Positivity of holographic energy

Piotr T. Chruściel*
Beijing Institute of Mathematical Sciences and Applications, Huairou, China, and Center for Theoretical Physics of the Polish Academy of Sciences, Warsaw, Poland†

Raphaela Wutte‡
Mathematical Sciences and STAG Research Centre, University of Southampton, Highfield, SO17 1BJ Southampton, United Kingdom

We prove positivity of a weighted holographic energy for four-dimensional spacetimes with negative cosmological constant whose conformal boundary at infinity is conformally static and admits either spherical sections, or toroidal sections with compatible spin structure.

CONTENTS

I. INTRODUCTION

Existence of lower bounds for energy-type expressions has fundamental importance for well-posedness of every theory, and is closely related to the global behaviour of solutions of the equations.

In the context of the AdS/CFT correspondence, a natural notion of energy is the holographic one [60], see also [5, 16, 30, 39, 63], which can be defined for spacetimes that locally approach Anti-de Sitter at infinity. Such spacetimes have a timelike conformal boundary, $\mathscr{I}$. Given a vector field X on $\mathscr{I}$, a section S of $\mathscr{I}$, and a metric g as made precise below, we write $Q[S,X](g)$ for the holographic charge of g defined as

$$Q[S,X](g) = -\int_S t^A{}_B X^B ds_A \,, \tag{I.1}$$

where $t^A{}_B$ is the holographic energy-momentum tensor of g. The holographic energy is obtained by choosing $X=\partial_t$. This energy is reasonably well understood when the metric at $\mathscr{I}$ is conformal to an ultrastatic metric with Einstein space-sections, in which case the holographic energy essentially coincides [63, Equation (17)] with the more usual hyperbolic energy [1, 4, 17, 38]. For these last metrics several positivity and rigidity theorems are available [13, 14, 21, 24–26, 33–35, 41, 42, 44, 45, 61, 65], but nothing has been known so far beyond these cases.

The aim of this letter is to point out that a suitably weighted holographic energy (see (II.40) below) is positive for all four-dimensional solutions $(\mathscr{M},g)$ of the Einstein equations with a negative cosmological constant, with a conformally static conformal infinity [52], and with sources satisfying the dominant energy condition. Furthermore we either assume spherical sections of $\mathscr{I}$, or toroidal sections of $\mathscr{I}$ with trivial induced spin structure on conformal infinity. Finally, we suppose that $\mathscr{M}$ contains a complete spacelike hypersurface $\mathcal{S}$ either without boundary [53], or with a compact boundary. If nonempty, each component of the boundary should be either outer trapped or marginally outer trapped.

The proof, inspired by [16], consists of showing how to adapt the Witten argument to such metrics.

Recall that the flagship result in this context, going back to [35], is positivity of energy for initial data sets which asymptote to a background with asymptotic imaginary Killing spinors and which have Birmingham-Kottler asymptotics [11] [54] (cf. [26, 65]; more recent developments can be found in [18, 24] and references therein). Further milestones include proofs of negative bounds from below for solutions with toroidal [13, 14], or higher genus [45], sections of conformal infinity.

II. THE PROOF

We consider four-dimensional spacetimes $(\mathscr{M},g)$ solving the Einstein equations with a negative cosmological constant, and with matter fields satisfying the usual positivity conditions. We suppose existence of a conformal completion *à la* Penrose, with a smooth conformal metric at the conformal boundary. When the matter fields decay sufficiently fast [55], under mild supplementary asymptotic conditions the arguments in [31, 37] show that the spacetime metric g can be written in the form

$$g = x^{-2}\big(dx^2+\big(\mathring{\gamma}_{AB}+x^2 \overset{(2)}{\gamma}{}_{AB}+x^3 \overset{(3)}{\gamma}{}_{AB} + O(x^4)\big)dy^A dy^B\big)\,, \tag{II.1}$$

with a smooth Lorentzian 3-dimensional metric $\mathring{\gamma}_{AB}$ and smooth tensor fields $\overset{(2)}{\gamma}{}_{AB}$, $\overset{(3)}{\gamma}{}_{AB}$, all satisfying $\partial_x\mathring{\gamma}_{AB} =$

* pchrusciel@cft.edu.pl
† homepage.univie.ac.at/piotr.chrusciel
‡ rwutte@hep.itp.tuwien.ac.at

$0 = \partial_x \overset{(2)}{\gamma}_{AB} = \partial_x \overset{(3)}{\gamma}_{AB}$. Here

$$\mathscr{I} := \{x = 0\}$$

is the conformal boundary at infinity, and

$$(y^A, x) \equiv (y^0, y^a, x) \equiv (t, y^a, x)$$

are local coordinates near $\mathscr{I}$. We assume that the time-coordinate t is globally defined and we set

$$\mathcal{S} = \{t = 0\}\,. \tag{II.2}$$

Recall that in the presence of a cosmological constant $\Lambda = -3$ the Witten equation reads

$$\gamma^j \hat{\nabla}_j \psi = 0\,, \tag{II.3}$$

where j runs over $\{1, 2, 3\}$, with

$$\hat{\nabla}_j \psi = \nabla_j \psi + \frac{i}{2}\gamma_j \psi\,.$$

Recall the Schrödinger-Lichnerowicz-Sen-Witten (SLSW) identity, for $\epsilon > 0$,

$$\int_{\mathcal{S}\setminus\{x\le\epsilon\}} \Big(|\hat{\nabla}\psi|^2 + \langle\psi, (\rho + J^i\gamma_i\gamma_0)\psi\rangle - |\gamma^j\hat{\nabla}_j\psi|^2\Big)\, d\mu = \Re \int_{\partial\left(\mathcal{S}\setminus\{x\le\epsilon\}\right)} B^i[\psi] dS_i\,, \tag{II.4}$$

where $d\mu$ is the metric measure on $\mathcal{S}$, and where ρ is the matter density, J^i the matter current, with the boundary integrand given by

$$B^i[\psi] = \langle \hat{\nabla}^i\psi + \gamma^i\gamma^j\hat{\nabla}_j\psi, \psi\rangle\,. \tag{II.5}$$

Here

$$\langle\psi, \phi\rangle := \psi^\dagger\phi\,,$$

where $\psi^\dagger$ denotes complex conjugation and transposition. We use the convention that $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$ with a Hermitian γ^0 and anti-Hermitian γ^i's, and

$$dS_i = \sqrt{\det g_{k\ell}}\, \partial_i \rfloor (dx^1 \wedge dx^2 \wedge dx^3)\,. \tag{II.6}$$

Now, when the matter fields satisfy the dominant energy condition, it follows from (II.4) with a spinor field satisfying (II.3) that

$$\Re \int_{\partial\mathcal{S}} B^i[\psi]\, dS_i \ge \int_{\mathcal{S}} |\hat{\nabla}\psi|^2 d\mu\,. \tag{II.7}$$

So, a finite boundary integral in (II.4) implies that $|\hat{\nabla}\psi|$ is square-integrable. Conversely, square-integrability of $|\hat{\nabla}\psi|$ and a finite contribution from the matter-fields volume-integral in (II.4) guarantees a finite boundary integral.

Letting the coordinates (x, y^A) near $\partial\mathcal{S}$ range over $[0, x_0] \times \partial\mathcal{S}$ we have

$$\int_{\mathcal{S}} |\hat{\nabla}\psi|^2 d\mu \ge \int_{x=0}^{x_0} \int_{\partial\mathcal{S}} |\hat{\nabla}\psi|^2 x^{-3} \sqrt{\det {}^3\tilde{g}}\, dx\, dy^1 dy^2\,, \tag{II.8}$$

where ${}^3\tilde{g}$ is the metric induced on the level sets of t by the unphysical spacetime metric

$$\tilde{g} = x^2 g\,.$$

The hypothesis of finiteness of the left-hand side of (II.7) implies that $|\hat{\nabla}\psi|$ must decay faster than x. In fact, for spinor fields ψ with the asymptotics below we must have

$$|\hat{\nabla}\psi| = O(x^{3/2})\,, \tag{II.9}$$

which is then necessary and sufficient for a finite Witten boundary-integral, i.e. the boundary integral in (II.4).

In order to evaluate the above explicitly we use an orthonormal coframe of the form $\theta^{\hat{3}} = x^{-1}dx$ and

$$\theta^{\hat{A}} = x^{-1}\big(\mathring{\theta}^{\hat{A}} + \frac{1}{2}\mathring{\gamma}^{\hat{A}\hat{B}}\big(x^2\overset{(2)}{\gamma}_{\hat{B}\hat{C}} + x^3\overset{(3)}{\gamma}_{\hat{B}\hat{C}} + O(x^4)\big)\mathring{\theta}^{\hat{C}}\big)\,, \quad \hat{A} \in \{0, 1, 2\}\,, \tag{II.10}$$

where $\{\mathring{\theta}^{\hat{A}}\}$ is an ON-coframe for $\mathring{\gamma}_{AB}$ which is independent of x, and we put hats over tetrad indices. The dual frame takes the form

$$e_{\hat{3}} = x\,\partial_x\,, \quad e_{\hat{A}} = x\big(\mathring{e}_{\hat{A}} + x^2 f_{\hat{A}}\big)\,, \tag{II.11}$$

where $\{\mathring{e}_{\hat{A}}\}$ is dual to $\{\mathring{\theta}^{\hat{A}}\}$, and where the $f_{\hat{A}}$'s are smooth on the conformally completed manifold. We have

$$f_{\hat{A}} = -\frac{1}{2}\big(\overset{(2)}{\gamma}_{\hat{A}\hat{B}} + x\overset{(3)}{\gamma}_{\hat{A}\hat{B}} + O(x^2)\big)\mathring{\gamma}^{\hat{B}\hat{C}}\mathring{e}_{\hat{C}} \tag{II.12}$$

The connection coefficients can be written as

$$\omega_{\hat{A}\hat{3}} = -\eta_{\hat{A}\hat{B}}\theta^{\hat{B}} + x^2 \underbrace{V_{\hat{A}\hat{B}}}_{\overset{(2)}{\gamma}_{\hat{A}\hat{B}} + \frac{3}{2}x\overset{(3)}{\gamma}_{\hat{A}\hat{B}} + O(x^2)} \theta^{\hat{B}}\,, \tag{II.13}$$

$$\omega_{\hat{A}\hat{B}} = \mathring{\omega}_{\hat{A}\hat{B}} + x^2\big(\underbrace{C_{\hat{A}\hat{B}}}_{O(x^2)}\theta^{\hat{3}} + C_{\hat{A}\hat{B}\hat{C}}\theta^{\hat{C}}\big)\,, \tag{II.14}$$

with

$$C_{\hat{A}\hat{B}\hat{C}} = x\mathring{\mathscr{D}}_{[\hat{B}}\overset{(2)}{\gamma}_{\hat{A}]\hat{C}} + x^2\mathring{\mathscr{D}}_{[\hat{B}}\overset{(3)}{\gamma}_{\hat{A}]\hat{C}} + O(x^3)\,, \tag{II.15}$$

where the $\mathring{\omega}_{\hat{A}\hat{B}}$ are connection one-forms associated with the frame $\mathring{\theta}^{\hat{A}}$, with

$$\overset{(2)}{\gamma}_{\hat{A}\hat{B}} = -\left(\mathring{R}_{\hat{A}\hat{B}} - \frac{\mathring{R}}{4}\mathring{\gamma}_{\hat{A}\hat{B}}\right) + o(1)\,, \tag{II.16}$$

where $\mathring{R}_{\hat{A}\hat{B}}$ is the Ricci tensor of $\mathring{\gamma}_{\hat{A}\hat{B}}$, where $o(1)$ is meant as $x \to 0$.

Following [16] we start with a formal solution ψ of (II.3) which, in a spin frame associated with (II.11), is assumed to have an asymptotic expansion of the form

$$\psi(x,y^A)=x^{-1/2}\big(\psi_{-\frac12}(y^A)+x\,\chi(x,y^A)\big)\,, \qquad \text{(II.17)}$$

where χ is a bounded polyhomogeneous spinor field on the conformally completed manifold. Using (II.13)-(II.14) we find

$$\begin{aligned}\hat\nabla_{\hat 3}\psi &\equiv \nabla_{e_{\hat 3}}\psi+\frac{i}{2}\gamma_{\hat 3}\psi\\ &= x^{3/2}\partial_x\chi-\frac12 x^{-1/2}(1-i\gamma_{\hat 3})\psi_{-\frac12}\\ &\quad-\frac{x^{3/2}}{4}C_{\hat A\hat B}\gamma^{\hat A}\gamma^{\hat B}\psi_{-\frac12}+\frac{i}{2}\gamma_{\hat 3}x^{1/2}\chi\\ &\quad+\frac12 x^{1/2}\chi-\frac{x^{\frac52}}{4}C_{\hat A\hat B}\gamma^{\hat A}\gamma^{\hat B}\chi\,, \qquad \text{(II.18)}\\ \hat\nabla_{\hat{\mathfrak a}}\psi &\equiv \nabla_{e_{\hat{\mathfrak a}}}\psi+\frac{i}{2}\gamma_{\hat{\mathfrak a}}\psi\\ &= x^{1/2}\mathring{\mathscr D}_{\hat e_{\hat{\mathfrak a}}}\psi_{-\frac12}+x^{3/2}\mathring{\mathscr D}_{\hat e_{\hat{\mathfrak a}}}\chi+x^{5/2}\mathring{\mathscr D}_{f_{\hat{\mathfrak a}}}\psi_{-\frac12}+x^{\frac72}\mathring{\mathscr D}_{f_{\hat{\mathfrak a}}}\chi\\ &\quad+\frac{x^{-1/2}}{2}i\gamma_{\hat{\mathfrak a}}(1-i\gamma^{\hat 3})\psi_{-\frac12}+\frac{x^{1/2}}{2}\gamma_{\hat{\mathfrak a}}(\gamma^{\hat 3}+i)\chi\\ &\quad-x^{3/2}\Big(\frac{V_{\hat{\mathfrak a}\hat B}}{2}\gamma^{[\hat B}\gamma^{\hat 3]}+\frac{C_{\hat B\hat C\hat{\mathfrak a}}}{4}\gamma^{[\hat B}\gamma^{\hat C]}\Big)\psi_{-\frac12}\\ &\quad-x^{\frac52}\Big(\frac{V_{\hat{\mathfrak a}\hat B}}{2}\gamma^{[\hat B}\gamma^{\hat 3]}+\frac{C_{\hat B\hat C\hat{\mathfrak a}}}{4}\gamma^{[\hat B}\gamma^{\hat C]}\Big)\chi\,, \qquad \text{(II.19)}\end{aligned}$$

where $\mathring{\mathscr D}$ is the spinor covariant derivative associated with the metric $\mathring\gamma_{AB}dy^Ady^B$.

Multiplying (II.18) by $x^{1/2}$ and passing in the resulting equation to the limit $x\to 0$, the fall-off requirement (II.9) provides a condition already pointed out in [16]:

$$\boxed{(1-i\gamma_{\hat 3})\psi_{-\frac12}=0}\,. \qquad \text{(II.20)}$$

From now on we assume that (II.20) holds.

Let us set

$$\psi_{\frac12}:=\chi|_{x=0}\,. \qquad \text{(II.21)}$$

Demanding that the terms of order $x^{1/2}$ in (II.18) vanish we get

$$(1+i\gamma_{\hat 3})\psi_{\frac12}=0\,. \qquad \text{(II.22)}$$

From (II.19) at order $x^{1/2}$ we have

$$0=\mathring{\mathscr D}_{\hat e_{\hat{\mathfrak a}}}\psi_{-\frac12}+\frac{i}{2}\gamma_{\hat{\mathfrak a}}(1-i\gamma^{\hat 3})\psi_{\frac12}=\mathring{\mathscr D}_{\hat e_{\hat{\mathfrak a}}}\psi_{-\frac12}+i\gamma_{\hat{\mathfrak a}}\psi_{\frac12}\,, \qquad \text{(II.23)}$$

where we used (II.22).

Note that if χ vanishes at $\{x=0\}$ we obtain

$$\mathring{\mathscr D}_{\hat e_{\hat{\mathfrak a}}}\psi_{-\frac12}=0\,. \qquad \text{(II.24)}$$

It then follows that $\mathring\gamma$ is flat, a case which has already been covered elsewhere [13, 14, 27, 50].

Multiplying (II.23) with $\gamma^{\hat{\mathfrak a}}$ we get

$$\boxed{\psi_{\frac12}=-\frac{i}{2}\gamma^{\hat{\mathfrak a}}\mathring{\mathscr D}_{\hat e_{\hat{\mathfrak a}}}\psi_{-\frac12}}\,, \qquad \text{(II.25)}$$

as already pointed out in [16, Equation (5.15)]. Plugging (II.25) back into (II.23) yields

$$\Big(\delta_{\hat{\mathfrak a}}{}^{\hat{\mathfrak b}}+\frac12\gamma_{\hat{\mathfrak a}}\gamma^{\hat{\mathfrak b}}\Big)\mathring{\mathscr D}_{\hat e_{\hat{\mathfrak b}}}\psi_{-\frac12}=0 \quad\Longleftrightarrow\quad \boxed{\gamma^{\hat{\mathfrak b}}\gamma^{\hat{\mathfrak a}}\mathring{\mathscr D}_{\hat e_{\hat{\mathfrak b}}}\psi_{-\frac12}=0}\,. \qquad \text{(II.26)}$$

We are ready now to prove our claim. Suppose that the conformal metric at $\mathscr I$ contains a static metric in its class. We can choose a conformal representative which is ultrastatic,

$$\mathring\gamma=-dt^2+\mathring\gamma_{ab}dx^adx^b\,,\quad \partial_t\mathring\gamma_{ab}=0\,. \qquad \text{(II.27)}$$

Then $\mathring{\mathscr D}_{\hat e_{\hat{\mathfrak b}}}\psi_{-\frac12}$ is the spinor derivative associated with the metric induced on the level sets of t within $\mathscr I$, and (II.26) is the two-dimensional twistor equation. Since this equation is conformally invariant, the space of solutions of this equation is essentially the same for all metrics on a two-dimensional sphere $\mathbb S^2$, and for all metrics on $\mathbb T^2$ carrying its trivial spin structure, cf. [36, Note A.2.2].

Indeed, for any smooth metric $\mathring\gamma_{ab}$ on a two-dimensional compact manifold we can write (cf., e.g., [51])

$$\mathring\gamma_{ab}=e^{2u}\widetilde{\mathring\gamma}_{ab}\,, \qquad \text{(II.28)}$$

where $\widetilde{\mathring\gamma}$ has constant scalar curvature in $\{0,\pm2\}$, for some smooth function $u(x^a)$. Letting

$$\widetilde{\psi_{-\frac12}}=e^{\frac u2}\psi_{-\frac12}\,, \qquad \text{(II.29)}$$

we have

$$\gamma^{\hat b}\gamma^{\hat a}\widetilde{\mathring{\mathscr D}}_{\widetilde e_{\hat b}}\widetilde{\psi_{-\frac12}}=e^{-\frac u2}\gamma^{\hat b}\gamma^{\hat a}\mathring{\mathscr D}_{e_{\hat b}}\psi_{-\frac12}\,. \qquad \text{(II.30)}$$

For further use we note that if

$$X^A=(\psi_{-\frac12})^\dagger\gamma^0\gamma^A\psi_{-\frac12}\,, \qquad \text{(II.31)}$$

then after the conformal rescaling (II.28) we will have

$$\widetilde X^A:=(\widetilde{\psi_{-\frac12}})^\dagger\gamma^0\gamma^A\widetilde{\psi_{-\frac12}}=e^uX^A\,. \qquad \text{(II.32)}$$

Given a solution of (II.26) we can define $\psi_{\frac12}$ using (II.25). In order to satisfy (II.23) we need

$$\psi_{\frac12}=-i\gamma_{\hat 1}\mathring{\mathscr D}_{\hat e_{\hat 1}}\psi_{-\frac12}=-i\gamma_{\hat 2}\mathring{\mathscr D}_{\hat e_{\hat 2}}\psi_{-\frac12}\,, \qquad \text{(II.33)}$$

which is satisfied when (II.26) holds.

We claim, now, that when $\mathscr I$ has spherical topology, then a solution ψ of (II.3) exists for any asymptotic values of $\psi_{-\frac12}$ and of $\psi_{\frac12}$ as just described; in the toroidal case this remains true provided the spin structure on $\mathcal S$

induces the trivial spin structure on $\mathscr{I}\cap\mathcal{S}$ [56]. For this we write

$$\psi(x,y^A)=x^{-1/2}\psi_{-\frac12}(y^A)+x^{1/2}\psi_{\frac12}(y^A)+\phi(x,y^A)\,,$$

with $\phi\in L^2(\mathcal{S})$. We thus need

$$\gamma^j\hat\nabla_j\phi=-\gamma^j\hat\nabla_j\big(x^{-1/2}\psi_{-\frac12}+x^{1/2}\psi_{\frac12}\big)\,. \tag{II.34}$$

If the right-hand side is in $L^2(\mathcal{S})$, one shows by standard methods that a solution $\phi\in L^2(\mathcal{S})$ exists, with the property that the contribution of ϕ to the boundary term vanishes (cf., e.g., [6, 26]).

For any such solution the boundary integral in (II.4) is positive and finite,

$$0\le\Re\int_S B^i[\psi]dS_i<\infty\,, \tag{II.35}$$

where we used S for $\mathcal{S}\cap\mathscr{I}$.

One can use the results in [46, 49] to show that near $\{x=0\}$ the solution ψ so obtained takes the form

$$\psi(x,y^A)=x^{-1/2}\phi(x,y^A)+x^{5/2}\log x\,\phi_{\log}(x,y^A)\,, \tag{II.36}$$

for some smooth-up-to-boundary fields ϕ and $\phi_{\log}$, with

$$\phi_{\log}(0,y^A)=-\frac{i}{4}\overset{(2)}{\gamma}{}_{\hat{\mathfrak{a}}\hat B}\gamma^{\hat{\mathfrak{a}}}\gamma^{\hat B}\gamma^{\hat{\mathfrak{c}}}\mathring{\mathscr{D}}_{\hat e_{\hat{\mathfrak{c}}}}\psi_{-\frac12}\,. \tag{II.37}$$

In particular ψ has the asymptotic behaviour needed for the calculations in [16] (the log term gives a vanishing contribution to the boundary integral because its leading coefficient is orthogonal to $\psi_{-\frac12}$). It is shown in the last reference that when (II.20)-(II.22) hold the Witten boundary integral, before passing to the limit $x\to0$, is proportional to

$$\begin{aligned}x^{-1}\Re\int_{\mathcal{S}\cap\{x=\epsilon\}}&\sqrt{\det\mathring\gamma_{ab}}\,(\psi_{-\frac12})^\dagger\Big[(\gamma^{\hat{\mathfrak{a}}}\mathring{\mathscr{D}}_{\hat e_{\hat{\mathfrak{a}}}})^2\\&+(\mathring R_{\hat{\mathfrak{b}}\hat A}-\frac{\mathring R}{4}\mathring\gamma_{\hat{\mathfrak{b}}\hat A})\gamma^{\hat{\mathfrak{b}}}\gamma^{\hat A}\Big]\psi_{-\frac12}\\&+Q[\mathcal{S}\cap\mathscr{I},X](g)+o(1)\,,\end{aligned} \tag{II.38}$$

with

$$X^A=(\psi_{-\frac12})^\dagger\gamma^{\hat0}\gamma^A\psi_{-\frac12}\,. \tag{II.39}$$

This seemingly leads to a divergent boundary term, which would contradict (II.35). However, Equation (II.4) shows that the divergent part of (II.38) must integrate out to zero; in fact, a commutation calculation shows that the singular part of the integrand vanishes when (II.26) holds. From what has been said, using (II.38) we obtain

$$Q_{CW}[S,\bar X](g):=-\int_S t^A{}_B\widetilde X^Be^{-u}dS_A\ge0\,, \tag{II.40}$$

where e^{2u} is the conformal factor defined in (II.28). Note that this factor can be chosen to be equal to one when $\mathring\gamma_{ab}$ has constant scalar curvature, recovering in this case the already-known positivity results. In case of a spherical $\mathscr{I}$, letting ψ run over all possible solutions of the equations above one concludes, as in [15, 27, 50], that in the zero-space-momentum frame the energy m, center of mass $\vec c$, and angular momentum $\vec j$ satisfy

$$m\ge\sqrt{|\vec c|^2+|\vec j|^2+2|\vec c\times\vec j|}\,, \tag{II.41}$$

where $\vec c\times\vec j$ is the vector product, while $|\vec j|=\sqrt{(j^1)^2+(j^2)^2+(j^3)^2}$, etc. For a toroidal $\mathscr{I}$ with trivial induced spin structure one has instead

$$m\ge|\vec j|\,,\qquad|\vec j|:=\sqrt{(j^1)^2+(j^2)^2}\,; \tag{II.42}$$

see [27] for definitions and details. Here the global charges are calculated by using in (II.40) the Killing vectors $\widetilde X^A$ of Anti-de Sitter space in the spherical case, or of the quotient thereof in the toroidal case.

The question then arises, which metrics saturate the inequality in (II.40) with $\widetilde X=\partial_t$. Then, assuming that the conformal metric at $\mathscr{I}$ is the same as for Anti-de Sitter, it is shown in [42, Corollary 9.4] that the initial-data hypersurface $\mathcal{S}$ embeds isometrically into Anti-de Sitter spacetime. More generally, under our assumptions it follows from (II.4), with ψ satisfying (II.3), that the right-hand side of (II.40) vanishes if and only if the matter contribution vanishes and there exists along $\mathcal{S}$ a spinor field satisfying $\hat\nabla_j\psi=0$. Deforming $\mathcal{S}$ within a development $\mathscr{M}$ of the data on $\mathcal{S}$, assuming there is one, we find that $\mathscr{M}$ admits an imaginary Killing spinor. Theorem 5.1 of [10] leads to the conclusion that the spacetime metric is a Siklos wave [47, 62] (which appears in the list given in [10, Theorem 5.1] as pp-manifold) (compare [8, 48]) [57]. Whether or not globally well-behaved such metrics exist in four spacetime dimensions requires further investigation.

In our positivity claim we assumed a conformally static infinity because this provides a natural zero-energy reference. But, as already pointed out by Cheng and Skenderis [16], it would be of interest to determine all backgrounds for which the Witten argument provides positive charges. It follows from our calculations above that positivity also holds for metrics with the same conformal infinity as Siklos waves, for which the positive charge is associated with a null conformal Killing vector on $\mathscr{I}$.

To make things precise, recall that the Siklos waves take the form

$$x^{-2}(-2duds+f(s,x,y)ds^2+dx^2+dy^2)\,, \tag{II.43}$$

where

$$f(s,x,y)=\mathring f(s,y)+\frac{x^2}{2}\partial_y^2\mathring f(s,y)+x^3\overset{(3)}{f}(s,y)+\dots\,, \tag{II.44}$$

and have an imaginary Killing spinor. They induce on $\mathscr{I}$ the metric

$$\mathring\gamma=-2duds+\mathring f(s,y)ds^2+dy^2 \tag{II.45}$$

which has vanishing Ricci scalar, with the only non-vanishing component of the Ricci tensor being $\mathring{R}_{ss} = -\frac{1}{2}\partial_y^2\mathring{f}$, and the only non-zero component of the Cotton tensor being $\mathring{C}_{ss} = \frac{1}{2}\partial_y^3\mathring{f}$. Hence the boundary metric is conformally flat if and only if $\partial_y^3\mathring{f} \equiv 0$. Every Lorentzian metric g with the conformal structure on $\mathscr{I}$ given by (II.45), which satisfies the dominant energy condition and which is vacuum to sufficiently high order, on a manifold which has a partial Cauchy surface $\mathcal{S}$ inducing the trivial spin structure on $\mathcal{S}\cap\mathscr{I}$, has positive charge $Q[\mathcal{S}\cap\mathscr{I},\partial_u](g)$. A similar statement holds for higher-dimensional Siklos waves.

Let $\mathring{g}$ be static and let g and $\mathring{g}$ share the same conformal structure on $\mathscr{I}$. When both metrics are, say, vacuum it was shown in [29] that the functional

$$Q[S,\partial_t](g) - Q[S,\partial_t](\mathring{g}) \tag{II.46}$$

is a Noether charge (compare [43, 60]) equal to the Hamiltonian energy of g relative to $\mathring{g}$. Recall that a perturbative analysis of the positivity of mass goes back to [1], where a quadratic expansion of the energy around the Anti-de Sitter metrics confirmed positivity for nearby metrics; compare [7]. Now, the calculations in that last reference have been carried out for perturbations of any static metrics, in any dimension. In particular the calculations there are directly relevant to static vacuum metrics near the Anti-de Sitter metric, constructed in [3, 22], which are parameterised by conformally static conformal structures. These are stationary points of energy, thus have a Taylor expansion in terms of the perturbations of the metric which starts with quadratic terms. Since the second-derivative operator of the energy functional is strictly positive at the Anti-de Sitter metric, and depends continuously upon the metric, the arguments in [7], including the possibility of realising a suitable gauge, prove positivity of Hamiltonian mass [58] near these metrics, and hence of (II.46), in all spacetime dimensions $n+1 \geq 3$ [59].

In $3+1$-dimensions we hence obtain two independent inequalities, namely (II.40) with $\widetilde{X} = \partial_t$ and

$$Q[S,\partial_t](g) \geq Q[S,\partial_t](\mathring{g}) \tag{II.47}$$

when a vacuum metric g is close to a vacuum static background $\mathring{g}$, both sharing the same induced geometry at $\mathscr{I}$, and when $\mathring{g}$ itself is close to the AdS metric. Under these restrictions, equality in (II.47) occurs only when $g = \mathring{g}$.

Note that we have been concentrating on spacetime dimension four, because the requirement of convergence, as $\epsilon \to 0$, of the volume integral of (II.4) in spacetime dimension $n+1$ requires $|\nabla\psi| = O(x^{\frac{n-1}{2}+\epsilon})$ for some $\epsilon > 0$, which might impose more conditions on the boundary geometry when $n > 3$. It is tempting to conjecture that positivity of Q_{CW} holds in all spacetime dimensions $n+1 \geq 4$ for all conformally static $\mathscr{I}$'s with sections which carry solutions of the $(n-1)$-dimensional twistor equation:

$$\Big(\delta_{\hat{\mathfrak{a}}}^{\hat{\mathfrak{b}}} + \frac{1}{(n-1)}\gamma_{\hat{\mathfrak{a}}}\gamma^{\hat{\mathfrak{b}}}\Big)\mathring{\mathscr{D}}_{\hat{e}_{\hat{\mathfrak{b}}}}\psi_{-\frac{1}{2}} = 0\,; \tag{II.48}$$

cf. e.g. [9, 36]. The fact that the conjecture is true in $(3+1)$-dimensions has been shown above; it is true in $(4+1)$-dimensions by calculations identical to the above together with the last paragraph of [16, Section 6].

III. CONCLUSIONS

In this letter we established positivity of the weighted holographic energy (II.40) for four-dimensional spacetimes with or without black hole boundaries, with conformally static infinity, with spherical cross-sections of infinity, or with toroidal cross-sections with a compatible spin structure. We further established positivity of a holographic charge associated with a null conformal Killing vector for conformal boundary geometries induced by a Siklos wave in all dimensions. Our argument exploits existence of solutions to the twistor equation on sections of the conformal boundary. The same method applies to prove positivity for all $(4+1)$-dimensional spacetimes which possess a conformally static boundary geometry at infinity with sections which admit solutions of the three-dimensional twistor equation. We conjecture that a similar statement holds true in all spacetime dimension greater than five.

Since

$$Q_{CW}[S,\bar{X}](g) \equiv Q[S,e^{-u}\bar{X}](g)\,, \tag{III.1}$$

one can instead think of Q_{CW} as the usual holographic charge with respect to a rescaled vector field. Now, the divergence theorem shows that $Q[S,\partial_t](g)$ is independent of S when ∂_t is a conformal Killing vector of the boundary geometry because t_{AB} is transverse and traceless in odd space dimensions. But $Q_{CW}[S,\partial_t](g)$ will depend upon S in general because $e^{-u}\partial_t$ is not a conformal Killing vector on $\mathscr{I}$ unless u is constant. This implies that Q_{CW} can be radiated away, which can be considered as a desirable feature, with positivity providing an upper bound on the amount of charge that can be emitted.

Our result should be contrasted with the analysis in [40], where it is proved that $(3+1)$-dimensional *static* vacuum conformally smooth metrics with conformally compact static slices without boundary have *negative or zero holographic* energy; recall that many such metrics exist near Anti-de Sitter spacetime by [2, 3]. It is rather counterintuitive that the introduction of the conformal factor arising from the solutions of the twistor equation changes the sign of the charge integrals, as follows from our analysis. This implies in particular that the integrand of (I.1) can have a constant sign for such metrics only if it vanishes.

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[52] A construction of large classes of static vacuum solutions can be found in [3]. A class of negative-mass solutions (which do not satisfy the hypotheses of our positivity theorem here) can be found in [20, 23].

[53] Note that this is compatible with any number of asymptotic ends, where the asymptotic behaviour is only restricted by the requirement of completeness of the metric induced on $\mathcal{S}$.

[54] In the physics literature, imaginary Killing spinors are often referred to simply as Killing spinors.

[55] In the context of the AdS/CFT correspondence, the case of matter fields which do not decay at infinity is also of interest, see [12, 32, 64] for a discussion of the Witten boundary integral in this context. We do not consider this case here.

[56] Our argument does not provide any information about higher genus two-dimensional manifolds, as no non-trivial solutions of (II.26) exist there.

[57] Every imaginary Killing spinor is a twistor spinor, whence the relevance of [10], which is concerned with twistor spinors. Theorem 5.1 of [10] shows that the metric must be a Siklos metric, since twistor spinors on Fefferman metrics listed there are never imaginary Killing spinors [47].

[58] We are grateful to Klaus Kröncke for pointing this out.

[59] See [19, 28] for $n = 2$.

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