Unawareness of Theorems*

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Abstract

This paper provides a set-theoretic model of knowledge and unawareness. A new property called Awareness Leads to Knowledge shows that unawareness of theorems not only constrains an agent's knowledge, but also, can impair his reasoning about what other agents know. For example, in contrast to Li (2009), Heifetz et al. (2006) and the standard model of knowledge, it is possible that two agents disagree on whether another agent knows a particular event. The model follows Aumann (1976) in defining common knowledge and characterizing it in terms of a self evident event, but departs in showing that no-trade theorems do not hold.

JEL-Classifications: C70, C72, D80, D82.

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1 Introduction

1.1 Motivation and outline

A common assumption in economics is that agents who participate in a model perceive the "world" the same way the analyst does. This means that they understand how the model works, they know all the relevant theorems and they do not miss any dimension of the problem they are facing. In essence, agents are as educated and as intelligent as the analyst and they can make the best decision, given their information and preferences.

Modelling unawareness aims at relaxing this assumption, so that agents may perceive a more simplified version of the world. Intuitively, there are many instances where agents of different perception coexist in the same market. In the stock market, for example, one

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can find investors who are highly educated about how the market and the economy work, together with investors whose understanding is much more limited.

One way in which we might hope to capture differences between these two types of investors is by attributing differences to asymmetric information. However, Dekel et al. (1998) showed that the standard model of knowledge, introduced into economics by Aumann (1976), cannot accommodate unawareness.[1] Moreover, it can be criticized on the grounds that it only models a highly sophisticated and rational agent, who is aware of everything, knows all the possible theorems that can be derived and has no constraints on the number of calculations he can perform.

This paper provides a model of knowledge and awareness, where agents may not know some of the relevant theorems, may be unaware of some of the dimensions of the world and thus can make mistakes. Moreover, the paper does not depart from the set-theoretic approach of Aumann (1976) and its advantages, while aiming at giving a better insight into the connection between awareness and knowledge.

Consider the following comparative statics exercise, where an agent gains awareness. He becomes aware of new events, and some of these he may subsequently know. This effect of awareness on knowledge is well described by other papers. The second, less immediate connection is that more awareness can lead to awareness of new theorems, which connect answers to different questions. As a result, more awareness can lead to knowing an event that the agent previously was aware of but did not know. Or equivalently, what one *is unaware of*, may constrain his knowledge about events he *is aware of*. This less immediate connection is not accommodated in the other papers that model unawareness - it is expressed in this model by the property Awareness Leads to Knowledge.

The implications of this property in a multi agent setting can be stark. The unaware agent 1 may falsely conclude that agent 2 does not know an event, when in fact agent 2 knows it, because he knows a theorem beyond 1's awareness.

It is worthwhile noting that these mistakes in reasoning about others (due to unawareness of theorems) can be accommodated by the standard model of knowledge or the extensions discussed below, only if we allow for false beliefs. But allowing for false beliefs permits all kinds of mistakes. For instance, it allows for agents to make numerical mistakes. The purpose of this paper is to isolate and study this specific type of mistake due to unawareness of theorems, without relaxing the assumption that agents are otherwise rational.

In order to overcome the impossibility result of Dekel et al. (1998), the paper follows the approach of Heifetz et al. (2009) and Li (2009) of introducing multiple state spaces. However, it retains the set-theoretic nature exhibited also in the standard model of knowledge and as a result, familiar notions naturally extend here. For instance, common knowledge is characterized in terms of a self evident event, just like in the standard model. Moreover, there is a well defined notion of a common state space. This is the state space that everyone is aware of and this is common knowledge. As the following discussion on no-trade theorems reveals, results that are true for the unique state space of the standard model are also true when stated for the common state space of this model, but fail to hold in general.

A natural question is whether agents can agree to disagree and trade in an environment

[1] An overview of the standard model of knowledge is given in Rubinstein (1998). A more philosophical treatment is given in Hintikka (1962).

with unawareness. In the standard model of knowledge this is not possible, if we assume a common prior - Aumann (1976) shows that common knowledge of posteriors implies they are identical. In this model it is shown that the same result is true for common priors and posteriors defined on the common state space. However, an example with two agents i and j demonstrates that although the posteriors defined on the common state space are common knowledge and identical, i's actual posterior is different and beyond j's reasoning because agent i is aware of a theorem that j is unaware of. As a result, the two agents can agree to disagree and trade.

Intuition for this result can be obtained if we interpret common knowledge of posteriors as the outcome of the following procedure. Initially the posteriors are different. Agent i announces his posterior and j updates his information and announces a possibly different posterior. Then, i can update and announce a different posterior, which triggers a new round of updating.[2] Geanakoplos and Polemarchakis (1982) show in the standard model that if the state space is finite, then after finitely many steps the agents will agree on their posteriors. A necessary condition for this result is that partitions are common knowledge. This is true in this model, but only for the common state space. Hence, updating of information due to other agents' actions or announcements still takes place in an environment with unawareness, but it is constrained by what is commonly known that everyone is aware of. As a result, agents can engage in trade when the differences in their posteriors stem from asymmetric information acquired by theorems that others are unaware of.

An example

Consider the following example, which has been cited numerous times in the literature on unawareness. Sherlock Holmes and Dr. Watson are investigating a crime where a horse was stolen from a stable and the keeper was killed. The question they want to answer is whether there was an intruder in the stable. Holmes is the highly sophisticated and intelligent agent who has already solved the mystery, while Watson struggles to keep up. Watson is unable to answer the question because he is unaware that the dog did not bark, and therefore he is also unaware of the theorem that no barking implies no intruder.

Using the example, we can distinguish three features of unawareness. The first is a restricted perception of the world, which limits the agent's reasoning and subsequently what he can potentially know, or know that he does not know. Watson does not know that the dog did not bark, and he does not know that he does not know. He also cannot reason whether Holmes knows whether or not the dog barked. The possibility of the dog not barking simply never crosses his mind - he is unaware of it.

Watson is already aware of the possibility of an intruder, but he does not know whether there was one or not. Although the information about the dog not barking is available to him, he is simply unaware of it. The second feature of unawareness is that available information cannot always be used by the agent. In other words, what Watson is unaware of, constrains his knowledge about events he is aware of.

The third feature of unawareness is that it constrains an agent's ability to reason about the knowledge of others. Unawareness of the theorem "no barking implies no intruder" results

[2] In Galanis (2010) we provide a dynamic version of the current model in order to formalize this story.

not only in Watson not knowing whether there was an intruder, but also in believing that Holmes does not know. In fact, Watson may be aware of many other ways (or theorems) in which Holmes could have known (for example, because he asked a police officer), but Watson has correctly deduced that none of these ways were employed. He therefore inevitably concludes that Holmes does not know whether there was an intruder. In other words, Watson's expressive power is not rich enough to include Holmes' knowledge of no intruder through the specific theorem "no barking implies no intruder". Moreover, Watson, within the bounds of his awareness, is not making a mistake.

However, Watson does make a mistake outside the bounds of his awareness. In particular, Watson underestimates Holmes' knowledge because he falsely believes one of the following. First, that he is aware of everything. Second, that whatever he is unaware of, Holmes is also unaware of. Third, that Holmes is aware of something (e.g. the dog) that Watson is unaware of, but it is of no use to Holmes to determine whether there was an intruder (e.g. because Holmes does not know whether the dog barked or not).[3]

To conclude the example, Holmes and Watson are exposed to the same information and the standard model would specify that they have the same state space and the same partition. However, Watson's reasoning is limited in three ways. First, his expressive power is poorer than Holmes', limiting the events that he knows and the events he knows that he does not know. Second, Watson misses information because he is unaware of its existence and cannot make the necessary deductions. As a result, his knowledge about an event he is aware of is constrained by a theorem that he is unaware of. Finally, Watson incorrectly deduces that Holmes does not know whether there was an intruder. This is not a result of a logical mistake, but of Watson's constrained reasoning, due to his unawareness of the theorem "no barking implies no intruder".

Suppose Holmes pointed out to Watson that the incident of the dog is important. Once Watson becomes aware of the dog, he can collect the information of the dog not barking that was always available to him, become aware of the theorem "no barking implies no intruder", and answer the question whether there was an intruder. Increased awareness can lead to increased knowledge about questions that one was already aware of.

1.2 Related literature

Unawareness and unforeseen contingencies have been studied using decision theoretic and epistemic approaches. In terms of decision theory, Kreps (1979) was the first to model preference for flexibility and derive a subjective state space. Kreps (1992) interpreted this as a model of unforeseen contingencies. Ghirardato (2001), Mukerji (1997), Skiadas (1997) and Nehring (1999) also model unforeseen contingencies. Dekel et al. (2001) derives an essentially unique subjective state space and shows that its size provides a measure of the agent's uncertainty about future contingencies. Sagi (2006) provides a definition for a subjective

[3]These three types of false beliefs cannot be captured by the present model (the same is true for other set-theoretic models in the literature) because knowledge and awareness always refer to specific events. That is, the model is not rich enough to capture notions such as "everything" or "something". Halpern and Rêgo (2009a), Board and Chung (2006), Sillari (2008) and Halpern and Rêgo (2009b) construct syntactic (logic-theoretic) models of agents who can reason about being unaware of "something".

state space that is topologically unique. Epstein et al. (2007) highlights the connection between ambiguity and coarse perceptions by axiomatizing a multiple priors utility without an objective state space. Higashi and Hyogo (forthcoming) axiomatizes a non-Archimedean model with subjective states. Rustichini (2002) extends the preference for flexibility model to many periods and axiomatizes an additive representation over a subjective state space. Finally, Kochov (2011) provides, in a dynamic setting, a definition of unforeseen contingencies that is based on preferences and shows that his model is behaviorally distinct from recursive models of ambiguity.

Models of knowledge (and of unawareness) are either syntactic or semantic (set-theoretic). The two approaches are equivalent, but syntactic models are widely used by logicians and computer scientists, while set-theoretic ones are more common in the economics literature, following Aumann (1976). Beginning with Fagin and Halpern (1988), there has been a stream of syntactic models, namely Halpern (2001), Modica and Rustichini (1994, 1999), Halpern and Rêgo (2008), Halpern and Rêgo (2009a), Heifetz et al. (2008a), Board and Chung (2006) and Galanis (forthcoming). Applications with unawareness have been provided by Modica et al. (1998), Ewerhart (2001), Feinberg (2004, 2005, 2009), Sadzik (2006), Čopič and Galeotti (2007), Li (2006b), Heifetz et al. (2009), Heifetz et al. (2008b), von Thadden and Zhao (2008), Zhao (2008), Filiz-Ozbay (2008), Ozbay (2008), Galanis (2010) and Halpern and Rêgo (2006).

Geanakoplos (1989) provides one of the first set-theoretic models that deals with unawareness, by using non-partitional information structures, defined on a standard state space. However, Dekel et al. (1998) proposes three intuitive properties for unawareness and shows that they are incompatible with the use of a standard state space. Addressing this impossibility result has been achieved with two different approaches. The first is by arguing against one of the properties (Chen et al. (2009)), or by relaxing them (Xiong (2007)). The second is by introducing multiple state spaces, one for each state of awareness. This approach was initiated by Li (2009) and Heifetz et al. (2006) (HMS from now on) and is being followed by the present paper.[4]

Before illustrating the differences between these models, recall that the standard model of knowledge (Aumann (1976)) specifies a unique state space Ω and a possibility correspondence P^i which maps states in Ω to subsets of Ω. The interpretation is that for any $\omega \in \Omega$, the set $P^i(\omega)$ denotes the states that the agent considers possible when ω has occurred. In contrast, modelling unawareness using multiple state spaces leads to a possibility correspondence that maps states of any possible state space to subsets of possibly different state spaces. The reason is that awareness varies with the state. For example, suppose that state $\omega \in \Omega$ specifies that the agent's awareness is different, so that if ω occurs, the agent's state space is Ω', not Ω. Then, the set of states that the agent considers possible, $P^i(\omega)$, is a subset of Ω' and not of Ω. As a result, a model with unawareness has to impose axioms on the possibility correspondence P^i, restricting what it prescribes across different state spaces.

One of the main differences between this model and the set-theoretic models of Li (2009) and HMS is that weaker restrictions are imposed here on what the possibility correspondence P^i prescribes across state spaces.

Li assumes a possibility correspondence P^i, just as in the standard model, which maps

[4] Halpern and Rêgo (2008) and Heifetz et al. (2008a) provide syntactic foundations of HMS.

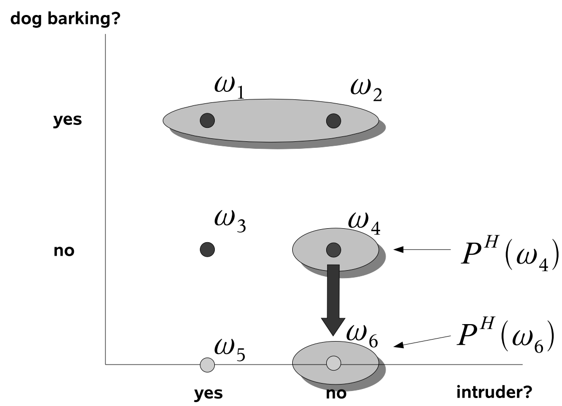

Figure 1: Projecting knowledge downwards.

full states to subsets of the full state space Ω^*, which, in Li's terminology, is the most complete state space.[5] For each full state $\omega^* \in \Omega^*$, $P^i(\omega^*)$ denotes the set of full states that the agent would consider possible if he were fully aware. In Li's terminology, $P^i(\omega^*)$ denotes the agent's factual information. If the agent is not fully aware at ω^*, so that his state space is different from the full state space, then what he actually perceives as possible is the projection of $P^i(\omega^*)$ onto the state space that he is aware of. Similarly, when i reasons about j's knowledge, he projects j's full state partition to i's state space. HMS follow a similar approach. Their property "Projections Preserve Knowledge" requires that if the agent considers states in $P^i(\omega)$ to be possible at ω, then at the projection of ω to a more limited state space S he considers possible the projection of $P^i(\omega)$ to S. In essence, these two properties place a restriction on what the possibility correspondence can prescribe across different state spaces.

In order to illustrate why these two properties are restrictive, recall the Holmes example depicted in Figure 1. The two relevant dimensions or questions of the problem are "Did the dog bark?" and "Was there an intruder?". Holmes is always aware of both questions, so his subjective state space is the full state space, containing the four states $(\omega_1, \omega_2, \omega_3, \omega_4)$ on the plane. At state ω_4, which specifies that there was no intruder and no barking, Holmes knows that there is no intruder because he knows that the dog did not bark and he is also aware of and knows the theorem "no barking implies no intruder". Hence, $P^H(\omega_4) = \omega_4$. At states ω_1 and ω_2 the dog barks and Holmes does not know whether there was an intruder, hence $P^H(\omega_1) = P^H(\omega_2) = \{\omega_1, \omega_2\}$. Finally, state ω_3 is impossible.[6]

Watson is only aware of the question "Was there an intruder?". His subjective state space consists of states ω_5 and ω_6 on the horizontal axis. Since he never knows whether there is an

[5] Since the full state space Ω^* is the most complete state space, only an agent who is fully aware, is also aware of Ω^*. A full state ω^* is an element of the full state space.

[6] Formally, the full state space only contains three states, $S^* = \{\omega_1, \omega_2, \omega_4\}$, since ω_3 is impossible and hence nonexistent. We have included ω_3 in the graph and in the example just for illustrative purposes, to highlight the connection with the theorem "no barking implies no intruder".

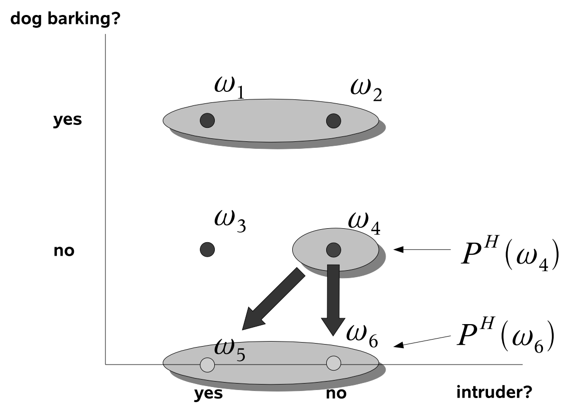

Figure 2: Allowing for unawareness of theorems.

intruder, his information is trivial. Formally, $P^W(\omega) = \{\omega_5, \omega_6\}$ for $\omega \in \{\omega_1, \omega_2, \omega_4, \omega_5, \omega_6\}$.

How will Watson reason at ω_6 about Holmes' knowledge ($P^H(\omega_6)$)? As modeled by Li (2009) and HMS, this is determined by projecting $P^H(\omega_4) = \omega_4$ and $P^H(\omega_2) = \{\omega_1, \omega_2\}$ to the lower state space, because both ω_4 and ω_2 project to ω_6. This is clearly not possible because the first projection yields $\{\omega_6\}$ while the second yields $\{\omega_5, \omega_6\}$. Therefore, we cannot simultaneously have the following. First, Holmes does not know there was no intruder in one state, while he knows in another state because of a theorem that Watson is always unaware of. Second, Watson is certain that Holmes never knows whether there was an intruder.[7]

In order to accommodate the example so that Watson reasons that Holmes does not know whether there is an intruder, we have to abandon projections.[8] When Watson reasons about Holmes at ω_6, he is unaware of the theorem "no barking implies no intruder" and therefore he cannot reason that Holmes is aware of it. We model this by having $P^H(\omega_6) = P^H(\omega_5) = \{\omega_5, \omega_6\}$, so that Watson reasons that Holmes does not know. This is depicted in Figure 2.

The example suggests that unawareness can restrict Watson's reasoning about Holmes' knowledge, concerning an event that both are aware of. This is not captured in other papers that model unawareness. Moreover, Watson makes no mistake within the bounds of his awareness. It is true that with Watson's awareness, Holmes would not know that there is no intruder and Watson can reason only up to his awareness. Essentially, there are two different views of Holmes' knowledge. The first belongs to Holmes and the second to Watson.

[7] It is possible to construct an example, within the HMS and Li (2009) framework, that satisfies the first requirement only. But in that case Watson will be aware that there is a possibility that Holmes knows, although he cannot determine whether this possibility has occurred. This version of the story seems natural because everyone "knows" that Holmes is very smart. However, in my model I also allow for the second requirement which is more plausible if we talk about two "regular" agents, where it is not "commonly known" that one is always smarter than the other.

[8] The only other way of accommodating the example is by allowing for false beliefs. But, as was argued before, this carries the excess baggage of allowing for any kind of false beliefs, even those unrelated to unawareness.

This is formally captured in this model by creating one knowledge operator for each state space S. If Watson's state space is S, then his view of Holmes' knowledge is given by knowledge operator K^i_S. Holmes' state space is S', so his view of Holmes' knowledge is given by $K^i_{S'}$. Moreover, S' is "more expressive" than S. In the model this is captured by having a partial order $\preceq$ on the collection of state spaces. The relationship between the two different views about knowledge is given by the property Awareness Leads to Knowledge. It states that if S' is more expressive than S, then $K^i_{S'}$ gives a better description of one's knowledge than K^i_S. HMS specify one knowledge operator K^i so that there is always one objective view of Holmes' knowledge.

A few clarifications are in order. First, is it the case that these multiple knowledge operators can somehow be incorporated in frameworks that have been proposed by other papers? For instance, is the present model a generalization of HMS? The answer is no. In Galanis (forthcoming) we construct a syntactic model which provides a complete and sound axiomatization of the present model. By comparing it with the syntactic model of Heifetz et al. (2008a), (which is a complete and sound axiomatization of HMS) we show that the model of the present paper is neither a generalization nor a weakening of HMS.

Second, agents are not allowed to make mistakes within the bounds of their awareness. In other words, the axiom of knowledge, when interpreted locally, is not violated, meaning that if an agent knows an event, then the event is true. The axiom is not violated because we allow for multiple knowledge operators, one for each state space. In the example, Watson knows the complement (negation) of the event $K^H_S(\{\omega_6\})$, (Holmes does not know ω_6), where $\{\omega_6\}$ is the event "there is no intruder". The axiom of knowledge says that the complement of $K^H_S(\{\omega_6\})$ must be true, but it does not specify that the complement of $K^H_{S'}(\{\omega_6\})$ should also be true, because Watson is unaware of that event and hence he does not know it. Therefore, although Watson makes a mistake outside the bounds of his awareness (from the perspective of the analyst), within these bounds he remains correct.

Third, outside the bounds of his awareness Watson is only allowed to "make mistakes" about the ignorance of others and only when he lacks awareness that the others have. We show that formally by defining global operator K^i and showing that the axiom of knowledge is violated only when i reasons about j's ignorance, that is, only when $\neg K^j$ is included in the chain of interactive reasoning.

The paper is organized as follows. Section 2 introduces the model and proves the main properties of the knowledge and awareness operators. Section 3 describes the multi-agent model; in particular, common knowledge is defined and characterized in terms of a self evident event. Section 4 examines no-trade theorems. Proofs are contained in the Appendix.

2 The Model

2.1 Preliminaries

Consider a complete lattice of disjoint state spaces $\mathcal{S} = \{S_a\}_{a \in A}$ and denote by $\Sigma = \cup_{a \in A} S_a$ the union of these state spaces. A state ω is an element of some state space S. Let $\preceq$ be a partial order on $\mathcal{S}$. For any S, $S' \in \mathcal{S}$, $S \preceq S'$ means that S' is more expressive than S. Moreover, there is a surjective projection $r^{S'}_S : S' \to S$. Projections are required to commute. If $S \preceq S' \preceq S''$ then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$. If $\omega \in S'$, denote $\omega_S = r_S^{S'}(\omega)$ and $\omega^{S''} = \{\omega' \in S'' : r_{S'}^{S''}(\omega') = \omega\}$. If $E \subseteq S'$, denote by $E_S = \{\omega_S : \omega \in E\}$ the restriction of E on S and by $E^{S''} = \bigcup\{\omega^{S''} : \omega \in E\}$ the enlargement of E on S''. Let $g(S) = \{S' : S \preceq S'\}$ be the collection of state spaces that are at least as expressive as S. For a set $E \subseteq S$, denote by $E^{\uparrow} = \bigcup_{S' \in g(S)} E^{S'}$ the enlargements of E to all state spaces which are at least as expressive as S. Let r_S^S be the identity for any $S \in \mathcal{S}$.

Consider a possibility correspondence $P^i : \Sigma \to 2^{\Sigma} \setminus \emptyset$ with the following properties:

(0) Confinedness: If $\omega \in S$ then $P^i(\omega) \subseteq S'$ for some $S' \preceq S$.

(1) Generalized Reflexivity: $\omega \in (P^i(\omega))^{\uparrow}$ for every $\omega \in \Sigma$.

(2) Stationarity: $\omega' \in P^i(\omega)$ implies $P^i(\omega') = P^i(\omega)$.

(3) Projections Preserve Ignorance: If $\omega \in S'$ and $S \preceq S'$ then $(P^i(\omega))^{\uparrow} \subseteq (P^i(\omega_S))^{\uparrow}$.

(4) Projections Preserve Awareness: If $\omega \in S'$, $\omega \in P^i(\omega)$ and $S \preceq S'$ then $\omega_S \in P^i(\omega_S)$.

(5) Projections Preserve Knowledge: If $S \preceq S' \preceq S''$, $\omega \in S''$ and $P^i(\omega) \subseteq S'$ then $(P^i(\omega))_S = P^i(\omega_S)$.

The setting above is identical to that of HMS. The first difference with the present model is that we drop the last axiom, Projections Preserve Knowledge (PPK). To argue against PPK, consider the example in the introduction. There are two different state spaces, $S' = S'' = \{\omega_1, \omega_2, \omega_4\}$ and $S = \{\omega_5, \omega_6\}$. At ω_4, Holmes is aware of the theorem "no barking implies no intruder" and he knows that there is no intruder. Hence, $P^H(\omega_4) = \{\omega_4\}$. Since the projection of ω_4 to S is ω_6, PPK implies that $P^H(\omega_6) = \{\omega_6\}$. As was argued in the introduction, this is restrictive. In order to allow for $P^H(\omega_6) = \{\omega_5, \omega_6\}$, we drop PPK.

2.2 Events, awareness and knowledge

Dropping PPK means that we also need to change the definitions of knowledge, awareness and events. In the setting of HMS, a set $E \subseteq \Sigma$ is an event if it is of the form $F^{\uparrow}$, where $F \subseteq S$ for some state space $S \in \mathcal{S}$. The negation of E, denoted $\neg E$, is defined as $(S \setminus F)^{\uparrow}$. Hence, an event in the setting of HMS contains states lying in different state spaces. For an event E, knowledge of E is defined to be $K^i(E) = \{\omega \in \Sigma : P^i(\omega) \subseteq E\}$. With PPK, $K^i(E)$ and $\neg K^i(E)$ are also events, so $K^i \neg K^j K^k(E)$, for example, is well defined.

However, if we drop PPK then $K^i(E)$ and $\neg K^i(E)$ may not be events. Consider the following example with two state spaces, $S' = \{\omega_1', \omega_2', \omega_3'\}$ and $S = \{\omega_1, \omega_2, \omega_3\}$, where $S \prec S'$. Moreover, ω_1' projects to ω_1, ω_2' projects to ω_2 and ω_3' projects to ω_3. The agent's possibility correspondence is such that $P^i(\omega') = \{\omega'\}$ for $\omega' \in S'$, $P^i(\omega_1) = P^i(\omega_2) = \{\omega_1, \omega_2\}$, $P^i(\omega_3) = \{\omega_3\}$. The possibility correspondence satisfies all properties except for PPK. Consider the event $E = \{\omega_2, \omega_3, \omega_2', \omega_3'\}$. Then, $K^i(E) = \{\omega_3, \omega_2', \omega_3'\}$ is not an event and hence $\neg K^i(E)$ is not defined.

As was suggested in the introduction, dropping PPK allows for differences in awareness to imply different views about knowledge. In the example, Watson's view of Holmes' knowledge is different from Holmes' view. Hence, together with the objective description of an agent's knowledge there are also several subjective descriptions, one for each state of awareness. Formally, for each state space $S \in \mathcal{S}$, $K^i_S(E)$ captures the subjective (or local) description of the agent's knowledge about E.

Moreover, allowing for different views of knowledge requires that we also change the definition of an event. The reason is that $K^i_S(E)$ describes "knowledge of E, with the vocabulary of state space S". Since we want $K^i_S(E)$ to be an event, we require that an event is a subset of some state space. Hence, contrary to HMS, an event does not contain states lying in different state spaces.

Formally, an *event* is a pair (E, S), where $E \subseteq S$ and $S \in \mathcal{S}$. The negation of (E, S), defined by $\neg(E, S) = (S \setminus E, S)$, is the complement of E with respect to S. Let $\mathcal{E} = \{(E, S) : E \subseteq S, S \in \mathcal{S}\}$ be the set of all events. We write E as a shorthand for (E, S) and $\emptyset_S$ as a shorthand for $(\emptyset, S)$. For each event E, let $S(E)$ be the state space of which it is a subset. An event E "inherits" the expressiveness of the state space of which it is a subset. Hence, we can extend $\preceq$ to a partial order $\preceq_0$ on $\mathcal{E}$ in the following way: $E \preceq_0 E'$ if and only if $S(E) \preceq S(E')$. Abusing notation, we write $\preceq$ instead of $\preceq_0$.

Before defining knowledge, we need to define awareness. For any event E, for any state space S such that $S \succeq E$, define

$$A^i_S(E) = \{\omega \in S : E \preceq P^i(\omega)\}$$ [9]

to be the event which describes, with the vocabulary of S, that the agent is aware of event E. The agent is aware of an event whenever his possibility set resides in a state space that is rich enough to express event E. Unawareness is defined as the negation of awareness. More formally, for any state space S such that $S \succeq E$, the event $U^i_S(E)$ describes, with the vocabulary of S, that the agent is unaware of E:

$$U^i_S(E) = \neg A^i_S(E) = (S \setminus A^i_S(E), S).$$

Let $\Omega^i : \Sigma \to \mathcal{S}$ be such that for any $\omega \in \Sigma$, $\Omega^i(\omega) = S$ if and only if $P^i(\omega) \subseteq S$. $\Omega^i(\omega)$ denotes the most expressive universal event that the agent is aware of at ω. We can therefore interpret $\Omega^i(\omega)$ as agent i's state space at ω. An agent knows an event E if he is aware of it and in all the states he considers possible, E is true. Formally, for any event E, for any state space S such that $S \succeq E$, define

$$K^i_S(E) = \{\omega \in A^i_S(E) : P^i(\omega) \subseteq E^{\Omega^i(\omega)}\}$$ [10]

to be the event which describes, with the vocabulary of S, that the agent knows event E.

Since there are many knowledge operators, one for each state space S, a natural question that arises is how do we determine which is used on every instance. As the motivating

[9] An equivalent definition is $\tilde{A}_S(E) = \{\omega \in S : P^i(\omega) \subseteq S'$ with $S' \succeq S(E)\}$. This is exactly the definition of the awareness operator in HMS, except that here it is restricted to states in S.

[10] Note that conceptually it is not possible to define $A^i_S(E)$, $U^i_S(E)$ and $K^i_S(E)$ when $S \not\succeq E$. The reason is that one cannot describe the awareness and knowledge of E in a language that cannot describe E. Such a definition would create other problems. To give an example, suppose that we defined $A^i_S(E) = (\emptyset, S)$ and $S \prec E$. Then, $U^i_S(E) = (S, S)$. If an agent is aware of S but not of E at $\omega \in S$, we would have that $\omega \in K^i_S U^i_S(E)$, which is undesirable.

example (and Proposition 1 below) shows, if $S' \succ S$ then $K^i_{S'}$ gives a more accurate picture of i's knowledge than K^i_S does. Hence, if j's (most complete) state space is S, she will use K^i_S to describe i's knowledge. See Section 2.4 for a discussion of interactive reasoning.

2.3 Awareness Leads to Knowledge

The next property is the most important departure from other models dealing with unawareness.

Proposition 1. Awareness Leads to Knowledge
If $E \preceq S \preceq S'$ *then* $K^i_S(E) \subseteq (K^i_{S'}(E))_S \cap A^i_S(E)$.

This feature is new. On the one hand, the standard model assumes an agent who is aware of everything and knows all relevant theorems. On the other hand, the property Projections Preserve Knowledge of HMS implies that $K^i_S(E) = (K^i_{S'}(E))_S \cap A^i_S(E)$. Nothing is lost by describing knowledge in less expressive state spaces.

The condition $E \preceq S \preceq S'$ ensures that S and S' are rich enough to describe the agent's knowledge and awareness of E, so that $K^i_S(E)$, $K^i_{S'}(E)$ and $A^i_S(E)$ are well defined. The property says that state spaces which are more expressive give a more complete description of the agent's knowledge. In other words, whatever we capture by describing knowledge with S, we can capture by describing knowledge with the more expressive S'. But the converse is not true.

Recall the example in the introduction. On the one hand, Holmes is aware of S' and state $\omega \in S'$ specifies that the dog did not bark, there is no intruder, and because of the theorem "no barking implies no intruder", Holmes knows event E, "there is no intruder". Hence, $\omega \in K^H_{S'}(E)$. On the other hand, Watson is aware of S, and his limited perception of the truth is ω_S, specifying that there is no intruder and that Holmes is aware of E, so $\omega_S \in A^H_S(E)$. The property allows for $\omega_S \notin K^H_S(E)$, so that according to Watson's limited view, Holmes does not know E at ω_S.

As was argued in the introduction, the reason behind Watson's reasoning about Holmes is Watson's unawareness of the theorem "no barking implies no intruder". Intuitively, if a state space is more complete then it may also include more "theorems", and in effect contain more ways in which an agent can know an event. Conversely, what an agent is unaware of constrains his knowledge about events he is aware of. In the multi-agent case an agent's limited awareness may lead to incomplete reasoning about other agents' knowledge. See Section 3.1 for further discussion and illustration of this property in the multi-agent context.

The Awareness Leads to Knowledge property can best be understood as a comparative statics property. It specifies what happens to the description of knowledge as we move to richer vocabularies. It does not, however, specify what would happen if the agent's awareness were higher because the model is static and the full state that has occurred has determined everything about the agents' awareness and knowledge. In other words, Holmes' knowledge is independent of Watson's awareness and the vocabulary that he uses. But Watson's perception of Holmes' knowledge is not.

2.4 Interactive reasoning

The Awareness Leads to Knowledge property shows that the description of knowledge depends on the state space/vocabulary used for that description. In other words, if $S'' \succ S'$ then $K^i_{S''}(E)$ is not just the enlargement of $K^i_{S'}(E)$ but can be a very different event qualitatively.

This is one of the main differences between the present model and those of HMS and Li (2009) and has direct implications for modelling interactive knowledge. In particular, the sentence "i knows that j knows E" has only one interpretation in those models (and in the standard model) because there is one knowledge operator. In this model however, one needs to specify in which state space j's knowledge of E is described. In other words, because $K^j_{S''}(E)$ is not just the enlargement of $K^j_{S'}(E)$, event $K^i_S K^j_{S''}(E)$ is qualitatively different from $K^i_S K^j_{S'}(E)$.

However, one can also interpret sentence "i knows E" as describing i's knowledge of E in all possible state spaces. We define

$$K^i(E) = \bigcup_{S \succeq E} K^i_S(E),$$

which contains all descriptions of i's knowledge of event E, $K^i_S(E)$, for all state spaces that are rich enough to describe E. Similarly, we define i's ignorance of E as $\neg K^i(E) = \bigcup_{S \succeq E} \neg K^i_S(E)$, her awareness $A^i(E) = \bigcup_{S \succeq E} A^i_S(E)$ and her unawareness $U^i(E) = \bigcup_{S \succeq E} U^i_S(E)$.[11] Note that $K^i(E)$ is not an event anymore as it contains states from different state spaces. If property PPK was assumed in the model then $K^i(E)$ would be an event according to the definition of Heifetz et al. (2006). Here, however, this is not true because PPK is violated, as is also shown in the motivating example.

Still, we can define interactive knowledge using these **global** operators by saying that "i knows that j knows E at ω" if for all states that i considers possible, $P^i(\omega)$, j knows E. Therefore, we define that $\omega \in K^i K^j(E)$ if $P^i(\omega) \subseteq K^j(E)$, which means that $P^i(\omega) \subseteq K^j_S(E)$ for some state space S.

We generalize the above for higher orders of interactive reasoning among any finite number of agents. Formally, if E' is not an event and therefore contains states from different state spaces, define knowledge,

$$K^i(E') = \{\omega \in \Sigma : \omega \succeq \omega' \text{ for some } \omega' \in E' \text{ and } P^i(\omega) \subseteq E'\},$$

ignorance,

$$\neg K^i(E') = \{\omega \in \Sigma : \omega \succeq \omega' \text{ for some } \omega' \in E' \text{ and } P^i(\omega) \nsubseteq E'\},$$

awareness,

$$A^i(E') = \{\omega \in \Sigma : \omega \succeq \omega' \text{ for some } \omega' \in E' \text{ and } \Omega^i(\omega) \succeq \omega' \text{ for some } \omega' \in E'\}$$

[11] Note that if $S \succeq E$ and $K^i_S(E)$ is defined but empty, then $\emptyset_S \in K^i(E)$.

and unawareness,

$$U^i(E') = \{\omega \in \Sigma : \omega \succeq \omega' \text{ for some } \omega' \in E' \text{ and } \Omega^i(\omega) \not\succeq \omega' \text{ for all } \omega' \in E'\}.$$

State ω belongs to $K^i(E')$ if two conditions are satisfied. First, ω can describe E', that is, $\omega \succeq \omega'$ for some state $\omega' \in E'$. Second, all the states considered possible at ω are contained in E'. The other operators are interpreted similarly. We make the convention that if $\omega \in S$ satisfies the first condition but no state in S satisfies the second condition, then $\emptyset_S \in K^i(E)$, and similarly for the other operators.[12] Hence, the sentence "i knows that j knows that k knows that l knows event E" is described by $K^iK^jK^kK^l(E)$, and similarly for higher orders or sentences that involve ignorance, awareness and unawareness. The difference of these global operators with the local operators K^i_S, A^i_S is that K^i gives the description of i's knowledge in all state spaces that can describe E.

2.5 Properties of the operators

The next two Theorems verify properties that have been proposed in the literature, or are generalizations of properties of the standard model. The first Theorem describes properties of the local operators K^i_S, $\neg K^i_S$, A^i_S, U^i_S, whereas the second Theorem describes the properties of the global operators K^i, $\neg K^i$, A^i and U^i. Note that the same properties hold in the multi-agent model as well.

2.5.1 Local operators

Theorem 1. *Suppose E, F are events and $E, F \preceq S$. Then,*

1. **Subjective Necessitation** *For all $\omega \in S$, $\omega \in K^i_S(\Omega^i(\omega))$.*
2. **Monotonicity** $E^{S(E)\vee S(F)} \subseteq F^{S(E)\vee S(F)}$, $F \preceq E \implies K^i_S(E) \subseteq K^i_S(F)$.
3. **Conjunction** $K^i_S(E) \cap K^i_S(F) = K^i_S\left(E^{S(E)\vee S(F)} \cap F^{S(E)\vee S(F)}\right)$.
4. **The Axiom of Knowledge** $K^i_S(E) \subseteq E^S$.
5. **The Axiom of Transparency** $\omega \in K^i_S(E) \iff \omega \in K^i_S(K^i_{\Omega^i(\omega)}(E))$.
6. **The Axiom of Wisdom** $\omega \in A^i_S(E) \cap \neg K^i_S(E) \iff \omega \in K^i_S(A^i_{\Omega^i(\omega)}(E) \cap \neg K^i_{\Omega^i(\omega)}(E))$.
7. **Plausibility** $U^i_S(E) \subseteq \neg K^i_S(E) \cap \neg K^i_S \neg K^i_S(E)$.
8. **Strong Plausibility** $U^i_S(E) \subseteq \neg K^i_S(E) \cap \neg K^i_S \neg K^i_S(E) \cap \ldots \cap \neg K^i_S \neg K^i_S \ldots \neg K^i_S(E)$.
9. **AU Introspection** $U^i_S(E) \subseteq U^i_S U^i_S(E)$.
10. **KU Introspection** $K^i_S U^i_S(E) = \emptyset_S$.
11. **Symmetry** $U^i_S(E) = U^i_S(\neg E)$.

[12] Note that if $\emptyset_S \in E'$ and $\Omega^i(\omega) \succeq S$, then $\omega \in A^i(E')$.

12. **AA-Self Reflection** $\omega \in A^i_S(E) \iff \omega \in A^i_S(A^i_{\Omega^i(\omega)}(E))$.

13. **AK-Self Reflection** $\omega \in A^i_S(E) \iff \omega \in A^i_S(K^i_{\Omega^i(\omega)}(E))$.

14. **A-Introspection** $\omega \in A^i_S(E) \iff \omega \in K^i_S(A^i_{\Omega^i(\omega)}(E))$.

The first six properties are generalizations of the standard properties. Some of these generalizations are proposed by Li (2009). Plausibility, strong plausibility, AU introspection and KU introspection are proposed by Dekel et al. (1998). Symmetry, AA-Self reflection, AK-Self reflection and A-Introspection are proposed by Modica and Rustichini (1999) and Halpern (2001).

The most interesting property is the axiom of knowledge, which specifies that whenever an agent knows an event, then this event is true. Note that the property applies to each knowledge operator, K^i_S, for each state space S. However, what we require here is still weaker than the equivalent property in HMS, which applies the axiom of knowledge to a unique knowledge operator K^i. The reason is that we allow for the case (as in the Watson example) that agent i knows $\neg K^j_S(E)$, he is unaware of S' and both events $\neg K^j_S(E)$ and $K^j_{S'}(E)$ are true.

Subjective necessitation states that at any state ω, the agent knows his state space, which is $\Omega^i(\omega)$. Monotonicity says that if at ω the agent knows event E, he is aware of F and E implies F, then he knows F. These two events may be subsets of different state spaces, so the usual notion of implication, $E \subseteq F$, is not defined. Li (2009) has proposed a generalized version of implication: The event E implies the event F if the enlargement of E to the join of spaces $S(E)$ and $S(F)$ is a subset of the respective enlargement of F. Conjunction states that the agent knows events E and F if and only if he knows that E and F have occurred. If E and F are subsets of different state spaces then their conjunction is the intersection of their enlargements to the meet of state spaces $S(E)$ and $S(F)$.

The next two properties generalize the axioms of transparency and wisdom. The axiom of transparency states that the agent knows an event E at ω if and only if he knows that he knows it at ω. Moreover, this is equivalent to knowing $K^i_{\Omega^i(\omega)}(E)$, which is the event "the agent knows event E", expressed in the awareness of the agent at ω. The axiom of wisdom is similar. The agent is aware of but does not know event E if and only if he knows that he is aware of and does not know it.

Plausibility states that if the agent is unaware of an event, then he does not know it, and he does not know that he does not know it. Strong Plausibility extends the result for any higher order of not knowing that he does not know. AU Introspection specifies that if the agent is unaware of an event, then he is unaware that he is unaware of it. KU Introspection states that the agent cannot know that he is unaware of an event E.

Symmetry states that if an agent is unaware of an event, then he is also unaware of its negation. Properties AA-Self Reflection, AK-Self Reflection and A-Introspection say that equivalent conditions for an agent to be aware of an event is that he is aware that he is aware of it, he is aware that he knows it and he knows that he is aware of it.

2.5.2 Global operators

Say that set E has a **base** if there exists $\omega \in E$ such that for all $\omega' \in E$, $\omega' \succeq \omega$.[13] If ω belongs to state space S we call S the base of E and denote it as $b(E)$. An event has a base by construction. Moreover, it is straightforward to show that if set E has a base, then $X^i(E)$ has a base as well, where $X^i = K^i, \neg K^i, A^i, U^i$. Therefore, only sets with a base can describe interactive reasoning. For a set E with base $b(E)$, its negation is defined as $\neg E = \{\omega \in \Sigma : \omega \notin E \text{ and } \omega \succeq b(E)\}$.

A set E is **expanding** if for state spaces $S \succ S'$, whenever $\omega \in S' \cap E$ we have $\omega^S \subseteq E$. If a set is expanding then whatever is described in a lower state space, it is also described in a higher state space. Note that $A^i(E)$ and $U^i(E)$ are always expanding sets and if E is an event or an expanding set then $K^i(E)$ is expanding, but $\neg K^i(E)$ might not be.[14] In the following Theorem, when it is not specified whether E is an event, the property is proven for both cases.

Theorem 2. *Suppose that E and F have bases.*[15]

1. **Subjective Necessitation** $K^i(\Sigma) = \Sigma$.

2. **Monotonicity** *If E,F are not events then* $E \subseteq F \implies K^i(E) \subseteq K^i(F)$*. If E, F are events then* $E^{S(E)\vee S(F)} \subseteq F^{S(E)\vee S(F)}$, $F \preceq E \implies K^i(E) \subseteq K^i(F)$.

3. **Conjunction** *If E,F are not events then* $K^i(E) \cap K^i(F) = K^i(E \cap F)$*. If E, F are events then* $K^i(E) \cap K^i(F) = K^i\left(E^{S(E)\vee S(F)} \cap F^{S(E)\vee S(F)}\right)$.

4. **The Axiom of Knowledge** *If E is an event then* $K^i(E) \subseteq \bigcup_{S \succeq E} E^S$*. If E is not an event then, for any $j \in I$,* $K^i A^j(E) \subseteq A^j(E)$ *and* $K^i U^j(E) \subseteq U^j(E)$*. If it is expanding then* $K^i K^j(E) \subseteq K^j(E)$.

5. **The Axiom of Transparency** $K^i(E) = K^i K^i(E)$.

6. **The Axiom of Wisdom** $A^i(E) \cap \neg K^i(E) \subseteq K^i(A^i(E) \cap \neg K^i(E))$.

7. **Plausibility** $U^i(E) \subseteq \neg K^i(E) \cap \neg K^i(\neg K^i(E))$.

8. **Strong Plausibility** $U^i(E) \subseteq \neg K^i(E) \cap \neg K^i \neg K^i(E) \cap \ldots \cap \neg K^i \neg K^i \ldots \neg K^i(E)$.

9. **AU Introspection** $U^i(E) \subseteq U^i U^i(E)$.

10. **KU Introspection** $K^i U^i(E) = \emptyset$.

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[13] Note that it could be that $\omega = \emptyset_S$, for some S.

[14] For $U^i(E)$, suppose $\omega \in U^i(E) \cap S'$ and $S \succ S'$. Then, $\omega \succeq \omega'$ for some $\omega' \in E$ and $P^i(\omega) \not\succeq \omega'$ for all $\omega' \in E$. Take $\omega'' \in S \succ S'$ such that $\omega''_{S'} = \omega$ and suppose $P^i(\omega'') \succeq \omega'$ for some $\omega' \in E$. From generalized reflexivity, $\omega''_{\Omega^i(\omega'')} \in P^i(\omega'') \succ P^i(\omega)$. From PPA and because $\omega''_{S'} = \omega$ we have that $P^i(\omega) \subseteq S' \succeq \omega'$ for some $\omega' \in E$, a contradiction. Therefore, $\omega^S \subseteq U^i(E)$. For $K^i(E)$, suppose E is expanding and $\omega \in K^i(E) \cap S'$, $S \succ S'$. Then, $P^i(\omega) \subseteq E$. From PPI and because E is expanding we have that $P^i(\omega') \subseteq E$ for each $\omega' \in \omega^S$, therefore $\omega^S \subseteq K^i(E)$.

[15] The fact that E, F have bases is only used in the proof of the axiom of knowledge.

11. **Symmetry** $U^i(E) = U^i(\neg E)$.

12. **AA-Self Reflection** $A^i(E) = A^i A^i(E)$.

13. **AK-Self Reflection** $A^i(E) = A^i K^i(E)$.

14. **A-Introspection** $A^i(E) = K^i A^i(E)$.

The interpretation for the properties of the global operators is similar. The most interesting property is the axiom of knowledge, as it shows more clearly how this model differs from other models of unawareness. The property says that if E is an event and agent i knows it, then E is true. Therefore, locally the agent does not make mistakes. However, suppose that E is not an event but has a base, so it can describe interactive reasoning. For example, suppose $E = \neg K^k A^i A^l K^j(E')$ where E' is an event. Then, it is not necessarily the case that E is true, even though i knows E. For a counter example, consider Figure 2. Set $E = \{\omega_1, \omega_2, \omega_5, \omega_6\}$ contains all states that describe that Holmes does not know there was no intruder. That is, $E = \neg K^H(E')$, where $E' = \{\omega_6\}$. At ω_4, Watson considers ω_5 and ω_6 possible, as $P^W(\omega_4) = \{\omega_5, \omega_6\}$. Hence, $\omega_4 \in K^W \neg K^H(E')$. However, $\omega_4 \notin \neg K^H(E)$ because Holmes knows E at ω_4. Therefore, $K^W \neg K^H(E') \nsubseteq \neg K^H(E')$. Therefore, the agent errs in her assessment about the reasoning of others.

Does this mean that the model allows for all possible mistakes? The axiom of knowledge shows that this is not the case, as it restricts the mistakes in interactive reasoning that are allowed. In particular, only if we have $\neg K^j$ somewhere in the chain of interactive reasoning can we have failure of the axiom of knowledge, because only then the set may not be expanding. In all other cases, when an agent knows something about the awareness, unawareness and knowledge of others, then it is true. This result confirms the intuition explained in the motivating example and expressed in the property Awareness Leads to Knowledge.

3 Multi-agent model

3.1 Unawareness and reasoning about others

In a multi-agent context, the property Awareness Leads to Knowledge implies that i's limited awareness may impair his reasoning about j's knowledge. For example, it may be that while i is aware of E, he wrongly deduces that j does not know it, exactly because i is unaware of the theorem that led j to know E. This clearly distinguishes the present approach from that of Li (2009) and HMS, which do not allow for such information processing errors.

To illustrate, suppose that agent i's state space is S^i, while agent j's state space is S^j and $S^j \preceq S^i$, so that i is more aware than j. They are both reasoning whether agent k knows event E. Suppose that both i and j are informed that the true state has occurred. That is, i is informed that ω has occurred, while j is informed that ω_{S^j} has occurred, which is the projection of ω to the more limited state space. Moreover, suppose that $\omega \in K^k_{S^i}(E)$ but $\omega_{S^j} \notin K^k_{S^j}(E)$, which is permitted by the Awareness Leads to Knowledge property. Since i knows that ω has occurred and j knows that ω_{S^j} has occurred, it is the case that i knows that k knows event E, while j knows that k does not know event E! Agents i and j disagree on what k knows.

It is important to emphasize that j's information processing error about k's knowledge is due to j's unawareness, not due to j's logical mistakes. Agent j is not excluding the true state, he merely perceives a limited version of the truth.

The standard model of knowledge excludes the possibility of two agents disagreeing about what a third agent knows. To be more precise, it can never be that i knows that k knows an event, while j knows that k does not know this event. Clearly, if this were to happen then one agent would be wrong, and the axiom of knowledge would be violated. Li (2009) and HMS also exclude such a possibility, because they assume that i's view of j's knowledge is the projection of P^j to i's state space. On the contrary, the present model allows for such a possibility without violating the axiom of knowledge, because knowledge is defined "locally", for each state space.

Consider the following example which illustrates how two agents can disagree on what a third agent knows. Suppose that agent k is inside a basement with no windows, and that it is raining. Agent j is informed that k is inside the basement, so he reasons that because k cannot see what is happening outside, he does not know that it is raining, and j knows that this is the case. On the other hand, agent i is aware of and knows the existence of a computer in the basement, connected with a camera outside the building. If he is informed that k is also aware of and knows this, then he can reason that k can see whether it is raining by checking the computer. Moreover, he knows that this is the case. Concluding, the more aware agent i knows that k knows that it is raining, while the less aware agent j knows that k does not know whether it is raining.

It is worth emphasizing that the source of the two agents' *disagreement stems from their different awareness, not from their different information.* Had j been aware of the possibility of a computer in the basement, even if he did not know whether it is connected with a camera or whether k was aware of it, would enable him to say that he did not know whether k knows that it is raining. In that case, i and j would not disagree, but i would have more information. It is precisely the fact that j is unaware of the possibility of the computer that makes him know that k does not know that it is raining. Moreover, j is not making any mistakes within the bounds of his awareness, because it is true that with this limited awareness, k would not know whether it rained. Finally, this disagreement can only occur if what one agent is unaware of, constrains his knowledge about what he is aware of, so that the "Awareness Leads to Knowledge" property is necessary.

3.2 Common knowledge

We define common knowledge using the global operators and show that there is an equivalent definition using the possibility correspondences, just like in the standard model.

Definition 1. *Event $E \preceq S$ is common knowledge among agents $i = 1, \ldots, I$ at $\omega \in S$ if for any $n \in \mathbb{N}$ and any sequence of agents $i_1, \ldots, i_n$, $\omega \in K^{i_1} K^{i_2} \ldots K^{i_n}(E)$.*

Note that in the standard model, event $K^i K^j(E)$ is the set of states ω such that $P^j(P^i(\omega)) \subseteq E$, where $P^i(E') = \bigcup_{\omega' \in E'} P^i(\omega')$ is the set of states that i considers possible if the truth lies in E'.[16] In words, i knows that j knows E if and only if all the states that i considers possible that j considers possible are contained in event E.

We provide a similar equivalence in Proposition 2. In particular, suppose E is an event and E' is a set of states (not necessarily an event). Slightly abusing notation, write $E' \succeq E$ if for every state $\omega \in E'$, $\{\omega\} \succeq E$. Moreover, if $E' \succeq S$, write $E'_S = \{\omega_S : \omega \in E'\}$ for the set of states that project to S.[17]

Proposition 2. *Event $E \preceq S$ is common knowledge among agents $i = 1, \ldots, I$ at $\omega \in S$ if and only if for any $n \in \mathbb{N}$ and any sequence of agents $i_1, \ldots, i_n$, $P^{i_n} \ldots P^{i_1}(\omega) \succeq E$ and $(P^{i_n} \ldots P^{i_1}(\omega))_{S(E)} \subseteq E$.*[18]

This is a direct generalization of the standard definition of common knowledge, which specifies that, for any sequence $i_1 \ldots i_n$ of agents, $P^{i_n} \ldots P^{i_1}(\omega) \subseteq E$. It says that all the states that i_1 considers possible that i_2 considers possible that $\ldots i_n$ considers possible are in a higher state space than $S(E)$ and when projected to $S(E)$ they are contained in E.

For each state space S, the event "Event E is common knowledge", denoted $CK_S(E)$, is the set of states $\omega \in S$ such that E is common knowledge at ω. If S is the uppermost state space then $CK_S(E)$ is the most complete description of common knowledge of E.[19] If S is not the uppermost state space then $CK_S(E)$ is a subjective description of common knowledge. In other words, $CK_S(E)$ denotes what the agent would consider as the expression of common knowledge of E if his state space was S. One of the main results of the paper is that more complete state spaces give a better description of one's knowledge. Similarly, they give a better description of common knowledge. This property is expressed in the following Lemma.

Lemma 1. *If $E \preceq S \preceq S'$ then $CK_S(E) \subseteq (CK_{S'}(E))_S$.*

3.3 Common knowledge of awareness

Recall that $\Omega^i(\omega)$ is the most expressive universal event that agent i is aware of at ω (his state space). What is i's view of j's most expressive universal event? Define $\Omega^{ij}(\omega)$ to be the most expressive universal event that i knows that j is aware of:

$$\Omega^{ij}(\omega) = \bigwedge_{\omega' \in P^i(\omega)} \Omega^j(\omega').$$[20]

[16] For details, see Geanakoplos (1992).

[17] Note that $P^i(E') = \bigcup_{\omega \in E'} P^i(\omega)$ is not necessarily an event.

[18] For simplicity, we write $P^{i_n} \ldots P^{i_1}(\omega)$ instead of $P^{i_n}(P^{i_{n-1}}(\ldots P^{i_1}(\omega)))$ from now on.

[19] Such a space exists because $\mathcal{S}$ is assumed to be a complete lattice.

[20] Note that we can have, for example, $P^i(\omega) = \{\omega, \omega'\}$, $\Omega^j(\omega) = S$, $\Omega^j(\omega') = S'$ and S, S' are not comparable. Then, $\Omega^{ij}(\omega) = S \wedge S'$, although i is certain that j's awareness is higher than $S \wedge S'$. However, $\Omega^{ij}(\omega)$ represents the most expressive universal event that i knows that j is aware of and this is $S \wedge S'$, even though he also knows that j is more aware than that (but cannot specify the direction). This is similar to a situation in the standard model with non-partitional structures. For example, suppose that the state space is $S = \{\omega, \omega_1, \omega_2\}$. Agent i has the partition $P^i(\omega) = P^i(\omega_1) = \{\omega, \omega_1\}$ and $P^i(\omega_2) = \{\omega_2\}$. Agent j has

Note that state $\omega' \in \Omega^i(\omega)$ specifies that the most expressive universal event that j is aware of is $\Omega^j(\omega')$. But i does not necessarily know what state has occurred - he only knows that one state in $P^i(\omega)$ has occurred. $\Omega^{ij}(\omega)$ is therefore the meet of all the most expressive universal events that, according to i's knowledge, j could be aware of.

We can now define the most expressive universal event that, at ω, i knows that j knows that k is aware of to be

$$\Omega^{ijk}(\omega) = \bigwedge_{\omega' \in P^j(P^i(\omega))} \Omega^k(\omega').$$

Adding more agents to the sequence can easily be accommodated. For $n \geq 2$, define

$$\Omega^{i_1 \dots i_n}(\omega) = \bigwedge_{\omega' \in P^{i_{n-1}} \dots P^{i_1}(\omega)} \Omega^{i_n}(\omega')$$

to be the most expressive universal event that i_1 knows that i_2 knows that ... that i_n is aware of at ω. The following Lemma shows that there is an equivalent definition.

Lemma 2. *For any sequence $i_1, \dots, i_n$,*

$$\Omega^{i_1 \dots i_n}(\omega) = \bigwedge_{\omega' \in P^{i_1}(\omega)} \Omega^{i_2 \dots i_n}(\omega').$$

Define $\Omega^\wedge(\omega)$ to be the meet of all state spaces $\Omega^{i_1 \dots i_n}(\omega)$, for any sequence $i_1, \dots, i_n$, $n \in \mathbb{N}$:

$$\Omega^\wedge(\omega) = \bigwedge_{\substack{i_1 \dots i_n \\ n \in \mathbb{N}}} \Omega^{i_1 \dots i_n}(\omega).$$

Lemma 3. *$\Omega^\wedge(\omega)$ is common knowledge at $\omega \in S$. Moreover, if $E \in \mathcal{E}$ is common knowledge at ω then $E \preceq \Omega^\wedge(\omega)$.*

The Lemma states that each state ω specifies a universal event $\Omega^\wedge(\omega)$, that every agent is aware of and this fact is common knowledge. Moreover, $\Omega^\wedge(\omega)$ is the most complete universal event with this property, because any event E that is common knowledge at ω can be expressed within the vocabulary of $\Omega^\wedge(\omega)$. We can therefore interpret $\Omega^\wedge(\omega)$ as the "common" state space at ω.

In the standard model, the set of states that are reachable from ω (the union of $P^{i_n} \dots P^{i_1}(\omega)$ for any sequence $i_1 \dots i_n$ of agents) is partitioned by each agent's possibility correspondence. In the present model, the set E of states that are reachable from ω contains states from different state spaces. But if we assume that $(\mathcal{S}, \preceq)$ is well-founded, then $\Omega^\wedge(\omega) = \Omega^{i_1 \dots i_n}(\omega)$ for some finite sequence $i_1 \dots i_n$ of agents.[21] The set $E \cap \Omega^\wedge(\omega)$ is partitioned by each agent's possibility correspondence, just like in the standard model. Moreover, every state in $E \setminus \Omega^\wedge(\omega)$, when projected to $\Omega^\wedge(\omega)$, is contained in $E \cap \Omega^\wedge(\omega)$. Conversely, any state space containing a subset which is partitioned by each agent's possibility correspondence is equal to $\Omega^\wedge(\omega)$ for some ω.

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a non-partitional structure, so that $P^j(\omega) = P^j(\omega_2) = \{\omega, \omega_2\}$ and $P^j(\omega_1) = \{\omega_1, \omega_2\}$. Then, the smallest event that, at ω, i knows that j knows is $\{\omega, \omega_1, \omega_2\}$, even though i realizes that j knows more than that, without being able to specify the direction.

[21] See Section 3.4 for an explanation of the well-founded assumption.

3.4 Characterizing common knowledge

In the standard model an event E^* is common knowledge at ω if and only if there is an event E which is self evident for all agents, it contains ω and is a subset of E^*. The following Theorem provides a similar characterization of common knowledge in an environment with unawareness. The definition of a self evident event is given below, and it is a direct analog of the standard definition. Recall that if E is an event then $S(E)$ is the state space of which it is a subset.

Definition 2. *Event E is self evident for $i \in I$ if $E \subseteq K^i_{S(E)}(E)$. If E is self evident for all $i \in I$, then it is called public.*

An event E is self evident for agent $i \in I$ if whenever it happens, the agent knows it. It is public if everyone knows it.

The following Theorem provides necessary and sufficient conditions for event E^* to be common knowledge, namely that there is a public event E whose enlargement to state space S contains $\omega \in S$, it is more expressive than E^* ($E^* \preceq E$) and it is a subset of the enlargement of E^* to S. To prove it we make the additional assumption that $(\mathcal{S}, \preceq)$ is well-founded, so that any non-empty subset X of $\mathcal{S}$ contains a $\preceq$-minimal element.[22] [23] Meier and Schipper (2010) was the first paper to introduce well-founded unawareness structures to prove a no-trade theorem.

Theorem 3. *Event E^* is common knowledge at $\omega \in S$ if and only if there exists a public event E such that $E^* \preceq E \preceq S$ and $\omega \in E^S \subseteq E^{*S}$.*

4 No-trade theorems

The standard model of knowledge specifies that asymmetric information alone cannot explain trade. In this section we provide, with an example and a Theorem, an explanation of why agents with asymmetric information *and* asymmetric awareness can engage in trade.

The literature on no-trade theorems stems from the well known result of Aumann (1976) that if agents have common priors and their posteriors about an event are common knowledge, then these posteriors must be identical. This section shows that in an environment with unawareness the same result is true only for common priors and posteriors which are defined on the "common" state space, which is the state space that not only everyone is aware of, but it is also common knowledge that everyone is aware of. However, as the property Awareness Leads to Knowledge suggests, state spaces which carry more awareness give a more complete description of one's knowledge and posteriors. An example with two agents shows that although the posteriors defined on this "common" state space are common knowledge and therefore identical, there still can be trade because one agent's higher awareness implies that his actual posterior is different and beyond the other agent's reasoning.

Since the result of Aumann (1976) requires a common prior, we also impose one here. In particular, we let π be a "common" prior on the most complete state space S^*. The prior

[22] This means that there is a $S \in X$ such that, for all $S' \in X$, if $S' \preceq S$ then $S' = S$.

[23] I thank a referee for pointing out the well-founded assumption.

on each state space S is then the marginal of π on S. The interpretation is that two agents, with possibly different state spaces, always agree on the prior probability of events they are both aware of.[24]

Let π be a prior on the most complete state space S^* and assume, for simplicity, that there are, at most, countably many states in S^*. The prior μ on state space S, where $S \preceq S^*$ is the marginal of π on S. That is, for $\omega \in S$, $\mu(\omega) = \pi(\omega_{S^*})$, where ω_{S^*} is the set of states in S^* that project to ω. Suppose that at ω' the agent's awareness is $\Omega^i(\omega') = S$. Let E be an event that the agent is aware of, that is, $E \preceq S$. His posterior about E is

$$q^i(E)(\omega') = \frac{\mu(P^i(\omega') \cap E^S)}{\mu(P^i(\omega'))},$$

where E^S is the set of states in S that project to E.

Let $I = \{i, j\}$ and suppose μ is the prior on $\Omega^\wedge(\omega)$, the most complete state space that is commonly known at ω that both agents are aware of. Let $E \subseteq \Omega^\wedge(\omega)$ be an event. Agent i's posterior about E at $\omega' \in \Omega^\wedge(\omega)$ is given by $q^i(E)(\omega')$. Event E^* specifies that both agents are aware of E and that i's posterior is q^i, while j's posterior is q^j:

$$E^* = \left\{\omega' \in \Omega^\wedge(\omega) : q^i(E)(\omega') = q^i, q^j(E)(\omega') = q^j\right\}.$$

The following Theorem gives conditions under which common knowledge of posteriors implies they are equal, reproducing the result of Aumann (1976) for the common state space. We assume, as in Aumann (1976), that $\mu(P^i(\omega'') \cap P^j(\omega'')) > 0$ for all $\omega'' \in \Omega^\wedge(\omega)$.

Theorem 4. *Suppose that E^* is common knowledge at ω, so that it is common knowledge that i's posterior is q^i and j's posterior is q^j. Then, we have $q^i = q^j$.*

Theorem 4 states that if the posteriors defined on the common state space are common knowledge, they are identical. The following example shows that if an agent's awareness is bigger than the common one, then his actual posterior may be different and beyond the other agent's reasoning. Hence, agents can agree to disagree and trade.

Example

Recall the example in the introduction, depicted in Figure 3.

There are two agents, Holmes and Watson. There are two state spaces, $S^* = \{\omega_1, \omega_2, \omega_4\}$ and $S = \{\omega_5, \omega_6\}$.[25] The union of S and S^* is Σ. Watson is always unaware of the extra dimension and his possibility correspondence is such that $P^W(\omega) = \{\omega_5, \omega_6\}$, for all $\omega \in \Sigma$. Holmes' possibility correspondence is as follows:

$$P^H(\omega_1) = P^H(\omega_2) = \{\omega_1, \omega_2\},$$

$$P^H(\omega_4) = \{\omega_4\},$$

[24] For a general discussion on the interpretation of common priors in an environment with unawareness, see section 4.1 in Heifetz et al. (2009).

[25] State ω_3 is impossible so we do not include it in S^*.

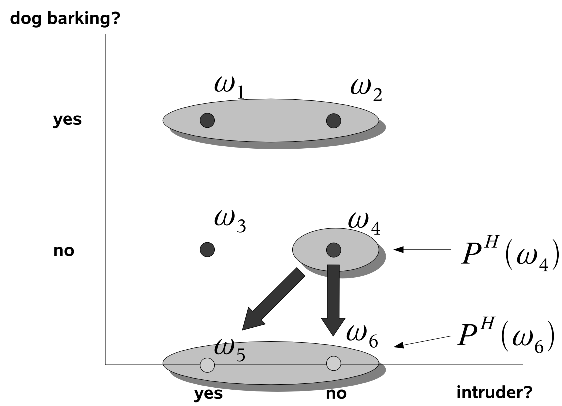

Figure 3: Holmes bets there was no intruder

$$P^H(\omega_5) = P^H(\omega_6) = \{\omega_5, \omega_6\}.$$

At ω_4, Holmes is aware of both dimensions and because he knows the theorem "no barking implies no intruder" and he receives information about the dog not barking, he is able to deduce that there is no intruder. However, Watson, being unaware of the dog, reasons that Holmes' possibility correspondence is $P^H(\omega_5) = P^H(\omega_6) = \{\omega_5, \omega_6\}$.

Let the common prior π on S^* be such that $\pi(\omega_1) = 1/2$ and $\pi(\omega_2) = \pi(\omega_4) = 1/4$. The common state space is S. Then, the common prior μ on S is such that $\mu(\omega_5) = \mu(\omega_6) = 1/2$. Suppose that Holmes and Watson bet on whether there is an intruder, that is, on the occurrence of event $E = \{\omega_6\}$. The posterior of Holmes about E at ω is $q^H(E)(\omega)$, while Watson's is $q^W(E)(\omega)$. Note that different state spaces give different descriptions of posteriors.

As discussed in Section 3.3, an event can be common knowledge only if it is expressed in S.[26] At ω_4, event $\{\omega_5, \omega_6\}$ is common knowledge, specifying that Holmes' posterior is $q^H(E)(\omega_5) = q^H(E)(\omega_6)$, while Watson's posterior is $q^W(E)(\omega_5) = q^W(E)(\omega_6)$. In accordance with Theorem 4, $q^H(E)(\omega_5) = q^H(E)(\omega_6) = q^W(E)(\omega_5) = q^W(E)(\omega_6) = 1/2$.

For Watson this is the end of the story, since he is unaware of the extra dimension of the dog not barking. However, Holmes is more aware. At ω_4, he knows that there is no intruder and hence he is willing to bet. His posterior about E at ω_4 is 1. Hence, although the posteriors described in S are common knowledge and equal, Holmes' "actual" posterior is different.

Discussion

Theorem 4 shows that whenever the posteriors defined on the common state space are common knowledge, they are identical. Nevertheless, the example showed that if Holmes is more aware, his true posterior may be different and beyond the other agent's reasoning.

[26] Or a less complete state space, which does not exist in this example.

Hence, agents can agree to disagree and trade. Note that we could have easily specified that also Watson was more aware in other dimensions that Holmes is unaware of. In that case, his true posterior would also be beyond Holmes' reasoning. But this was not necessary in order to have trade.

Intuition for this result can be obtained if we interpret the equality of the posteriors as the outcome of the following procedure, described in the context of the standard model of knowledge by Geanakoplos and Polemarchakis (1982). Suppose that initially Holmes and Watson have different posteriors about E, and in particular Holmes has a posterior above a half and wants to buy, while Watson has a posterior below a half, and wants to sell. Suppose that they meet and they announce their posteriors and their willingness to trade. Holmes can then use Watson's announcement in order to further refine what he knows, by taking the intersection of his own information with the set of states that describe a posterior below a half for Watson. Holmes can now announce a possibly different posterior which reflects his new information, while Watson can use Holmes' announcement to further refine his own information. Geanakoplos and Polemarchakis (1982) shows that the agents will eventually agree on the posteriors.

A necessary condition for this result is that partitions are common knowledge, which is true in the standard model. It is also true in this model but only for state spaces that is common knowledge that everyone is aware of. Therefore, the updating of the posteriors that was described above can only refer to such a common state space. If Holmes is more aware, then announcing his "true" posterior or his willingness to buy will be of no value to Watson, because he is simply unaware of the states that would enable Holmes to make these announcements. As a result, Watson cannot further refine his own knowledge. Updating of information due to other agents' actions or announcements still takes place in an environment with unawareness, but it is constrained by what is common knowledge that everyone is aware of. Hence, agents can engage in trade when the differences in their posteriors stem from asymmetric information acquired by theorems that others are unaware of.[27]

Concluding, we need to emphasize that the purpose of the example is not to show that there *can* be trade. This can easily be shown within the framework of the standard model, by assuming that agents have different priors or that they make mistakes. Unawareness is a special type of a mistake because the agent is completely rational within the bounds of his awareness and he cannot recognize that he has committed one, unless his awareness increases. Therefore, the purpose of the example is to isolate this particular type of mistake (unawareness of theorems) and use it to provide an interesting or plausible story of why (otherwise rational) agents might trade.

HMS and Heifetz et al. (2009) also provide alternative examples of speculative trade in an environment with unawareness. In their setting, an owner contemplates selling his firm to a potential buyer. The "common" state space specifies that the value of the firm can be either 100 or 80. The owner is aware of a possible lawsuit that could decrease the firm's

[27] In Galanis (2010) we construct a dynamic version of the present model and examine how agents update their awareness, when they exchange their posteriors. The mechanism is that if an agent hears an announcement that he did not expect (it was a zero probability event for him), he is able to increase his awareness as much as necessary, so as to rationalize the announcement. We then determine the direction of the awareness updating.

value by 20, but not of a possible novelty that could increase its value by 20. The potential buyer is aware of the novelty but not of the lawsuit. It is shown that "there is common certainty of preference to trade, but each player strictly prefers to trade".

Both examples in Heifetz et al. (2009) and in the present paper specify a "common" state space where agents are indifferent between trading or not, but are willing to trade in their respective, more complete, state spaces.[28] But the reason is different. In the example of Heifetz et al. (2009), differences in awareness imply differences in the perception of the payoff relevant state space and wrong reasoning about the other agent's perception. The owner thinks that there are three payoff relevant states, yielding 60, 80 and 100, while the buyer considers payoff relevant states yielding 80, 100 and 120. Both think that the other agent's payoff relevant states yield only 80 and 100.

In the example of the present paper, differences in awareness imply differences in posteriors about events of the common state space and wrong inferences about the other agent's posterior. But both agents agree on the deterministic payoffs of the common state space, that is, they agree on what constitutes the payoff relevant state space, which specifies whether there is an intruder. Watson's posterior about the event "there is no intruder" is 1/2, while he wrongly deduces that Holmes' posterior about the same event is also 1/2.

A Appendix

Proof of Proposition 1.

First we prove that if $S \preceq S'$, then $(K^i_S(E))^{S'} \subseteq K^i_{S'}(E)$. Suppose $\omega \in (K^i_S(E))^{S'}$. Then, $\omega_S \in K^i_S(E)$, which implies that $E \preceq P^i(\omega_S)$ and $P^i(\omega_S) \subseteq E^{\Omega^i(\omega_S)}$. Projections Preserve Ignorance implies that $E \preceq P^i(\omega_S) \preceq P^i(\omega)$ and $P^i(\omega) \subseteq (P^i(\omega_S))^{\Omega^i(\omega)} \subseteq E^{\Omega^i(\omega)}$. Hence, $\omega \in K^i_{S'}(E)$. Finally, $(K^i_S(E))^{S'} \subseteq K^i_{S'}(E)$ implies $K^i_S(E) \subseteq (K^i_{S'}(E))_S$. Also, note that $K^i_S(E) \subseteq A^i_S(E)$. That the other direction is not necessarily true is shown by the main example.

□

Proof of Theorem 1.

1. **Subjective Necessitation** Suppose $\omega \in S$. Confinedness implies that $P^i(\omega) \subseteq S'$ for some $S' \preceq S$. Since $\Omega^i(\omega) = S'$ we have $\omega \in K^i_S(\Omega^i(\omega))$.

2. **Monotonicity** Suppose $\omega \in K^i_S(E)$. Then, $E \preceq P^i(\omega)$ and $P^i(\omega) \subseteq E^{\Omega^i(\omega)}$. Also, $F \preceq P^i(\omega)$ which implies $S(E) \vee S(F) \preceq \Omega^i(\omega)$ and $E^{\Omega^i(\omega)} \subseteq F^{\Omega^i(\omega)}$. Therefore, $\omega \in K^i_S(F)$.

3. **Conjunction** We have that $E \preceq P^i(\omega)$ and $F \preceq P^i(\omega)$ if and only if $S(E) \vee S(F) \preceq P^i(\omega)$. Also, $P^i(\omega) \subseteq E^{\Omega^i(\omega)}$ and $P^i(\omega) \subseteq F^{\Omega^i(\omega)}$ if and only if $P^i(\omega) \subseteq E^{\Omega^i(\omega)} \cap F^{\Omega^i(\omega)} = (E^{S(E)\vee S(F)} \cap F^{S(E)\vee S(F)})^{\Omega^i(\omega)}$. The latter equality follows because $\omega_1 \in (E^{S(E)\vee S(F)} \cap F^{S(E)\vee S(F)})^{\Omega^i(\omega)} \iff \{\omega_1\}^{S(E)\vee S(F)} \in E^{S(E)\vee S(F)} \cap F^{S(E)\vee S(F)} \iff \omega_1 \in E^{\Omega^i(\omega)} \cap F^{\Omega^i(\omega)}$.

[28] In the example of this paper only one agent's state space is more expressive than the common state space. This can easily be extended to an example where this is true for both agents.

4. **The Axiom of Knowledge** $\omega \in K_S^i(E)$ implies $E \preceq P^i(\omega)$ and $P^i(\omega) \subseteq E^{\Omega^i(\omega)}$. Generalized Reflexivity implies $\omega_{\Omega^i(\omega)} \in P^i(\omega)$. Hence, $\omega_{\Omega^i(\omega)} \in E^{\Omega^i(\omega)}$, which implies $\omega \in E^S$.

5. **The Axiom of Transparency** Suppose $\omega \in K_S^i(E)$. Then, $E \preceq P^i(\omega)$ and $P^i(\omega) \subseteq E^{\Omega^i(\omega)}$. We have to show that $P^i(\omega) \subseteq K^i_{\Omega^i(\omega)}(E)$, or that $\omega_1 \in P^i(\omega)$ implies $E \preceq P^i(\omega_1)$ and $P^i(\omega_1) \subseteq E^{\Omega^i(\omega_1)}$. From Stationarity we have that $\omega_1 \in P(\omega)$ implies $P^i(\omega_1) = P^i(\omega)$ and $\Omega^i(\omega) = \Omega^i(\omega_1)$. Hence, $E \preceq P^i(\omega_1)$ and $P^i(\omega_1) \subseteq E^{\Omega^i(\omega_1)}$. Suppose $\omega \in K_S^i K^i_{\Omega^i(\omega)}(E)$. Then, $P^i(\omega) \subseteq K^i_{\Omega^i(\omega)}(E)$. From Generalized Reflexivity we have $\omega_{\Omega^i(\omega)} \in P^i(\omega)$ and from the proof of Proposition 1 we have $(K^i_{\Omega^i(\omega)}(E))^S \subseteq K_S^i(E)$. Therefore, $\omega \in K_S^i(E)$.

6. **The Axiom of Wisdom** Suppose $\omega \in A_S^i(E) \cap \neg K_S^i(E)$. Then, $E \preceq P^i(\omega)$ and $P^i(\omega) \nsubseteq E^{\Omega^i(\omega)}$. We need to show that $P^i(\omega) \subseteq A^i_{\Omega^i(\omega)}(E) \cap \neg K^i_{\Omega^i(\omega)}(E)$. Suppose $\omega_1 \in P^i(\omega)$. Stationarity implies that $P^i(\omega_1) = P^i(\omega)$. Hence, $E \preceq P^i(\omega_1)$ and $P^i(\omega_1) \nsubseteq E^{\Omega^i(\omega_1)}$, which imply that $\omega_1 \in A^i_{\Omega^i(\omega)}(E) \cap \neg K^i_{\Omega^i(\omega)}(E)$.

   Suppose $\omega \in K_S^i(A^i_{\Omega^i(\omega)}(E) \cap \neg K^i_{\Omega^i(\omega)}(E))$. Then, $P^i(\omega) \subseteq A^i_{\Omega^i(\omega)}(E) \cap \neg K^i_{\Omega^i(\omega)}(E)$. Since $A^i_{\Omega^i(\omega)}(E)$ is defined only if $E \preceq \Omega^i(\omega)$, we have that $\omega \in A_S^i(E)$. It remains to show that $\omega \in \neg K_S^i(E)$, or that $P^i(\omega) \nsubseteq E^{\Omega^i(\omega)}$. We know that for all $\omega_1 \in P^i(\omega)$, $\omega_1 \in \neg K^i_{\Omega^i(\omega)}(E)$, which implies that $P^i(\omega_1) \nsubseteq E^{\Omega^i(\omega)}$. Since $P^i(\omega) = P^i(\omega_1)$, we have that $P^i(\omega) \nsubseteq E^{\Omega^i(\omega)}$.

8. **Strong Plausibility** Suppose $\omega \in U_S^i(E)$. By definition, we have $E \preceq S$ and $E \npreceq P^i(\omega)$ which imply $S \npreceq P^i(\omega)$. Hence, $\omega \in \neg K_S^i(E) \cap \neg K_S^i(\neg K_S^i(E)) \cap \ldots \cap \neg K_S^i(\neg K_S^i(\ldots \neg K_S^i(E)))$.

9. **AU Introspection** Suppose $\omega \in U_S^i(E)$. By definition, we have $E \preceq S$ and $E \npreceq P^i(\omega)$ which imply $S \npreceq P^i(\omega)$ and $\omega \in U_S^i(U_S^i(E))$.

10. **KU Introspection** Suppose $\omega \in K_S^i(U_S^i(E))$. Then, $S \preceq P^i(\omega)$ and from Confinedness and $\omega \in S$ we have $P^i(\omega) \preceq S$ and $P^i(\omega) \subseteq U_S^i(E)$. Generalized Reflexivity implies that $\omega \in U_S^i(E)$, which implies $E \npreceq P^i(\omega)$. But this contradicts that $E \preceq S$.

11. **Symmetry** Follows since by definition $E \preceq \neg E$ if and only if $\neg E \preceq E$.

12. **AA-Self Reflection** $\omega \in A_S^i(E)$ implies $E \preceq S$ and $E \preceq P^i(\omega)$. Therefore, $A_S^i(A^i_{\Omega^i(\omega)}(E))$ is well defined and $\Omega^i(\omega) \preceq P^i(\omega)$ implies $\omega \in A_S^i(A^i_{\Omega^i(\omega)}(E))$. For the other direction, suppose that $\omega \in A_S^i(A^i_{\Omega^i(\omega)}(E))$. Since $A^i_{\Omega^i(\omega)}(E)$ is defined only if $E \preceq \Omega^i(\omega)$, we have that $\omega \in A_S^i(E)$.

13. **AK-Self Reflection** The proof is similar.

14. **A-Introspection** $\omega \in A_S^i(E)$ implies $E \preceq S$ and $E \preceq P^i(\omega)$, so we just have to show that $P(\omega) \subseteq A^i_{\Omega^i(\omega)}(E)$. Suppose that $\omega_1 \in P(\omega)$. Stationarity implies $P^i(\omega) = P^i(\omega_1)$, so we have $E \preceq P^i(\omega_1)$ and $\omega_1 \in A^i_{\Omega^i(\omega)}(E)$. For the other direction, suppose that

$\omega \in K^i_S(A^i_{\Omega^i(\omega)}(E))$. This implies that $\omega \in A^i_S(A^i_{\Omega^i(\omega)}(E))$ and $\omega \in A^i_S(E)$ follows from AA-Self Reflection.

□

Proof of Theorem 2. For ease of notation, let $S = S(\{\omega\})$ whenever ω is used in the proof.

1. **Subjective Necessitation** One direction is obvious so suppose $\omega \in \Sigma$. Then $P^i(\omega) \subseteq \Sigma$ and $\omega \in K^i(\omega)$.

2. **Monotonicity** Suppose E, F are not events. Then, $\omega \in K^i(E) \implies P^i(\omega) \subseteq E \subseteq F \implies \omega \in K^i(F)$. Suppose E, F are events. From 2. of Theorem 1, we have that $\omega \in K^i(E)$ implies $\omega \in K^i_S(E) \subseteq K^i_S(F)$, hence $\omega \in K^i(F)$.

3. **Conjunction** Suppose E, F are not events. Then, $\omega \in K^i(E) \cap K^i(F) \iff P^i(\omega) \subseteq E \cap F \iff \omega \in K^i(E \cap F)$. Suppose E, F are events. From 3. of Theorem 1 we have that $\omega \in K^i(E) \cap K^i(F) \iff \omega \in K^i_S(E) \cap K^i_S(F) \iff \omega \in K^i_S(E^{S(E)\vee S(F)} \cap F^{S(E)\vee S(F)}) \iff \omega \in K^i(E^{S(E)\vee S(F)} \cap F^{S(E)\vee S(F)})$.

4. **The Axiom of Knowledge** If E is an event, then from 4. of Theorem 1 we have $K^i(E) = \bigcup_{S \succeq E} K^i_S(E) \subseteq \bigcup_{S \succeq E} E^S$. Suppose E is not an event but has a base and is expanding. Suppose $\omega \in K^i K^j(E)$, which implies $\omega \succeq \omega'$ for some $\omega' \in K^j(E)$ and $P^i(\omega) \subseteq K^j(E)$. By transitivity of $\succeq$ we have that $\omega \succeq \omega''$ for some $\omega'' \in E$. By generalized reflexivity, $\omega_{\Omega^i(\omega)} \in K^j(E)$. Because E is expanding, $K^j(E)$ is expanding as well (see footnote 14) and $\omega \in K^j(E)$.

   Suppose $\omega \in K^i A^j(E)$, which implies $P^i(\omega) \subseteq A^j(E)$. From generalized reflexivity we have $\omega_{\Omega^i(\omega)} \in A^j(E)$ and $\Omega^j(\omega_{\Omega^i(\omega)}) \succeq \omega'$ for some $\omega' \in E$. Because $\Omega^j(\omega) \succeq \Omega^j(\omega_{\Omega^i(\omega)})$ we have $\omega \in A^j(E)$. Suppose $\omega \in K^i U^j(E)$, which implies that $P^i(\omega) \subseteq U^j(E)$. Because E has a base, let $\omega' \in S$ be such that $\omega'' \succeq \omega'$ for all $\omega'' \in E$. By generalized reflexivity, $\omega_{\Omega^i(\omega)} \succeq S$ and $\Omega^j(\omega_{\Omega^i(\omega)}) \not\succeq S$. Suppose $\Omega^j(\omega) \succeq S$. By PPA, stationarity and generalized reflexivity, $\Omega^j(\omega_S) = S$. Because $\omega_{\Omega^i(\omega)} \succeq \omega_S$, we must have $\Omega^j(\omega_{\Omega^i(\omega)}) \succeq \Omega^j(\omega_S)$, which contradicts $\Omega^j(\omega_{\Omega^i(\omega)}) \not\succeq S$. Therefore, $\Omega^j(\omega) \not\succeq S$ and $\omega \in U^j(E)$.

5. **The Axiom of Transparency** If E is an event, then $\omega \in K^i(E) \iff \omega \in K^i_S(E)$. From 5. in Theorem 1 this is equivalent to $P^i(\omega) \subseteq K^i_{\Omega^i(\omega)}(E)$. Because $K^i_{\Omega^i(\omega)}(E) \subseteq K^i(E)$ we have $P^i(\omega) \subseteq K^i(E)$ which is equivalent to $\omega \in K^i K^i(E)$. If E is not an event, then $\omega \in K^i(E) \implies P^i(\omega) \subseteq E$. From Stationarity, we have that if $\omega' \in P^i(\omega)$ then $P^i(\omega') = P^i(\omega)$ and $\omega' \in K^i(E)$. Hence, $P^i(\omega) \subseteq K^i(E)$ and $\omega \in K^i K^i(E)$. Conversely, $\omega \in K^i K^i(E)$ implies $\omega \succeq \omega'$ for some $\omega' \in K^i(E)$ and $P^i(\omega) \subseteq K^i(E)$. By transitivity of $\succeq$ and stationarity we have that $\omega \succeq \omega''$ for some $\omega'' \in E$ and $P^i(\omega) \subseteq E$, hence $\omega \in K^i(E)$.

6. **The Axiom of Wisdom** If E is an event, then $\omega \in A^i(E) \cap \neg K^i(E) \implies \omega \in A^i_S(E) \cap \neg K^i_S(E)$. From 6. in Theorem 1 we have $\omega \in K^i_S(A^i_{\Omega(\omega)}(E) \cap \neg K^i_{\Omega(\omega)}(E))$ and $P^i(\omega) \subseteq A^i_{\Omega(\omega)}(E) \cap \neg K^i_{\Omega(\omega)}(E) \subseteq A^i(E) \cap \neg K^i(E)$. Therefore, $\omega \in K^i(A^i(E) \cap \neg K^i(E))$. If

E is not an event, then $\omega \in A^i(E) \cap \neg K^i(E) \implies \Omega^i(\omega) \succeq \omega'$ for some $\omega' \in E$ and $P^i(\omega) \nsubseteq E$. This implies that $P^i(\omega) \subseteq A^i(E)$. Moreover, for each $\omega' \in P^i(\omega)$, from stationarity we have $P^i(\omega') = P^i(\omega)$ and therefore $P^i(\omega') \nsubseteq E$. Because $\omega' \in A^i(E)$, we have that $\omega' \succeq \omega''$ for some $\omega'' \in E$. Therefore, $\omega' \in \neg K^i(E)$ and $P^i(\omega) \subseteq \neg K^i(E)$. Hence, $P^i(\omega) \subseteq A^i(E) \cap \neg K^i(E)$ and $\omega \in K^i(A^i(E) \cap \neg K^i(E))$.

8. **Strong Plausibility** Suppose E is an event. $\omega \in U^i(E) \implies \omega \in U^i_S(E)$ and $\Omega^i(\omega) \not\succeq E$. From 8. in Theorem 1 we have $\omega \in \neg K^i_S(E)$ and $\omega \in \neg K^i(E)$. Because $\Omega^i(\omega) \not\succeq E$ and $\omega \succeq \omega'$ for some $\omega' \in E$, we have that $P^i(\omega) \nsubseteq \neg K^i(E)$. Hence, $\omega \in \neg K^i \neg K^i(E)$. Similarly, because $\Omega^i(\omega) \cap \neg K^i \neg K^i(E) = \emptyset$, we have $P^i(\omega) \nsubseteq \neg K^i \neg K^i(E)$ and $\omega \in \neg K^i \neg K^i \neg K^i(E)$. The same is true for higher orders.

   Suppose E is not an event. Then, $\omega \in U^i(E)$ implies that for all $\omega' \in E$, $P^i(\omega) \not\succeq \omega'$ and $\omega \succeq \omega'$ for some $\omega' \in E$. Hence, $\omega \in \neg K^i(E)$. Moreover, if $\omega'' \in \Omega^i(\omega)$ then $\omega'' \notin \neg K^i(E)$, which implies that $P^i(\omega) \nsubseteq \neg K^i(E)$ and $\omega \in \neg K^i \neg K^i(E)$. Similarly for higher orders.

9. **AU Introspection** Suppose E is an event. Then, $\omega \in U^i(E) \implies \Omega^i(\omega) \not\succeq E$ and $\omega \succeq E$. This implies that for all $\omega' \in \Omega^i(\omega)$, $\omega' \not\succeq E$. Because for all $\omega'' \in U^i(E)$ we have $\omega'' \succeq E$, it must be that $\Omega^i(\omega) \not\succeq \omega''$. Hence, $\omega \in U^i U^i(E)$. Suppose E is not an event. Then, $\omega \in U^i(E)$ implies that for all $\omega' \in E$, $\Omega^i(\omega) \not\succeq \omega'$ and $\omega \succeq \omega'$ for some $\omega' \in E$. First, we have $\omega \succeq \omega$ for $\omega \in U^i(E)$. Second, suppose there exists $\omega' \in U^i(E)$ such that $\Omega^i(\omega) \succeq \omega'$. Because $\omega' \in U^i(E)$ we have $\omega' \succeq \omega''$ for some $\omega'' \in E$. By transitivity of $\succeq$ we have $\Omega^i(\omega) \succeq \omega''$, which is a contradiction. Therefore, $\Omega^i(\omega) \not\succeq \omega'$ for all $\omega' \in U^i(E)$ and $\omega \in U^i U^i(E)$.

10. **KU Introspection** Suppose E is an event. Then, $\omega \in K^i U^i(E) \implies P^i(\omega) \subseteq U^i(E) \implies P^i(\omega) \subseteq U^i_{\Omega^i(\omega)}(E) \implies \omega \in K^i_S U^i_{\Omega^i(\omega)}(E)$, which is a contradiction, because we simultaneously have $\Omega^i(\omega) \succeq E$ and $\Omega^i(\omega) \not\succeq E$. Suppose E is not an event and $\omega \in K^i U^i(E)$. Then, $P^i(\omega) \subseteq U^i(E)$ and from PPA and generalized reflexivity we have that $\omega_{\Omega^i(\omega)} \in P^i(\omega)$. This implies that $\Omega^i(\omega_{\Omega^i(\omega)}) \not\succeq \omega'$ for all $\omega' \in E$ and $\omega_{\Omega^i(\omega)} \succeq \omega'$ for some $\omega' \in E$. This is a contradiction because $\omega_{\Omega^i(\omega)} \in \Omega^i(\omega_{\Omega^i(\omega)})$.

11. **Symmetry** Suppose E is an event. Then, $\omega \in U^i(E) \iff \omega \in U^i_S(E) = U^i_S(\neg E) \iff \omega \in U^i(\neg E)$, using 11. in Theorem 1. Suppose E is not an event. By the definition of $\neg E$, for any state space S, $E \cap S \neq \emptyset$ if and only if $\neg E \cap S \neq \emptyset$, (noting that $\emptyset_S \in S$). Therefore, the result is immediate.

12. **AA-Self Reflection** Suppose E is an event. Then, $\omega \in A^i(E) \implies \omega \in A^i_S(E) \implies \Omega^i(\omega) \succeq E \implies \omega_{\Omega^i(\omega)} \succeq E \implies \omega_{\Omega^i(\omega)} \in A^i(E)$. Because $\Omega^i(\omega) \succeq \omega_{\Omega^i(\omega)}$ we have $\omega \in A^i A^i(E)$. Conversely, $\omega \in A^i A^i(E) \implies \Omega^i(\omega) \succeq \omega'$, for some $\omega' \in A^i(E)$. Because $\omega' \succeq E$ we have $\Omega^i(\omega) \succeq E$ and $\omega \in A^i(E)$. Suppose E is not an event. Then, $\omega \in A^i(E)$ implies that $\Omega^i(\omega) \succeq \omega'$ for some $\omega' \in E$. Because $\Omega(\omega_{\Omega^i(\omega)}) = \Omega^i(\omega)$ we have $\omega_{\Omega^i(\omega)} \in A^i(E)$. Because $\Omega^i(\omega) \succeq \omega_{\Omega^i(\omega)}$ we have $\omega \in A^i A^i(E)$. Conversely, $\omega \in A^i A^i(E)$ implies $\omega \succeq \omega'$ for some $\omega' \in A^i(E)$, hence $\omega' \succeq \omega''$, for $\omega'' \in E$. By transitivity we have the result.

13. **AK-Self Reflection** Suppose E is an event. Then, $\omega \in A^i(E) \implies \omega \in A^i_S(E) \implies \Omega^i(\omega) \succeq E$. Because $K^i_{\Omega^i(\omega)}(E) \subseteq K^i(E)$ (even if $K^i_{\Omega^i(\omega)}(E)$ is empty) and $\Omega^i(\omega) \succeq K^i_{\Omega^i(\omega)}(E)$ we have $\omega \in A^i K^i(E)$. Conversely, $\omega \in A^i K^i(E) \implies \Omega^i(\omega) \succeq \omega'$ for some $\omega' \in K^i(E)$. This implies that $\Omega^i(\omega) \succeq \Omega^i(\omega') \succeq E$ and $\omega \in A^i(E)$. Suppose E is not an event. Then, $\omega \in A^i(E)$ implies that $\Omega^i(\omega) \succeq \omega'$ for some $\omega' \in E$. Let $\omega' \in S'$. Then, either $\emptyset_{S'} \in K^i(E)$ or there exists $\omega'' \in S'' \cap K^i(E)$. Because $\Omega^i(\omega) \succeq \emptyset_{S''} \sim \omega''$ we have $\omega \in A^i K^i(E)$. Conversely, $\omega \in A^i K^i(E) \implies \Omega^i(\omega) \succeq \omega'$ for $\omega' \in K^i(E)$ and $\Omega^i(\omega') \succeq \omega''$ for $\omega'' \in E$. By transitivity, $\omega \in A^i(E)$.

14. **A-Introspection** Suppose E is an event. Then, $\omega \in A^i(E) \implies \omega \in A^i_S(E) \implies \Omega^i(\omega) \succeq E \implies P^i(\omega) \succeq E \implies P^i(\omega) \subseteq A^i_{\Omega^i(\omega)}(E) \subseteq A^i(E) \implies \omega \in K^i A^i(E)$. Conversely, $\omega \in K^i A^i(E) \implies P^i(\omega) \subseteq A^i_{S'}(E)$ for some S'. This implies that $P^i(\omega) \succeq S' \succeq E \implies \omega \in A^i(E)$. Suppose E is not an event. Then, $\omega \in A^i(E)$ implies that $\Omega^i(\omega) \succeq \omega'$ for some $\omega' \in E$, which implies that $\Omega^i(\omega) \subseteq A^i(E)$. Because $P^i(\omega) \subseteq \Omega^i(\omega)$ we have $\omega \in K^i A^i(E)$. Conversely, $\omega \in K^i A^i(E)$ implies $P^i(\omega) \subseteq A^i(E) \implies P^i(\omega) \succeq \omega'$ for some $\omega' \in A^i(E)$ and $\omega' \succeq \omega''$ for $\omega'' \in E$. By transitivity we have $\omega \in A^i(E)$.

□

Proof of Proposition 2. We need to show that $\omega \in K^{i_1} \dots K^{i_n}(E)$ if and only if $P^{i_n} \dots P^{i_1}(\omega) \succeq E$ and $(P^{i_n} \dots P^{i_1}(\omega))_{S(E)} \subseteq E$. For $n = 2$, $\omega \in K^{i_1} K^{i_2}(E) \iff P^{i_1}(\omega) \subseteq K^{i_2}(E) \iff P^{i_1}(\omega) \subseteq K^{i_2}_{\Omega^i(\omega)}(E) \iff \bigcup_{\omega' \in P^{i_1}(\omega)} P^{i_2}(\omega') \succeq E$ and $(\bigcup_{\omega' \in P^{i_1}(\omega)} P^{i_2}(\omega'))_{S(E)} \subseteq E \iff P^{i_2} P^{i_1}(\omega) \succeq E$ and $(P^{i_2} P^{i_1}(\omega))_{S(E)} \subseteq E$.

Suppose that for $n = k$, $\omega \in K^{i_1} \dots K^{i_k}(E)$ if and only if $P^{i_k} \dots P^{i_1}(\omega) \succeq E$ and $(P^{i_k} \dots P^{i_1}(\omega))_{S(E)} \subseteq E$. Then, we have $\omega \in K^{i_1} \dots K^{i_{k+1}}(E) \iff P^{i_1}(\omega) \subseteq K^{i_2} \dots K^{i_{k+1}}(E) \iff$ for all $\omega' \in P^{i_1}(\omega)$, $P^{i_{k+1}} \dots P^{i_2}(\omega') \succeq E$ and $(P^{i_{k+1}} \dots P^{i_2}(\omega'))_{S(E)} \subseteq E$. This is equivalent to $\bigcup_{\omega' \in P^{i_1}(\omega)} P^{i_{k+1}} \dots P^{i_2}(\omega) \succeq E$ and $(\bigcup_{\omega' \in P^{i_1}(\omega)} P^{i_{k+1}} \dots P^{i_2}(\omega))_{S(E)} \subseteq E$. Finally, this is equivalent to $P^{i_{k+1}} \dots P^{i_1}(\omega) \succeq E$ and $(P^{i_{k+1}} \dots P^{i_1}(\omega))_{S(E)} \subseteq E$.

□

Lemma 4. *Suppose $S' \succeq S$, $\omega \in S'$ and take any sequence $i_1, \dots, i_n$. Then, for any $\omega' \in P^{i_n} \dots P^{i_1}(\omega)$ there exists $S'' \preceq \omega'$ such that $\omega'_{S''} \in P^{i_n} \dots P^{i_1}(\omega_S)$. Moreover, $\Omega^{i_1 \dots i_n}(\omega) \succeq \Omega^{i_1 \dots i_n}(\omega_S)$.*

Proof. For $n = 1$, PPI implies that $(P^{i_1}(\omega))^{\uparrow} \subseteq (P^{i_1}(\omega_S))^{\uparrow}$ hence the result follows.

Suppose the claim is true for $n = k$. Recall that

$$P^{i_{k+1}} \dots P^{i_1}(\omega) = \bigcup_{\omega' \in P^{i_k} \dots P^{i_1}(\omega)} P^{i_{k+1}}(\omega').$$

Take $\omega' \in P^{i_{k+1}} \dots P^{i_1}(\omega)$. Then, $\omega' \in P^{i_{k+1}}(\omega'')$ for some $\omega'' \in P^{i_k} \dots P^{i_1}(\omega)$. From the induction hypothesis we know that there exists $S'' \preceq \omega''$ such that $\omega''_{S''} \in P^{i_k} \dots P^{i_1}(\omega_S)$. From PPI we have $(P^{i_{k+1}}(\omega''))^{\uparrow} \subseteq (P^{i_{k+1}}(\omega''_{S''}))^{\uparrow}$. The second result follows from the definition of $\Omega^{i_1 \dots i_n}(\omega)$ and PPI.

□

Proof of Lemma 1. We first show that $(CK_S(E))^{S'} \subseteq CK_{S'}(E)$. Suppose that $\omega \in (CK_S(E))^{S'}$. Then, $\omega_S \in CK_S(E)$ which implies that for any sequence $i_1, \dots, i_n$ we have $P^{i_n} \dots P^{i_1}(\omega_S) \succeq E$ and $(P^{i_n} \dots P^{i_1}(\omega_S))_{S(E)} \subseteq E$. From Lemma 4 and PPI we have $P^{i_n} \dots P^{i_1}(\omega) \succeq E$ and $(P^{i_n} \dots P^{i_1}(\omega))_{S(E)} \subseteq E$, hence $\omega \in CK_{S'}(E)$. Finally, $(CK_S(E))^{S'} \subseteq CK_{S'}(E)$ implies $CK_S(E) \subseteq (CK_{S'}(E))_S$. That the other direction is not necessarily true is shown by the following example. There are two agents, equipped with possibility correspondences identical to that of Holmes, of the original example. Then, $\{\omega_6\}$ is common knowledge at ω_4, but not at ω_6. $\square$

Proof of Lemma 2. First, note that the two definitions are equivalent for $k = 2$. Suppose that the claim holds for all sequences where $k = n - 1$. Using this induction hypothesis we have that

$$\bigwedge_{\omega_1 \in P^{i_1}(\omega)} \Omega^{i_2 \dots i_n}(\omega_1) = \bigwedge_{\omega_1 \in P^{i_1}(\omega)} \bigwedge_{\omega_2 \in P^{i_2}(\omega_1)} \cdots \bigwedge_{\omega_{n-1} \in P^{i_{n-1}}(\omega_{n-2})} \Omega^{i_n}(\omega_{n-1})$$

and that, by definition,

$$\Omega^{i_1 i_2 \dots i_n}(\omega) = \bigwedge_{\omega' \in P^{i_{n-1}} \dots P^{i_1}(\omega)} \Omega^{i_n}(\omega').$$

The result follows because the right hand sides of the two equations are identical. $\square$

Proof of Lemma 3. Take any sequence $i_1, \dots, i_n$. By definition, $\Omega^\wedge(\omega) \preceq \Omega^{i_1 \dots i_n}(\omega)$ and we also have $(P^{i_n} \dots P^{i_1}(\omega))_{\Omega^\wedge(\omega)} \subseteq \Omega^\wedge(\omega)$. For the second claim, suppose that $E \in \mathcal{E}$ is common knowledge at ω. Then, $E \preceq \Omega^{i_1 \dots i_n}(\omega)$ for any sequence $i_1, \dots, i_n$ and $E \preceq \Omega^\wedge(\omega)$. $\square$

Proof of Theorem 3. Suppose E^* is common knowledge at ω. This means that for every sequence $i_1, \dots, i_n$, $P^{i_n} \dots P^{i_1}(\omega) \succeq E^*$ and $(P^{i_n} \dots P^{i_1}(\omega))_{S(E^*)} \subseteq E^*$. Because $(\mathcal{S}, \preceq)$ is well-founded, any non-empty subset of $\mathcal{S}$ contains a $\preceq$-minimal state space. If that subset consists of all state spaces S such that $P^{i_n} \dots P^{i_1}(\omega) \subseteq S$, for some sequence $i_1, \dots, i_n$, then the $\preceq$-minimal state space is $\Omega^\wedge(\omega)$. This means that $P^{i_n} \dots P^{i_1}(\omega) \subseteq \Omega^\wedge(\omega)$ for some sequence $i_1, \dots, i_n$ of agents and $S(E^*) \preceq \Omega^\wedge(\omega)$. We therefore have that, for any $i_1, \dots, i_n$, $P^{i_n} \dots P^{i_1}(\omega_{\Omega^\wedge(\omega)}) \subseteq \Omega^\wedge(\omega)$ and $(P^{i_n} \dots P^{i_1}(\omega_{\Omega^\wedge(\omega)}))_{S(E^*)} \subseteq E^*$. Define E to be the union of all such $P^{i_n} \dots P^{i_1}(\omega_{\Omega^\wedge(\omega)})$. Hence, $E \subseteq \Omega^\wedge(\omega)$ is an event. By Generalized Reflexivity we have $\omega_{\Omega^\wedge(\omega)} \in E$, hence $\omega \in E^S \subseteq E^{*S}$ and $E^* \preceq E \preceq S$. To show that E is public, suppose $\omega' \in E$ and fix $i \in I$. Then, for some sequence $i_1, \dots, i_n$, $\omega' \in P^{i_n} \dots P^{i_1}(\omega_{\Omega^\wedge(\omega)})$. Because $P^i P^{i_n} \dots P^{i_1}(\omega_{\Omega^\wedge(\omega)}) \succeq \Omega^\wedge(\omega)$ and $(P^i P^{i_n} \dots P^{i_1}(\omega_{\Omega^\wedge(\omega)}))_{S(E^*)} \subseteq E^*$ we have $\omega' \in K^i_{\Omega^\wedge(\omega)}(E^*)$.

Conversely, suppose that there exists public event E such that $E^* \preceq E \preceq S$ and $\omega \in E^S \subseteq E^{*S}$. We first prove the following Lemma.

Lemma 5. *Event E is common knowledge at ω.*

Proof. We need to show that for any sequence $i_1, \dots, i_n$ of agents, $P^{i_n} \dots P^{i_1}(\omega) \succeq E$ and $(P^{i_n} \dots P^{i_1}(\omega))_{S(E)} \subseteq E$. The proof is by induction.

- For $n = 1$, since E is self evident for i_1 and from the proof of Proposition 1 we have $\omega \in E^S \subseteq \left(K^{i_1}_{S(E)}(E)\right)^S \subseteq K^{i_1}_S(E)$. Hence, $E \preceq P^{i_1}(\omega)$ and $(P^{i_1}(\omega))_{S(E)} \subseteq E$.

- Suppose that for $n = k$, $P^{i_k} \dots P^{i_1}(\omega) \succeq E$ and $(P^{i_k} \dots P^{i_1}(\omega))_{S(E)} \subseteq E$.

- For $n = k+1$, we need to show that $P^{i_{k+1}} \dots P^{i_1}(\omega) \succeq E$ and $(P^{i_{k+1}} \dots P^{i_1}(\omega))_{S(E)} \subseteq E$. By definition,
$$P^{i_{k+1}} \dots P^{i_1}(\omega) = \bigcup_{\omega' \in P^{i_k} \dots P^{i_1}(\omega)} P^{i_{k+1}}(\omega').$$
From the induction hypothesis, for any $\omega' \in P^{i_k} \dots P^{i_1}(\omega)$, we have $\omega'_{S(E)} \in E \subseteq K^{i_{k+1}}_{S(E)}(E)$, which, from PPI, implies that $E \preceq P^{i_{k+1}}(\omega')$ and $(P^{i_{k+1}}(\omega'))_{S(E)} \subseteq E$.

□

Since $E^* \preceq E$ and $E^S \subseteq E^{*S}$, we have that $E \subseteq E^{*S(E)}$. Fix a sequence $i_1, \dots, i_n$ of agents. We have $E^* \preceq E \preceq P^{i_n} \dots P^{i_1}(\omega)$ and $(P^{i_n} \dots P^{i_1}(\omega))_{S(E)} \subseteq E \subseteq E^{*S(E)}$, hence $(P^{i_n} \dots P^{i_1}(\omega))_{S(E^*)} \subseteq E^*$.

□

Proof of Theorem 4. From Theorem 3, there exists a public event E' such that $E^* \preceq E'$ and $\omega \in E'^S \subseteq E^{*S}$. Its proof (and because $E^* \subseteq \Omega^\wedge(\omega)$) also shows that $E' \subseteq \Omega^\wedge(\omega)$, which implies that $E' \subseteq E^*$. We need to show that $E' = \bigcup_{\omega \in E'} P^i(\omega)$. Generalized Reflexivity implies $E' \subseteq \bigcup_{\omega \in E'} P^i(\omega)$. For the opposite direction, since E' is a public event, $\omega \in E'$ implies $P^i(\omega) \subseteq E'$. Therefore, $E' = \bigcup_{\omega \in E'} P^i(\omega)$, and by symmetry $E' = \bigcup_{\omega \in E'} P^j(\omega)$.

The next step is to show that E' is partitioned by P^i. First, since E' is public, for any $\omega' \in E'$, $\Omega^\wedge(\omega) \preceq \Omega^i(\omega') \preceq \Omega^\wedge(\omega)$. Generalized Reflexivity and Stationarity imply that if $\omega', \omega'' \in E'$ then either $P^i(\omega') = P^i(\omega'')$ or $P^i(\omega') \cap P^i(\omega'') = \emptyset$. The rest of the proof is identical to that of Aumann (1976).

Agent i's posterior at $\omega' \in E'$ is
$$q^i(\omega') = \frac{\mu(P^i(\omega') \cap E)}{\mu(P^i(\omega'))}.$$
Since $q^i(\omega') = q^i$ for all $\omega' \in E'$ we can sum over the disjoint partition cells of E' and derive $\mu(E')q^i = \mu(E' \cap E)$. Similarly for agent j we have $\mu(E')q^j = \mu(E' \cap E)$ and therefore $q^i = q^j$.

□

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