left(\frac{k}{2}\sqrt{\frac{u}{r}}\right)^{-n} I_n\left(k\sqrt{\frac{u}{r}}\right)\psi_n(x)\right] \tag{6.1.26}$$

Therefore,

$$\begin{aligned}\phi^{(\mathscr{I}^+)}(u,r,k) =&\phi^{(\mathscr{I}^+)}_{rad}(u,r,k)+\phi^{(\mathscr{I}^+)}_{sub}(u,r,k)\\ =&r^{(1-d)/2}\int\frac{d\omega}{2\pi}\exp\left(-i\omega u+\frac{ik^2}{4\omega r}\right)\phi_0^{(\mathscr{I}^+)}(\omega,k)\\ &+r^{(1-d)/2}\left[\sum_{n=1}^{\infty}n!\,r^{-n}\left(\frac{k}{2}\sqrt{\frac{u}{r}}\right)^{-n}I_n\left(k\sqrt{\frac{u}{r}}\right)\psi_n^{(\mathscr{I}^+)}(x)\right],\qquad r>0.\end{aligned}\tag{6.1.27}$$

To get the solution around $\mathscr{I}^-$ we replace $r\to -r$ in the expression above

$$\begin{aligned}\phi^{(\mathscr{I}^-)}(u,r,k) =&\phi^{(\mathscr{I}^-)}_{rad}(u,r,k)+\phi^{(\mathscr{I}^-)}_{sub}(u,r,k)\\ =&r^{(1-d)/2}\int\frac{d\omega}{2\pi}\exp\left(-i\omega u-\frac{ik^2}{4\omega r}\right)\phi_0^{(\mathscr{I}^-)}(\omega,k)\\ &+r^{(1-d)/2}\left[\sum_{n=1}^{\infty}n!\,r^{-n}\left(\frac{k}{2}\sqrt{\frac{u}{r}}\right)^{-n}J_n\left(k\sqrt{\frac{u}{r}}\right)\psi_n^{(\mathscr{I}^-)}(x)\right],\qquad r>0.\end{aligned}\tag{6.1.28}$$

A factor of $i^{(1-d)}$ has been absorbed into the integration constants $\phi_0^{(\mathscr{I}^-)}$, and $\psi_n^{(\mathscr{I}^-)}$.

6.1.1.2 Closed form exact solution

Under the Fourier transform to momentum space (5.4.53), the massless scalar field equation of motion (6.1.10) takes the form

$$\left(-\frac{k^2}{2r^2}+i\omega\frac{(d-1)}{r}+2i\omega\partial_r\right)\phi(r,\omega,k_i)=0\tag{6.1.29}$$

The solution to this is

$$\phi(r,\omega,k_i)=r^{(1-d)/2}\exp\left(\frac{ik^2}{4r\omega}\right)\phi_0(\omega,k_i)\tag{6.1.30}$$

We can get a better idea of the solution by working in the coordinates (u,v,x) where $dv=-\frac{dr}{r^2}$, in which the flat metric takes the form

$$ds^2=\frac{2}{v^2}(dudv+dx^adx_a).\tag{6.1.31}$$

The Minkowski metric in these coordinates is related to the light cone metric $d\tilde{s}^2=dudv+dx_adx^a$ by a Weyl transformation, i.e., $d\tilde{s}^2=\Omega^2ds^2$ with $\Omega=|v|/\sqrt{2}$.

If $\phi(u,v,y)$ is a solution of the massless scalar equation of motion in the Minkowski metric and $\tilde{\phi}(u,v,y)$ in the light cone metric, we have

$$\tilde{\phi}(u,v,y)=\left(\frac{v}{\sqrt{2}}\right)^{(1-d)/2}\phi(u,v,y)\tag{6.1.32}$$

$$\tilde{\Box}\tilde{\phi}(u,v,x)=(4\partial_u\partial_v+\partial_a\partial^a)\tilde{\phi}(u,v,x)=0\tag{6.1.33}$$

We use a separable ansatz $\tilde{\phi}(u,v,x) = U(u)V(v,x)$, which implies

$$\frac{U'}{U} = -\frac{\partial_a\partial^a V(v,x)}{4\partial_v V(v,x)} = -i\omega \tag{6.1.34}$$

solving which we get,

$$\tilde{\phi}_{rad}(u,v,x) = \int_{-\infty}^{\infty} d\omega\, e^{-i\left(u\,\omega + v\,\frac{\partial_a\partial^a}{4\omega}\right)} f(\omega,x) = \int_{-\infty}^{\infty} d\omega\, e^{-i\left(u\,\omega + \frac{\partial_a\partial^a}{4r\omega}\right)} f(\omega,x) \tag{6.1.35}$$

is the radiative solution.

We could have also looked for a separable solution of form $\tilde{\phi}(u,v,x) = U(u,x)V(v)$ which gives the solution

$$\tilde{\phi}_{sub}(u,v,x) = \int_{-\infty}^{\infty} d\omega\, e^{-i\left(v\,\omega + u\,\frac{\partial_a\partial^a}{4\omega}\right)} g(\omega,x) \tag{6.1.36}$$

is the subradiative solution. Notice that it is related to the radiative solution by the interchange $u \leftrightarrow v$.

Adding both, the full solution is

$$\int_{-\infty}^{\infty} d\omega \left[e^{-i\left(u\,\omega + v\,\frac{\partial_a\partial^a}{4\omega}\right)} f(\omega,x) + e^{-i\left(v\,\omega + u\,\frac{\partial_a\partial^a}{4\omega}\right)} g(\omega,x) \right] \tag{6.1.37}$$

which depends on two independent functions $f(\omega,x)$ and $g(\omega,x)$.

6.1.2 Solution with Δ arbitrary and $\partial_u\phi_0(u,x) = 0$

We will now solve for solutions with Δ unfixed satisfying $\partial_u\phi_0(u,x) = 0$.

6.1.2.1 Series solution

The recursive relation (6.1.14) implies

$$\phi_n(u,r,x) = \int_0^u du' \left(-\frac{\nabla^2\phi_{n-1}}{[-2\beta + 4n]} \right) + \eta_n(x), \qquad \beta = (-2\Delta + d - 1). \tag{6.1.38}$$

Let us explicitly write down a few of the terms,

$$
\begin{aligned}
\phi_1(u,x) =& u\left(\frac{1}{[-2\beta+4]}\right)(-\nabla^2)\phi_0(x)+\eta_1(x)\\
\phi_2(u,x) =& \frac{u^2}{2}\left(\frac{1}{[-2\beta+4][-2\beta+8]}\right)(-\nabla^2)^2\phi_0+u\left(\frac{1}{[-2\beta+8]}\right)(-\nabla^2)\eta_1(x)+\eta_2(x)\\
\phi_3(u,x) =& \frac{u^3}{3!}\left(\frac{1}{[-2\beta+4][-2\beta+8][-2\beta+12]}\right)(-\nabla^2)^3\phi_0\\
&+\frac{u^2}{2}\left(\frac{1}{[-2\beta+8][-2\beta+12]}\right)(-\nabla^2)^2\,\eta_1(x)\\
&+u\left(\frac{1}{[-2\beta+12]}\right)(-\nabla^2)\eta_2(x)+\eta_3(x)\\
&\vdots\\
\phi_n(u,x) =& \frac{u^n}{n!}\left(\frac{1}{[-2\beta+4][-2\beta+8]\cdots[-2\beta+4n]}\right)(-\nabla^2)^n\phi_0(x)\\
&+\left(\eta_n(x)+\sum_{m=1}^{n-1}\left(\frac{1}{[-2\beta+4(n-m+1)][-2\beta+4(n-m+2)]\cdots[-2\beta+4n]}\right)\right.\\
&\left.\frac{u^m}{m!}\left(-\nabla^2\right)^m\eta_{n-m}(x)\right)
\end{aligned}
\tag{6.1.39}
$$

We have the full solution

$$
\phi(u,r,x)=\phi_\Delta(u,r,x)+\phi_{sub\,\Delta}(u,r,x), \tag{6.1.40}
$$

with

$$
\begin{aligned}
\phi_\Delta(u,r,x) =& \frac{1}{r^\Delta}\left[\phi_0(x)+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{1}{[-2\beta+4]\cdots[-2\beta+4n]}\right)\left(-\nabla^2\,\frac{u}{r}\right)^n\phi_0(x)\right]\\
=& \frac{1}{r^\Delta}\Gamma\left(1-\frac{\beta}{2}\right)\left(\frac{1}{2}\sqrt{-\nabla^2\frac{u}{r}}\right)^{\beta/2} I_{-\beta/2}\left(\sqrt{-\nabla^2\frac{u}{r}}\right)\phi_0(x)
\end{aligned}
\tag{6.1.41}
$$

This is not a valid solution when β is even, since the series has poles for $n=\beta/2$. The general solution valid for all β is given by,

$$
\begin{aligned}
\phi_\Delta(u,r,x) =& \frac{1}{r^\Delta}\left(\frac{1}{2}\sqrt{-\nabla^2\frac{u}{r}}\right)^{\beta/2}\left[K_{\beta/2}\left(\sqrt{-\nabla^2\frac{u}{r}}\right)A(x)+I_{\beta/2}\left(\sqrt{-\nabla^2\frac{u}{r}}\right)B(x)\right],\\
& r>0.
\end{aligned}
\tag{6.1.42}
$$

or equivalently in momentum space,

$$
\phi_\Delta(u,r,k)=\frac{1}{r^\Delta}\left(\frac{k}{2}\sqrt{\frac{u}{r}}\right)^{\beta/2}\left[K_{\beta/2}\left(k\sqrt{\frac{u}{r}}\right)A(k)+I_{\beta/2}\left(k\sqrt{\frac{u}{r}}\right)B(k)\right],\qquad r>0. \tag{6.1.43}
$$

For integer $n \geq 0$, the modified Bessel function of the second kind $K_n(z)$ has the expansion:

$$\begin{aligned} K_n(z) = &\frac{1}{2}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(-1)^k\left(\frac{z}{2}\right)^{2k-n} \\ &+(-1)^{n+1}\log\left(\frac{z}{2}\right)I_n(z) \\ &+(-1)^n\frac{1}{2}\sum_{k=0}^{\infty}\frac{\psi(k+1)+\psi(n+k+1)}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k+n}, \end{aligned} \tag{6.1.44}$$

where $I_n(z)$ is the modified Bessel function of the first kind, and $\psi(x)$ is the digamma function.

To relate $A(x)$ and $B(x)$ to the boundary data $\phi_0(x)$, define

$$z := \sqrt{-\nabla^2\frac{u}{r}}, \qquad \nu = \frac{\beta}{2}. \tag{6.1.45}$$

For small z,

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}, \qquad I_\nu(z) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^{\nu}. \tag{6.1.46}$$

Hence,

$$\left(\frac{z}{2}\right)^{\nu}K_\nu(z) \sim \frac{1}{2}\Gamma(\nu), \qquad \left(\frac{z}{2}\right)^{\nu}I_\nu(z) \sim \mathcal{O}(z^{2\nu}). \tag{6.1.47}$$

Therefore, the leading behaviour is

$$\phi_\Delta(u,r,x) \sim \frac{1}{r^\Delta}\cdot\frac{1}{2}\Gamma\left(\frac{\beta}{2}\right)A(x), \tag{6.1.48}$$

which must match $\frac{1}{r^\Delta}\phi_0(x)$. This fixes

$$A(x) = \frac{2}{\Gamma(\beta/2)}\,\phi_0(x). \tag{6.1.49}$$

The function $B(x)$ is not fixed by the boundary data and corresponds to the independent subleading mode. For non-even β, the special solution proportional to $I_{-\beta/2}$ is recovered by choosing

$$B(x) = \Gamma\left(1-\frac{\beta}{2}\right)\phi_0(x). \tag{6.1.50}$$

This can be seen by using the identity

$$K_\nu(z) = \frac{\pi}{2\sin(\pi\nu)}\left[I_{-\nu}(z) - I_\nu(z)\right]. \tag{6.1.51}$$

in (6.1.42), setting the coefficient of $I_{\beta/2}$ to zero, and using the relation (6.1.49).

From (6.1.39), the subleading part of the solution is given by

$$\phi_{sub\,\Delta}(u,r,x) = \frac{1}{r^{\Delta}}\Bigg[\sum_{n=1}^{\infty}\frac{\eta_n}{r^n} + \sum_{n=2}^{\infty}\frac{1}{r^n}\sum_{m=1}^{n-1}$$
$$\left(\frac{1}{[-2\beta+4(n-m+1)][-2\beta+4(n-m+2)]\cdots[-2\beta+4n]}\right)\frac{u^m}{m!}\left(-\nabla^2\right)^m \eta_{n-m}(x)\Bigg] \quad (6.1.52)$$

This can be reorganised as,

$$\phi_{sub\,\Delta}(u,r,x) = \frac{1}{r^{\Delta}}\Bigg[\sum_{n=1}^{\infty}\frac{\eta_n}{r^n} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m!}$$
$$\left(\frac{1}{[-2\beta+4(n+m)][-2\beta+4(n+m-1)]\cdots[-2\beta+4(n+1)]}\right)\left(-\nabla^2\frac{u}{r}\right)^m\left(\frac{\eta_n}{r^n}\right)\Bigg]$$
$$= \frac{1}{r^{\Delta}}\left[\sum_{n=1}^{\infty}\frac{\Gamma(n+1-\beta/2)}{r^n}\left(\frac{1}{2}\sqrt{-\nabla^2\frac{u}{r}}\right)^{-n+\beta/2} I_{n-\beta/2}\left(\sqrt{-\nabla^2\frac{u}{r}}\right)\eta_n(x)\right] \quad (6.1.53)$$

This is ill-defined for β even. A general solution valid for all β is

$$\phi_{sub\Delta}(u,r,x) = \frac{1}{r^{\Delta}}\sum_{n=1}^{\infty}\frac{1}{r^n}\left(\frac{1}{2}\sqrt{-\nabla^2\frac{u}{r}}\right)^{-n+\beta/2}$$
$$\left[K_{n-\beta/2}\left(\sqrt{-\nabla^2\frac{u}{r}}\right)C_n(x) + I_{n-\beta/2}\left(\sqrt{-\nabla^2\frac{u}{r}}\right)D_n(x)\right] \quad (6.1.54)$$

or equivalently in momentum space,

$$\phi_{sub\Delta}(u,r,k) = \frac{1}{r^{\Delta}}\sum_{n=1}^{\infty}\frac{1}{r^n}\left(\frac{k}{2}\sqrt{\frac{u}{r}}\right)^{-n+\beta/2}$$
$$\left[K_{n-\beta/2}\left(k\sqrt{\frac{u}{r}}\right)C_n^{(\mathscr{I}^+)}(k) + I_{n-\beta/2}\left(k\sqrt{\frac{u}{r}}\right)D_n(k)\right] \quad (6.1.55)$$

Therefore, the most general solution is

$$\phi^{(\mathscr{I}^+)}(u,r,k) = \phi_{\Delta}^{(\mathscr{I}^+)}(u,r,k) + \phi_{sub\Delta}^{(\mathscr{I}^+)}(u,r,k),$$
$$= \frac{1}{r^{\Delta}}\sum_{n=0}^{\infty}\frac{1}{r^n}\left(\frac{k}{2}\sqrt{\frac{u}{r}}\right)^{-n+\beta/2}\left[K_{n-\beta/2}\left(k\sqrt{\frac{u}{r}}\right)C_n^{(\mathscr{I}^+)}(k) + I_{n-\beta/2}\left(k\sqrt{\frac{u}{r}}\right)D_n^{(\mathscr{I}^+)}(k)\right],$$
$$r>0. \quad (6.1.56)$$

We can get the solution around $\mathscr{I}^-$ by replacing $r \to -r$ in the solutions above. Let

$$z := k\sqrt{\frac{u}{r}} \quad (6.1.57)$$

Under $r \to -r$, we have $z \to iz$. Using

$$I_\nu(iz) = i^\nu J_\nu(z),$$
$$K_\nu(iz) = -\frac{\pi i}{2}e^{-i\pi\nu/2}\left[J_\nu(z) - i\,Y_\nu(z)\right], \quad (6.1.58)$$

we have

$$\begin{aligned}&\phi^{(\mathscr{I}^-)}(u,r,k) = \phi_{\Delta}^{(\mathscr{I}^-)}(u,r,k) + \phi_{sub\Delta}^{(\mathscr{I}^-)}(u,r,k),\\ =&\frac{1}{r^{\Delta}}\sum_{n=0}^{\infty}\frac{1}{r^n}\left(\frac{k}{2}\sqrt{\frac{u}{r}}\right)^{-n+\beta/2}\left[J_{n-\beta/2}\left(k\sqrt{\frac{u}{r}}\right)C_n^{(\mathscr{I}^-)}(k) + Y_{n-\beta/2}\left(k\sqrt{\frac{u}{r}}\right)D_n^{(\mathscr{I}^-)}(k)\right],\\ &r>0.\end{aligned} \tag{6.1.59}$$

where,

$$\begin{aligned}A_n &= e^{i\pi\left(-\Delta-\frac{n}{2}-\frac{\beta}{4}\right)}, \qquad \nu_n = n-\frac{\beta}{2},\\ C_n^{(\mathscr{I}^-)}(k) &= A_n\left[-\frac{\pi i}{2}\, e^{-i\pi\nu_n/2}\, C_n^{(\mathscr{I}^+)}(k) + i^{\nu_n}\, D_n^{(\mathscr{I}^+)}(k)\right],\\ D_n^{(\mathscr{I}^-)}(k) &= -\,A_n\,\frac{\pi}{2}\, e^{-i\pi\nu_n/2}\, C_n^{(\mathscr{I}^+)}(k).\end{aligned} \tag{6.1.60}$$

We also notice that we can get the subradiative solution ϕ_{sub} (6.1.26) from $\phi_{sub\Delta}$, by setting $\Delta = (d-1)/2$ *i.e.*, $\beta = 0$.

6.1.2.2 Closed form exact solution

From the above expressions, we see that asymptotically, ϕ can depend upon u and r in the combination $\frac{u^m}{r^n}$ for $m = 0, 1, \cdots, n$. To find a general solution with these dependencies, we assume a general ansatz of the form

$$\phi = r^{-\Delta}\sum_{q=0}^{\infty} r^{-q} f_q\left(\frac{u}{r}\right) \tag{6.1.61}$$

Substituting it in the ϕ equation of motion and demanding the coefficient of r^{-q-2} to vanish, we find

$$x f_q''(x) + (\alpha+1+q) f_q'(x) - \frac{k^2}{4} f_q(x) = 0 \quad ; \qquad \alpha = \Delta - \frac{d-1}{2} = -\frac{\beta}{2}, \tag{6.1.62}$$

where $x = \frac{u}{r}$ and the derivatives are with respect to x.

Around $\mathscr{I}^+$, the solution is given by

$$f_q = \left(\frac{k}{2}\sqrt{\frac{u}{r}}\right)^{-\alpha-q}\left[A_q(k) K_{\alpha+q}\left(k\sqrt{\frac{u}{r}}\right) + B_q(k) I_{\alpha+q}\left(k\sqrt{\frac{u}{r}}\right)\right], \qquad r>0, \tag{6.1.63}$$

where $\alpha = \Delta - \frac{d-1}{2} = -\frac{\beta}{2}$.

The general solution can be written as

$$\begin{aligned}\phi^{(\mathscr{I}^+)}(u,r,k) =& r^{-\Delta}\sum_{q=0}^{\infty} r^{-q}\left(\frac{k}{2}\sqrt{\frac{u}{r}}\right)^{-\alpha-q}\\ &\left[A_q^{(\mathscr{I}^+)}(k) K_{\alpha+q}\left(k\sqrt{\frac{u}{r}}\right) + B_q^{(\mathscr{I}^+)}(k) I_{\alpha+q}\left(k\sqrt{\frac{u}{r}}\right)\right], \; r>0.\end{aligned} \tag{6.1.64}$$

We can get the solution around $\mathscr{I}^-$ by replacing $r \to -r$ in the solution above. We have

$$\begin{aligned}\phi^{(\mathscr{I}^-)}(u,r,k) =& r^{-\Delta}\sum_{q=0}^{\infty} r^{-q}\left(\frac{k}{2}\sqrt{\frac{u}{r}}\right)^{-\alpha-q} \\ &\left[A_q^{(\mathscr{I}^-)}(k)J_{\alpha+q}\left(k\sqrt{\frac{u}{r}}\right)+B_q^{(\mathscr{I}^-)}(k)Y_{\alpha+q}\left(k\sqrt{\frac{u}{r}}\right)\right],\ r>0. \end{aligned} \tag{6.1.65}$$

where,

$$\begin{aligned} N_q &= e^{i\pi\left(-\Delta-\frac{q}{2}+\frac{\alpha}{2}\right)}, \qquad \nu_q = q+\alpha, \\ A_q^{(\mathscr{I}^-)}(k) &= N_q\left[-\frac{\pi i}{2}\, e^{-i\pi\nu_q/2}\, A_q^{(\mathscr{I}^+)}(k) + i^{\nu_q}\, B_q^{(\mathscr{I}^+)}(k)\right], \\ B_q^{(\mathscr{I}^-)}(k) &= -\,N_q\,\frac{\pi}{2}\, e^{-i\pi\nu_q/2}\, A_q^{(\mathscr{I}^+)}(k). \end{aligned} \tag{6.1.66}$$

Note that we have taken q to be integer in the above solutions. However, if q is not integer, the solution is still given by the above expressions without the sum over q.

6.2 Plane waves, soft modes

6.2.1 Plane wave solution

In Cartesian coordinates $\{X^0, X^a, X^{d+1}\}$, $a \in \{1, 2, \cdot, d-1\}$, the solution to the free Klein Gordon equation of motion in flat space is a plane wave

$$\phi_p(X^\mu) = \exp(\pm i\, P\cdot X) = \exp\left[\pm\, i(-\omega_p X^0 + p_a X^a + p_{d+1}X^{d+1})\right];\ a = 1, \cdots, d-1, \tag{6.2.67}$$

where $P^\mu = \left\{\omega_p, p^a, p^{d+1}\right\}$, with the on-shell condition

$$\omega_p^2 = p^2 + p_{d+1}^2 + m^2, \qquad \text{with} \qquad \omega_p \in \mathbb{R}_{\geq 0}. \tag{6.2.68}$$

In defining $P\cdot X$ we have used the metric $\eta_{\mu\nu} = \{-,+,+,\cdots,+\}$.

Since the plane waves form a basis, the most general solution is obtained by integrating over the momentum P

$$\phi(u, x^a, r) = \int \frac{d^d P}{(2\pi)^d\, 2\omega_p}\left[a_p\, e^{iP\cdot X} + b_p^\dagger\, e^{-iP\cdot X}\right] \tag{6.2.69}$$

Using the coordinate transformation between Cartesian and flat null coordinates in (1.3.135), we can write

$$\begin{aligned}\phi_p(u,r,\mathbf{x}) =& \exp(i\, P.X) = \exp[-i\,\bar q\, u - ir(q + \bar q\, x^2 - \sqrt{2}\, p_a x^a)], \\ =& \exp\left[-i\,\bar q\, u - ir\,\bar q\left(x - \frac{p}{\sqrt{2}\,\bar q}\right)^2\right]\exp\left[-ir\left(q - \frac{p^2}{2\bar q}\right)\right]. \end{aligned} \tag{6.2.70}$$

where

$$\bar{q} = \frac{\omega_p + p_{d+1}}{\sqrt{2}}, \qquad q = \frac{\omega_p - p_{d+1}}{\sqrt{2}}. \tag{6.2.71}$$

with the on-shell condition

$$2q\,\bar{q} = p^2 + m^2 \tag{6.2.72}$$

Since $p^2 + m^2 \geq 0$, the on-shell condition above implies

$$q\,\bar{q} \geq 0, \qquad \text{or} \qquad \omega_p^2 - p_{d+1}^2 \geq 0 \tag{6.2.73}$$

Since $\omega_p \geq 0$, this implies,

$$q \geq 0, \qquad \bar{q} \geq 0. \tag{6.2.74}$$

Using the parametrisation above, we can rewrite the plane wave as

$$\phi_p(u, r, \mathbf{x}) = \exp\left[-i\,\bar{q}\,u - ir\,\bar{q}\left(x - \frac{p}{\sqrt{2}\,\bar{q}}\right)^2\right]\exp\left[-ir\,\frac{m^2}{2\bar{q}}\right], \qquad \bar{q} > 0. \tag{6.2.75}$$

For $m = 0$, this reduces to

$$\phi_p(u, r, \mathbf{x}) = \exp\left[-i\,\bar{q}\,u - ir\,\bar{q}\left(x - \frac{p}{\sqrt{2}\,\bar{q}}\right)^2\right] \tag{6.2.76}$$

This is indeed a solution to the massless equation of motion in flat null coordinates (6.1.10)

$$\Box\phi_p = \left(\frac{1}{2r^2}\partial_a\partial^a - \frac{(d-1)}{r}\partial_u - 2\partial_u\partial_r\right)\phi_p(u, r, x^a) = 0 \tag{6.2.77}$$

Let us now Fourier transform $u \to \omega$ and $x_a \to k_a$

$$\begin{aligned}\phi_p(\omega, r, k) &= \int_M du\,d^{d-1}x\,e^{i\omega u + ik_a x^a}\,\phi_p(u, r, x)\\ &= \left(\frac{\pi}{i\bar{q}}\right)^{(d-1)/2} e^{ik_a s^a}(2\pi)\delta(\omega - \bar{q})\,r^{(1-d)/2}\exp\left(\frac{ik^2}{4\,\bar{q}\,r}\right), \qquad s^a = \frac{p^a}{\sqrt{2}\,\bar{q}}.\end{aligned} \tag{6.2.78}$$

In (6.2.76), the null momenta is parametrised as

$$P^\mu = \left(\omega_p, p^a, p^{d+1}\right) = \frac{1}{\sqrt{2}}\left(\left(\bar{q} + \frac{p^2}{2\bar{q}}\right), \sqrt{2}\,p^a, \left(\bar{q} - \frac{p^2}{2\bar{q}}\right)\right). \tag{6.2.79}$$

where $\bar{q}$ and p^a are the free parameters. We could equivalently choose the parametrisation where $\omega_n = \bar{q}$, $x'^a = \frac{p^a}{\sqrt{2}\,\bar{q}} = \frac{p^a}{2\,\omega_n}$

$$P^\mu = \frac{\omega_n}{\sqrt{2}}\left(1 + x'^2,\, 2\,x'^a,\, 1 - x'^2\right) \tag{6.2.80}$$

then (6.2.76) can be written more compactly as

$$\phi_p(u, r, \mathbf{x}) = \exp\left[-i\,\omega_n\,u - ir\,\omega_n\left(x - x'\right)^2\right], \qquad \omega_n \geq 0. \tag{6.2.81}$$

The most general solution is given by

$$
\begin{aligned}
&\phi(u, x^a, r)\\
=&\int \frac{d^dP}{(2\pi)^d\, 2\omega_p}\left[a_p\, e^{iP\cdot X} + b_p^\dagger\, e^{-iP\cdot X}\right]\\
=&\int \frac{d^dP}{(2\pi)^d\,\sqrt{2}\,\omega_n(1+x'^2)}\, a_p \exp\left[-i\,\omega_n\, u - ir\,\omega_n\,(x-x')^2\right] + \text{h.c}\\
=&\frac{2^{(d-3)/2}}{(2\pi)^d}\int d\omega_n\, d^{d-1}x'\,\omega_n^{d-2}\,\left(a_p(\omega_n, x'^a)e^{-i\,\omega_n\,\left[u+r(x-x')^2\right]} + b_p^\dagger(\omega_n, x'^a)e^{i\,\omega_n\,\left[u+r(x-x')^2\right]}\right),\\
&\omega_n \geq 0.
\end{aligned}
\tag{6.2.82}
$$

where we have used

$$
d^dP = 2^{(d-2)/2}\,\omega_n^{d-1}\,(1+x'^2)\,d\omega_n\,d^{d-1}x' \tag{6.2.83}
$$

We could also express this in terms of the momentum space variables (ω, k^a) using the momentum space expression for the plane wave in (6.2.78) and the null parametrisation (6.2.79)

$$
\begin{aligned}
&\phi(\omega, k^a, r)\\
=&\int \frac{d^dP}{(2\pi)^d\,2\omega_p}\left(\int_M du\, d^{d-1}x\, e^{i\omega u + ik_a x^a}\left[a_p(\bar q, p^a)\, e^{iP\cdot X} + b_p^\dagger(\bar q, p^a)\, e^{-iP\cdot X}\right]\right)\\
=&\int \frac{d^dP}{(2\pi)^d\,\sqrt{2}\left(\bar q + \frac{p^2}{2\bar q}\right)}\, a_p \left(\frac{\pi}{i\bar q}\right)^{(d-1)/2} e^{ik_a s^a}\,(2\pi)\delta(\omega-\bar q)\, r^{(1-d)/2}\exp\left(\frac{ik^2}{4\,\bar q\, r}\right) + \text{h.c},\\
=&\int \frac{d^{d-1}p\, d\bar q}{(2\pi)^d\, 2\,\bar q}\, a_p\left(\frac{\pi}{i\bar q}\right)^{(d-1)/2} e^{ik_a s^a}\,(2\pi)\delta(\omega-\bar q)\, r^{(1-d)/2}\exp\left(\frac{ik^2}{4\,\bar q\, r}\right) + \text{h.c},\ s^a = \frac{p^a}{\sqrt{2}\,\bar q},\\
=&\frac{1}{2}\,\left(\frac{\pi}{i}\right)^{(d-1)/2}\int \frac{d^{d-1}p}{(2\pi)^{d-1}}\, e^{ik_a s^a}\, a_p\,\omega^{-(d+1)/2}\, r^{(1-d)/2}\exp\left(\frac{ik^2}{4\,\omega\, r}\right) + \text{h.c},\ s^a = \frac{p^a}{\sqrt{2}\,\omega} = x'^a,\\
=&2^{(d-3)/2}\,\left(\frac{\pi}{i}\right)^{(d-1)/2}\int \frac{d^{d-1}x'}{(2\pi)^{d-1}}\, e^{ik_a x'^a}\, a_p(\omega, x'^a)\,\omega^{\frac{(d-3)}{2}}\, r^{(1-d)/2}\exp\left(\frac{ik^2}{4\,\omega\, r}\right) + \text{h.c},\\
=&\frac{2^{(d-3)/2}}{(2\pi)^{d-1}}\,\left(\frac{\pi}{i}\right)^{(d-1)/2}\,\omega^{\frac{(d-3)}{2}}\, r^{(1-d)/2}\\
&\left[a_p(\omega, k^a)\,\exp\left(\frac{ik^2}{4\,\omega\, r}\right) + (-1)^{\frac{d-1}{2}}\, b_p^\dagger(\omega, k^a)\,\exp\left(-\frac{ik^2}{4\,\omega\, r}\right)\right],
\end{aligned}
\tag{6.2.84}
$$

with $\omega > 0$, where in the third equality we have used

$$
d^dP = \frac{1}{\sqrt{2}}\left(1 + \frac{p^2}{2\bar q^2}\right) d^{d-1}p\, d\bar q \tag{6.2.85}
$$

This matches the radiative solution we obtained directly in flat null coordinates (6.1.30) with the identification

$$
\begin{aligned}
\phi_0^{(\mathscr{I}^+)}(\omega, k) &= \frac{2^{(d-3)/2}}{(2\pi)^{d-1}}\,\left(\frac{\pi}{i}\right)^{(d-1)/2}\, a_p(\omega, k^a)\,\omega^{\frac{(d-3)}{2}}\\
\phi_0^{(\mathscr{I}^-)}(\omega, k) &= \frac{2^{(d-3)/2}}{(2\pi)^{d-1}}\,\left(-\frac{\pi}{i}\right)^{(d-1)/2}\, b_p^\dagger(\omega, k^a)\,\omega^{\frac{(d-3)}{2}}.
\end{aligned}
\tag{6.2.86}
$$

6.2.2 Asymptotic boundary limit

We will now take the boundary limit $r \to \infty$ of these solutions. We multiply ϕ by $r^{(d-1)/2}$ in order to obtain a finite $r \to \infty$ limit. We get

$$\begin{aligned}\phi_1(u, x^a) &= \lim_{r\to\infty} r^{(d-1)/2}\,\phi(u, x^a, r)\\ &= A_d \int_0^\infty d\omega\,\omega^{(d-3)/2}\left[a_p(\omega, x)\,e^{-i\omega u} + (-1)^{\frac{d-1}{2}} b_p^\dagger(\omega, x)\,e^{i\omega u}\right]\\ &\quad + B_d \int d^{d-1}x'\,\frac{\left[\omega^{d-2}\left\{a_p(\omega, x') + (-1)^{\frac{d-1}{2}} b_p^\dagger(\omega, x')\right\}\right]_{\omega=0}}{(x-x')^{2(d-2)}}\\ &= \phi_{1h} + \phi_{1soft}\end{aligned} \tag{6.2.87}$$

The more detailed derivation of this limit is given in Appendix (D.2) [316]. Notice that ϕ_{1soft} is independent of u.

Similarly, if we start from the momentum space expression (6.2.84) and naively take the $r \to \infty$ limit, we get

$$\begin{aligned}\phi_1(\omega, k^a) &= \lim_{r\to\infty} r^{(d-1)/2}\,\phi(\omega, k^a, r)\\ &= \frac{2^{(d-3)/2}}{(2\pi)^{d-1}}\left(\frac{\pi}{i}\right)^{(d-1)/2}\omega^{\frac{(d-3)}{2}}\left[a_p(\omega, k^a) + (-1)^{\frac{d-1}{2}} b_p^\dagger(\omega, k^a)\right]\end{aligned} \tag{6.2.88}$$

This is missing the soft part. Fourier transforming the position space result in (6.2.87), gives

$$\begin{aligned}\phi_1(\omega, k) &= (2\pi) A_d\,\Theta(\omega)\,\omega^{(d-3)/2}\left[a_p(\omega, k^a) + (-1)^{\frac{d-1}{2}} b_p^\dagger(\omega, k^a)\right]\\ &\quad + (2\pi)\,C_d\,\delta(\omega)\,k^{d-3}\left[\omega^{d-2}\left\{a_p(\omega, k^a) + (-1)^{\frac{d-1}{2}} b_p^\dagger(\omega, k^a)\right\}\right]_{\omega=0}\\ &= \phi_{1h} + \phi_{1soft},\end{aligned} \tag{6.2.89}$$

where,

$$C_d = B_d\,2^{3-d}\,\pi^{(d-1)/2}\frac{\Gamma((3-d)/2)}{\Gamma(d-2)}. \tag{6.2.90}$$

6.3 Flat space holography at Null Infinity

In this section, we aim to extend the gauge/gravity correspondence approach, which has been extensively studied in the context of AdS/CFT duality, to the case of the duality between a massless bulk scalar field formulated in flat Minkowski space and a boundary conformal Carrollian (CCFT) theory living at null infinity. This duality manifests itself at the level of symmetries, the Poincaré algebra with vanishing second Casimir is isomorphic to the conformal Carroll algebra. It would therefore be interesting to explore whether, in this different context, the on-shell bulk action can generate CCFT correlators analogously to the standard AdS/CFT correspondence. We shall investigate this duality starting from the simplest setup: a free scalar field in the bulk and the corresponding CCFT defined at null infinity, $\mathcal{J}^{\pm}$.

6.3.1 Boundary Terms

6.3.1.1 Determining boundary terms

The boundary terms in an action are determined by demanding that the variational principle is well defined. Basically, this means that on using the equations of motion, the first order variation of the action must vanish including any boundary term. We consider a general scalar field ϕ described by the action

$$S = -\int d^{d+1}x\sqrt{-g}\Big[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\Big] \tag{6.3.91}$$

Its first order variation is given by

$$\begin{aligned}
\delta S &= -\int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\Big[g^{\mu\nu}\partial_\mu\delta\phi\partial_\nu\phi + V'(\phi)\delta\phi\Big] \\
&= -\int_{\mathcal{M}} d^{d+1}x\Big[-\partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\Big)\delta\phi + \partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\delta\phi\Big) + \sqrt{-g}V'(\phi)\delta\phi\Big] \\
&= \int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\Big[\Box\phi - V'(\phi)\Big]\delta\phi - \int_{\mathcal{M}} d^{d+1}x\ \partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\delta\phi\Big) \\
&= \int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\Big[\Box\phi - V'(\phi)\Big]\delta\phi - \int_{\partial\mathcal{M}} d\Sigma_\mu\Big(g^{\mu\nu}\partial_\nu\phi\delta\phi\Big)
\end{aligned} \tag{6.3.92}$$

where we have used the identities

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\Big) \tag{6.3.93}$$

and

$$\int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\nabla_\mu V^\mu = \int_{\partial\mathcal{M}} d\Sigma_\mu V^\mu \tag{6.3.94}$$

where the surface element on the boundary is defined by

$$d\Sigma_\mu = n\varepsilon_{\mu\nu_1\cdots\nu_d}e_1^{\nu_1}\cdots e_d^{\nu_d}dy^1\cdots dy^d \tag{6.3.95}$$

where n denotes the orientation and $\varepsilon_{\mu\nu_1\cdots\nu_d} = \sqrt{-g}\epsilon_{\mu\nu_1\cdots\nu_d}$ (with $\epsilon_{\mu\cdots}$ denoting the levi civita symbol). The coordinates $\{y^a\}$ parametrize the boundary surface $\partial\mathcal{M}$ and e_a^ν is defined by

$$e_a^\nu = \frac{\partial x^\nu}{\partial y^a} \tag{6.3.96}$$

Now, usually, we demand that the variation of the field (i.e. $\delta\phi$) vanish at the boundary $\partial\mathcal{M}$. In that case, the boundary term in (6.3.92) would also vanish and we would get the standard equation of motion. However, if somehow the boundary variation of the field does not vanish, then we shall need to add a boundary term in the original action whose variation would cancel the boundary term present in (6.3.92). This situation arises if we are interested in scattering solutions in Minkowski space-time. In this case, the fields approach free fields near infinity. The null infinities $\mathscr{I}^+$ and $\mathscr{I}^-$ play an important role in this. Near $\mathscr{I}^-$, the fields should behave as incoming and near $\mathscr{I}^+$, field should behave as outgoing. To ensure this, we demand that the variation of the outgoing field near $\mathscr{I}^-$ vanishes and the variation of the incoming mode near $\mathscr{I}^+$ should vanish. If we denote the incoming and outgoing modes by ϕ^- and ϕ^+ respectively (so that we represent ϕ as $\phi = \phi^+ + \phi^-$), then the scattering problem requires

$$\begin{aligned} \delta\phi^+ &= 0 \qquad \text{at} \quad \mathscr{I}^- \\ \delta\phi^- &= 0 \qquad \text{at} \quad \mathscr{I}^+ \end{aligned} \tag{6.3.97}$$

Note that we are not demanding that the incoming mode ϕ^- vanishes at $\mathscr{I}^+$ and the outgoing mode ϕ^+ vanishes at $\mathscr{I}^-$. These may be non zero. The constraint is only on the variations of these modes. Thus, we have only fixed part of the boundary data. In particular, $\delta\phi^+\big|_{\mathscr{I}^+}$ and $\delta\phi^-\big|_{\mathscr{I}^-}$ are not demanded to vanish. These are the free data which denote the variation of outgoing and incoming radiation at $\mathscr{I}^+$ and $\mathscr{I}^-$ respectively and are needed to have non zero radiation on the boundary.

We shall now apply the above considerations to our case. First we need to determine the boundary terms. For this, we first find the boundary surface elements $d\Sigma_\mu$. In the flat null coordinates we are working in, the range of coordinates are

$$-\infty < u < \infty \quad ; \qquad -\infty < r < \infty \quad ; \qquad -\infty < x^a < \infty \quad ; \tag{6.3.98}$$

Hence, the boundary components of space-time are given by

$$\begin{aligned} u &= \pm U \qquad \text{with} \quad U \to \infty \\ r &= \pm R \qquad \text{with} \quad R \to \infty \\ |x^a| &= L \qquad \text{with} \quad L \to \infty \end{aligned} \tag{6.3.99}$$

For $\mu = r$, parametrizing the surface by $(y_1, \cdots, y^d) = (u, x^1, \cdots, x^{d-1})$, the surface element is given by

$$\begin{aligned} d\Sigma_r &= n\varepsilon_{r,u,1,2,\cdots,d-1} e^u_1 e^1_2 e^2_3 du dx^1 \cdots dx^{d-1} \\ &= n\sqrt{-g}\epsilon_{r,u,1,2,\cdots,d-1} du d^{d-1}x \\ &= 2^{\frac{d-1}{2}} n|r|^{d-1} du d^{d-1}x \end{aligned} \tag{6.3.100}$$

On the outer surface $r = R$, the outward normal is ∂_r (hence $n = 1$), whereas on the surface $r = -R$, the outward normal is $-\partial_r$ (hence $n = -1$). Similarly, we have

$$d\Sigma_u = -2^{\frac{d-1}{2}} n|r|^{d-1} dr d^{d-1}x \tag{6.3.101}$$

Again for the outer surface $u = U$, we have $n = 1$ whereas for the surface $u = -U$, we have $n = -1$. Using these, we find

$$\begin{aligned} \delta S &= -\int d\Sigma_r g^{ru}\partial_u\phi\delta\phi \; - \int d\Sigma_u g^{ur}\partial_r\phi\delta\phi \\ &= 2^{\frac{d-1}{2}} \lim_{R\to\infty} \left[\int_{r=R} du d^{d-1}x \, r^{d-1}\partial_u\phi\delta\phi - \int_{r=-R} du d^{d-1}x \, r^{d-1}\partial_u\phi\delta\phi\right] \\ &\quad -2^{\frac{d-1}{2}} \lim_{U\to\infty} \left[\int_{u=U} dr d^{d-1}x \, |r|^{d-1}\partial_r\phi\delta\phi - \int_{u=-U} dr d^{d-1}x \, |r|^{d-1}\partial_r\phi\delta\phi\right] \end{aligned} \tag{6.3.102}$$

The above expression should be understood with a regulator $-U \le u \le U$, $-R \le r \le R$, which is removed at the end by setting $U \to \infty$, $R \to \infty$. The integrals over $r = \pm R$, which become $\mathscr{I}^\pm$ as $R \to \infty$, extend over $u \in [-U, U]$. Their endpoints at $u = \pm U$ correspond to the top and bottom of $\mathscr{I}^{\pm}$. Similarly, the integrals over $u = \pm U$ extend over $r \in [-R, R]$, and their endpoints at $r = \pm R$ correspond to the same four codimension-two corner surfaces,

$$(r, u) = (R, U), \quad (R, -U), \quad (-R, U), \quad (-R, -U).$$

These corners are common boundaries of the codimension-one pieces $r = \pm R$ and $u = \pm U$.

Note that we would have obtained the same result if we had directly done the integration by parts by explicitly writing the indices in the 3rd line of (6.3.92). Following the paper [317], we assume that the variation of the fields vanishes at $u = \pm\infty$. Hence, the last two terms in the above variation would vanish. On the other hand, for the first two terms, writing the asymptotic expansion as

$$\phi(r, u, x) = r^{\frac{1-d}{2}} \phi_1(u, x) + \cdots \tag{6.3.103}$$

we get

$$\delta S = 2^{\frac{d-1}{2}} \int_{\mathscr{I}^+} du d^{d-1}x\, \partial_u\phi_1\delta\phi_1 - 2^{\frac{d-1}{2}} \int_{\mathscr{I}^-} du d^{d-1}x\, \partial_u\phi_1\delta\phi_1 \tag{6.3.104}$$

Now, the two terms can be expressed as the Klein-Gordon inner product. E.g., for the 1st term, we have

$$\begin{aligned} 2^{\frac{d-1}{2}} \int_{\mathscr{I}^+} du d^{d-1}x\, \partial_u\phi_1\delta\phi_1 &= \frac{1}{2} 2^{\frac{d-1}{2}} \int_{\mathscr{I}^+} du d^{d-1}x\, [\partial_u\phi_1\delta\phi_1 - \phi_1\partial_u\delta\phi_1] \\ &\equiv -(\phi_1, \delta\phi_1)_{\mathscr{I}^+} \end{aligned} \tag{6.3.105}$$

Similarly, the 2nd term can be expressed as the Klein Gordan inner product to get

$$\delta S = -(\phi_1, \delta\phi_1)_{\mathscr{I}^+} + (\phi_1, \delta\phi_1)_{\mathscr{I}^-} \tag{6.3.106}$$

Since the variation of the action is not zero, we need to add some boundary terms to the original action whose variation could cancel the above terms. To find these boundary terms, we decompose the field in the incoming and outgoing modes to get

$$\begin{aligned} (\phi_1, \delta\phi_1)_{\mathscr{I}^+} &= (\phi_1^+ + \phi_1^-, \delta\phi_1^+ + \delta\phi_1^-)_{\mathscr{I}^+} \\ &= (\phi_1^+, \delta\phi_1^+)_{\mathscr{I}^+} + (\phi_1^-, \delta\phi_1^-)_{\mathscr{I}^+} + (\phi_1^+, \delta\phi_1^-)_{\mathscr{I}^+} + (\phi_1^-, \delta\phi_1^+)_{\mathscr{I}^+} \\ &= (\phi_1^-, \delta\phi_1^+)_{\mathscr{I}^+} \end{aligned} \tag{6.3.107}$$

In going to the last line, we have used the fact that the first and second term in second line vanishes by using the definition of the inner product (see appendix (D.3)). The second and third term also vanish due to the boundary condition (6.3.97). Now, noting that $(\delta\phi_1^-, \phi_1^+)_{\mathscr{I}^+} = 0$ and

$$\delta(A, B) = (\delta A, B) + (A, \delta B) \tag{6.3.108}$$

we can express (6.3.107) as

$$(\phi_1, \delta\phi_1)_{\mathscr{I}^+} = \delta(\phi_1^-, \phi_1^+)_{\mathscr{I}^+} \tag{6.3.109}$$

Similarly, we have

$$(\phi_1, \delta\phi_1)_{\mathscr{I}^-} \;\; = \;\; \delta(\phi_1^+, \phi_1^-)_{\mathscr{I}^-} \tag{6.3.110}$$

Thus, we finally have

$$\delta S = -\delta(\phi_1^-, \phi_1^+)_{\mathscr{I}^+} + \delta(\phi_1^+, \phi_1^-)_{\mathscr{I}^-} \tag{6.3.111}$$

Clearly, to cancel these terms, we need to add the following boundary terms to the action

$$\begin{aligned} S_{\partial\mathcal{M}} &= (\phi_1^-, \phi_1^+)_{\mathscr{I}^+} - (\phi_1^+, \phi_1^-)_{\mathscr{I}^-} \\ &= (\phi_1^-, \phi_1)_{\mathscr{I}^+} - (\phi_1^+, \phi_1)_{\mathscr{I}^-} \end{aligned} \tag{6.3.112}$$

where, in going to the second line, we have used $(\phi_1^\pm, \phi_1^\pm) = 0$. Writing explicitly, we have

$$\begin{aligned} S_{\partial\mathcal{M}} &= 2^{\frac{d-3}{2}} \int_{\mathscr{I}^+} du d^{d-1}x \; \left[\phi_1^- \partial_u \phi_1 - \phi_1 \partial_u \phi_1^-\right] \\ &\quad -2^{\frac{d-3}{2}} \int_{\mathscr{I}^-} du d^{d-1}x \; \left[\phi_1^+ \partial_u \phi_1 - \phi_1 \partial_u \phi_1^+\right] \\ &= 2^{\frac{d-3}{2}} \lim_{r\to\infty} \int_{\mathscr{I}^+} du d^{d-1}x \; r^{d-1} \left[\phi^- \partial_u \phi - \phi \partial_u \phi^-\right] \\ &\quad -2^{\frac{d-3}{2}} \lim_{r\to\infty} \int_{\mathscr{I}^-} du d^{d-1}x \; r^{d-1} \left[\phi^+ \partial_u \phi - \phi \partial_u \phi^+\right] \end{aligned} \tag{6.3.113}$$

The full well-defined action is thus given by

$$S = -\int d^{d+1}x\sqrt{-g}\Big[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\Big] + S_{\partial\mathcal{M}} \tag{6.3.114}$$

6.3.1.2 Classical on-shell action

The classical on-shell action would now be given by

$$\begin{aligned} S_{cl} &= -\int d^{d+1}x\,\sqrt{-g}\,\left[\frac{1}{2}g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + V\right] + S_{\partial\mathcal{M}} \\ &= \int d^{d+1}x\,\sqrt{-g}\Big[-\frac{1}{2\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\phi\partial_\nu\phi) + \frac{1}{2}\phi\Box\phi - V\Big] + S_{\partial\mathcal{M}} \\ &= \int d^{d+1}x\,\sqrt{-g}\Big[\frac{1}{2}\phi V' - V\Big] + S_{\partial\mathcal{M}} \\ &\quad -2^{\frac{d-3}{2}}\int d^{d-1}x\,dr\,\Big[|r|^{d-1}\phi\,\partial_r\phi\Big]_{u=-\infty}^{u=+\infty} - 2^{\frac{d-3}{2}}\int d^{d-1}x\,du\,\Big[|r|^{d-1}\phi\,\partial_u\phi\Big]_{r=-\infty}^{r=+\infty} \end{aligned} \tag{6.3.115}$$

The boundary terms of the above on-shell action are given by (focusing on the terms on $\mathscr{I}^{\pm}$)

$$\begin{aligned}
S^{on}_{\partial\mathcal{M}} &= -2^{\frac{d-3}{2}}\int d^{d-1}x\,du\left[|r|^{d-1}\phi\partial_u\phi\right]_{r=-\infty}^{r=+\infty} + S_{\partial\mathcal{M}} \\
&= -2^{\frac{d-3}{2}}\int_{\mathscr{I}^+} d^{d-1}x\,du\left[(\phi_1^+ + \phi_1^-)\partial_u\phi_1\right] + 2^{\frac{d-3}{2}}\int_{\mathscr{I}^-} d^{d-1}x\,du\left[(\phi_1^+ + \phi_1^-)\partial_u\phi_1\right] \\
&\quad +2^{\frac{d-3}{2}}\int_{\mathscr{I}^+} dud^{d-1}x\ \left[\phi_1^-\partial_u\phi_1 - \phi_1\partial_u\phi_1^-\right] - 2^{\frac{d-3}{2}}\int_{\mathscr{I}^-} dud^{d-1}x\ \left[\phi_1^+\partial_u\phi_1 - \phi_1\partial_u\phi_1^+\right] \\
&= -2^{\frac{d-3}{2}}\int_{\mathscr{I}^+} dud^{d-1}x\ \left[\phi_1\partial_u\phi_1^- + \phi_1^+\partial_u\phi_1\right] + 2^{\frac{d-3}{2}}\int_{\mathscr{I}^-} dud^{d-1}x\ \left[\phi_1\partial_u\phi_1^+ + \phi_1^-\partial_u\phi_1\right]
\end{aligned} \tag{6.3.116}$$

Hence,

$$\begin{aligned}
S_{cl} &= \int d^{d+1}x\,\sqrt{-g}\Big[\frac{1}{2}\phi V' - V\Big] - 2^{\frac{d-3}{2}}\int d^{d-1}x\,dr\left[|r|^{d-1}\phi\partial_r\phi\right]_{u=-\infty}^{u=+\infty} \\
&\quad -2^{\frac{d-3}{2}}\int_{\mathscr{I}^+} dud^{d-1}x\ \left[\phi_1\partial_u\phi_1^- + \phi_1^+\partial_u\phi_1\right] + 2^{\frac{d-3}{2}}\int_{\mathscr{I}^-} dud^{d-1}x\ \left[\phi_1\partial_u\phi_1^+ + \phi_1^-\partial_u\phi_1\right]
\end{aligned} \tag{6.3.117}$$

6.4 Source (J)

We will show that the CCFT correlators are generated by the on-shell action. For a correlator involving the boundary CCFT operator $O(u,x)$ (with scaling dimension Δ), the corresponding source is given by the null boundary value of the classical solution *with the correct corresponding scaling dimensions*. In this section, we discuss the scaling dimension of the source for the operator $O(u,x)$.

We use $[A]$ to denote the mass dimension of A and $[A]_{sc}$ to denote the scaling dimension of A under dilatations. Consider a Carroll CFT operator $O(u,\mathbf{x})$ with conformal scaling dimension Δ i.e., $[O(u,x)]_{sc} = \Delta$. From the Dilatation Ward identity in position space (5.3.32), this implies

$$O(\lambda\,u,\lambda\,x_a) = \lambda^{-\Delta}\,O(u,x_a). \tag{6.4.118}$$

We also note that $[u] = -1$ and $[u]_{sc} = -1$, while $[x_a] = 0$ and $[x_a]_{sc} = -1$. From,

$$O(\omega,k_a) = \int du\,d^{d-1}x\,e^{i(\omega u + \mathbf{k}\cdot\mathbf{x})}\,O(u,\mathbf{x}), \tag{6.4.119}$$

we can read off

$$[O(\omega,\mathbf{k})] = \Delta - 1, \qquad [O(\omega,\mathbf{k})]_{sc} = \Delta - d. \tag{6.4.120}$$

We also have

$$\begin{aligned}
[\omega] &= 1, & [\omega]_{sc} &= 1, \\
[k_a] &= 0, & [k_a]_{sc} &= 1, \\
[\delta(\omega)] &= -1, & [\delta(\omega)]_{sc} &= -1, \\
[\delta(k_a)] &= 0, & [\delta(k_a)]_{sc} &= -1.
\end{aligned} \tag{6.4.121}$$

From the linear source coupling

$$\int \frac{d\omega\, d^{d-1}k}{(2\pi)^d} J(-\omega,-\mathbf{k})O(\omega,\mathbf{k}) \qquad \text{or} \qquad \int du\, d^{d-1}x\, J(-u,-\mathbf{x})O(u,\mathbf{x}) \tag{6.4.122}$$

we have,

$$\begin{aligned} [J(\omega,\mathbf{k})] &= -\Delta, & [J(\omega,\mathbf{k})]_{sc} &= -\Delta, \\ [J(u,\mathbf{x})] &= 1-\Delta. & [J(u,\mathbf{x})]_{sc} &= d-\Delta. \end{aligned} \tag{6.4.123}$$

The on-shell action is dimensionless. This implies

$$\begin{aligned} [\phi_0(\omega,\mathbf{k})] &= -1, & [\phi_0(\omega,\mathbf{k})]_{sc} &= \frac{-(d+1)}{2}, \\ [\phi_0(u,\mathbf{x})] &= 0, & [\phi_0(u,\mathbf{x})]_{sc} &= \frac{(d-1)}{2}. \end{aligned} \tag{6.4.124}$$

We now define the source $J(\omega,\mathbf{k})$,

$$J(\omega,\mathbf{k}) = \omega^a\, |k|^b\, \phi_0(\omega,\mathbf{k}) \tag{6.4.125}$$

From the scaling dimensions of $J(\omega,\mathbf{k})$ (6.4.123) and $\phi_0(\omega,\mathbf{k})$ (6.4.124), this implies

$$a+b = -\Delta + \frac{d}{2} + \frac{1}{2} \tag{6.4.126}$$

We choose a source with $b=0$ and $a = -\Delta + \frac{d}{2} + \frac{1}{2}$ i.e.,

$$J_\Delta(\omega,\mathbf{k}) = \omega^{-\Delta+\frac{d}{2}+\frac{1}{2}}\, \phi_0(\omega,\mathbf{k}) \tag{6.4.127}$$

6.5 Two-point function

We assume that the fields at $u = \pm\infty$ vanish. Further, for the discussion of the two-point function, we can ignore the interactions. The on-shell action is given by

$$S_{cl} \;=\; -2^{\frac{d-3}{2}} \int_{\mathscr{I}^+} dud^{d-1}x\, \left[\phi_1\partial_u\phi_1^- + \phi_1^+\partial_u\phi_1\right] + 2^{\frac{d-3}{2}} \int_{\mathscr{I}^-} dud^{d-1}x\, \left[\phi_1\partial_u\phi_1^+ + \phi_1^-\partial_u\phi_1\right] \tag{6.5.128}$$

Further from (6.2.87),

$$\phi_1^\pm = \phi_{1h}^\pm + \phi_{1soft}^\pm \tag{6.5.129}$$

and $\partial_u\phi_{1soft}^\pm = 0$, we have

$$S_{cl} \;=\; S_{cl}^E + S_{cl}^M \tag{6.5.130}$$

where

$$S_{cl}^E = -2^{\frac{d-1}{2}} \int_{\mathscr{I}^+} du d^{d-1}x \; [\phi_1^+ \partial_u \phi_{1h}^-] + 2^{\frac{d-1}{2}} \int_{\mathscr{I}^-} du d^{d-1}x \; [\phi_1^- \partial_u \phi_{1h}^+] \tag{6.5.131}$$

$$\begin{aligned} S_{cl}^M &= -2^{\frac{d-3}{2}} \int_{\mathscr{I}^+} du d^{d-1}x \; [\phi_1^- \partial_u \phi_{1h}^- + \phi_1^+ \partial_u \phi_{1h}^+] \\ &\quad +2^{\frac{d-3}{2}} \int_{\mathscr{I}^-} du d^{d-1}x \; [\phi_1^+ \partial_u \phi_{1h}^+ + \phi_1^- \partial_u \phi_{1h}^-] \, . \end{aligned} \tag{6.5.132}$$

6.5.1 Electric branch

Taking into account (6.4.127), we choose the boundary fields $J_\Delta^\pm$ as the sources of the dual Carrollian operators

$$\begin{aligned} &N_1 \, \omega^{\frac{(d-3)}{2}} \, a(\omega, k) = \phi_0^+(\omega, k) \equiv \omega^{\Delta_1 - \frac{d+1}{2}} \, J_{\Delta_1}^+(\omega, k) \quad ; \quad r \to \infty \\ &(-1)^{(d-1)/2} N_1 \, \omega^{\frac{(d-3)}{2}} \, b^\dagger(\omega, k) = \phi_0^-(\omega, k) \equiv \omega^{\Delta_2 - \frac{d+1}{2}} \, J_{\Delta_2}^-(\omega, k) \quad ; \quad r \to -\infty \end{aligned} \tag{6.5.133}$$

with

$$N_1 = \frac{2^{(d-3)/2}}{(2\pi)^{d-1}} \left(\frac{\pi}{i}\right)^{(d-1)/2} \tag{6.5.134}$$

where, the first equality follows from (6.2.86). For the sources above, the only non trivial contribution is from S_{cl}^E. We have

$$\begin{aligned} S_{cl}^E &= -2^{\frac{d-1}{2}} \int_{\mathscr{I}^+} \frac{d\omega \, d^{d-1}k}{(2\pi)^d} \; [\phi_{1h}^+(i\omega)\phi_{1h}^-] \\ &= N_2 \int_{\mathscr{I}^+} \frac{d\omega \, d^{d-1}k}{(2\pi)^d} \; \left[\omega^{d-2} \, a(\omega, k^a) \, b^\dagger(\omega, k^a)\right] \\ &= N \int_{\mathscr{I}^+} \frac{d\omega \, d^{d-1}k}{(2\pi)^d} \; \omega^{\Delta_1 + \Delta_2 - d} J_{\Delta_1}^+(\omega, \mathbf{k}) J_{\Delta_2}^-(\omega, \mathbf{k}) \end{aligned} \tag{6.5.135}$$

where $N = -i \, 2^{\frac{d-1}{2}} A_d^2 (2\pi)^2 / N_1^2$.

$$\begin{aligned} \langle O_{\Delta_1}(\omega_1, k_1) O_{\Delta_2}(\omega_2, k_2) \rangle &= \frac{(2\pi)^d \, \delta}{\delta J_{\Delta_1}^+(-\omega_1, -\mathbf{k}_1)} \frac{(2\pi)^d \, \delta}{\delta J_{\Delta_2}^-(-\omega_2, -\mathbf{k}_2)} S_{cl}^E \\ &= (2\pi)^d \, N \, \omega_2^{\Delta_1 + \Delta_2 - d} \, \delta(\omega_1 + \omega_2) \, \delta^{d-1}(\mathbf{k}_1 + \mathbf{k}_2) \end{aligned} \tag{6.5.136}$$

This matches exactly the electric branch of the CCFT two-point function (5.4.98), derived using the Ward identities.

In S_{cl}^E, the integrand $\phi_{1soft}^+(i\omega)\phi_{1h}^-$, does not contribute due to the explicit ω dependence from the integrand and the sources which is accompanied by a $\delta(\omega)$. Although we have only used the integrand around $\mathscr{I}^+$ in S^E_{cl}, a similar contribution with corresponding sources also contribute around $\mathscr{I}^-$.

6.5.2 Magnetic branch

Taking into account (6.4.125), we choose the boundary fields J^+_Δ and S^+_Δ as sources, where S^+_Δ is given by

$$d(k^a) = \Big[\omega^{d-2} a(\omega, k^a)\Big]_{\omega=0} = S^+_\Delta(\omega, k^a)\,\left(k^2\right)^{\Delta-d+2}\omega^{\beta/2}; \qquad \beta = -2\Delta + d - 1. \tag{6.5.137}$$

For the sources S^+_Δ and J^+_Δ, the only non trivial contribution is from S^M_{cl}:

$$\begin{aligned} S^M_{cl} &= -2^{\frac{d-3}{2}} \int_{\mathscr{I}^+} \frac{d\omega\, d^{d-1}k}{(2\pi)^d}\, \left[\phi^+_{1soft}(i\omega)\phi^+_{1h}\right] \\ &= N_3 \int_{\mathscr{I}^+} \frac{d\omega\, d^{d-1}k}{(2\pi)^d}\, \omega^{(d-1)/2}\,\delta(\omega)\, k^{d-3}\, d(k^a)\, a(\omega, k^a), \\ &= M \int_{\mathscr{I}^+} \frac{d\omega\, d^{d-1}k}{(2\pi)^d}\, k^{2\Delta-d+1}\,\delta(\omega)\, J^+_\Delta(\omega, k^a)\, S^+_\Delta(\omega, k^a), \end{aligned} \tag{6.5.138}$$

where $M = -i\, 2^{\frac{d-3}{2}}\, A_d\, C_d\, (2\pi)^2/N_1$.

$$\begin{aligned} \langle O_\Delta(\omega_1, k_1) O_\Delta(\omega_2, k_2)\rangle &= \frac{(2\pi)^d\,\delta}{\delta J^+_\Delta(-\omega_1, -\mathbf{k}_1)}\frac{(2\pi)^d\,\delta}{\delta S^+_\Delta(-\omega_2, -\mathbf{k}_2)} S^M_{cl} \\ &= (2\pi)^d\, M\, k_2^{2\Delta-d+1}\,\delta(\omega_1)\,\delta(\omega_1+\omega_2)\,\delta^{d-1}(\mathbf{k}_1+\mathbf{k}_2) \end{aligned} \tag{6.5.139}$$

This matches exactly the magnetic branch of the CCFT two-point function (5.4.98), derived using the CCFT Ward identity.

If we choose the sources to be J^-_Δ and S^-_Δ (corresponding to $a \to (-1)^{(d-1)/2} b^\dagger$ in (6.5.137)), the integrand $\phi^-_1 \partial_u \phi^-_{1h}$ in S^M_{cl} would contribute. We have also considered only the contribution from $\mathscr{I}^+$ in S^M_{cl}; a similar contribution with corresponding sources also contributes around $\mathscr{I}^-$.

6.6 Discussion

We have shown that the classical on-shell action written in flat null coordinates adapted to null infinity, with the right sources and boundary conditions suitable for scattering, generates the CCFT two and three-point correlators derived using the CCFT Ward identities in (6).

The analysis also gives a relation between the boundary value of the classical solution and the creation and annihilation operators, thereby giving a relation between the CCFT correlators and S-matrices.

A non-trivial check of the proposal would be to consider interactions and similarly derive the three-point correlator and check if these match the results of the Ward identity.

We can derive such an explicit holographic dictionary for theories with gravity in the bulk. We can do this by understanding the right boundary expansion of the metric, fall-off conditions, and sources, learning from the scalar case. In parallel, by solving the Carrollian CFT spin-2 Ward identities, we can predict the two and three-point correlators expected from the holographic analysis.

Using the symmetry and structure of the correlators, we can explore the analytic properties of CCFT correlators, such as locality, causality, and unitarity. Also, by transforming certain well-known Yang-Mills and gravity amplitudes to the Carrollian basis, we can explore the implications of the bulk analytic properties of amplitudes in the CCFT correlators. This includes the manifestation of asymptotic flat space symmetries in the dual Carrollian CFT, giving us insights into flat space gravity. We can also explore the implications in the other direction: how do Carrollian conformal block expansions manifest in amplitudes? Using these analytic properties, we can explore a CCFT bootstrap. Since CCFTs are $c \to 0$ limits of CFTs and are related to the S-matrix by a simple transformation, the presence of CFT and S-matrix bootstrap suggests a CCFT bootstrap.

The explorations so far in flat Carrollian holography have mainly focused on null boundaries $\mathscr{I}^{\pm}$. This is relevant for the massless bulk fields. However, for the massive bulk fields, we also need to take into account the timelike and spacelike components of the flat space boundaries [314]. The theories living on these boundary components are still very poorly understood compared to the theories on the null boundaries.

A complementary approach to Carrollian holography is the flat limit (vanishing cosmological constant $\Lambda \to 0$) of the well-understood correlators in AdS/CFT to reproduce flat amplitudes (that are related to the Carrollian correlators by a simple integral transformation). In section (1.3.3), we have discussed the relation between the Carroll limit of the CFT on the conformal boundary of AdS and the flat limit. In chapter (7), following [153], we will explore the flat limit of AdS/CFT correlators in momentum space to yield corresponding flat amplitudes.

Chapter 7

From AdS to Flat: Massive Spin-2 Fields

7.1 Introduction

Massive higher spin fields are expected to play an important role in the classification of consistent gravitational theories (see [144, 145] and references therein for review and literature on the higher spin fields in general backgrounds, including AdS). In particular, it can be argued that an infinite tower of massive higher spin fields are needed for the gravitational theories to be consistent with causality at high energies [146]. However, the requirement of causality only forces the inclusion of a higher spin field above a certain energy scale. In an effective field theory (EFT), an interacting theory of a single massive higher spin field can be consistent below some energy scale when the particle masses are large compared to the exchanged momenta. Such consistent EFTs of massive higher spin fields can be useful for some purposes, e.g., for describing the higher spin resonances in colliders.

In spite of the utility of massive higher spin fields, it is very tricky to construct a consistent EFT of single massive higher spin fields. An EFT obtained by truncating the degrees of freedom above some energy scale will, in general, still be inconsistent in a given background. A classic example of this is provided by the electromagnetic couplings of the massive higher spin fields in the flat spacetimes. It is possible to write down the kinetic terms for the massive higher spin fields in flat spacetimes [318, 319]. However, including interactions is subtle. E.g., the minimal electromagnetic coupling of massive higher spin fields (with spins greater than 1) introduces new unphysical degrees of freedom and leads to superluminal propagation of signals. This is the so called Velo-Zwanziger instability [320–331]. The current understanding is that these instabilities appear when we use the EFT beyond its domain of validity [331]. The analysis involving spin-2 fields indicates that the propagation of the correct number of degrees of freedom requires constraints on the gyromagnetic ratio. However, the gyromagnetic ratio required for this purpose, which is $\frac{1}{2}$, is in contradiction with the value 2 required for the unitary behaviour of Compton scattering of these fields in UV [332] (see [333] for a recent discussion of the gyromagnetic ratio of massive higher spin fields). As argued in [331], this apparent contradiction can be resolved if we note that the extra degrees of freedom arise only if we push the theory to higher orders in the perturbation theory. Further, the restoration of causality requires adding specific nonlinear terms [331].

A consistent interacting theory involving a single higher massive spin field and gravity is also very difficult to construct. It was argued in [146] that 3-point higher derivative graviton couplings not present in Einstein's theory would lead to non causal propagation of gravitons in non trivial backgrounds, and to restore causality, one needs to include an infinite tower of massive higher spin fields. On the boundary CFT side, this is reflected, e.g., in the analysis of 4 point function involving two stress tensor and 2 scalar fields. In the Regge limit, this correlator can get contributions from the exchange of stress energy tensors. This contribution is controlled by the 3-point function involving the stress energy tensor and contains causality violating terms [334]. One can show that including non-conserved spin-2 operators changes the Regge behaviour, but it is not enough to cancel the causality violating terms. One needs to include an infinite tower of non-conserved higher spin operators in the spectrum [334].

The root cause of the above issue is the application of the EFT beyond its regime of validity. E.g., in the Camanho-Edelstein-Maldacena-Zhiboedov (CEMZ) analysis, the acausal behaviour arises if the higher derivative terms become important in the EFT regime and dominate Einstein's term [146]. This shows that we can talk about the EFT of single massive higher spin fields coupled to gravity at least in some regimes. This is the spirit in which we shall work. However, even if the causality problem is taken care of, putting massive higher spin fields on a generic curved background also has other problems, such as non unitary behaviour. Even at the free field level, in a curved background, the derivatives of the equations of motion may involve second and higher derivatives of the massive fields and may no longer remain constraint equations. This changes the expected number of degrees of freedom of the massive higher spin field. It is possible to prevent the constraint equations from becoming dynamical by putting suitable restrictions on the curved background. In particular, in the $d+1$ dimensional Einstein spaces satisfying the condition $R_{MN} = \frac{R}{d+1} G_{MN}$ (where R_{MN} and G_{MN} denote the Ricci and metric tensors respectively), one can prevent the constraint equations from becoming dynamical [335, 336].

Due to the above feature of AdS, it is a useful background for studying EFTs of massive higher spin fields and their interactions. The cubic couplings of higher spin fields have been studied in [337, 338]. However, a systematic analysis of their correlators using the AdS/CFT dictionary is missing. In particular, the holographic renormalization of massive higher spin fields, which is needed to deal with the boundary UV divergences, is missing. Such results would be useful for those analyses which require correlators of massive higher spin fields, such as the study of causality properties. For developing the tools for massive higher spins in AdS, the spin-2 case is a useful starting point (see, e.g., [334]). Motivated by this, in this chapter, we shall consider some aspects of massive spin-2 fields in the AdS background (see, e.g., [339, 340] for reviews on massive spin-2 fields). Since it satisfies the conditions of Einstein spaces, one can write down the kinetic term for the massive spin-2 fields in this background. It turns out that there is a one-parameter family of allowed kinetic terms in Einstein spaces parametrized by a number ξ [335]. We shall work with the simplest choice of this parameter, namely $\xi = 1$. For this value, one obtains the expected AdS relation $m^2L^2 = \Delta(\Delta - d)$, for the massive spin-2 fields. Our main goal will be to develop the momentum space tools for studying massive spin-2 fields in AdS. In particular, we shall study the free field solution of the massive spin-2 field and obtain its bulk-to-boundary (Btb) propagator in the momentum space. This will then be used to perform the holographic renormalization of the massive spin-2 field and compute its one and two point functions.

We shall also consider the interaction of complex massive spin-2 fields with the gauge field in AdS. We shall use the AdS/CFT prescription and holographic renormalization to study the 3-point correlator in this theory. To obtain the CFT 3-point function, we only need to turn on the boundary sources infinitesimally which are switched off after computing the correlators. Hence, the backreaction due to the matter fields can be ignored. Further, since the field strength of the gauge field will be taken to be very small, the causal propagation of the fields will not be affected, and we won't have to worry about the Velo-Zwanziger causality problem.

Our focus will be on the 3-point correlator of two massive spin-2 fields and a gauge field. This will be a function of the bulk couplings. For the massive spin-2 field interacting with a gauge field, involving up to 3 derivatives, there are 5 bulk couplings: minimal coupling g, gyromagnetic coupling α, and 3 higher derivative couplings β_1, β_2, and β_3. The higher derivative couplings are expected to play an important role in the Regge limit [146, 334]. However, in this work, we shall mostly focus on the minimal and gyromagnetic couplings. The 3-point function computed in the bulk would be a function of these couplings. We shall map these to the CFT coefficients which appear in the 3-point CFT correlator of two non-conserved spin-2 and a conserved spin-1 current in the boundary theory. We shall be using the momentum space CFT techniques for the non-conserved CFT operators in our analysis, following [147, 295].

We shall also consider the flat limit of the above AdS 3-point function. The polarizations of the massive spin-2 field in the flat limit will be dictated by the Btb propagator. In the flat limit, we shall show that the AdS 3-point function involving two massive complex spin-2 fields and a gauge field matches the expected 3-point function in the flat space-time. We shall follow the approach developed in [147] for taking the flat limit of CFT correlators in the momentum space.

The rest of the chapter is organised as follows. In section 7.2, we review some results which are needed in this work. In particular, we summarise the momentum space CFT 3-point function of a conserved current and two spin-2 non-conserved operators following [295]. We also review the analysis of massive spin-2 fields in a curved background having constant curvature, following [335, 336]. In section 7.3, we shall analyze the free massive spin-2 field in AdS and derive its Btb propagator in the Fefferman Graham coordinates. In section 7.4, we shall consider the holographic renormalization of the massive spin-2 field and compute one and two point functions. In section 7.5, we shall compute the bulk 3-point function involving two massive spin-2 fields and a gauge field. We shall also match this with the boundary result, obtaining relations between the bulk gyromagnetic coupling and the boundary OPE coefficients. In section 7.6, we shall consider the flat limit of the AdS Btb propagator and the 3-point function using the momentum space CFT techniques and show that we obtain the expected 3-point function in the flat space. We shall end with some discussion in section 7.7. The appendices contain some useful results and details of some calculations done in this work. In appendix E.1, we summarise our notations and conventions and note some useful identities. In appendix E.2, we shall note the detailed expression of the form factors which appear in the CFT 3-point function of two non-conserved spin-2 operators and a conserved current in terms of the triple K integrals. In appendix E.3, we shall give the expressions of the Btb propagator of the massive spin-2 field in the Poincare coordinates. In appendices E.4 and E.5, we shall give some details of the bulk 3-point function and the flat limit computations respectively. Finally, in appendix E.6, we shall review the expected 3-point function of two massive spin-2 fields and a gauge field in the flat space.

7.2 Review

7.2.1 CFT correlators involving spin-2 operators

In this section, we summarise the momentum space CFT 3-point function involving non-conserved spin-2 operators and a conserved current following [295]. Extracting the momentum conserving delta function, we can write

$$\mathcal{A}_3^{\mu_1\mu_2\mu_3\mu_4\mu_5} \quad = \quad (2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\,\langle\langle\mathcal{O}_1^{*\mu_1\mu_2}(\mathbf{p}_1)\mathcal{J}^{\mu_3}(\mathbf{p}_2)\mathcal{O}_3^{\mu_4\mu_5}(\mathbf{p}_3)\rangle\rangle. \qquad (7.2.1)$$

The double bracket notation just means that we have extracted the overall momentum conserving delta function. The operators $\mathcal{O}_i$ have conformal dimensions Δ_i and spin-2. Below, we shall only consider the case where $\Delta_1 = \Delta_3 \equiv \Delta$ since in the bulk they would be dual to the massive spin-2 particles of same mass. The conserved current $\mathcal{J}$ has conformal dimension $d-1$. For stating the results, it is convenient to make use of the auxiliary polarization tensors and avoid the explicit Lorentz indices as in [295]. Hence, we express the reduced correlator as

$$\mathcal{A}_3(\mathbf{p}_1,\mathbf{p}_2) \quad \equiv \quad \langle\langle\epsilon_1\cdot\mathcal{O}_1^*(\mathbf{p}_1)\epsilon_2\cdot J(\mathbf{p}_2)\epsilon_3\cdot\mathcal{O}_3(\mathbf{p}_3)\rangle\rangle\,, \qquad (7.2.2)$$

where $\epsilon\cdot\mathcal{O} = \epsilon^{\mu\nu}\mathcal{O}_{\mu\nu}$ and $\epsilon\cdot J = \epsilon^\mu J_\mu$. Since we shall be working with the symmetric traceless operators $\mathcal{O}_i$, we shall write $\epsilon^{\mu\nu} = \epsilon^\mu\epsilon^\nu$ and impose the condition $\epsilon^\mu\epsilon_\mu = 0$. For more details on use of auxiliary polarization tensors in this context, see e.g., [295, 341, 342]. Next, we split the reduced correlator in terms of the transverse and longitudinal parts [295, 306] and write

$$\begin{aligned}\mathcal{A}_3(p_1,p_2) \quad &= \quad \langle\langle\epsilon_1\cdot\mathcal{O}^*(\mathbf{p}_1)\epsilon_2\cdot j(\mathbf{p}_2)\epsilon_3\cdot\mathcal{O}(\mathbf{p}_3)\rangle\rangle \;+\; \frac{\epsilon_2\cdot p_2}{p_2^2}\langle\langle\epsilon_1\cdot\mathcal{O}^*(\mathbf{p}_1)p_{2\nu}J^\nu(\mathbf{p}_2)\epsilon_3\cdot\mathcal{O}(\mathbf{p}_3)\rangle\rangle \\ &\equiv \quad \mathcal{A}^\perp + \mathcal{A}^{||}\,. \qquad (7.2.3)\end{aligned}$$

The $\mathcal{A}^\perp$ denotes the transverse and $\mathcal{A}^{||}$ denotes the longitudinal part. The j^μ denotes the transverse part of the current defined as $j^\mu = J^\nu\,{\pi_2}_\nu^{\;\mu}$, where the projector $\pi^{\mu\nu}$ is defined as

$$\pi_2^{\mu\nu} = \delta^{\mu\nu} - \frac{p_2^\mu p_2^\nu}{p_2^2} \qquad ; \qquad p_2^\mu\,\pi_{2\,\mu\nu} = 0 \qquad \Longrightarrow \qquad p_\mu j^\mu = 0\,. \qquad (7.2.4)$$

The transverse part $\mathcal{A}^\perp$ can be expressed as

$$\mathcal{A}^\perp(\mathbf{p}_1,\mathbf{p}_2) \quad = \quad (\epsilon_2\cdot\pi_2\cdot p_1)A + (\epsilon_2\cdot\pi_2\cdot\epsilon_1)B_1 + (\epsilon_2\cdot\pi_2\cdot\epsilon_3)B_2\,, \qquad (7.2.5)$$

with the functions A, B_1 and B_2 parametrized as

$$\begin{aligned}
A &= z^2 A_0^{(0,0)} + \zeta_2\xi_2 z A_1^{(0,0)} + \frac{1}{2}\zeta_2^2\xi_2^2 A_2^{(0,0)} + \zeta_1\xi_2 z A_1^{(0,1)} + \zeta_2\xi_1 z A_1^{(1,0)} + \zeta_1\xi_1 z A_1^{(1,1)} \\
&\quad + \frac{1}{2}\zeta_1\zeta_2\xi_2^2 A_2^{(0,1)} + \frac{1}{4}\zeta_1^2\xi_2^2 A_2^{(0,2)} + \frac{1}{2}\zeta_2^2\xi_1\xi_2 A_2^{(1,0)} + \frac{1}{2}\zeta_1\zeta_2\xi_1\xi_2 A_2^{(1,1)} + \frac{1}{4}\zeta_1^2\xi_1\xi_2 A_2^{(1,2)} \\
&\quad + \frac{1}{4}\zeta_2^2\xi_1^2 A_2^{(2,0)} + \frac{1}{4}\zeta_1\zeta_2\xi_1^2 A_2^{(2,1)} + \frac{1}{8}\zeta_1^2\xi_1^2 A_2^{(2,2)}, \\
B_1 &= \xi_1 z B_{1;0}^{(1,0)} + \xi_2 z B_{1;0}^{(0,0)} + \zeta_2\xi_2\xi_1 B_{1;1}^{(1,0)} + \zeta_1\xi_2\xi_1 B_{1;1}^{(1,1)} + \zeta_2\xi_2^2 B_{1;1}^{(0,0)} + \zeta_1\xi_2^2 B_{1;1}^{(0,1)} \\
&\quad + \frac{1}{2}\zeta_2\xi_1^2 B_{1;1}^{(2,0)} + \frac{1}{2}\zeta_1\xi_1^2 B_{1;1}^{(2,1)}, \\
B_2 &= \zeta_1 z B_{2;0}^{(0,1)} + \zeta_2 z B_{2;0}^{(0,0)} + \zeta_2\zeta_1\xi_2 B_{2;1}^{(0,1)} + \zeta_2\zeta_1\xi_1 B_{2;1}^{(1,1)} + \zeta_2^2\xi_2 B_{2;1}^{(0,0)} + \zeta_2^2\xi_1 B_{2;1}^{(1,0)} \\
&\quad + \frac{1}{2}\zeta_1^2\xi_2 B_{2;1}^{(0,2)} + \frac{1}{2}\zeta_1^2\xi_1 B_{2;1}^{(1,2)}.
\end{aligned} \tag{7.2.6}$$

The tensor structures z, ξ_i and ζ_i are defined as

$$z = \epsilon_1\cdot\epsilon_3 \quad , \quad \xi_1 = \epsilon_3\cdot p_2 \quad , \quad \xi_2 = \epsilon_3\cdot(p_1+p_2) \quad , \quad \zeta_1 = \epsilon_1\cdot p_2 \quad , \quad \zeta_2 = \epsilon_1\cdot p_1. \tag{7.2.7}$$

The 30 form factors $A_n^{(p,q)}$, $B_{1;n}^{(p,q)}$ and $B_{2;n}^{(p,q)}$ appearing in (7.2.6) depend on the magnitudes of the three momenta[1] $p_j = |\mathbf{p}_j| = \sqrt{\mathbf{p}_j^2}$ $(j = 1, 2, 3)$ and can be obtained by solving the Ward identities. They are given in terms of the triple K integrals. E.g., the form factor $A_0^{(0,0)}$ is given by

$$A_0^{(0,0)} = a_0^{(0,0)} J_{1\{0,0,0\}} - a_1^{(1,1)} J_{2\{0,1,0\}} + \frac{1}{4} a_2^{(2,2)} J_{3\{0,2,0\}}, \tag{7.2.8}$$

where the triple K integrals $J_{N\{k_1,k_2,k_3\}}$ are defined by (see [148, 295, 306, 343] for more details and useful properties of these integrals)

$$J_{N\{k_1,k_2,k_3\}}(p_1,p_2,p_3) \equiv \int_0^\infty dx\, x^{\frac{d}{2}+N-1} \prod_{i=1}^3 p_i^{\Delta_i-\frac{d}{2}+k_i} K_{\Delta_i-\frac{d}{2}+k_i}(xp_i)\,. \tag{7.2.9}$$

The coefficients $a_0^{(0,0)}, a_1^{(1,1)}$ and $a_2^{(2,2)}$ appearing in (7.2.8) are independent of momenta but depend upon the conformal dimension Δ and the CFT spacetime dimension d.

The other form factors in (7.2.6) are summarized in appendix E.2. The 3-point function of the conserved current and two spin-2 operators having the same conformal dimension Δ is given in terms of 5 independent parameters. This means that not all the coefficients $a_p^{(i,j)}$ are independent. In (7.2.6), a total of 30 coefficients appear. The Ward identities fix 25 of them in terms of the remaining 5 independent coefficients. The relations between these 30 coefficients is also given in

[1] In equation (7.2.5), the momentum conserving delta function has been used to express $p_3^\mu = -p_1^\mu - p_2^\mu$.

appendix E.2.

The two point function of the spin-2 operators in momentum space is given by [295, 344]

$$\langle\langle \epsilon_1 \cdot \mathcal{O}^*(\mathbf{p})\epsilon_2 \cdot \mathcal{O}(-\mathbf{p})\rangle\rangle \;=\; \epsilon_1^{\mu\sigma}\epsilon_2^{\nu\rho}\left[a_0\delta_{\mu\nu}\delta_{\rho\sigma} + a_1\frac{\delta_{\mu\nu}p_\rho p_\sigma}{p^2} + a_2\frac{p_\mu p_\nu p_\rho p_\sigma}{2p^4}\right]p^{2\Delta-d} \tag{7.2.10}$$

The coefficients a_1 and a_2 are given in terms of a_0 as

$$a_1 = -\frac{2(2\Delta-d)}{\Delta}a_0 \qquad ; \qquad a_2 = \frac{2(2\Delta-d)(2\Delta-d-2)}{\Delta(\Delta-1)}a_0 \tag{7.2.11}$$

Since we only use the conformal properties to derive the above correlators, the two point function in (7.2.10) reduces to the 2-point function of conserved stress energy tensor if we specialise the conformal dimension to $\Delta = d$. To see this, we note that we can express the 2-point function in (7.2.10) in the form

$$\langle\langle \epsilon_1 \cdot \mathcal{O}^*(\mathbf{p})\epsilon_2 \cdot \mathcal{O}(-\mathbf{p})\rangle\rangle \;=\; \epsilon_1^{\mu\sigma}\epsilon_2^{\nu\rho}a_0\left[\Pi_{\mu\sigma,\nu\rho}(p) + \frac{(d-\Delta)p_\mu p_\nu}{\Delta(\Delta-1)(d-1)p^2}\Big(E\,\delta_{\sigma\rho} + F\frac{p_\sigma p_\rho}{p^2}\Big)\right] \tag{7.2.12}$$

where, $E = 2(d-1)(\Delta-1), F = (2\Delta - 3d\Delta + d^2 + d - 2)$ and

$$\Pi_{\mu\sigma,\nu\rho}(p) = \frac{1}{2}\Big[\pi_{\mu\nu}(p)\pi_{\sigma\rho}(p) + \pi_{\mu\rho}(p)\pi_{\nu\sigma}(p)\Big] - \frac{1}{d-1}\pi_{\mu\sigma}(p)\pi_{\nu\rho}(p). \tag{7.2.13}$$

The second term in (7.2.12) vanishes for the conserved spin-2 currents satisfying $\Delta = d$ and the expression reduces to the 2-point function of stress energy tensor in momentum space (see, e.g., [306]).

The Ward identities also relate the CFT coefficients appearing in 2 and 3-point functions. For the above described 2 and 3 point functions, the relation is given by

$$a_0 = 2^{\frac{d}{2}-6}\,\frac{(d-2\Delta)}{g\,(d-2)}\,\Gamma\left(\frac{2\Delta-d}{2}\right)\Gamma\left(\frac{d-2\Delta}{2}\right)\Gamma\left(\frac{d}{2}\right)\,\Big[4a_0^{(0,0)} + (d-2)(-4a_1^{(1,1)} + da_2^{(2,2)})\Big] \tag{7.2.14}$$

This identity is also a consequence of the relation between the longitudinal part of the 3-point function in (7.2.3) and the 2-point function of the spin-2 operators. The generating functional of the boundary CFT correlators involving the spin-2 operators and a conserved spin-1 current is given by

$$Z[A_{(0)}, \phi_{(0)}, \phi^*_{(0)}] = \int \mathcal{D}\Phi\;\exp\Big[-S_{CFT} - \int d^dx\Big(\mathcal{J}^\mu A_{(0)\mu} + \mathcal{O}^{*\mu\nu}\phi_{(0)\mu\nu} + \mathcal{O}^{\mu\nu}\phi^*_{(0)\mu\nu}\Big)\Big] \tag{7.2.15}$$

The $A_{(0)\mu}, \phi_{(0)\mu\nu}$ and $\phi^*_{(0)\mu\nu}$ are the sources for the CFT operators $\mathcal{J}^\mu, \mathcal{O}^{*\mu\nu}$ and $\mathcal{O}^{\mu\nu}$, respectively and determine the boundary conditions of the corresponding bulk fields in the AdS CFT correspondence. The generating function is invariant under the $U(1)$ transformation

$$\delta A_{(0)\mu}(\mathbf{x}) = \partial_\mu \lambda(\mathbf{x}) \; ; \quad \delta\phi_{(0)\mu\nu}(\mathbf{x}) = ig\lambda(\mathbf{x})\phi_{(0)\mu\nu}(\mathbf{x}) \; ; \quad \delta\phi^*_{(0)\mu\nu}(\mathbf{x}) = -ig\lambda(\mathbf{x})\phi^*_{(0)\mu\nu}(\mathbf{x}) \tag{7.2.16}$$

This invariance gives the conservation Ward identity

$$\partial^\mu \langle \mathcal{J}^\mu(\mathbf{x})\rangle_s = ig\left(\phi_{(0)\mu\nu}(\mathbf{x})\langle \mathcal{O}^{*\mu\nu}(\mathbf{x})\rangle_s - \phi^*_{(0)\mu\nu}(\mathbf{x})\langle \mathcal{O}^{\mu\nu}(\mathbf{x})\rangle_s\right) \,, \tag{7.2.17}$$

where the expectation value in the above equation are valid in the presence of the sources. If we differentiate w.r.t. $\phi_{(0)\mu_1\mu_2}(\mathbf{x}_1), \phi_{(0)\mu_4\mu_5}(\mathbf{x}_3)$ and Fourier transforming to momentum space gives,

$$\begin{aligned}&\langle\langle \mathcal{O}^{*\mu_1\mu_2}(\mathbf{p}_1)\, \mathbf{p}_{2\mu_3}\mathcal{J}^{\mu_3}(\mathbf{p}_2)\, \mathcal{O}^{\mu_4\mu_5}(\mathbf{p}_3)\rangle\rangle \\ &\quad = \;\; g\langle\langle \mathcal{O}^{*\mu_1\mu_2}(-\mathbf{p}_3)\, \mathcal{O}^{\mu_4\mu_5}(\mathbf{p}_3)\rangle\rangle \; - \; g\langle\langle \mathcal{O}^{*\mu_1\mu_2}(\mathbf{p}_1)\, \mathcal{O}^{\mu_4\mu_5}(-\mathbf{p}_1)\rangle\rangle\end{aligned} \tag{7.2.18}$$

The relation (7.2.14) is consistent with the above equation.

7.2.2 Massive spin-2 fields in curved backgrounds

In this subsection, we review the Lagrangian formulation of the free massive spin-2 field in $(d+1)$ dimensional curved backgrounds following [335, 336]. As mentioned in the introduction, at the free field level, one can describe the massive higher spin fields in a Lagrangian framework in the flat spacetimes. However, if we put the fields on a curved background, the constraint equations may involve second derivatives of the massive field and may no longer remain constraint equations. Fortunately, in the Einstein spaces satisfying the condition $R_{MN} = \frac{R}{d+1}G_{MN}$, the constraint equations are prevented from becoming dynamical. In [335, 336], a detailed analysis is done for the real massive spin-2 field. Generalizing this to complex massive spin-2 fields is straightforward. The free field action of these fields in the Einstein spaces depends on a single real parameter, ξ, and takes the form

$$\begin{aligned}S = \int d^{d+1}x\, \sqrt{G}\Big[&\frac{1}{2}\nabla_M\phi^*_{NP}\nabla^M\phi^{NP} - \nabla_M\phi^*{}_{NP}\nabla^P\phi^{NM} - \frac{1}{2}\nabla_M\phi^*\nabla^M\phi + \frac{1}{2}\nabla_M\phi^{*MN}\nabla_N\phi \\ &+ \frac{1}{2}\nabla_M\phi^{MN}\nabla_N\phi^* - \frac{\xi}{d+1}R\,\phi^*_{MN}\phi^{MN} - \frac{1-2\xi}{2(d+1)}R\,\phi^*\,\phi + \frac{m^2}{2}(\phi^*_{MN}\phi^{MN} - \phi^*\phi)\Big]\end{aligned} \tag{7.2.19}$$

The ϕ^*_{MN} is the complex conjugate of ϕ_{MN} and will be treated as an independent field. The $\phi = G^{MN}\phi_{MN}$ denotes the trace of the spin-2 field. By varying with respect to ϕ^*_{MN}, we obtain the equation of motion for ϕ_{MN}. The variation of the action is given by

$$\delta S = -\frac{1}{2}\int_{\mathcal{M}} d^{d+1}x\,\sqrt{G}\,E_{MN}\,\delta\phi^{*MN} + \int_{\partial\mathcal{M}}\sqrt{\gamma}\,n_M\delta B^M \tag{7.2.20}$$

where the B^M appearing in the boundary term is given by

$$B^M = \frac{1}{2}\Big(\phi^*{}_{NP}\nabla^M\phi^{NP} - 2\phi^*{}_{NP}\nabla^P\phi^{NM} - \phi^*\nabla^M\phi + \phi^{*MN}\nabla_N\phi + \phi^*\nabla_N\phi^{NM}\Big) \tag{7.2.21}$$

and the equation of motion is $E_{MN}=0$ where

$$\begin{aligned} E_{MN} =& \nabla^2\phi_{MN} - 2\nabla^P\nabla_{(N}\phi_{M)P} - G_{MN}\nabla^2\phi + \nabla_M\nabla_N\phi + G_{MN}\nabla^P\nabla^Q\phi_{PQ} \\ &+ \frac{2\xi}{d+1}R\,\phi_{MN} + \frac{1-2\xi}{(d+1)}R\,\phi\, G_{MN} - m^2\,(\phi_{MN} - \phi\, G_{MN}) \end{aligned} \tag{7.2.22}$$

The trace of the above equation gives

$$\nabla_M\nabla_N\phi^{MN} - \Box\phi + \frac{d(1-2\xi)+1}{(d+1)(d-1)}R\,\phi + d\,m^2\phi = 0. \tag{7.2.23}$$

On the other hand, the single and double divergences (i.e. acting with ∇^M) give the following subsidiary conditions

$$\begin{aligned} \nabla^M E_{MN} \;=\;& m^2(\nabla_N\phi - \nabla_M\phi^M_{\ N}) + R_{NM}(\nabla^M\phi - 2\nabla_P\phi^{MP}) + \frac{2\xi}{d+1}\,(\phi_{MN}\nabla^M R + R\nabla^M\phi_{MN}) \\ &+ \frac{1-2\xi}{d+1}\,(\phi\nabla_N R + R\nabla_N\phi) - \phi^{MP}(\nabla_P R_{NM} + \nabla^Q R_{NMPQ}) \end{aligned} \tag{7.2.24}$$

$$\begin{aligned} &\nabla^M\nabla^N E_{MN} \\ =\;& m^2(\Box\phi - \nabla_M\nabla_N\phi^{MN}) + R^{MN}(\nabla_M\nabla_N\phi - 2\nabla_M\nabla^P\phi_{NP}) + \nabla_M R^{MN}(\nabla_N\phi - 2\nabla^P\phi_{NP}) \\ &+ \frac{1}{d+1}\Big\{2\xi\Big[(\nabla^M\nabla^N R)\phi_{MN} + 2(\nabla^M R)\nabla^N\phi_{MN} + R\,\nabla^M\nabla^N\phi_{MN}\Big] \\ &+ (1-2\xi)\Big[(\nabla^2 R)\phi + 2(\nabla^N R)(\nabla_N\phi) + R\,\nabla^2\phi\Big]\Big\} + \phi^{MN}(\nabla^P\nabla^Q R_{MQNP} - \nabla^P\nabla_N R_{MP}) \\ &- \nabla^M\phi^{NP}(\nabla_P R_{MN} - \nabla^Q R_{NMPQ}) = 0 \end{aligned} \tag{7.2.25}$$

For the Einstein spaces satisfying the condition $R_{MN} = \frac{R}{d+1}G_{MN}$, the terms involving the double derivatives of ϕ_{MN} in the above constraint equations vanish and they simplify. They take their simplest form for $\xi = 1$ and are given by

$$\Box\phi_{MN} + 2R^P{}_M{}^Q{}_N\,\phi_{PQ} - m^2\phi_{MN} = 0, \qquad \phi^M{}_M = 0, \quad \nabla^M\phi_{MN} = 0\,. \tag{7.2.26}$$

In the AdS background, these equations become

$$\left(\Box - m^2 + \frac{2}{L^2}\right)\phi_{MN} = 0 \qquad ; \qquad \phi^M{}_M = 0 = \nabla^M\phi_{MN}\,. \tag{7.2.27}$$

7.3 Massive spin-2 fields in AdS

The free field action of a complex massive spin-2 field in $(d+1)$ dimensions, describing the expected $(d+2)(d-1)/2$ complex propagating degrees of freedom on an Einstein background, is given in section 7.2.2. Specializing to AdS$_{d+1}$ and working with the choice $\xi = 1$ for the free parameter, the complex massive spin-2 field in the AdS background can be described by the action

$$\begin{aligned} S \;=\;& \int d^{d+1}x\,\sqrt{G}\Big[\frac{1}{2}\nabla_M\phi^*_{NP}\nabla^M\phi^{NP} - \nabla_M\phi^*{}_{NP}\nabla^P\phi^{NM} - \frac{1}{2}\nabla_M\phi^*\nabla^M\phi + \frac{1}{2}\nabla_M\phi^{*MN}\nabla_N\phi \\ &+ \frac{1}{2}\nabla_M\phi^{MN}\nabla_N\phi^* + \frac{d}{L^2}(\phi^*_{MN}\phi^{MN} - \frac{1}{2}\phi^*\phi) + \frac{m^2}{2}(\phi^*_{MN}\phi^{MN} - \phi^*\phi)\Big], \end{aligned} \tag{7.3.28}$$

where the indices M, N etc. run from 0 to d.

For doing the bulk calculations and matching with the boundary CFT, we shall Fourier transform the boundary directions as

$$f(u, \mathbf{k}) = \int d^d x \, e^{-ik\cdot x} \, f(u, \mathbf{x}), \tag{7.3.29}$$

where f can be any bulk quantity and may have arbitrary spacetime index structure. The u denotes the bulk coordinate. We shall use both position and momentum space expressions in our derivations. To go from one to other, we need to use the replacement rule $\partial_\mu \to ik_\mu$.

7.3.1 Free field solution

Below, we shall be performing the holographic renormalization of the massive spin-2 field. For this, we need to analyse its free field equations of motion. The free field equation of motion for ϕ^*_{MN} is given by

$$\left(\Box - m^2 + \frac{2}{L^2}\right)\phi_{MN} = 0 \quad ; \qquad \nabla_M \phi^M{}_N = 0 = G^{MN}\phi_{MN}. \tag{7.3.30}$$

The constraint conditions $\nabla_M \phi^M{}_N = 0$ in the AdS background can be written more explicitly in the Fefferman Graham coordinates $\{\rho, x^\mu\}$ (see (E.1.3) for details on metric variables in these coordinates) as

$$(N=\rho) : \qquad \frac{4\rho^2}{L^2}\partial_\rho \phi_{\rho\rho} + \frac{\rho}{L}\delta^{\mu\nu}\partial_\mu \phi_{\nu\rho} + \frac{2(4-d)}{L^2}\rho\phi_{\rho\rho} + \frac{1}{2L}\delta^{\sigma\nu}\phi_{\sigma\nu} = 0, \tag{7.3.31a}$$

$$(N=\mu) : \qquad \frac{4\rho^2}{L^2}\partial_\rho \phi_{\rho\mu} + \frac{\rho}{L}\delta^{\nu\sigma}\partial_\nu \phi_{\sigma\mu} + \frac{2(2-d)}{L^2}\rho\phi_{\rho\mu} = 0. \tag{7.3.31b}$$

On the other hand, the tracelessness condition $G^{MN}\phi_{MN} = 0$ gives

$$\frac{4\rho}{L}\phi_{\rho\rho} + \delta^{\mu\nu}\phi_{\mu\nu} = 0. \tag{7.3.32}$$

The auxiliary conditions (7.3.31a) and (7.3.32) can be combined to obtain

$$iL\delta^{\mu\nu}k_\mu \phi_{\nu\rho} - 2(d-3)\phi_{\rho\rho} + 4\rho\partial_\rho \phi_{\rho\rho} = 0. \tag{7.3.33}$$

This form of the constraint will be useful later.

Using the above conditions along with (E.1.8), the equation of motion for different components in the Fefferman Graham coordinates are given by

$$(\rho\rho) \quad : \qquad 4\rho^2\partial_\rho^2\phi_{\rho\rho} + 2(6-d)\rho\partial_\rho\phi_{\rho\rho} - (\rho L k^2 + m^2L^2 + 2d - 4)\phi_{\rho\rho} = 0\,, \tag{7.3.34a}$$

$$(\mu\rho) \quad : \qquad 4\rho^2\partial_\rho^2\phi_{\mu\rho} + 2(6-d)\rho\partial_\rho\phi_{\mu\rho} - (\rho L k^2 + m^2L^2 + 2d - 4)\phi_{\mu\rho} = 4i\rho k_\mu \phi_{\rho\rho}\,, \tag{7.3.34b}$$

$$\begin{aligned}(\mu\nu) \quad : \qquad & 4\rho^2\partial_\rho^2\phi_{\mu\nu} + 2(6-d)\rho\partial_\rho\phi_{\mu\nu} - (\rho L k^2 + m^2L^2 + 2d - 4)\phi_{\mu\nu} \\ & = 4i\rho(k_\mu\phi_{\nu\rho} + k_\nu\phi_{\mu\rho}) - \frac{8\rho}{L}\delta_{\mu\nu}\phi_{\rho\rho}\,.\end{aligned} \tag{7.3.34c}$$

The general solution of (7.3.34a) is given by

$$\phi_{\rho\rho} \quad = \quad c_1(\mathbf{k})\rho^{\frac{d-4}{4}}K_\beta(k\sqrt{\rho L}) + c_2(\mathbf{k})\rho^{\frac{d-4}{4}}I_\beta(k\sqrt{\rho L})\,, \tag{7.3.35}$$

where I_β and K_β are the modified Bessel functions of the first and second kind, respectively, and k denotes the magnitude of the momenta k^μ. In the limit $\rho \to \infty$ (which corresponds to deep AdS interior), the Bessel function $I_\beta(k\sqrt{\rho L})$ diverges. Hence, we discard this solution, obtaining

$$\phi_{\rho\rho}(\mathbf{k},\rho) \quad = \quad c_1(\mathbf{k})\rho^{\frac{d-4}{4}}K_\beta(k\sqrt{\rho L})\,. \tag{7.3.36}$$

The order β is related to the mass of the field and the conformal dimension of the dual operator as

$$\beta = \sqrt{m^2L^2 + \frac{d^2}{4}} \equiv \Delta - \frac{d}{2} \qquad \Longrightarrow \qquad m^2L^2 = \Delta(\Delta - d)\,. \tag{7.3.37}$$

Next, the solution of (7.3.34b) which is regular as $\rho \to \infty$ is given by

$$\phi_{\mu\rho}(\mathbf{k},\rho) \quad = \quad b_\mu(\mathbf{k})\rho^{\frac{d-4}{4}}K_\beta(k\sqrt{\rho L}) - \frac{2ic_1(\mathbf{k})}{k\sqrt{L}}k_\mu\rho^{\frac{d-2}{4}}K_{\beta+1}(k\sqrt{\rho L})\,, \tag{7.3.38}$$

where b_μ is at this point an arbitrary vector. Imposing the auxiliary conditions (7.3.31a) and (7.3.32), we find the relation between b_μ and c_1

$$c_1(\mathbf{k}) = -\frac{ik^\mu b_\mu L}{2\beta - d + 2}\,. \tag{7.3.39}$$

Finally, the solution of (7.3.34c) is given by

$$\begin{aligned}\phi_{\mu\nu}(\mathbf{k},\rho) \quad = \quad & a_{\mu\nu}(\mathbf{k})\rho^{\frac{d-4}{4}}K_\beta(k\sqrt{\rho L}) - \frac{2i(k_\mu b_\nu + k_\nu b_\mu)}{k\sqrt{L}}\rho^{\frac{d-2}{4}}K_{\beta+1}(k\sqrt{\rho L}) \\ & + \frac{4c_1(\mathbf{k})}{kL^{\frac{3}{2}}}\delta_{\mu\nu}\rho^{\frac{d-2}{4}}K_{\beta+1}(k\sqrt{\rho L}) - \frac{4c_1(\mathbf{k})}{k^2L}k_\mu k_\nu \rho^{\frac{d}{4}}K_{\beta+2}(k\sqrt{\rho L})\,,\end{aligned} \tag{7.3.40}$$

where $a_{\mu\nu}(\mathbf{k})$ is a symmetric tensor. Substituting this solution in (7.3.31b), we find

$$b_\mu(\mathbf{k}) = \frac{iL\,k^\nu a_{\mu\nu}}{d - 2\beta} \qquad \Longrightarrow \qquad c_1(\mathbf{k}) = -\frac{k^\mu k^\nu a_{\mu\nu}L^2}{(2\beta - d + 2)(2\beta - d)}\,. \tag{7.3.41}$$

Similarly, substituting in (7.3.32), we find $\delta^{\mu\nu}a_{\mu\nu} = 0$ (where, we raise the indices of the momenta using the boundary metric $\delta^{\mu\nu}$). Thus, we have determined the free field solution of massive spin-2 fields in AdS in terms of a single traceless symmetric tensor $a_{\mu\nu}(\mathbf{k})$.

7.3.2 Bulk-to-boundary Propagator

Using the free field solution of the massive spin two field ϕ_{MN} (which is determined by the symmetric traceless tensor $a_{\mu\nu}$), we can obtain its bulk-to-boundary (Btb) propagator. For this, we need to relate $a_{\mu\nu}(\mathbf{k})$ with the boundary value of the field $\phi_{\mu\nu}$. Using standard expressions for the modified Bessel function K_β, we find that near the boundary $\rho \to 0$, the solutions behave as

$$\lim_{\rho\to 0}\phi_{\rho\rho} = \frac{2^{\beta-1}\Gamma(\beta)k^{-\beta}L^{2-\frac{\beta}{2}}}{(d-2\beta)(2\beta-d+2)}k^\mu k^\nu a_{\mu\nu}\rho^{-\frac{\beta}{2}+\frac{d}{4}-1} \;\equiv\; \phi_{(0)\rho\rho}(\mathbf{k})\,\rho^{-\frac{\beta}{2}+\frac{d}{4}-1}, \tag{7.3.42a}$$

$$\lim_{\rho\to 0}\phi_{\rho\mu} = \frac{i2^{\beta-1}\Gamma(\beta)L^{1-\frac{\beta}{2}}}{(d-2\beta)k^\beta}\left(\delta_\mu{}^\sigma k^\nu - \frac{4\beta k^\sigma k^\nu k_\mu}{k^2(2\beta+2-d)}\right)a_{\sigma\nu}\rho^{-\frac{\beta}{2}+\frac{d}{4}-1} \;\equiv\; \phi_{(0)\rho\mu}\,\rho^{-\frac{\beta}{2}+\frac{d}{4}-1}, \tag{7.3.42b}$$

$$\begin{aligned}\lim_{\rho\to 0}\phi_{\mu\nu} &= 2^{\beta-1}\Gamma(\beta)k^{-\beta}L^{-\frac{\beta}{2}}\left[\delta_\mu{}^\sigma\delta_\nu{}^\tau + \frac{4\beta}{k^2(d-2\beta)}(\delta_\nu{}^\tau k_\mu k_\sigma + \delta_\mu{}^\tau k_\nu k_\sigma)\right.\\ &\quad\left. - \frac{8\beta k^\sigma k^\tau}{k^2(d-2\beta)(d-2-2\beta)}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}(2+2\beta)\right)\right]a_{\sigma\tau}\rho^{-\frac{\beta}{2}+\frac{d}{4}-1}\\ &\equiv \phi_{(0)\mu\nu}(\mathbf{k})\,\rho^{-\frac{\beta}{2}+\frac{d}{4}-1}.\end{aligned} \tag{7.3.42c}$$

For relating the integration constant $a_{\mu\nu}(\mathbf{k})$ with the boundary value $\phi_{(0)\mu\nu}(\mathbf{k})$ of the field, we must invert the above expressions. To do this, we first decompose $a_{\mu\nu}(\mathbf{k})$ as

$$a_{\mu\nu} \;=\; b_{\mu\nu} + t_\mu k_\nu + t_\nu k_\mu + b k_\mu k_\nu + c\delta_{\mu\nu}. \tag{7.3.43}$$

Substituting this in (7.3.42c), we find that the terms proportional to $\delta_{\mu\nu}$ and the terms linear and quadratic in the momenta k_μ can be gotten rid of by choosing

$$t_\mu \;=\; -\frac{4\beta b_{\nu\mu}k^\nu}{k^2(d+2\beta)} \quad;\quad b = \frac{16(\beta-1)\beta k_\mu k_\nu b^{\mu\nu}}{k^4(2\beta+d-2)(2\beta+d)} \quad;\quad c = \frac{8\beta k_\mu k_\nu b^{\mu\nu}}{k^2(2\beta+d-2)(2\beta+d)}. \tag{7.3.44}$$

Then, the near boundary behaviour of $\phi_{\mu\nu}$ takes the form

$$\lim_{\rho\to 0}\phi_{\mu\nu}(\rho,\mathbf{k}) = 2^{\beta-1}\Gamma(\beta)k^{-\beta}L^{-\frac{\beta}{2}}b_{\mu\nu}(\mathbf{k})\,\rho^{-\frac{\beta}{2}+\frac{d}{4}-1} \;\equiv\; \hat{\phi}^{(0)}_{\mu\nu}(\mathbf{k})\left(\frac{\rho}{L}\right)^{-\frac{\beta}{2}+\frac{d}{4}-1}. \tag{7.3.45}$$

The boundary behaviours of $\phi_{\rho\rho}$ and $\phi_{\rho\mu}$ also simplify to

$$\lim_{\rho\to 0}\phi_{\rho\rho}(\rho,\mathbf{k}) = -\frac{2^{\beta-1}b_{\mu\nu}(\mathbf{k})k^\nu k^\nu\Gamma(\beta)k^{-\beta}L^{2-\frac{\beta}{2}}}{(2\beta+d)(2\beta+d-2)}\rho^{-\frac{\beta}{2}+\frac{d}{4}-1}, \tag{7.3.46a}$$

$$\lim_{\rho\to 0}\phi_{\rho\mu}(\rho,\mathbf{k}) = \frac{i2^{\beta-1}b_{\mu\nu}(\mathbf{k})k^\nu\Gamma(\beta)k^{-\beta}L^{1-\frac{\beta}{2}}}{2\beta+d}\rho^{-\frac{\beta}{2}+\frac{d}{4}-1}. \tag{7.3.46b}$$

Contracting (7.3.43) with $\delta^{\mu\nu}$ and demanding it to be zero, we see that $b_{\mu\nu}(\mathbf{k})$ must also be traceless, i.e., $\delta^{\mu\nu}b_{\mu\nu}(\mathbf{k}) = 0$. Using (7.3.43)-(7.3.45), we can express $a_{\mu\nu}$ in terms of the boundary value of $\phi_{\mu\nu}$ namely, $\hat{\phi}^{(0)}_{\mu\nu}$ and the solutions can be expressed as

$$\phi_{\mu\nu} \;=\; K_{\mu\nu}{}^{\tau\sigma}\,\hat{\phi}^{(0)}_{\tau\sigma} \quad;\quad \phi_{\rho\mu} \;=\; K_{\rho\mu}{}^{\tau\sigma}\,\hat{\phi}^{(0)}_{\tau\sigma} \quad;\quad \phi_{\rho\rho} \;=\; K_{\rho\rho}{}^{\tau\sigma}\,\hat{\phi}^{(0)}_{\tau\sigma}. \tag{7.3.47}$$

where ${K_{MN}}^{\tau\sigma}$ denote the components of the Btb propagator. We have

$$
\begin{aligned}
{K_{\mu\nu}}^{\tau\sigma}(\rho,\mathbf{k}) = \frac{L^{-\gamma}}{\omega}\rho^{\frac{d-4}{4}}\Bigg[&\delta^\tau_\mu\delta^\sigma_\nu K_\beta(k\sqrt{\rho L}) - \frac{4\sqrt{\rho L}\delta_{\mu\nu}k^\tau k^\sigma}{k(d+2\beta)(d+2\beta-2)}K_{\beta-1}(k\sqrt{\rho L}) \\
&+ \frac{4\sqrt{\rho L}k^\tau\delta_{(\mu}{}^\sigma k_{\nu)}}{k(d+2\beta)}K_{\beta-1}(k\sqrt{\rho L}) + \frac{4k_\mu k_\nu k^\tau k^\sigma}{k^4(d+2\beta)(d+2\beta-2)} \\
&\Big\{(k^2\rho L + 4\beta(\beta-1))K_\beta(k\sqrt{\rho L}) - 2k(\beta-1)\sqrt{\rho L}K_{\beta+1}(k\sqrt{\rho L})\Big\}\Bigg], \qquad (7.3.48a)
\end{aligned}
$$

$$
{K_{\rho\mu}}^{\tau\sigma}(\rho,\mathbf{k}) = \frac{iL^{1-\gamma}k^\tau}{\omega(d+2\beta)}\left(\delta_\mu^\sigma K_\beta(k\sqrt{\rho L}) + \frac{2\sqrt{\rho L}k^\sigma k_\mu}{k(2\beta+d-2)}K_{\beta-1}(k\sqrt{\rho L})\right)\rho^{\frac{d-4}{4}}, \qquad (7.3.48b)
$$

$$
{K_{\rho\rho}}^{\tau\sigma}(\rho,\mathbf{k}) = -\frac{L^{2-\gamma}k^\tau k^\sigma}{\omega(d+2\beta)(d+2\beta-2)}\rho^{\frac{d-4}{4}}K_\beta(k\sqrt{\rho L}), \qquad (7.3.48c)
$$

where we have defined $\omega \equiv 2^{\beta-1}\Gamma(\beta)k^{-\beta}$ and $\gamma = -\beta + \frac{d}{4} - 1$.

For the computations later, it will be useful to consider the free field solution and the Btb propagator in different bulk coordinates. The expressions of Btb propagator in the Poincaré coordinates (E.1.2) is given in appendix E.3. In general, if we use B_{MN} as a proxy for both the bulk fields ϕ_{MN} and the bulk-to-boundary propagator ${K_{MN}}^{\mu\nu}$, then under a change of coordinate $u(\rho)$ with boundary coordinates x^μ unchanged, we have

$$
\begin{aligned}
B_{uu}(u,\mathbf{k}) &= \left(\frac{\partial\rho}{\partial u}\right)^2 B_{\rho\rho}(\rho(u),\mathbf{k}), \\
B_{u\mu}(u,\mathbf{k}) &= \left(\frac{\partial\rho}{\partial u}\right) B_{\rho\mu}(\rho(u),\mathbf{k}), \\
B_{\mu\nu}(u,\mathbf{k}) &= B_{\mu\nu}(\rho(u),\mathbf{k}).
\end{aligned}
$$

7.4 Holographic renormalization of massive spin-2 field

In this section, we shall study the holographic renormalization of the massive spin-2 field. We shall follow the approach described in [147] for massive spin-1 field (for a review on holographic renormalization, see [345]). For spin-2 and higher, an additional complication arises due to the greater number of components along the bulk direction. The equations of motion relate the bulk components with the boundary components. For the holographic renormalization, we need to perturbatively solve the equations of motion near the boundary $\rho \to 0$. As is standard from the procedure of holographic renormalization, coefficients of certain powers of ρ act as vacuum expectation value (VEV) and sources. These can be identified by solving the equations of motion for the x^μ independent configurations of $\phi_{\rho\rho}, \phi_{\rho\mu}, \phi_{\mu\nu}$. In this case, the equations of motion are given by (7.3.34a) - (7.3.34c) with ks set to zero. More specifically, we have

$$
4\rho^2\partial_\rho^2\phi_{MN} + 2(6-d)\rho\partial_\rho\phi_{MN} - (m^2L^2 + 2d - 4)\phi_{MN} = -\frac{8\rho}{L}\delta_{M\mu}\delta_{N\nu}\phi_{\rho\rho}, \qquad (7.4.49)
$$

with the solutions given by

$$
\phi_{MN} = c_{MN}\rho^{\frac{d}{2}-\frac{\Delta}{2}-1} + d_{MN}\rho.^{\frac{\Delta}{2}-1}. \qquad (7.4.50)
$$

The coefficients c_{MN} and d_{MN} are integration constants. The coefficient of $\rho^{\frac{d}{2}-\frac{\Delta}{2}-1}$ plays the role of source and the coefficient of $\rho^{\frac{\Delta}{2}-1}$ plays the role of VEV. This also shows that the sources will appear at $O(\rho^{\frac{d}{2}-\frac{\Delta}{2}-1})$ whereas VEVs appear at $O(\rho^{\frac{\Delta}{2}-1})$ in the asymptotic solution. With this in mind, we will now start analyzing the full equations (i.e. with the x^μ dependence reinserted) asymptotically.

7.4.1 Asymptotic solutions

For the holographic renormalization, we need to solve for ϕ_{MN} using the equations of motion (7.3.34a)-(7.3.34c), order by order in ρ around $\rho = 0$. Since the leading power is $\frac{d}{2} - \frac{\Delta}{2} - 1$, we shall factor out $\rho^{\frac{d}{2}-\frac{\Delta}{2}-1}$ from ϕ_{MN} and write

$$\phi_{MN}(\rho, \mathbf{x}) = \left(\frac{\rho}{L}\right)^{\frac{d}{2}-\frac{\Delta}{2}-1} \hat{\phi}_{MN}(\rho, \mathbf{x})\,. \tag{7.4.51}$$

Substituting this in the equations of motion, at the leading order, we recover the relation between mass m and the conformal dimension Δ, namely $m^2L^2 = \Delta(\Delta - d)$. For subleading orders, the $\hat{\phi}_{MN}$ satisfy the equations

$$4\rho^2\partial_\rho^2\hat{\phi}_{\rho\rho}(\rho) + 2\rho(d-2\Delta+2)\partial_\rho\hat{\phi}_{\rho\rho}(\rho) - \left(\Delta(d-\Delta) + k^2L\rho + L^2m^2\right)\hat{\phi}_{\rho\rho}(\rho) = 0\,, \tag{7.4.52a}$$

$$4\rho^2\partial_\rho^2\hat{\phi}_{\mu\rho}(\rho) + 2\rho(d-2\Delta+2)\partial_\rho\hat{\phi}_{\mu\rho}(\rho) - \left(\Delta(d-\Delta) + k^2L\rho + L^2m^2\right)\hat{\phi}_{\mu\rho}(\rho) - 4i\rho k_\mu\hat{\phi}_{\rho\rho} = 0\,, \tag{7.4.52b}$$

$$\begin{aligned}&4\rho^2\partial_\rho^2\hat{\phi}_{\mu\nu}(\rho) + 2\rho(d-2\Delta+2)\partial_\rho\hat{\phi}_{\mu\nu}(\rho) - \left(\Delta(d-\Delta) + k^2L\rho + L^2m^2\right)\hat{\phi}_{\mu\nu}(\rho)\\ &- 4i\rho(k_\mu\hat{\phi}_{\nu\rho} + k_\nu\hat{\phi}_{\mu\rho}) + \frac{8\rho\delta_{\mu\nu}\hat{\phi}_{\rho\rho}}{L} = 0\,.\end{aligned} \tag{7.4.52c}$$

The asymptotic solutions of these equations are given by

$$\hat{\phi}_{MN}(\rho, \mathbf{x}) = \sum_{j=0}^{\lfloor\Delta-\frac{d}{2}\rfloor}\left(\frac{\rho}{L}\right)^j\hat{\phi}^{(2j)}_{MN}(\mathbf{x}) + \left(\frac{\rho}{L}\right)^{\Delta-\frac{d}{2}}\left(\hat{\phi}^{(2\Delta-d)}_{MN}(\mathbf{x}) + \delta_{\Delta,\frac{d}{2}+n}\chi^{(2\Delta-d)}_{MN}(\mathbf{x})\log\left(\frac{\rho}{L}\right)\right)\,. \tag{7.4.53}$$

The symbol $\lfloor x\rfloor$ denotes the integer part of x. The logarithmic terms only appear if the quantity $\Delta - \frac{d}{2}$ is an integer n. The coefficients are given by

$$\begin{aligned}
\hat{\phi}^{(2j)}_{\mu\nu} &= C_j\Big[(kL)^{2j}\hat{\phi}^{(0)}_{\mu\nu} + 4ijk^{2(j-1)}L^{2j-1}(k_\mu\hat{\phi}^{(0)}_{\nu\rho} + k_\nu\hat{\phi}^{(0)}_{\mu\rho})\\
&\quad -8j(kL)^{2(j-1)}\left(\delta_{\mu\nu} + 2(j-1)\frac{k_\mu k_\nu}{k^2}\right)\hat{\phi}^{(0)}_{\rho\rho}\Big]\,,\\
\chi^{(2\Delta-d)}_{\mu\nu} &= N\Big[(kL)^{2n}\hat{\phi}^{(0)}_{\mu\nu} + 4ink^{2(n-1)}L^{2n-1}(k_\mu\hat{\phi}^{(0)}_{\nu\rho} + k_\nu\hat{\phi}^{(0)}_{\mu\rho})\\
&\quad -8n(kL)^{2(n-1)}\left(\delta_{\mu\nu} + 2(n-1)\frac{k_\mu k_\nu}{k^2}\right)\hat{\phi}^{(0)}_{\rho\rho}\Big]\,,\\
\hat{\phi}^{(2j)}_{\mu\rho} &= C_j\left((kL)^{2j}\hat{\phi}^{(0)}_{\mu\rho}(x) + 4ij(kL)^{2(j-1)}Lk_\mu\hat{\phi}^{(0)}_{\rho\rho}(x)\right)\,,\\
\chi^{(2\Delta-d)}_{\mu\rho} &= N\left((Lk)^{2n}\hat{\phi}^{(0)}_{\mu\rho} + 4in(Lk)^{2(n-1)}Lk_\mu\hat{\phi}^{(0)}_{\rho\rho}\right)\,,\\
\hat{\phi}^{(2j)}_{\rho\rho} &= C_j(kL)^{2j}\hat{\phi}^{(0)}_{\rho\rho}(x)\,,\\
\chi^{(2\Delta-d)}_{\rho\rho} &= N(Lk)^{2n}\hat{\phi}^{(0)}_{\rho\rho}\,,
\end{aligned} \tag{7.4.54a}$$

where, $n = \Delta - \frac{d}{2}$ and we have defined

$$C_j = \frac{1}{2^j j!(2j+d-2\Delta)\dots(4+d-2\Delta)(2+d-2\Delta)} \quad ; \quad N \equiv \frac{(-1)^{n-1}}{2^{2n}\Gamma(n)\Gamma(n+1)} . \qquad (7.4.55)$$

The constraint equation (7.3.33) can be used to solve for the coefficients $\{\hat{\phi}^{(0)}_{\rho\rho}, \hat{\phi}^{(2\Delta-d)}_{\rho\rho}\}$ in terms of $\{\hat{\phi}^{(0)}_{\mu\rho}, \hat{\phi}^{(2\Delta-d)}_{\mu\rho}\}$. Plugging (7.4.51) into (7.3.33) and solving the equation order by order in ρ, we get

$$\hat{\phi}^{(0)}_{\rho\rho} = \frac{iLk^\mu}{2(\Delta-1)}\hat{\phi}^{(0)}_{\mu\rho} , \qquad (7.4.56a)$$

$$\chi^{(2\Delta-d)}_{\rho\rho} = \frac{-iLk^\mu}{2(\Delta-d+1)}\chi^{(2\Delta-d)}_{\mu\rho} , \qquad (7.4.56b)$$

$$\hat{\phi}^{(2\Delta-d)}_{\rho\rho} = -N\frac{ik^\mu L(kL)^{2n}}{(\Delta-d+1)(\Delta-1)}\hat{\phi}^{(0)}_{\mu\rho} - \frac{ik^\mu L}{2(\Delta-d+1)}\hat{\phi}^{(2\Delta-d)}_{\mu\rho} . \qquad (7.4.56c)$$

The Auxiliary condition (7.3.31b) can be used to solve for the coefficients $\{\hat{\phi}^{(0)}_{\mu\rho}, \hat{\phi}^{(2\Delta-d)}_{\mu\rho}\}$ in terms of $\{\hat{\phi}^{(0)}_{\mu\nu}, \hat{\phi}^{(2\Delta-d)}_{\mu\nu}\}$. Plugging (7.4.51) into (7.3.31b) and solving the equations order by order in ρ, we get

$$\hat{\phi}^{(0)}_{\mu\rho} = \frac{iLk^\nu}{2\Delta}\hat{\phi}^{(0)}_{\mu\nu} , \qquad (7.4.57a)$$

$$\chi^{(2\Delta-d)}_{\mu\rho} = \frac{-iLk^\nu}{2(\Delta-d)}\chi^{(2\Delta-d)}_{\mu\nu} , \qquad (7.4.57b)$$

$$\hat{\phi}^{(2\Delta-d)}_{\mu\rho} = \frac{ik^\nu L}{2(d-\Delta)}\hat{\phi}^{(2\Delta-d)}_{\mu\nu} + N\frac{k^{2(n-1)}L^{2n}ik^\nu L}{\Delta(\Delta-d)(\Delta-1)}\Big(2nk^\sigma k_\mu \hat{\phi}^{(0)}_{\sigma\nu} - k^2(\Delta-1)\hat{\phi}^{(0)}_{\mu\nu}\Big) . \qquad (7.4.57c)$$

The above equations can be used to express the $(\rho\rho)$ components in terms of $(\mu\nu)$ components and we obtain

$$\hat{\phi}^{(0)}_{\rho\rho} = -\frac{L^2}{4\,\Delta(\Delta-1)}\,k^\mu k^\nu\,\hat{\phi}^{(0)}_{\mu\nu} , \qquad (7.4.58a)$$

$$\chi^{(2\Delta-d)}_{\rho\rho} = -\frac{L^2}{4(\Delta-d+1)(\Delta-d)}\,k^\mu k^\nu\,\chi^{(2\Delta-d)}_{\mu\nu} , \qquad (7.4.58b)$$

$$\hat{\phi}^{(2\Delta-d)}_{\rho\rho} = -\frac{L^2k^\mu k^\nu}{4(\Delta-d+1)(\Delta-d)}\left[\hat{\phi}^{(2\Delta-d)}_{\mu\nu} - N\,\frac{2(2\Delta-2d+1)}{\Delta(\Delta-1)}\,(kL)^{2\Delta-d}\hat{\phi}^{(0)}_{\mu\nu}\right] . \qquad (7.4.58c)$$

Finally, using the relations between $(\rho\rho), (\mu\rho)$ and $(\mu\nu)$ components, we can express the coefficients $\hat{\phi}^{(2j)}_{\rho\rho}, \hat{\phi}^{(2j)}_{\mu\rho}$ and $\hat{\phi}^{(2j)}_{\mu\nu}$ in terms of the single coefficient $\hat{\phi}^{(0)}_{\mu\nu}$ as

$$\hat{\phi}^{(2j)}_{\mu\nu} = C_j\left[(kL)^{2j}\hat{\phi}^{(0)}_{\mu\nu} - \frac{2j\,(kL)^{2(j-1)}L^2k^\sigma}{\Delta}\left(k_\mu\hat{\phi}^{(0)}_{\nu\sigma} + k_\nu\hat{\phi}^{(0)}_{\mu\sigma} - \left(\delta_{\mu\nu} + 2(j-1)\frac{k_\mu k_\nu}{k^2}\right)\frac{k^\tau\hat{\phi}^{(0)}_{\sigma\tau}}{\Delta-1}\right)\right] , \qquad (7.4.59a)$$

$$\hat{\phi}^{(2j)}_{\mu\rho} = \frac{iLC_j\,k^\sigma}{2\Delta}\left[(kL)^{2j}\hat{\phi}^{(0)}_{\mu\sigma} - \frac{2j}{\Delta-1}\,k^{2(j-1)}L^{2j}k_\mu k^\tau\hat{\phi}^{(0)}_{\sigma\tau}\right] , \qquad (7.4.59b)$$

$$\hat{\phi}^{(2j)}_{\rho\rho} = -\frac{C_j}{4\Delta(\Delta-1)}k^{2j}L^{2(j+1)}\,k^\mu k^\nu\hat{\phi}^{(0)}_{\mu\nu} . \qquad (7.4.59c)$$

So far we have determined the expansion coefficients in terms of two undetermined quantities $\hat{\phi}^{(0)}_{\mu\nu}$ and $\hat{\phi}^{(2\Delta-d)}_{\mu\nu}$. Using the auxiliary condition (7.3.32), we also get following constraints on these quantities

$$\delta^{\mu\nu}\hat{\phi}^{(0)}_{\mu\nu} = 0 \quad ; \qquad \delta^{\mu\nu}\chi^{(2\Delta-d)}_{\mu\nu} \ = \ 0 \quad ; \qquad \delta^{\mu\nu}\hat{\phi}^{(2j)}_{\mu\nu} + 4\hat{\phi}^{(2j-2)}_{\rho\rho} = 0\,. \qquad (7.4.60)$$

where the 3rd identity is valid for all j from 1 to $\lfloor \Delta - \frac{d}{2} \rfloor$. For the non integer Δ, we also have a relation

$$\delta^{\mu\nu}\hat{\phi}^{(2\Delta-d)}_{\mu\nu} \ = \ 0\,. \qquad (7.4.61)$$

7.4.2 Matching with exact solution

The full analytic solutions for $\phi_{\rho\rho}$, $\phi_{\mu\rho}$ and $\phi_{\mu\nu}$ has one undetermined tensor $a_{\mu\nu}$, while the asymptotic solution is determined in terms of two constant tensors $\hat{\phi}^{(0)}_{\mu\nu}$ and $\hat{\phi}^{(2\Delta-d)}_{\mu\nu}$. Using the full analytic solution, we can determine the relation between $\hat{\phi}^{(0)}_{\mu\nu}$ and $\hat{\phi}^{(2\Delta-d)}_{\mu\nu}$. This will be useful later for computing the two point function of spin-2 CFT operators holographically.

Using the relation between c_1, b_μ and $a_{\mu\nu}$ in (7.3.41), the solution of $\phi_{\mu\nu}$ can be expressed in terms of the single tensor $a_{\mu\nu}$ as

$$\begin{aligned}\phi_{\mu\nu} \ &= \ \rho^{\frac{d-4}{4}}\Bigg[a_{\mu\nu}K_\beta(k\sqrt{L\rho}) + \frac{1}{k^2(d-\Delta)(\Delta-d+1)}\Bigg(-\,a_{\sigma\tau}k^\sigma k^\tau k_\mu k_\nu L\rho\, K_{\beta+2}(k\sqrt{L\rho}) \\ &\quad k\sqrt{L\rho}\,K_{\beta+1}(k\sqrt{L\rho})\,k^\sigma\Big\{a_{\sigma\tau}k^\tau\delta_{\mu\nu} + (\Delta-d+1)(a_{\sigma\mu}k_\nu + a_{\sigma\nu}k_\mu)\Big\}\Bigg)\Bigg]\,. \qquad (7.4.62)\end{aligned}$$

Below, we shall be interested in the non integer conformal dimension Δ. For this, the order of Bessel function- which is $\Delta - d/2$- will also be non integer. The expansion of the Bessel function $K_\nu(x)$ near zero for non integer order is given by

$$\begin{aligned}\lim_{x\to 0} K_\nu(x) \ &= \ \left[\Gamma(\nu)2^{\nu-1}\,x^{-\nu} - \frac{\Gamma(\nu)2^{\nu-3}}{\nu-1}x^{2-\nu} + \frac{\Gamma(\nu)2^{\nu-6}}{(\nu-1)(\nu-2)}x^{4-\nu} + \cdots\right] \\ &\quad + \left[\Gamma(-\nu)2^{-\nu-1}\,x^{\nu} + \frac{\Gamma(-\nu)2^{-\nu-3}}{\nu+1}x^{2+\nu} + \frac{\Gamma(-\nu)2^{-\nu-6}}{(\nu+1)(\nu+2)}x^{4+\nu} + \cdots\right], \qquad (7.4.63)\end{aligned}$$

with the dots denoting further Frobenius terms in x- their form can be extrapolated by extending the pattern visible in the first few terms. Using this expression in (7.4.62) gives,

$$\phi_{\mu\nu} \ = \ \left(\frac{\rho}{L}\right)^{\frac{d}{2}-\frac{\Delta}{2}-1}\left[\hat{\phi}^{(0)}_{\mu\nu} + \cdots + \left(\frac{\rho}{L}\right)^{\left(\Delta-\frac{d}{2}\right)}\hat{\phi}^{(2\Delta-d)}_{\mu\nu} + \cdots\right], \qquad (7.4.64)$$

where

$$\hat{\phi}^{(0)}_{\mu\nu} = L^{\frac{3d}{4}-\Delta-1}\, 2^{\beta} k^{-\beta}\Bigg[2^{-1}\Gamma(\beta)\delta^{\sigma}_{(\mu}\delta^{\tau}_{\nu)} + \frac{1}{k^2(d-\Delta)(\Delta-d+1)}$$

$$\Big(\Gamma(\beta+1)\left\{k^{\sigma}k^{\tau}\delta_{\mu\nu} + (\Delta-d+1)k^{\sigma}(\delta^{\tau}_{\mu}k_{\nu}+\delta^{\tau}_{\nu}k_{\mu})\right\} - \Gamma(\beta+2)2k^{-2}k^{\sigma}k^{\tau}k_{\mu}k_{\nu}\Big)\Bigg]a_{\sigma\tau}\,, \tag{7.4.65a}$$

$$\hat{\phi}^{(2\Delta-d)}_{\mu\nu} = L^{\Delta-\frac{d}{4}-1}\, 2^{-1-\beta} k^{\beta}\Gamma(-\beta)a_{\mu\nu}\,. \tag{7.4.65b}$$

We want to express $\hat{\phi}^{(2\Delta-d)}_{\mu\nu}$ in terms of $\hat{\phi}^{(0)}_{\mu\nu}$. To do this, we first express $a_{\mu\nu}$ in terms of $\hat{\phi}^{(0)}_{\mu\nu}$. This can be done by decomposing $a_{\mu\nu}$ as

$$a_{\mu\nu} \;=\; c_{\mu\nu} + \ell_{\mu}k_{\nu} + \ell_{\nu}k_{\mu} + qk_{\mu}k_{\nu} + r\delta_{\mu\nu}\,. \tag{7.4.66}$$

Substituting this in (7.4.65), we find that the terms proportional to $\delta_{\mu\nu}, k_{\mu}, k_{\nu}$ and $k_{\mu}k_{\nu}$ can be gotten rid of by choosing

$$\ell_{\mu} = \frac{(d-2\Delta)}{k^2\Delta}c_{\mu\nu}k^{\nu} \;\; ; \quad q = \frac{(d-2\Delta)(d-2\Delta+2)}{k^4\Delta(\Delta-1)}c_{\mu\nu}k^{\mu}k^{\nu} \;\; ; \quad r = -\frac{(d-2\Delta)}{k^2\Delta(\Delta-1)}c_{\mu\nu}k^{\mu}k^{\nu} \tag{7.4.67}$$

and we get the following relation

$$c_{\mu\nu} = \frac{2\beta}{L^{\frac{3d}{4}-\Delta-1}2^{\beta}k^{-\beta}\Gamma(\beta+1)}\hat{\phi}^{(0)}_{\mu\nu}\,. \tag{7.4.68}$$

Using equations (7.4.65) - (7.4.68), we get following relation between $\hat{\phi}^{(2\Delta-d)}_{\mu\nu}$ and $\hat{\phi}^{(0)}_{\mu\nu}$

$$\hat{\phi}^{(2\Delta-d)}_{\mu\nu} \;=\; A\,\mathcal{O}_{\mu\nu}{}^{\sigma\tau}\,\hat{\phi}^{(0)}_{\sigma\tau}\,, \tag{7.4.69}$$

where $\mathcal{O}_{\mu\nu}{}^{\sigma\tau}$ is a tensor symmetric in indices (μ,ν) as well as in (σ,τ) and is given by

$$\mathcal{O}_{\mu\nu}{}^{\sigma\tau} \;=\; \delta^{\tau}_{(\mu}\delta^{\sigma}_{\nu)} + B\,k_{\mu}k_{\nu}k^{\sigma}k^{\tau} + C\left(\delta^{\tau}_{\mu}k^{\sigma}k_{\nu} + \delta^{\tau}_{\nu}k^{\sigma}k_{\mu}\right) + D\,k^{\tau}k^{\sigma}\delta_{\mu\nu}\,, \tag{7.4.70}$$

where parentheses over indices denote symmetrization and where the constants A, B, C, D are given by

$$A = \left(2^{-1}k\,L\right)^{2\beta}\frac{\Gamma(-\beta)}{\Gamma(\beta)}, \quad B = \frac{(d-2\Delta)(d+2-2\Delta)}{k^4\,\Delta(\Delta-1)}, \quad C = \frac{d-2\Delta}{k^2\,\Delta}, \quad D = -\frac{(d-2\Delta)}{k^2\,\Delta(\Delta-1)}\,. \tag{7.4.71}$$

Defining $\mathcal{O}^{\mu\nu\sigma\tau} = \delta^{\mu\alpha}\delta^{\nu\beta}\mathcal{O}_{\alpha\beta}{}^{\sigma\tau}$, we see that it is symmetric under the exchange $\mu\nu \leftrightarrow \sigma\tau$ except for the term involving the coefficient D. To remedy this, we first note that the condition $\delta^{\sigma\tau}\hat{\phi}^{(0)}_{\sigma\tau} = 0$ implies that we can add a term proportional to $\delta^{\sigma\tau}$ to $\mathcal{O}_{\mu\nu}{}^{\sigma\tau}$ as

$$\hat{\phi}^{(2\Delta-d)}_{\mu\nu} \;=\; A\Big(O_{\mu\nu}{}^{\sigma\tau} + Q_{\mu\nu}\,\delta^{\sigma\tau}\Big)\,\hat{\phi}^{(0)}_{\sigma\tau} \;\equiv\; P_{\mu\nu}{}^{\sigma\tau}\,\hat{\phi}^{(0)}_{\sigma\tau}\,. \tag{7.4.72}$$

where $Q_{\mu\nu}$ is an arbitrary symmetric tensor. Since it is arbitrary, we can appropriately choose it to ensure that $\mathcal{O}^{\mu\nu\sigma\tau}$ is symmetric under the exchange of $\mu\nu \leftrightarrow \sigma\tau$. Choosing $Q_{\mu\nu} = Dk_{\mu}k_{\nu} +$

$Q(k^2)\delta_{\mu\nu}$, we get

$$P_{\mu\nu}{}^{\sigma\tau} = A\left[\delta^{(\sigma}_{(\mu}\delta^{\tau)}_{\nu)} + B\, k_\mu k_\nu k^\sigma k^\tau + C\delta^{(\sigma}_{(\mu}k^{\tau)}k_{\nu)} + D\left(k^\sigma k^\tau \delta_{\mu\nu} + k_\mu k_\nu \delta^{\sigma\tau}\right) + Q(k^2)\,\delta_{\mu\nu}\delta^{\sigma\tau}\right]. \tag{7.4.73}$$

This satisfies the desired relation $P^{\mu\nu\sigma\tau} = P^{\sigma\tau\mu\nu}$. Next, the condition $\delta^{\mu\nu}\hat{\phi}^{(2\Delta-d)}_{\mu\nu} = 0$ implies

$$\delta_{\mu\nu}\, P^{\mu\nu\sigma\tau} = \frac{Q\, d(\Delta-1)\Delta - d + \Delta^2 + \Delta}{\Delta(\Delta-1)}\,\delta^{\sigma\tau} = 0\,, \tag{7.4.74}$$

which fixes

$$Q = \frac{d - \Delta^2 - \Delta}{d\,\Delta(\Delta-1)}\,. \tag{7.4.75}$$

7.4.3 Regularized action and counter terms

The classical action evaluated on-shell in AdS diverges and hence requires regularization. It is done by putting the boundary at $z = L\epsilon$ (where ϵ is an infinitesimal dimensionless quantity). Doing this, we obtain the following regularized action

$$\begin{aligned}
S_{reg} &= \int_{\rho\geq L\epsilon} d^{d+1}x\,\sqrt{G}\Big[\frac{1}{2}\nabla_M\phi^*_{NP}\nabla^M\phi^{NP} - \nabla_M\phi^*{}_{NP}\nabla^P\phi^{NM} + \frac{1}{2}\nabla_M\phi^{*MN}\nabla_N\phi \\
&\quad + \frac{1}{2}\nabla_M\phi^{MN}\nabla_N\phi^* - \frac{1}{2}\nabla_M\phi^*\nabla^M\phi + \frac{d}{L^2}(\phi^*_{MN}\phi^{MN} - \frac{1}{2}\phi^*\phi) + \frac{m^2}{2}(\phi^*_{MN}\phi^{MN} - \phi^*\phi)\Big] \\
&= \frac{1}{2}\int_{\rho=L\epsilon} d^dx\,\sqrt{\gamma}\; n_M\phi^*{}_{NP}\nabla^M\phi^{NP} \\
&= \frac{-1}{4}\int_{\rho=L\epsilon} d^dx\,\left(\frac{L}{\rho}\right)^{\frac{d}{2}+1}\phi^*{}_{NP}\nabla^\rho\phi^{NP}\,.
\end{aligned} \tag{7.4.76}$$

In going to the second equality, we have used integration by parts and the bulk equations of motion (7.3.30). We have also used Ricci identities corresponding to spin-2 fields. For example, we have

$$[\nabla_M, \nabla^{(P}]\,\phi^{N)M} - \nabla^{(P}\nabla_M\phi^{N)M} = R^{(N}{}_{QM}{}^{P)}\phi^{QM} + R^{(P}{}_Q\,\phi^{N)Q} \;=\; \frac{1}{L^2}\Big(G^{PN}\phi - (d+1)\phi^{PN}\Big)\,, \tag{7.4.77}$$

where the final equality is valid only in AdS.

In going to the third equality in (7.4.76), we have used the fact that the boundary hypersurface is defined by $\rho =$ constant. Hence, the unit normal space like vector and the induced boundary metric are given by

$$n_M = -\frac{L}{2\rho}\delta^\rho_M \quad ; \qquad \gamma_{\mu\nu}(x) = L\frac{\delta_{\mu\nu}}{\rho} \qquad , \qquad \sqrt{\gamma} = \left(\frac{L}{\rho}\right)^{\frac{d}{2}}. \tag{7.4.78}$$

Now, using the asymptotic solution, we can express the regularized action in the form

$$S_{reg} = -\int d^dx\left[d_{(0)}\epsilon^{\frac{d}{2}-\Delta} + d_{(2)}\epsilon^{\frac{d}{2}-\Delta+1} + \cdots + d_{(2\Delta-d)}\log(\epsilon) + \cdots\right], \tag{7.4.79}$$

where

$$d_{(2i)} = \frac{1}{2L}(16a_i + 8b_i + c_i) \quad ; \qquad 0 \le i < \Delta - \frac{d}{2}\,, \tag{7.4.80a}$$

$$d_{(2\Delta-d)} = \frac{1}{2L}\left((d-\Delta)\hat{\phi}^{(0)}_{\beta\nu}\hat{\chi}^{(2\Delta-d)*}_{\mu\alpha}\delta^{\mu\nu}\delta^{\alpha\beta} + \Delta\hat{\chi}^{(2\Delta-d)}_{\beta\nu}\hat{\phi}^{(0)*}_{\mu\alpha}\delta^{\mu\nu}\delta^{\alpha\beta}\right)\,, \tag{7.4.80b}$$

with

$$a_i = \sum_{\substack{j,k\ge 0\\ j+k=i-2}}^{\lfloor \Delta-\frac{d}{2}\rfloor} (d-\Delta+2j+2)\hat{\phi}^{(2j)}_{\rho\rho}\hat{\phi}^{(2k)*}_{\rho\rho}\,, \tag{7.4.81a}$$

$$b_i = \sum_{\substack{j,k\ge 0\\ j+k+1=i}}^{\lfloor \Delta-\frac{d}{2}\rfloor} (d-\Delta+2j+1)\hat{\phi}^{(2j)}_{\rho\nu}\hat{\phi}^{(2k)*}_{\rho\mu}\delta^{\mu\nu}\,, \tag{7.4.81b}$$

$$c_i = \sum_{\substack{j,k\ge 0\\ j+k=i}}^{\lfloor \Delta-\frac{d}{2}\rfloor} (d-\Delta+2j)\hat{\phi}^{(2j)}_{\beta\nu}\hat{\phi}^{(2k)*}_{\mu\alpha}\delta^{\mu\nu}\delta^{\alpha\beta}\,. \tag{7.4.81c}$$

To express the regularized action in the boundary covariant form, we need to invert the series and express $\hat{\phi}^{(0)}_{MN}$ in terms of ϕ_{MN}. Upto $O(\rho^j)$, the inverted relations are given by

$$\hat{\phi}^{(0)}_{\rho\rho} = \left(\frac{\rho}{L}\right)^{\frac{\Delta-d+2}{2}} \sum_{r=0}^{j} b_r (L^2 k_\gamma^2)^r \phi_{\rho\rho}\,, \tag{7.4.82a}$$

$$\hat{\phi}^{(0)}_{\mu\rho} = \left(\frac{\rho}{L}\right)^{\frac{\Delta-d+2}{2}} \sum_{r=0}^{j} b_r \left[(L^2k_\gamma^2)^r\phi_{\mu\rho} + 4r(L^2k_\gamma^2)^{r-1}\rho i k_\mu \phi_{\rho\rho}\right]\,, \tag{7.4.82b}$$

$$\begin{aligned}\hat{\phi}^{(0)}_{\mu\nu} = \left(\frac{\rho}{L}\right)^{(\frac{\Delta-d+2}{2})} \sum_{r=0}^{j} b_r \Bigg[&(L^2k_\gamma^2)^r\phi_{\mu\nu} + 4r(L^2k_\gamma^2)^{r-1}\rho i(k_\mu\phi_{\nu\rho} + k_\nu\phi_{\mu\rho}) \\ &- 8r(L^2k_\gamma^2)^{r-2}(k_\gamma^2\gamma_{\mu\nu} + 2(r-1)k_\mu k_\nu)\rho^2\phi_{\rho\rho}\Bigg]\,,\end{aligned} \tag{7.4.82c}$$

where the coefficients b_r are recursively defined as

$$b_k \quad = \quad -\sum_{\substack{m+n=k\\ m\ge 1; n\ge 0}} C_m b_n \quad ; \qquad b_0 = 1\,. \tag{7.4.83}$$

We have also defined $k_\gamma^2 = \frac{\rho}{L}k^2$. This is a useful notation since in the position space, k_γ^2 becomes the boundary Laplacian $\Box_\gamma = \gamma^{\mu\nu}\partial_\mu\partial_\nu$.

In (7.4.82c), $\hat{\phi}^{(0)}_{\mu\nu}$ is in terms of all the components $\phi_{\mu\nu}, \phi_{\mu\rho}$ and $\phi_{\rho\rho}$. To write down the counter term action, it is useful to express $\hat{\phi}^{(0)}_{\mu\nu}$ completely in terms of $\phi_{\mu\nu}$. It is given by

$$\begin{aligned}\hat{\phi}^{(0)}_{\mu\nu} = \left(\frac{\rho}{L}\right)^{\frac{\Delta-d}{2}+1}\Bigg[\phi_{\mu\nu} - C_1\Bigg\{&\left(\frac{\rho}{L}\right)(kL)^2\phi_{\mu\nu} - \left(\frac{\rho}{L}\right)\left(\frac{2L^2}{\Delta}\right)k^\sigma(k_\mu\phi_{\nu\sigma} + k_\nu\phi_{\mu\sigma}) \\ &+ \left(\frac{\rho}{L}\right)^2\gamma_{\mu\nu}\frac{2L^2}{\Delta(\Delta-1)}k^\sigma k^\tau\phi_{\sigma\tau}\Bigg\} + \cdots\Bigg]\,,\end{aligned} \tag{7.4.84}$$

where dots denote higher order terms. Using the above expression, we can express the regularized action in a covariant form. Noting that the role of counterterms is to get rid of the divergent terms in the regularized action when we take the limit $\epsilon \to 0$, it follows that the counterterms are simply given by the negative of the divergent terms in the regularized action. Noting this, the counter term action can be obtained to be (in position space)

$$S_{\rm ct} = -S_{\rm reg} = \int d^d x\, \sqrt{\gamma}\, \left[\frac{d-\Delta}{2L}\, \phi_{\beta\alpha}\phi^*_{\mu\nu}\, \gamma^{\mu\alpha}\gamma^{\nu\beta} + \cdots \right] , \tag{7.4.85}$$

The renormalised action is the sum of regularized and counterterm actions, i.e.,

$$S_{\rm ren} = \lim_{\epsilon\to 0} \big(S_{\rm reg} + S_{\rm ct} \big) \, . \tag{7.4.86}$$

7.4.4 One and two point functions

The one point function of the boundary operator $O^{\mu\nu}$ which is dual to the bulk field $\phi^*_{\mu\nu}$ is given by

$$\langle O^{\mu\nu}(\mathbf{k}) \rangle \quad = \quad \frac{\delta S_{ren}}{\delta \hat{\phi}^{*(0)}_{\mu\nu}(-\mathbf{k})} = \lim_{\epsilon\to 0} \frac{1}{\epsilon^{\frac{\Delta+2}{2}}} \frac{1}{\sqrt{\gamma}} \frac{\delta(S_{reg}+S_{ct})}{\delta {\phi^*}_{\mu\nu}(-\mathbf{k})} \, . \tag{7.4.87}$$

Now, we have

$$\frac{\delta S_{reg}}{\delta {\phi^*}_{\mu\nu}} = -\sqrt{\gamma}\gamma^{\mu\alpha}\gamma^{\nu\beta}\Big(\frac{1}{L}\phi_{\beta\alpha} + \frac{\rho}{L}\partial_\rho \phi_{\beta\alpha} \Big)\Big|_{\rho=L\epsilon} , \tag{7.4.88a}$$

$$\frac{\delta S_{ct}}{\delta {\phi^*}_{\mu\nu}} = \frac{1}{2L}\sqrt{\gamma}\gamma^{\mu\alpha}\gamma^{\nu\beta}\Big\{ (d-\Delta)\phi_{\beta\alpha} + 2C_1 L^2 k_\gamma^2 \phi_{\beta\alpha} + \epsilon 8 C_1 L (ik_\beta \phi_{\alpha\rho} + ik_\alpha \phi_{\beta\rho}) \Big\}\Big|_{\rho=L\epsilon} . \tag{7.4.88b}$$

Using these in (7.4.87) and taking the boundary limit, we obtain the one point function

$$\langle O^{\mu\nu}(\mathbf{k}) \rangle = -\frac{1}{L}\left(\Delta - \frac{d}{2}\right) \delta^{\mu\alpha}\delta^{\nu\beta} \hat{\phi}^{(2\Delta-d)}_{\alpha\beta}(\mathbf{k}) \; = \; -\frac{1}{L}\left(\Delta - \frac{d}{2}\right) \delta^{\mu\alpha}\delta^{\nu\beta}\, {P_{\alpha\beta}}^{\sigma\tau}(\mathbf{k})\, \hat{\phi}^{(0)}_{\sigma\tau}(\mathbf{k}) \, , \tag{7.4.89}$$

where we have used the relation (7.4.72).

The two-point function can now be derived by differentiating the one point function with respect to the source $\hat{\phi}^{(0)}_{\sigma\tau}$

$$\big\langle O^{*\sigma\tau}(\mathbf{p}) O^{\mu\nu}(\mathbf{k}) \big\rangle = -\, (2\pi)^d \frac{\delta \left\langle O^{\mu\nu}(\mathbf{k}) \right\rangle}{\delta \hat{\phi}^{(0)}_{\sigma\tau}(-\mathbf{p})} = \frac{1}{L}\left(\Delta - \frac{d}{2}\right) (2\pi)^d \delta^d(\mathbf{p}+\mathbf{k}) P^{\mu\nu\sigma\tau}(\mathbf{k}) \, , \tag{7.4.90}$$

where we have used $\delta^{\mu\nu} k_\nu = k^\mu$.

Using the expression of $P^{\mu\nu\sigma\tau}$ given earlier, we can express the two point function as

$$\begin{aligned} &\big\langle O^{*\sigma\tau}(\mathbf{p}) O^{\mu\nu}(\mathbf{k}) \big\rangle \\ &= \quad (2\pi)^d \delta^d(\mathbf{p}+\mathbf{k}) a_0 \left[\Pi_{\mu\sigma,\nu\tau} + \frac{(d-\Delta)}{\Delta(\Delta-1)(d-1)} \frac{k_\mu k_\nu}{k^2} \left(E\, \delta_{\sigma\tau} + F\, \frac{k_\sigma\, k_\tau}{k^2} \right) \right] k^{2\Delta-d} \, , \end{aligned} \tag{7.4.91}$$

where E, F and $\Pi_{\mu\nu,\sigma\tau}$ are defined below equation (7.2.12) and a_0 is an overall constant given by

$$a_0 = \frac{1}{L}\left(\Delta - \frac{d}{2}\right)\left(\frac{L}{2}\right)^{2\Delta-d} \frac{\Gamma\left(\frac{d}{2}-\Delta\right)}{\Gamma\left(\Delta-\frac{d}{2}\right)} . \tag{7.4.92}$$

The CFT two-point function of the non-conserved spin-2 operators in the momentum space has been studied in [295] and reviewed in section 7.2.1. The above expression (7.4.91) matches precisely with the expected result given in [295]. Thus, our results for the CFT two point function of massive spin-2 operators computed using the holographic renormalization are in agreement with the expected boundary CFT results.

7.5 3-point function from bulk theory

7.5.1 Bulk theory

In this section we consider a d dimensional CFT 3-point function $\left\langle O^{*\alpha\beta} J^{\kappa} O^{\gamma\delta}\right\rangle$ of a $U(1)$ conserved current J^μ and two spin-2 operators $O^{\mu\nu}$ and $O^{*\mu\nu}$ using the AdS/CFT prescription. This 3-point function on the boundary can be evaluated from a $d+1$ dimensional bulk effective action in AdS whose cubic coupling terms are linear in the gauge field A_M and quadratic in massive spin-2 fields, ϕ_{MN}. The most general such action in Euclidean signature describing the interaction between a $U(1)$ gauge field and a complex massive spin-2 field in $d+1$ dimension up to 3 derivative terms can be written as

$$\begin{aligned} S &= \int d^{d+1}x\, \sqrt{G}\Big[-\frac{1}{16\pi G_N}(R-2\Lambda) + \frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}D_M\phi^*_{NP}D^M\phi^{NP} - D_M\phi^*{}_{NP}D^P\phi^{NM} \\ &- \frac{1}{2}D_M\phi^* D^M\phi + \frac{1}{2}D_M\phi^{*MN}D_N\phi + \frac{1}{2}D_M\phi^{MN}D_N\phi^* + \frac{d}{L^2}(\phi^*_{MN}\phi^{MN} - \frac{1}{2}\phi^*\phi) \\ &+ \frac{m^2}{2}(\phi^*_{MN}\phi^{MN} - \phi^*\phi) - i\alpha g\phi^{*PM}F_{MN}\phi^N_{\ P} + ig\beta_1 F^{MN}\left(\nabla_M\phi^*_{PQ}\nabla^P\phi_N^{\ Q} - \nabla_M\phi_{PQ}\nabla^P\phi^{*\ Q}_N\right) \\ &+ ig\beta_2 F^{MN}\left(\nabla_M\phi^*_{PQ}\nabla_N\phi^{PQ}\right) + ig\beta_3 F^{MN}\left(\nabla_P\phi^*_{MQ}\nabla^Q\phi_N^{\ P} - \nabla_P\phi_{MQ}\nabla^Q\phi^{*\ P}_N\right)\Big], \end{aligned} \tag{7.5.93}$$

where the indices M, N etc. run from 0 to d and Λ is the cosmological constant. The $F_{MN} = \partial_M A_N - \partial_N A_M$ is the field strength of the gauge field A_M and we have defined

$$\phi \;=\; G^{MN}\phi_{MN} \quad ; \qquad D_M = \nabla_M + igA_M , \tag{7.5.94}$$

with ∇_M being the covariant derivative with respect to the Levi-Civita connection. The ϕ^*_{MN} is the complex conjugate of ϕ_{MN} and will be treated as an independent field. The cubic terms are parametrized by five independent parameters, namely $g, \alpha, \beta_1, \beta_2, \beta_3$, matching the number of independent parameters that we find in the CFT analysis of corresponding 3 point function. One of these parameters is the gauge coupling constant g, and it multiples the terms introduced by minimal coupling. The α denotes the gyromagnetic coupling, whereas β_i (for $i = 1, 2, 3$) denote the higher derivative couplings. One way to arrive at the higher derivative terms is to write down the most general cubic action involving the 3 derivatives and then use the lower order equations of motion to eliminate the redundant terms. This was explained in detail for the massive spin-1 field in [147].

Below, we shall only consider the minimal and gyromagnetic couplings. A systematic analysis of the higher derivative couplings for the massive spin-2 field requires causality analysis and will be postponed for a future work. Further, as mentioned in the introduction, we shall ignore the back reaction due to the presence of massive fields by working with the infinitesimal sources. An infinitesimal energy momentum tensor will also avoid the Velo-Zwanziger problem. The gauge field equation derived from (7.5.93) in the AdS background is given by

$$\nabla_M F^{MN} = J^N \qquad \Longrightarrow \qquad \Big(\nabla_M \nabla^M + \frac{d}{L^2}\Big) A_N - \nabla_N \nabla_M A^M = J_N \,, \tag{7.5.95}$$

with the source current given by

$$\begin{aligned} &J^N \\ &= -\frac{ig}{2}\phi^*_{MP}\nabla^N\phi^{MP} + \frac{ig}{2}\nabla^N\phi^*_{MP}\phi^{MP} + ig\phi^*_{MP}\nabla^P\phi^{MN} - ig\nabla_M\phi^{*N}_{P}\,\phi^{PM} \\ &+ \frac{ig}{2}\phi^*\nabla^N\phi - \frac{ig}{2}\nabla^N\phi^*\,\phi - \frac{ig}{2}\phi^{*MN}\nabla_M\phi + \frac{ig}{2}\nabla_M\phi^{*MN}\phi + \frac{ig}{2}\phi^{MN}\nabla_M\phi^* - \frac{ig}{2}\nabla_M\phi^{MN}\phi^* \\ &+ 2i\alpha g\nabla_M(\phi^{*P[M}\phi^{N]}_{\;\;P}) - 2ig\beta_1\nabla_M\left(\nabla^{[M}\phi^*_{PQ}\nabla^{|P|}\phi^{N]Q} - \nabla^{[M}\phi_{PQ}\nabla^{|P|}\phi^{*N]Q}\right) \\ &- 2ig\beta_2\nabla_M\left(\nabla^{[M}\phi^*_{PQ}\nabla^{N]}\phi^{PQ}\right) - 2ig\beta_3\nabla_M\left(\nabla_P\phi^{*[M}_{\;\;Q}\nabla^{|Q|}\phi^{N]P} - \nabla_P\phi^{[M}_{\;\;Q}\nabla^{|Q|}\phi^{*N]P}\right). \end{aligned} \tag{7.5.96}$$

For computing the 3-point function, we shall only need the $O(g)$ terms in the above current. Since the trace of the spin-2 field vanishes at the zeroth order in the coupling g, we can set the trace ϕ to zero in the above expression.

For computing the 3-point function, it is convenient to gauge fix $A_M(z,k)$. For similar analysis involving massive spin-1 fields, this was done in detail in [147]. We shall follow the same approach here working in the axial gauge $A_0(z,k) = 0$. The perturbative classical solution of the gauge field upto $O(g)$ is given, in momentum space, by [147]

$$A_\mu(z,\,\mathbf{k}) = \mathbb{K}_\mu^{\;\nu}(z,\mathbf{k})A_{(0)\nu}(\mathbf{k}) + \int dw\sqrt{G}\;\mathcal{G}_{\mu\nu}(z,\,w;\,\mathbf{k})\,J^\nu(w,\,\mathbf{k})\,, \tag{7.5.97}$$

where $A_{(0)\mu}(\mathbf{k})$ denotes the boundary value of the gauge field and $\mathbb{K}_\mu^{\;\nu}(z,\mathbf{k})$ and $\mathcal{G}_{\mu\nu}(z,w;k)$ denote the bulk-to-boundary and bulk-to-bulk propagators of the gauge field respectively. In the Poincaré coordinates, their expressions are given by [147]

$$\mathbb{K}_{\mu\nu}(z,\mathbf{k}) = c_0(k)z^{\frac{d-2}{2}}K_{\frac{d}{2}-1}(zk)\pi_{\mu\nu} \;+\; \frac{k_\mu k_\nu}{k^2}\,, \tag{7.5.98a}$$

$$\mathcal{G}_{\mu\nu}(z,w;\mathbf{k}) = -L\begin{cases} (\hat{z}\hat{w})^{\frac{d}{2}-1}I_{\frac{d}{2}-1}(kz)K_{\frac{d}{2}-1}(kw)\pi_{\mu\nu} + \frac{\hat{z}^{d-2}}{d-2}\frac{k_\mu k_\nu}{k^2}, & \text{if } z < w\,, \\ (\hat{z}\hat{w})^{\frac{d}{2}-1}I_{\frac{d}{2}-1}(kw)K_{\frac{d}{2}-1}(kz)\pi_{\mu\nu} + \frac{\hat{w}^{d-2}}{d-2}\frac{k_\mu k_\nu}{k^2}, & \text{if } z > w\,, \end{cases} \tag{7.5.98b}$$

where $\hat{z} = \frac{z}{L}$ and $\pi_{\mu\nu}$ denotes the transverse projector

$$\pi_{\mu\nu} = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \quad ; \quad \delta^{\mu\nu} k_\mu \pi_{\nu\sigma} = 0 \quad ; \quad \pi_{\mu\nu}\, \delta^{\nu\tau} \pi_{\tau\sigma} = \pi_{\mu\sigma}\,. \tag{7.5.99}$$

with $k^2 = \delta^{\mu\nu}\, k_\mu\, k_\nu$.

We shall use (7.5.97) to determine the 3-point function via holographic renormalization. Thus, we shall only need the free field classical solutions of the massive spin-2 fields, which can be expressed in terms of the Btb propagator as

$$\phi_{MN}(z, \mathbf{k}) = K_{MN}{}^{\mu\nu}(z, \mathbf{k})\, \hat{\phi}^{(0)}_{\mu\nu}(\mathbf{k}) \quad ; \quad \phi^*_{MN}(z, \mathbf{k}) = \bar{K}_{MN}{}^{\mu\nu}(z, \mathbf{k})\, \hat{\phi}^{*(0)}_{\mu\nu}(\mathbf{k})\,. \tag{7.5.100}$$

The expressions of the Btb propagators $K_{MN}{}^{\mu\nu}(z; \mathbf{k})$ and $\bar{K}_{MN}{}^{\mu\nu}(z; \mathbf{k})$ in the Poincaré coordinates are given in appendix E.3. The $\hat{\phi}^{(0)}_{\mu\nu}$ and $\hat{\phi}^{*(0)}_{\mu\nu}$ are related to the boundary values of the bulk fields $\phi_{\mu\nu}$ and $\phi^*_{\mu\nu}$, respectively. Note that we only need to specify the boundary components of the massive fields. The component involving the bulk direction, namely $\phi_{\mu z}$ and ϕ_{zz} get fixed in terms of the boundary components as discussed in section 7.3. The bulk fields ϕ_{MN} and ϕ^*_{MN} are dual to the non-conserved spin-2 boundary CFT operators with their mass m related to the conformal dimension Δ of the boundary CFT operators by the relation $L^2\, m^2 = \Delta(\Delta - d)$. On the other hand, the bulk field A_M is dual to a boundary conserved current having conformal dimension $d - 1$.

7.5.2 Three point function

In this subsection, we compute the 3-point function involving a gauge field and two massive spin-2 fields in the bulk and compare with the CFT 3-point function described in section 7.2.1. The desired bulk 3-point function can be computed using the procedure of holographic renormalization following the approach described in [147] in the momentum space for two massive spin-1 fields and one gauge field. Here, we need to follow the same steps, replacing the massive spin-1 fields with the massive spin-2 fields. The first step is to obtain the 1 point function of the bulk gauge field. This would be a function of the sources of all the fields in the theory. The functional derivative of this one point function with respect to the boundary sources of the massive spin-2 fields will give the desired 3-point function. The one point function of the bulk gauge field in the axial gauge $A_0 = 0$ is given in boundary momentum space by [147]

$$\langle \mathcal{J}^\mu(\mathbf{p}_2) \rangle = \delta^{\tau\lambda} \int_0^\infty d\sigma \sqrt{G}\, \mathbb{K}_{\lambda\kappa}(\sigma; \mathbf{p}_2) J^\kappa_{(0)}(\sigma, \mathbf{p}_2)\,. \tag{7.5.101}$$

Taking the functional derivative of the above expression with respect to the sources, we obtain

$$\begin{aligned} &\langle O^{*\alpha\beta}(\mathbf{p}_1) \mathcal{J}^\kappa(\mathbf{p}_2) O^{\gamma\delta}(\mathbf{p}_3) \rangle \\ &= \delta^{\kappa\mu} (2\pi)^d (2\pi)^d \frac{\delta^2}{\delta \hat{\phi}^{(0)}_{\alpha\beta}(-\mathbf{p}_1) \delta \hat{\phi}^{*(0)}_{\gamma\delta}(-\mathbf{p}_3)} \int_0^\infty d\sigma \sqrt{G}\, \mathbb{K}_{\mu\nu}(\sigma; \mathbf{p}_2) J^\nu_{(0)}(\sigma, \mathbf{p}_2)\,, \end{aligned} \tag{7.5.102}$$

where $\hat{\phi}^{(0)}_{\mu\nu}$ and $\hat{\phi}^{*(0)}_{\mu\nu}$ are related to the boundary conditions of the bulk massive spin-2 fields ϕ_{MN} and ϕ^*_{MN}, respectively. In AdS, these act as the sources of the boundary operators $\mathcal{O}^*_{\mu\nu}$ and

$\mathcal{O}_{\mu\nu}$ respectively. The $J^{\nu}_{(0)}$ is the boundary component of the gauge field current J^N evaluated on the free spin-2 field and is a function of the sources $\hat{\phi}^{(0)}_{\mu\nu}$ and $\hat{\phi}^{*(0)}_{\mu\nu}$. Its expression is given in equation (E.4.26). Here, we only consider the minimal and gyromagnetic terms, setting the higher derivative couplings β_i to zero. For a detailed expansion of the current and contribution to the 3-point function from each interaction term, see Appendix E.4. The contribution to the 3-point function from each term in the current can be organized in terms of the triple K integrals (7.2.9). For stating the results, we shall again work with an index free notation multiplying both sides of (7.5.102) with the auxiliary polarization tensors. Using the various triple-K identities (see, e.g., [295, 306] for the identities involving the triple K integrals), the transverse part of the desired 3-point function, after a tedious calculation, is found to be

$$\langle\langle \epsilon_1 \cdot \mathcal{O}^*(\mathbf{p_1})\epsilon_2 \cdot j(\mathbf{p}_2)\epsilon_3 \cdot \mathcal{O}(\mathbf{p}_3)\rangle\rangle = (\epsilon_2 \cdot \pi_2 \cdot p_1)A + (\epsilon_2 \cdot \pi_2 \cdot \epsilon_1)B_1 + (\epsilon_2 \cdot \pi_2 \cdot \epsilon_3)B_2\,, \tag{7.5.103}$$

with the functions A, B_1 and B_2 having the same functional form as in (7.2.5) and (7.2.6) (see appendix E.2 for detailed expressions in terms of the triple K integrals). However, the momentum independent coefficients are now given in terms of the bulk parameters as

$$\begin{aligned}
&a_0^{(0,0)} = -gC; \qquad a_1^{(0,1)} = gC\,\frac{(-1+\alpha)}{\Delta}; \qquad a_1^{(1,0)} = -gC\,\frac{(-1+\alpha)}{\Delta},\\
&a_2^{(0,2)} = gC\,\frac{4\alpha}{(\Delta-1)\Delta}; \qquad a_2^{(2,0)} = gC\,\frac{4\alpha}{(\Delta-1)\Delta}; \qquad a_2^{(1,1)} = -gC\,\frac{4\alpha}{\Delta^2},\\
&a_1^{(0,0)} = gC\,\frac{2\left((-2+d)\alpha-2\Delta\right)(\Delta-1)}{\Delta^2}; \qquad a_2^{(0,1)} = gC\,\frac{2\left(d+3d\alpha+2\alpha(\Delta-2)-2\Delta\right)}{\Delta^2},\\
&a_2^{(1,0)} = -gC\,\frac{2\left(d+3d\alpha+2\alpha(\Delta-2)-2\Delta\right)}{\Delta^2},\\
&a_2^{(0,0)} = gC\,\frac{2(\Delta-2)(-2+d+2\Delta)(d+2d\alpha-2(2\alpha+\Delta))}{(\Delta-1)\Delta^2},\\
&a_1^{(1,1)} = 0; \qquad a_2^{(2,2)} = 0; \qquad a_2^{(1,2)} = 0; \qquad a_2^{(2,1)} = 0,
\end{aligned}$$

$$\begin{aligned}
&b_{1;1}^{(0,0)} = gC\,\frac{(-2+\Delta)\left[3d^2(1+\alpha)+6d(-2+\alpha(-4+\Delta)+\Delta)-2(-1+\Delta)(\Delta+\alpha(12+\Delta))\right]}{3\Delta^2},\\
&b_{1;1}^{(2,1)} = 0; \qquad b_{1;0}^{(1,0)} = -gC\,\frac{(1+\alpha)(-1+\Delta)}{\Delta}; \qquad b_{1;1}^{(1,1)} = -gC\,\frac{(1+\alpha)}{2\Delta^2},\\
&b_{1;1}^{(2,0)} = gC\,\frac{2(1+\alpha)}{\Delta}; \qquad b_{1;0}^{(0,0)} = gC\,\frac{(1+\alpha)(d-2\Delta)(-1+\Delta)}{\Delta},\\
&b_{1;1}^{(0,1)} = gC\,\frac{(1+\alpha)(-1+d-\Delta)}{\Delta}; \qquad b_{1;1}^{(1,0)} = -gC\,\frac{2(1+\alpha)(-1+\Delta)(d+\Delta)}{\Delta^2},
\end{aligned}$$

$$\begin{aligned}
&b_{2;1}^{(0,0)} = gC\,\frac{(-2+\Delta)\left[3d^2(1+\alpha)+6d(-2+\alpha(-4+\Delta)+\Delta)-2(-1+\Delta)(\Delta+\alpha(12+\Delta))\right]}{3\Delta^2},\\
&b_{2;1}^{(1,2)} = 0; \qquad b_{2;0}^{(0,1)} = gC\,\frac{(1+\alpha)(-1+\Delta)}{\Delta}; \qquad b_{2;1}^{(1,1)} = -gC\,\frac{(1+\alpha)}{2\Delta^2},\\
&b_{2;1}^{(0,2)} = gC\,\frac{2(1+\alpha)}{\Delta}; \qquad b_{2;0}^{(0,0)} = gC\,\frac{(1+\alpha)(d-2\Delta)(-1+\Delta)}{\Delta},\\
&b_{2;1}^{(1,0)} = -gC\,\frac{(1+\alpha)(-1+d-\Delta)}{\Delta}; \qquad b_{2;1}^{(0,1)} = gC\,\frac{2(1+\alpha)(-1+\Delta)(d+\Delta)}{\Delta^2},
\end{aligned} \tag{7.5.104}$$

where, we have defined

$$C = -\frac{2^{2-\frac{d}{2}}}{\Gamma\left(\frac{d}{2}-1\right)}\left[\frac{2^{\frac{d}{2}+1-\Delta}}{\Gamma\left(\Delta-\frac{d}{2}\right)}\right]^2 L^{2\Delta-d-1}\,. \tag{7.5.105}$$

As discussed in appendix E.2, among the 30 form factors given above, only 5 are independent which we have chosen to be $a_0^{(0,0)}$, $a_1^{(1,1)}$ $a_2^{(2,2)}$, $a_2^{(1,1)}$ and $b_{1;1}^{(1,1)}$. It is easy to verify that the above coefficients satisfy the relation between the different coefficients given in (E.2.12)-(E.2.16). This is a non trivial check of the relation between the bulk gyromagnetic coupling of massive spin-2 fields and the CFT parameters.

In (7.5.103), we have only determined the part of the 3-point function which gets contribution from the transverse part of the gauge field (first term in equation (7.2.3)). The longitudinal part is given by the second term in (7.2.3). By equation (7.2.18), the longitudinal part can be related to the two point function of the spin-2 operators. We shall now show that the longitudinal part of our 3-point function can also be determined from the bulk analysis and is consistent with (7.2.18).

Focussing on odd d, we start by noting that the 1-point function of the divergence of the boundary current is given in terms of the bulk current as [147]

$$\langle\langle p_{2\mu}\mathcal{J}^\mu(\mathbf{p}_2)\rangle\rangle = -\frac{2}{L}\,\delta^{\mu\nu}\,\left(\frac{d}{2}-1\right)p_{2\mu}A_\nu^{(d-2)}\,, \tag{7.5.106}$$

where $A_\nu^{(d-2)}$ is the coefficient of $\rho^{\frac{d-2}{2}}$ in the asymptotic expansion of the gauge field

$$A_\mu(\rho,\mathbf{x}) \;=\; \sum_{j=0}^{\frac{d-3}{2}}\left(\frac{\rho}{L}\right)^j\,A_\mu^{(2j)}(\mathbf{x}) \;+\; \left(\frac{\rho}{L}\right)^{\frac{d-2}{2}}\,A_\mu^{(d-2)}(\mathbf{x})+\ldots \tag{7.5.107}$$

where the explicit expressions of $A_\mu^{(2j)}(x)$ can be found in [147].

Now, for the action in (7.5.93), the divergence of the gauge field can be computed, up to $O(g)$, via asymptotic analysis to be

$$\partial^\mu A_\mu^{(d-2)} \;=\; \frac{2ig}{(d-2)}\left(\Delta-\frac{d}{2}\right)\left(\phi_{\mu\nu}^{*(0)}\phi^{\mu\nu(2\Delta-d)}-\phi_{\mu\nu}^{(0)}\phi^{*\mu\nu(2\Delta-d)}\right)\,, \tag{7.5.108}$$

In momentum space, this gives

$$\begin{aligned}
&(d-2)\delta^{\mu\nu}p_\mu A_\nu^{(d-2)}(\mathbf{p})\\
&= g\left(\Delta-\frac{d}{2}\right)\int\frac{d^dk}{(2\pi)^d}\delta^{\mu\alpha}\delta^{\nu\beta}\left[\hat{\phi}_{\mu\nu}^{*(0)}(\mathbf{k})\hat{\phi}_{\alpha\beta}^{(2\Delta-d)}(\mathbf{p}-\mathbf{k})-\hat{\phi}_{\mu\nu}^{(0)}(\mathbf{k})\hat{\phi}_{\alpha\beta}^{*(2\Delta-d)}(\mathbf{p}-\mathbf{k})\right],\\
&= g\left(\Delta-\frac{d}{2}\right)\int\frac{d^dk}{(2\pi)^d}\delta^{\mu\alpha}\delta^{\nu\beta}P_{\alpha\beta}{}^{\gamma\delta}(\mathbf{p}-\mathbf{k})\left[\hat{\phi}_{\mu\nu}^{*(0)}(\mathbf{k})\hat{\phi}_{\gamma\delta}^{(0)}(\mathbf{p}-\mathbf{k})-\hat{\phi}_{\mu\nu}^{(0)}(\mathbf{k})\hat{\phi}_{\gamma\delta}^{*(0)}(\mathbf{p}-\mathbf{k})\right]
\end{aligned} \tag{7.5.109}$$

Using (7.5.106) and (7.5.109), we now get

$$\begin{aligned}
&\Big\langle \mathcal{O}^{*\gamma\delta}(\mathbf{p}_1) p_{2\mu} \mathcal{J}^{\mu}(\mathbf{p}_2) \mathcal{O}^{\sigma\tau}(\mathbf{p}_3) \Big\rangle \\
=\;& -\frac{1}{L}(d-2)\delta^{\mu\nu} p_{2\mu} \frac{\delta^2 A_\nu^{(d-2)}(\mathbf{p}_2)}{\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_1)\hat{\phi}^{*(0)}_{\sigma\tau}(-\mathbf{p}_3)}(2\pi)^d(2\pi)^d \\
=\;& -\frac{1}{L} g\left(\Delta - \frac{d}{2}\right)\Big[P^{\sigma\tau\gamma\delta}(\mathbf{p}_1) - P^{\sigma\tau\gamma\delta}(\mathbf{p}_3)\Big](2\pi)^d \delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \\
=\;& g\Big[\langle\langle \mathcal{O}^{*\sigma\tau}(-\mathbf{p}_3)\mathcal{O}^{\gamma\delta}(\mathbf{p}_3)\rangle\rangle - \langle\langle \mathcal{O}^{*\sigma\tau}(\mathbf{p}_1)\mathcal{O}^{\gamma\delta}(-\mathbf{p}_1)\rangle\rangle\Big](2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\,,
\end{aligned} \tag{7.5.110}$$

where we have used equation (7.4.90) for the 2-point function.

The above result is consistent with the transverse Ward identity (7.2.18) as expected. Further, it is easy to check that the coefficients of 2 and 3-points given in (7.4.92) and (7.5.104) respectively, which are obtained using the holographic renormalization procedure, satisfy the expected CFT relation (7.2.14). For even d, the same steps can be repeated and we obtain the same result (7.5.110).

7.6 Flat limit

As is well known, in the flat limit, the AdS correlators morph into the flat space correlators [123, 124, 127–129, 137, 142, 299, 346–354]. Representations such as position, Mellin or momentum provide different perspectives of the flat limit analysis. In particular, for analyzing energy poles, analytic structure of correlators and for dealing with fields with spins, the momentum representation is quite convenient and powerful [129, 147, 300]. Moreover, for connecting with the CFT data, momentum space is the cleanest approach [147].

In this section, we shall consider the flat limit of the CFT 3-point function discussed in the previous section. Following [147], for taking the flat limit of AdS correlators in the momentum space, a convenient approach is to work in the bulk coordinate τ defined by $z = L\, e^{\tau/L}$ with the metric given by

$$ds^2_{AdS_{d+1}} = d\tau^2 + e^{-2\tau/L}\delta_{\mu\nu}dx^\mu dx^\nu\,. \tag{7.6.111}$$

At the leading order in L, it reduces to the flat metric in Euclidean signature. By Wick rotating $\tau = it$, we can obtain the corresponding Lorentzian signature metric.

In subsection 7.6.1, we shall consider the flat limit of the Btb propagator of the massive spin-2 field. This will dictate the polarization of the massive spin-2 field in the flat space. In subsection 7.6.2, we shall take the flat limit of the 3-point function.

7.6.1 Flat limit of massive spin-2 Btb propagator

The Btb propagator of the massive spin-2 fields in the boundary momentum space and the bulk Poincaré coordinate z is given in (E.3.23). In the τ coordinate, the different components of the propagator can be obtained in the standard way as

$$K_{\tau\tau}{}^{\alpha\beta}(\tau;\mathbf{k}) = \left(\tfrac{\partial z}{\partial \tau}\right)^2 K_{zz}{}^{\alpha\beta}(z(\tau);\mathbf{k}), \quad K_{\tau\nu}{}^{\alpha\beta}(\tau;\mathbf{k}) = \left(\tfrac{\partial z}{\partial \tau}\right) K_{z\nu}{}^{\alpha\beta}(z(\tau);\mathbf{k}). \tag{7.6.112}$$

and $K_{\mu\nu}{}^{\alpha\beta}(\tau;\mathbf{k}) = K_{\mu\nu}{}^{\alpha\beta}(z(\tau);\mathbf{k})$. To take the flat limit, we need to know how various quantities behave as we take the limit $L \to \infty$. The relation $\Delta(\Delta - d) = m^2L^2$ implies that a finite mass in this limit can be obtained only if we simultaneously send the conformal dimension Δ to infinity along with L. More precisely, in the limit $L \to \infty$, we have

$$\Delta = mL + \frac{d}{2} + O\left(\frac{1}{L}\right). \tag{7.6.113}$$

The Btb propagator also involves the Bessel function of second kind whose order depends upon the conformal dimension Δ and the argument depends upon the bulk coordinate z (or equivalently τ and L). Hence, both the order as well as arguments of the Bessel function must be taken to be large. This is known as the uniform expansion and gives [147]

$$K_{\Delta-\frac{d}{2}+s}(kz) = \left(\frac{\pi}{2EL}\right)^{\frac{1}{2}} \left(\frac{k}{m+E}\right)^{-mL-s} e^{-EL-E\tau}\left(1 + O\left(\frac{1}{L}\right)\right), \tag{7.6.114}$$

where $E = \sqrt{k^2+m^2}$ denotes the energy of the particle in the flat limit.

Using the above relations, the different components of the bulk-to-boundary propagators can be easily worked out which in turn gives the flat limit of the classical solutions. To leading order, we have

$$\phi_{\tau\tau}(\tau,\mathbf{k}) = K_{\tau\tau}{}^{\alpha\beta}(\tau,\mathbf{k})\,\hat{\phi}^{(0)}_{\alpha\beta}(\mathbf{k}) \quad\to\quad \frac{e^{-E(\tau+L)}}{Z_\phi^{1/2}}\left(\frac{ik^\alpha}{m}\right)\left(\frac{ik^\beta}{m}\right)\hat{\phi}^{(0)}_{\alpha\beta}(\mathbf{k}), \tag{7.6.115a}$$

$$\phi_{\tau\nu}(\tau,\mathbf{k}) = K_{\tau\nu}{}^{\alpha\beta}(\tau,\mathbf{k})\,\hat{\phi}^{(0)}_{\alpha\beta}(\mathbf{k}) \quad\to\quad \frac{e^{-E(\tau+L)}}{Z_\phi^{1/2}}\left(\frac{ik^\alpha}{m}\right)\tilde{\pi}^\beta{}_\nu\,\hat{\phi}^{(0)}_{\alpha\beta}(\mathbf{k}), \tag{7.6.115b}$$

$$\phi_{\mu\nu}(\tau,\mathbf{k}) = K_{\mu\nu}{}^{\alpha\beta}(\tau,\mathbf{k})\,\hat{\phi}^{(0)}_{\alpha\beta}(\mathbf{k}) \quad\to\quad \frac{e^{-E(\tau+L)}}{Z_\phi^{1/2}}\tilde{\pi}^\alpha{}_{(\mu}\,\tilde{\pi}^\beta{}_{\nu)}\,\hat{\phi}^{(0)}_{\alpha\beta}(\mathbf{k}). \tag{7.6.115c}$$

where

$$\tilde{\pi}_{\mu\nu} = \delta_{\mu\nu} + \frac{k_\mu k_\nu}{m(m+E)}\quad , \qquad \frac{1}{Z_\phi^{1/2}} = \left(\frac{2^{-mL}\,e^{mL}\,(L(m+E))^{mL}}{(EL)^{1/2}\,(mL)^{mL-\frac{1}{2}}}\right). \tag{7.6.116}$$

The above expressions give the classical Euclidean profile of the massive spin-2 field in the flat limit

$$\phi^{\text{flat}}_{ab}(\tau,\mathbf{k}) = \Phi_{ab}(\mathbf{k})\,e^{-E(\tau+L)}, \tag{7.6.117}$$

where the indices a, b run over the $(d+1)$ dimensional flat directions and we have defined

$$\Phi_{ab} = \left(\frac{ik_\alpha}{m}\frac{ik_\beta}{m}\phi^{(0)\alpha\beta}_{\ \ R}\,,\ \frac{ik_\alpha}{m}\tilde{\pi}_{\beta\nu}\phi^{(0)\alpha\beta}_{\ \ R}\,,\ \tilde{\pi}_{\alpha\mu}\,\tilde{\pi}_{\beta\nu}\,\phi^{(0)\alpha\beta}_{\ \ R} \right) \quad , \qquad \phi^{(0)\mu\nu}_{\ \ R} = \frac{\hat{\phi}^{(0)\mu\nu}}{Z_\phi^{1/2}}. \qquad (7.6.118)$$

The factor $1/Z_\phi^{1/2}$ diverges as we take the limit $L \to \infty$. Therefore, in this limit we also need to take the AdS source $\hat{\phi}^{(0)\mu\nu} \to 0$ to keep $\hat{\phi}^{(0)\mu\nu}_{\ \ R}$ finite. This is possible since $\hat{\phi}^{(0)\mu\nu}$ is a free parameter. The uplifted Euclidean momenta for the $(d+1)$ dimensional flat spin-2 massive field is

$$q^a = (\pm iE, k^\mu) \ , \qquad q^2 = \delta^{ab}q_a q_b = -m^2\,. \qquad (7.6.119)$$

The $\pm$ here corresponds to incoming vs outgoing fields.

We can express Φ_{ab} in a basis of suitable polarization tensors. For this, we first introduce a basis $\epsilon^{(r)}_{\mu\nu}$ for $\phi^{(0)}_{\mu\nu\,R}$. The symmetry and tracelessness property of $\phi^{(0)}_{\mu\nu\,R}$ requires

$$\epsilon^{(r)}_{\mu\nu} = \epsilon^{(r)}_{\nu\mu} \quad , \qquad \delta^{\mu\nu}\epsilon^{(r)}_{\mu\nu} = 0, \qquad \mu, \nu \in \{1, 2, \cdots, d\}. \qquad (7.6.120)$$

The above conditions imply that the basis $\epsilon^{(r)}_{\mu\nu}$ satisfying these conditions has dimension $(\frac{d(d+1)}{2} - 1)$ and hence $r \in \{1, 2, \cdots, \frac{d(d+1)}{2} - 1\}$. As will be clear shortly, we shall also choose $\epsilon^{(r)}_{\mu\nu}$ to be orthonormal, i.e.,

$$\delta^{\mu\alpha}\delta^{\nu\beta}\epsilon^{(r_1)}_{\mu\nu}\epsilon^{(r_2)}_{\alpha\beta} = \delta^{r_1 r_2}, \qquad r_1, r_2 \in \left\{1, 2, \cdots, \frac{d(d+1)}{2} - 1\right\}. \qquad (7.6.121)$$

In terms of $\epsilon^{(r)}_{\mu\nu}$, we write

$$\phi^{(0)}_{\mu\nu\,R} = \sum_{r=1}^{\frac{d(d+1)}{2}-1} \phi^{(r)}\,\epsilon^{(r)}_{\mu\nu}\,. \qquad (7.6.122)$$

We can now express the flat space field Φ_{ab} as

$$\Phi_{ab} = \sum_{r=1}^{\frac{d(d+1)}{2}-1} \phi^{(r)}\,\varepsilon^{(r)}_{ab}\,, \qquad (7.6.123)$$

where we have introduced the Euclidean $(d+1)$ dimensional flat polarisation tensors $\varepsilon^{(r)}_{ab}$ as

$$\varepsilon^{(r)}_{ab} = \left(\frac{ik^\alpha}{m}\frac{ik^\beta}{m}\epsilon^{(r)}_{\alpha\beta}\,,\ \frac{ik^\alpha\tilde{\pi}^\beta{}_\nu}{m}\epsilon^{(r)}_{\alpha\beta}\,,\ \tilde{\pi}^\alpha{}_\mu\,\tilde{\pi}^\beta{}_\nu\,\epsilon^{(r)}_{\alpha\beta} \right). \qquad (7.6.124)$$

Using the definition of $\varepsilon^{(r)}_{ab}$ and the properties of $\epsilon^{(r)}_{\mu\nu}$, we can show that $\varepsilon^{(r)}_{ab}$ has the expected properties, namely it is symmetric and traceless and satisfies orthonormality, i.e.,

$$\varepsilon^{(r)}_{ab} = \varepsilon^{(r)}_{ba}\,, \qquad \delta^{ab}\varepsilon^{(r)}_{ab} = 0, \qquad \delta^{ac}\delta^{bd}\varepsilon^{(r_1)}_{ab}\,\varepsilon^{(r_2)}_{cd} = \delta^{r_1 r_2}\,. \qquad (7.6.125)$$

Further, the Polarization tensors are also transverse and satisfy the completeness relation for massive spin-2 fields, i.e.,

$$\varepsilon^{(r)}_{ab}\, q^b = 0\,, \tag{7.6.126}$$

and

$$\sum_{r=1}^{\frac{d(d+1)}{2}-1} \varepsilon^{(r)}_{ab}\,\varepsilon^{(r)}_{cd} = \frac{\pi_{ac}\pi_{bd}+\pi_{ad}\pi_{bc}}{2} - \frac{1}{d}\pi_{ab}\pi_{cd} \quad , \qquad \pi_{ab} = \delta_{ab} + \frac{q_a q_b}{m^2}. \tag{7.6.127}$$

For a derivation of the completeness relation for massive spin-2 particles, see e.g., [355].

It is also possible to construct the spin-2 polarization tensor in terms of the tensor product of spin-1 polarization vectors. Suppose $\varepsilon^{(\lambda)}_a$ denote the spin-1 polarization vectors (see [147] for the construction of spin-1 polarization vectors in terms of the AdS quantities in flat limit). Define the object

$$\epsilon^{(\lambda_1\lambda_2)}_{ab} = \frac{\varepsilon^{(\lambda_1)}_a\varepsilon^{(\lambda_2)}_b + \varepsilon^{(\lambda_2)}_a\varepsilon^{(\lambda_1)}_b}{2} - \frac{\pi_{ab}}{d}\varepsilon^{(\lambda_1)}_c\varepsilon^{(\lambda_2)c}\,, \tag{7.6.128}$$

where $\lambda_1, \lambda_2 \in (1, \cdots, d)$ and the spin-1 polarizations are given by [147]

$$\varepsilon^{(\lambda)}_a = \left(\frac{ik^\nu}{m}\epsilon^{(\lambda)}_\nu, \tilde{\pi}_\mu{}^\nu\epsilon_\nu{}^{(\lambda)}\right) \quad ; \qquad \epsilon^{(\lambda)}_\nu = \delta^\lambda_\nu\,. \tag{7.6.129}$$

Using the identities $\pi_{ab}q^a = 0$ and $\delta^{ab}\pi_{ab} = d$, we can easily check that $\epsilon^{(\lambda_1\lambda_2)}_{ab}$ is symmetric, transverse and traceless.

Using the spin-1 completeness relation $\varepsilon^{(\lambda)}_a\varepsilon^{(\lambda')*a} = \delta^{\lambda\lambda'}$, one can show that the objects $\epsilon^{(\lambda_1\lambda_2)}_{ab}$ satisfy the relation

$$\epsilon^{(\lambda_1\lambda_2)}_{ab}\epsilon^{(\lambda'_1\lambda'_2)*ab} = \frac{\delta^{\lambda_1\lambda'_1}\delta^{\lambda_2\lambda'_2}+\delta^{\lambda_1\lambda'_2}\delta^{\lambda_2\lambda'_1}}{2} - \frac{1}{d}\delta^{\lambda_1\lambda_2}\delta^{\lambda'_1\lambda'_2}\,. \tag{7.6.130}$$

The spin-2 polarization can now be defined in terms of $\epsilon^{(\lambda_1\lambda_2)}_{ab}$ as

$$\varepsilon^{(r)}_{ab} = \sum_{\lambda_1,\lambda_2=1}^{d} U^{(r)}_{\lambda_1\lambda_2}\epsilon^{(\lambda_1\lambda_2)}_{ab}\,, \tag{7.6.131}$$

where, the matrix elements $U^{(r)}_{\lambda_1\lambda_2}$ are essentially the Clebsch Gordan (CG) coefficients in going from $\epsilon^{(\lambda_1\lambda_2)}_{ab}$ basis to $\varepsilon^{(r)}_{ab}$ basis. A canonical choice is $U^{(r)}_{\lambda_1\lambda_2} = \langle 1\lambda_1; 1\lambda_2|2r\rangle$, i.e. Clebsch–Gordan coefficients[2] for the standard angular momentum problem $1\otimes 1 \to 2$ (in $D = 4$ this is same as usual angular momentum addition with $\lambda \in \{+1, 0, -1\}$ and $r \in \{+2, +1, 0, -1, -2\}$). In

[2] We use the standard notation $\langle j_1 m_1; j_2 m_2|JM\rangle$ to denote the Klebsch-Gordan coefficients for adding the angular momenta j_1 and j_2 with z components m_1 and m_2 respectively to produce the angular momentum J with z component M.

general, we have following properties of $U^{(r)}_{\lambda_1\lambda_2}$

$$U^{(r)}_{\lambda_1\lambda_2} = U^{(r)}_{\lambda_2\lambda_1} \qquad \text{(symmetric)}, \tag{7.6.132a}$$

$$\delta^{\lambda\lambda'} U^{(r)}_{\lambda_1\lambda_2} = 0 \qquad \text{(traceless)}, \tag{7.6.132b}$$

$$\sum_{\lambda,\lambda'=1}^{d} U^{(r)}_{\lambda_1\lambda_2}\,(U^{(r')}_{\lambda_1\lambda_2})^* = \delta^{rr'} \qquad \text{(orthonormality on symmetric traceless subspace)}. \tag{7.6.132c}$$

For Clebsch–Gordan coefficients, (7.6.132a) follows from the exchange symmetry $\langle j_1 m_1; j_2 m_2|JM\rangle = (-1)^{j_1+j_2-J}\langle j_2 m_2; j_1 m_1|JM\rangle$ with $j_1 = j_2 = 1$ and $J = 2$, while (7.6.132b) expresses orthogonality to the singlet ($J = 0$) state and (7.6.132c) is the standard CG orthogonality relation.

Using the properties of $U^{(r)}_{\lambda_1\lambda_2}$, we can easily show that the polarization defined by (7.6.131) satisfy the orthonormality and completeness relations. For orthonormality, we note that

$$\begin{aligned}
\varepsilon^{(r)}_{ab}\,\varepsilon^{(r')*ab} &= \sum_{\lambda,\lambda'}\sum_{\kappa,\kappa'} U^{(r)}_{\lambda\lambda'}\,(U^{(r)'}_{\kappa\kappa'})^*\,\epsilon^{(\lambda\lambda')}_{ab}\,\epsilon^{(\kappa\kappa')*ab} \\
&= \sum_{\lambda,\lambda'} U^{(r)}_{\lambda\lambda'}\,(U^{(r)'}_{\lambda\lambda'})^* \\
&= \delta^{rr'}\,,
\end{aligned} \tag{7.6.133}$$

where in going to the second equality, we have used equation (7.6.130), (7.6.132a) and (7.6.132b). In going to the last equality, we have used (7.6.132c).

The completeness relation also follows if we note that $\sum_s U^s_{\lambda\lambda'}\,(U^s_{\kappa\kappa'})^*$ acts as the projector onto the symmetric traceless subspace and hence it satisfies

$$\sum_r U^{(r)}_{\lambda\lambda'}\,(U^{(r)}_{\kappa\kappa'})^* \equiv P_{\lambda\lambda',\kappa\kappa'} = \frac{1}{2}\Big(\delta^{\lambda\kappa}\delta^{\lambda'\kappa'} + \delta^{\lambda\kappa'}\delta^{\lambda'\kappa}\Big) - \frac{1}{d}\,\delta^{\lambda\lambda'}\delta^{\kappa\kappa'}. \tag{7.6.134}$$

Using the above identity together with the definition (7.6.128) and the completeness relation of spin-1 polarizations, it follows that the spin-2 completeness relation (7.6.127) is satisfied.

The analysis done so far recovers the flat space massive spin-2 field in the Euclidean signature. However, we can also Wick rotate the results to the Lorentzian signature. The Lorentzian time t is related to the Euclidean time τ by $t = -i\tau$, i.e., $x^0_L = -ix^0_E$ where the subscript L and E correspond to the Lorentzian and Euclidean respectively. The Wick rotated $(d+1)$ dimensional momenta and polarisation are given by[3]

$$\begin{aligned}
q^0_L &= -iq^0_E = E, \qquad q^\mu_L = q^\mu_E; \\
\varepsilon^{00(r)}_{L} &= (-i)^2\varepsilon^{00(r)}_{E}, \qquad \varepsilon^{0\nu(r)}_{L} = (-i)\varepsilon^{0\nu(r)}_{E}, \qquad \varepsilon^{\mu\nu(r)}_{L} = \varepsilon^{\mu\nu(r)}_{E}.
\end{aligned} \tag{7.6.135}$$

[3]In general, under a Wick rotation to Minkowski signature, any $(n+k)$- rank Euclidean tensor $T^{0\cdots0,\mu_1\cdots\mu_k}$ with n contravariant 0-components, picks up n factors of $(-i)$, i.e.,

$$T_L^{\overbrace{0\ldots0}^{n\text{ times}}\,\mu_1\ldots\mu_k} = (-i)^n\,T_E^{\overbrace{0\ldots0}^{n\text{ times}}\,\mu_1\ldots\mu_k}.$$

Thus, the Lorentzian signature momenta and the polarization tensors in the AdS flat limit are given by

$$q_L^a \;=\; (\pm E, k^\mu) \quad ; \qquad \varepsilon_L^{ab(r)} \;=\; \left(\frac{k^\alpha}{m}\frac{k^\beta}{m}\epsilon^{(r)}_{\alpha\beta}\;,\; \frac{k^\alpha \tilde{\pi}^{\beta\nu}}{m}\tilde{\pi}^{\alpha\nu}\epsilon^{(r)}_{\alpha\beta}\;,\; \tilde{\pi}^{\alpha\mu}\,\tilde{\pi}^{\beta\nu}\,\epsilon^{(r)}_{\alpha\beta}\right) \tag{7.6.136}$$

The Wick rotated Lorentzian $(d+1)$ dimensional flat polarisations also satisfy conditions (7.6.125), (7.6.126), (7.6.127) with $\delta^{ab} \to \eta^{ab}$.

The flat limit of the classical profile of the gauge field and its bulk-to-boundary propagators were given in [147]. Here, we just note that in the flat limit $L \to \infty$ the transverse component of the massless spin-1 gauge field becomes

$$A^{\perp}_M \to A^{\text{flat}\,\perp}_a = \mathbb{A}_a e^{-k(L+\tau)}\,, \tag{7.6.137}$$

where

$$\mathbb{A}_a = \left(0, \pi_{\mu\nu} a^{\nu}{}^{(0)}_R\right) \quad ; \qquad a^{\nu}{}^{(0)}_R = \frac{a^{\nu(0)}}{Z_A^{1/2}}. \tag{7.6.138}$$

The factor Z_A denotes a renormalization factor whose explicit expression can be found in [147]. For $d > 3$, as $L \to \infty$, $\frac{1}{Z_A^{1/2}} \to \infty$, therefore we need to also take $a^{\nu(0)} \to 0$ to keep $a^{\nu}{}^{(0)}_R$ finite. In the flat limit, the Euclidean $(d+1)$ dimensional momentum for the massless spin-1 gauge field is identified with

$$q^a = (\pm ik, k^\mu) \quad ; \qquad q^2 = \delta_{ab} q^a q^b = 0\,. \tag{7.6.139}$$

The basis for the polarization vector in terms of the AdS quantities can be written as in (7.6.129).

7.6.2 Flat limit of CFT 3-point function

The Btb propagators dictate the flat space polarizations. In the bulk AdS calculation, the 3-point function in the momentum space is expressed as sum of terms involving the triple K integrals. In the large L and large Δ limit, these triple K integrals take a simple form and give rise to the energy conserving delta functions [147]

$$J_{N\{k_i\}} \xrightarrow{\textbf{flat limit}} \left(\frac{\pi}{2}\right)^{3/2} L^{\frac{d-5}{2}+N}\, \frac{(m+E_1)^{mL+k_1}\, p_2^{\frac{d-3}{2}+k_2}\, (m+E_2)^{mL+k_3}}{(E_1\,E_3)^{1/2}}\, (2\pi i)\, \delta(E_1+E_2+E_3)\,. \tag{7.6.140}$$

We see that in the large L limit, the energy conserving delta function emerges naturally. We can apply the above identity to the triple K integrals appearing in the 3-point function involving the spin-1 conserved current and two spin-2 non-conserved operators given in section 7.5.2. Alternatively, it is also possible to directly work with the 3-point function expressed as integrals over the modified Bessel function. Both approaches give the same final result. Here, we shall follow the second approach. We give the details of the calculations in appendix E.5.2 and state the main results here. In the large L, Δ limit, the AdS 3-point function takes the form

$$\lim_{L\to\infty} \sqrt{Z_{\phi_1} Z_A Z_{\phi_3}}\, \epsilon^1_{\mu_1\mu_2}\epsilon^2_{\mu_3}\epsilon^3_{\mu_4\mu_5} A_3^{\mu_1\mu_2;\mu_3,\mu_4\mu_5} = (2\pi)^{d+1}\delta^{d+1}(q_1+q_2+q_3)\,\varepsilon^1_{ab}\varepsilon^2_c\varepsilon^3_{de}\mathcal{M}_3^{ab;c;de}\,. \tag{7.6.141}$$

In the above expression, ε_{ab} and ε_a are the flat space polarizations of massive spin-2 and massless spin-1 fields constructed out of the d dimensional auxiliary polarizations $\epsilon_{\mu\nu}$ and ϵ_μ as described in the previous subsection. The $A_3^{\mu_1\mu_2;\mu_3;\mu_4\mu_5}$ is the momentum space CFT 3-point function and $\mathcal{M}_3^{ab;c;de}$ is the corresponding flat space scattering amplitude. The $q^a = (\pm E, p^\mu)$ denote the flat space momenta. The spatial momentum conserving delta function is present in the CFT 3-point function itself. The energy conserving delta function $\delta(E_1+E_2+E_3)$ arises in the flat limit after integration over the bulk coordinate (see appendix E.5.2). Putting the expression of $\mathcal{M}_3$ in (7.6.141), we obtain

$$\begin{aligned}
&\lim_{L\to\infty} \sqrt{Z_A Z_{\phi_1} Z_{\phi_3}}\,\epsilon_1^{(r)}{}_{\alpha\beta}\,\epsilon_\kappa^{(\lambda)}\,\epsilon_3^{(r)}{}_{\gamma\delta}\,\langle O^{*\alpha\beta}(\mathbf{p}_1)\,J^\kappa(\mathbf{p}_2)\,O^{\gamma\delta}(\mathbf{p}_3)\rangle \\
&= -\,ig\,(2\pi)\delta(E_1+E_2+E_3)(2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \\
&\quad\left[\frac{1}{2}(\varepsilon_{1ab}\,\varepsilon_3^{ab})(q_3-q_1)\cdot\varepsilon_2 + \Big\{(1+\alpha)\,\varepsilon_2^a\,\varepsilon_{1ab}\,\varepsilon_3^{bc}q_{2c} - (1\leftrightarrow 3)\Big\}\right].
\end{aligned} \tag{7.6.142}$$

By comparing this d-dimensional CFT correlator in flat limit with the corresponding expected $(d+1)$ dimensional flat amplitude reviewed in appendix E.6, we see that they match exactly provided we identify the flat space couplings $\hat{g}$ and the gyromagnetic ratio $\hat{\alpha}$ with their AdS counterparts g and α respectively at tree level. This shows that the flat space limit of the CFT correlator involving non-conserved massive spin-2 operators correctly reproduces the interacting part of the corresponding tree level flat-space S-matrix.

7.7 Discussion

In this chapter, we have analysed $U(1)$ charged massive spin-2 fields in the context of holography. In particular, we have studied the holographic renormalization of massive spin-2 field and used it to compute the 2 point function in momentum space. The expression matches exactly with the expected CFT result. We have also computed the AdS 3-point correlator involving two complex massive spin-2 fields and a gauge field. Our analysis gives the relation between the gyromagnetic coupling in the bulk and the boundary OPE coefficients. We have also reproduced the expected 3-point function involving the gauge and massive spin-2 fields in the flat space by taking a suitable flat limit of AdS.

The 3-point analysis we have done can't fix the value of the gyromagnetic ratio. One needs to analyse the 4-point function for that. In particular, a bootstrap type of analysis may help analyse the gyromagnetic coupling in the bulk. The relation between the bulk coupling and the boundary OPE coefficients, we have obtained in this work, will be useful for such an analysis. In this work, we have not considered the higher derivative couplings involving the massive spin-2 fields. At the level of the the 3-point function, it will be straightforward to include their contributions. They play an important role in the analysis of causality behaviour in theories of gravity coupled to massive higher spin fields. We have also worked with a specific choice 1 of the parameter ξ which parametrises the space of possible kinetic terms of massive spin-2 fields in AdS. This gave the expected relation between mass m of the bulk massive spin-2 field and the conformal dimension Δ of the dual boundary operator. However, since all values of ξ are expected to give a consistent kinetic term in AdS, it would be interesting to explore the role played by other values of ξ in the holographic setting.

From the analysis done in [147], and in this work, we also see how to perform the holographic renormalization of any massive higher spin field in AdS. For the symmetric traceless tensors, the analysis will be very similar to the massive spin-2 case. The auxiliary conditions will relate the solutions of field components having all boundary indices with the components having one or more bulk indices. The calculation of two point function of such fields is expected to reproduce the result given in [295, 344] for arbitrary symmetric traceless massive higher spin fields in the momentum space. For the mixed symmetry fields, the analysis is expected to be more involved. In particular, the auxiliary conditions will be more complicated, and one may have to analyse the conditions on a case by case basis. The calculations of correlators involving the massive higher spin fields using holographic renormalization in an EFT approach is expected to proceed along similar lines. However, if we want to move beyond EFT, the analysis will become more involved due to the inclusion of all massive higher spin fields. It would be interesting to use the momentum space CFT techniques to explore this aspect.

In this work, we have considered the flat limit of only the bulk theory. The bulk limit $L \to \infty$ corresponds to a Carrollian limit in which we send the speed of light c to zero. on the boundary [253]. The boundary CFT correlators reduce to Carrollian conformal correlators living on the null boundary in Minkowski spacetime in this limit. This has been analysed for the scalar fields in position [247, 253] as well as momentum spaces [151]. However, there have been very few studies of Carrollian theories for higher spin fields. The ultra relativistic Carrollian limit of CFTs provides a useful starting point to study the theories on the flat space null surfaces (for a discussion on limitations to this approach, see [151]). Since momentum space provides a powerful and systematic framework for taking the Carrollian limit, it would be useful to consider the Carrollian limit of the results in this chapter.

Chapter 8

Conclusions

In chapter (2), we have computed entanglement entropy using the holographic Ryu-Takayanagi prescription, for fields on flat$_{d\geq3}$, and AdS$_{d\geq3}$ backgrounds. We have also analysed certain features of entanglement entropy in AdS$_{d\geq3}$ black hole backgrounds. Using these results, we have analysed the presence of islands in higher-dimensional AdS black holes. The results of this chapter seem to suggest that islands might not be a generic feature of all field theories for evaporating black holes in higher dimensions.

Prompted by this observation, we pose the following questions: how generic are islands in the case of a higher-dimensional black hole evaporation? Should the field theories and their spectrum on these backgrounds satisfy certain conditions for the presence of islands? In chapter (3), we address these questions. We use the Quantum extremal surface/ island prescription to derive particular conditions on the entanglement entropy of quantum fields on a black hole background, for the presence of islands. Since the entanglement entropy of fields on these black hole backgrounds captures details of the fields, such as the degree of entanglement across subregions, therefore it imposes conditions on the allowed spectrum of radiation described by these fields. These conditions are sensitive to the scaling of entanglement entropy with a subregion on black hole backgrounds, rather than entanglement entropy itself.

In chapter (4), using the replica trick we systematically analyze the properties of entanglement entropy, including its scaling with a subregion (4.4.52), (4.5.74), on static black hole backgrounds. We explore the analytic $(q-1)$ expansion of Rényi entropy S_q and its variations. For conformal fields, the scaling behavior depends on the replica stress tensor. The geometry of the entangling region is captured by the metric on the replica space. On static backgrounds we can uniquely and explicitly solve for the replica stress tensor expectation value using the replica stress tensor conservation equations (4.6.176), (4.6.185), (4.6.186) and trace equation (4.6.183) in the case of a CFT. Therefore, we can completely determine the scaling of the entanglement entropy (see (4.5.115)) for a CFT on a static black hole background. Our setup also applies to generic variations, from symmetry transformations to variations of the background metric or entangling region (4.4.52).

The approach presented in this chapter provides a way to calculate the scaling of Rényi and entanglement entropies (and the quantities themselves up to a constant) for general states and subregions on arbitrary backgrounds. It also sets up the methodology required to calculate replica stress tensor correlations in any state given the corresponding stress tensor correlations on the base manifold. We have presented explicit computations for vacuum and quasithermal states for a subregion bounded by two planes in flat space using the replica metric corresponding to an unsquashed cone (4.3.26). We have also given explicit computations in spherical and cylindrical entangling boundaries in flat space.

Islands provide a semiclassical prescription for the fine-grained entropy of Hawking radiation. They account for the additional degrees of freedom required to purify the radiation and restore unitarity during black hole evaporation. However, they are motivated through quantum corrections to the black hole entropy. They do not provide the microscopic mechanism through which the information is preserved during black hole evaporation. A description of the microscopic mechanism will also give us clarity on the microscopic degrees of freedom that make up the island. A related observation is that semiclassical physics is expected to fail near singularities and at final stages of black hole evaporation. Hence an accepted complete picture of how the island story transitions into full quantum gravity is missing. Although, the quantum corrections via islands provide us hints beyond a naive semiclassical analysis, to address the above outstanding issues and really understand black hole evaporation and other gravitational effects completely, we need to gain a better understanding of how to consistently quantize the microscopic degrees of freedom of gravity coupled to other quantum fields. The loss of unitarity is a feature of partial quantization via semiclassical analysis. A consistent UV complete quantum description is expected to preserve unitarity by construction. An important open direction is to include dynamical gravitons and account for quantum gravitational backreaction, which will give us a better understanding of black hole evaporation and gravity.

Realistic black hole evaporation is intrinsically time-dependent: as Hawking radiation is emitted, the black hole loses mass, so the background geometry and the quantum state evolve in time. Static geometries are controlled approximations, and not exact descriptions of evaporation. The island prescription is substantially less controlled in time dependent settings than in static or eternal black hole backgrounds. The main difficulty is that the generalized entropy functional must now be extremized in a geometry that is itself evolving, with a quantum state that is not time-translation invariant. As a result, the area term that depends on the geometry, and the matter entanglement entropy term are explicitly time dependent, and hence the same is also true for the location of candidate QES surface(s) .

We also note that in dynamical spacetimes there is generally no natural global static slice. Hence the relevant surface is covariantly extremal, not a minimal surface on a preferred time slice. We have already observed that the holographic HRT prescription (1.2.5) explicitly characterizes the relevant surface by vanishing null expansions rather than by ordinary spatial minimality. Time-dependent setups often exhibit multiple competing QES saddles and sharp phase transitions between them, which can be much more intricate as compared to the static approximation. Interesting first steps in this direction could follow from considering de Sitter type backgrounds.

Besides black hole evaporation, in many body quantum systems, time-dependent entanglement entropy is a key diagnostic of non-equilibrium quantum dynamics, because it directly measures the spreading of quantum correlations and information in processes such as quenches, transport, thermalization, and localization. Currently, entanglement entropy is calculated in time dependent settings using the replica trick (in special cases), using numerics in lattice systems, and holographically using the HRT prescription (1.2.5). An open problem is to get a better handle on these computations for general time dependent settings.

Many semiclassical analyses assume *quasi-static* adiabatic black hole evaporation. In this approximation, the spacetime may be treated as a sequence of instantaneous stationary configurations making the computation of matter entanglement entropy on these backgrounds more tractable. In addition to time dependence, black hole evaporation is non adiabatic and the background geometry, quantum state, and generalized entropy evolve on comparable timescales, rendering the quasi-static approximation unreliable.

A modest but challenging next step would be to analyse islands in stationary black hole backgrounds. To do this we would need to extend the computation of entanglement entropy in these backgrounds. One way of approaching this would be to make use of the generating boost or rotation transforms that relate stationary and static black holes. For example, the defining equations for the replica stress tensor in the stationary case will be related to those in the static case by the corresponding generating transformations.

A further motivation, along the lines of the previous discussion, to explore beyond just the island paradigm is the observation by [53]. They argue that a key premise behind many Page curve narratives: the idea that the gravitational Hilbert space factorizes cleanly into an interior and exterior along the radial direction, is not correct in quantum gravity because the algebra of asymptotic observables is complete (in both AdS and asymptotically flat settings). They contend that, in a standard theory of gravity, information therefore does not literally emerge from the black hole at late times in the way the Page curve story suggests, rather, it is already encoded in the asymptotic algebra, and (in principle) the entire interior can be reconstructed from exterior data. They further explain that one can still obtain Page curves by effectively removing the Hamiltonian from the accessible exterior algebra: interpreted operationally as a detector with a "blind spot" that cannot measure part of the asymptotic region, or (in special discussions of null infinity) by discarding the Hamiltonian despite its physical accessibility, so that the resulting Page curve tracks a redistribution of information between measured and unmeasured degrees of freedom rather than fundamental information recovery.

Relatedly, in earlier work [95, 231–233] argue that in theories with long-range gravity where Gauss law holds, islands which by definition are disconnected from the asymptotic region of spacetime do not constitute consistent entanglement wedges. This is because the energy of an excitation localized to the island can be detected from outside the island. It was argued that this can be resolved if islands appear in conjunction with a massive graviton.

Taken together, these observations strengthen the case for going beyond the island paradigm and for understanding more directly how information is organized in a UV complete theory of quantum gravity.

We could straightforwardly extend our methodology to compute entanglement entropy to Carrollian and celestial field theories, relevant for holography in asymptotically flat spacetimes. Our computation uses the field theoretic replica trick and does not rely on holography or on the nature of the asymptotics of the spacetime. The trace and conservation equations will be modified, reflecting the underlying symmetry structure of these theories [224–229]. It would be interesting to explore entanglement in such theories, and relate to the proposed holographic descriptions.

Recently, several notions of entanglement from spatial to temporal bipartitions of quantum systems have emerged as a novel frontier in quantum many-body physics, quantum field theory and gravity [356]. Beyond the von Neumann entropy, several useful information-theoretic measures that quantify correlations, such as mutual information, entanglement negativity, reflected entropy, relative entropy, and Rényi entropies, are also being actively explored. Our methodology can be generalized to facilitate the computation of these measures.

Explicit computations of S_{QFT} in higher dimensions also facilitate exploration of the quantum focusing conjecture [230], with the possibility of refining bounds further.

In chapter (5) we have solved the momentum space *Carrollian CFT Ward identities* for the 2 and 3-point correlators. Unlike standard conformal field theories, the Ward identities of Carrollian conformal theories admit multiple structures reflecting the fact that the Carrollian manifold has a null direction. The 2-point function has two independent structures known as electric and magnetic branches. On the other hand, the 3-point function admits 5 independent branches. For the 2-point function, both the branches are completely fixed upto some normalization constant. On the other hand, for the 3-point functions, four of the branches are fixed by symmetries, whereas one of the branches is only fixed up to a function of ratios of the Carrollian energies. For specific values of the conformal dimensions, the three-point functions in momentum space exhibit logarithmic behaviour. This has no analogue in position space and instead originates from singularities in the Fourier transform relating position and momentum space correlators. We have also reproduced these from the Carrollian limit of the corresponding CFT correlators. If we put the CFT on the AdS boundary, then their Carrollian limit $c \to 0$ can be interpreted as the flat limit of AdS from the boundary perspective.

We can similarly analyse the 4-point Carrollian conformal correlators by solving the CCFT Ward identities. 4-point conformal correlators in position space are fixed up to a function of cross ratios and admit a conformal block expansion. Each conformal block corresponds to a projector involving an operator with a particular conformal dimension and its descendants. Since Carrollian CFTs are Carroll limits ($c \to 0$) of CFTs, we expect a similar constraint on at least one branch of the 4-point Carrollian CFT correlator.

Carrollian Conformal theories on null infinities, that are dual to scattering in asymptotically flat spacetimes (AFS), have a very different symmetry group as compared to usual Poincaré invariant QFTs. Consequently, they have very different analytic properties. In particular, it will be interesting to explore how these map to the well-known analytic properties of traditional QFTs. In momentum space representation, these properties translate into analytic behavior of the flat amplitude. For example, locality requires some kind of polynomial boundedness of amplitudes for theories involving a finite number of derivatives. Causality is ensured by behavior under the analytic continuation of the amplitudes. Unitarity shows up in various dispersion relations and cutting rules of the amplitudes.

This hints that we can get better insights into the analytic properties of CCFTs by analyzing them in momentum space. Moreover, relations between asymptotic symmetry Ward identities, soft theorems, and memory effects are naturally expressed when these are considered in momentum space. Hence, a momentum space formulation of the flat space boundary dynamics can provide the most direct bridge between these aspects of the bulk gravitational physics.

In chapter (6), we have proposed a generating functional for boundary CCFT correlation functions, analogous to the on-shell action that generates boundary QFT connected graphs in AdS/CFT. The usual path integral in flat space with the right choice of source that places the scattering fields directly on null infinities bypasses the requirement of LSZ and reproduces the Carrollian correlators. We have explored this in the simple setup of interacting scalar fields in the bulk. By starting with these in flat space written in flat null coordinates, the on-shell action, with the right boundary conditions for the classical solutions at null infinity and a suitable choice of source, generates the Carrollian correlators derived earlier. The formalism naturally gives an explicit relation between the CCFT correlators and S matrices.

An important step further would be to derive such an explicit holographic dictionary for theories with gravity in the bulk. We can do this by understanding the right boundary expansion of the metric, fall-off conditions, and sources, learning from the scalar case. Solving the Carrollian CFT spin-2 Ward identities will predict the 2 and 3-point correlators expected from the holographic analysis.

Using the symmetry and structure of the correlators, we can explore the analytic properties of CCFT correlators, such as locality, causality, and unitarity. By transforming Yang-Mills and gravity amplitudes to the Carrollian basis, we can explore the implications of the analytic properties of amplitudes in the CCFT correlators. This includes the manifestation of asymptotic flat space symmetries in the dual Carrollian CFT, giving us insights into flat space gravity. We can also explore the implications in the other direction: how do Carrollian conformal block expansions manifest in amplitudes? Using these analytic properties, we can explore a CCFT bootstrap. Since CCFTs are $c \to 0$ limits of CFTs and are related to the S-matrix by a simple integral transformation, the presence of CFT and S-matrix bootstrap suggests a CCFT bootstrap.

Further, it has been shown by Britto, Cachazo, Feng, and Witten that certain tree-level amplitudes factorise into lower-point on-shell amplitudes [357, 358]. The proof of these BCFW recursion relations involves deforming the momentum by a complex shift and reconstructing the tree amplitude as a sum over its on-shell factorization residues (products of lower-point amplitudes) at the resulting poles. Carrollian CFT correlators are expected to admit similar recursion relations, and momentum space facilitates their derivation. These CCFT relations can also be explored by transforming the S-matrix BCFW, or by a flat limit ($\Lambda \to 0$) of the AdS BCFW recursions [359].

The explorations so far in flat Carrollian holography have mainly focused on null boundaries $\mathscr{I}^{\pm}$. This is relevant for the massless bulk fields. However, for the massive bulk fields, we also need to take into account the timelike and spacelike components of the flat space boundaries [314]. The theories living on these boundary components are still very poorly understood compared to the theories on the null boundaries.

[229] proposed a natural Carrollian structure "blowup of timelike infinity" (**Ti**) exists around timelike infinity. The generators of **Ti** have a non-zero Casimir and can capture massive scattering. We intend to study correlators on this Carrollian structure. In the flat limit (vanishing cosmological constant $\Lambda \to 0$), Euclidean AdS reproduces **Ti**. By considering the flat limit of correlators formed by operator insertions on a Euclidean AdS glued to a Lorentzian cylinder [360], we intend to reproduce these massive **Ti** correlators. We also plan to develop a holographic

proposal with the on-shell action with appropriate sources as a generating functional, along the lines of the massless case.

In chapter (7), we have analysed a bulk effective field theory in AdS containing a U(1)-charged massive spin-2 field coupled to a gauge field. After holographic renormalization, we have computed the one, two, and three-point functions involving two massive spin-2 fields and one gauge field. Matching with the CFT 3-point correlator of two non-conserved spin-2 operators and a conserved current, we have explicitly mapped the bulk minimal and gyromagnetic couplings and the boundary OPE data.

A complementary approach to Carrollian holography is the flat limit (vanishing cosmological constant $\Lambda \to 0$) of the well-understood correlators in AdS/CFT to reproduce flat amplitudes (that are related to the Carrollian correlators by a simple integral transformation). In section (1.3.2) for the scalar case, and in chapter (7) for higher spin fields, we have shown that tree-level 2 and 3-point momentum space AdS correlators involving massive fields reduce to corresponding massive flat amplitudes under the flat limit (taken such that the bulk mass is kept fixed). While the Carroll limit ($c \to 0$) of the holographic CFTs works only for the massless case, the flat limit works for both massless and massive fields.

We can further explore the flat limit of higher-point AdS correlators and loop-level contributions. We can analyse how the renormalization of UV divergences through conformal regulators in AdS reduces to flat space regulators and renormalization. The flat limit is also closely related to the Carroll limit $c \to 0$ of CFT correlators, as we noticed in section (1.3.3), which we can explore further through these examples.

In the top-down derivation of AdS/CFT, one considers Type IIB string theory with N coincident D3 branes. The same setup admits two complementary low energy descriptions: in the open string description, the massless modes of strings ending on the branes give a 4d conformal gauge theory, $\mathcal{N} = 4$ super–Yang–Mills; in the closed string description, the branes are heavy, charged sources for the massless closed string fields, and the resulting near horizon/ decoupling limit isolates a curved throat geometry $\mathrm{AdS}_5 \times S^5$. The decoupled sector can then be described equivalently either as the 4d gauge theory or as string theory propagating on $\mathrm{AdS}_5 \times S^5$.

Motivated by analogous top-down ideas for flat space quantum gravity (for example, matrix theory formulations of M-theory in flat space), we can investigate a UV complete route to holography in asymptotically flat spacetimes. Compactifying a 10-dimensional string theory on a 6-dimensional torus T^6 yields an effective 4-dimensional Minkowski spacetime at long distances. Using such a setup we can determine the resulting asymptotic symmetry structure and its conserved charges: while one expects the Bondi-van der Burg-Metzner-Sachs (BMS) group (and possible extensions) to govern the universal gravitational sector, the compact T^6 Kaluza-Klein and form-field sector is expected to induce additional large gauge symmetries, new families of charges, and corresponding Ward identities acting on four-dimensional observables. The BMS group can be understood as the group of conformal Carroll transformations preserving the Carrollian structure of null infinity $\mathscr{I}^{\pm}$. This provides a concrete top-down route toward a Carrollian (null boundary) description of the compactified string theory in terms of boundary currents and charges.

Appendix A

Appendix to Chapter 2

A.1 Analysis of the RT surface

The area functional (2.5.59) is minimized by solving the differential equation

$$\cosh(\sigma)x(\sigma)^2\left(\cosh(\sigma)x''(\sigma)+3\sinh(\sigma)x'(\sigma)\right) \\ +2\sinh(\sigma)\cosh^3(\sigma)x'(\sigma)^3+x(\sigma)^3=0. \tag{A.1.1}$$

The RT surface is the solution $x(\sigma)$ that has a turning point at (x_0,σ_0) in the bulk and intersects the boundary at $(\sigma\to\infty,x_1)$ and $(\sigma\to\infty,x_2=x_1+L)$. We expect two branches of solution corresponding to whether the solution intersects the boundary at x_1 or x_2: $x_a(\sigma)$ and $x_b(\sigma)$. Hence, we have the boundary conditions

$$\begin{aligned} &x_a(\infty)=x_1, \quad x_b(\infty)=x_2 \\ &x_a(\sigma_0)=x_b(\sigma_0)=x_0 \\ &x_a'(\sigma_0)=x_b'(\sigma_0)=\infty. \end{aligned} \tag{A.1.2}$$

Doing a change of coordinates $u=e^{-2\sigma}$, the differential equation becomes

$$(u-1)u(u+1)^3x'(u)^3+\frac{1}{2}(u+1)x(u)^2\big(2u(u+1)x''(u) \\ +(5u-1)x'(u)\big)+x(u)^3=0 \tag{A.1.3}$$

where the conformal boundary now is at $u=0$. Further changing coordinates to $x(u)=e^{f(u)}$ we get,

$$u(u+1)^2f''(u)+\frac{1}{2}\left(5u^2+4u-1\right)f'(u)+(u-1)u(u+1)^3f'(u)^3 \\ +u(u+1)^2f'(u)^2+1=0. \tag{A.1.4}$$

We can immediately notice that the resulting differential equation depends only on $f''(u)$ and $f'(u)$. Hence, we can now split the second-order ODE into two first-order ODEs:

$$f'(u) = g(u) \tag{A.1.5}$$

$$\begin{aligned} &u(u+1)^2 g'(u) + \frac{1}{2}\left(5u^2+4u-1\right) g(u) + (u-1)u(u+1)^3 g(u)^3 \\ &+ u(u+1)^2 g(u)^2 + 1 = 0. \end{aligned} \tag{A.1.6}$$

Solving (A.1.6) we get the implicit relation for $g(u)$

$$\frac{\sqrt{\frac{u-1}{u}}\left(\frac{(2(u+1)ug(u)-u+1)\,{}_2F_1\left(\frac{1}{4},1;\frac{3}{2};-\frac{(-2(u+1)g(u)u+u-1)^2}{u\left((u^2-1)g(u)+2\right)^2}\right)}{(u^2-1)g(u)+2}+u-1\right)}{2\sqrt{1-u}\sqrt[4]{-\frac{(-2(u+1)ug(u)+u-1)^2}{u\left((u^2-1)g(u)+2\right)^2}-1}} = C_1 \tag{A.1.7}$$

where C_1 is the integration constant. Reinstating the coordinates $x(u)$ we have

$$g(u) = f'(u) = \frac{\partial(\log[x(u)])}{\partial u} = \frac{x'(u)}{x(u)} \tag{A.1.8}$$

Substituting this back in (A.1.7) and imposing the boundary condition at the turning point $x'(u_0 = e^{-2\sigma_0}) = \infty$ we fix C_1 in terms of $u_0 = e^{-2\sigma_0}$:

$$C_1(u_0) = -\frac{\sqrt{\frac{u_0-1}{u_0}}\left(2u_0\,{}_2F_1\left(\frac{1}{4},1;\frac{3}{2};-\frac{4u_0}{(u_0-1)^2}\right)+(u_0-1)^2\right)}{2(1-u_0)^{3/2}\sqrt[4]{-\frac{(u_0+1)^2}{(u_0-1)^2}}} \tag{A.1.9}$$

So C_1 encodes the information about only the turning point. Now considering (A.1.7) and (A.1.8), we have the general relation

$$g(u) = \frac{x'(u)}{x(u)} = P(u, C_1(u_0)) \tag{A.1.10}$$

for a general function $P(u, C_1(u_0))$. Solving for $x(u)$ gives us

$$x(u) = C_2 e^{\int du P(u,C_1)} \tag{A.1.11}$$

where C_2 is the second integration constant that acts as an overall scaling. This can also be observed from the differential equation for $x(u)$ (A.1.3) where we see that $C_2 x(u)$ is a solution if $x(u)$ is a solution. Below we will see that the asymptotic analysis of $u \to 0$ shows that $e^{\int du P(u,C_1)} \to 1$ as $u \to 0$.

Now, imposing the boundary condition $x(0) = x_1, x_2$ along with $x'(u_0) = \infty$, we get two branches of solutions, one with $C_2 = x_1$ and the other with $C_2 = x_2$. C_2 is independent of the choice of C_1. In other words, C_2 only captures where the curve intersects the boundary and is independent of C_1 which only captures information about the turning point u_0.

Close to the boundary, we can write down the following ansatz for a particular $g(u)$:

$$g(u) = \sum_{n=0}^{\infty} a_n u^n. \tag{A.1.12}$$

Plugging this ansatz in (A.1.6) and solving perturbatively order by order for a_n we get

$$g(u) = \sum_{n=0}^{\infty} 2u^{2n} = \frac{2}{1-u^2}. \tag{A.1.13}$$

This is a particular solution for $g(u)$. Reinstating the coordinates $x(u) = e^{\int dug(u)}$ we get a one-parameter family of solutions for $x(u)$

$$x(u) = C_3 \left(\frac{1+u}{1-u}\right). \tag{A.1.14}$$

From our previous analysis of the full solution for $x(u)$, we see that this particular solution corresponds to a choice of the integration constant $C_1(u_0)$. C_3 in this particular solution is the scaling constant.

A.1.1 Asymptotic analysis at the boundary

Consider expanding the particular solution (A.1.13) near the boundary.
Since $0 < u \leq u_0 < 1$ a natural expansion parameter for a perturbative series is any function $f(u)$ such that $0 < f(u) < 1$. We choose the expansion parameter $f(u) = q = \sqrt{u}$ and consider an ansatz for $g(u)$ of the form

$$g(u) = \frac{2}{1-u^2} + q \sum_{n=0}^{\text{order}} h_n q^n. \tag{A.1.15}$$

We can plug this ansatz into the differential equation for $g(u)$ (A.1.6) and solve for the h_n's order by order perturbatively. We have listed the h_n's up to h_6 below:

$$h_0 = k, \tag{A.1.16}$$
$$h_1 = 0, \tag{A.1.17}$$
$$h_2 = 5k, \tag{A.1.18}$$
$$h_3 = (10k^2)/3, \tag{A.1.19}$$
$$h_4 = k(28 + k^2)/2, \tag{A.1.20}$$
$$h_5 = (80k^2)/3, \tag{A.1.21}$$
$$h_6 = 30k + (305k^3)/18 \tag{A.1.22}$$

where k is the integration constant. Reinstating the coordinates $x(u) = e^{\int dug(u)}$ we get,

$$x(u; k, C_2) = C_2 \frac{1+u}{1-u} e^{\frac{2}{3}ku^{3/2}\left(1+3\sum_{n=2}^{\text{order}} \frac{h_n}{k} u^{n/2}\right)} \tag{A.1.23}$$

Since C_2 is just the scaling constant and $x(u) \to C_2$ as $u \to 0$, therefore $C_2 = x_1, x_2$. These two choices along with corresponding choices for the constant $k = k_1, k_2$ gives two branches of solutions, $x_a(u; x_1, k_1)$ and $x_b(u; x_2, k_2)$, on which the matching boundary conditions at the turning point have to be imposed to fix $k_1(x_1, x_2)$ and $k_2(x_1, x_2)$.

A.1.2 Divergent terms in the regulated entropy

Since the divergent contributions to the entropy functional come from close to the boundary, we can extract this divergence by simply considering the asymptotic series solution of $x(u)$ around the boundary, up to suitable orders that capture the full divergent behavior.

In $x(u)$ coordinates the entropy functional (2.5.59) takes the form

$$S_{\text{reg}} = \frac{1}{4G_4} \int_0^{2\pi} dy \left(\int_\epsilon^{u_0} du \mathcal{L}(u, x_a(u; k_1, x_1)) + \int_{u_0}^{\epsilon} du \mathcal{L}(u; x_b(u; k_2, x_2)) \right), \tag{A.1.24}$$

$$\mathcal{L}(u) = \frac{-1}{4u} \sqrt{\frac{\ell_4^4 (u+1)^2 \phi^2 \left(u(u+1)^2 x'(u)^2 + x(u)^2\right)}{u x(u)^4}} \tag{A.1.25}$$

where $x_a(u; k_1, x_1)$ and $x_b(u; k_2, x_2')$ are the two branches intersecting the boundary at x_1, x_2 respectively.

Substituting the asymptotic series solution of $x(u; k, C_2)$ around the boundary (A.1.23) into $\mathcal{L}(u)$, and expanding around $u = 0$ gives

$$\mathcal{L}(u; k, C_2) = \frac{\phi \ell_4^2}{C_2} \left(\frac{-1}{4u^{3/2}} - \frac{1}{4u^{1/2}} - \frac{k}{3} - \frac{k^2}{8} u^{1/2} - \frac{4y}{3} u - \frac{125k^2}{72} u^{3/2} \right) + \mathcal{O}(u^2). \tag{A.1.26}$$

Only the first term $\frac{\phi \ell_4^2}{C_2} \left(\frac{-1}{4u^{3/2}} \right)$ in $\mathcal{L}(u; k, C_2)$ contributes to the divergence in the entanglement entropy. Since we are considering the series solution of $x(u)$ around the boundary from where the divergent contributions reside, more terms in the asymptotic series for $x(u)$ will not give additional contributions to the divergence.

Plugging $\mathcal{L}(u)$ back into the entropy functional (A.1.24), we get the divergent contribution to the entanglement entropy in full generality given by

$$S_{\text{div}} = \frac{\pi \phi \ell_4^2}{4G_4 \sqrt{\epsilon}} \left(\frac{1}{x_2} - \frac{1}{x_1} \right) \tag{A.1.27}$$

A.2 Coordinate transformations

Global coordinates: Consider the global metric

$$ds^2 = -\left(1 + \frac{r^2}{\ell_3^2}\right) d\tau^2 + \left(1 + \frac{r^2}{\ell_3^2}\right)^{-1} dr^2 + r^2 d\theta^2 \tag{A.2.28}$$

where $-\pi \le \theta < \pi$. By putting $r = \ell_3 \sinh(\rho)$ we obtain the metric

$$ds^2 = -\cosh^2(\rho) d\tau^2 + \ell_3^2 d\rho^2 + \ell_3^2 \sinh^2(\rho) d\theta^2 \tag{A.2.29}$$

Now, letting $\rho = \sinh^{-1}(\tan(\tilde{\rho}/\ell_3))$ and $\theta = \tilde{\theta}/\ell_3$ we get

$$ds^2 = \frac{1}{\cos^2\left(\frac{\tilde{\rho}}{\ell_3}\right)}\left(-d\tau^2 + d\tilde{\rho}^2 + \sin^2\left(\frac{\tilde{\rho}}{\ell_3}\right)d\tilde{\theta}^2\right). \tag{A.2.30}$$

Poincaré coordinates: The metric in Poincaré coordinates reads

$$ds^2 = \frac{\ell_3^2}{x^2}(-dt^2 + dx^2 + dy^2) \tag{A.2.31}$$

and can be achieved via the transformations

$$\sqrt{\ell_3^2 + r^2}\cos(\ell_3\tau) = \frac{\ell_3\alpha^2 + \ell_3(-t^2 + x^2 + y^2)}{2\alpha x}, \tag{A.2.32}$$

$$\sqrt{\ell_3^2 + r^2}\sin(\ell_3\tau) = \frac{\ell_3 t}{x_p}, \tag{A.2.33}$$

$$r\sin\theta = \frac{\ell_3 y}{x}, \tag{A.2.34}$$

$$-r\cos\theta = \frac{-\ell_3\alpha^2 + \ell_3(-t^2 + x^2 + y^2)}{2\alpha x} \tag{A.2.35}$$

where α is an arbitrary real number corresponding to a particular isometry of AdS.

Appendix B

Appendix to Chapter 4

B.1 Notations

Lorentzian Metric signature: $(-,+,+,+)$
Entanglement entropy of a QFT: S_{QFT}
Entanglement entropy of a CFT: S_{CFT}

Base vs Replica Manifold		
	Base/Original Manifold	Replica q-Manifold
Manifold	M_1	M_q
Metric	$g_{\mu\nu}$ or $g_{\mu\nu}^{[0]}$	$g_{\mu\nu}^{(q)}$ (around endpoint)
Stress Tensor expectation	$\langle T^{\mu\nu}\rangle$ or $\langle T^{\mu\nu}\rangle^{[0]}$	$\langle T^{\mu\nu}\rangle_q$
State-dependent part of stress tensor expectation	$t^{\mu\nu}$ or $t^{\mu\nu[0]}$	$t^{\mu\nu(q)}$
Geometric part of stress tensor expectation	$\chi_{\mu\nu}$ or $\chi_{\mu\nu}^{[0]}$	${\chi_{\mu\nu}}^{(q)}$
Anomaly	$\mathcal{A}[g_{\mu\nu}]$	$\mathcal{A}[g_{\mu\nu}^{(q)}]$
Christoffel	$\Gamma^{\mu}_{\nu\rho}$ or ${\Gamma^{\mu}_{\nu\rho}}^{[0]}$	${\Gamma^{\mu}_{\nu\rho}}^{(q)}$
Covariant derivative	∇_μ	$\nabla_\mu^{(q)}$
$O(q-1)$ correction to the stress tensor expectation	-	$\langle T^{\mu\nu}\rangle^{[1]}$
$O(q-1)$ correction to the state-dependent part of stress tensor expectation	-	$t^{\mu\nu[1]}$
$O(q-1)$ correction to the Anomaly	-	$\mathcal{A}^{[1]}[g_{\mu\nu}^{(q)}]$

$$\langle T^{\mu\nu}\rangle_q = \langle T^{\mu\nu}\rangle + (q-1)\,\langle T^{\mu\nu}\rangle^{[1]} + \mathcal{O}((q-1)^2), \tag{B.1.1}$$

$$\langle T^{\mu\nu}\rangle_q = t^{\mu\nu(q)} + \chi^{\mu\nu(q)}, \tag{B.1.2}$$

$$t^{\mu\nu(q)} = t^{\mu\nu[0]} + (q-1)t^{\mu\nu[1]} + \mathcal{O}((q-1)^2)). \tag{B.1.3}$$

B.2 State and (reduced) density matrix in a QFT

In this section, we will review the literature on how to construct states and density matrices in the path integral formalism [201].

Path integrals and states: Consider a generic set of fields denoted by $\phi_i(x)$. A Lorentzian Path integral defines a transition amplitude under unitary time evolution from a state $|\phi_1(x)\rangle$ to a state $\langle\phi_2(x)|$.

$$\langle\phi_2(\vec{x})|e^{-i(t_2-t_1)H}|\phi_1(\vec{x})\rangle = \int_{\phi(\vec{x},t_1)=\phi_1(\vec{x})}^{\phi(\vec{x},t_2)=\phi_2(\vec{x})} D\phi e^{iS[\phi]} = \langle\phi_2(\vec{x})|\psi\rangle. \tag{B.2.4}$$

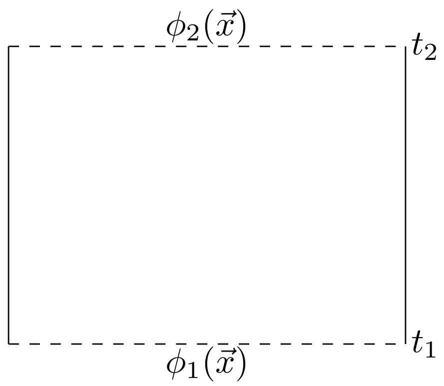


FIGURE B.1: Lorentzian Path integral

This can be interpreted as the overlap/ inner product of state $|\psi\rangle = e^{-i(t_2-t_1)H}|\phi_1(\vec{x})\rangle$ with $|\phi_2(\vec{x})\rangle$. The state so defined satisfies the Schrodinger equation by construction.

We can Wick rotate to the Euclidean time ($t \to t' = -i\tau$), to get the Euclidean path integral

$$\langle\phi_2(\vec{x})|e^{-(\tau_2-\tau_1)H}|\phi_1(\vec{x})\rangle = \int_{\phi(\vec{x},\tau_1)=\phi_1(\vec{x})}^{\phi(\vec{x},\tau_2)=\phi_2(\vec{x})} D\phi e^{-S_E[\phi]} = \langle\phi_2(\vec{x})|\psi\rangle. \tag{B.2.5}$$

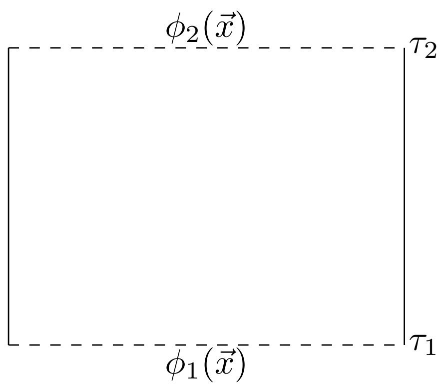


FIGURE B.2: Euclidean Path integral

This gives a transition amplitude as well, but the imaginary time evolution is not unitary i.e., $e^{-(\tau_2-\tau_1)H}$ is not a unitary operator. This can be viewed as an overlap of states. The transition amplitude can again be interpreted as the overlap with $|\phi_2(\vec{x})\rangle$ of the state

$$|\psi\rangle = e^{-(\tau_2-\tau_1)H}|\phi_1(\vec{x})\rangle. \tag{B.2.6}$$

If we now define a *cut* as a spatial slice (codimension-1 surface) of the Euclidean manifold, then one can formally think of a Euclidean path integral with one set of boundary conditions and one open cut (i.e., cut with unspecified boundary condition) as a quantum state i.e.,

$$|\psi\rangle = e^{-(\tau_2-\tau_1)H}|\phi_1(\vec{x})\rangle = \int_{\phi(\vec{x},\tau_1)=\phi_1(\vec{x})} D\phi e^{-S_E[\phi]}. \tag{B.2.7}$$

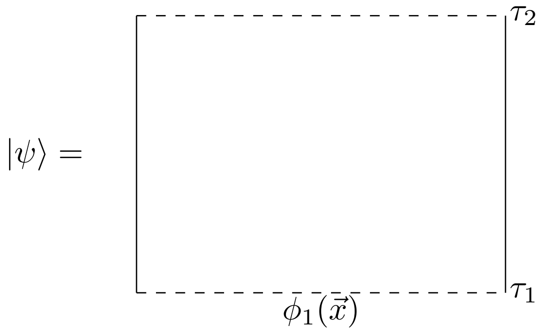


FIGURE B.3: Path integral with one open cut - state

The data on the top cut ($\tau = \tau_2$) is left unspecified. The functional $|\psi\rangle$ along with the top boundary condition given by $\langle\phi_2(\vec{x})|$ gives the complex numbers $\langle\phi_2(\vec{x})|\psi\rangle$. States are defined on a spatial slice and do not care about Lorentzian vs Euclidean path integral. The state defined above by a Euclidean path integral is a state in the Hilbert space of the Lorentzian theory.

Density matrix using Path integrals: A density matrix is an operator which takes a bra and a ket and produces a complex number. Thus any path integral with two identically shaped open cuts defines a density matrix (unnormalised).

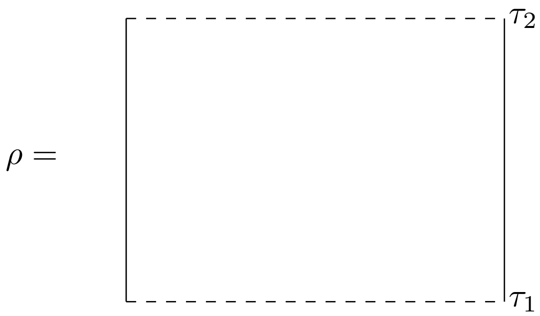


FIGURE B.4: An example of a path integral with two identical open cuts - density matrix

Consider the density matrix corresponding to the state defined by (B.2.7) i.e.,

$$\rho = |\psi\rangle\langle\psi|. \tag{B.2.8}$$

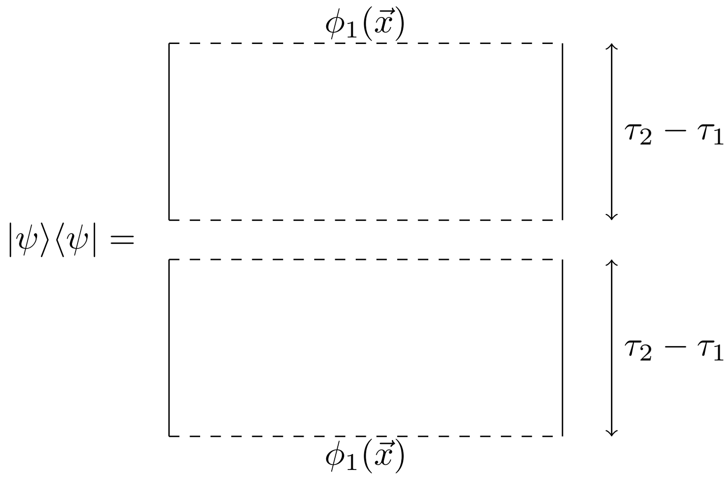


FIGURE B.5: Density matrix corresponding to a particular state

If $\phi_2(x)$ and $\phi_3(x)$ are the respective boundary conditions at the two open cuts, we have,

$$\langle\phi_3(\vec{x})|\rho|\phi_2(\vec{x})\rangle = \langle\phi_3(\vec{x})|\psi\rangle\langle\psi|\phi_2(\vec{x})\rangle. \tag{B.2.9}$$

This can be computed using the path integral expressions given by (B.2.5).

The trace of the density matrix

$$Z = Tr\rho = \sum_{\phi(\vec{x})}\langle\phi(\vec{x})|\rho|\phi(\vec{x})\rangle = \sum_{\phi(\vec{x})}\langle\psi|\phi(\vec{x})\rangle\langle\phi(\vec{x})|\psi\rangle = \langle\psi|\psi\rangle, \tag{B.2.10}$$

is the path integral with the two open cuts having the same boundary condition $\phi(\vec{x})$ and summed over all possible field configurations $\phi(\vec{x})$ i.e., gluing the two open cuts.

Example 1: The vacuum/ ground state: Let us start with a state $|X\rangle$. Expanding in the energy eigenstate $|n\rangle$, we have,

$$|X\rangle = \sum_n x_n|n\rangle; \quad H|n\rangle = E_n|n\rangle. \tag{B.2.11}$$

We can define the (unnormalised) ground state by doing a path integral that extends to infinity in one direction i.e., by evolving over a long Euclidean time

$$e^{-\tau H}|X\rangle \approx e^{-\tau E_0}x_0|0\rangle; \quad \tau \to \infty. \tag{B.2.12}$$

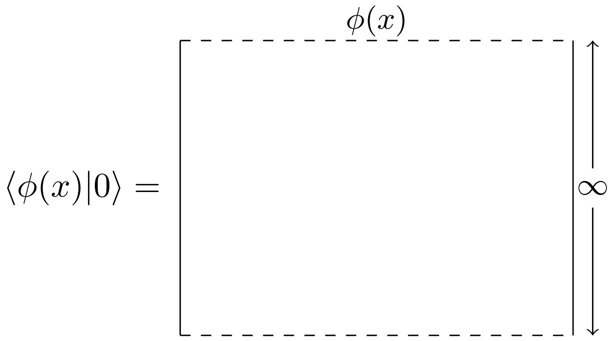


FIGURE B.6: Vacuum state

The trace of the vacuum density matrix

$$
\begin{aligned}
Z = Tr\rho = \sum_{\phi(\vec{x})} \langle \phi(\vec{x})|\rho_{vacuum}|\phi(\vec{x})\rangle &= \sum_{\phi(\vec{x})} \langle\, 0|\phi(\vec{x})\rangle\langle\phi(\vec{x})|0\rangle = \langle 0|0\rangle \\
= \int D\phi e^{-S_E[\phi]}. & \qquad \text{(B.2.13)}
\end{aligned}
$$

FIGURE B.7: Trace of the vacuum density matrix

The path integral on the lower half plane produces $|0\rangle$, the path integral on the upper half plane produces $\langle 0|$, and summing over all possible boundary conditions $\phi(\vec{x})$ in the middle at $\tau = 0$, glues the half-planes together giving the trace.

Example 2: The thermal density matrix: Consider

$$\rho_{thermal} = e^{-\beta H}. \qquad \text{(B.2.14)}$$

We have,

$$
\begin{aligned}
\langle \phi_2(\vec{x})|\rho_{thermal}|\phi_1(\vec{x})\rangle &= \langle \phi_2(\vec{x})|e^{-\beta H}|\phi_1(\vec{x})\rangle \\
&= \int_{\phi(x,\tau_1)=\phi_1(\vec{x})}^{\phi(x,\tau_2)=\phi_2(\vec{x})} D[\phi] e^{-S_E}. \qquad \text{(B.2.15)}
\end{aligned}
$$

The trace

$$\begin{aligned} Z(\beta) = Tre^{-\beta H} &= \sum_{\phi(\vec{x})} \langle \phi(\vec{x})|e^{-\beta H}|\phi(\vec{x})\rangle \\ &= \sum_{\phi(\vec{x})} \int_{\phi(x,\tau_1)=\phi(\vec{x})}^{\phi(x,\tau_2)=\phi(\vec{x})} D[\phi]e^{-S_E} = \int_{S_1^\tau} D[\phi]e^{-S_E}, \end{aligned} \tag{B.2.16}$$

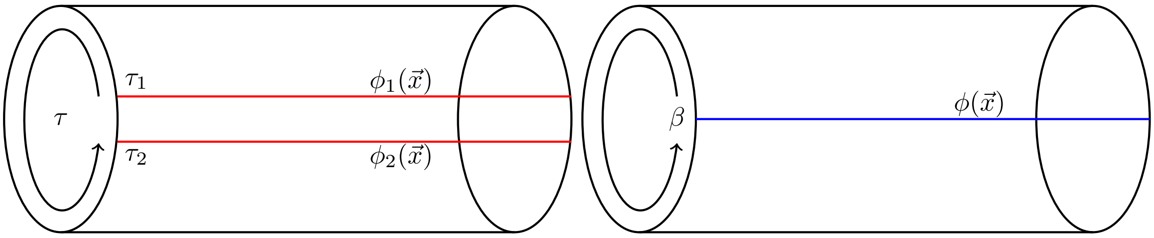


FIGURE B.8: Thermal density matrix and its trace

is an Euclidean path integral over a cylinder of time length β[38]. The boundary conditions above and the sum over all possible field configuration $\phi(\vec{x})$ in the path integral, has the effect of sewing together edges along τ_1 and τ_2 forming a cylinder of circumference $\tau_2 - \tau_1 = \beta$. [1]

Reduced density matrix: The reduced density matrix $\rho_A = Tr_{\bar{A}}\rho$, corresponding to a spatial interval A on a fixed τ slice is the path integral with the two open cuts having the same boundary condition $\phi(\vec{x})$ and summed over all possible field configurations $\phi(\vec{x})$ only along the complement region $\bar{A}$ i.e., gluing the two open cuts in ρ only along the complement region $\bar{A}$. $\langle\phi_2(\vec{x})|\rho_A|\phi_1(\vec{x})\rangle$ has the following path integral field boundary conditions along the open cuts

$$\begin{aligned} &\phi(\text{cut } 1) = \phi_1(\vec{x}) \ \forall \text{ x }, \\ &\phi(\text{cut } 2) = \phi_2(\vec{x}) \text{ if } x \in A \ , \\ &\phi(\text{cut } 2) = \phi_1(\vec{x}) \text{ if } x \notin A \ , \end{aligned} \tag{B.2.17}$$

and summed over all possible $\phi_1(\vec{x})$.

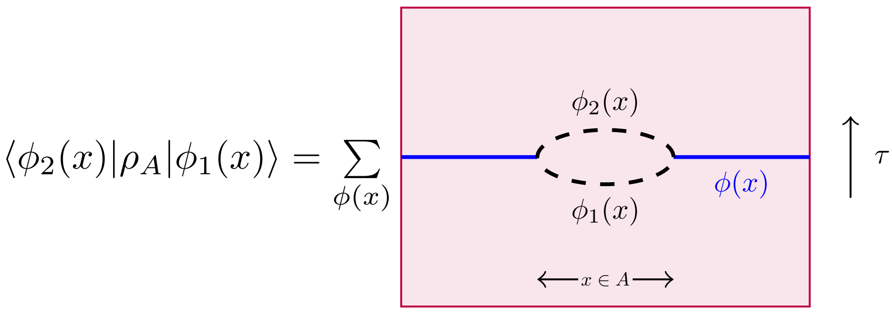


FIGURE B.9: Reduced density matrix corresponding to a region A

[1] β is a placeholder for the coordinate length $(\tau_2 - \tau_1)$. The physical length depends on the metric.

Appendix C

Appendix to Chapter 5

C.1 Conventions and useful identities

Throughout chapter (6), we have worked in Minkowski spacetime of dimension $d+1$. Hence, the boundary Carrollian CFT theory lives on the d dimensional manifold $R \times R^{d-1}$. The position space coordinates of the Carrollian CFT are denoted by $(u, \mathbf{x})$. The corresponding momentum space coordinates are denoted by $(\omega, \mathbf{p})$. The magnitude of the spatial momenta $\mathbf{p}_a$ is denoted by k_a. The dimension of the Anti de Sitter spacetime is also taken to be $d+1$ with its dual Lorentzian CFT being d dimensional.

We have considered Carrollian CFTs in both position as well as momentum spaces. Our convention for the Fourier transform is

$$f(\omega) = \int_{-\infty}^{\infty} dx e^{i\omega x} f(x) \qquad \Longrightarrow \qquad f(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega x} f(\omega) \tag{C.1.1}$$

and its generalization to higher dimensions. In the above convention, we have following delta function identities

$$\delta^D(\mathbf{k}) = \int \frac{d^D\mathbf{x}}{(2\pi)^D} e^{i\mathbf{k}\cdot\mathbf{x}} \qquad \text{and} \qquad \delta^D(\mathbf{x}) = \int \frac{d^D\mathbf{x}}{(2\pi)^D} e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{C.1.2}$$

For various derivations, we need the Fourier transform of the power law functions. For D dimensional spacelike vectors, the basic identity is given by

$$\int d^D x \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{|\mathbf{x}|^{\Delta}} \quad = \quad \frac{\pi^{D/2} 2^{D-\Delta}}{\Gamma(\frac{\Delta}{2})} \Gamma\Big(\frac{D-\Delta}{2}\Big) |\mathbf{p}|^{\Delta - D} \tag{C.1.3}$$

This can be inverted to obtain

$$\frac{1}{|\mathbf{x}|^{\Delta}} = \frac{(\pi)^{\frac{D}{2}}\ 2^{D-\Delta}}{\Gamma\Big(\frac{\Delta}{2}\Big)} \Gamma\Big(\frac{D-\Delta}{2}\Big) \int \frac{d^D\mathbf{k}}{(2\pi)^D}\ e^{-i\mathbf{k}\cdot\mathbf{x}} |\mathbf{k}|^{\Delta - D} \tag{C.1.4}$$

For 3-point function in momentum space (Carrollian) CFTs, we encounter the so called triple-K integrals $I_{\alpha(\beta_1\beta_2\beta_3)}$. It is defined by

$$I_{\alpha(\beta_1\beta_2\beta_3)}(k_1,k_2,k_3) \quad = \quad \int_0^\infty dy\, y^\alpha \prod_{a=1}^{3}(k_a)^{\beta_a}K_{\beta_a}(k_a y) \tag{C.1.5}$$

and k_i denotes the magnitude of the momenta k_i^μ in some dimension D and K_β denotes the modified bessel function of the second kind. A useful integral representation of the triple K integral is obtained by Fourier transforming the position space 3-point function in D dimensions and is given by [306]

$$\begin{aligned}&\int \frac{d^D k}{(2\pi)^D}\frac{1}{|\mathbf{p}_2+\mathbf{k}|^{2\tau_1}|\mathbf{p}_1-\mathbf{k}|^{2\tau_2}|\mathbf{k}|^{2\tau_3}}\\ &= C(\tau_a) I_{\frac{D-2}{2}\left(\frac{D}{2}-\tau_2-\tau_3,\frac{D}{2}-\tau_1-\tau_3,\frac{D}{2}-\tau_1-\tau_2\right)}(k_1,k_2,k_3)\end{aligned} \tag{C.1.6}$$

where the constant $C(\tau_a)$ is given by

$$C(\tau_a) \quad = \quad \frac{\pi^{-\frac{D}{2}}2^{4-\frac{3D}{2}}\Gamma\left(\frac{D}{2}-\tau_1\right)\Gamma\left(\frac{D}{2}-\tau_2\right)\Gamma\left(\frac{D}{2}-\tau_3\right)}{\Gamma(\tau_1)\Gamma(\tau_2)\Gamma(\tau_3)\Gamma(d-\tau_1)\Gamma(d-\tau_2)\Gamma(d-\tau_3)\Gamma(d-\tau_1-\tau_2-\tau_3)} \tag{C.1.7}$$

For more details about these integrals, see, e.g., [306].

C.2 Lorentzian CFT correlators

C.2.1 2-point function

The two-point function of scalar operators with the same conformal dimension Δ in Lorentzian space can be obtained from the corresponding Euclidean expression by performing a Wick rotation $t \to it$. This prescription is, however, ambiguous and leads to two distinct Lorentzian Wightman functions ($\epsilon > 0$)[1]:

$$\begin{aligned}\langle \mathcal{O}(t_1,\mathbf{x}_1)\,\mathcal{O}(t_2,\mathbf{x}_2)\rangle \quad &= \quad \frac{C_2(\Delta)}{[-(c\,t_{12}-i\epsilon)^2+|\mathbf{x}_{12}|^2]^\Delta}\,,\\ \langle \mathcal{O}(t_2,\mathbf{x}_2)\,\mathcal{O}(t_1,\mathbf{x}_1)\rangle \quad &= \quad \frac{C_2(\Delta)}{[-(c\,t_{12}+i\epsilon)^2+|\mathbf{x}_{12}|^2]^\Delta}\,.\end{aligned} \tag{C.2.8}$$

The Lorentzian time-ordered two-point function then takes the form (see, e.g., [253])

$$\langle T\{\mathcal{O}(t_1,\mathbf{x}_1)\,\mathcal{O}(t_2,\mathbf{x}_2)\}\rangle = \frac{C_2(\Delta)}{\left[-c^2t_{12}^2+|\mathbf{x}_{12}|^2+i\epsilon\right]^\Delta}\,. \tag{C.2.9}$$

These correlation functions differ only by the $i\epsilon$ prescription.

[1] We follow the prescription given in Ref. [309]

C.2.2 3-point function

Upto an overall normalization (which may depend on d and Δ_a), the CFT Wightman 3-point function in the position space is given by the following $i\epsilon$ prescription (see, e.g., [361])

$$\begin{aligned}&\langle \mathcal{O}(t_1,\mathbf{x}_1)\,\mathcal{O}(t_2,\mathbf{x}_2)\,\mathcal{O}(t_3,\mathbf{x}_3)\rangle\\ &= \frac{1}{(x_{12}^2+i\epsilon\,t_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(x_{23}^2+i\epsilon t_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}(x_{13}^2+i\epsilon\,t_{13})^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}}\end{aligned}$$

with $x^2=-t^2+|\mathbf{x}|^2$.

On the other hand, the time ordered Lorentzian CFT three-point function is given by a different $i\epsilon$ prescription (see, e.g., [253, 361])

$$\Big\langle T\left(\mathcal{O}(x_1)\,\mathcal{O}(x_2)\,\mathcal{O}(x_3)\right)\Big\rangle = \frac{1}{(x_{12}^2+i\epsilon)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(x_{23}^2+i\epsilon)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}(x_{13}^2+i\epsilon)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}} \tag{C.2.10}$$

We want to find the momentum representation of the above time ordered correlator. For our purposes, it will be sufficient to obtain an integral representation of the correlator in the momentum space. This will turn out to be enough for the discussion of Carrollian limit in section 5.5.2.1. To obtain this representation, we first consider the Fourier transform of (C.2.10).

$$\begin{aligned}&\mathcal{A}(p_1,p_2,p_3)\\ &= \int d^dx_1\,d^dx_2,d^dx_3\prod_{i=1}^{3}e^{-ip_i\cdot x_i},\big\langle T\left(\mathcal{O}(x_1)\,\mathcal{O}(x_2)\,\mathcal{O}(x_3)\right)\big\rangle\\ &= (2\pi)^d\delta^{(d)}(p_1+p_2+p_3)\int\frac{d^dx_{13}\,d^dx_{23}\;\;e^{-ip_1\cdot x_{13}-ip_2\cdot x_{23}}}{\left[(x_{13}-x_{23})^2+i\epsilon\right]^{\beta_3}(x_{23}^2+i\epsilon)^{\beta_1}(x_{13}^2+i\epsilon)^{\beta_2}}.\end{aligned} \tag{C.2.11}$$

with $\beta_c=\Delta_{ab}=\frac{\Delta_a+\Delta_b-\Delta_c}{2}$, ($a$, b, $c\in\{1,2,3\}$ with a,b,c distinct).

We now consider the following representation of unity

$$1=\int d^dy\,\delta^{(d)}\left[y-(x_{13}-x_{23})\right]=\int d^dy\int\frac{d^dk}{(2\pi)^d}\,e^{-ik\cdot[y-(x_{13}-x_{23})]}\,. \tag{C.2.12}$$

Inserting this inside the integral in (C.2.11) and changing variables $x_{13}\to x_1$, $x_{23}\to x_2$, $y\to x_3$, we obtain

$$\mathcal{A}(p_1,p_2,p_3)=(2\pi)^d\delta^d(p_1+p_2+p_3)\int\frac{d^dk}{(2\pi)^d}\int\prod_{a=1}^{3}dx_a\,\frac{e^{-i(p_1-k)\cdot x_1-i(p_2+k)\cdot x_2-ik\cdot x_3}}{(x_3^2+i\epsilon)^{\beta_3}(x_2^2+i\epsilon)^{\beta_1}(x_1^2+i\epsilon)^{\beta_2}} \tag{C.2.13}$$

Finally, by using the Fourier-transform identity (see, e.g., equation (2.3) of Ref.[308])

$$\int d^dx\,\frac{e^{-ip\cdot x}}{[x^2+i\epsilon]^{\Delta}}=\mathcal{N}_\Delta(p^2-i\epsilon)^{\Delta-\frac{d}{2}}\qquad,\qquad\mathcal{N}_\Delta=-i\frac{\pi^{\frac{d}{2}}\,\Gamma[\frac{d}{2}-\Delta]}{2^{2\Delta-d}\Gamma[\Delta]} \tag{C.2.14}$$

we obtain the following momentum-space representation

$$\mathcal{A}_3(p_1,p_2,p_3) \quad = \quad \delta^d(p_1+p_2+p_3)\,\mathcal{N}(\Delta_1\,,\Delta_2\,,\Delta_3)\left\langle\!\!\left\langle \mathcal{O}_{\Delta_1}(p_1)\,\mathcal{O}_{\Delta_2}(p_2)\,\mathcal{O}_{\Delta_3}(p_3)\right\rangle\!\!\right\rangle \tag{C.2.15}$$

where we have defined

$$\left\langle\!\!\left\langle \mathcal{O}_{\Delta_1}(p_1)\,\mathcal{O}_{\Delta_2}(p_2)\,\mathcal{O}_{\Delta_3}(p_3)\right\rangle\!\!\right\rangle = \int \frac{d^dk}{(2\pi)^d}\,\frac{1}{(k^2-i\epsilon)^{\frac{d}{2}-\beta_3}\,[(p_2+k)^2-i\epsilon]^{\frac{d}{2}-\beta_1}\,[(p_1-k)^2-i\epsilon]^{\frac{d}{2}-\beta_2}} \tag{C.2.16}$$

and

$$\mathcal{N}(\Delta_1\,,\Delta_2\,,\Delta_3) = \mathcal{N}\prod_{i=1}^{3}\left[-i\frac{\pi^{\frac{d}{2}}\,\Gamma[\frac{d}{2}-\Delta_i]}{2^{2\Delta_i-d}\Gamma[\Delta_i]}\right] \tag{C.2.17}$$

C.3 Delta distribution identities

In solving the Ward identities, we need to deal with the derivatives of momentum conserving delta functions such as $\delta^d(\sum_{a=1}^n p_a^\mu)$. In this appendix, we describe the method and summarize various identities involving the derivatives of these delta functions. We shall be working in the Carrollian dimension d with indices μ,ν running from 0 to $d-1$ whereas i,j running from 1 to $d-1$. We start by defining the "total" momenta

$$P^\mu = \sum_{a=1}^{n} p_a^\mu \tag{C.3.18}$$

The derivative of the above delta function with respect to a momentum component p_a^μ is given by

$$\frac{\partial}{\partial p_a^\mu}\delta^d(\sum_{b=1}^n p_b^\mu) = \left(\frac{\partial\sum_b p_b^\mu}{\partial p_a^\mu}\right)\frac{\partial}{\partial P^\mu}\delta^d(P^\mu) = \frac{\partial}{\partial P^\mu}\delta^d(P^\mu) \tag{C.3.19}$$

The spacetime index μ is not summed over in the above expression. The above identity implies

$$\frac{\partial}{\partial p_a^\mu}\delta^d(\sum_{b=1}^n p_b^\mu) = \frac{\partial}{\partial p_c^\mu}\delta^d(\sum_{b=1}^n p_b^\mu) \quad ; \qquad \forall a,c \tag{C.3.20}$$

This shows that we can take the derivative of the momentum conserving delta function with respect to the momentum of any field p_a and it will give the same result. This is very useful. E.g., suppose, we have an expression $\sum_{a=1}^n p_a^\mu \frac{\partial}{\partial p_a^\mu}$ (where μ index is summed over) which we want to simplify. We consider an arbitrary test function $f(P)$ and the integral

$$\begin{aligned}
\int d^dP\, f(P) \sum_{a=1}^{n} p_a^\mu \frac{\partial}{\partial p_a^\mu}\, \delta^d(P) \;&=\; \int d^dP\, f(P) \sum_{a=1}^{n} p_a^\mu \frac{\partial}{\partial p_n^\mu}\, \delta^d(P) \\
&=\; \int d^dP\, f(P)\, P^\mu\, \frac{\partial}{\partial p_n^\mu}\, \delta^d(P) \\
&=\; \int d^dP\, f(P)\, P^\mu\, \frac{\partial}{\partial P^\mu} \delta^d(P) \\
&=\; -\int d^dP\, P^\mu\, \frac{\partial f(P)}{\partial P^\mu}\, \delta^d(P) - \int d^dP\, f(P) \frac{\partial}{\partial P^\nu} P^\nu\, \delta^d(P) \\
&=\; -d \int d^dP\, f(P)\, \delta^d(P) \qquad \text{(C.3.21)}
\end{aligned}$$

The first term in the second last line vanishes since $\delta^d(P)$ imposes $P^\mu = 0$. Equation (C.3.21) gives the delta function identity

$$\sum_{a=1}^{n} p_a^\mu \frac{\partial}{\partial p_a^\mu}\, \delta^d(P) \;=\; -d\, \delta^d(P) \qquad \text{(C.3.22)}$$

Two useful special cases of this identity are

$$\sum_{a=1}^{n} \omega_a \frac{\partial}{\partial \omega_a} \delta\Big(\sum_{a=1}^{n} \omega_a\Big) = -\delta\Big(\sum_{a=1}^{n} \omega_a\Big) \quad ; \qquad \sum_{a=1}^{n} p_a^i \frac{\partial}{\partial p_a^i} \delta^{d-1}\Big(\sum_{a=1}^{n} \mathbf{p}_a\Big) = -(d-1)\delta^{d-1}\Big(\sum_{a=1}^{n} \mathbf{p}_a\Big) \qquad \text{(C.3.23)}$$

For dealing with special conformal Ward identity, we need delta function identities with more than one derivatives. To see how to deal with such identities, consider the following test integral

$$\begin{aligned}
\int d^dP\, f(P) \sum_{a=1}^{n} \omega_a \frac{\partial}{\partial p_i^a} \frac{\partial}{\partial p_a^i} \delta^d\Big(\sum_{b=1}^{n} p_b\Big) \;&=\; \int d^dP\, f(P)\, P^0 \frac{\partial^2}{\partial P_i \partial P^i} \delta^d(P) \\
&=\; \int d^dP\, \delta^d(P)\, P^0 \frac{\partial^2}{\partial P_i \partial P^i} f(P) \qquad \text{(C.3.24)}
\end{aligned}$$

In going to the 2nd equality, we have used $\frac{\partial P^0}{\partial P^i} = 0$. The last line vanishes since the delta-function imposes $P^0 = 0$. This gives the delta function identity

$$\sum_{a=1}^{n} \omega_a \frac{\partial}{\partial p_i^a} \frac{\partial}{\partial p_a^i} \delta^d\Big(\sum_{b=1}^{n} p_b\Big) \;=\; 0 \qquad \text{(C.3.25)}$$

Some other useful identities which can be proved similarly are

$$\sum_{a=1}^{n} p_i^a \frac{\partial^2}{\partial p_{ja} \partial p_a^j} \delta^{d-1}(\mathbf{p}_1 + \cdots + \mathbf{p}_n) \;=\; -2 \frac{\partial}{\partial P^i} \delta^{d-1}(\mathbf{P}) \qquad \text{(C.3.26)}$$

$$\sum_{a=1}^{n} p_{aj} \frac{\partial^2}{\partial p_j^a \partial p_a^i} \delta^{d-1}(\mathbf{p}_1 + \cdots + \mathbf{p}_n) \;=\; -d \frac{\partial}{\partial P^i} \delta^{d-1}(\mathbf{P}) \qquad \text{(C.3.27)}$$

C.4 Fourier transform of position space Carrollian correlators

In this appendix, we consider the Fourier transform of the 2 and 3 point position space Carrollian conformal correlators which were reviewed in section 5.3. While doing the Fourier transform, we shall consider generic values of Δ_a for which the Fourier transform exists. For some special values of Δ_a, such as the one discussed at the end of section 5.4, there may be divergence issues requiring appropriate regularizations.

C.4.1 2-point function

The 2-point function of scalar operators in the d dimensional Carrollian conformal theories in the position space was given in equation (5.3.38)

$$\langle\mathcal{O}(\mathbf{x}_1,u_1)\mathcal{O}(\mathbf{x}_2,u_2)\rangle=\frac{C_1}{|\mathbf{x}_{12}|^{\Delta_1+\Delta_2}}\delta_{\Delta_1,\Delta_2}+\frac{C_2}{|u_{12}|^{\Delta_1+\Delta_2-d+1}}\delta(\mathbf{x}_{12}) \tag{C.4.28}$$

The first term is the magnetic branch whereas the second term is electric branch. The Fourier transform of the magnetic branch is given by

$$\begin{aligned}
&\Big\langle\mathcal{O}_1(\omega_1,\mathbf{p}_1)\mathcal{O}_2(\omega_2,\mathbf{p}_2)\Big\rangle_M\\
&= \int du_1du_2e^{iu_1\omega_1+iu_2\omega_2}\int d^{d-1}x_1d^{d-1}x_2\,\frac{C_1}{|\mathbf{x}_{12}|^{\Delta_1+\Delta_2}}\delta_{\Delta_1,\Delta_2}\,e^{i\mathbf{p}_1\cdot\mathbf{x}_1+i\mathbf{p}_2\cdot\mathbf{x}_2}\\
&= C_1\delta_{\Delta_1,\Delta_2}(2\pi)^2\delta(\omega_1)\delta(\omega_2)(2\pi)^{d-1}\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2)\int d^{d-1}y\frac{e^{i\mathbf{p}_1\cdot\mathbf{y}}}{\mathbf{y}^{2\Delta_1}}\\
&= C_1\delta_{\Delta_1,\Delta_2}(2\pi)^2(2\pi)^{d-1}\delta(\omega_1)\delta(\omega_2)\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2)\frac{2^{d-1-2\Delta_1}\pi^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2}-\Delta_1)}{\Gamma(\Delta_1)}|\mathbf{p}_1|^{2\Delta_1-d+1}
\end{aligned} \tag{C.4.29}$$

In going to the 2nd equality, we have made a change of variable $(\mathbf{x}_1,\mathbf{x}_2)\to(\mathbf{y},\mathbf{x}_2)$ where $\mathbf{y}=\mathbf{x}_1-\mathbf{x}_2$. In going to the 3rd equality, we have used the identity (C.1.3).

Next, the Fourier transform of the electric branch is given by

$$\begin{aligned}
&\Big\langle\mathcal{O}_1(\omega_1,\mathbf{p}_1)\mathcal{O}_2(\omega_2,\mathbf{p}_2)\Big\rangle_E\\
&= \int du_1du_2e^{iu_1\omega_1+iu_2\omega_2}\frac{C_2}{|u_{12}|^{\Delta_1+\Delta_2-d+1}}\int d^{d-1}x_1d^{d-1}x_2\,\,e^{i\mathbf{p}_1\cdot\mathbf{x}_1+i\mathbf{p}_2\cdot\mathbf{x}_2}\delta^{d-1}(\mathbf{x}_{12})\\
&= \int du_1du_2e^{iu_1\omega_1+iu_2\omega_2}\frac{C_2}{|u_{12}|^{\Delta_1+\Delta_2-d+1}}\int d^{d-1}x_1\,\,e^{i(\mathbf{p}_1+\mathbf{p}_2)\cdot\mathbf{x}_1}\\
&= C_2(2\pi)^{d-1}\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2)\int du_2e^{i(\omega_1+\omega_2)u_2}\int dve^{iv\omega_1}\frac{1}{|v|^{\Delta_1+\Delta_2-d+1}}\\
&= C_2(2\pi)(2\pi)^{d-1}\delta(\omega_1+\omega_2)\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2)2^{d-\Delta_1-\Delta_2}\frac{(\pi)^{\frac{1}{2}}\Gamma(\frac{d-\Delta_1-\Delta_2}{2})}{\Gamma(\frac{\Delta_1+\Delta_2-d+1}{2})}|\omega_1|^{\Delta_1+\Delta_2-d}
\end{aligned} \tag{C.4.30}$$

In going to the last equality, we have used the integral

$$\begin{aligned}\int_{-\infty}^{\infty} dy e^{iv\omega_1}\frac{1}{|v|^{\Delta_1+\Delta_2-d+1}} &= \int_{-\infty}^{0} dv e^{iv\omega_1}\frac{1}{|v|^{\Delta_1+\Delta_2-d+1}} + \int_{0}^{\infty} dv e^{iv\omega_1}\frac{1}{|v|^{\Delta_1+\Delta_2-d+1}}\\ &= 2\int_0^{\infty} dv \frac{\cos(v\omega_1)}{|v|^{\Delta_1+\Delta_2-d+1}}\\ &= 2\Gamma(d-\Delta_1-\Delta_2)\sin\left(\frac{1}{2}\pi\left(-d+\Delta_1+\Delta_2+1\right)\right)|\omega_1|^{\Delta_1+\Delta_2-d}\end{aligned}$$

The above integral converges for $0 < d-\Delta_1-\Delta_2 < 1$. The sine factor can be replaced in terms of the Gamma functions by using the identities

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad ; \quad \Gamma(z)\Gamma\Big(z+\frac{1}{2}\Big) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z) \tag{C.4.31}$$

C.4.2 3-point function

The 3-point position space Carrollian conformal correlators were given in equation (5.3.41).

$$\begin{aligned}&\left\langle \mathcal{O}_1(u_1,\mathbf{x}_1)\mathcal{O}_2(u_2,\mathbf{x}_2)\mathcal{O}_3(u_3,\mathbf{x}_3)\right\rangle\\ &\quad= G_1(u_1,u_2,u_3)\delta^{d-1}(\mathbf{x}_{12})\delta^{d-1}(\mathbf{x}_{23}) + G_2(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) + G_3(\mathbf{x}_{12},u_{23})\delta^{d-1}(\mathbf{x}_{23})\\ &\quad\quad + G_4(\mathbf{x}_{23},u_{31})\delta^{d-1}(\mathbf{x}_{31}) + G_5(\mathbf{x}_{31},u_{12})\delta^{d-1}(\mathbf{x}_{12})\end{aligned} \tag{C.4.32}$$

where, the form of functions G_i are given in section 5.3.2. There are a total of 5 branches in the 3-point function. We consider each of them one by one denoting the corresponding Fourier transform by the subscript G_i. The Fourier transform of the 1st term in (C.4.32) is given by (denoting $\alpha = 2d-2-\Delta_t$)

$$\begin{aligned}&\Big\langle \mathcal{O}_1(\omega_1,\mathbf{p}_1)\mathcal{O}_2(\omega_2,\mathbf{p}_2)\mathcal{O}_3(\omega_3,\mathbf{p}_3)\Big\rangle_{G_1}\\ &= \int\prod_{a=1}^{3} du_a d^{d-1}\mathbf{x}_a\, e^{i(\omega_1u_1+\omega_2u_2+\omega_3u_3)}e^{i(\mathbf{p}_1\cdot\mathbf{x}_1+\mathbf{p}_2\cdot\mathbf{x}_2+\mathbf{p}_3\cdot\mathbf{x}_3)}u_{23}^{\alpha}f\left(\frac{u_{13}}{u_{23}}\right)\delta^{d-1}(\mathbf{x}_{12})\delta^{d-1}(\mathbf{x}_{23})\\ &= \int d^{d-1}\mathbf{x}_3\, e^{i(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\cdot\mathbf{x}_3}\int dU\, e^{i(\omega_1+\omega_2+\omega_3)U}\int drdt\, e^{i(\omega_1t+\omega_2r)}r^{\alpha}f\left(\frac{t}{r}\right)\\ &= (2\pi)^{d-1}\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)2\pi\delta(\omega_1+\omega_2+\omega_3)\int dr\, r^{\alpha+1}e^{-i\omega_2r}\int dy e^{i\omega_1ry}f(y)\\ &= (2\pi)^{d-1}\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)(2\pi)\delta(\omega_1+\omega_2+\omega_3)\int dr\, r^{\alpha+1}e^{i\omega_2r}\hat{f}(\omega_1r)\\ &= (2\pi)^{d-1}\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)(2\pi)\delta(\omega_1+\omega_2+\omega_3)\omega_1^{-\alpha-2}\int ds\, s^{\alpha+1}e^{i\frac{\omega_2}{\omega_1}s}\hat{f}(s)\\ &= (2\pi)^{d-1}\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)(2\pi)\delta(\omega_1+\omega_2+\omega_3)\omega_1^{\Delta_t-2d}\hat{G}_1\left(\frac{\omega_2}{\omega_1}\right)\end{aligned} \tag{C.4.33}$$

In going to the second equality, we have performed the x_1 and x_2 integrals using the delta functions and have made a change of variable $u_3 = U, u_{23} = r$ and $u_{13} = t$. In going to the 3rd equality, we have made a change of variable $t = yr$. Finally, in going to the 5th equality, we have made a change of variable $\omega_1 r = s$.

The expression (C.4.33) agrees with (5.4.99). On the other hand, if we consider example (5.3.46), its Fourier transform is given by

$$\begin{aligned}
&\Big\langle \mathcal{O}_1(\omega_1,\mathbf{p}_1)\mathcal{O}_2(\omega_2,\mathbf{p}_2)\mathcal{O}_3(\omega_3,\mathbf{p}_3)\Big\rangle_{G_1} \\
&=(2\pi)^{d-1}\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\sum_{p,q,r} B_{pqr}\int du_1 du_2 du_3 \frac{e^{i(u_1\omega_1+u_2\omega_2+u_3\omega_3)}}{|u_{12}|^p|u_{23}|^q|u_{31}|^r} \\
&=(2\pi)^{d-1}\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)(2\pi)\delta(\omega_1+\omega_2+\omega_3) \\
&\quad \sum_{p,q,r} B_{pqr}C_{pqr}\int \frac{d\tau}{2\pi}|\tau|^{p-1}|\omega_2-\tau|^{q-1}|\omega_1+\tau|^{r-1} \qquad \text{(C.4.34)}
\end{aligned}$$

The sum is over values of p,q,r satisfying the condition $p+q+r = \Delta_1+\Delta_2+\Delta_3-2d+2$. In going to the second equality, we have made a change of variable $u_{13} = x\,, u_{23} = y, u_3 = z$ and used the identity (C.1.4) for $D = 1$. The constant C_{pqr} is given by

$$C_{pqr} = \frac{(\pi)^{\frac{3}{2}}\ 2^{3-p-q-r}}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{r}{2}\right)}\Gamma\Big(\frac{1-p}{2}\Big)\Gamma\Big(\frac{1-q}{2}\Big)\Gamma\Big(\frac{1-r}{2}\Big) \qquad \text{(C.4.35)}$$

The integral in (C.4.34) can be shown to be a triple K integral and the final result is given by

$$\begin{aligned}
&\Big\langle \mathcal{O}_1(\omega_1,\mathbf{p}_1)\mathcal{O}_2(\omega_2,\mathbf{p}_2)\mathcal{O}_3(\omega_3,\mathbf{p}_3)\Big\rangle_{G_1} \\
&= (2\pi)^{d-1}\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)(2\pi)\delta(\omega_1+\omega_2+\omega_3)\sum_{p,q,r}\frac{2^{\frac{11}{2}-p-q-r}B_{pqr}}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})\Gamma(\frac{r}{2})} \\
&\quad \frac{\pi}{\Gamma\left(\frac{p+q+r-1}{2}\right)} I_{-\frac{1}{2}\{\frac{q+r-1}{2},\frac{p+r-1}{2},\frac{p+q-1}{2}\}}(\omega_1,\omega_2,\omega_3)
\end{aligned}$$

While evaluating the integrals to obtain the above expression, we have assumed that p,q and r are less than 1. Hence, for values of p,q,r not satisfying this condition, the above expression should be taken as analytically continued expression.

The Fourier transform of the 2nd term in (C.4.32) is given by

$$\begin{aligned}
&\Big\langle \mathcal{O}_1(\omega_1,\mathbf{p}_1)\mathcal{O}_2(\omega_2,\mathbf{p}_2)\mathcal{O}_3(\omega_3,\mathbf{p}_3)\Big\rangle_{G_2} \\
&= C_{123}\int\prod_{a=1}^{3} du_a d^{d-1}\mathbf{x}_a\ \frac{e^{i(\omega_1u_1+\omega_2u_2+\omega_3u_3)}e^{i(\mathbf{p}_1\cdot\mathbf{x}_1+\mathbf{p}_2\cdot\mathbf{x}_2+\mathbf{p}_3\cdot\mathbf{x}_3)}}{|\mathbf{x}_{12}|^{\Delta_1+\Delta_2-\Delta_3}|\mathbf{x}_{23}|^{\Delta_2+\Delta_3-\Delta_1}|\mathbf{x}_{31}|^{\Delta_3+\Delta_1-\Delta_2}} \\
&= C_{123}(2\pi)^3\delta(\omega_1)\delta(\omega_2)\delta(\omega_3)\int\prod_{a=1}^{3} d^{d-1}\mathbf{x}_a\ \frac{e^{i(\mathbf{p}_1\cdot\mathbf{x}_1+\mathbf{p}_2\cdot\mathbf{x}_2+\mathbf{p}_3\cdot\mathbf{x}_3)}}{|\mathbf{x}_{12}|^{\Delta_1+\Delta_2-\Delta_3}|\mathbf{x}_{23}|^{\Delta_2+\Delta_3-\Delta_1}|\mathbf{x}_{31}|^{\Delta_3+\Delta_1-\Delta_2}} \\
&= (2\pi)^3\delta(\omega_1)\delta(\omega_2)\delta(\omega_3)C_{123}\frac{(\pi)^{\frac{3(d-1)}{2}}\ 2^{3(d-1)-\Delta_t}}{\Gamma\left(\frac{\delta_1}{2}\right)\Gamma\left(\frac{\delta_2}{2}\right)\Gamma\left(\frac{\delta_3}{2}\right)}\Gamma\Big(\frac{d-1-\delta_1}{2}\Big)\Gamma\Big(\frac{d-1-\delta_2}{2}\Big)\Gamma\Big(\frac{d-1-\delta_3}{2}\Big) \\
&\quad (2\pi)^{d-1}\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\int\frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}}\ \frac{1}{|\mathbf{k}|^{d-1-\delta_3}|\mathbf{p}_2+\mathbf{k}|^{d-1-\delta_1}|\mathbf{p}_1-\mathbf{k}|^{d-1-\delta_2}} \qquad \text{(C.4.36)}
\end{aligned}$$

We have defined $\delta_1 = \Delta_2 + \Delta_3 - \Delta_1, \delta_2 = \Delta_1 + \Delta_3 - \Delta_2$ and so on. In going to the 3rd equality, we have first used the translation invariance to extract the $d-1$ dimensional momentum conserving delta function and for the factors in denominator, we have used the identity (C.1.4) for $D = d-1$. The expression in (C.4.36) (apart from the factor $\delta(\omega_1)\delta(\omega_2)\delta(\omega_3)$) is basically the momentum space 3-point function of 3 scalar operators in $d-1$ dimensional Euclidean CFT. The integral in (C.4.36) can again be shown to be proportional to a triple-K integral with the final result given by

$$\begin{aligned}
&\Big\langle \mathcal{O}_1(\omega_1,\mathbf{p}_1)\mathcal{O}_2(\omega_2,\mathbf{p}_2)\mathcal{O}_3(\omega_3,\mathbf{p}_3)\Big\rangle_{G_2} \\
&= \frac{(2\pi)^3\delta(\omega_1)\delta(\omega_2)\delta(\omega_3)(2\pi)^{d-1}\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)C_{123}\pi^{d-1}2^{4+\frac{3d-3}{2}-\Delta_t}}{\Gamma\Big(\frac{\Delta_t-d+1}{2}\Big)\Gamma\Big(\frac{\Delta_1+\Delta_2-\Delta_3}{2}\Big)\Gamma\Big(\frac{\Delta_2+\Delta_3-\Delta_1}{2}\Big)\Gamma\Big(\frac{\Delta_3+\Delta_1-\Delta_2}{2}\Big)} \\
&\quad I_{\frac{d-3}{2}\{\Delta_1-\frac{d-1}{2},\Delta_2-\frac{d-1}{2},\Delta_3-\frac{d-1}{2}\}}(p_1,p_2,p_3)
\end{aligned} \tag{C.4.37}$$

Next, for the 3rd term in (C.4.32), we have

$$\begin{aligned}
&\Big\langle \mathcal{O}_1(\omega_1,\mathbf{p}_1)\mathcal{O}_2(\omega_2,\mathbf{p}_2)\mathcal{O}_3(\omega_3,\mathbf{p}_3)\Big\rangle_{G_3} \\
&= C_3\int\prod_{a=1}^{3}du_a d^{d-1}\mathbf{x}_a\,\frac{e^{i(\omega_1u_1+\omega_2u_2+\omega_3u_3)}e^{i(\mathbf{p}_1\cdot\mathbf{x}_1+\mathbf{p}_2\cdot\mathbf{x}_2+\mathbf{p}_3\cdot\mathbf{x}_3)}}{|\mathbf{x}_{12}|^{2\Delta_1}|u_{23}|^{\Delta_2+\Delta_3-\Delta_1-d+1}}\delta^{d-1}(\mathbf{x}_{23}) \\
&= (2\pi)^2\delta(\omega_1)\delta(\omega_2+\omega_3)(2\pi)^{d-1}\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)C_3 \\
&\quad\int d^{d-1}\mathbf{z}\frac{e^{i\mathbf{p}_1\cdot\mathbf{z}}}{|\mathbf{z}|^{2\Delta_1}}\int dv\frac{e^{iv\omega_2}}{|v|^{\Delta_2+\Delta_3-\Delta_1-d+1}} \\
&= (2\pi)^2\delta(\omega_1)\delta(\omega_2+\omega_3)(2\pi)^{d-1}\delta^{d-1}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)C_3\pi^{\frac{d}{2}}2^{2d-1-\Delta_t} \\
&\quad\frac{\Gamma\Big(\frac{d-1-2\Delta_1}{2}\Big)\Gamma\Big(\frac{d+\Delta_1-\Delta_2-\Delta_3}{2}\Big)}{\Gamma(\Delta_1)\Gamma\Big(\frac{\Delta_2+\Delta_3-\Delta_1-d+1}{2}\Big)}(p_1)^{2\Delta_1-d+1}(\omega_2)^{\Delta_2+\Delta_3-\Delta_1-d}
\end{aligned} \tag{C.4.38}$$

In going to the 2nd equality, we have performed integrations over u_1 and $\mathbf{x}_3$ and then made a change of variables $u_3 = t, u_{23} = v$ and $\mathbf{x}_2 = \mathbf{y}, \mathbf{x}_{12} = \mathbf{z}$. In going to the 3rd equality, we have used identity (C.1.3) for $D = d-1$ and $D = 1$.

The Fourier transform of 4th and 5th terms in (5.3.41) is exactly analogous to the previous case and can be written down by permuting the indices in (C.4.38).

C.5 Direct evaluation of equation (5.118)

In this appendix we provide an alternative derivation of the integral in equation (5.5.118). The method involves decomposing the integration range into three distinct regions, identified by the phase of the integrand, and evaluating the contribution of each region separately. This alternative derivation gives an independent consistency check of the result derived in Section 5.5.1.2.

Following the cuts shown in figure 5.2, the integral can be decomposed into three separate contributions as

$$I(\alpha,\,|\mathbf{p}|) = \left[\int_{-\infty}^{-c|\mathbf{p}|} + \int_{-c|\mathbf{p}|}^{c|\mathbf{p}|} + \int_{c|\mathbf{p}|}^{\infty}\right] \frac{d\omega}{[(-\omega + c\,|\mathbf{p}| - i0^+)(\omega + c|\mathbf{p}| - i0^+)]^\alpha}\,, \quad \alpha = \frac{d}{2} - \Delta \tag{C.5.39}$$

The integrand between $[-c|\mathbf{p}|,\, c|\mathbf{p}|]$ is analytic with no branch cuts and can be evaluated in the standard way

$$\begin{aligned} \int_{-c|\mathbf{p}|}^{c|\mathbf{p}|} \frac{d\omega}{[(-\omega + c\,|\mathbf{p}|)(\omega + c|\mathbf{p}|)]^\alpha} &= (2c|\mathbf{p}|)^{1-2\alpha} \int_0^1 dt\, t^{-\alpha}\,(1-t)^{-\alpha} \\ &= (2c|\mathbf{p}|)^{1-2\alpha}\, B(1-\alpha,\,1-\alpha) \end{aligned} \tag{C.5.40}$$

The two tails of the integral contribute with two distinct phase factors along the real axis

$$\begin{aligned} I_{\text{tails}}(\alpha, |\mathbf{p}|) &\equiv \left[\int_{-\infty}^{-c|\mathbf{p}|} + \int_{c|\mathbf{p}|}^{\infty}\right] \frac{d\omega}{[(-\omega + c\,|\mathbf{p}| - i0^+)(\omega + c|\mathbf{p}| - i0^+)]^\alpha} \\ &= e^{i\pi\alpha} \int_{-\infty}^{-c|\mathbf{p}|} d\omega\, |-\omega^2 + c^2|\mathbf{p}|^2|^{-\alpha} + e^{i\pi\alpha} \int_{c|\mathbf{p}|}^{\infty} d\omega\, |-\omega^2 + c^2|\mathbf{p}|^2|^{-\alpha} \\ &= 2e^{i\pi\alpha} \int_{c|\mathbf{p}|}^{\infty} d\omega\, [\omega^2 - c^2|\mathbf{p}|^2]^{-\alpha} \\ &= 2\,e^{i\pi\alpha}\, (2c|\mathbf{p}|)^{1-2\alpha} B(1-\alpha, 2\alpha - 1) \end{aligned} \tag{C.5.41}$$

where, in going to the last equality, we have made a change of variable $x = \omega - c|\mathbf{p}|$ and then rescaled the x variable as $x = 2c|\mathbf{p}|\, t$.

Now, using the reflection formula and the duplication formula for the Gamma functions given in equation (C.4.31) and the trigonometric relation $\sin\!\big(\pi(\alpha - \frac{1}{2})\big) = -\cos(\pi\alpha)$, we can obtain the following identity

$$B(1-\alpha, 2\alpha - 1) = -\frac{1}{2\cos\pi\alpha}\, B(1-\alpha, 1-\alpha) \tag{C.5.42}$$

Using the above identity in (C.5.41) and combining all the contributions, we find the desired result

$$\begin{aligned} I(\alpha,\,|\mathbf{p}|) &= \left[2e^{i\pi\alpha} - 2\cos\pi\alpha\right] (2c|\mathbf{p}|)^{1-2\alpha}\, B(1-\alpha,\, 2\alpha - 1) \\ &= 2i\sin\pi\alpha\, (2c|\mathbf{p}|)^{1-2\alpha}\, B(1-\alpha,\, 2\alpha - 1) \end{aligned} \tag{C.5.43}$$

This agrees with the result obtained in equation (5.5.120).

Appendix D

Appendix to Chapter 6

D.1 Conventions for flat null coordinates

In the flat–null ("flat Bondi") coordinates (1.3.135) with

$$ds^2 = -2\,du\,dr + 2\,r^2\delta_{ij}\,dx^i dx^j. \tag{D.1.1}$$

The future-directed generators are

$$\text{on } \mathscr{I}^+ :\ +\partial_u$$
$$\text{on } \mathscr{I}^- :\ -\partial_u. \tag{D.1.2}$$

We *define* positive/negative frequency with respect to the future-directed generator on the relevant boundaries as

$$\mathscr{I}^+ :\quad \phi_1^+(u,x) = \int_0^\infty d\omega\, a_{\text{out}}(\omega,x)\, e^{-i\omega u}, \qquad \phi_1^-(u,x) = \int_0^\infty d\omega\, a^\dagger_{\text{out}}(\omega,x)\, e^{+i\omega u}; \tag{D.1.3}$$

$$\mathscr{I}^- :\quad \phi_1^+(u,x) = \int_0^\infty d\omega\, a_{\text{in}}(\omega,x)\, e^{+i\omega u}, \qquad \phi_1^-(u,x) = \int_0^\infty d\omega\, a^\dagger_{\text{in}}(\omega,x)\, e^{-i\omega u}. \tag{D.1.4}$$

Thus, with a single u:

- On $\mathscr{I}^+$: ϕ^+ is *outgoing* (the "out" data), ϕ^- is incoming-from-future.
- On $\mathscr{I}^-$: ϕ^+ is *incoming* (the "in" data), ϕ^- is outgoing-to-past.

In all cases $\phi_1 = \phi_1^+ + \phi_1^-$. The boundary conditions imposed are

$$\delta\phi_1^-\big|_{\mathscr{I}^+} = 0, \qquad \delta\phi_1^+\big|_{\mathscr{I}^-} = 0 \tag{D.1.5}$$

The interpretation of the above conditions is that on past null infinity, we prepare the "in" data

$$\phi^+ = f_{in} \qquad \text{on} \quad \mathscr{I}^- \tag{D.1.6}$$

Since we want to keep this fixed, we set its variation $\delta\phi^+$ to zero on $\mathscr{I}^-$. For the physical scattering, we should set $\phi^- = 0$ to avoid any outgoing radiation to $\mathscr{I}^-$. On $\mathscr{I}^+$, we have no control over outgoing data. It is already fixed by the bulk dynamics and whatever is incoming. Hence, we don't fix the outgoing data ϕ^+ on $\mathscr{I}^+$.

Hence, $\delta\phi_1^-\big|_{\mathscr{I}^+} = 0$ keeps the chosen "in" data while extremizing the path integration whereas $\delta\phi_1^+\big|_{\mathscr{I}^-} = 0$ forbids the incoming radiation from the future null infinity. To summarize,

- On $\mathscr{I}^+$: the incoming sector is held fixed; only *outgoing* variations $\delta\phi_1^{(+)}$ are allowed.
- On $\mathscr{I}^-$: the *incoming* sector (the "in" data) is held fixed; only variations of the outgoing-to-past sector $\delta\phi_1^{(-)}$ are allowed. Fixing the value of $\delta\phi^-$ to zero is not a variational problem, but it is a physical choice.

D.2 Boundary limits: Position space null infinity limit and hard/soft split

Let us start with the position space solution to the free massive scalar field derived in (6.2.82)

$$\phi(u, x^a, r) = \frac{2^{(d-3)/2}}{(2\pi)^d} \int d\omega_n \, d^{d-1}x' \, \omega_n^{d-2} \, a_p(\omega_n, x'^a) \exp\left[-i\,\omega_n\, u - ir\,\omega_n \left(x - x'\right)^2\right] + \text{h.c} \tag{D.2.7}$$

We will now evaluate the null boundary limit of this solution, given by

$$\Phi = \lim_{r\to\infty} r^{(d-1)/2}\,\phi \tag{D.2.8}$$

Let us change variables $\rho = (x - x')^2$ and $y = \sqrt{\rho}\, n$ with $n \in S^{d-2}$

$$\begin{aligned} &r^{(d-1)/2}\,\phi \\ =&\frac{2^{(d-3)/2}\, r^{(d-1)/2}}{(2\pi)^d} \frac{1}{2} \int_{S^{d-2}} d\Omega \int_0^\infty d\rho\, \rho^{(d-3)/2} \int_0^\infty d\omega\, \omega^{d-2}\, a_p(\omega, x - \sqrt{\rho}\, n) e^{-i\omega u} e^{-ir\omega\rho} + \text{h.c} \end{aligned} \tag{D.2.9}$$

Let us also define $m = (d-1)/2$, and

$$j = 4\,\frac{\omega}{\omega_0}\,\rho, \qquad \frac{\omega}{\omega_0} - \rho = s, \qquad j \geq 0,\ -\infty < s < \infty. \tag{D.2.10}$$

which implies

$$\begin{aligned} \frac{\omega}{\omega_0} &= \frac{\sqrt{j+s^2}+s}{2}, \qquad \rho = \frac{\sqrt{j+s^2}-s}{2}, \\ d\omega\, d\rho &= \frac{\omega_0}{4\sqrt{j+s^2}}\, dj\, ds. \end{aligned} \tag{D.2.11}$$

Defining,

$$F(\omega, \rho) \int_{S^{d-2}} d\Omega\, \omega^{d-2}\, a_p(\omega, x - \sqrt{\rho}\, n)\, e^{-i\omega u}, \tag{D.2.12}$$

we have

$$
\begin{aligned}
&r^{(d-1)/2}\,\phi = \\
&\frac{r^{(d-1)/2}\,\omega_0}{4\,(2\pi)^d}\frac{1}{2}\int_{S^{d-2}} d\Omega \int_0^\infty dj \int_{-\infty}^{\infty} ds \left(\frac{\left(\sqrt{j+s^2}-s\right)^{(d-3)/2}}{\sqrt{j+s^2}}\right. \\
&\left. F\left(\omega_0\,\frac{\sqrt{j+s^2}+s}{2},\,\frac{\sqrt{j+s^2}-s}{2}\right) e^{-irj/4} + \text{h.c}\right)
\end{aligned} \tag{D.2.13}
$$

For the purpose of taking the $r \to \infty$ limit after scaling with $r^{(d-1)/2}$, we will use,

$$
e^{irj/4} = \left(\frac{4}{-ir}\right) \partial_j (e^{irj/4}) \tag{D.2.14}
$$

and integrate by parts m times

$$
\begin{aligned}
\Phi = \lim_{r\to\infty} r^{(d-1)/2}\,\phi =& \frac{\omega_0\,(4\,i)^m}{4\,(2\pi)^d}\frac{1}{2}\int_{-\infty}^{\infty} ds \left(\partial_j^{(m-1)} \left[\frac{\left(\sqrt{j+s^2}-s\right)^{m-1}}{\sqrt{j+s^2}} F\, e^{-irj/4}\right]_0^\infty \right. \\
&\left. + \Phi_R = \int_0^\infty dj\, \partial_j^m \left[\frac{\left(\sqrt{j+s^2}-s\right)^{m-1}}{\sqrt{j+s^2}} F\right] e^{-irj/4}\right) + \text{h.c}
\end{aligned} \tag{D.2.15}
$$

where $m = (d-1)/2$. Using the Riemann-Lebesgue lemma, the Φ_R term vanishes. We have

$$
\begin{aligned}
\Phi =& \lim_{r\to\infty} r^{(d-1)/2}\,\phi \\
=& \frac{\omega_0\,(4\,i)^m}{4\,(2\pi)^d}\frac{1}{2}\int_{S^{d-2}} d\Omega \int_{-\infty}^{\infty} ds \left(\partial_j^{(m-1)} \left[\frac{\left(\sqrt{j+s^2}-s\right)^{m-1}}{\sqrt{j+s^2}} F\, e^{-irj/4}\right]\right)\Bigg|_{j=0}^{\infty} + \text{h.c}
\end{aligned} \tag{D.2.16}
$$

The contribution leading in r comes from $j = 0$ and the $j = \infty$ terms can be discarded.

$$
\begin{aligned}
&\Phi = \lim_{r\to\infty} r^{(d-1)/2}\,\phi = \\
&\frac{\omega_0\,(4\,i)^m}{4\,(2\pi)^d}\frac{1}{2}\int_{-\infty}^{\infty} ds \left(\partial_j^{(m-1)} \left[\frac{\left(\sqrt{j+s^2}-s\right)^{m-1}}{\sqrt{j+s^2}} F\, e^{-irj/4}\right]\right)\Bigg|_{j=0} + \text{h.c} \\
&= \frac{\omega_0\,(4\,i)^m}{4\,(2\pi)^d}\frac{1}{2}\int_{-\infty}^{\infty} ds \left(\partial_j^{(m-1)} \left(\sqrt{j+s^2}-s\right)^{m-1} \left[\frac{1}{\sqrt{j+s^2}} F\, e^{-irj/4}\right]\right)\Bigg|_{j=0} + \text{h.c} \\
&= N_d \int_{-\infty}^{\infty} ds\, \frac{1}{(i|s|)^m}\, F(\omega_0 \Theta(s) s, -\Theta(-s) s) + \text{h.c}, \\
&= N_d \int_0^\infty ds \left(\frac{F(\omega_0 s, 0)}{(i|s|)^m} - \frac{F(0,s)}{(i|s|)^m}\right) + \text{h.c}
\end{aligned} \tag{D.2.17}
$$

where

$$
N_d = \frac{\omega_0\,(4\,i)^m}{4\,(2\pi)^d}\frac{1}{2}\,\partial_j^{(m-1)} \left(\sqrt{j+s^2}-s\right)^{m-1}\Big|_{j=0} (i|s|)^{m-1} \tag{D.2.18}
$$

We have,

$$F(\omega,0)=\int_{S^{d-2}}d\Omega\,\omega^{d-2}\,a_p(\omega,x)\,e^{-i\omega u}=Vol_{S^{d-2}}\,\omega^{d-2}\,a_p(\omega,x)\,e^{-i\omega u} \tag{D.2.19}$$

which implies

$$\begin{aligned}\int_0^\infty ds\,\frac{F(\omega_0 s,0)}{(i\,s)^{(d-1)/2}}&=\Big(-i^{(d-1)/2}\,Vol_{S^{d-2}}\Big)\,\omega_0^{(d-3)/2}\int_0^\infty \omega_0 ds\,(\omega_0\,s)^{(d-3)/2}\,a_p(\omega_0 s,x)\,e^{-i\omega_0 su}\\&=\Big(-i^{(d-1)/2}\,Vol_{S^{d-2}}\Big)\,\omega_0^{(d-3)/2}\int_0^\infty d\omega\,\omega^{(d-3)/2}\,a_p(\omega,x)\,e^{-i\omega u}\end{aligned} \tag{D.2.20}$$

We also parametrise $(x'-x)=\sqrt{s}\,n=a\,n$ where $s=(x-x')^2$ and $n\in S^{d-2}$. This implies

$$d^{d-1}x'=a^{d-2}da\,d\Omega_{d-2}=\frac{1}{2}s^{(d-3)/2}ds\,d\Omega_{d-2} \tag{D.2.21}$$

We have

$$\begin{aligned}\int_0^\infty ds\,\frac{F(0,s)}{(is)^{(d-1)/2}}=&(-i)^{(d-1)/2}\int_0^\infty ds\int_{S^{d-2}}d\Omega\,\frac{1}{s^{(d-1)/2}}\,\Big[\omega^{d-2}\,a_p(\omega,x')\Big]_{\omega=0}\\=&2\,(-i)^{(d-1)/2}\int d^{d-1}x'\,\frac{\big[\omega^{d-2}\,a_p(\omega,x')\big]_{\omega=0}}{(x-x')^{2(d-2)}}\end{aligned} \tag{D.2.22}$$

Using (D.2.20), (D.2.22), in (D.2.17), we get

$$\begin{aligned}\Phi=&A_d\int_0^\infty d\omega\,\omega^{(d-3)/2}a_p(\omega,x)\,e^{-i\omega u}+B_d\int d^{d-1}x'\,\frac{\big[\omega^{d-2}\,a_p(\omega,x')\big]_{\omega=0}}{(x-x')^{2(d-2)}}+\text{h.c},\\=&\Phi_h+\Phi_{soft}.\end{aligned} \tag{D.2.23}$$

where,

$$A_d=N_d\,\Big(-i^{(d-1)/2}\,Vol_{S^{d-2}}\Big)\,\omega_0^{(d-3)/2},\qquad B_d=-2N_d\,(-i)^{(d-1)/2} \tag{D.2.24}$$

D.3 Vanishing of same sign terms in inner product

We have used the fact that the inner product between the terms with same frequencies vanish, i.e.,

$$(\phi_1^+,\delta\phi_1^+)_{\mathscr{I}^+}=0,\qquad(\phi_1^-,\delta\phi_1^-)_{\mathscr{I}^-}=0.$$

To prove this, lets consider the first term and write the fields and its variation in the fourier space as

$$\phi_1^+(u,x)=\int_0^\infty d\omega\;a(\omega,x)\,e^{-i\omega u},\qquad \delta\phi_1^+(u,x)=\int_0^\infty d\omega'\;b(\omega',x)\,e^{-i\omega' u}. \tag{D.3.25}$$

Note that we have taken the range of integration over ω from 0 to ∞ since we have taken care of negative frequencies by defining the negative frequency modes ϕ^- explicitly keeping ω always positive. Inserting (D.3.25) into the definition of inner product gives

$$\begin{aligned}(\phi_1^+, \delta\phi_1^+)_{\mathscr{I}^+} &= 2^{\frac{d-3}{2}} \int_{-\infty}^{\infty} du \int d^2x \int_0^{\infty} d\omega \int_0^{\infty} d\omega' \left[\phi_1^+ \left(\partial_u \delta\phi_1^+\right) - \left(\partial_u \phi_1^+\right) \delta\phi_1^+\right] \\ &= 2^{\frac{d-3}{2}} \int d^2x \int_0^{\infty} d\omega \int_0^{\infty} d\omega' \left(- i\omega' + i\omega\right) a(\omega, x)\, b(\omega', x) \int_{-\infty}^{\infty} du \; e^{-i(\omega+\omega')u} \\ &= 2^{\frac{d-3}{2}} i \int d^2x \int_0^{\infty} d\omega \int_0^{\infty} d\omega' \left(\omega - \omega'\right) a(\omega, x)\, b(\omega', x)\, 2\pi\delta(\omega + \omega'). \quad \text{(D.3.26)}\end{aligned}$$

For $\omega, \omega' \geq 0$, the delta function $\delta(\omega + \omega')$ only has support at $\omega = \omega' = 0$, where the prefactor $(\omega - \omega')$ vanishes. Hence

$$(\phi_1^+, \delta\phi_1^+)_{\mathscr{I}^+} = 0. \quad \text{(D.3.27)}$$

The same steps can be repeated at $\mathscr{I}^-$ to show that

$$(\phi_1^-, \delta\phi_1^-)_{\mathscr{I}^-} = 0. \quad \text{(D.3.28)}$$

In the above derivation, we have assumed that the zero frequency modes $a(0, x)$, $b(0, x)$ are well behaved.

Appendix E

Appendix to Chapter 7

E.1 Conventions and useful identities

In this section, we summarise our conventions and note some useful identities used in this work. We closely follow the conventions of [147] for the momentum space CFT. In particular, we shall denote the $(d+1)$-dimensional AdS indices by $M, N, P\ldots$ which run from 0 to d. The d dimensional boundary indices will be denoted by the Greek letters $\mu, \nu, \alpha, \beta, \sigma, \cdots$. These run from 1 to d. The boundary position and momentum variables would be denoted by boldface letters (e.g., $\mathbf{x}$ and $\mathbf{p}$). The indices of $(d+1)$-dimensional flat space would be denoted by $a, b, c, \cdots$, which would range from 1 to $(d+1)$. The symmetrization would include a factor of $\frac{1}{n!}$. E.g., for 2nd rank tensor, we have

$$P_{(\mu}\, Q_{\nu)} = \frac{1}{2}\Big(P_\mu\, Q_\nu + P_\nu\, Q_\mu\Big). \tag{E.1.1}$$

The AdS metric in the Fefferman Graham and the Poincaré coordinates is given by

$$\text{Poincaré} \quad : \quad ds^2 = \frac{L^2}{z^2}\big(dz^2 + \delta_{\mu\nu}\, dx^\mu\, dx^\nu\big) \quad ; \quad \sqrt{G} = \left(\frac{L}{z}\right)^{d+1}, \tag{E.1.2}$$

$$\text{Fefferman Graham} \quad : \quad ds^2 = L^2\frac{d\rho^2}{4\rho^2} + L\frac{\delta_{\mu\nu}\, dx^\mu\, dx^\nu}{\rho} \quad ; \quad \sqrt{G} = \frac{1}{2}\left(\frac{L}{\rho}\right)^{\frac{d+2}{2}} \tag{E.1.3}$$

The two coordinates are related by $\rho = \frac{z^2}{L}$. For the AdS metric denoted G_{MN}, the various curvature quantities are given by

$$R_{MNPQ} = \frac{G_{MQ}\, G_{NP} - G_{MP}G_{NQ}}{L^2} \quad ; \quad R_{MN} = -\frac{d}{L^2}\, G_{MN} \quad ; \quad R = -\frac{d(d+1)}{L^2}. \tag{E.1.4}$$

In our derivations, we need to work with covariant derivatives of the spin-2 fields in AdS. Below, we note the expressions of two covariant derivatives acting on the spin-2 fields in the Fefferman

Graham coordinates. We start with the following fully contracted covariant derivatives

$$
\begin{aligned}
\nabla^M\nabla^N\phi_{MN} &= \frac{4}{L^4}(d-4)^2\rho^2\phi_{\rho\rho}+\frac{(2-d)}{L^3}\rho\delta^{\mu\nu}\phi_{\mu\nu}+\frac{16(5-d)}{L^4}\rho^3\partial_\rho\phi_{\rho\rho}+\frac{2\rho^2}{L^3}\delta^{\mu\nu}\partial_\rho\phi_{\mu\nu}\\
&\quad+\frac{4(3-d)}{L^3}\rho^2\delta^{\mu\nu}\partial_\mu\phi_{\nu\rho}+\frac{16\rho^4}{L^4}\partial_\rho\partial_\rho\phi_{\rho\rho}+\frac{8\rho^3}{L^3}\delta^{\mu\nu}\partial_\mu\partial_\rho\phi_{\rho\nu}+\frac{\rho^2}{L^2}\delta^{\mu k}\delta^{\nu l}\partial_\mu\partial_\nu\phi_{kl}\,,\\
\nabla^M\nabla_M\phi &= \frac{4}{L^2}\left(1-\frac{d}{2}\right)\rho\partial_\rho\phi+\frac{4\rho^2}{L^2}\partial_\rho\partial_\rho\phi+\frac{\rho}{L}\delta^{\mu\nu}\partial_\mu\partial_\nu\phi\,. \qquad \text{(E.1.5)}
\end{aligned}
$$

Components of $\nabla_M\nabla_N\phi$

$$
\begin{aligned}
(\rho\rho):&\qquad \nabla_\rho\nabla_\rho\phi=\partial_\rho^2\phi+\frac{1}{\rho}\partial_\rho\phi,\\
(\mu\rho):&\qquad \nabla_\mu\nabla_\rho\phi=\partial_\mu\partial_\rho\phi+\frac{1}{2\rho}\partial_\mu\phi,\\
(\mu\nu):&\qquad \nabla_\mu\nabla_\nu\phi=\partial_\mu\partial_\nu\phi-2\delta_{\mu\nu}\partial_\rho\phi. \qquad \text{(E.1.6)}
\end{aligned}
$$

Components of $\nabla^P\nabla_M\phi_{PN}$

$$
\begin{aligned}
(\rho\rho):\quad \nabla^P\nabla_\rho\phi_{P\rho} &= \frac{4\rho^2}{L^2}\partial_\rho\partial_\rho\phi_{\rho\rho}+\frac{\rho}{L}\delta^{\mu\nu}\partial_\mu\partial_\rho\phi_{\nu\rho}+\frac{2}{L^2}\Big(8-\frac{5d}{2}\Big)\phi_{\rho\rho}+\frac{2}{L}\delta^{\mu\nu}\partial_\mu\phi_{\nu\rho}\\
&\quad+\frac{3\delta^{\mu\nu}\phi_{\mu\nu}}{4\rho L}+\frac{2(10-d)}{L^2}\rho\partial_\rho\phi_{\rho\rho}+\frac{1}{2L}\delta^{\mu\nu}\partial_\rho\phi_{\mu\nu},\\
(\mu\rho):\quad \nabla^P\nabla_\mu\phi_{P\rho} &= \frac{4\rho^2}{L^2}\partial_\rho\partial_\mu\phi_{\rho\rho}+\frac{\rho}{L}\delta^{kl}\partial_\mu\partial_k\phi_{l\rho}+\frac{2\rho}{L^2}\partial_\rho\phi_{\mu\rho}+\frac{1}{2L}\delta^{jk}\partial_j\phi_{\mu k}\\
&\quad+\frac{2(4-d)}{L^2}\rho\partial_\mu\phi_{\rho\rho}+\frac{2}{L^2}\Big(\frac{1}{2}-d\Big)\phi_{\mu\rho}+\frac{1}{2L}\delta^{jk}\partial_\mu\phi_{jk},\\
(\rho\mu):\quad \nabla^P\nabla_\rho\phi_{P\mu} &= \frac{4\rho^2}{L^2}\partial_\rho^2\phi_{\rho\mu}+\frac{\rho}{L}\delta^{jk}\partial_j\partial_\rho\phi_{\mu k}+\frac{3}{2L}\delta^{jk}\partial_k\phi_{\mu j}+\frac{2(7-d)}{L^2}\rho\partial_\rho\phi_{\mu\rho}\\
&\quad+\frac{1}{L^2}(5-4d)\phi_{\mu\rho},\\
(\mu\nu):\quad \nabla^P\nabla_\mu\phi_{P\nu} &= \frac{4\rho^2}{L^2}\partial_\rho\partial_\mu\phi_{\rho\nu}+\frac{\rho}{L}\delta^{kl}\partial_\mu\partial_k\phi_{l\nu}-\frac{8\rho^2}{L^3}\delta_{\mu\nu}\partial_\rho\phi_{\rho\rho}-\frac{2\rho}{L^3}\delta_{\mu\nu}\delta^{kl}\partial_k\phi_{l\rho}\\
&\quad+\frac{2(2-d)}{L^2}\rho\partial_\mu\phi_{\rho\nu}-\frac{(d+1)}{L^2}\phi_{\mu\nu}+\frac{4(d-3)}{L^3}\rho\delta_{\mu\nu}\phi_{\rho\rho}. \qquad \text{(E.1.7)}
\end{aligned}
$$

Components of $\nabla^P\nabla_P\phi_{MN}$

$$
\begin{aligned}
(\rho\rho):\qquad \nabla^P\nabla_P\phi_{\rho\rho} &= \frac{4\rho^2}{L^2}\partial_\rho^2\phi_{\rho\rho}+\frac{\rho}{L}\delta^{\mu\nu}\partial_\mu\partial_\nu\phi_{\rho\rho}-\frac{2(3d-8)}{L^2}\phi_{\rho\rho}+\frac{2}{L}\delta^{\mu\nu}\partial_\mu\phi_{\nu\rho}\\
&\quad+\frac{\delta^{\mu\nu}\phi_{\mu\nu}}{2\rho L}-\frac{2(d-10)}{L^2}\rho\partial_\rho\phi_{\rho\rho},\\
(\mu\rho):\qquad \nabla^P\nabla_P\phi_{\mu\rho} &= \frac{4\rho^2}{L^2}\partial_\rho^2\phi_{\mu\rho}+\frac{\rho}{L}\delta^{kl}\partial_k\partial_l\phi_{\mu\rho}-\frac{2(2d-3)}{L^2}\phi_{\mu\rho}-\frac{2(d-8)}{L^2}\rho\partial_\rho\phi_{\mu\rho}\\
&\quad-\frac{4\rho}{L^2}\partial_\mu\phi_{\rho\rho}+\frac{1}{L}\delta^{jk}\partial_j\phi_{\mu k},\\
(\mu\nu):\qquad \nabla^P\nabla_P\phi_{\mu\nu} &= \frac{4\rho^2}{L^2}\partial_\rho\partial_\rho\phi_{\mu\nu}+\frac{\rho}{L}\delta^{kl}\partial_k\partial_l\phi_{\mu\nu}-\frac{4\rho}{L^2}\partial_\mu\phi_{\nu\rho}-\frac{4\rho}{L^2}\partial_\nu\phi_{\mu\rho}\\
&\quad-\frac{2(d-6)}{L^2}\rho\partial_\rho\phi_{\mu\nu}-\frac{2(d-1)}{L^2}\phi_{\mu\nu}+\frac{8\rho}{L^3}\delta_{\mu\nu}\phi_{\rho\rho}.
\end{aligned}
\tag{E.1.8}
$$

E.2 Form factors of 3-point CFT correlator

In this section, we summarise the CFT$_d$ 3-point function of a conserved current with conformal dimension $d-1$ and two non-conserved spin-2 operators having arbitrary but same conformal dimension Δ. The expressions will be given in terms of the triple-K integral

$$
J_{N\{\beta_1,\beta_2,\beta_3\}}(k_1,k_2,k_3)\equiv\int_0^\infty dz\, z^{\frac{d}{2}+N-1}\prod_{i=1}^{3}k_i^{\Delta_i-\frac{d}{2}+\beta_i}\,K_{\Delta_i-\frac{d}{2}+\beta_i}(zk_i)\,. \tag{E.2.9}
$$

The transverse part of the 3 point function of two spin-2 operators and a conserved vector current is given by (see equation (7.2.5))

$$
\mathcal{A}^{\perp}_{2,J,2} \;=\; (\epsilon_2\cdot\pi_2\cdot p_1)A+(\epsilon_2\cdot\pi_2\cdot\epsilon_1)B_1+(\epsilon_2\cdot\pi_2\cdot\epsilon_3)B_2\,, \tag{E.2.10}
$$

where, the functions A, B_1 and B_2 are parametrized as

$$
\begin{aligned}
A &= z^2A_0^{(0,0)}+\zeta_2\xi_2 zA_1^{(0,0)}+\frac{1}{2}\zeta_2^2\xi_2^2A_2^{(0,0)}+\zeta_1\xi_2 zA_1^{(0,1)}+\zeta_2\xi_1 zA_1^{(1,0)}+\zeta_1\xi_1 zA_1^{(1,1)}\\
&\quad+\frac{1}{2}\zeta_1\zeta_2\xi_2^2A_2^{(0,1)}+\frac{1}{4}\zeta_1^2\xi_2^2A_2^{(0,2)}+\frac{1}{2}\zeta_2^2\xi_1\xi_2A_2^{(1,0)}+\frac{1}{2}\zeta_1\zeta_2\xi_1\xi_2A_2^{(1,1)}+\frac{1}{4}\zeta_1^2\xi_1\xi_2A_2^{(1,2)}\\
&\quad+\frac{1}{4}\zeta_2^2\xi_1^2A_2^{(2,0)}+\frac{1}{4}\zeta_1\zeta_2\xi_1^2A_2^{(2,1)}+\frac{1}{8}\zeta_1^2\xi_1^2A_2^{(2,2)}\,,\\[1ex]
B_1 &= \xi_1 zB_{1;0}^{(1,0)}+\xi_2 zB_{1;0}^{(0,0)}+\zeta_2\xi_2\xi_1B_{1;1}^{(1,0)}+\zeta_1\xi_2\xi_1B_{1;1}^{(1,1)}+\zeta_2\xi_2^2B_{1;1}^{(0,0)}+\zeta_1\xi_2^2B_{1;1}^{(0,1)}\\
&\quad+\frac{1}{2}\zeta_2\xi_1^2B_{1;1}^{(2,0)}+\frac{1}{2}\zeta_1\xi_1^2B_{1;1}^{(2,1)}\,,\\[1ex]
B_2 &= \zeta_1 zB_{2;0}^{(0,1)}+\zeta_2 zB_{2;0}^{(0,0)}+\zeta_2\zeta_1\xi_2B_{2;1}^{(0,1)}+\zeta_2\zeta_1\xi_1B_{2;1}^{(1,1)}+\zeta_2^2\xi_2B_{2;1}^{(0,0)}+\zeta_2^2\xi_1B_{2;1}^{(1,0)}\\
&\quad+\frac{1}{2}\zeta_1^2\xi_2B_{2;1}^{(0,2)}+\frac{1}{2}\zeta_1^2\xi_1B_{2;1}^{(1,2)}\,,
\end{aligned}
$$

with

$$
z=\epsilon_1\cdot\epsilon_3,\qquad \xi_1=\epsilon_3\cdot p_2,\qquad \xi_2=\epsilon_3\cdot(p_1+p_2),\qquad \zeta_1=\epsilon_1\cdot p_2,\qquad \zeta_2=\epsilon_1\cdot p_1\,. \tag{E.2.11}
$$

The 30 form factors appearing above are given by [295]

$$
\begin{aligned}
A_2^{(2,2)} &= a_2^{(2,2)} J_{5\{0,0,0\}} \\
A_2^{(2,1)} &= a_2^{(2,1)} J_{4\{-1,0,0\}} - a_2^{(2,2)} J_{5\{-1,1,0\}} \\
A_2^{(1,2)} &= a_2^{(1,2)} J_{4\{0,0,-1\}} + a_2^{(2,2)} J_{5\{0,1,-1\}} \\
A_2^{(2,0)} &= a_2^{(2,0)} J_{3\{-2,0,0\}} - a_2^{(2,1)} J_{4\{-2,1,0\}} + \frac{1}{2} a_2^{(2,2)} J_{5\{-2,2,0\}} \\
A_2^{(0,2)} &= a_2^{(0,2)} J_{3\{0,0,-2\}} + a_2^{(1,2)} J_{4\{0,1,-2\}} + \frac{1}{2} a_2^{(2,2)} J_{5\{0,2,-2\}} \\
A_2^{(1,1)} &= a_2^{(1,1)} J_{3\{-1,0,-1\}} - a_2^{(2,2)} J_{5\{-1,2,-1\}} + \big(a_2^{(2,1)} - a_2^{(1,2)}\big) J_{4\{-1,1,-1\}} \\
A_2^{(1,0)} &= \Big(a_2^{(2,0)} - a_2^{(1,1)}\Big) J_{3\{-2,1,-1\}} + \frac{1}{2} a_2^{(2,2)} J_{5\{-2,3,-1\}} + a_2^{(1,0)} J_{2\{-2,0,-1\}} \\
&\quad + \Big(\frac{1}{2} a_2^{(1,2)} - a_2^{(2,1)}\Big) J_{4\{-2,2,-1\}} \\
A_2^{(0,1)} &= \Big(a_2^{(1,1)} - a_2^{(0,2)}\Big) J_{3\{-1,1,-2\}} - \frac{1}{2} a_2^{(2,2)} J_{5\{-1,3,-2\}} + a_2^{(0,1)} J_{2\{-1,0,-2\}} \\
&\quad + \Big(\frac{1}{2} a_2^{(2,1)} - a_2^{(1,2)}\Big) J_{4\{-1,2,-2\}} \\
A_2^{(0,0)} &= a_2^{(0,0)} J_{1\{-2,0,-2\}} + \frac{1}{4} a_2^{(2,2)} J_{5\{-2,4,-2\}} + \Big(\frac{1}{2} a_2^{(0,2)} + \frac{1}{2} a_2^{(2,0)} - a_2^{(1,1)}\Big) J_{3\{-2,2,-2\}} \\
&\quad + \frac{1}{2} \big(a_2^{(1,2)} - a_2^{(2,1)}\big) J_{4\{-2,3,-2\}} + \Big(a_2^{(1,0)} - a_2^{(0,1)}\Big) J_{2\{-2,1,-2\}}
\end{aligned}
$$

$$
\begin{aligned}
A_1^{(1,1)} &= a_1^{(1,1)} J_{3\{0,0,0\}} - \frac{1}{2} a_2^{(2,2)} J_{4\{0,1,0\}} \\
A_1^{(1,0)} &= a_1^{(1,0)} J_{2\{-1,0,0\}} - \Big(a_1^{(1,1)} + \frac{1}{2} a_2^{(2,1)}\Big) J_{3\{-1,1,0\}} + \frac{1}{2} a_2^{(2,2)} J_{4\{-1,2,0\}} \\
A_1^{(0,1)} &= a_1^{(0,1)} J_{2\{0,0,-1\}} + \Big(a_1^{(1,1)} - \frac{1}{2} a_2^{(1,2)}\Big) J_{3\{0,1,-1\}} - \frac{1}{2} a_2^{(2,2)} J_{4\{0,2,-1\}} \\
A_1^{(0,0)} &= -\Big(a_1^{(1,1)} + \frac{1}{2} a_2^{(2,1)} - \frac{1}{2} a_2^{(1,2)}\Big) J_{3\{-1,2,-1\}} + \big(a_1^{(1,0)} - a_1^{(0,1)} - \frac{1}{2} a_2^{(1,1)}\big) J_{2\{-1,1,-1\}} \\
&\quad + a_1^{(0,0)} J_{1\{-1,0,-1\}} + \frac{1}{2} a_2^{(2,2)} J_{4\{-1,3,-1\}} \\
A_0^{(0,0)} &= a_0^{(0,0)} J_{1\{0,0,0\}} - a_1^{(1,1)} J_{2\{0,1,0\}} + \frac{1}{4} a_2^{(2,2)} J_{3\{0,2,0\}}
\end{aligned}
$$

$$
\begin{aligned}
B_{2;0}^{(0,1)} &= b_{2;0}^{(0,1)} J_{1\{0,0,0\}} + \left(\frac{1}{2}a_2^{(2,1)} - b_{2;1}^{(1,1)} - b_{2;1}^{(1,2)}\right) J_{2\{0,1,0\}} + \left(\frac{1}{2}a_2^{(2,1)} - b_{2;1}^{(1,1)}\right) J_{2\{0,0,1\}} \\
&\quad -\frac{1}{2}a_2^{(2,2)} J_{3\{1,1,0\}} + \left(\frac{1}{2}a_2^{(2,1)} - b_{2;1}^{(1,1)} + a_1^{(1,1)}\right) J_{2\{1,0,0\}} \\
B_{2;0}^{(0,0)} &= b_{2;0}^{(0,0)} J_{0\{-1,0,0\}} - \left(a_1^{(1,1)} + \frac{1}{2}a_2^{(2,1)}\right) J_{2\{0,1,0\}} + b_{2;0}^{(0,1)} J_{1\{-1,0,1\}} \\
&\quad +\big(b_{2;0}^{(0,1)} + a_1^{(1,0)}\big) J_{1\{0,0,0\}} + \frac{1}{2}a_2^{(2,2)} J_{3\{0,2,0\}} + b_{2;1}^{(1,2)} J_{2\{-1,2,0\}} \\
B_{1;0}^{(1,0)} &= b_{1;0}^{(1,0)} J_{1\{0,0,0\}} + \left(\frac{1}{2}a_2^{(1,2)} + b_{1;1}^{(1,1)} - b_{1;1}^{(2,1)}\right) J_{2\{0,1,0\}} + \left(\frac{1}{2}a_2^{(1,2)} + b_{1;1}^{(1,1)}\right) J_{2\{1,0,0\}} \\
&\quad +\frac{1}{2}a_2^{(2,2)} J_{3\{0,1,1\}} + \left(\frac{1}{2}a_2^{(1,2)} + b_{1;1}^{(1,1)} - a_1^{(1,1)}\right) J_{2\{0,0,1\}} \\
B_{1;0}^{(0,0)} &= b_{1;0}^{(0,0)} J_{0\{0,0,-1\}} - \left(a_1^{(1,1)} - \frac{1}{2}a_2^{(1,2)}\right) J_{2\{0,1,0\}} - \big(b_{1;0}^{(1,0)} + a_1^{(0,1)}\big) J_{1\{0,0,0\}} \\
&\quad -b_{1;0}^{(1,0)} J_{1\{1,0,-1\}} + \frac{1}{2}a_2^{(2,2)} J_{3\{0,2,0\}} - b_{1;1}^{(2,1)} J_{2\{0,2,-1\}}
\end{aligned}
$$

$$
\begin{aligned}
&B_{2;1}^{(1,2)} = b_{2;1}^{(1,2)} J_{3\{0,0,0\}} + \frac{1}{2}a_2^{(2,2)} J_{4\{1,0,0\}} \\
&B_{2;1}^{(1,1)} = b_{2;1}^{(1,1)} J_{3\{0,0,0\}} - \left(b_{2;1}^{(1,2)} - b_{2;1}^{(1,1)} + \frac{1}{2}a_2^{(2,1)}\right) J_{3\{-1,1,0\}} + \big(b_{2;1}^{(1,1)} - \frac{1}{2}a_2^{(2,1)}\big) J_{3\{-1,0,1\}} \\
&- \frac{1}{2}a_2^{(2,2)} J_{4\{0,1,0\}} \\
&B_{2;1}^{(1,0)} \\
&= b_{2;1}^{(1,0)} J_{1\{-2,0,0\}} + \frac{1}{2}b_{2;1}^{(1,1)} J_{3\{0,0,0\}} + \frac{1}{2}a_2^{(2,1)} J_{3\{-2,1,1\}} + \frac{1}{2}b_{2;1}^{(1,1)} J_{3\{-2,0,2\}} + b_{2;1}^{(1,1)} J_{3\{-1,0,1\}} \\
&+ \left(\frac{1}{2}a_2^{(2,1)} - \frac{1}{2}b_{2;1}^{(1,1)} + \frac{1}{2}b_{2;1}^{(1,2)}\right) J_{3\{-2,2,0\}} + \frac{1}{4}a_2^{(2,2)} J_{4\{-1,2,0\}} + \frac{1}{2}a_2^{(2,0)} J_{2\{-1,0,0\}} \\
&B_{2;1}^{(0,1)} \\
&= b_{2;1}^{(0,1)} J_{1\{-1,0,-1\}} - \frac{1}{2}a_2^{(2,2)} J_{4\{0,2,-1\}} + \frac{1}{2}a_2^{(1,1)} J_{2\{0,0,-1\}} + \left(b_{2;1}^{(1,2)} + b_{2;1}^{(1,1)} - \frac{1}{2}a_2^{(1,2)}\right) J_{3\{0,1,-1\}} \\
&+ \left(b_{2;1}^{(1,1)} - \frac{1}{2}a_2^{(2,1)}\right) J_{3\{-1,2,-1\}} + \left(b_{2;1}^{(1,2)} + b_{2;1}^{(1,1)} - \frac{1}{2}a_2^{(2,1)}\right) J_{3\{-1,1,0\}} - b_{2;1}^{(0,2)} J_{2\{-1,1,-1\}} \\
&B_{2;1}^{(0,2)} = b_{2;1}^{(0,2)} J_{2\{0,0,-1\}} - b_{2;1}^{(1,2)} J_{3\{0,0,0\}} + \frac{1}{2}a_2^{(2,2)} J_{4\{1,1,-1\}} + \left(\frac{1}{2}a_2^{(1,2)} - b_{2;1}^{(1,2)}\right) J_{3\{1,0,-1\}}
\end{aligned}
$$

$$
\begin{aligned}
&B^{(0,0)}_{2;1}\\
=&b^{(0,0)}_{2;1}J_{0\{-2,0,-1\}}-\frac{1}{2}a^{(1,0)}_{2}J_{1\{-2,0,0\}}+\Big(\frac{1}{4}a^{(1,2)}_{2}-\frac{1}{2}b^{(1,1)}_{2;1}-\frac{1}{2}b^{(1,2)}_{2;1}\Big)J_{3\{-1,2,-1\}}\\
&+\Big(\frac{1}{2}a^{(2,0)}_{2}-\frac{1}{2}a^{(1,1)}_{2}\Big)J_{2\{-1,1,-1\}}+\Big(\frac{1}{2}a^{(2,1)}_{2}-\frac{1}{2}b^{(1,1)}_{2;1}-\frac{1}{2}b^{(1,2)}_{2;1}\Big)J_{3\{-2,2,0\}}+\frac{1}{4}a^{(2,2)}_{2}J_{4\{-1,3,-1\}}\\
&+\Big(b^{(1,0)}_{2;1}-\frac{1}{2}a^{(1,0)}_{2}-b^{(0,1)}_{2;1}\Big)J_{1\{-2,1,-1\}}+\frac{1}{2}b^{(0,2)}_{2;1}J_{2\{-2,2,-1\}}+\Big(\frac{1}{2}a^{(2,1)}_{2}-\frac{2}{3}b^{(1,1)}_{2;1}\Big)J_{3\{-2,3,-1\}}\\
&-\frac{1}{2}b^{(1,1)}_{2;1}J_{3\{0,0,0\}}-\frac{1}{2}b^{(1,1)}_{2;1}J_{3\{-1,0,1\}}-\frac{1}{6}b^{(1,1)}_{2;1}J_{3\{1,0,-1\}}-\frac{1}{6}b^{(1,1)}_{2;1}J_{3\{-2,0,2\}}
\end{aligned}
$$

$$
\begin{aligned}
B^{(2,1)}_{1;1} &= b^{(2,1)}_{1;1}J_{3\{0,0,0\}}-\frac{1}{2}a^{(2,2)}_{2}J_{4\{0,0,1\}}\\
B^{(1,1)}_{1;1} &= b^{(1,1)}_{1;1}J_{3\{0,0,0\}}+\Big(b^{(2,1)}_{1;1}+b^{(1,1)}_{1;1}+\frac{1}{2}a^{(1,2)}_{2}\Big)J_{3\{0,1,-1\}}+\big(b^{(1,1)}_{1;1}+\frac{1}{2}a^{(1,2)}_{2}\big)J_{3\{1,0,-1\}}\\
&\quad -\frac{1}{2}a^{(2,2)}_{2}J_{4\{0,1,0\}}\\
B^{(0,1)}_{1;1} &= b^{(0,1)}_{1;1}J_{1\{0,0,-2\}}-\frac{1}{2}b^{(1,1)}_{1;1}J_{3\{0,0,0\}}+\frac{1}{2}a^{(1,2)}_{2}J_{3\{1,1,-2\}}-\frac{1}{2}b^{(1,1)}_{1;1}J_{3\{2,0,-2\}}-b^{(1,1)}_{1;1}J_{3\{1,0,-1\}}\\
&\quad +\Big(\frac{1}{2}a^{(1,2)}_{2}+\frac{1}{2}b^{(1,1)}_{1;1}+\frac{1}{2}b^{(2,1)}_{1;1}\Big)J_{3\{0,2,-2\}}-\frac{1}{4}a^{(2,2)}_{2}J_{4\{0,2,-1\}}-\frac{1}{2}a^{(0,2)}_{2}J_{2\{0,0,-1\}}\\
B^{(1,0)}_{1;1} &= b^{(1,0)}_{1;1}J_{1\{-1,0,-1\}}+\frac{1}{2}a^{(2,2)}_{2}J_{4\{-1,2,0\}}-\frac{1}{2}a^{(1,1)}_{2}J_{2\{-1,0,0\}}\\
&\quad +\Big(b^{(2,1)}_{1;1}-b^{(1,1)}_{1;1}-\frac{1}{2}a^{(2,1)}_{2}\Big)J_{3\{-1,1,0\}}-\Big(b^{(1,1)}_{1;1}+\frac{1}{2}a^{(1,2)}_{2}\Big)J_{3\{-1,2,-1\}}\\
&\quad +\Big(b^{(2,1)}_{1;1}-b^{(1,1)}_{1;1}-\frac{1}{2}a^{(1,2)}_{2}\Big)J_{3\{0,1,-1\}}+b^{(2,0)}_{1;1}J_{2\{-1,1,-1\}}\\
B^{(2,0)}_{1;1} &= b^{(2,0)}_{1;1}J_{2\{-1,0,0\}}+b^{(2,1)}_{1;1}J_{3\{0,0,0\}}+\frac{1}{2}a^{(2,2)}_{2}J_{4\{-1,1,1\}}+\Big(-\frac{1}{2}a^{(2,1)}_{2}+b^{(2,1)}_{1;1}\Big)J_{3\{-1,0,1\}}
\end{aligned}
$$

$$
\begin{aligned}
&B^{(0,0)}_{1;1}\\
=&b^{(0,0)}_{1;1}J_{0\{-1,0,-2\}}+\frac{1}{2}a^{(0,1)}_{2}J_{1\{0,0,-2\}}-\Big(\frac{1}{4}a^{(2,1)}_{2}+\frac{1}{2}b^{(1,1)}_{1;1}-\frac{1}{2}b^{(2,1)}_{1;1}\Big)J_{3\{-1,2,-1\}}\\
&+\Big(\frac{1}{2}a^{(0,2)}_{2}-\frac{1}{2}a^{(1,1)}_{2}\Big)J_{2\{-1,1,-1\}}+\Big(-\frac{1}{2}a^{(1,2)}_{2}-\frac{1}{2}b^{(1,1)}_{1;1}+\frac{1}{2}b^{(2,1)}_{1;1}\Big)J_{3\{0,2,-2\}}+\frac{1}{4}a^{(2,2)}_{2}J_{4\{-1,3,-1\}}\\
&+\Big(-b^{(0,1)}_{1;1}+\frac{1}{2}a^{(0,1)}_{2}+b^{(1,0)}_{1;1}\Big)J_{1\{-1,1,-2\}}+\frac{1}{2}b^{(2,0)}_{1;1}J_{2\{-1,2,-2\}}-\Big(\frac{1}{2}a^{(1,2)}_{2}+\frac{2}{3}b^{(1,1)}_{1;1}\Big)J_{3\{-1,3,-2\}}\\
&-\frac{1}{2}b^{(1,1)}_{1;1}J_{3\{0,0,0\}}-\frac{1}{2}b^{(1,1)}_{1;1}J_{3\{1,0,-1\}}-\frac{1}{6}b^{(1,1)}_{1;1}J_{3\{-1,0,1\}}-\frac{1}{6}b^{(1,1)}_{1;1}J_{3\{2,0,-2\}}
\end{aligned}
$$

The above form factors are in terms of 30 momentum independent coefficients. However, for $\Delta_1=\Delta_3\equiv\Delta$, only 5 of them are independent while the rest can be expressed in terms of the independent ones. The expression of the 25 coefficients in terms of the 5 independent ones are given by [295]

$$
\begin{aligned}
a_1^{(0,1)} &= \frac{1}{8}(d+2)d^2(\Delta+1)a_2^{(2,2)} + \left(-\frac{3d^2}{4} - \frac{1}{2}d(\Delta+2) + \Delta + 1\right) a_1^{(1,1)} + \left(\frac{d}{2\Delta} + \frac{1}{\Delta}\right) a_0^{(0,0)} \\
&\quad + \frac{1}{8}(d-2)\Delta a_2^{(1,1)} - d\Delta b_{1;1}^{(1,1)} \\
a_1^{(0,0)} &= \frac{(\Delta-1)\left(d^2-2d+4\Delta\right) a_0^{(0,0)}}{\Delta^2} + \frac{d^2\left(d^2-4\right)\left(\Delta^2-1\right) a_2^{(2,2)}}{4\Delta} + \frac{1}{4}(d-2)^2(\Delta-1)a_2^{(1,1)} \\
&\quad - \frac{(d-2)(\Delta-1)\left(3d^2+2d(\Delta+1)-4\Delta\right) a_1^{(1,1)}}{2\Delta} - 2d(d-2)(\Delta-1)b_{1;1}^{(1,1)} \qquad \text{(E.2.12)}
\end{aligned}
$$

$$
\begin{aligned}
a_2^{(1,2)} &= \frac{d^2(\Delta+1)a_2^{(2,2)}}{4\Delta} - \frac{(3d+2\Delta-2)a_1^{(1,1)}}{2\Delta} + \frac{a_0^{(0,0)}}{\Delta^2} + \frac{1}{4}a_2^{(1,1)} - 2b_{1;1}^{(1,1)} \\
a_2^{(0,2)} &= \frac{d\left(d^2+2d-4\right)(\Delta+1)a_2^{(2,2)}}{4(\Delta-1)} - \frac{\left(3d^2+2d(\Delta+1)-8(\Delta+1)\right) a_1^{(1,1)}}{2(\Delta-1)} + \frac{da_0^{(0,0)}}{(\Delta-1)\Delta} \\
&\quad + \frac{(d-4)\Delta a_2^{(1,1)}}{4(\Delta-1)} - \frac{2d\Delta b_{1;1}^{(1,1)}}{\Delta-1} \\
a_2^{(0,1)} &= \frac{\left(3d^2+2d(2\Delta-5)+8\Delta\right)}{2\Delta^2} a_0^{(0,0)} + \frac{(d+2)d^2(\Delta+1)(3d+4\Delta-8)a_2^{(2,2)}}{8\Delta} \\
&\quad + \left(-\frac{9d^3}{4\Delta} + d^2\left(\frac{5}{\Delta} - \frac{9}{2}\right) + d\left(-2\Delta + \frac{3}{\Delta} + 5\right) + 4(\Delta-1)\right) a_1^{(1,1)} \\
&\quad + \frac{1}{8}\Big(3d^2+4d\Delta-18d-8\Delta+16\Big)a_2^{(1,1)} + d(-3d-4\Delta+6)b_{1;1}^{(1,1)} \\
a_2^{(0,0)} &= \frac{\left(d^2-4\right) d^2\left(\Delta^2-\Delta-2\right)(d+4\Delta-5)a_2^{(2,2)}}{4(\Delta-1)\Delta} - \frac{2(d-2)d(\Delta-2)(d+4\Delta-4)b_{1;1}^{(1,1)}}{\Delta-1} \\
&\quad - \frac{(d-2)(\Delta-2)\left(14d^2(\Delta-1)+3d^3+8d\left(\Delta^2-\Delta-1\right)-16(\Delta-1)\Delta\right) a_1^{(1,1)}}{2(\Delta-1)\Delta} \\
&\quad + \frac{(\Delta-2)\left(4d^2(\Delta-2)+d^3+d(12-8\Delta)+8(\Delta-1)\Delta\right) a_0^{(0,0)}}{(\Delta-1)\Delta^2} \\
&\quad + \frac{(d-2)(\Delta-2)\left(d^2+4d(\Delta-2)-8\Delta+8\right) a_2^{(1,1)}}{4(\Delta-1)} \qquad \text{(E.2.13)}
\end{aligned}
$$

$$
a_1^{(1,0)} = -a_1^{(0,1)} \quad ; \quad a_2^{(2,1)} = -a_2^{(1,2)} \quad ; \quad a_2^{(2,0)} = a_2^{(0,2)} \quad ; \quad a_2^{(1,0)} = -a_2^{(0,1)} \qquad \text{(E.2.14)}
$$

$$
\begin{aligned}
b_{1;0}^{(1,0)} &= \left(1-\frac{d}{2}-\frac{1}{\Delta}\right)a_0^{(0,0)}+\frac{1}{4}\Big(2d(3+\Delta^2)+3d^2\Delta-4(\Delta-1)^2\Big)a_1^{(1,1)}-\frac{1}{8}\Delta\left(2+(d-2)\Delta\right)a_2^{(1,1)}\\
&\quad -\frac{1}{8}d^2(\Delta+1)((d+2)\Delta+2)a_2^{(2,2)}+d\Delta^2 b_{1;1}^{(1,1)}
\end{aligned}
$$

$$
\begin{aligned}
b_{1;0}^{(0,0)} &= \frac{1}{8}d^2\left(\Delta^2-1\right)\left(d^2-2d\Delta-4(\Delta+2)\right)a_2^{(2,2)}-\frac{1}{8}(\Delta-1)\Delta\left(-d^2+2d(\Delta+2)-4\Delta\right)a_2^{(1,1)}\\
&\quad +\frac{1}{4}(\Delta-1)\left(d^2(4\Delta+6)-3d^3+4d(\Delta^2+2\Delta+2)-8(\Delta-1)\Delta\right)a_1^{(1,1)}\\
&\quad +\frac{(\Delta-1)\left(d^2-2d(\Delta+2)+4\Delta\right)a_0^{(0,0)}}{2\Delta}+d\Delta\left(-d\Delta+d+2\Delta^2-2\right)b_{1;1}^{(1,1)}\\
b_{1;1}^{(2,1)} &= \frac{1}{4}(3d+2\Delta+6)a_1^{(1,1)}-\frac{1}{8}d(d+4)(\Delta+1)a_2^{(2,2)}-\frac{a_0^{(0,0)}}{2\Delta}-\frac{1}{8}\Delta a_2^{(1,1)}+\Delta b_{1;1}^{(1,1)}
\end{aligned}
$$

$$
\begin{aligned}
b_{1;1}^{(2,0)} &= \left(-\frac{3d^2}{4}+d(3-2\Delta)-\Delta^2+1\right)a_1^{(1,1)}+\frac{1}{8}d(\Delta+1)\left(d^2+2d(\Delta-1)+8(\Delta-1)\right)a_2^{(2,2)}\\
&\quad +\left(\frac{d}{2\Delta}-\frac{3}{\Delta}+1\right)a_0^{(0,0)}+\frac{1}{8}\Delta(d+2\Delta-6)a_2^{(1,1)}-\Delta(d+2\Delta-2)b_{1;1}^{(1,1)}
\end{aligned}
$$

$$
\begin{aligned}
b_{1;1}^{(1,0)} &= -\frac{(d+2)d^2\left(\Delta^2-1\right)(d+2\Delta-4)a_2^{(2,2)}}{8\Delta}+\left(\frac{d^2}{2\Delta^2}-\frac{d^2}{2\Delta}-\frac{3d}{\Delta^2}+\frac{4d}{\Delta}-d-\frac{2}{\Delta}+2\right)a_0^{(0,0)}\\
&\quad +\frac{(\Delta-1)\left(4d^2(2\Delta-3)+3d^3+4d\left(\Delta^2-3\Delta-1\right)-8(\Delta-1)\Delta\right)}{4\Delta}a_1^{(1,1)}\\
&\quad -\frac{1}{8}(\Delta-1)\left(d^2+2d(\Delta-3)-4\Delta\right)a_2^{(1,1)}+d(\Delta-1)(d+2\Delta-2)b_{1;1}^{(1,1)}
\end{aligned}
$$

$$
\begin{aligned}
b_{1;1}^{(0,1)} &= \frac{1}{16}(d-2)d^3(\Delta+1)a_2^{(2,2)}+\left(\frac{d^2}{4\Delta}-\frac{3d}{2\Delta}+\frac{2}{\Delta}\right)a_0^{(0,0)}+\frac{1}{16}\left(d^2-6d+8\right)\Delta a_2^{(1,1)}\\
&\quad -\frac{1}{8}(d-2)\left(3d^2+2d(\Delta-3)-8\Delta+8\right)a_1^{(1,1)}+\frac{1}{2}\Delta\left(-d^2+2d+4\Delta-4\right)b_{1;1}^{(1,1)}
\end{aligned}
$$

$$
\begin{aligned}
b_{1;1}^{(0,0)} &= \frac{(d+2)d^2\left(\Delta^2-\Delta-2\right)\left(d^2+4d(\Delta-3)-16\Delta+24\right)a_2^{(2,2)}}{16\Delta}\\
&\quad +\frac{d(\Delta-2)\left(d^2+2d(2\Delta-7)-24\Delta+32\right)a_0^{(0,0)}}{4\Delta^2}\\
&\quad -\frac{1}{8\Delta}\Big[2d^3\left(7\Delta^2-32\Delta+36\right)+4d^2\left(2\Delta^3-23\Delta^2+53\Delta-30\right)+3d^4(\Delta-2)\\
&\quad -16d(\Delta-2)^2(3\Delta+1)+64\Delta\left(\Delta^2-3\Delta+2\right)\Big]a_1^{(1,1)}\\
&\quad +\frac{1}{16}(d-4)(\Delta-2)\left(d^2+2d(2\Delta-5)-8\Delta+8\right)a_2^{(1,1)}\\
&\quad +\frac{1}{6}(\Delta-2)\left(-6d^2(2\Delta-5)-3d^3+48d(\Delta-1)+8(\Delta-1)\Delta\right)b_{1;1}^{(1,1)}
\end{aligned}
\tag{E.2.15}
$$

The corresponding coefficients for B_2 are given in terms of the coefficients of B_1 as

$$\begin{aligned}
b_{2;0}^{(0,1)} &= -b_{1;0}^{(1,0)} \quad ; \quad b_{2;0}^{(0,0)} = b_{1;0}^{(0,0)} \\
b_{2;1}^{(1,2)} &= -b_{1;1}^{(2,1)} \quad ; \quad b_{2;1}^{(0,2)} = b_{1;1}^{(2,0)} \quad ; \quad b_{2;1}^{(1,1)} = b_{1;1}^{(1,1)} \\
b_{2;1}^{(1,0)} &= -b_{1;1}^{(0,1)} \quad ; \quad b_{2;1}^{(0,1)} = -b_{1;1}^{(1,0)} \quad ; \quad b_{2;1}^{(0,0)} = b_{1;1}^{(0,0)}
\end{aligned} \tag{E.2.16}$$

E.3 Massive spin-2 Btb propagator in Poincaré Coordinates

In section 7.3, we obtained the bulk-to-boundary propagator in the Fefferman Graham coordinates. Here, we shall give the expressions in the Poincaré coordinates which are useful in the evaluation of the 3-point function. Before giving the results in the Poincaré coordinates, we first note some useful relations involving the massive spin-2 Btb propagator. The fields ϕ_{MN} and ϕ^*_{MN} are treated as independent fields and their Fourier transforms along the boundary directions are defined in the same way, i.e.,

$$\phi_{MN}(z,\mathbf{k}) = \int d^dx\, e^{-ik\cdot x}\phi_{MN}(z,\mathbf{x}) \quad ; \qquad \phi^*_{MN}(z,\mathbf{k}) = \int d^dx\, e^{-ik\cdot x}\phi^*_{MN}(z,\mathbf{x})\,. \tag{E.3.17}$$

For the Btb propagators we have

$$K_{MN}{}^{\mu\nu}(z,\mathbf{k}) = \int d^dx\, e^{-ik\cdot(x-y)}K_{MN}{}^{\mu\nu}(z,\mathbf{x};\mathbf{y})\,, \tag{E.3.18}$$

with the Btb propagator defined by

$$\phi_{MN}(z,\mathbf{k}) = K_{MN}{}^{\mu\nu}(z,\mathbf{k})\hat{\phi}^{(0)}_{\mu\nu}(\mathbf{k})\,. \tag{E.3.19}$$

The field ϕ^*_{MN} satisfies the same equation of motion as ϕ_{MN} and hence has the same bulk-to-boundary propagator, i.e.,

$$\phi^*_{MN}(z,\mathbf{k}) = \bar{K}_{MN}{}^{\kappa\gamma}(z,\mathbf{k})\hat{\phi}^{*(0)}_{\kappa\gamma}(\mathbf{k}) \quad ; \qquad \bar{K}_{MN}{}^{\kappa\gamma}(z,\mathbf{k}) = K_{MN}{}^{\kappa\gamma}(z,\mathbf{k})\,. \tag{E.3.20}$$

The Btb propagator also satisfies the following relations

$$\begin{aligned}
K_{\mu\nu}{}^{\kappa\gamma}(z,-\mathbf{k}) &= K_{\mu\nu}{}^{\kappa\gamma}(z,\mathbf{k}) = K^*_{\mu\nu}{}^{\kappa\gamma}(z,\mathbf{k})\,, \\
K_{zz}{}^{\kappa\gamma}(z,-\mathbf{k}) &= K_{zz}{}^{\kappa\gamma}(z,k) = K^*_{zz}{}^{\kappa\gamma}(z,\mathbf{k})\,, \\
K_{z\mu}{}^{\kappa\gamma}(z,-\mathbf{k}) &= -K_{z\mu}{}^{\kappa\gamma}(z,\mathbf{k}) = K^*_{z\mu}{}^{\kappa\gamma}(z,\mathbf{k})\,.
\end{aligned} \tag{E.3.21}$$

The Btb propagator in the Poincaré z coordinates is related to the Btb propagator in the Fefferman Graham ρ coordinates as follows

$$\begin{aligned}
K_{zz}{}^{\tau\sigma}(z;\mathbf{k}) &= \left(\frac{\partial\rho}{\partial z}\right)^2 K_{\rho\rho}{}^{\tau\sigma}(\rho(z);\mathbf{k}),\\
K_{z\mu}{}^{\tau\sigma}(z;\mathbf{k}) &= \left(\frac{\partial\rho}{\partial z}\right) K_{\rho\mu}{}^{\tau\sigma}(\rho(z);\mathbf{k}),\\
K_{\mu\nu}{}^{\tau\sigma}(z;\mathbf{k}) &= K_{\mu\nu}{}^{\tau\sigma}(\rho(z);\mathbf{k}).
\end{aligned} \tag{E.3.22}$$

By explicitly solving the equations in the Poincaré coordinates or using the above relations, we find that the Btb propagators for a massive spin-2 field in the Poincaré coordinates are given by

$$\begin{aligned}
K_{\mu\nu}{}^{\tau\sigma}(z;\mathbf{k}) &= \frac{L^{-\gamma}}{\alpha_1}\frac{z^{\frac{d-4}{2}}}{L^{\frac{d-4}{4}}}\Bigg[\delta^\tau_\mu\delta^\sigma_\nu K_\beta(kz) - \frac{4z\delta_{\mu\nu}k^\tau k^\sigma}{k(d+2\beta)(d+2\beta-2)}K_{\beta-1}(kz) + \frac{4zk^\tau\delta_{(\mu}{}^{\sigma}k_{\nu)}}{k(d+2\beta)}K_{\beta-1}(kz)\\
&\quad + \frac{4k_\mu k_\nu k^\tau k^\sigma}{k^4(d+2\beta)(d+2\beta-2)}\Big\{(k^2\rho L + 4\beta(\beta-1))K_\beta(kz) - 2k(\beta-1)zK_{\beta+1}(kz)\Big\}\Bigg],\\
K_{z\mu}{}^{\tau\sigma}(z;\mathbf{k}) &= \frac{2\,L^{-\gamma}\,ik^\tau}{\alpha_1(d+2\beta)}\left(\delta^\sigma_\mu K_\beta(kz) + \frac{2zk^\sigma k_\mu}{k(2\beta+d-2)}K_{\beta-1}(kz)\right)\frac{z^{\frac{d-2}{2}}}{L^{\frac{d-4}{4}}},\\
K_{zz}{}^{\tau\sigma}(z;\mathbf{k}) &= -\frac{4\,L^{-\gamma}\,k^\mu k^\nu}{\alpha_1(d+2\beta)(d+2\beta-2)}\frac{z^{\frac{d}{2}}}{L^{\frac{d-4}{4}}}K_\beta(kz)\,,
\end{aligned} \tag{E.3.23}$$

where

$$\alpha_1 = 2^{\beta-1}\Gamma(\beta)k^{-\beta}L^{-\frac{\beta}{2}} \quad ; \qquad \gamma = \frac{d}{4} - \frac{\beta}{2} - 1 \quad ; \qquad \beta = \Delta - \frac{d}{2}. \tag{E.3.24}$$

E.4 Some details of 3-point function computation

The one point function of the gauge field computed using the holographic renormalization procedure is given by [147]

$$\langle \mathcal{J}^\mu(\mathbf{x})\rangle = \delta^{\mu\tau}\int d^dy\,dw\sqrt{G}\,\mathbb{K}_\tau{}^\nu(w;\mathbf{x},\mathbf{y})J_\nu(\mathbf{y},w)\,. \tag{E.4.25}$$

Fourier transforming the above expression, the 3-point function involving the conserved current and two non-conserved operators is given by taking the derivative with respect to the sources of the boundary operators as

$$\begin{aligned}
\langle O^{*\alpha\beta}(\mathbf{p}_1)\mathcal{J}^\kappa(\mathbf{p}_2)O^{\gamma\delta}(\mathbf{p}_3)\rangle &= (2\pi)^d(2\pi)^d\frac{\delta^2\langle\mathcal{J}^\kappa(\mathbf{p}_2)\rangle}{\delta\hat{\phi}^{*(0)}_{\alpha\beta}(-\mathbf{p}_1)\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_3)}\\
&= \frac{\delta^{\kappa\mu}(2\pi)^d(2\pi)^d\delta^2}{\delta\hat{\phi}^{*(0)}_{\alpha\beta}(-\mathbf{p}_1)\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_3)}\int_0^\infty d\sigma\sqrt{G}\,\mathbb{K}_{\mu\nu}(\sigma;\mathbf{p}_2)J^\nu_{(0)}(\sigma,\mathbf{p}_2)\,.
\end{aligned}$$

The $J^\nu_{(0)}$ is the boundary component of the gauge field current J^N evaluated on the free spin-2 field and can be expressed in the form

$$\begin{aligned} J^\nu_{(0)} \;=\; ig\Bigg[\Bigg\{&\frac{1}{2}\phi_{MP}\partial^\nu\phi^{*MP} - G^{00}\phi_{M0}\partial_0\phi^{*M\nu} - G^{\eta\sigma}\phi_{M\eta}\partial_\sigma\phi^{*M\nu} + \frac{2z}{L^2}\left(\phi^{*0\nu}\phi_{00} + \phi^{*\mu\nu}\phi_{\mu 0}\right) \\ &-\frac{\alpha}{\sqrt{G}}\partial_0\left(\sqrt{G}\,G^{00}\phi_{0P}\phi^{*P\nu}\right) - \alpha G^{\mu\sigma}\partial_\mu\left(\phi_{\sigma P}\phi^{*P\nu}\right)\Bigg\} - \phi_{MN}\leftrightarrow\phi^*_{MN}\Bigg] \;+\; O(\partial^3)\,, \end{aligned} \tag{E.4.26}$$

where we have used the auxiliary condition $\phi = 0$, which is valid at the free field level.

We shall now give the results of taking the functional derivatives for each of the terms in $J^\nu_{(0)}$ above. We consider each of the 6 terms inside the bracket in (E.4.26) (together with the corresponding terms obtained by the interchange $\phi_{MN}\leftrightarrow\phi^*_{MN}$).

First term: $J_1^\nu(x^0,x^\mu) = \frac{ig}{2}\left(\phi_{MK}\partial^\nu\phi^{*MK} - \phi^*{}_{MK}\partial^\nu\phi^{MK}\right)$

The Fourier transform of the first term gives

$$\begin{aligned} J_1^\nu(x^0,\mathbf{p}_2) &= \frac{ig}{2}\int d^dx e^{-ip_2\cdot x}\left(\phi_{MK}\partial^\nu\phi^{*MK} - \phi^*{}_{MK}\partial^\nu\phi^{MK}\right) \\ &= \frac{ig}{2}G^{MN}G^{KS}G^{\nu\lambda}\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}(2\pi)^d\delta^d(\mathbf{k}_2+\mathbf{k}_1-\mathbf{p}_2) \\ &\quad\Bigg[K_{MK}{}^{\gamma\delta}(\mathbf{k}_2)(ik_{1\lambda})\bar{K}_{NS}{}^{\alpha\beta}(\mathbf{k}_1) \\ &\quad -\bar{K}_{MK}{}^{\alpha\beta}(\mathbf{k}_1)ik_{2\lambda}K_{NS}{}^{\gamma\delta}(\mathbf{k}_2)\Bigg]\hat{\phi}^{(0)*}_{\alpha\beta}(\mathbf{k}_1)\hat{\phi}^{(0)}_{\gamma\delta}(\mathbf{k}_2)\,. \end{aligned} \tag{E.4.27}$$

For each term appearing in the sum over indices, the result of the integration is given by

$$\begin{aligned} &\frac{(2\pi)^d(2\pi)^d\delta^2}{\delta\hat{\phi}^*{}^{(0)}_{\alpha\beta}(-\mathbf{p}_1)\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_3)}\left(\frac{ig}{2}\phi_{00}\partial^\nu\phi^{*00} - (\phi^*\leftrightarrow\phi)\right) \\ &= -\frac{ig}{2}G^{00}G^{00}G^{\nu\lambda}(2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \\ &\quad\left[K_{00}{}^{\gamma\delta}(\mathbf{p}_3)ip_{1\lambda}K_{00}{}^{\alpha\beta}(\mathbf{p}_1) - K_{00}{}^{\alpha\beta}(\mathbf{p}_1)ip_{3\lambda}K_{00}{}^{\gamma\delta}(\mathbf{p}_3)\right], \end{aligned}$$

$$\begin{aligned} &\frac{(2\pi)^d(2\pi)^d\delta^2}{\delta\hat{\phi}^*{}^{(0)}_{\alpha\beta}(-\mathbf{p}_1)\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_3)}\left(ig\phi_{\mu 0}\partial^\nu\phi^{*\mu 0} - (\phi^*\leftrightarrow\phi)\right) \\ &= -ig\,G^{\mu\eta}G^{00}G^{\nu\lambda}(2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \\ &\quad\left[K_{\mu 0}{}^{\gamma\delta}(\mathbf{p}_3)ip_{1\lambda}K_{\eta 0}{}^{\alpha\beta}(\mathbf{p}_1) - K_{\mu 0}{}^{\alpha\beta}(\mathbf{p}_1)ip_{3\lambda}K_{\eta 0}{}^{\gamma\delta}(\mathbf{p}_3)\right], \end{aligned}$$

$$
\begin{aligned}
&\frac{(2\pi)^d(2\pi)^d\delta^2}{\delta\hat{\phi}^{*(0)}_{\alpha\beta}(-\mathbf{p}_1)\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_3)}\left(\frac{ig}{2}\phi_{\mu\kappa}\partial^\nu\phi^{*\mu\kappa}-(\phi^*\leftrightarrow\phi)\right)\\
&\quad= -\frac{ig}{2}G^{\mu\eta}G^{\kappa\sigma}G^{\nu\lambda}(2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\\
&\qquad\left[K_{\mu\kappa}{}^{\gamma\delta}(\mathbf{p}_3)ip_{1\lambda}K_{\eta\sigma}{}^{\alpha\beta}(\mathbf{p}_1)-K_{\mu\kappa}{}^{\alpha\beta}(\mathbf{p}_1)ip_{3\lambda}K_{\eta\sigma}{}^{\gamma\delta}(\mathbf{p}_3)\right],
\end{aligned}
$$

Second term: $J_2^\nu(x^0,x^\mu) = -ig\left(G^{00}\phi_{M0}\partial_0\phi^{*M\nu} - G^{00}\phi^*_{M0}\partial_0\phi^{M\nu}\right)$

The Fourier transform of the second term gives

$$
\begin{aligned}
J_2^\nu(x^0,\mathbf{p}_2) &= -igG^{00}\int d^dx e^{-ip_2.x}\left(\phi_{M0}\partial_0(G^{MK}G^{\lambda\nu}\phi^*_{K\lambda})-\phi^*_{M0}\partial_0(G^{MK}G^{\lambda\nu}\phi_{K\lambda})\right)\\
&= -igG^{00}\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}(2\pi)^d\delta^d(\mathbf{k}_2+\mathbf{k}_1-\mathbf{p}_2)\\
&\quad\left[K_{M0}{}^{\gamma\delta}(\mathbf{k}_2)\partial_0\left(G^{MK}G^{\lambda\nu}\bar{K}_{K\lambda}{}^{\alpha\beta}(\mathbf{k}_1)\right)\right.\\
&\quad\left.-\bar{K}_{M0}{}^{\alpha\beta}(\mathbf{k}_1)\partial_0\left(G^{MK}G^{\lambda\nu}K_{K\lambda}{}^{\gamma\delta}(\mathbf{k}_2)\right)\right]\hat{\phi}^{(0)*}_{\alpha\beta}(\mathbf{k}_1)\hat{\phi}^{(0)}_{\gamma\delta}(\mathbf{k}_2)\,, \qquad \text{(E.4.28)}
\end{aligned}
$$

For different terms, the result of the integration is given by

$$
\begin{aligned}
&\frac{(2\pi)^d(2\pi)^d\delta^2}{\delta\hat{\phi}^{*(0)}_{\alpha\beta}(-\mathbf{p}_1)\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_3)}\left(-igG^{00}\phi_{00}\partial_0\phi^{*0\nu}-(\phi^*\leftrightarrow\phi)\right)\\
&\quad= igG^{00}(2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\\
&\qquad\left[K_{00}{}^{\gamma\delta}(\mathbf{p}_3)\partial_0\left(G^{00}G^{\lambda\nu}K_{0\lambda}{}^{\alpha\beta}(\mathbf{p}_1)\right)-K_{00}{}^{\alpha\beta}(\mathbf{p}_1)\partial_0\left(G^{00}G^{\lambda\nu}K_{0\lambda}{}^{\gamma\delta}(\mathbf{p}_3)\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&\frac{(2\pi)^d(2\pi)^d\delta^2}{\delta\hat{\phi}^{*(0)}_{\alpha\beta}(-\mathbf{p}_1)\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_3)}\left(-igG^{00}\phi_{\mu0}\partial_0\phi^{*\mu\nu}-(\phi^*\leftrightarrow\phi)\right)\\
&\quad= igG^{00}(2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\\
&\qquad\left[K_{\mu0}{}^{\gamma\delta}(\mathbf{p}_3)\partial_0\left(G^{\mu\kappa}G^{\lambda\nu}K_{\kappa\lambda}{}^{\alpha\beta}(\mathbf{p}_1)\right)-K_{\mu0}{}^{\alpha\beta}(\mathbf{p}_1)\partial_0\left(G^{\mu\kappa}G^{\lambda\nu}K_{\kappa\lambda}{}^{\gamma\delta}(\mathbf{p}_3)\right)\right],
\end{aligned}
$$

Third term: $J_3^\nu(x^0,x^\mu) = -ig\left(G^{\eta\sigma}\phi_{M\eta}\partial_\sigma\phi^{*M\nu} - G^{\eta\sigma}\phi^*_{M\eta}\partial_\sigma\phi^{M\nu}\right)$

The Fourier transform of the 3rd term gives

$$\begin{aligned}
J_3^\nu(x^0,\mathbf{p}_2) &= -igG^{\eta\sigma}G^{\lambda\nu}G^{MK}\int d^dx e^{-ip_2.x}\left(\phi_{M\eta}\partial_\sigma\phi^*_{K\lambda} - \phi^*_{M\eta}\partial_\sigma\phi_{K\lambda}\right) \\
&= -igG^{\eta\sigma}G^{\lambda\nu}G^{MK}\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}(2\pi)^d\delta^d(\mathbf{k}_2+\mathbf{k}_1-\mathbf{p}_2) \\
&\quad \Bigg[K_{M\eta}{}^{\gamma\delta}(\mathbf{k}_2)(ik_{1\sigma})\bar{K}_{K\lambda}{}^{\alpha\beta}(\mathbf{k}_1) \\
&\quad -\bar{K}_{M\eta}{}^{\alpha\beta}(\mathbf{k}_1)ik_{2\sigma}K_{K\lambda}{}^{\gamma\delta}(\mathbf{k}_2)\Bigg]\hat{\phi}^{(0)*}_{\alpha\beta}(\mathbf{k}_1)\hat{\phi}^{(0)}_{\gamma\delta}(\mathbf{k}_2)\,, \qquad \text{(E.4.29)}
\end{aligned}$$

The different terms of the above expression are given by

$$\begin{aligned}
&\frac{(2\pi)^d(2\pi)^d\delta^2}{\delta\hat{\phi}^{*(0)}_{\alpha\beta}(-\mathbf{p}_1)\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_3)}\left(-igG^{\eta\sigma}\phi_{0\eta}\partial_\sigma\phi^{*0\nu} - (\phi^*\leftrightarrow\phi)\right) \\
&= igG^{\eta\sigma}G^{\lambda\nu}G^{00}(2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \\
&\quad \left[K_{0\eta}{}^{\gamma\delta}(\mathbf{p}_3)(ip_{1\sigma})K_{0\lambda}{}^{\alpha\beta}(\mathbf{p}_1) - K_{0\eta}{}^{\alpha\beta}(\mathbf{p}_1)(ip_{3\sigma})K_{0\lambda}{}^{\gamma\delta}(\mathbf{p}_3)\right],
\end{aligned}$$

$$\begin{aligned}
&\frac{(2\pi)^d(2\pi)^d\delta^2}{\delta\hat{\phi}^{*(0)}_{\alpha\beta}(-\mathbf{p}_1)\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_3)}\left(-igG^{\eta\sigma}\phi_{\mu\eta}\partial_\sigma\phi^{*\mu\nu} - (\phi^*\leftrightarrow\phi)\right) \\
&= igG^{\eta\sigma}G^{\lambda\nu}G^{\mu\kappa}(2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \\
&\quad \left[K_{\mu\eta}{}^{\gamma\delta}(\mathbf{p}_3)(ip_{1\sigma})K_{\kappa\lambda}{}^{\alpha\beta}(\mathbf{p}_1) - K_{\mu\eta}{}^{\alpha\beta}(\mathbf{p}_1)(ip_{3\sigma})K_{\kappa\lambda}{}^{\gamma\delta}(\mathbf{p}_3)\right],
\end{aligned}$$

Fourth term: $J_4^\nu(x^0,x^\mu) = ig\left(\phi^{*MN}\phi_{MQ} - \phi^{MN}\phi^*_{MQ}\right)$

The Fourier transform of the 4th term gives

$$\begin{aligned}
J_4^\nu(x^0,\mathbf{p}_2) &= ig\,G^{NS}G^{MK}\int d^dx e^{-ip_2.x}\left(\phi^*_{KS}\phi_{MQ} - \phi_{KS}\phi^*_{MQ}\right) \\
&= ig\,G^{NS}G^{MK}\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}(2\pi)^d\delta^d(\mathbf{k}_2+\mathbf{k}_1-\mathbf{p}_2)\Bigg[\bar{K}_{KS}{}^{\alpha\beta}(\mathbf{k}_1)K_{MQ}{}^{\gamma\delta}(\mathbf{k}_2) \\
&\quad -K_{KS}{}^{\gamma\delta}(\mathbf{k}_2)\bar{K}_{MQ}{}^{\alpha\beta}(\mathbf{k}_1)\Bigg]\hat{\phi}^{(0)*}_{\alpha\beta}(\mathbf{k}_1)\hat{\phi}^{(0)}_{\gamma\delta}(\mathbf{k}_2)\,, \qquad \text{(E.4.30)}
\end{aligned}$$

The different terms of the above expression are given by

$$\frac{(2\pi)^d(2\pi)^d\delta^2}{\delta\hat{\phi}^{*(0)}_{\alpha\beta}(-\mathbf{p}_1)\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_3)}\left(ig\,\phi^{*0\nu}\phi_{00}-(\phi^*\leftrightarrow\phi)\right)$$
$$=-ig\,G^{00}G^{\nu\lambda}(2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\left[K_{0\lambda}{}^{\alpha\beta}(\mathbf{p}_1)K_{00}{}^{\gamma\delta}(\mathbf{p}_3)-K_{0\lambda}{}^{\gamma\delta}(\mathbf{p}_3)K_{00}{}^{\alpha\beta}(\mathbf{p}_1)\right],$$

$$\frac{(2\pi)^d(2\pi)^d\delta^2}{\delta\hat{\phi}^{*(0)}_{\alpha\beta}(-\mathbf{p}_1)\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_3)}\left(ig\,\phi^{*\mu\nu}\phi_{\mu0}-(\phi^*\leftrightarrow\phi)\right)$$
$$=-ig\,G^{\mu\kappa}G^{\nu\lambda}(2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\left[K_{\kappa\lambda}{}^{\alpha\beta}(\mathbf{p}_1)K_{\mu0}{}^{\gamma\delta}(\mathbf{p}_3)-K_{\kappa\lambda}{}^{\gamma\delta}(\mathbf{p}_3)K_{\mu0}{}^{\alpha\beta}(\mathbf{p}_1)\right],$$

Fifth term: $J_5^\nu(x^0,x^\mu)=-\frac{ig\,\alpha}{\sqrt{G}}\partial_0\left(\sqrt{G}G^{00}\left(\phi_{0P}\phi^{*P\nu}-\phi^*_{0P}\phi^{P\nu}\right)\right)$

The Fourier transform of the 5th term gives

$$\begin{aligned}
J_5^\nu(x^0,\mathbf{p}_2) &= -\frac{ig\,\alpha}{\sqrt{G}}\int d^dx e^{-ip_2.x}\partial_0\left(\sqrt{G}G^{00}G^{\lambda\nu}G^{MP}\left(\phi_{0P}\phi^*_{M\lambda}-\phi^*_{0P}\phi_{M\lambda}\right)\right)\\
&= -\frac{ig\,\alpha}{\sqrt{G}}\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}(2\pi)^d\delta^d(\mathbf{k}_2+\mathbf{k}_1-\mathbf{p}_2)\partial_0\\
&\quad\left[\sqrt{G}G^{00}G^{\lambda\nu}G^{MP}\left(K_{0P}{}^{\gamma\delta}(\mathbf{k}_2)\bar{K}_{M\lambda}^{\alpha\beta}(\mathbf{k}_1)\right.\right.\\
&\quad\left.\left.-\bar{K}_{0P}{}^{\alpha\beta}(\mathbf{k}_1)K_{M\lambda}^{\gamma\delta}(\mathbf{k}_2)\right)\right]\hat{\phi}^{(0)*}_{\alpha\beta}(\mathbf{k}_1)\hat{\phi}^{(0)}_{\gamma\delta}(\mathbf{k}_2)\,,
\end{aligned}\tag{E.4.31}$$

The different terms are given by

$$\begin{aligned}
&\frac{(2\pi)^d(2\pi)^d\delta^2}{\delta\hat{\phi}^{*(0)}_{\alpha\beta}(-\mathbf{p}_1)\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_3)}\left[-\frac{ig\,\alpha}{\sqrt{G}}\partial_0\left(\sqrt{G}G^{00}\left(\phi_{00}\phi^{*0\nu}-\phi^*_{00}\phi^{0\nu}\right)\right)\right]\\
&= \frac{ig\,\alpha}{\sqrt{G}}(2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\\
&\quad\partial_0\left(\sqrt{G}G^{00}G^{00}G^{\lambda\nu}\left[K_{0\lambda}{}^{\alpha\beta}(\mathbf{p}_1)K_{00}{}^{\gamma\delta}(\mathbf{p}_3)-K_{0\lambda}{}^{\gamma\delta}(\mathbf{p}_3)K_{00}{}^{\alpha\beta}(\mathbf{p}_1)\right]\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{(2\pi)^d(2\pi)^d\delta^2}{\delta\hat{\phi}^{*(0)}_{\alpha\beta}(-\mathbf{p}_1)\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_3)}\left[-\frac{ig\,\alpha}{\sqrt{G}}\partial_0\left(\sqrt{G}G^{00}\left(\phi_{\mu0}\phi^{*\mu\nu}-\phi^*_{\mu0}\phi^{\mu\nu}\right)\right)\right]\\
&= \frac{ig\,\alpha}{\sqrt{G}}(2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\\
&\quad\partial_0\left(\sqrt{G}G^{\mu\kappa}G^{00}G^{\lambda\nu}\left[K_{\kappa\lambda}{}^{\alpha\beta}(\mathbf{p}_1)K_{\mu0}{}^{\gamma\delta}(\mathbf{p}_3)-K_{\kappa\lambda}{}^{\gamma\delta}(\mathbf{p}_3)K_{\mu0}{}^{\alpha\beta}(\mathbf{p}_1)\right]\right),
\end{aligned}$$

Sixth term: $J_6^\nu(x^0,x^\mu) = -ig\,\alpha\, G^{\kappa\sigma}\partial_\kappa\left(\phi_{\sigma M}\phi^{*M\nu} - \phi^*_{\sigma M}\phi^{M\nu}\right)$

Finally, the Fourier transform of the 6th term is

$$\begin{aligned}
J_6^\nu(x^0,\mathbf{p}_2) &= -G^{MD}G^{\nu\lambda}\int d^dx e^{-ip_2x}\left(ig\alpha G^{\kappa\sigma}\partial_\kappa\left(\phi_{\sigma M}\phi^*{}_{D\lambda} - \phi^*_{\sigma M}\phi_{D\lambda}\right)\right) \\
&= -G^{MD}G^{\nu\lambda}\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}(2\pi)^d\delta^d(\mathbf{k}_2+\mathbf{k}_1-\mathbf{p}_2)\Big[ig\alpha G^{\kappa\sigma}i(k_2+k_1)_\kappa \\
&\quad \Big(K_{\sigma M}{}^{\gamma\delta}(\mathbf{k}_2)\bar{K}_{D\lambda}{}^{\alpha\beta}(\mathbf{k}_1) - \bar{K}_{\sigma M}{}^{\alpha\beta}(\mathbf{k}_1)K_{D\lambda}{}^{\gamma\delta}(\mathbf{k}_2)\Big)\Big]\hat{\phi}^{(0)}{}^*_{\alpha\beta}(k_1)\hat{\phi}^{(0)}_{\gamma\delta}(\mathbf{k}_2)\,,
\end{aligned} \tag{E.4.32}$$

with different terms given by

$$\begin{aligned}
&\frac{(2\pi)^d(2\pi)^d\delta^2}{\delta\hat{\phi}^*{}^{(0)}_{\alpha\beta}(-\mathbf{p}_1)\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_3)}\left[-ig\alpha G^{\kappa\sigma}\partial_\kappa\left(\phi_{\sigma 0}\phi^{*0\nu} - \phi^*_{\sigma 0}\phi^{0\nu}\right)\right] \\
&= ig\alpha\, G^{\kappa\sigma}G^{00}G^{\nu\lambda}(2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)i(p_1+p_3)_\kappa \\
&\quad \Big(K_{\sigma 0}{}^{\gamma\delta}(\mathbf{p}_3)K_{0\lambda}{}^{\alpha\beta}(\mathbf{p}_1) - K_{\sigma 0}{}^{\alpha\beta}(p_1)K_{0\lambda}{}^{\gamma\delta}(\mathbf{p}_3)\Big),
\end{aligned}$$

$$\begin{aligned}
&\frac{(2\pi)^d(2\pi)^d\delta^2}{\delta\hat{\phi}^*{}^{(0)}_{\alpha\beta}(-\mathbf{p}_1)\delta\hat{\phi}^{(0)}_{\gamma\delta}(-\mathbf{p}_3)}\left[-ig\alpha G^{\kappa\sigma}\partial_\kappa\left(\phi_{\sigma\mu}\phi^{*\mu\nu} - \phi^*_{\sigma\mu}\phi^{\mu\nu}\right)\right] \\
&= ig\alpha\, G^{\kappa\sigma}G^{\mu\eta}G^{\nu\lambda}(2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)i(p_1+p_3)_\kappa \\
&\quad \Big(K_{\sigma\mu}{}^{\gamma\delta}(\mathbf{p}_3)K_{\eta\lambda}{}^{\alpha\beta}(\mathbf{p}_1) - K_{\sigma\mu}{}^{\alpha\beta}(\mathbf{p}_1)K_{\eta\lambda}{}^{\gamma\delta}(\mathbf{p}_3)\Big)\,,
\end{aligned}$$

E.5 Some details of flat limit computations

In this section, we give some details of the flat space limit. In taking the flat limit, we need to deal with the derivative of the Btb propagators. In subsection E.5.1, we shall obtain an expression for the flat limit of the derivative of the Btb propagators. In subsection E.5.2, we shall determine the contributions of different terms in the expression of the 3-point function to the flat limit.

E.5.1 Flat limit of derivatives of the Btb propagators

To obtain the flat limit of the derivatives of the Btb propagators, we start with the following identity satisfied by the modified Bessel function of the second kind

$$\partial_z(z^mK_n(kz)) = (m-n)z^{m-1}K_n(kz) - kz^mK_{n-1}(kz), \tag{E.5.33}$$

Schematically, the general form of the bulk-to-boundary propagator in (E.3.23) has the form

$$K_{MN}{}^{AB}(z,\mathbf{k}) = \sum_n Q_{MN}{}^{AB}(n)\,z^{a(n)}\,K_{b(n)}(kz)\,, \tag{E.5.34}$$

where n is a summation index, Qs are independent of z, $a(n)$ is a d dependent real number that does *not* depend on any other parameter, while $b(n)$ depends on $\beta = \Delta - \frac{d}{2}$. For the flat limit, the dependence on z is important, and hence the exact form of the Qs or their tensor structure is not relevant. We also note that the large L behaviours of $a(n)$ and $b(n)$ are given by

$$a(n) \;=\; \mathcal{O}(1) \quad ; \qquad b(n) \;=\; mL + \mathcal{O}(1). \tag{E.5.35}$$

Now, taking the derivative of (E.5.34) and applying (E.5.33) gives

$$\frac{d}{dz}\left(K_{MN}{}^{AB}\right) = \sum_n Q_{MN}{}^{AB}(n)\left[(a(n)-b(n))\; z^{a(n)-1}K_{b(n)}(kz) - kz^{a(n)}K_{b(n)-1}(kz)\right]. \tag{E.5.36}$$

We now recall that in the large L limit, the modified Bessel function of the second kind behaves as [147]

$$K_{\Delta-\frac{d}{2}+c}(kz) = \left(\frac{\pi}{2EL}\right)^{\frac{1}{2}}\left(\frac{k}{m+E}\right)^{-mL-c} e^{-EL-E\tau}\left(1+O\left(\frac{1}{L}\right)\right). \tag{E.5.37}$$

The only c dependence is in the exponent of the second term. Additionally, we also note that in the large L limit, we have

$$z = Le^{\frac{\tau}{L}} = L+\tau+O\left(\frac{1}{L}\right). \tag{E.5.38}$$

Using the above identities together with (E.5.35) gives, in the large L limit

$$\begin{aligned}\frac{d}{dz}\left(K_{MN}{}^{AB}\right) &= \sum_n Q_{MN}{}^{AB}L^{a(n)}\left(-mK_{b(n)}(kz)-kK_{b(n)-1}(kz)\right)+O\left(\frac{1}{L}\right)\\ &= -E\sum_n Q_{MN}{}^{AB}L^{a(n)}\left(\frac{\pi}{2EL}\right)^{\frac{1}{2}}\left(\frac{k}{m+E}\right)^{-b(n)} e^{-E(L+\tau)}+O\left(\frac{1}{L}\right),\end{aligned} \tag{E.5.39}$$

where, in going to the second line, we have used equation (E.5.37) and the identity

$$m+\frac{k^2}{m+E} \;=\; m+\frac{E^2-m^2}{m+E} \;=\; E\,. \tag{E.5.40}$$

Next, following the same manipulations, we note that applying the flat limit directly to the bulk-to-boundary propagators gives

$$K_{MN}{}^{AB} = \sum_n Q_{MN}{}^{AB}L^{a(n)}\left(\frac{\pi}{2EL}\right)^{\frac{1}{2}}\left(\frac{k}{m+E}\right)^{-b(n)} e^{-E(L+\tau)}+O\left(\frac{1}{L}\right). \tag{E.5.41}$$

Comparing (E.5.39) with (E.5.41), we see that, at the leading order in L, the two expressions agree apart from a factor of $-E$. In other words, if $X_{MN}{}^{AB}$ is the flat limit of the bulk-to-boundary propagators and $Y_{MN}{}^{AB}$ is the flat limit of its derivative, then in coordinate space, we simply have

$$Y_{MN}{}^{AB} \;=\; \partial_\tau X_{MN}{}^{AB}+\mathcal{O}\left(\frac{1}{L}\right) \;=\; \partial_z X_{MN}{}^{AB}+\mathcal{O}\left(\frac{1}{L}\right), \tag{E.5.42}$$

where, in writing the second equality, we have used

$$\lim_{L\to\infty} \partial_z = \lim_{L\to\infty} e^{-\tau/L}\partial_\tau = \partial_\tau\,, \tag{E.5.43}$$

which shows that in the flat limit $L \to \infty$, taking the derivative with respect to z and τ coincide.

The above result can be Wick rotated to Lorentzian signature by replacing $E \to -iE$ (or $\tau = it$) which gives

$$Y^{(L)\,AB}_{MN} = (-iE)X^{(L)\,AB}_{MN} \;+\; \mathcal{O}\left(\frac{1}{L}\right). \tag{E.5.44}$$

E.5.2 Flat limit of the 3-point function

In this subsection, we consider each term of the current described in appendix E.4 and consider their contribution to the bulk 3-point function in the flat limit. We shall work in the bulk τ coordinates. The 3-point function involves an integration over the bulk coordinate. In the limit $L \to \infty$, the integrand contains a factor $e^{-\tau(E_1+E_2+E_3)}$ where $E_{1,3} = \sqrt{p_{1,3}^2 + m^2} \equiv -iq^0_{1,3}$ and $E_2 = p_2 \equiv -iq^0_2$ (see equations (7.6.119) and (7.6.139)). This gives the desired energy conserving delta function as

$$\begin{aligned}\int_{-\infty}^{\infty} d\tau\, e^{-\tau(E_1+E_2+E_3)}e^{-L(E_1+E_2+E_3)} &= \int_{-\infty}^{\infty} d\tau\, e^{i\tau(q^0_1+q^0_2+q^0_3)}e^{iL(q^0_1+q^0_2+q^0_3)} \\ &= 2\pi\delta(q^0_1+q^0_2+q^0_3) \\ &= (2\pi i)\,\delta(E_1+E_2+E_3)\end{aligned} \tag{E.5.45}$$

where we have set the exponent involving L to 1 on the support of the delta function and the last equality corresponds to the Lorentzian signature.

We now consider the $L \to \infty$ limit of each term in appendix E.4. The contributions to the flat limit are as follows

First term: $J_1^\nu(x^0, x^\mu) = \frac{ig}{2}\left(\phi_{MK}\partial^\nu\phi^{*MK} - \phi^*{}_{MK}\partial^\nu\phi^{MK}\right)$

The 3 terms of J_1^ν contribute as follows

$$\begin{aligned}&\lim_{L,\Delta\to\infty} \epsilon_{1\alpha\beta}\,\epsilon_\kappa\,\epsilon_{3\gamma\delta}\,\langle O^{*\alpha\beta}(\mathbf{p}_1)J^\kappa_{1,1}(\mathbf{p}_2)O^{\gamma\delta}(\mathbf{p}_3)\rangle \\ &= -g\,\epsilon_{1\alpha\beta}\,\epsilon_\kappa\,\epsilon_{3\gamma\delta}\int_{-\infty}^{\infty} d\tau\left(\frac{k_1^\alpha\,k_1^\beta\,k_3^\gamma\,k_3^\delta\,(k_3^\nu - k_1^\nu)\,\pi^\kappa{}_\nu}{2\,m^4\,\sqrt{Z_A Z_{\phi_1} Z_{\phi_3}}}\,e^{-(E_1+E_2+E_3)(\tau+L)}\right)(2\pi)^d\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \\ &= -i(2\pi)^{d+1}\delta(E_1+E_2+E_3)\delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\frac{g}{2\sqrt{Z_A Z_{\phi_1} Z_{\phi_3}}}(\varepsilon_{100}\,\varepsilon_3^{00})\{(q_3-q_1)\cdot\varepsilon_2\}\end{aligned} \tag{E.5.46}$$

$$
\begin{aligned}
&\lim_{L,\Delta\to\infty} \epsilon_{1\alpha\beta}\, \epsilon_\kappa\, \epsilon_{3\gamma\delta} \langle O^{*\alpha\beta}(\mathbf{p}_1) J^\kappa_{1,2}(\mathbf{p}_2) O^{\gamma\delta}(\mathbf{p}_3)\rangle \\
&= \quad -g\,(2\pi)^d \delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\, \epsilon_{1\alpha\beta}\, \epsilon_\kappa\, \epsilon_{3\gamma\delta} \frac{-1}{\sqrt{Z_A Z_{\phi_1} Z_{\phi_3}}} \\
&\qquad \int_{-\infty}^{\infty} d\tau \left[\left(\frac{k_1^\alpha \tilde{\pi}^\beta_{1\,\mu} + k_1^\beta \tilde{\pi}^\alpha_{1\,\mu}}{2m} \right) \left(\frac{k_3^\gamma \tilde{\pi}_3^{\delta\mu} + k_3^\delta \tilde{\pi}_3^{\gamma\mu}}{2m} \right) (k_3^\nu - k_1^\nu) \pi^\kappa{}_\nu \right] e^{-(E_1+E_2+E_3)(\tau+L)} \\
&= \quad -(2\pi)^{d+1}\, \delta(E_1+E_2+E_3) \delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \frac{ig}{\sqrt{Z_A Z_{\phi_1} Z_{\phi_3}}} (\varepsilon_{10\mu}\, \varepsilon_3^{0\mu})(q_3-q_1)\cdot\varepsilon_2
\end{aligned} \tag{E.5.47}
$$

$$
\begin{aligned}
&\lim_{L,\Delta\to\infty} \epsilon_{1\alpha\beta}\, \epsilon_\kappa\, \epsilon_{3\gamma\delta}\, \langle O^{*\alpha\beta}(\mathbf{p}_1) J^\kappa_{1,3}(\mathbf{p}_2) O^{\gamma\delta}(\mathbf{p}_3)\rangle \\
&= \quad -g\,(2\pi)^d \delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\, \epsilon_{1\alpha\beta}\, \epsilon_\kappa\, \epsilon_{3\gamma\delta} \frac{1}{8\,\sqrt{Z_A Z_{\phi_1} Z_{\phi_3}}} \\
&\qquad \int_{-\infty}^{\infty} d\tau \bigg(\pi^\kappa{}_\nu (k_3^\nu - k_1^\nu) \Big((\tilde{\pi}_1^{\alpha m} \tilde{\pi}_1^{\beta a} + \tilde{\pi}_1^{\alpha a} \tilde{\pi}_1^{\beta m})(\tilde{\pi}^\gamma_{3\,m} \tilde{\pi}^\delta_{3a} + \tilde{\pi}^\gamma_{3\,a} \tilde{\pi}^\delta_{3m}) \Big)\, e^{-(E_1+E_2+E_3)(\tau+L)} \bigg) \\
&= \quad -(2\pi)^{d+1}\, \delta(E_1+E_2+E_3) \delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \frac{ig}{2\,\sqrt{Z_A Z_{\phi_1} Z_{\phi_3}}} (\varepsilon_{1\mu\nu}\, \varepsilon_3^{\mu\nu}) \{(q_3-q_1)\cdot\varepsilon_2\}
\end{aligned} \tag{E.5.48}
$$

Summing the contributions (E.5.46), (E.5.47), and (E.5.48), we obtain

$$
\lim_{L,\Delta\to\infty} \epsilon_{1\alpha\beta}\, \epsilon_\kappa\, \epsilon_{3\gamma\delta}\, \langle O^{*\alpha\beta}(\mathbf{p}_1) J^\kappa_1(\mathbf{p}_2) O^{\gamma\delta}(\mathbf{p}_3)\rangle \;=\; -i\,\frac{g}{2} \Upsilon\, \varepsilon_{1ab}\, \varepsilon_3^{ab} (q_3-q_1)\cdot\varepsilon_2 \tag{E.5.49}
$$

where, we have defined

$$
\Upsilon \equiv (2\pi)^{d+1}\, \delta(E_1+E_2+E_3) \delta^d(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \frac{1}{\sqrt{Z_A Z_{\phi_1} Z_{\phi_3}}} \tag{E.5.50}
$$

Second term: $J_2^\nu(x^0, x^\mu) = -\,ig \left(G^{00} \phi_{M0} \partial_0 \phi^{*M\nu} - G^{00} \phi^*_{M0} \partial_0 \phi^{M\nu} \right)$

For the J_2 contribution, we follow the same procedure. In this case, we need to use the flat limit of the derivative of the Btb propagators derived in (E.5.44). In the flat limit, the two terms of J_2^ν give the following contributions

$$
\lim_{L,\Delta\to\infty} \epsilon_{1\alpha\beta}\, \epsilon_\kappa\, \epsilon_{3\gamma\delta}\, \langle O^{*\alpha\beta}(\mathbf{p}_1) J^\kappa_{2,1}(\mathbf{p}_2) O^{\gamma\delta}(\mathbf{p}_3)\rangle = -\,ig\Upsilon \Big(\varepsilon_2^\mu\, \varepsilon^1_{\mu 0}\, \varepsilon_3^{00}\, q_0^1 - (1 \leftrightarrow 3) \Big) \tag{E.5.51}
$$

$$
\lim_{L,\Delta\to\infty} \epsilon_{1\alpha\beta}\, \epsilon_\kappa\, \epsilon_{3\gamma\delta}\, \langle O^{*\alpha\beta}(\mathbf{p}_1) J^\kappa_{2,2}(\mathbf{p}_2) O^{\gamma\delta}(\mathbf{p}_3)\rangle = -ig\Upsilon \left(\varepsilon_2^\mu\, \varepsilon^1_{\mu\nu}\, \varepsilon_3^{\nu 0}\, q_0^1 - (1 \leftrightarrow 3) \right) \tag{E.5.52}
$$

In writing the above expressions, we have used $q^1_M = (-E, k_\mu) = (-E, k^\mu)$ for the flat space momenta in the Lorentzian signature.

Third term: $J_3^\nu(x^0, x^\mu) = -ig\left(G^{\eta\sigma}\phi_{M\eta}\partial_\sigma\phi^{*M\nu} - G^{\eta\sigma}\phi^*_{M\eta}\partial_\sigma\phi^{M\nu}\right)$

The two terms of J_3^ν give

$$\lim_{L,\Delta\to\infty}\epsilon_{1\alpha\beta}\,\epsilon_\kappa\,\epsilon_{3\gamma\delta}\,\langle O^{*\alpha\beta}(\mathbf{p}_1)J^\kappa_{3,1}(\mathbf{p}_2)O^{\gamma\delta}(\mathbf{p}_3)\rangle = -ig\Upsilon\left(\varepsilon_2^\mu\,\varepsilon^1_{\mu 0}\,\varepsilon_3^{0\sigma}\,q^1_\sigma - (1\leftrightarrow 3)\right),$$

$$\lim_{L,\Delta\to\infty}\epsilon_{1\alpha\beta}\,\epsilon_\kappa\,\epsilon_{3\gamma\delta}\,\langle O^{*\alpha\beta}(\mathbf{p}_1)J^\kappa_{3,2}(\mathbf{p}_2)O^{\gamma\delta}(\mathbf{p}_3)\rangle = -ig\Upsilon\left(\varepsilon_2^\mu\,\varepsilon^1_{\mu\nu}\,\varepsilon_3^{\nu\sigma}\,q^1_\sigma - (1\leftrightarrow 3)\right), \quad \text{(E.5.53)}$$

Fourth term: $J_4^\nu(x^0, x^\mu) = ig\left(\phi^{*MN}\phi_{MQ} - \phi^{MN}\phi^*_{MQ}\right)$

The two terms of J_4^ν give

$$\lim_{L,\Delta\to\infty}\epsilon_{1\alpha\beta}\,\epsilon_\kappa\,\epsilon_{3\gamma\delta}\,\langle O^{*\alpha\beta}(\mathbf{p}_1)J^\kappa_{4,1}(\mathbf{p}_2)O^{\gamma\delta}(\mathbf{p}_3)\rangle = \lim_{L\to\infty}\Upsilon\left[\frac{g}{L}\,\varepsilon_2^\mu\,\varepsilon^1_{\mu 0}\varepsilon_3^{00} - (1\leftrightarrow 3)\right] = 0,$$
$$\lim_{L,\Delta\to\infty}\varepsilon_{1\alpha\beta}\,\varepsilon_\kappa\,\varepsilon_{3\gamma\delta}\,\langle O^{*\alpha\beta}(\mathbf{p}_1)J^\kappa_{4,2}(\mathbf{p}_2)O^{\gamma\delta}(\mathbf{p}_3)\rangle = \lim_{L\to\infty}\Upsilon\left[\frac{g}{L}\,\varepsilon_2^\mu\,\varepsilon^1_{\mu\sigma}\,\varepsilon_3^{\sigma 0} - (1\leftrightarrow 3)\right] = 0. \quad \text{(E.5.54)}$$

This shows that the flat limit contribution from J_4 is subleading in L compared to the contributions from J_1, J_2 and J_3 (as well as J_5 and J_6 given below) and thus vanishes in the flat limit. This is not surprising because the J_4 piece comes *only* from the Christoffel connection terms in the action. Hence, in the flat limit, one would expect their contributions to vanish, which is precisely what happens.

Summing the contributions of J_2, J_3 and J_4, we get

$$\begin{aligned}&\lim_{L,\Delta\to\infty}\epsilon_{1\alpha\beta}\,\epsilon_\kappa\,\epsilon_{3\gamma\delta}\,\langle O^{*\alpha\beta}(\mathbf{p}_1)\left(J_2^\kappa(\mathbf{p}_2) + J_3^\kappa(\mathbf{p}_2) + J_4^\kappa(\mathbf{p}_2)\right)O^{\gamma\delta}(\mathbf{p}_3)\rangle\\ =\;& -ig\Upsilon\left(\varepsilon_2^a\,\varepsilon^1_{ab}\,\varepsilon_3^{bc}\,q^1_c - (1\leftrightarrow 3)\right)\end{aligned} \quad \text{(E.5.55)}$$

where have used the axial gauge condition $\varepsilon_2^0 = 0$.

Fifth term: $J_5^\nu(x^0, x^\mu) = -\frac{ig\,\alpha}{\sqrt{G}}\partial_0\left(\sqrt{G}G^{00}\left(\phi_{0P}\phi^{*P\nu} - \phi^*_{0P}\phi^{P\nu}\right)\right)$

The two terms of J_5^ν give

$$\lim_{L,\Delta\to\infty}\epsilon_{1\alpha\beta}\,\epsilon_\kappa\,\epsilon_{3\gamma\delta}\,\langle O^{*\alpha\beta}(\mathbf{p}_1)J^\kappa_{5,1}(\mathbf{p}_2)O^{\gamma\delta}(\mathbf{p}_3)\rangle = -ig\Upsilon\left(\alpha\,\varepsilon_2^\mu\,\varepsilon^1_{\mu 0}\,\varepsilon_3^{00}\,(q^1_0 + q^3_0) - (1\leftrightarrow 3)\right),$$

$$\lim_{L,\Delta\to\infty}\epsilon_{1\alpha\beta}\,\epsilon_\kappa\,\epsilon_{3\gamma\delta}\,\langle O^{*\alpha\beta}(\mathbf{p}_1)J^\kappa_{5,2}(\mathbf{p}_2)O^{\gamma\delta}(\mathbf{p}_3)\rangle = -ig\Upsilon\left(\alpha\,\varepsilon_2^\mu\,\varepsilon^1_{\mu\nu}\,\varepsilon_3^{\nu 0}\,(q^1_0 + q^3_0) - (1\leftrightarrow 3)\right), \quad \text{(E.5.56)}$$

Sixth term: $J_6^\nu(x^0, x^\mu) = -ig\,\alpha\, G^{\kappa\sigma}\partial_\kappa\left(\phi_{\sigma M}\phi^{*M\nu} - \phi^*_{\sigma M}\phi^{M\nu}\right)$

The two terms of J_6^ν give

$$\lim_{L,\Delta\to\infty} \epsilon_{1\alpha\beta}\,\epsilon_\kappa\,\epsilon_{3\gamma\delta}\,\langle O^{*\alpha\beta}(\mathbf{p}_1)J^\kappa_{6,1}(\mathbf{p}_2)O^{\gamma\delta}(\mathbf{p}_3)\rangle = -ig\Upsilon\left(\alpha\,\varepsilon_2^\mu\,\varepsilon^1_{\mu 0}\,\varepsilon_3^{0\sigma}\,(q^1_\sigma + q^3_\sigma) - (1\leftrightarrow 3)\right),$$

$$\lim_{L,\Delta\to\infty} \epsilon_{1\alpha\beta}\,\epsilon_\kappa\,\epsilon_{3\gamma\delta}\,\langle O^{*\alpha\beta}(\mathbf{p}_1)J^\kappa_{6,2}(\mathbf{p}_2)O^{\gamma\delta}(\mathbf{p}_3)\rangle = -ig\Upsilon\left(\alpha\,\varepsilon_2^\mu\,\varepsilon^1_{\mu\nu}\,\varepsilon_3^{\nu\sigma}\,(q^1_\sigma + q^3_\sigma) - (1\leftrightarrow 3)\right). \tag{E.5.57}$$

Summing the contributions from J_5 and J_6, we obtain

$$\lim_{L,\Delta\to\infty} \epsilon_{1\alpha\beta}\,\epsilon_\kappa\,\epsilon_{3\gamma\delta}\,\langle O^{*\alpha\beta}(\mathbf{p}_1)\left(J_5(\mathbf{p}_2)^\kappa + J_6^\kappa(\mathbf{p}_2)\right)O^{\gamma\delta}(\mathbf{p}_3)\rangle = -\,ig\Upsilon\left(\alpha\,\varepsilon_2^M\,\varepsilon^1_{ab}\,\varepsilon_3^{bc}\,q^1_c - (1\leftrightarrow 3)\right),$$

where we also used the orthogonality relation $\varepsilon_3^{NS}\,p^3_S = 0$.

Summing all the contributions from J_1 through J_6, we finally obtain

$$\begin{aligned}
&\lim_{L,\Delta\to\infty} \epsilon_{1\alpha\beta}\,\epsilon_\kappa\,\epsilon_{3\gamma\delta}\,\langle O^{*\alpha\beta}(\mathbf{p}_1)\,J^\kappa(\mathbf{p}_2)\,O^{\gamma\delta}(\mathbf{p}_3)\rangle \\
= \;& -\frac{ig}{\sqrt{Z_A Z_{\phi_1} Z_{\phi_3}}}\,(2\pi)^{d+1}\delta^{d+1}(q_1+q_2+q_3) \\
&\left[\frac{1}{2}(\varepsilon_{1ab}\,\varepsilon_3^{ab})\{(q_3-q_1)\cdot\varepsilon_2\} + \left\{(1+\alpha)\,\varepsilon_2^a\,\varepsilon^1_{ab}\,\varepsilon_3^{bc}q^1_c - (1\leftrightarrow 3)\right\}\right].
\end{aligned} \tag{E.5.58}$$

In the above expression, the first term comes from J_1, the α independent term inside the curly bracket comes from J_2 and J_3, whereas α dependent term comes from J_5 and J_6.

E.6 Expected 3-point amplitude in flat space

In this section, we review the 3-point function involving a $U(1)$ charged massive spin-2 field and the massless gauge field in the flat Minkowski spacetime. The 3-point Feynman amplitude is kinematically forbidden for real momenta. One may consider complex momenta or masses of incoming and outgoing massive fields to be different. For more details on this point, see Appendix H of [147]. Below, we summarise the expected 3-point function in $d+1$ dimensional flat spacetimes.

The action describing the complex massive spin-2 field and a masssless field in flat spacetimes is given by[1] (see, e.g., [362] for the one and two derivative terms)

$$
\begin{aligned}
S \;=\; & \int d^{d+1}x \left[\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} D_a \phi^*_{bc} D^a \phi^{bc} - D_a \phi^*{}_{bc} D^c \phi^{ba} - \frac{1}{2} D_a \phi^* D^a \phi \right. \\
& + \frac{1}{2} D_a \phi^{*ab} D_b \phi + \frac{1}{2} D_a \phi^{ab} D_b \phi^* + \frac{m^2}{2} (\phi^*_{ab} \phi^{ab} - \phi^* \phi) - i\, \hat{\alpha}\, \hat{g}\, \phi^{*ca} F_{ab} \phi^b{}_c \\
& + i\beta_1 \hat{g}\, F^{ab} \left(\partial_a \phi^*_{cd} \partial^c \phi_b{}^d - \partial_a \phi_{cd} \partial^c \phi^{*d}_b \right) + i\beta_3 \hat{g}\, F^{ab} \left(\partial_c \phi^*_{ad} \partial^d \phi_b{}^c - \partial_c \phi_{ad} \partial^d \phi^{*c}_b \right) \\
& \left. + i\beta_2 \hat{g}\, F^{ab} \left(\partial_a \phi^*_{cd} \partial_b \phi^{cd} \right) \right], \qquad \text{(E.6.60)}
\end{aligned}
$$

where $D_a = \partial_a + i\hat{g} A_a$.

The AdS action (7.5.93) reproduces exactly the above action in the flat limit $L \to \infty$. The 3-point interaction vertex involving the massless field A^a having momentum and polarization q_1^a and ε_1^a respectively and the massive spin-2 fields ϕ_{ab} and ϕ^*_{ab} with momenta and polarizations $(q_2, \varepsilon_2^{ab})$ and $(q_3, \varepsilon_3^{ab})$ is given by (see also [362] for the β_i independent terms)

$$
\begin{aligned}
& V_{e;ab;cd}(q_1, q_2, q_3) \\
= \; & \frac{i\hat{g}}{2} \left[\frac{1}{2} (\eta_{ca}\eta_{db} + \eta_{cb}\eta_{da} - 2\eta_{cd}\eta_{ba})(q_2 - q_3)_e + \frac{1}{2} \eta_{ec}\eta_{ab}(q_2 - q_3)_d + \frac{1}{2} \eta_{ed}\eta_{ab}(q_2 - q_3)_c \right. \\
& + \frac{1}{2} \eta_{ea}\eta_{cd}(q_2 - q_3)_b + \frac{1}{2} \eta_{eb}\eta_{cd}(q_2 - q_3)_a - \frac{1}{2} \eta_{ec}(\eta_{db} q_{2a} + \eta_{ad} q_{2b}) - \frac{1}{2} \eta_{ed}(\eta_{cb} q_{2a} + \eta_{ca} q_{2b}) \\
& + \frac{1}{2} \eta_{ea}(q_{3c}\eta_{db} + \eta_{cb} q_{3d}) + \frac{1}{2} \eta_{eb}(q_{3c}\eta_{da} + \eta_{ca} q_{3d}) + \frac{(1+\hat{\alpha})}{2} \Big\{ \eta_{ea}(q_{1d}\eta_{cb} + q_{1c}\eta_{db}) \\
& + \eta_{eb}(q_{1c}\eta_{da} + q_{1d}\eta_{ca}) - \eta_{ed}(q_{1a}\eta_{cb} + q_{1b}\eta_{ca}) - \eta_{ec}(q_{1a}\eta_{db} + q_{1b}\eta_{ad}) \Big\} \\
& + 2\beta_1 \mathcal{F}^{mn}{}_e(q_1) \left\{ q_{2(c}\eta_{d)(a}\eta_{b)n} q_{3m} - q_{3(a}\eta_{b)(c}\eta_{d)n} q_{2m} \right\} + 2\beta_2 \mathcal{F}^{mn}{}_e(q_1) q_{3m} q_{2n} \eta_{c(a}\eta_{b)d} \\
& \left. + 2\beta_3 \mathcal{F}^{mn}{}_e(q_1) \left\{ \eta_{m(c} q_{2d)} \eta_{n(a} q_{3b)} - \eta_{m(a} q_{3b)} \eta_{n(c} q_{2d)} \right\} \right], \qquad \text{(E.6.61)}
\end{aligned}
$$

where

$$
\mathcal{F}^{mne}(q) = q^m \eta^{ne} - q^n \eta^{me}. \qquad \text{(E.6.62)}
$$

[1]The 3rd term of the action in (E.6.60) is expressed slightly differently in [362]. However, using the identity

$$
D_a \phi^*{}_{bc} D^c \phi^{ba} \;=\; D_a \phi^{*ab} D^c \phi_{cb} + i\hat{g} \phi^{*ca} F_{ab} \phi^b{}_c\,, \qquad \text{(E.6.59)}
$$

the kinetic terms in (E.6.60) can be shown to agree with the expression used in [362]. The gyromagnetic interaction terms also agree if we identify $1 + \hat{\alpha}$ above with the gyromagnetic ratio given in [362].

Contracting the above expression with the polarisation tensors and using their properties, we get the desired result

$$\begin{aligned}
A_3 &= \varepsilon_1^e\,\varepsilon_2^{ab}\,\varepsilon_3^{cd}\,V_{e;ab;cd}(q_1,q_2,q_3)\\
&= i\hat{g}\left[\frac{1}{2}(\varepsilon_{2ab}\,\varepsilon_3^{ab})\{(q_3-q_1)\cdot\varepsilon_1\} + \Big\{(1+\hat{\alpha})\,\varepsilon_1^a\,\varepsilon_{2ab}\,\varepsilon_3^{bc}q_{1c} - (1\leftrightarrow 3)\Big\}\right]\\
&\quad + i\hat{g}\,(q_1^m\varepsilon_1^n - q_1^n\varepsilon_1^m)\Big[\beta_1\,\{q_{3m}(q_2\cdot\varepsilon_3\cdot\varepsilon_2)_n - q_{2m}(q_3\cdot\varepsilon_2\cdot\varepsilon_3)_n\} + \beta_2\,q_{3m}q_{2n}\,(\varepsilon_2\cdot\varepsilon_3)\\
&\quad + \beta_3\,\{(\varepsilon_3\cdot q_2)_m\,(\varepsilon_2\cdot q_3)_n - (\varepsilon_3\cdot q_2)_n\,(\varepsilon_2\cdot q_3)_m\}\Big]\,. \qquad \text{(E.6.63)}
\end{aligned}$$

The flat limit result obtained in the previous section matches exactly with the above expression for $\beta_i = 0$.

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