DYNAMICS OF RECTANGULAR CURVED PLATES

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ABSTRACT

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The free vibration characteristics of uniform singly curved rectangular plates are analysed by solving the differential equations of motion by using the method of Kantorovich. The effects of applied static membrane stresses are also included in the analysis. Subsequently the method is extended to the solution of uniform doubly curved plates of constant curvatures. Among other facts there emerged from the analysis that there could be more than one frequency for a given nodal system of the transverse displacement.

A doubly curved rectangular finite element in the Cartesian co-ordinate system is developed. The effects of applied static membrane stress on the stiffness of the element are taken into account. The results obtained by this element are found to converge monotonically to the Kantorovich solutions when the size of the elements is progressively reduced. The formulation is capable of predicting degenerate modes of vibrations and critical stress system over which buckling may occur in plates.

The finite element analysis is also applied to study the behaviour of curved plates under the action of random and static loading.

Finally, a singly curved clamped rectangular plate which is machined out of a block of aluminium is tested to verify the results of free vibrations obtained by the method of Kantorovich and the finite element method. The tests confirm the analytically obtained solutions.
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LIST OF SYMBOLS

- for Chapters I to IV -

Symbols that are not defined here are explained as they appear.

L  Length along a straight edge
b  Chord length of a curved section
h  Constant thickness of the plate
R  Constant radius of curvature of the plate
θ  Total central angle
m  Number of axial half waves
n  Number of circumferential half waves
ρ  Weight of the plate/volume
g  Acceleration due to gravity
E  Young's modulus of elasticity
ν  Poisson's ratio
x₁  Axial coordinate
x  Dimensionless axial coordinate, x₁/R
ϕ  Circumferential coordinate
x̅  L/R
u,v, w  Longitudinal, circumferential and radial displacement component respectively
f₁,f₂, f₃  Variations of u, v and w respectively along the direction of x
φ₁,φ₂, φ₃  Variations of u, v and w respectively along the direction of ϕ
D  Extensional rigidity  Eh/(1 - ν²)
K  Flexural rigidity  Eh³/12(1 - ν²)
I  Moment of inertia of a beam of unit width also sometimes unit matrix
k  Bending stiffness  = K/DR² = h²/12R²
f  Frequency of vibration C.P.S.
ω  Circular frequency  = 2πf.
Δ  Frequency factor  = R²ρhw²/Dg = R²ρω²(1 - ν²)/Eg

(i)
D(Δ) Frequency determinant
A_u, A_v, A_w Frequency factor corresponding to predominant motion along u, v and w respectively
q_x, q_φ Membrane tensile stresses along x and φ respectively
q_x = p_x/D, q_φ = p_r R/D
P_x, P_r Applied axial tension per unit circumferential length and radial internal pressure respectively.

(..)'
\[
R \frac{\partial}{\partial x_1} (..) \equiv \frac{\partial}{\partial x}(..)
\]

(..)'
\[
\frac{\partial}{\partial \phi}(..)
\]

[..] Usually reference at end or determinant of a square matrix
[..] Rectangular matrix
{ } Column matrix (if used in matrix notation)
i Usually \sqrt{-1}

- for Chapters V to VII -

Symbols that are not defined here are explained as they appear

t Time
x, y, z Co-ordinate system defined in Figure 18
L_x, L_y Length of the plate along x and y respectively
a, b Length of an element along x and y respectively
h Thickness of the plate or of an element
R Radius of a spherical surface
k_x, k_y Principal curvatures of the plate or an element
k_xy Third or twisting curvature of the plate or of an element
ε_x, ε_y Extensions of the middle surface along x and y respectively
ε_xy Shear strain of the middle surface
X_x, X_y Change of curvatures
X_xy Change of twisting curvature

(ii)
Components of displacements of a point (ref. Figure 18)

\( f_1, f_2, f_3 \) Variation of \( u, v \) and \( w \) respectively along \( x \)

\( g_1, g_2, g_3 \) Variation of \( u, v \) and \( w \) respectively along \( y \)

\( E \) Modulus of elasticity

\( \rho \) Weight of the plate per unit volume

\( \nu \) Poisson's ratio

\( f \) Frequency C.P.S.

\( g \) Acceleration due to gravity

\( \omega \) Circular frequency \( = 2\pi f \)

\( \Omega \) Frequency parameter \( = \rho \omega^2 (1 - \nu^2)/Eg \)

\( D(\Omega) \) Determinantal equation for determination of \( \Omega \)

\( D \) Extensional rigidity \( = Eh/(1 - \nu^2) \)

\( K \) Flexural rigidity \( = Dh^2/12 \)

\( m, n \) Number of halfwaves along \( x \) and \( y \) respectively

\( N_x, N_y \) Stress resultants along \( x \) and \( y \) respectively

\( N_{xy}, N_{yx} \) Shear stress resultants

\( M_x, M_y \) Stress couples

\( M_{xy}, M_{yx} \) Twisting moments

\( Q_x, Q_y \) Transverse shear

\( s_x, s_y \) Applied inplane direct stresses

\( s_{xy} \) Applied inplane shear stress

\( f_h \) Maximum rise of a plate over its base plane

\( S \) Total strain energy

\( S_0, S_a \) Strain energy without inplane stresses applied

\( S_I \) Strain energy due to inplane stresses

\( S_m, S_b, S_{mb}, S_{cmb} \) Components of strain energy

\( e_x, e_y, e_{xy} \) Strain at a point inside the plate

\( \sigma_x, \sigma_y, \sigma_{xy} \) Stress at a point inside the plate

\[ \frac{\partial f(\ldots)}{\partial x} \]

\[ \frac{\partial f(\ldots)}{\partial y} \]  

(iii)
|..| Reference at end or determinant of a square matrix
[..] Rectangular or square matrices
[..]⁻¹ Inverse of a square matrix
[..]ᵀ Transpose of a rectangular matrix
[[..]⁻¹]ᵀ Transpose of the inverse of a square matrix
{..} Column matrix if used in matrix notation
{..}ᵀ Transpose of a column matrix
i Usually \(\sqrt{-1}\)

The address of different matrices in the finite element section are defined as they appear.
CHAPTER I
INTRODUCTION

Curved plates and shells are widely used as structural forms. Aerospace structures like rockets, artificial satellites, aircraft fuselages, aircraft wings and tailplanes are constructed of basic curved plate elements. Curved plates are also used in civil engineering construction, usually when large areas are to be covered without intermediate supports. Massive structures like arch and buttress dams may also be treated as curved plates. In mechanical engineering, curved plates are extensively used in machine manufacture. Structurally, the fan blades of rotors and engines as well as the hulls of ships and submarines are also curved plates.

Since these types of structures have to withstand vibration induced stresses of various degrees during their useful life, it is essential to know their dynamic behaviour for the safety and economy of their design. There are various methods of determining this structural behaviour which may broadly be classified as (i) analytical and (ii) experimental. Whichever approach is taken, the knowledge of the properties of the construction material to a reasonable degree of accuracy is essential. In analytical methods, the mathematical modelling of the problem incorporates various simplifying assumptions and idealisations. Some of these idealisations involve the properties
of the materials. The following two major assumptions are used throughout this thesis regarding the properties of the materials:

(a) the material is isotropic and homogeneous,
(b) the stress-strain characteristic is linear within the range of working stresses.

In practice these two assumptions seem to be interpreted quite liberally including such materials as brick, masonry and concrete whose properties hardly satisfy them. However, materials such as timber, foam, rubber, and plastics cannot be accommodated within these two assumptions.

So far as mathematical analysis is concerned, it is advantageous to have a unified theory that will deal with all types of plates. It may be realized that such a theory will be too complicated to use in practice, to obtain data, with undue amount of complications. Therefore, different theories are available to deal with different types of plates. We do not intend to go into the detail of these theories. We will just indicate the theories we are interested in, together with some of the assumptions inherent in their derivations.

The plates are assumed to be thin in this thesis. Because of this assumption usually the effect of normal and shearing stresses on bending of the plate is neglected (see Timoshenko [99, p.98]), and it is usual to consider only the state of plane stress in surfaces which are at constant distance from the middle surface. Because of the thinness of the plate, sometimes the variation of strains across the thickness of the plate is assumed to be linear providing it facilitates formulation and computation without appreciable loss of accuracy. A plate is assumed
to be thin when the ratio between the maximum thickness and the minimum radius is less than or equal to $1/30$ (Vlasov [3, p.337]).

Only small displacement theory is used here. "In the small displacement theory, the displacements are assumed so small as to allow linearization of all governing equations of the solid body except the stress-strain relations. Consequently, the equations of equilibrium, the strain-displacement relations and the boundary conditions are reduced to linearized form in small displacement theory" (Washizu [100, p.3]). Therefore, it is seen that the small displacement theory is associated with linearity. The magnitude of the smallness is a relative term. Timoshenko and Woinowsky-Krieger [99] have defined this smallness, once in relation to the thickness of the plate and once in relation to the other dimensions. In the first definition the displacements are required to be small in comparison with the thickness of the plate [99, p.48]. "In cases in which the deflections are no longer small in comparison with the thickness of the plate but are still small as compared with the other dimensions, the analysis of the problem must be extended to include the strain of the middle plane of the plate" (Timoshenko and Woinowsky-Krieger [99, p.396]). Both these definitions of small deflection theory are used in this thesis.

Sometimes it becomes relatively easier to formulate and solve curved plate problems if it is assumed that the plate is shallow, together with the other assumptions mentioned above. A rectangular curved plate is defined as shallow when the ratio between its maximum rise over the base plane to its minimum base side does not exceed
1/5 (Vlasov [3, p.343]). Wherever this assumption is valid, the geometry of the curved plate may be defined adequately by the geometry of the plane with a considerable amount of simplification. Advantage of shallowness is taken here whenever it is found to be suitable.

A great deal of work is being done towards the solution of the problems of vibration of flat plates. Comparatively little work is being done towards the solution of the problems of vibration of curved plates. This is probably due to the relative difficulties in solving the differential equations of motion. There are practically no analytical solutions to the problem except for a few special cases. Singly curved rectangular panels with constant thickness are the simplest forms of curved plates. Yet, only a few attempts are being made to solve the problems of free vibrations of such plates. One of the simplest methods of dealing with this problem is the Rayleigh-Ritz method. This method will yield accurate solutions under certain conditions. The solutions with the Rayleigh-Ritz method always depend upon the prescribed displacements. The method tends to become complicated when the solutions are successively improved by the procedure of Ritz.

To use the Rayleigh-Ritz method the kinetic-potential of the plate is required. The kinetic-potential may be derived, when, (i) the equation of the middle surface together with its variation of thickness, if any, (ii) the strain-displacement relations and (iii) suitable laws of variation of strain across the thickness are known.

Love [2] has given a general theory of plates and shells. Because of the unsymmetrical character of the expressions for the
change of curvatures many authors have criticised this theory and many have tried to rectify them. Langhaar [7] appears to be one of the first authors to point out the discrepancies in Love's equations. He himself has derived the strain energy expression for a general thin shell removing the so-called discrepancies in Love's theory. Like the first approximation of Love, Langhaar has assumed linear variation of strains across the thickness. Koiter [9] did not object to the assumption of linear variation of strain across the thickness used by Langhaar, but criticised Langhaar on the ground that his corrections to Love's theory had only small practical importance. However, later Langhaar and Carver [33] have asserted that the approximation for linear variation of strain across the thickness is a questionable approximation on the ground that the membrane strains and the bending strains often have quite different orders of magnitude. Therefore, relatively small inaccuracies in one of the strain components may completely obliterate the other. Osgood and Joseph [27] have also pointed out the inconsistencies in Love's equations of equilibrium and presented a new set of equations. Following Love, Timoshenko [99] has also given the theory of curved plates and shells similar to those of Love.

Vlasov ([3, pp.289-297] and [4]) has also commented on the Love-Timoshenko equations. Vlasov has pointed out that the missing terms in the Love-Timoshenko equations due to the use of approximate expressions for the change of curvatures, which have rendered the structure of the equations unsymmetrical, are of the same order of
smallness as some of the bending terms which are retained. Again Vlasov contends that because of the use of the assumption of linear variation of strain across the thickness of the plate, some terms arising out of the presence of the curvatures could not be accounted for, which are of the same order of magnitude as some of the moment terms. However, he points out that the maximum error due to these discrepancies does not exceed 5% of the maximum stress. "We believe, however, that the system of differential equations with a symmetrical matrix is in full agreement with the basic laws of energostatics of solid elastic bodies". Concerning the use of these equations to the solution of problems of free vibrations this is what Vlasov has to comment: "The problem of the natural vibrations of a shell with a symmetrical matrix of the basic differential equations is always reduced to a secular equation which yields real values for the spectrum of vibrations of all frequencies. On the other hand, the equations of Love, Galerkin and others could yield imaginary values for vibrations of higher frequencies due to the asymmetry of the basic matrix, which does not agree with the law of the conservation of energy". Usually, the higher frequencies of vibrations are of not much practical use to structural engineers. Despite all the criticisms and shortcomings the set of equations of Love-Timoshenko is one of the most popular sets and is very often used. The strain energy expression for a constant thickness circular cylindrical shell using Love-Timoshenko's equations with linear variation of strains across the thickness may be found in Warburton [22, pp.186-194]. The critics of Love usually do not
mention about his so-called second approximation where the variation of strains across the thickness is assumed to be quadratic.

There are many more sets of equations, the details of which we do not intend to discuss. Vlasov [4] has given a few sets of equations to suit a particular plate depending upon whether it is thin or medium thick. However, we shall concentrate on a set of equations in curvilinear co-ordinates of a general thin shell, given by Vlasov [3], which are claimed to be free of the discrepancies found in Love's equations. The equations for a uniform circular cylindrical shell obtained from this set agree completely with a set of equations derived by Flügge [6]. Neither Vlasov nor Flügge has given the actual strain energy expression. Accurate strain energy expression using these strain-displacement relations may be found in Bleich and DiMaggio [11], Warburton [20] and a similar one in Miller [12]. Kennard [10], has retained a large number of small terms in deriving a strain energy expression of a circular cylindrical shell. We have derived here two strain energy expressions based upon Vlasov's [3 and 5] equations for a general curved plate. One of them is in curvilinear co-ordinates based upon quadratic variation of strain across the thickness of the plate and the other is in Cartesian co-ordinates based upon linear variation of strains across the thickness of the shell. We shall discuss their relative merits later in another chapter. Now we shall briefly review the work done in analysing the vibrations of rectangular curved plates using the energy expressions in conjunction with the Rayleigh-Ritz method. From this discussion we exclude all plates whose boundaries do not form a rectangle. This makes
the discussion much simpler, because very little is being done in this field except in singly curved uniform plates.

Ballentine et al [47] have analysed both simply supported and clamped panels. Love's strain energy expression with linear variation of strain across the thickness is being used. The displacement functions in both the directions are prescribed by characteristic vibration functions of straight uniform beam and its derivatives. The comparison between the theoretical and the experimental results shows that the experimental results are bounded between the simply supported and the clamped solutions. The results obtained are for one term Rayleigh-Ritz solution. The effects of the inplane inertias on the out of plane motions are retained.

Sewall [14] has also used exactly the same procedures as in reference [47] in analysing singly curved rectangular panels. However, he has used Sanders' strain-displacement relations, which are more or less similar to the ones given by Flügge. A general uncoupled shallow-shell frequency equation is derived and is being used to compute results. The major assumption in it is that the effects of the inplane inertias on the transverse vibration are neglected. We shall comment on this assumption later in another chapter. The comparison of theoretical and experimental frequencies shows that some of the experimental frequencies for an apparently clamped plate lie below the calculated simply supported frequencies. In another case, however, some of the experimental frequencies lie over the calculated clamped plate frequencies.
Webster [32] has used the strain energy expression given by Warburton [20] using Flügge's so-called exact strain-displacement relations. The displacements are prescribed by truncated double power series. The method of solution is essentially the same as the Rayleigh-Ritz method. The range of panel geometry, for which the expression for frequencies of a clamped plate given by Sewall, gives accurate results (only for the lowest frequency), is defined. From the results given by Webster, it may be seen that the approximate solutions given by Sewall are valid only for a very limited range of panel geometry. However, except for this limited range, Sewall's solutions are much higher than those of Webster. The one term Rayleigh-Ritz approximation is known to yield, in general, solutions higher than the exact ones. Since Webster has used more terms in his polynomial approximations, there are enough reasons to expect his solutions to be quite accurate.

The classical energy method in conjunction with the Rayleigh-Ritz method will not be used in a direct manner in this thesis. Instead, we choose to solve the so-called 'exact set' of partial differential equations of motion given by Flügge [6].

In Chapter II these differential equations are reduced to a set of ordinary differential equations by using the method of Kantorovich [1].

In Chapter III, these ordinary differential equations of motion are solved employing various methods. From the comparison of the results, a method is selected which is employed later.

In Chapter IV the free vibrations of uniform singly curved
rectangular panels in the presence of bi-axial tensile membrane stresses are analysed. Differential equations of motion are solved using more or less the same procedures that are used in Chapters II and III.

In Chapters II, III and IV, emphasis is placed on the solution of clamped panels though other boundary conditions are also considered, particularly the simply supported ones. The analysis is general, and the plates need not be shallow.

The methods of Chapters II and III are used in Chapter V to analyse the free vibrations of rectangular uniform doubly curved plates. In that chapter the plates are assumed to be shallow, their curvatures are assumed to be constant, the third being assumed to be zero.

A rectangular doubly curved shallow (not necessarily the plate) finite element is developed in Chapter VI. All the three curvatures and the thickness over an element are assumed to be constant. The effects of applied inplane stresses (both compressive and tensile including shear) on the stiffness of the element are taken into account. The relevant consistent mass matrix is also developed.

Subsequently, the finite element method is used to analyse various plates and to compare results obtained with the method of Kantorovich, both with and without inplane stresses.

The finite element developed in Chapter VI is used in Chapter VII to analyse statics and random vibrations of curved plates. The relevant load and stress matrices are developed in this chapter.
The computational aspects of the various methods employed in this thesis are briefly discussed in Chapter VIII.

The results of an experimental investigation into the free vibration characteristics of a singly curved rectangular clamped plate are given in Chapter IX. Rapid frequency excitation, together with a newly developed method of analysing the test data, are used.

In Chapter X, the general conclusions of the findings of the research are summarised.
CHAPTER II

REDUCTION OF THE GOVERNING PARTIAL DIFFERENTIAL EQUATIONS
TO ORDINARY DIFFERENTIAL EQUATIONS -
THE METHOD OF KANTOROVICH

2.1 Introduction

It has been discussed in Chapter I that the energy principle in conjunction with the Rayleigh-Ritz method is being used in the analysis of free vibrations of singly curved rectangular plates. The one term Rayleigh-Ritz approximation is inadequate except for simply supported plates (ref. [14, 47]). The solutions for other boundary conditions may be improved by increasing the number of terms in the expansions of the displacement functions. Sometimes it may be lengthy and tedious to find the number of terms necessary in the expansions of the displacements to yield convergent solutions. Therefore, the energy approach with direct application of the Rayleigh-Ritz type of analysis will not be considered here. Instead, attempts will be made to solve the differential equations of motion. No previous attempts in this direction are noticed except for the case of complete cylinders (see Forsberg [13] and Warburton [20]).

The theory of constant thickness, circular cylindrical shells is one of the most widely studied subjects in the mechanics of continua. There is more than one theory which leads to the question of relative accuracy and applicability of the different theories to specific structural problems.

Reissner [17], derived the two dimensional system of equations for stresses and deformations of thin elastic shells from the theory of elasticity by means of the calculus of variation. Naghdi [18] applied Reissner's variational theorem to derive a set of shell equations where he included the deformations due to transverse shear and normal stresses.

Most of the differential equations differ in their bending terms. This is due to the simplifying assumptions made in the derivation of these equations. These equations may be divided into two broad groups. One group is known as the "simple set" and the other is known as the "exact set". It may again be emphasised that the simple and the exact sets differ only in some small bending terms. Both Vlasov and Flügge have given two sets of equations each, one of which is 'simple' and the other is 'exact'. The simple set of equations given by Vlasov and Flügge are identical to the set of equations known as Donnel-Jenkins equations. The simple set of equations is usually more popular with structural engineers for their
simplicity. However, Flügge's exact set of equations will be used here.

It is worthwhile to examine briefly the merits of both sets of equations given by Flügge. First, both sets are symmetrical in structure with respect to the partial differential operators (see Vlasov [3, pp.292-297]). The main appeal of the simple set is not only its simplicity, but, in certain cases, it will lead to the same type of solutions as the exact set. For reference both sets of equations are given below. Flügge's notations are used. The external forces are replaced by reversed mass acceleration terms using d'Alembert's principle. The geometry of the plate is shown in Figure 1. The forces and the moments in a shell element are shown in Figure 2.

The exact set of equations of Flügge are as follows.

\[ u'' + \frac{1-v}{2} (1+k)u'' + \frac{1+v}{2} v'' + vw' + k\left[-w''' + \frac{1-v}{2} w''\right] - \frac{R^2}{D} \frac{\rho h}{g} \frac{\partial^2 u}{\partial t^2} = 0, \]

\[ \frac{1+v}{2} u'' + v'' + \frac{1-v}{2} (1 + 3k) v'' + w' + k\left[-\frac{3-v}{2} w''\right] - \frac{R^2}{D} \frac{\rho h}{g} \frac{\partial^2 v}{\partial t^2} = 0, \]

\[ \nu u'' + k\left[\frac{1-v}{2} u''' - u''\right] + v' - \frac{3-v}{2} kv''' + w' + k\left[v'''' + 2w''' + w' + 2w'' + w\right] + \frac{R^2}{D} \frac{\rho h}{g} \frac{\partial^2 w}{\partial t^2} = 0. \]
The simple set is

\[ u'' + \frac{1-v}{2} u'' + \frac{1+v}{2} v'' + vv' - \frac{R^2 \rho h}{g} \frac{\partial^2 u}{\partial t^2} = 0, \]

\[ \frac{1+v}{2} u'' + v'' + \frac{1-v}{2} v'' + w' - \frac{R^2 \rho h}{g} \frac{\partial^2 v}{\partial t^2} = 0, \]

\[ vu' + v' + w + k [w'' + 2w''' + w'''' + w'''''' + \cdots] + \frac{R^2 \rho h}{g} \frac{\partial^2 w}{\partial t^2} = 0. \]

2.2(a-c)

It may be seen that there are large qualitative differences between the two sets of equations. The differences are only in the terms associated with \( k \). The equations 2.2 (a-b) are completely free of them. The contribution to bending by \( u \) and \( v \) is eliminated in equation 2.2(c). In terms of the energy of the plate, both the sets account correctly for the membrane part of the energy. But the simple set of equations does not account for the bending part of the energy as adequately as the exact set.

Therefore, if the state of deformation is purely or nearly flexural then it is necessary to use the exact set. On the other hand, if the deformations are primarily of extensional type, either set of equations may be used. Flügge [6, p221] has cautioned not to rely too much upon the simple set of equations for solutions of problems with large bending. For further discussions on the two sets of equations, reference may be made to Forsberg [13].
2.2 Order of the Equations

For determination of the constants of integration, the number of prescribed boundary conditions require to be equal to the order of the system of equations. The present problem may be treated as a combination of a pair of two point boundary value problems, one in the variable x and the other in \( \phi \). For the solution of each of these two-point boundary value problems, the order of the system of differential equations in each of the variables must be even so that half of the conditions may be prescribed at each edge.

Both the exact (equation 2.1(a-c)) and the simple (equation 2.2(a-c)) set of equations are of order eight with respect to \( \phi \). Therefore, four boundary conditions are required at each end \( \phi=0 \) and \( \phi=\bar{\phi} \). Similarly the simple set of equations is also of order eight with respect to x so that four conditions are required at each of the edges \( x=0 \) and \( x=\bar{x} \). However, due to the presence of the term \( ku'''' \) in the equation 2.1(c), the exact set of equations are of order nine with respect to x. Under this situation, it will not be possible to prescribe the boundary conditions correctly. This difficulty may be overcome by eliminating \( u \) from the governing equations. However, for further discussions it will be assumed that both the sets of equations are of order eight with respect to both \( \phi \) and x.

The small term \( ku'''' \) is contributed to the equilibrium equation by the transverse shear \( Q_x \). Although \( Q_x \) is small in magnitude, it might contain large derivatives. If the apparently small term \( ku'''' \) is neglected, then the system of equations will no longer be symmetrical.
In order to preserve the symmetry of the system \( kw''' \) in equation 2.1(a) will also have to be neglected when \( ku''' \) is neglected.

It may be pointed out that when shearing deformations are included the order of the equations will be ten with respect to both \( x \) and \( \phi \).

Henceforth, only the exact set of equations 2.1(a-c) will be considered.

### 2.3 The Boundary Conditions and the Stress Resultants

The boundary conditions discussed in the previous section may be formulated in terms of the displacements and their derivatives. Alternatively, they may be formulated in terms of the forces and their derivatives or in a mixed way. In the present analysis they are always formulated in terms of the displacements and their derivatives.

The following are the stress displacement relations.

**Direct stress resultants:**

\[
N_{\phi} = \frac{D}{R}(v'' + w + vu') + \frac{K}{R^3} (w + w'').
\]

\[
N_x = \frac{D}{R}(u' + vv' + vw) - \frac{K}{R^3} (w'').
\]

**In plane shear stress resultants:**

\[
N_{\phi x} = \frac{D}{R} \frac{1-v}{2} (u' + v') + \frac{K}{R^3} \frac{1-v}{2} (u' + w '').
\]

\[
N_{x\phi} = \frac{D}{R} \frac{1-v}{2} (u' + v') + \frac{K}{R^3} \frac{1-v}{2} (v' - w '').
\]
Bending stress resultants (moments):

\[ M_\phi = \frac{K}{R^2} (w + w'' + vw''') \]

\[ M_x = \frac{K}{R^2} (w'' + vw''' - u' - vv') . \]

Twisting stress resultants (moments):

\[ M_{\phi x} = \frac{K}{R^2} (1-v)(w'' + \frac{1}{2}u' - \frac{1}{2}v'). \]

\[ M_{x\phi} = \frac{K}{R^2} (1-v)(w'' - v'). \]

Transverse shear stress resultants:

\[ Q_\phi = \frac{K}{R^3} (w' + w'' + w''' - (1-v) v''') . \]

\[ Q_x = \frac{K}{R^3} (w'' + w''' - u' + \frac{1-v}{2} u'' - \frac{1+v}{2} v''') . \]

Effective shear stress resultant:

\[ T_\phi = N_{\phi x} . \]

\[ T_x = N_{x\phi} - \frac{M_{x\phi}}{R} = \frac{D}{R} \{ u' + (1 + 3k) v' + 3k w''' \} . \]

Effective transverse force:

\[ \bar{T}_\phi = Q_\phi + \frac{M_{\phi x}}{R} . \]

\[ \bar{T}_x = Q_x + \frac{M_{x\phi}}{R} = \frac{D}{R} \{ w'' + (2-v) w''' - u'' + \frac{1-v}{2} u''' - \frac{3-v}{2} v''' \} . \]

The boundary conditions that are most commonly encountered in practice are:

\[ -18- \]
(a) clamped,
(b) simply supported,
(c) free edge.

Boundary conditions at edges $x = \text{constant}$ and at $\phi = \text{constant}$ are given below.

(a) clamped:

$$u = v = w = w' = 0, \quad \text{at } x = \text{constant}. $$

$$u = v = w = w' - v = 0, \quad \text{at } \phi = \text{constant}. $$

(b) simply supported:

$$v = w = N_x = M_x = 0, \quad \text{at } x = \text{constant}.$$  \hspace{1cm} 2.4(a-f)

$$u = w = N_\phi = M_\phi = 0, \quad \text{at } \phi = \text{constant}. $$

(c) free edge:

$$N_x = M_x = T_x = \bar{T}_x = 0, \quad \text{at } x = \text{constant}. $$

$$N_\phi = M_\phi = T_\phi = \bar{T}_\phi = 0, \quad \text{at } \phi = \text{constant}. $$

Similarly, other boundary equations may be formulated to reflect the actual edge conditions.

2.4 Methods of Solution of Free Vibration Problems

In general, the methods of solutions may be analytical, numerical, experimental or a mixture of all these methods. The numerical methods such as the finite element technique or the experimental techniques are not considered in this chapter. Consideration will be given only to semi-numerical methods.
A purely analytical solution is obtainable only when all the four edges of the plate are simply supported. Exact solutions to the problem may be obtained only when any pair of opposite edges are simply supported. Otherwise, the solutions will always be approximate. The approximate methods for such solutions will at the best be semi-numerical in character.

Such intermediate approximate analytical methods rest upon the works of notable mathematicians such as Ritz, Galerkin and Kantorovich. It may be emphasized that if the two dimensional problem is reduced to a one-dimensional one it becomes easier to solve. The method due to Kantorovich [1, pp.304-357] may be used to reduce the partial differential equations to ordinary differential equations. Then, the ordinary differential equations may be solved by various methods, like the transfer matrices, matrix progression, line solutions, Runge-Kutta and others.

2.5 Kantorovich's Method or the Method of Reduction to Ordinary Equations

This method occupies a position between the exact solution of the problem and the method of Rayleigh-Ritz and Galerkin (Kantorovich [1, p.304]). In the method of Rayleigh-Ritz, complete functions are assumed for the displacements. These assumed functions are substituted into the expression for the Kinetic-Potential $\Lambda$ which is a double integral. The problem then reduces to the determination of the undetermined constants in the assumed functions, such that $\Lambda$ is minimised. This minimum condition leads to the final set
of algebraic equations. However, this type of solution obtained by assuming functions strongly depends upon the assumed functions (Kerr |34|).

In the method of Kantorovich only part of the functions are assumed. For the sake of definiteness assume that $f_1(x)$, $f_2(x)$ and $f_3(x)$ are assumed to be known. After performing the integrations with respect to $x$ the expression of $\Lambda$ will contain undetermined functions of one variable which are $g_1(\phi)$, $g_2(\phi)$ and $g_3(\phi)$. The problem now reduces to finding these functions such that $\Lambda$ is a minimum. The condition that $\Lambda$ is a minimum with respect to the undetermined functions yields a set of linear homogeneous ordinary differential equations.

This approach to the method of Kantorovich need not be followed in practice if the differential equations of motion are already available. In such cases, Kantorovich [1] has suggested a more convenient way of obtaining the ordinary differential equations. Following the method of Kantorovich the exact set of equations 2.1(a-c) are reduced to a set of ordinary differential equations as follows.

The equations 2.1(a-c) may be written in operational form as follows:

\[
\begin{align*}
L_1(u,v,w) - \frac{R^2}{D} \frac{\partial h}{g} \frac{\partial^2 u}{\partial t^2} &= 0 , \\
L_2(u,v,w) - \frac{R^2}{D} \frac{\partial h}{g} \frac{\partial^2 v}{\partial t^2} &= 0 , \\
L_3(u,v,w) + \frac{R^2}{D} \frac{\partial h}{g} \frac{\partial^2 w}{\partial t^2} &= 0 ,
\end{align*}
\]

where $L_1$, $L_2$ and $L_3$ are partial differential operators.
If the boundary of the plate coincides with a rectangle, the solutions for equation 2.5(a-c) may be written as follows.

\[ u = f_1(x) g_1(\phi) e^{i\omega t} , \]
\[ v = f_2(x) g_2(\phi) e^{i\omega t} , \]
\[ w = f_3(x) g_3(\phi) e^{i\omega t} . \]

where \( f_1(x), f_2(x) \) and \( f_3(x) \) are functions of \( x \) only and \( g_1(\phi), g_2(\phi) \) and \( g_3(\phi) \) are functions of \( \phi \) only. Depending upon availability and convenience either the functions along \( x \) or the functions along \( \phi \) may be prescribed. In either case the displacement functions must satisfy the given edge conditions. (Henceforth the \( x \) and \( \phi \) inside the parentheses will in general be omitted). Reduction of equation 2.5(a-c) to ordinary equations is as follows, when \( f_1, f_2 \) and \( f_3 \) are prescribed.

Substituting for \( u, v \) and \( w \) from equation 2.6(a-c) into equation 2.5(a-c) and integrating according to the method of Kantorivich, the following equations are obtained.

\[
\int_{x=0}^{x} \left[ L_1(f_1 g_1, f_2 g_2, f_3 g_3) + \Delta f_1 g_1 \right] f_1 \, dx = 0 ,
\]

\[
\int_{x=0}^{x} \left[ L_2(f_1 g_1, f_2 g_2, f_3 g_3) + \Delta f_2 g_2 \right] f_2 \, dx = 0 ,
\]

\[
\int_{x=0}^{x} \left[ L_3(f_1 g_1, f_2 g_2, f_3 g_3) - \Delta f_3 g_3 \right] f_3 \, dx = 0 ,
\]

where \( \Delta = \frac{R \rho \omega^2}{D g} \).
Similarly when \( g_1, g_2 \) and \( g_3 \) are prescribed the reduced equations become the following.

\[
\begin{align*}
\theta = 0 & \quad \left[ L_1 (f_1 g_1, f_2 g_2, f_3 g_3) + \Delta f_1 g_1 \right] g_1 d\phi = 0, \\
\phi = 0 & \quad \left[ L_2 (f_1 g_1, f_2 g_2, f_3 g_3) + \Delta f_2 g_2 \right] g_2 d\phi = 0, \\
\theta = 0 & \quad \left[ L_3 (f_1 g_1, f_2 g_2, f_3 g_3) - \Delta f_3 g_3 \right] g_3 d\phi = 0.
\end{align*}
\]

2.8(a-c)

After performing the term by term integrations and making necessary simplifications the equations 2.7(a-c) and 2.8(a-c) become,

\[
\begin{align*}
L_{a1} (g_1, g_2, g_3) + \Delta g_1 &= 0, \\
L_{a2} (g_1, g_2, g_3) + \Delta g_2 &= 0, \\
L_{a3} (g_1, g_2, g_3) - \Delta g_3 &= 0;
\end{align*}
\]

2.9(a-c)

and

\[
\begin{align*}
L_{b1} (f_1, f_2, f_3) + \Delta f_1 &= 0, \\
L_{b2} (f_1, f_2, f_3) + \Delta f_2 &= 0, \\
L_{b3} (f_1, f_2, f_3) - \Delta f_3 &= 0;
\end{align*}
\]

2.10(a-c)

respectively.

In these equations, \( L_{a1}, L_{a2}, L_{a3}, L_{b1}, L_{b2} \) and \( L_{b3} \) are linear ordinary differential operators with constant coefficients. Thus the original partial differential equations are reduced to two sets of ordinary differential equations. If the functions along \( x \)
are prescribed then equations 2.9(a-c) are used to find the appropriate functions along \( \phi \). Similarly, equations 2.10(a-c) are used to find the functions along \( x \) when the functions along \( \phi \) are prescribed.

The full equations 2.9(a-c) and 2.10(a-c) and their coefficients are given in Appendix 1 and 2 respectively.

The equations 2.9(a-c) and 2.10(a-c) may be obtained directly from the consideration of the theory of structures. The first, second and third of the equations 2.5(a-c) represent the motion of the plate along the directions of \( u \), \( v \) and \( w \) respectively. Therefore, the equations 2.5(a), 2.5(b) and 2.5(c) are primarily functions of \( u \), \( v \) and \( w \) respectively.

Due to the prescription of the approximate functions there will be residual loads \( E_u \), \( E_v \) and \( E_w \) along the directions of \( u \), \( v \) and \( w \) respectively. After substitution of the prescribed functions the equations 2.5(a-c) will be the following.

\[
L_1(f_1g_1, f_2g_2, f_3g_3) + \Delta f_1g_1 = E_u,
\]
\[
L_2(f_1g_1, f_2g_2, f_3g_3) + \Delta f_2g_2 = E_v,
\]
\[
L_3(f_1g_1, f_2g_2, f_3g_3) - \Delta f_3g_3 = E_w.
\]

Since the system is required to execute the motions unaffected by these residual loads, the work done by these loads must be zero. This condition is equivalent to either
\[
\begin{align*}
\int_0^x E_u f_1 dx &= \int_0^x E_v f_2 dx = \int_0^x E_w f_3 dx = 0, \quad 2.12(a) \\
\text{or} \quad \int_0^\theta E_u g_1 d\phi &= \int_0^\theta E_v g_2 d\phi = \int_0^\theta E_w g_3 d\phi = 0. \quad 2.12(b)
\end{align*}
\]

It may be seen that fulfilment of these conditions will lead to the equations 2.9(a-c) and 2.10(a-c). The residual loads \( E_u, E_v \)
and \( E_w \) may be treated as errors of substitution of the approximate functions. The approximate functions may be considered as the 'weight functions' in minimising these errors.

The method may be applied to solution of plate problems when the boundaries of the plate do not form a rectangle. In such cases one of the variables will be dependent on the other. The functions may then be written in one of the following two forms if the boundary curves are analytically definable in \( x \) and \( \phi \).

\[
w_j = 1, 2, 3 = f_j(x) g_j(x, \phi) e^{i \omega t}, \quad 2.13(a-b)
\]

or

\[
w_j = 1, 2, 3 = f_j(x, \phi) g_j(\phi) e^{i \omega t},
\]

where \( w_1 = u, w_2 = v \) and \( w_3 = w \).

For more complicated boundaries of the plate, it may be necessary to set up different functions at different parts of the plate. The whole plate may then be accounted for by establishing continuity of all the actions (the relevant unknowns) across the boundaries of the different parts.
2.6 Selection of the Direction for Reduction

The selection of the direction for reduction of the partial differential equations to ordinary differential equations for specific problems will depend upon the following considerations.

The first consideration is of course the accuracy and the reliability of the solutions. The solutions by this method will depend upon the prescribed functions. Application of the Rayleigh-Ritz method with characteristic beam vibration functions have yielded acceptably accurate solutions for flat plates. This suggests that it may require less efforts and at the same time yield dependable solutions if the straight line direction is chosen for reduction. This will also give the added advantage that the plate need not be shallow for the solution of the unknown functions.

If the width of the plate varies along the direction of x then equations 2.10(a-c) have to be used. Similarly, if the width varies along the direction of φ then equations 2.9(a-c) have to be used.

Sometimes end conditions of a pair of opposite edges may also dictate the choice of the direction. For example, if a pair of opposite edges are simply supported and the equations are reduced across them, the exact solutions to the problem may be obtained. Alternatively, if the other direction is chosen, approximate solutions to the problem may be obtained with little effort.
2.7 Refinement of the Solutions

The solutions by the method of Kantorovich may be refined to any desired degree of accuracy by prescribing a family of functions instead of one. The representation of the functions when \( N \) functions of the families are prescribed, is as follows:

\[
\begin{align*}
    u &= (f_{1a}g_{1a} + f_{1b}g_{1b} + f_{1c}g_{1c} + \ldots + f_{1N}g_{1N}) e^{i\omega t}, \\
    v &= (f_{2a}g_{2a} + f_{2b}g_{2b} + f_{2c}g_{2c} + \ldots + f_{2N}g_{2N}) e^{i\omega t}, \\
    w &= (f_{3a}g_{3a} + f_{3b}g_{3b} + f_{3c}g_{3c} + \ldots + f_{3N}g_{3N}) e^{i\omega t}.
\end{align*}
\]

2.14(a-c)

These are known as the first, second, \( \ldots \), \( N \)th approximation, depending upon the number of functions of the families prescribed.

It is possible to use different approximations for \( u \), \( v \) and \( w \) such as the first approximation for \( u \), the second for \( v \) and the third for \( w \), as shown below.

\[
\begin{align*}
    u &= f_{1a}g_{1a} e^{i\omega t}, \\
    v &= (f_{2a}g_{2a} + f_{2b}g_{2b}) e^{i\omega t}, \\
    w &= (f_{3a}g_{3a} + f_{3b}g_{3b} + f_{3c}g_{3c}) e^{i\omega t}.
\end{align*}
\]

2.15(a-c)

The use of the second approximation for \( u \), \( v \) and \( w \) with the reduction in the direction of \( x \) will be demonstrated here. The functional representations of \( u \), \( v \) and \( w \) are done with the first two terms of equations 2.14(a), (b) and (c) respectively. Then the reduction to ordinary equations will take the following forms.
\[ \int_{x_0}^{x_1} \left[ L_1(u,v,w) + \Delta u \right] f_{1a} \, dx = 0 , \]
\[ \int_{x_0}^{x_1} \left[ L_1(u,v,w) + \Delta u \right] f_{1b} \, dx = 0 , \]
\[ \int_{x_0}^{x_1} \left[ L_2(u,v,w) + \Delta v \right] f_{2a} \, dx = 0 , \]
\[ \int_{x_0}^{x_1} \left[ L_2(u,v,w) + \Delta v \right] f_{2b} \, dx = 0 , \]
\[ \int_{x_0}^{x_1} \left[ L_3(u,v,w) \right] f_{3a} \, dx = 0 , \]
\[ \int_{x_0}^{x_1} \left[ L_3(u,v,w) \right] f_{3b} \, dx = 0 . \]

Thus the number of equations have doubled for the second approximation and will be trebled for the third and so on. Again, all these prescribed functions must satisfy the given boundary conditions. In order to avoid numerical difficulties, the individual functions in each of the family of functions should be orthogonal or nearly orthogonal to all other functions in that family.

An extension to the method of Kantorovich is suggested by Kerr [34]. The idea is to remove, as far as practicable, the dependence of the solutions on the assumed functions. Half of this dependence is removed by the method of Kantorovich. Kerr argues that a recursive method may be developed so that the original unknown functions, now determined by the method of Kantorovich, may be used
to redetermine the functions which were originally prescribed. The repeated application of the process should lead more or less to the exact solutions.

The first approximation of Kantorovich is used here, without incorporating any means of refining the solution. This is done because of computer storage limitation and turn round time when the method was first developed. Therefore, the solutions obtained here with this method will always be an upper bound to the exact solutions, except in the case of some special boundary conditions.

2.8 Reduction of the Boundary Conditions

Sometimes the boundary conditions may also require reduction similar to the partial differential equation. Where the reductions are necessary and how to apply these reductions will be discussed in this section.

Suppose the edge \( x = 0 \) is clamped. The required boundary conditions are

\[
\begin{align*}
  u &= v = w = w' = 0.
\end{align*}
\]

These conditions are equivalent to

\[
\begin{align*}
  u &= f_1 g_1 = 0, \\
  v &= f_2 g_2 = 0, \\
  w &= f_3 g_3 = 0, \\
  w' &= f'_3 g_3 = 0. 
\end{align*}
\]

Since these boundary conditions are to be satisfied irrespective of \( \phi \) the necessary boundary conditions are

\[
\begin{align*}
  f_1 = f_2 = f_3 = f'_3 = 0. 
\end{align*}
\]

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For the second approximation these conditions will be

$$f_{1a} = f_{1b} = f_{2a} = f_{2b} = f_{3a} = f_{3b} = f'_{3a} = f'_{3b} = 0. \quad 2.19$$

Under such conditions no reductions are necessary.

Next, consider an edge $x = \text{constant}$, where $M_x$ is required to vanish.

This condition is equivalent to

$$w'' + vw''' - u' - vv'' = 0 \quad 2.20(a-b)$$

or

$$f'_{3} g_{3} + v f_{3} g_{3}''' - f'_{1} g_{1} - v f_{2} g_{2}''' = 0.$$ 2.20(a)

Each of the displacements has a contribution in this equation. Since it is a moment $w$ has the greatest influence on it. Accordingly the weighting function will be a component of $w$. In this case the reduction will take the form

$$f'_{3} \int_{0}^{\theta} g_{3} g_{3} d\phi + v f_{3} \int_{0}^{\theta} g_{3} g_{3}''' d\phi - f'_{1} \int_{0}^{\theta} g_{1} g_{3} d\phi - v f_{2} \int_{0}^{\theta} g_{2} g_{3}''' d\phi = 0 \quad 2.21(a)$$

In the case of the second approximation the same condition will produce two boundary conditions of the form

$$\int_{0}^{\theta} M_{x} g_{3a} d\phi = \int_{0}^{\theta} M_{x} g_{3b} d\phi = 0 \quad 2.21(b)$$

Thus, a component of $w$ may be taken as a weight function for all the moments and the transverse shears $Q_x$ and $Q_\phi$. Similarly, for $N_x$, $N_{\phi x}$
and $T_\phi$ the weighting function will be a component of $u$; and for $N_\phi$, $N_{x\phi}$, $T_x$ it will be a component of $v$. For $T_\tilde{\phi}$ and $T_\tilde{x}$ the weighting function will be a component of $w$.

Also it may be seen that the number of boundary conditions increases for higher order approximations in accordance with the increase of the order of the systems of governing equations.
CHAPTER III

SOLUTION OF THE ORDINARY DIFFERENTIAL EQUATIONS
- THE EIGENVALUES AND THE EIGENVECTORS -

3.1 Introduction

The application of the method of Kantorovich to the partial differential equations of motion has yielded a set of linear homogeneous ordinary differential equations with constant coefficients. For free vibrations, the boundary conditions are also homogeneous and linear. Some of the coefficients of the set of ordinary differential equations and their fundamental sets of solutions depend upon the real frequency parameter $\Delta$.

There are several methods of solving such a set of ordinary differential equations. Since it is not known which is the best method of solving these equations, we select two of the simplest methods. They are (a) Flügge's method of integration which we call 'the general method of solution', and (b) matrix methods of integration. In the general method of solution the frequency determinant is formed once with complex numbers and once with real numbers. Three different versions of the matrix method are considered. They are (a) a single step integration, (b) a matrix progression method which is a multi-step integration method, and (c) a modified version of the matrix progression method. A description of each of these methods will now be given.

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3.2(a) General Method of Solution

The essential features of the method may be clarified by considering a second order equation.

Let the second order differential equation be

\[ y'' + p(x,\Delta) y' + q(x,\Delta) y = 0 \] \hspace{1cm} 3.1(a)

The boundary conditions at \( x=0 \) and at \( x = \bar{x} \) are

\[ a_1 y(0) + a_2 y'(0) = 0, \] \hspace{1cm} 3.1(b-c)

and

\[ b_1 y(\bar{x}) + b_2 y'(\bar{x}) = 0, \]

respectively. \( a_1, a_2, b_1 \) and \( b_2 \) are constants.

\( p(x,\Delta) \) and \( q(x,\Delta) \) are continuous functions of \( x \) and \( \Delta \) within the interval \( 0 \leq x \leq \bar{x} \), for all values of \( \Delta \). The general solution of equation 3.1(a) may be written as

\[ y = c_1 y_1(x,\Delta) + c_2 y_2(x,\Delta), \] \hspace{1cm} 3.2

where \( y_1 \) and \( y_2 \) are fundamental systems of solutions, \( c_1 \) and \( c_2 \) are free constants to be determined from the boundary conditions.

Differentiating equation 3.2 once the following equation is obtained

\[ y' = c_1 k_1 y'_1(x,\Delta) + c_2 k_2 y'_2(x,\Delta). \] \hspace{1cm} 3.3

\( k_1 \) and \( k_2 \) are constants as a result of the differentiation.

Substituting the values of \( y \) and \( y' \) from equation 3.2 and 3.3 respectively into the boundary conditions 3.1(b-c) the following two equations for
the determination of \( c_1 \) and \( c_2 \) are obtained.

\[
c_1\{a_1y_1(0,\Delta) + a_2k_1y_1'(0,\Delta)\} + c_2\{a_1y_2(0,\Delta) + a_2k_2y_2'(0,\Delta)\} = 0, \\
c_1\{a_1y_1(\bar{x},\Delta) + a_2k_1y_1'(\bar{x},\Delta)\} + c_2\{a_1y_2(\bar{x},\Delta) + a_2k_2y_2'(\bar{x},\Delta)\} = 0.
\]

\[3.4(a-b)\]

The set of two equations has non-trivial solutions for \( c_1 \) and \( c_2 \) if and only if the determinant \( D(\Delta) \) of the coefficients vanishes.

If such values of \( \Delta \) exist, which satisfy \( D(\Delta) = 0 \), then they are the eigenvalues of the problem. Corresponding to each eigenvalue, there is a non-vanishing solution of the boundary value problem which is the eigenfunction. The eigenfunctions cannot be determined uniquely, because, for a homogeneous problem, any multiple of a solution is also a solution. Therefore, the eigenfunctions can be determined only up to a multiplicative constant.

The method outlined above is exact in the mathematical sense. The general solution is obtained from the characteristic equation of the differential system and it is independent of the edge conditions.

Suggestion for the use of this method of solution for shell problems is credited to Flügge [6]. The method has been used successfully by him for the solution of edge load problems and later on by Forsberg [13] and Warburton [20] to analyse free vibrations of circular cylindrical shells.

This seems to be the first attempt at combining the method of Kantorovich reduction and the exact method of solution of the resulting
ordinary differential equations for the problem under consideration.

3.2(b) Application of the General Method of Solution to Equations 2.9(a-c)

The general method of solution described in the previous section will be applied for the solution of equations 2.9(a-c). The application to equation 2.10(a-c) is exactly similar.

Let \( \mu_1 \) be a parameter of solution, and also let \( \mu = \mu_1 / \theta \).

A solution of equation 2.9(a-c) may be written as

\[
\begin{align*}
\varphi_1 &= A e^{\mu_1} , \\
\varphi_2 &= B e^{\mu_1} , \\
\varphi_3 &= C e^{\mu_1} ,
\end{align*}
\]

where \( A, B \) and \( C \) are constants to be determined later. Substituting for \( \varphi_1, \varphi_2 \) and \( \varphi_3 \) from equation 3.5(a-c) into equation 2.9(a-c), the following equation is obtained.

\[
\begin{pmatrix}
\varphi_1 + a_{12} \mu^2 & a_{13} \mu & \varphi_1 + a_{16} \mu^2 \\
a_{21} \mu & \varphi_1 + a_{22} \mu^2 & \varphi_2 \\
\varphi_3 + a_{34} \mu^2 & \varphi_3 & \varphi_3 + a_{34} \mu^2 + a_{39} \mu
\end{pmatrix}
\begin{pmatrix}
A \\
B \\
C
\end{pmatrix} = 0
\]

(The co-efficients \( a_{j\ell} \) and \( p_{j\ell} \) are defined in Appendix 1).

For a non-trivial solution of the constants \( A, B \) and \( C \) the determinant of the coefficient must vanish, leading to the characteristic equation of the form
\[ T_1(\mu^2)^4 + T_2(\mu^2)^3 + T_3(\mu^2)^2 + T_4(\mu^2) + T_5 = 0. \] 3.7(a)

The coefficients \( T_1 \) to \( T_5 \) of this equation are as follows.

\[
T_1 = a_{12}^a a_{22}^a a_{39}^a \\
T_2 = a_{39}^a (p_{11}^a a_{22}^a + p_{21}^a a_{12}^a - a_{13}^a a_{21}^a) + a_{22}^a (a_{12}^a p_{34}^a - a_{16}^a a_{34}^a) \\
T_3 = p_{11}^a (p_{21}^a a_{39}^a + a_{22}^a p_{34}^a) + a_{12}^a (p_{21}^a p_{34}^a + a_{22}^a p_{32}^a - p_{22}^a p_{33}^a) + a_{13}^a (p_{22}^a a_{34}^a - a_{21}^a p_{34}^a) + a_{16}^a (a_{21}^a p_{33}^a - p_{31}^a a_{22}^a) - a_{34}^a (p_{12}^a a_{22}^a + p_{21}^a a_{16}^a) \\
T_4 = p_{11}^a (p_{21}^a p_{34}^a + a_{22}^a p_{32}^a - p_{22}^a p_{33}^a) + p_{21}^a (a_{12}^a p_{32}^a - a_{16}^a p_{31}^a) + a_{13}^a (p_{22}^a p_{31}^a - a_{21}^a p_{32}^a) + p_{12}^a (a_{21}^a p_{33}^a - p_{31}^a a_{22}^a) - p_{21}^a a_{34}^a \\
T_5 = p_{21}^a (p_{11}^a p_{32}^a - p_{12}^a p_{31}^a).
\]

The roots of equation 3.7(a) will depend on the boundary conditions at the edges \( x = \text{constant} \) and the prescribed modes of vibrations along the direction of \( x \). Also, they will depend upon the dimensions of the plate and the properties of its material. Therefore, it is not possible to predetermine the exact nature of the roots for every case. The precise knowledge of the nature of the roots greatly facilitates the general formulation of the frequency equation in terms of real numbers. It may be seen that equation 3.7(a) is quartic in \( \mu^2 \). Corresponding to every positive real root for \( \mu^2 \) there will be two real roots for \( \mu \) which are \( \pm \chi \). Similarly, corresponding to every negative real root for \( \mu^2 \) there will be two imaginary roots for \( \mu \) which are \( \pm i\tau \). However, for every complex root for \( \mu^2 \) its conjugate will also be a root for \( \mu^2 \). Therefore, for every complex root for \( \mu^2 \) there will be four complex roots for \( \mu \) of the form \( \chi + i\tau, \chi - i\tau, -\chi + i\tau \) and \( -\chi - i\tau \).
Corresponding to each root \( \mu_j (j = 1,8) \) of equation 3.7(a), there will be a solution of the three linear equations given by equation 3.6 of the form (using any two equations of 3.6).

\[
\begin{align*}
A &= \alpha_j C_j, \\
B &= \beta_j C_j, \\
C &= C_j;
\end{align*}
\]

where \( C_j \) is at present undetermined and \( \alpha_j \) and \( \beta_j \) are in general complex. The general solution of equations 2.9(a-c) is of the form

\[
\begin{align*}
\varepsilon_1 &= \sum_{j=1}^{8} \alpha_j C_j e^{\mu_j \phi}, \\
\varepsilon_2 &= \sum_{j=1}^{8} \beta_j C_j e^{\mu_j \phi}, \\
\varepsilon_3 &= \sum_{j=1}^{8} C_j e^{\mu_j \phi}.
\end{align*}
\]

Similar solutions may be written for \( f_1, f_2 \) and \( f_3 \) of equations 2.10(a-c).

The problem now reduces to finding the eigenvalues \( \lambda \) and the constants \( C_j \). Applying four boundary conditions at each edge, a system of eight homogeneous linear algebraic equations in the \( C_j (j = 1,8) \) are obtained.

Since the coefficients of the ordinary differential equations are functions of the frequency parameter \( \Delta \), then \( \mu_j, \alpha_j, \beta_j \) are all functions of \( \Delta \). Thus the coefficients of the eight linear equations are also functions of \( \Delta \). For a non-trivial solution, the determinant of coefficients \( D(\Delta) \) must vanish, thus giving an equation for the
determination of $\Delta$. The roots of this equation are the eigenvalues of the system, and the corresponding solutions of the eight linear equations yield the eigenfunctions.

3.3 Formulation of the Frequency Determinant with Complex Coefficients when $f_1$, $f_2$, and $f_3$ are prescribed.

Let the eight roots for $\mu$ of the characteristic equation 3.7(a) be denoted by $\lambda_{j=1,8}$. Further it is assumed that they are all complex and none of the roots is identically zero.

Using the first two equations of equation 3.6, $a_j$ of equation 3.7(b) may be expressed as follows.

$$a_j = \frac{-a_{16}a_{22}\lambda_j^4 + (a_{13}p_{22} - a_{16}p_{21} - a_{22}p_{12})\lambda_j^2 - p_{12}p_{21}}{a_{12}a_{22}\lambda_j^4 + (a_{22}p_{11} + a_{12}p_{21} - a_{13}a_{21})\lambda_j^2 - p_{11}p_{21}}.$$ 3.8

Similarly, using the same two equations of 3.6, $b_j$ of equation 3.7(c) may be expressed as follows.

$$b_j = \frac{(a_{21}a_{16} - a_{12}p_{22})\lambda_j^3 + (a_{21}p_{12} - p_{11}p_{22})\lambda_j}{a_{12}a_{22}\lambda_j^4 + (a_{22}p_{11} + a_{12}p_{21} - a_{13}a_{21})\lambda_j^2 - p_{11}p_{21}}.$$ 3.9

Then, $u$, $v$ and $w$ may be written in complex form as follows

$$u = f_1(x) g_1(\phi) = f_1(x) \sum_{j=1}^{8} \sum_{j=1}^{8} c_j \alpha_j e^{i\lambda_j \phi},$$

$$v = f_2(x) g_2(\phi) = f_2(x) \sum_{j=1}^{8} \sum_{j=1}^{8} c_j \beta_j e^{i\lambda_j \phi},$$

$$w = f_3(x) g_3(\phi) = f_3(x) \sum_{j=1}^{8} \sum_{j=1}^{8} c_j e^{i\lambda_j \phi}.$$ 3.10(a-c)
Expressing the displacements in this form it is reasonably easy to formulate the frequency determinant, depending upon the edge conditions at $\phi = \text{constant}$.

The frequency determinant when the edges $\phi=0$ and $\phi=\theta$ are clamped may be written as the following. The clamped edge conditions at $\phi=0$ and $\phi=\theta$ are

\[
\begin{align*}
\gamma_3 &= 0 \quad \text{from} \quad w = 0, \\
\gamma_1 &= 0 \quad \text{from} \quad u = 0, \\
\gamma_2 &= 0 \quad \text{from} \quad v = 0, \\
\gamma_3 &= 0 \quad \text{from} \quad w - v = 0. \\
\end{align*}
\]

From equation 3.11(a-d) in conjunction with 3.10(a-c) the following frequency determinant is obtained. For convenience $\gamma_{j=1...8} = e^{\lambda_j \theta}$ is used in the equation.

\[
\begin{vmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 \\
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 \\
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 \\
\alpha_{1\gamma_1} & \alpha_{2\gamma_2} & \alpha_{3\gamma_3} & \alpha_{4\gamma_4} & \alpha_{5\gamma_5} & \alpha_{6\gamma_6} & \alpha_{7\gamma_7} & \alpha_{8\gamma_8} \\
\beta_{1\gamma_1} & \beta_{2\gamma_2} & \beta_{3\gamma_3} & \beta_{4\gamma_4} & \beta_{5\gamma_5} & \beta_{6\gamma_6} & \beta_{7\gamma_7} & \beta_{8\gamma_8} \\
\lambda_{1\gamma_1} & \lambda_{2\gamma_2} & \lambda_{3\gamma_3} & \lambda_{4\gamma_4} & \lambda_{5\gamma_5} & \lambda_{6\gamma_6} & \lambda_{7\gamma_7} & \lambda_{8\gamma_8} \\
\end{vmatrix} = 0. 
\]

Similarly when the edge $\phi=0$ is clamped and the edge $\phi=\theta$ is simply supported the frequency determinant may be obtained in the following way.
At $\phi=0$ the edge conditions are shown to be (equation 3.11)

$$g_3 = g_1 = g_2 = g_3' = 0.$$  

The simply supported edge conditions at $\phi=0$ are

$$g_1 = 0 \quad \text{from} \quad u = 0,$$
$$g_3 = 0 \quad \text{from} \quad w = 0,$$

$$g_3 f_3 + g_3'' f_3 + vg_3 f_3' = 0 \quad \text{from} \quad M_\phi = 0,$$
$$g_1 f_1' + g_2 f_2 + (1 + k) g_3 f_3 + kg_3' f_3 = 0 \quad \text{from} \quad N_\phi = 0.$$  \hspace{1cm} 3.13(a-d)

Before reducing the last two equations of 3.13, the following simplifications may be carried out. Since $g_3 = 0$ at the boundary and $f_3$ and $f_3''$ are arbitrary the equations 3.13(c) reduces to $g_3'' = 0$. Since $g_1 = g_3 = g_3'' = 0$ at the boundary and $f_1'$, $f_2$ and $f_3$ are arbitrary, the equation 3.13(d) reduces to $g_2' = 0$. Therefore, the reduction of the last two boundary equations are not necessary in this case. The simply supported boundary conditions at a section $\phi = \text{constant}$ reduce to

$$g_3 = g_1 = g_2 = g_3' = 0.$$  \hspace{1cm} 3.14(a-d)

The frequency determinant for $\phi=0$ clamped and $\phi=0$

$$\lambda_i \delta$$

simply supported with $\gamma_j = 1 \ldots \delta = e^j$ is the following.
Similarly, the frequency determinant for any desired boundary conditions may be formed without any difficulty. Obviously most of the elements of the frequency determinant are complex. The computational aspect of the problem will be discussed later. Since the actual system is a real one without damping the eigenvalues of the complex system will all be real. When \( D(\Lambda) = 0 \) both the real and the imaginary parts of the value of the complex determinant must be zero.

Next, the frequency determinant will be written in real form with real coefficients.

3.4 Formulation of the Frequency Determinant with Real Coefficients when \( f_1, f_2 \) and \( f_3 \) are prescribed

The frequency determinant with real coefficients cannot be written in a compact form as the one with complex coefficients. It has been seen that only the following types of roots can occur.
(a) A pair of real roots
(b) A pair of imaginary roots
(c) Four complex roots.

Therefore, the solutions corresponding to these types of roots will be given.

(a) **A pair of real roots**

When two roots, \( \lambda_1 = \chi \) and \( \lambda_2 = -\chi \) are real, then it may be shown that the complex constants

\[
\alpha_1 = \bar{\alpha}_1, \quad \beta_1 = \bar{\beta}_1
\]
\[
\alpha_2 = \bar{\alpha}_1, \quad \beta_2 = -\bar{\beta}_1
\]

3.16(a-d)

where \( \bar{\alpha}_1 \) and \( \bar{\beta}_1 \) are real quantities.

Two of the eight solutions of \( \xi_3, \xi_1, \xi_2 \) and \( \xi_3^* \) will be as follows

\[
\xi_3 = c_1e^{\chi \phi} + c_2e^{-\chi \phi},
\]
\[
\xi_1 = \bar{\alpha}_1(c_1e^{\chi \phi} + c_2e^{-\chi \phi}),
\]
\[
\xi_2 = \bar{\beta}_1(c_1e^{\chi \phi} + c_2e^{-\chi \phi}),
\]
\[
\xi_3^* = \chi(c_1e^{\chi \phi} + c_2e^{-\chi \phi}).
\]

3.17(a-d)

Let \( \bar{c}_1 = c_1 + c_2 \) and \( \bar{c}_2 = c_1 - c_2 \).

The equation 3.17(a-d) may be written in real form as follows.
\[ g_3 = \overline{c}_1 \cosh \chi \phi + \overline{c}_2 \sinh \chi \phi \],
\[ g_1 = \overline{a}_1 (\overline{c}_1 \cosh \chi \phi + \overline{c}_2 \sinh \chi \phi) \],
\[ g_2 = \overline{b}_1 (\overline{c}_1 \sinh \chi \phi + \overline{c}_2 \cosh \chi \phi) \],
\[ g_3^* = \chi (\overline{c}_1 \sinh \chi \phi + \overline{c}_2 \cosh \chi \phi) . \]

It may be noted that \( g_3 \) and \( g_1 \) are similar in shapes. Similarly, \( g_2 \) and \( g_3^* \) are similar in shape. For symmetrical modes of vibrations \( \overline{c}_2 = 0 \), whereas for antisymmetrical modes of vibrations \( \overline{c}_1 = 0 \).

**(b) A pair of imaginary roots**

When two roots \( \lambda_1 = i \tau \) and \( \lambda_2 = -i \tau \) are imaginary, it may be shown that

\[ a_1 = \overline{a}_1 \quad \beta_1 = i \overline{b}_2 \],
\[ a_2 = \overline{a}_1 \quad \beta_2 = -i \overline{b}_2 \],

where \( \overline{a}_1 \) and \( \overline{b}_2 \) are real quantities.

Two of the eight solutions of \( g_3 \) may be written as

\[ g_3 = c_1 e^{i \tau \phi} + c_2 e^{-i \tau \phi} \]
\[ = (c_1 + c_2) \cos \tau \phi + i(c_1 - c_2) \sin \tau \phi \]
\[ = \overline{c}_1 \cos \tau \phi + \overline{c}_2 \sin \tau \phi , \]

where \( \overline{c}_1 = c_1 + c_2 \) and \( \overline{c}_2 = i(c_1 - c_2) \).

Similarly it may be shown that

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\[ g_1 = \bar{a}_1 (\bar{c}_1 \cos \tau \phi + \bar{c}_2 \sin \tau \phi) , \]
\[ g_2 = \bar{\beta}_2 (-\bar{c}_1 \sin \tau \phi + \bar{c}_2 \cos \tau \phi) , \]
\[ g_3^* = \tau (-\bar{c}_1 \sin \tau \phi + \bar{c}_2 \cos \tau \phi) . \]

3.19(b-d)

It may be noted again that the shape of \( g_1 \) is the same as that of \( g_3 \) and the shape of \( g_2 \) is the same as that of \( g_3^* \).

For symmetrical modes of vibrations \( \bar{c}_2 = 0 \) whereas for anti-symmetrical modes of vibrations \( \bar{c}_1 = 0 \).

(c) When a root is complex

When one root is complex, then in total there will be four complex roots as shown below.

\[ \lambda_1 = \chi + i\tau , \]
\[ \lambda_2 = -\chi - i\tau , \]
\[ \lambda_3 = \chi - i\tau , \]
\[ \lambda_3 = -\chi + i\tau . \]

3.20(a-d)

The corresponding \( a_j \) and \( \beta_j \) are as follows.

\[ a_1 = \bar{a}_1 + i\bar{\sigma}_2 , \]
\[ a_2 = \bar{a}_1 + i\bar{\sigma}_2 , \]
\[ a_3 = \bar{a}_1 - i\bar{\sigma}_2 , \]
\[ a_4 = \bar{a}_1 - i\bar{\sigma}_2 , \]
\[ \beta_1 = \bar{\beta}_1 + i\bar{\beta}_2 , \]
\[ \beta_2 = -\bar{\beta}_1 - i\bar{\beta}_2 , \]
\[ \beta_3 = \bar{\beta}_1 - i\bar{\beta}_2 , \]
\[ \beta_4 = -\bar{\beta}_1 + i\bar{\beta}_2 . \]

3.21(a-h)

In these expressions, the quantities \( \bar{a}_1, \bar{\sigma}_2, \bar{\beta}_1 \) and \( \bar{\beta}_2 \) are real quantities. Four of the eight solutions of \( g_3, g_1 \) and \( g_2 \) may be expressed as the following with \( \gamma_j=1,2,3,4 = e^{j\phi} \).
\[ g_3 = C_1 Y_1 + C_2 Y_2 + C_3 Y_3 + C_4 Y_4, \]
\[ g_1 = C_1 Y_1 + C_2 Y_2 + C_3 Y_3 + C_4 Y_4, \] 3.22(a-c)
\[ g_2 = C_1 Y_1 + C_2 Y_2 + C_3 Y_3 + C_4 Y_4. \]

Let
\[ \bar{c}_1 = C_1 + C_2 + C_3 + C_4, \]
\[ \bar{c}_2 = i(C_1 + C_2 - C_3 - C_4), \]
\[ \bar{c}_3 = C_1 - C_2 + C_3 - C_4, \] 3.23(a-d)
\[ \bar{c}_4 = i(C_1 - C_2 - C_3 + C_4). \]

Using relations 3.20, 3.21 and 3.23, the expressions 3.22 may be written in real form as follows.

\[ g_3 = \bar{c}_1 \cosh \chi \phi \cos \tau \phi + \bar{c}_2 \sinh \chi \phi \sin \tau \phi \]
\[ + \bar{c}_3 \sinh \chi \phi \cos \tau \phi + \bar{c}_4 \cosh \chi \phi \sin \tau \phi, \]
\[ g_1 = \bar{c}_1 (\bar{a}_1 \cosh \chi \phi \cos \tau \phi - \bar{a}_2 \sinh \chi \phi \sin \tau \phi) \]
\[ + \bar{c}_2 (\bar{a}_1 \sinh \chi \phi \sin \tau \phi + \bar{a}_2 \cosh \chi \phi \cos \tau \phi) \]
\[ + \bar{c}_3 (\bar{a}_1 \sinh \chi \phi \cos \tau \phi - \bar{a}_2 \cosh \chi \phi \sin \tau \phi) \]
\[ + \bar{c}_4 (\bar{a}_1 \cosh \chi \phi \sin \tau \phi + \bar{a}_2 \sinh \chi \phi \cos \tau \phi), \] 3.24(a-c)
\[ g_2 = \bar{c}_1 (\bar{b}_1 \sinh \chi \phi \cos \tau \phi - \bar{b}_2 \cosh \chi \phi \sin \tau \phi) \]
\[ + \bar{c}_2 (\bar{b}_1 \cosh \chi \phi \sin \tau \phi + \bar{b}_2 \sinh \chi \phi \cos \tau \phi) \]
\[ + \bar{c}_3 (\bar{b}_1 \cosh \chi \phi \cos \tau \phi - \bar{b}_2 \sinh \chi \phi \sin \tau \phi) \]
\[ + \bar{c}_4 (\bar{b}_1 \sinh \chi \phi \sin \tau \phi + \bar{b}_2 \cosh \chi \phi \cos \tau \phi). \]

Again it may be seen that \( g_3 \) is similar in shape with \( g_1 \) and \( g_2 \) is similar in shape with \( g_3 \).
For symmetrical modes of vibrations \( \bar{c}_3 = \bar{c}_4 = 0 \) and for antisymmetrical modes of vibration \( \bar{c}_1 = \bar{c}_2 = 0 \).

Thus knowing the functions \( g_1, g_2 \) and \( g_3 \) in real form the frequency determinant for any given boundary conditions may be formed.

3.5 Formulation of the Frequency Determinant with Real Coefficients when the Functions \( g_1, g_2 \) and \( g_3 \) are prescribed

The procedure is exactly similar to that described in the previous section. This time the ordinary differential equations 2.10(a-c) have to be used. Let \( \mu_1 \) be a parameter of solution and denote \( \mu = \mu_1/\bar{x} \). A solution of equation 2.10(a-c) may be written as

\[
\begin{align*}
    f_1 &= A e^{\mu x}, \\
    f_2 &= B e^{\mu x}, \\
    f_3 &= C e^{\mu x}, 
\end{align*}
\]  

3.25(a-c)

where \( A, B \) and \( C \) are constants to be determined. Substitution for \( f_1, f_2 \) and \( f_3 \) from equation 3.25(a-c) into equation 2.10(a-c) leads to the characteristic equation of order eight. Some of the eight roots \( (\lambda_j = 1 \ldots 8) \) of the characteristic equation may be real, some may be imaginary and some may be complex. As in the previous case, real and imaginary roots will appear in pairs and the complex roots will appear with their conjugates. Again \( A \) and \( B \) may be expressed in terms of \( C \) such that

\[
\begin{align*}
    A_j &= \alpha_j C_j, \\
    B_j &= \beta_j C_j, 
\end{align*}
\]  

3.26(a-b)
where $\alpha_j$ and $\beta_j$ are in general complex.

The displacements, $u$, $v$ and $w$ may be written in the complex form as follows:

$$u = f_1 g_1 = g_1(\phi) \sum_{j=1}^{\infty} C_j \alpha_j e^{\lambda_j x},$$

$$v = f_2 g_2 = g_2(\phi) \sum_{j=1}^{\infty} C_j \beta_j e^{\lambda_j x},$$

$$w = f_3 g_3 = g_3(\phi) \sum_{j=1}^{\infty} C_j e^{\lambda_j x}.$$  \hspace{1cm} 3.27(a-c)

In the following, the solutions will be written in the real form when the characteristic equation has

(a) a pair of real roots,

(b) a pair of imaginary roots,

and (c) four complex conjugate roots.

For the sake of brevity, only the essential steps will be given. The solutions in the complex form will not be written.

(a) A pair of real roots

When two roots $\lambda_1 = \mu$ and $\lambda_2 = -\mu$ are real then it may be shown that

$$\alpha_1 = \bar{\alpha}_1, \quad \beta_1 = \bar{\beta}_1,$$

$$\alpha_2 = -\bar{\alpha}_1, \quad \beta_2 = \bar{\beta}_1,$$  \hspace{1cm} 3.28(a-d)

where $\bar{\alpha}_1$ and $\bar{\beta}_1$ are real quantities. Two solutions of $f_3, f_1, f_2$ and $f'_3$ corresponding to these two real roots may be written in real form.

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as follows

\[ f_3 = \bar{c}_1 \cosh \mu x + \bar{c}_2 \sinh \mu x , \]
\[ f_1 = \bar{a}_1 (\bar{c}_1 \sinh \mu x + \bar{c}_2 \cosh \mu x) , \]
\[ f_2 = \bar{\beta}_1 (\bar{c}_1 \cosh \mu x + \bar{c}_2 \sinh \mu x) , \]
\[ f'_3 = \mu (\bar{c}_1 \sinh \mu x + \bar{c}_2 \cosh \mu x) . \]  \hspace{1cm} 3.29(a-d)

It may be seen that the shape of \( f_3 \) is the same as that of \( f_2 \) and that of \( f_1 \) is the same as the shape of \( f'_3 \). For symmetrical modes of vibrations \( \bar{c}_2 = 0 \) whereas for antisymmetrical modes of vibrations \( \bar{c}_1 = 0 \).

(b) A pair of imaginary roots

Let \( \lambda_1 = i\mu \) and \( \lambda_2 = -i\mu \) be the pair of imaginary roots.

It may be shown that

\[ a_1 = i\bar{a}_1 , \quad \beta_1 = \bar{\beta}_1 , \]
\[ a_2 = -i\bar{a}_1 , \quad \beta_2 = \bar{\beta}_1 \]  \hspace{1cm} 3.30(a-d)

where \( \bar{a}_1 \) and \( \bar{\beta}_1 \) are real. Two solutions of \( f_3, f_1, f_2 \) and \( f'_3 \) corresponding to these two imaginary roots are written in real form as follows.

\[ f_3 = \bar{c}_1 \cos \mu x + \bar{c}_2 \sin \mu x , \]
\[ f_1 = \bar{a}_1 (-\bar{c}_1 \sin \mu x + \bar{c}_2 \cos \mu x) , \]
\[ f_2 = \bar{\beta}_1 (\bar{c}_1 \cos \mu x + \bar{c}_2 \sin \mu x) , \]
\[ f'_3 = \mu (-\bar{c}_1 \sin \mu x + \bar{c}_2 \cos \mu x) , \]  \hspace{1cm} 3.31(a-d)

where \( \bar{c}_1 \) and \( \bar{c}_2 \) are constants.
It may be noted that the shapes of \( f_3 \) and \( f_2 \) are similar.
Similarly, the shapes of \( f_1 \) and \( f_1' \) are similar. For symmetrical modes of vibrations \( \bar{c}_2 = 0 \) whereas for antisymmetrical modes of vibrations \( \bar{c}_1 = 0 \).

(c) When four roots are complex

The roots \( \lambda_j \) and the constants \( a_j \) and \( \beta_j \) may be written as the following.

\[
\begin{align*}
\lambda_1 &= \tau + i\mu, \quad a_1 = \bar{a}_1 + i\bar{a}_2, \quad \beta_1 = \bar{\beta}_1 + i\bar{\beta}_2, \\
\lambda_2 &= -\tau + i\mu, \quad a_2 = -\bar{a}_1 - i\bar{a}_2, \quad \beta_2 = \bar{\beta}_1 + i\bar{\beta}_2, \\
\lambda_3 &= \tau - i\mu, \quad a_3 = \bar{a}_1 - i\bar{a}_2, \quad \beta_3 = \bar{\beta}_1 - i\bar{\beta}_2, \quad 3.32(a-2) \\
\lambda_4 &= -\tau + i\mu, \quad a_4 = -\bar{a}_1 + i\bar{a}_2, \quad \beta_4 = \bar{\beta}_1 - i\bar{\beta}_2, 
\end{align*}
\]

where \( \tau, \mu, \bar{a}_1, \bar{a}_2, \bar{\beta}_1 \) and \( \bar{\beta}_2 \) are real quantities. For solution in real form of \( f_3, f_1 \) and \( f_2 \) corresponding to the four complex roots may be written as follows.

\[
\begin{align*}
f_3 &= \bar{c}_1 \cosh \tau x \cos \mu x + \bar{c}_2 \sinh \tau x \sin \mu x \\
+ \bar{c}_3 \sinh \tau x \cos \mu x + \bar{c}_4 \cosh \tau x \sin \mu x, \\
f_1 &= \bar{c}_1(\bar{a}_1 \sinh \tau x \cos \mu x - \bar{a}_2 \cosh \tau x \sin \mu x) \\
+ \bar{c}_2(\bar{a}_1 \cosh \tau x \sin \mu x + \bar{a}_2 \sinh \tau x \cos \mu x) \\
+ \bar{c}_3(\bar{a}_1 \cosh \tau x \cos \mu x - \bar{a}_2 \sinh \tau x \sin \mu x) \\
+ \bar{c}_4(\bar{a}_1 \sinh \tau x \sin \mu x + \bar{a}_2 \cosh \tau x \cos \mu x), \\
f_2 &= \bar{c}_1(\bar{\beta}_1 \cosh \tau x \cos \mu x - \bar{\beta}_2 \sinh \tau x \sin \mu x) \\
+ \bar{c}_2(\bar{\beta}_1 \sinh \tau x \sin \mu x + \bar{\beta}_2 \cosh \tau x \cos \mu x) \\
+ \bar{c}_3(\bar{\beta}_1 \sinh \tau x \cos \mu x - \bar{\beta}_2 \cosh \tau x \sin \mu x) \\
+ \bar{c}_4(\bar{\beta}_1 \cosh \tau x \sin \mu x + \bar{\beta}_2 \sinh \tau x \cos \mu x), 
\end{align*}
\]

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For symmetrical modes of vibration \( \bar{c}_3 = \bar{c}_4 = 0 \), whereas for antisymmetrical modes of vibration \( \bar{c}_1 = \bar{c}_2 = 0 \). It may be seen that when \( f_3 \) is symmetrical in nature, \( f_2 \) is also symmetrical in nature but \( f_1 \) is antisymmetrical in nature.

When the roots of the characteristic equation are known there is no problem in formulating the frequency determinant according to the conditions at the edges.

Provisions have to be made for the three kinds of roots of the characteristic equation when the complete solutions of \( u, v \) and \( w \) are written.

### 3.6 Computation of the Eigenfunctions

The system equation may be written as

\[
[D(\Delta)]\{C\} = \{0\},
\]

with \( |D(\Delta)| = 0 \) when \( \Delta \) is an eigenvalue of the system. \([D(\Delta)]\) is an 8 x 8 square matrix and \( \{C\} \) is an 8 x 1 column matrix of the unknowns that are required to be evaluated. For the real system \( \{C\} = \{\bar{C}\} \).

Since 3.34 is a homogeneous system, the absolute values of \( \{C\} \) cannot be found. The relative values of \( \{C\} \) may be easily obtained by assuming any one component of \( \{C\} \) equal to unity. The procedure will fail to yield correct solutions if the assumed component of \( \{C\} \) is zero or nearly so. Under such a condition another component of \( \{C\} \) should be assumed to be unity.

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After evaluation of the complete vector \( \{C\} \) it is convenient to normalise the whole vector with respect to the largest component. Knowing \( \{C\} \), the constants for evaluation of \( u \) and \( v \), which may be written in vector form as \( \{A\} \) and \( \{B\} \) respectively, may be evaluated. Knowing \( \{A\} \), \( \{B\} \) and \( \{C\} \) the displacements \( u \), \( v \) and \( w \) at any point of the plate may be calculated. After calculating \( u \), \( v \) and \( w \) at all the required points, it is convenient to normalise them with respect to the largest of them. Once the eigenfunctions are known, their derivatives may be evaluated and from there the stress resultants may be computed if desired.

It is not feasible to develop an analytical solution. This can be conveniently obtained by a numerical iterative technique. The major steps in this method are indicated in Figure 3.

3.7 Matrix Method of Solution of the Reduced Equations

The ordinary differential equations given by equations 2.9(a–c) and equations 2.10(a–c) may be solved by matrix methods also. It has been discussed in section 2.2 that the original partial differential equations are of order nine with respect to \( x \). Therefore, the equations 2.10(a–c) are also of order nine. Unless they are transformed the equations 2.10(a–c) cannot be reduced to a set of eight first order equations. Since this reduction to a first order system is essential to the matrix method of solution to be developed, the equations 2.10(a–c) will no more be considered.
The ordinary differential equations 2.9(a-c) may be reduced to eight first order differential equations as shown in Appendix 3. In matrix form they become

\[
\frac{d\{G\}}{d\phi} = [A(8 \times 8)]\{G(8 \times 1)\} ,
\]

where \( \{G\} \) is the column of the unknowns from \( g_1 \) to \( g_8 \)
and \( [A] \) is the \( 8 \times 8 \) square matrix of constant coefficients.

The boundary conditions at the end \( \phi = 0 \) may be written as (See Appendix 3)

\[
\{G_0\} = [J_0(8 \times 4)]\{G^{**}(4 \times 1)\} .
\]

The boundary condition at the other end \( \phi = \theta \)
may be expressed as (see Appendix 3)

\[
[K_\theta(4 \times 8)]\{g_\theta(8 \times 1)\} = \{0\} .
\]

3.7(a) Single Step Integration of the First Order System

The solution of equation 3.35 may be written as

\[
\{G\} = e^{[A]\phi}\{G_0\} ,
\]

where \( e \) is exponentiation and \( e^{[A]\phi} \) may be written as

\[
e^{[A]\phi} = I + [A] \phi + \frac{[A]^2 \phi^2}{2} + \frac{[A]^3 \phi^3}{6} + \ldots .
\]

(I is a unit matrix).
Frazer, Duncan and Collar \[42, p.45\] have shown that the series 3.39 is absolutely and uniformly convergent for all values of \( \phi \). At the other end \( \phi = 0 \),

\[
\{G_0\} = e^{[A]0}[G_o].
\]  

Premultiplying equation 3.40 with \([K_0]\) and substituting for \([G_0]\) from 3.36 and together with the boundary condition of 3.37 the following equation is obtained.

\[
[K_0]\{G_0\} = [K_0]e^{[A]0}[J]\{G^{**}\} = \{0\}.
\]  

For \( \{G^{**}\} \neq \{0\} \), the determinant

\[
| [K_0] e^{[A]0}[J_o] | = 0.
\]  

The determinant of 3.42 is the frequency determinant \( D(\Lambda) \). The values of \( \Lambda \) which satisfy equation 3.42 are the eigenvalues of the system.

3.7(b) Matrix Progression Method of Integration

This method is due to Tottenham \[45\], and was successfully used by Tottenham and Desai \[46\] for the static analysis of arch dams. In this method the total interval of integration is divided into \( n_1 \) parts which may or may not be equal. The manner of division is shown below.

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & \text{\ldots} & n_1 \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\text{\ldots} \quad \theta_2 \quad \text{\ldots} \\
\hline
\end{array}
\]

\[
\begin{array}{c}
0 \quad \text{\ldots} \quad \theta \\
\hline
\end{array}
\]

-53-
Let the step length of a typical step \( j \) be \( \theta_j \) between the points \( j-1 \) and \( j \) where \( 0 \leq j \leq n_1 \). Also denote \( e^{[A] \theta_j} \) by \( [H_j] \). Now the solution at the end of the first step (point 1) may be written as

\[
\{G_1\} = [H_1]\{G_0\}.
\]

Shifting the origin from point 0 to point 1 the solution for point 2 may be written as

\[
\{G_2\} = [H_2]\{G_1\} = [H_2][H_1]\{G_0\}.
\]

The process may be continued until the end point \( n_1 \) is arrived at. At the point \( n_1 \) the solution becomes

\[
\{G_{n_1}\} = \{G_n\} = [H_{n_1}] \cdots [H_2][H_1]\{G_0\} \nonumber
\]

\[
= [\bar{H}]\{G_0\},
\]

where

\[
[\bar{H}] = [H_{n_1}] \cdots [H_2][H_1].
\]

Using the boundary conditions 3.36 and 3.37 the equations 3.45 may be written as the following.

\[
[K_\theta][\bar{H}][J_\circ]\{G^{**}\} = \{0\}.
\]

For \( \{G^{**}\} \neq \{0\} \), the determinant

\[
| [K_\theta][\bar{H}][J_\circ] | = 0.
\]
The determinant of 3.47 is the frequency determinant $D(\Delta)$. The values of $\Delta$ which satisfy equation 3.47 are the eigenvalues of the system.

3.7(c) **Modified Matrix Progression Technique**

This is also a multi-step integration technique. First the total integration interval is subdivided into $n_1$ subdivisions. Then the boundary conditions are brought forward from the beginning to the end of the step. Thus, the plate behind is replaced by an elastic medium. The originator of this idea seems to be Kalnins \cite{33 and 44}. However, it is found convenient to use the technique as developed by Tottenham which may be found in reference \cite{24}. The method is described as follows.

The interval $(0, \theta)$ is divided into $n_1$ equal sub-intervals of size $\theta_1 = \theta / n_1$.

At $\phi = \theta_1$

\[
\{G_1\} = e^{[A] \theta_1} \{G_0\} = [H_1] \{G_0\}. \tag{3.48}
\]

Introducing the boundary equation 3.36 gives

\[
\{G_1\} = [H_1][J_0] \{G^{**}\} = [F_1] \{G^{**}\}. \tag{3.49}
\]

Now partition this equation in the form

\[
\begin{pmatrix}
G^*(4 \times 1) \\
G^{**}(4 \times 1)
\end{pmatrix}
= 
\begin{pmatrix}
F^*(4 \times 4) \\
F^{**}(4 \times 4)
\end{pmatrix}
\begin{pmatrix}
G^{**}
\end{pmatrix}. \tag{3.50}
\]

-55-
From equation 3.50

\[ \{G_1^{**}\} = [F_{1}^{**}] \{G_0^{**}\} \]  \hspace{1cm} 3.51

Therefore

\[ \{G_0^{**}\} = [F_{1}^{**}]^{-1}\{G_0^{**}\} \]  \hspace{1cm} 3.52

Substituting 3.52 into 3.50 gives

\[ \{G_1\} = \begin{bmatrix} [F_1^{*}][F_{1}^{**}]^{-1} \\ I \end{bmatrix} \{G_0^{**}\} = [J_1] \{G_1^{**}\} \]  \hspace{1cm} 3.53

Similarly at \( \phi = \theta_2 \)

\[ \{G_2\} = e^{[A] \phi_2} \{G_0\} \]
\[ = [A] \phi_1 e^{[A] \phi_1} \{G_0\} \]
\[ = e^{[A] \phi_1} \{G_1\} \]  \hspace{1cm} 3.54

The above procedure can now be repeated to give

\[ \{G_2\} = [J_2] \{G_2^{**}\} \]  \hspace{1cm} 3.55

This process is repeated for all the sub-intervals. The final interval gives the equation

\[ \{G_{n_1}\} = [J_{n_1}] \{G_{n_1}^{**}\} = \{G_0\} \]  \hspace{1cm} 3.56
Premultiplying by \([K_0]\) and introducing the boundary equation 3.37 yields

\[
[K_0] \begin{bmatrix} J_{n_1} \\ n_1 \end{bmatrix} \{G^{**}\} = \{0\} \quad .
\]

3.57

Since the matrix \([J_{n_1}]\) is a function of the frequency parameter \(\Delta\), the frequency determinant is

\[
| [K_0] \begin{bmatrix} J_{n_1} \\ n_1 \end{bmatrix} | = 0 \quad .
\]

3.58

As in the previous method it is not feasible to develop an analytical solution to equation 3.58. A numerical iterative technique is adopted as indicated in Figure 4.

Corresponding to each eigenvalue there is a corresponding solution of equation 3.57 for \(\{G^{**}\}_{n_1}\). This is used to determine the eigenvectors as follows.

At the \(j^{th}\) step

\[
\{G_j^{**}\} = \begin{bmatrix} F^{**} \\ j+1 \end{bmatrix}^{-1} \{G_j^{**}\} ,
\]

3.59

and

\[
\{G_j\} = [J_j] \{G_j^{**}\} \quad .
\]

3.60

3.8 The Prescribed Displacement Functions

In the method of Rayleigh-Ritz, the assumed functions contain undetermined constants which may be selected to suit a particular problem. No such constants are available in the Kantorovich reduction procedure. Therefore, the prescribed functions need to represent the actual displacement.
functions as closely as possible. In the proposition of the method the only restriction imposed on the displacement functions is that they must satisfy the end conditions. Moreover, the assumed functions need to satisfy some more additional conditions which are peculiar to the problem under consideration.

$u$, $v$ and $w$ are components of the displacement of a point in the plate. They are related differentially by the differential equations of motion (equation 2.1(a-c)). Therefore, the variation of only one of them may be prescribed arbitrarily. The variation of the other two must be such that they are compatible with the prescribed variation of the other. The nature of these variations and their compatibility conditions may be deduced directly from the differential equations of motion or more conveniently from sections 3.4 and 3.5. In more complicated cases, they may be deduced from physical reasonings also.

In this section the terms "symmetrical" and "anti-symmetrical" are used in the sense that the shapes may approximately be described in the sense of "symmetry" and "anti-symmetry". However, these terms are strictly applicable in the mathematical sense when two opposite edges have identical edge conditions. From sections 3.4 and 3.5 the following relations between the shapes of the displacement functions may readily be given.

(i) when $f_1$ is symmetrical in shape $f_2$ and $f_3$ are anti-symmetrical in shape and vice versa.

(ii) $f_2$ is similar in shape with $f_3$ but differs in amplitudes. Similarly, $f_1$ is similar in shape with $f_3$ but differs in amplitude.
(iii) when \( g_1 \) and \( g_3 \) are symmetrical in shape \( g_2 \) is anti-symmetrical in shape and vice versa.
(iv) \( g_1 \) is similar in shape with \( g_3 \) but differs in amplitude. Similarly, \( g_2 \) is similar in shape with \( g_3 \) but different in amplitude.

The relations between the mode shapes discussed above are true except for rigid body modes. They are strictly true when two opposite edges have identical edge conditions, irrespective of the conditions at the other two edges.

The problem is correctly solved only when the computed eigenfunctions satisfy the given edge conditions and also satisfy the relations between the mode shapes, stated above. Suppose for a clamped plate \( f_3 \) is prescribed as symmetrical. Then \( f_2 \) also must be prescribed as symmetrical and \( f_1 \) as anti-symmetrical. Then, if the problem is correctly solved the computed mode shapes of \( g_1, g_2 \) and \( g_3 \) must satisfy the conditions (iii) and (iv) above.

It is worthwhile noting that the above four conditions merely state the relative shapes of the displacement functions. They do not define the shapes precisely. As an example, if \( f_3 \) is symmetrical with one half wave, \( f_2 \) may be symmetrical with either one or three half waves and \( f_1 \) may be antisymmetrical either with two or four half waves.

Thus there exists the possibility of more than one physically and mathematically admissible mode combination.

Since the primarily transverse modes of vibrations are required it is pertinent to define \( w \) adequately. This may be done approximately...
from the differential equations of motion 2.2(a-c). For the definition of \( f_3 \) let \( v, v \) and all the derivatives with respect to \( \phi \) vanish in equation 2.2(a-c). Then, the first and the second of the resulting equations may be neglected since they are primarily concerned with the inplane vibrations. In the third equation \( w \) may be neglected in comparison to \( w^{''''} \) and it becomes the eigenvalue problem.

\[
\begin{align*}
w^{''''} - \Delta_b \ w &= 0 \\
\text{or} \quad f^{''''} - \Delta_b \ f_3 &= 0,
\end{align*}
\]

3.61

where \( \Delta_b \) is the eigenvalue of the problem.

Equation 3.61 is the same as the classical beam vibration equation. Its eigenvalues and mode shapes are extensively tabulated by Bishop and Johnson [31] for different edge conditions. Denoting the beam vibration function in the \( x \) direction by \( F_{bx} \) the admissible and compatible displacement functions may approximately be defined as follows.

\[
\begin{align*}
f_1 &= F'_{bx} \\
f_2 &= F_{bx} \\
f_3 &= F_{bx}
\end{align*}
\]

3.62(a-c)

Similarly, in the curved direction the equations 2.2(a-c) reduce to a pair of equations defining the vibrations of a circular arch in \( v \) and \( w \). The resulting equation either in \( v \) or in \( w \) is of order 6. The eigenvalues and the mode shapes of this system are not readily available. In the absence of these functions, if the plate is shallow, the uniform beam vibration functions may be used. This
representation will not be quite adequate if the plate is not very shallow. Representing the beam function by $F_{b\phi}$ the approximate functional representation in the curved direction may be as follows.

\begin{align*}
\xi_1 &= F_{b\phi} , \\
\xi_2 &= F_{b\phi}' , \\
\xi_3 &= F_{b\phi}'' .
\end{align*} \quad 3.63(a-c)

Trial functions other than the beam functions are found to yield inferior solutions. Polynomials were not used since they tended to increase the order of the system of equations. Principally, two types of edge conditions are considered here. They are

(i) two opposite edges simply supported,
(ii) two opposite edges clamped.

The relevant assumed functions and the required integrals are given in Appendix 4. For reference the assumed functions and the required integrals when one edge is clamped and the opposite one is simply supported are also given in the same Appendix.

3.9 Treatment of Problems with a Pair of Opposite Edges Simply Supported

When a pair of opposite edges of the plate are simply supported, the exact solutions of the problem may be obtained. In such cases exact displacement functions exist for the simply supported edge conditions. Because of this the partial differential equations need not be reduced to ordinary differential equations. Even when reduced, the ordinary differential
equations are exact. The method of solution is as follows.

3.9(a) The Edges \( x = \text{constant} \) Simply Supported

The boundary conditions at \( x = 0 \) and at \( x = \bar{x} \) are

\[
\begin{align*}
v &= 0, \\
w &= 0, \\
u' + v v' + v w - k w'' &= 0, \quad (\text{from } N_x = 0), \\
w'' + v w'' - u' - v v' &= 0, \quad (\text{from } M_x = 0).
\end{align*}
\]

These conditions may be shown to reduce to

\[f_2 = f_3 = f_1' = f_3'' = 0.\] 3.65

These conditions and the governing partial differential equation 2.1(a-c) in terms of \( f_1 g_1, f_2 g_2 \) and \( f_3 g_3 \) are satisfied in the direction of \( x \) by the following functions

\[
\begin{align*}
f_1 &= \cos \frac{m \pi x}{\bar{x}}, \\
f_2 &= \sin \frac{m \pi x}{\bar{x}}, \\
f_3 &= \sin \frac{m \pi x}{\bar{x}}.
\end{align*}
\] 3.66(a-c)

Substitution of these functions directly into the governing equations 2.1(a-c) or to 2.7(a-c) leads to the ordinary equations 2.9(a-c) which are exact. Then the solutions of the ordinary equations may be obtained by using any of the methods discussed earlier. There is a quicker way
to obtain an approximate solution. The method is to use the reduced equations 2.10(a-c). It may be verified that the equations 2.10(a-c) and the boundary conditions 3.65 are satisfied by the following functions.

\[ f_1 = q_1 \cos \frac{m\pi x}{x} \]

\[ f_2 = q_2 \sin \frac{m\pi x}{x} \quad 3.67(a-c) \]

\[ f_3 = q_3 \sin \frac{m\pi x}{x} \]

where \( q_1, q_2 \) and \( q_3 \) are the amplitudes of vibrations to be determined.

Note that the equations 3.66(a-c) and 3.67(a-c) are the same except for the amplitudes. Substitution of 3.67(a-c) into 2.10(a-c) leads to the following frequency determinant.

\[
\begin{vmatrix}
 b_{11} + \Delta & b_{12} & b_{13} \\
 b_{21} & b_{22} + \Delta & b_{23} \\
 b_{31} & b_{32} & b_{33} - \Delta
\end{vmatrix} = 0.
\]

3.68

The lowest of the three roots of \( \Delta \) for each mode of vibration is the required \( \Delta_w \). The coefficients \( b_{ij} \) of equation 3.68 are given in Appendix 5. Once the eigenvalues are known, then the amplitudes \( q_1, q_2 \) and \( q_3 \) may easily be found.
3.9(b) The Edges $\phi = \text{constant}$ are Simply Supported

The simply supported boundary conditions at $\phi = 0$ may be written as follows

\[ u = 0, \]
\[ v = 0, \]
\[ v u' + v' + (1 + k) w + k w'' = 0, \quad \text{from} \quad N_\phi = 0 \]
\[ w + w'' + v w''' = 0. \]

These conditions reduce to (equation 3.14(a-d))

\[ g_1 = g_3 = g_2' = g_3'' = 0. \]

These conditions and the governing partial differential equations written in terms of $f_1 g_1$, $f_2 g_2$, and $f_3 g_3$ are satisfied in the direction of $\phi$ by the following functions.

\[ g_1 = \sin \frac{n \pi \phi}{6}, \]
\[ g_2 = \cos \frac{n \pi \phi}{6}, \]
\[ g_3 = \sin \frac{n \pi \phi}{6}. \]

3.71(a-c)

Substitution of the functions directly to the equations 2.1(a-c) or to 2.8(a-c) leads to the ordinary equations 2.10(a-c) which are exact. The ordinary differential equations may then be solved by using the general method of solution. The matrix methods may be used only when $f_1$ is eliminated from the system of equations which are otherwise of order nine. An alternative approximate method is to use the reduced equations 2.9(a-c).

The ordinary differential equations 2.9(a-c) and the boundary
conditions 3.70 are satisfied by the following functions

\[ g_1 = q_1 \sin \frac{n\phi}{\theta} \]
\[ g_2 = q_2 \cos \frac{n\phi}{\theta} \]
\[ g_3 = q_3 \sin \frac{n\phi}{\theta} \]

Substitution of these functions into equations 2.9(a-c) yields a frequency determinant of the form

\[
\begin{vmatrix}
 b_{11} + \Delta & b_{12} & b_{13} \\
 b_{21} & b_{22} + \Delta & b_{23} \\
 b_{31} & b_{32} & b_{33} - \Delta
\end{vmatrix} = 0.
\]

The constants \( b_{ij} \) being defined in Appendix 6. The lowest of the three roots of equation 3.73 is the eigenvalue of the primarily transverse mode of vibration.

3.9(c) Four Edges Simply Supported

The differential equations 2.1(a-c) and the boundary conditions 3.64(a-d) and 3.69(a-d) are satisfied by the following functions.
\[ u = A \cos \frac{m\pi x}{\lambda} \sin \frac{n\pi \phi}{\theta}, \]
\[ v = B \sin \frac{m\pi x}{\lambda} \cos \frac{n\pi \phi}{\theta}, \]
\[ w = C \sin \frac{m\pi x}{\lambda} \sin \frac{n\pi \phi}{\theta}. \]

Substitution of these functions into the set of differential equations 2.1(a-c) yields a frequency determinant of the form

\[
\begin{vmatrix}
  c_{11} + \Delta & c_{12} & c_{13} \\
  c_{12} & c_{22} + \Delta & c_{23} \\
  c_{13} & c_{23} & c_{33} + \Delta
\end{vmatrix} = 0. \quad 3.75
\]

the constants \( c_{ij} \) being defined in Appendix 6. Again the lowest of the three roots of equation 3.75 is the eigenvalue of the primarily transverse mode of vibration.

3.10 Discussion of the Results

The computation of the eigenvalues and the eigenfunctions does not present many problems when the frequency determinants are reducible to polynomial equations. In such cases, the eigenvalues and the eigenfunctions may be computed to very high degree of accuracy. On the other hand, when the frequency determinants are transcendental in nature, it is not always possible to obtain a very high degree of accuracy for the
solutions. This is mainly due to the loss of influence of the
circular sine and cosine functions on the solutions when the hyperbolic
sine and cosine functions have large relative magnitudes. Also, it is
very seldom possible to locate the zeros of the frequency determinant
exactly. Therefore, in order to establish the reliability of the
solutions, the following plate is selected for analysis.

\[ L = 17 \text{ in.} \]
\[ b = 17 \text{ in.} \]
\[ h = 0.02 \text{ in.} \]
\[ R = 96 \text{ in.} \]
\[ \nu = 0.33. \]

All edges of the plate are assumed to be clamped. The reduced
equations 2.9(a-c) are used for the analysis. The displacement functions
3.62(a-c) are used in the reduction. The general method of solutions
both with complex and real numbers are used in the analysis. Out of the
three matrix methods discussed in section 3.7 only the modified matrix
progression method could be applied successfully. The eigenvalues and
the eigenvectors are computed for \( m = 1 \) and \( 0 \leq \Delta \leq 1.0 \). The results
obtained are presented in Table 1. Before discussion of these results
it is pertinent to discuss the causes of failure of the other two matrix
methods to predict the eigenvalues and the eigenvectors.

The single step integration method outlined in section 3.7(a)
and the final frequency determinant of equation 3.42 are deceptively simple.
Although the procedure is mathematically correct, numerically it seems to be beset with unsurmountable problems. It was found impossible to obtain any solutions with this method. One of the problems is that for large values of $\phi = \theta$ some of the off diagonal elements of $[A]$ are very large compared to the others. This results in complete loss of accuracy. This is fully discussed by Kalnins [43 and 44]. Values of $D(\Delta)$ were always found to increase monotonically with the increase of the value of $\Delta$ showing no sign of existence of the eigenvalues, though within these values of $\Delta$ the general method of solutions produced eigenvalues.

Since the single step integration technique has failed, the matrix progression technique outlined in section 3.7(b) is used. Again no values of $\Delta$ could be found to satisfy equation 3.47. In this case, due to the subdivisions of the total integration length, the ill effects of the large off-diagonal elements are removed. Yet, no solutions could be obtained with the method. The reason for this is explained as follows.

Initial value differential equations always admit solutions. But two point homogeneous boundary value problems may admit no solutions other than the trivial ones [25, pp. 322-332]. But in the present case certainly there should be some solutions. There is another source of ill-conditioning in two point boundary value problems. It frequently so happens that the influence of the boundary conditions at one end becomes insignificant at the other end. This produces numerical sensitivity and often leads to serious ill-conditioning. Because of this, in the case of homogeneous problems, there will be no solutions and in the case of
inhomogeneous problems, the boundary conditions will not be adequately satisfied. This seems to have happened in this case also. To remove this type of ill-conditioning the method of solution suggested by Kalnins [43 and 44] may be used. Instead we have preferred to use the modified matrix progression technique as outlined in section 3.7(c).

Due to the subdivision of the total integration interval, the inaccuracy due to very large off-diagonal elements of matrix \([A]\) is removed. Since at each step the boundary conditions are transferred from the beginning to the end of the step, the effects of the boundary conditions are retained and the second cause of ill-conditioning is removed. Thus, only the modified matrix progression technique is found to be suitable.

Now we shall turn our attention to the results of Table 1.

At first sight, it appears that the modified matrix method has failed to detect two eigenvalues. But an examination of the eigenfunctions of these two eigenvalues shows that they do not satisfy the shape compatibility conditions discussed in section 3.8. This implies that they do not satisfy the original partial differential equations though they satisfy the given boundary conditions. Therefore, these two may be called pseudo eigenvalues. In both these cases \(n\) of \(v\) violates the conditions of shape compatibility and hence \(v\) is not compatible with the functions \(u\) and \(w\).

This analysis is confined to modes with nodal lines parallel to the edges. Though the plate under consideration is square in plane, its curvature is large enough to produce sufficient asymmetry which should prevent the degenerate modes of vibrations (Webster [32]).
The values of $D(\Delta)$ vs $\Delta$ are plotted in Figure 5 and Figure 6 for the general method of solution with real coefficients and the modified matrix progression method respectively. Figure 5 does not show any form of discontinuity in the function, and therefore cannot provide any explanation for the existence of the pseudo eigenvalues. Since the prescribed functions are approximate, the resulting ordinary differential equations are approximate. Therefore, all the solutions of these approximate ordinary differential equations may not be expected to be the true solutions of the original partial differential equations of motion. Moreover, the general method of solution, being an exact method of solving the set of ordinary differential equations, finds all the solutions including the pseudo ones. On the other hand, the matrix progression method, being an approximate one, somehow fails to find the pseudo solutions. No rational explanation for this could be advanced.

Thus, apart from these two pseudo eigenvalues, the three methods produce results which are in very close agreement. The complex formulation of the general solution is the easier to formulate, but the real formulation takes less computing time.

Further, it may be noticed that there are two eigenvalues with $w$ having the same $m$ and $n$. The possibility that this can happen is indicated in section 3.8. Additional evidence of this is supplied in Table 2 where the results of an "exact" solution of two plates are given. The curved edges of the plates are assumed to be simply supported and the straight edges clamped.
It may be emphasised that though \( v \) has the same \( m, n \) in each case, at least one of the values of \( m \) or \( n \) of \( u \) or \( v \) is different. Thus, they are distinctly different modes. The position of the nodal lines is different in each case.

A clearer picture of the phenomenon may be obtained by considering a pair of opposite edges of the plate which are simply supported. For definiteness consider the edges \( \phi = \) constant. From equation 3.71(b) and 3.71(c) \( g_2 \) and \( g_3 \) are seen to be cosine and sine functions respectively. The first four modes are depicted in Figure 7. Now consider edge conditions such that \( g_3 \) remains unchanged but \( g_2 \) is required to be zero at the edges. It may be visualized that in such a case \( g_2 \) may have at least two alternative mode shapes which are also depicted in Figure 7. In a particular problem \( g_2 \) may assume either of the two shapes in a particular mode of \( g_3 \) or may assume both with distinct frequencies. It is not possible to ascertain a priori which is the likeliest shape that \( g_2 \) will assume. Similarly, it cannot be ascertained without computing if both of them exist. If they exist it is not known in advance which one will yield the lower frequency. Also it appears that there is little possibility of determining in advance whether this can happen more than twice or also in every mode and in every plate. It is known that some of the lower modes may not appear in curved plates. However, there seems to be no reason to suspect that these "double modes" replace some of the lower modes that seem to have disappeared. The only certainty is that this phenomenon cannot happen in simply supported plates. For the rest, it is again uncertain. Similar arguments hold also in the case of \( f_3 \) and \( f_1 \).
The problem may also be viewed from the angle of Rayleigh-Ritz type of analysis where the mode shapes are uniquely prescribed (see Sewall [14]). These prescribed mode shapes only need to be admissible and also need to be compatible with one another. For a given pair of m and n of w theoretically there may be infinite numbers of variations of m and n of u and v which are compatible with w. Prescription of each set is legitimate and will yield a discrete frequency. Most of them may be of little practical importance but at the same time the possibility of their existence is acknowledged.

This phenomenon seems to have attracted very little attention from the analysts. Therefore, to decide whether a particular mode is a true one or not the following criteria may be established.

First the computed mode shapes have to satisfy the given edge conditions. In the case of more than one mode with w having the same (m,n) values, at least one of the half waves of u and v must be different between the modes which have otherwise identical mode shapes. Also they have to satisfy the following conditions strictly when the opposite edges have similar conditions, or in sense if they are not.

<table>
<thead>
<tr>
<th>$f_3$</th>
<th>$f_1$</th>
<th>$f_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>symmetrical</td>
<td>antisymmetrical</td>
<td>symmetrical</td>
</tr>
<tr>
<td>antisymmetrical</td>
<td>symmetrical</td>
<td>antisymmetrical</td>
</tr>
<tr>
<td>$\varepsilon_3$</td>
<td>$\varepsilon_2$</td>
<td>$\varepsilon_1$</td>
</tr>
<tr>
<td>----------------</td>
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<td>---------------</td>
</tr>
<tr>
<td>symmetrical</td>
<td>antisymmetrical</td>
<td>symmetrical</td>
</tr>
<tr>
<td>antisymmetrical</td>
<td>symmetrical</td>
<td>antisymmetrical</td>
</tr>
</tbody>
</table>

Obviously these conditions are not enforceable in the case of degenerate modes. Thus, for any given mode of $w$, $u$ and $v$ may have more than one admissible mode. Each change will produce a new mode and a discrete frequency. This clearly shows that unlike a flat plate, the modes of the curved plates may precisely be defined only when the mode shapes of $u$ and $v$ together with that of $w$ are also defined. If only the mode of $w$ is quoted, it may not be only imprecise but sometimes may also be confusing. Therefore, when more than one mode with the same $m$ and $n$ of $w$ appear, then it is necessary to quote the $m$ and $n$ of both $u$ and $v$ also.

Further comparisons between the real formulation of the general solution and the modified matrix method are shown in Table 3 for various radii and thicknesses. In this table all pseudo eigenvalues produced by the general solution are omitted. In all cases the two methods show close agreement. Henceforth, only the modified matrix method will be used since it does not produce pseudo eigenvalues.

In Table 4 a comparison is made between the present solution and two Rayleigh-Ritz solutions. The first is a one term beam function approximation whilst the second is a multi-term polynomial solution due to Webster [32]. In the latter case only the lowest frequency is available.
In this particular case there is very close agreement between the Webster and Kantorovich solutions, but the one term beam function approximation gives a much higher estimate for the frequency. Also included in the table are the frequencies when all the edges are simply supported and some experimental results by Rucker [15]. In all cases the experimental frequencies are bracketed by the one term beam function solution for clamped edges and the solution for simply supported edges. The Kantorovich method produces frequencies of similar order to the experimental values. In most cases the discrepancy is less than 10%. Only in the case of the lowest frequency is this exceeded. However, the experimental boundary conditions were not truly representative of a clamped boundary, resulting in the lowest frequency lying closer to the simply supported value than the clamped one.

Further comparisons between the theoretical and the experimental frequencies are presented in Table 5 for different radii and thicknesses. Similar trends to Table 4 are indicated.

However, it may be seen from Table 5 (R=48 in, h=0.02 in) that an experimental frequency is lower than the corresponding simply supported plate frequency. Another similar low frequency may be found in Sewall's Table IV for h=0.04 in. These seem to be under-estimates of the true frequencies. This is usually attributed to the inadequacy of the reproduction of clamped boundaries in experimental set up. But this type of inadequacy is expected to produce frequencies which may be at the most as low as the simply supported frequencies.
Since these experimental frequencies are lower than the corresponding simply supported frequencies the inadequacy of the boundary conditions alone is not responsible. Unless the supports are massive and infinitely rigid compared to the plate, at low frequency excitations, the plate together with its supports will respond. The response of the support system may be very small. But it may be sufficient to lower the apparent natural frequencies of the plate considerably. In such situations they may even be lower than the corresponding simply supported frequencies. Also, the effective dimensions of the test plate may be greater than they are actually assumed to be in the theoretical investigations.

Sewall has also reported experimental frequencies higher than the clamped plate frequencies. The one term Rayleigh-Ritz analysis used by Sewall is generally recognised to give higher estimate of the frequencies than the true ones. Moreover, Sewall has neglected the effects of the inplane inertias on $\Delta w$. It is generally assumed that this is a valid assumption for shallow plates. The author has analysed one square plate and one rectangular clamped plate. Both the plates are analysed, once retaining the effects of the inplane inertias and once neglecting them. Flügge's exact set of strain-displacement relations are used. The beam function approximations with Rayleigh-Ritz analysis were used in the same way as Sewall. For the square plate there were little differences in the results. But for the rectangular one it was found that when $m=n$ the frequencies were not much affected by the omission of the inplane inertias, though they tended to be slightly larger. But when $m \neq n$ the differences in some frequencies were as high as 22%. The frequencies with omission of the inplane
inertias for the higher modes were found to be lower (between 10 to 22%) than those without omitting them. This seems to agree with the difference observed by Sewall between the experimental results and the Rayleigh-Ritz solutions in case of the clamped rectangular plates for the higher frequencies. This example shows that the indiscriminate omission of the effects of the inplane inertias even in the case of shallow plates may produce misleading information.

Discrete frequency response tests carried out on two plates \(L = 17\text{ in}, \ b = 17\text{ in}, \ h = 0.02\text{ in} \) with \(R = 96\text{ in} \) and \(48\text{ in}\) by Rucker [15] show a fairly large number of peak responses. However, only a few of the modes of the frequencies of these peak responses could be identified. Since the experimental determination of the mode shapes depends upon the modes of \(w\) alone, it is not possible to determine all the different modes that may possibly be generated by the combinations of different modes of \(u\) and \(v\). Because of such combinations there may be large numbers of close natural frequencies. Though these frequencies may be determined reasonably well their mode shapes cannot be determined correctly since the influence of the other modes cannot be removed to determine a particular one. Under such conditions comparison of experimental and theoretical frequencies may lead to comparison of frequencies which are not necessarily frequencies of the same mode.

A comparison between theoretical and experimental lowest frequencies for rectangular plates \(11 \times 9\text{ in}\) of varying radii and thickness is presented in Table 6. In all cases the one term
Rayleigh-Ritz solution predicts higher values than the Kantorovich solution for clamped edges. In two of these cases the Rayleigh-Ritz solution does not predict the correct mode. In most cases the experimental frequencies are bounded by the Kantorovich clamped values and the ones for straight edges simply supported and curved edges clamped.

An examination of the theory presented in Chapter II reveals that the non-dimensional frequency parameter \( \Delta \) is a function of the parameters \( L/R, h/R, b/R, m, n, \nu \) which are also non-dimensional, that is

\[
\Delta = F\left(\frac{L}{R}, \frac{h}{R}, \frac{b}{R}, m, n, \nu\right) .
\]  

3.76

Since

\[
\frac{L}{R} = \frac{L}{b} \cdot \frac{b}{R} , \quad \frac{h}{R} = \frac{h}{b} \cdot \frac{b}{R} ,
\]

then the relationship 3.76 may be rewritten in the form

\[
\Delta = G\left(\frac{L}{b}, \frac{h}{b}, \frac{b}{R}, m, n, \nu\right) .
\]  

3.77

Reference 47 quotes the following ranges of values as being of practical interest

\[
0 \leq \frac{b}{R} \leq \pi
\]

\[
20 \leq \frac{L}{h} \leq 1000
\]  

3.78

\[
0.5 \leq \frac{b}{L} \leq 2.0
\]
Converting to the non-dimensional parameters used in 3.77 gives

\[ 0.5 \leq \frac{L}{b} \leq 2.0 \]

\[ 0.0005 \leq \frac{h}{b} \leq 0.1 \]

\[ 0 \leq \frac{b}{R} \leq \pi \]

Figure 8 shows how the lowest value of the frequency parameter for a square plate varies with the central angle. For clamped boundaries there is very close agreement between the Webster and Kantorovich solutions. However, the Rayleigh-Ritz solution using beam functions is only in agreement with these results for very small values of the central angle. The reason for the discrepancy is that a straight beam function is not a good approximation to the true mode shape in the curved direction when the central angle is large.

Figure 9 shows how the lowest frequency parameter of a shallow plate varies with aspect ratio L/b for various boundary conditions. These results have been obtained using the Kantorovich method.

Figure 10 shows how the lowest \( \Delta \) varies with variation of h/b of a clamped shallow square plate.

3.11 Conclusions

The method of Kantorovich has been used to reduce the partial differential equations of motion of a singly curved rectangular plate to a set of ordinary differential equations. The latter equations were
then solved by two different methods. The first solution, which
utilises the general solution of the equations, was found to produce
pseudo-eigenvalues. Moreover, it was observed that the method could
be used successfully only for a limited range of the aspect ratio.
Particularly for very short lengths, the solutions tend to become
very unstable and in such cases practically any value of $\Delta$ will give
$D(\Delta) = 0$. Similar trends are also observed for higher values of
curvature, thickness, $m$ and $\Delta$. However, the method is found to be quite
stable for $b/L > 0.75$, $0 < h/R < 0.001$, $1 < m < 2$ and $0 < \Delta < 1.0$. It is
found that the method cannot cope with values of $\Delta$ greater than 1, what-
ever the other parameters may be.

On the other hand, the second method which is the modified
matrix progression method, did not produce any pseudo-eigenvalues and
was found to work satisfactorily for all practical values of the
different parameters. Therefore this method is preferable.

It has been shown that the method agrees satisfactorily with
experiment if allowance is made for the inadequacy of the experiments.

Investigations have shown that a one term Rayleigh-Ritz
solution using beam functions gives accurate results only for very
shallow plates. However, a many term Rayleigh-Ritz solution using
polynomials can give accurate results for non shallow plates also,
provided a sufficient number of terms are taken. Increased accuracy
in the Rayleigh-Ritz method is obtained at the expense of computer
storage and time. In the Kantorovich method the required accuracy
is obtained at the expense of time only, which is minimal.
CHAPTER IV

ANALYSIS OF THE PRIMARILY TRANSVERSE VIBRATIONS OF CYLINDRICALLY CURVED RECTANGULAR PLATES IN THE PRESENCE OF BI-AXIAL TENSILE MEMBRANE STRESSES

4.1 Introduction

Stringer stiffened panels are used in many structures in the aerospace engineering, like aircraft wings and fuselages. The sheet of metal forming the skin panel is loaded by normal pressure in addition to end loads caused by the bending of the wings and the fuselage. The air pressure acting on the surface of the fuselage, wing and tailplane and the pressure load due to the pressurisation of the cockpits and cabins are the major sources of the normal pressure. Thus, the skin plates experience various types of loading inducing both compressive and tensile stresses.

Initially in this chapter we shall consider a plate as shown in Figure 1 with general values of the inplane stresses. Subsequently, a special case where the hoop stress and the axial stress are given by the corresponding pressures of a cylinder under internal pressure \( p \text{ } \) will be considered. In this chapter the stresses are assumed to be tensile.

The knowledge of vibration characteristics of such curved panels in the presence of bi-axial tensile stresses is of considerable practical interest to the aerospace engineers. A list of references of work on flat plates may be found in Weeks and Shideler [94] who have studied the effect of edge loadings on the vibration of rectangular
plates with various boundary conditions. Vlasov [3, p.537] has given the expressions for frequencies of free vibrations of a general curved plate in the presence of membrane stresses when all the edges of the plate are simply supported. Other investigations seem to be mainly concerned about buckling of complete cylinders or buckling of curved panels with a pair of opposite edges simply supported. No attempt at solving the free vibration problem in the presence of the membrane stresses of clamped curved panels is noticed. In this chapter we shall primarily be concerned with clamped and simply supported curved panels.

Though the following study is restricted to the presence of bi-axial membrane stresses only, it may easily be adapted to the following stress conditions also.

(i) Both the membrane stresses are compressive

(ii) One of the membrane stresses is tensile and the other is compressive

(iii) The membrane stress (tensile or compressive) is applied in one direction only.

The method cannot be applied without drastic modifications if inplane shear stresses are present. Therefore, the application of inplane shear stresses is not considered. Also, the membrane stresses are assumed to be uniformly applied.

4.2 Differential Equations of Motion

The equations of motion in the presence of inplane stresses may be obtained from the buckling equations for cylindrical shells given by
Flügge [6, p.422]. These equations contain all the terms in the equations 2.1(a-c) and additional terms reflecting the effect of the inplane stresses. These equations are

\[(1 + q_x)u'' + \left(\frac{1 - \nu}{2} (1 + k) + q_\phi\right)u'' + \frac{1 + \nu}{2} v'' + (\nu - q_\phi)w' \]
\[-k w''' + \frac{1 - \nu}{2} k w'''' - \frac{\rho R^2 h}{E D} \frac{\partial^2 u}{\partial t^2} = 0 ,\]
\[\frac{1 + \nu}{2} u'' + (1 + q_\phi)v'' + \left(\frac{1 - \nu}{2} (1 + 3k) + q_x\right) v'' + (1 + q_\phi)w' \]
\[\left(\frac{3 - \nu}{2}\right) k w'''' - \frac{\rho R^2 h}{E D} \frac{\partial^2 v}{\partial t^2} = 0 ,\]
\[(\nu - q_\phi)u' + \frac{1 - \nu}{2} k u'''' - ku'''' + (1 + q_\phi)v' - \frac{3 - \nu}{2} k v'''\]
\[+ (1 + k)w + k w''' + 2kw'''' + k w'''\]
\[+ (2k - q_\phi)w'''' - q_x w'' + \frac{\rho R^2 h^2}{E D} \frac{\partial^2 w}{\partial t^2} = 0 .\]

In these equations \(q_x\) and \(q_\phi\) are obtained from the applied axial tensile forces \(p_x\) per unit of circumference and \(p_r\) the applied internal normal pressure respectively.

\[q_x = \frac{p_x}{D} \quad \text{and} \quad q_\phi = \frac{p_r R}{D} \]

From equation 4.2(a-b) it may be seen that the axial and the hoop stresses may be applied arbitrarily. For a cylinder pressurized with a pressure \(p_r\), \(q_x\) is half of \(q_\phi\). However, it is convenient to express both \(q_x\) and \(q_\phi\) in terms of \(p_r\) such that \(q_x = c q_\phi\) where \(c\) is an arbitrary constant and \(q_\phi\) is given by equation 4.2(b).
4.3 Boundary Conditions

The panels forming the fuselage are elastically supported by the frames and the stringers. The elastic boundary conditions are intermediate between the clamped and the simply supported conditions. Therefore, instead of elastic support the panels will be considered to be fully clamped and fully simply supported.

Equations 4.1(a-c) are in exactly the same form as equations 2.1(a-c). This suggests that the same methods can be applied to obtain their solutions. The time part of the solutions for \( u, v \) and \( w \) are assumed to be \( e^{i\omega t} \).

4.4 All Edges Clamped

To obtain the solutions for the clamped plate, the Kantorovich method discussed in Chapter II may be employed. Further description of the method is unnecessary. Kantorovich's first approximation is used. It is desirable to reduce the equations parallel to the straight edges. It has been discussed in section 3.8 that the assumed displacement variations should be as close to the actual variations as possible. An approximate idea for the variation of \( w \) along \( x \) may be obtained from equation 4.1(c). It suggests that the approximate variation of \( w \) along \( x \) is similar to the variation of the normal displacement of a uniform beam in tension.

The equation of transverse vibrations \( w \) of such a beam may be written as

\[
\frac{d^4w}{dx^4} - \frac{P_b}{EI} \frac{d^2w}{dx^2} - \frac{Mw}{EI} = 0 , \quad 4.3(a)
\]
where $P_b$ is the applied axial tension and $M$ is mass per unit length. 
Non-dimensionalising equation 4.3(a) with the transformation ($\eta$ is non-dimensional co-ordinate)

$$x = \eta L ,$$

4.3(b)

the following equation is obtained.

$$\frac{d^4 \bar{w}}{d\eta^4} - \frac{P_b L^2}{EI} \frac{d^2 \bar{w}}{d\eta^2} - \frac{M L}{EI} \frac{d \bar{w}}{d\eta} = 0 .$$

4.3(c)

When the edges of the beam are simply supported, the eigenfunctions $\bar{w}$ are sine functions which do not vary with $P_b L^2/(EI)$. Weeks and Shideler [94] have shown that even for a clamped beam the eigenfunctions of 4.3(c) do not vary with $P_b L^2/(EI)$ provided it is positive (tensile).

Hence, for simplicity this parameter will be taken as unity giving

$$\frac{d^4 \bar{w}}{d\eta^4} - \frac{d^2 \bar{w}}{d\eta^2} - \Delta_b \bar{w} = 0 ,$$

4.4(a)

where $\Delta_b$ is eigenvalue of equation 4.4(d). Subsequently, only equation 4.4(d) will be considered. For any given edge conditions at $\eta = 0$ and at $\eta = 1$ the eigenfunction $\bar{w}$ will represent $f_3$. The variation of $f_1$ will be the same as the variation of $f'_3$ and the variation of $f_2$ will be the same as that of $f_3$. These functions are not tabulated, as is the case without axial loads. So for the case of a fully clamped beam they are derived in Appendix 7.

Applying the method of Kantorovich to equations 4.1(a-c) gives the set of ordinary equations (A7.5(a-c)) which are given in Appendix 7.
It will be seen that these equations are the same as equations Al.1(a-c) with modified coefficients. These coefficients are given in Appendix 7.

Since the reduced equations A7.5(a-c) are exactly the same in form as equations Al.1(a-c), theoretically all the methods that were employed to solve equations Al.1(a-c) should work for A7.5(a-c) also. But the general method of solution was found to be unsuitable for the following reasons.

At least there is always a pair of real roots of equal magnitude but opposite in sign of the characteristic equation. The modulus of them is so large that exponential with respect to the positive root tends to infinity whereas with respect to the negative one tends to zero. Due to this, some coefficients of the frequency determinant become excessively large compared to the others. This results in complete loss of accuracy of the solutions.

On the other hand, the modified matrix progression method described in Chapter III is found to be stable. Therefore, this is the only method that could be used in this case. The derivation of the first order system and the clamped boundary conditions are exactly the same as that for equations Al.1(a-c), which is shown in Appendix 3. The modified matrix progression method is given in section 3.7(c).

4.5 Treatment of Problems with a Pair of Opposite Edges Simply Supported

When the edges \( n=0 \) and \( n=1 \) are simply supported the eigenfunction of equation 4.4(d) is

\[
\bar{w} = \bar{w}_m \sin m\pi \eta \\
= \bar{w}_m \sin \frac{m\pi x}{L},
\]

4.5
where $\tilde{w}_m$ is the amplitude of vibration. Similarly, it may be shown that when a pair of opposite edges are simply supported, the problem may be solved exactly the same way as the corresponding problem without the in-plane stresses applied. The exact solutions of the problems may be obtained by finding the zeros of the transcendental frequency equation depending on the boundary conditions at the remaining edges. If the other pair of edges are also simply supported the exact solutions of the problem may be found easily. Thus the differential equations 4.1(a-c) and the boundary conditions 3.64(a-d) and 3.69(a-d) are satisfied by the functions given in equation 3.74(a-c). Substitution of these functions into the set of differential equations 4.1(a-c) leads to the frequency determinant of the form given in 3.75. The coefficients $C_{ij}$ are defined in Appendix 8. The lowest of the three roots of the frequency determinant is the eigenvalue of the primarily transverse mode of vibration.

If the straight edges of the plate are simply supported and the curved edges are clamped, the exact solutions of the problem may be obtained. An alternative approximate method is to use the reduced equations A7.5(a-c). The boundary conditions 3.69(a-d) and the reduced equations A7.5(a-c) are satisfied by the functions given in equation 3.72(a-c). Substitution of these functions into the reduced equations yields a frequency determinant of the form given in equation 3.73. The coefficient $b_{ij}$ of this determinantal equation is given in Appendix 8. The lowest of the three roots of equation 3.73 is the eigenvalue of the primarily transverse mode of vibration.
4.6 Discussion of the Results

Initially a trial analysis was performed on a clamped plate using the general solution technique. It was found that some of the roots of the eighth order polynomial were so large that no reliable solutions for eigenvalues and eigenvectors could be expected. Therefore, the modified matrix progression method was used and found to work satisfactorily. This confirms the findings in Chapter III that the modified matrix method is preferred.

When the membrane stresses are present, then the functional relationship 3.77 becomes

\[ \Delta = G \left( \frac{L}{b}, \frac{h}{b}, \frac{b}{R}, q_x, q_\phi, m, n, \nu \right) \] \hspace{1cm} 4.6

In this section the effect of varying the stress parameters \( q_x \) and \( q_\phi \) only will be considered, since variations in aspect ratio and curvature were studied in Chapter III.

The relative effect of \( q_x \) and \( q_\phi \) on \( \Delta_w \) was obtained by considering two plates with simply supported edges, one square and one rectangular. Their dimensions are as follows

(i) \( L = 17 \text{ in}, b = 17 \text{ in}, h = 0.02 \text{ in}, R = 96 \text{ in} \) and \( \nu = 0.33 \).

(ii) \( L = 11 \text{ in}, b = 9 \text{ in}, h = 0.028 \text{ in}, R = 96 \text{ in} \) and \( \nu = 0.33 \).

The results are valid for any other plates which have the same parameters in the functional relationship 4.6 as these two plates. For each plate the following cases were considered.
(i) \( q_x = q_\phi = 0 \)

(ii) \( q_x = 0, q_\phi = 10^{-5} \)

(iii) \( q_x = 5 \times 10^{-5}, q_\phi = 0 \)

(iv) \( q_x = 0, q_\phi = 5 \times 10^{-5} \)

(v) \( q_x = q_\phi = 5 \times 10^{-5} \).

The results for all these cases are presented in Figures 11 and 12 for \( m=1 \). It can be seen from these results that the variation of \( \Delta_w \) with \( q_x \) is very small. It may be noticed that the difference between the eigenvalues for case (iv) and (v) of the stresses is very small. Henceforth, \( \frac{1}{2} q_\phi \) will be taken as a representative value for \( q_x \). This coincides with the longitudinal stress in a pressurised cylinder.

From Figure 11 it may be seen that the variation of \( \Delta_w \) is very small with the variation of \( q_x \) and \( q_\phi \) for the \( 1,1 \) mode. With the rise of the internal pressure, the eigenvalues increase. The lowest frequency mode gradually changes from \( m,n = 1,3 \) towards \( m,n = 1,1 \). For the rectangular plate the lowest frequency mode is at \( 1,1 \) without pressurisation. Therefore, in this case the lowest frequency mode will always appear at \( m,n = 1,1 \) and will not change with the change of \( p_r \).

Figures 13 and 14 show the lowest eigenvalues of the square and the rectangular plate respectively for different values of \( m, q_x \) and \( q_\phi \). In all the cases the lowest eigenvalue is found to appear at \( m=1 \). Therefore in subsequent analysis usually only \( m=1 \) will be considered.

The variation of the lowest \( \Delta_w \) with internal pressure for three different plates is given in Figures 15, 16 and 17. In each case the boundary conditions considered are
(a) all edges simply supported
(b) straight edges simply supported and curved edges clamped
(c) all edges clamped.

Figures 15 and 16 are for a square plate with different curvatures and Figure 17 is for a rectangular plate.

In the initial stages of pressurisation the lowest frequency occurs in one of the higher order modes. As the pressure is increased the frequency increases and the mode number \( n \) decreases until it becomes unity. The variation of the frequency parameter with pressure is linear over the ranges where the mode number is constant.

The reliability of the clamped plate solutions may be assessed from the facts that the computed eigenfunctions satisfy the given boundary conditions and all the conditions of shape compatibility discussed in Chapter III. Also for very small values of \( q_x \) and \( q_\phi \) the solutions tend to the clamped plate solutions without the membrane stresses. For the clamped plate the variation of \( D(\Delta) \) with \( \Delta \) appears to be nearly a tangent function.

4.7 Conclusions

The Kantorovich matrix progression method was used to investigate the effect of applied tensile membrane stresses on the natural frequencies. It has been shown that the variation of the lowest frequency parameter with cylindrical pressure is a series of straight lines. The method may readily be adopted for the study of the behaviour of the plate under compressive membrane stresses also.
CHAPTER V
DOUBLY CURVED PLATES
- KANTOROVICH REDUCTION WITH MODIFIED MATRIX PROGRESSION -

5.1 Introduction

The analysis of the dynamic behaviour of doubly curved plates is of considerable practical interest. The shapes of structural components of aircraft, like the tailplane, which are designed from aerodynamic considerations, are in general not of simple form. Usually they are fabricated from small doubly curved plate elements. Plates of doubly curved shape are used in numerous civil engineering designs also. The resistance of a plate to the external loads and their distribution to the boundaries depend largely on the strength of the material of the plate in conjunction with its shape. Doubly curved plates use their special shape much more than flat or singly curved plates in resisting the imposed loads. Therefore, they are usually much thinner than flat or singly curved plates. This reduces material weight which may be an important consideration in design of aircraft.

Though doubly curved plates are of considerable practical use, their static and dynamic behaviour are not yet adequately investigated. Probably this is due to the complex nature of the differential equations governing this behaviour. The differential equations of a general doubly curved plate (the thickness and the curvatures may vary from point to point) may be found in Vlasov [3, pp.239-334]. These equations are quite complex

-90-
in their general form. Therefore, another set of equations also due to Vlasov [5] shall be used here for analysis.

This is a restricted set of equations which is applicable only to shallow plates. The assumptions involved in deriving the set of equations are mentioned in the next section. Also, an exactly similar set of equations is given by Marguerre which may be found in Reissner [21]. The aim of this Chapter is to deal with problems of free vibrations of uniform doubly curved shallow plates of constant curvatures using the method of Kantorovich [1].

5.2 Equations of Motion of Doubly Curved Shallow Plates

The equations of motion of a slightly curved plate in Cartesian co-ordinates are given by Vlasov [5]. The system is shown in Figure 18. The ratio between the maximum rise and the shortest length of the plate is small (less than 1/5), conforming with the assumption of shallow shells. It is implied that the geometry of the plate is not much different from the geometry of a flat plate, and the total, or "Gaussian", curvature is zero. Under these conditions the equations of Vlasov [5] for slightly curved arbitrary surfaces are as follows.

The strain-displacement relations:

\[ \varepsilon_x = u' - k_x w, \]
\[ \varepsilon_y = v' - k_y w, \]
\[ \varepsilon_{xy} = u' + v' - 2k_{xy} w, \]
\[ \chi_x = w'', \]
\[ \chi_y = w''', \]
the twist
\[ \chi_{xy} = - w''''. \]
The stress-strain relations are given by

\[ N_x = D(\varepsilon_x + \nu \varepsilon_y), \]
\[ N_y = D(\nu \varepsilon_x + \varepsilon_y), \]
\[ N_{xy} = N_{yx} = D(1 - \nu) \varepsilon_{xy}. \]

The moment-curvature relations are given by

\[ M_x = -K(x_x + \nu x_y), \]
\[ M_y = -K(x_y + \nu x_x), \]
\[ M_{xy} = -K(1 - \nu) x_{xy}. \]

The equations of motion are

\[ N'_x + N'_{xy} = \frac{\rho h}{g} \frac{\partial^2 u}{\partial t^2}, \]
\[ N'_y + N'_{xy} = \frac{\rho h}{g} \frac{\partial^2 v}{\partial t^2}, \]
\[ Q'_x + Q'_y + k_{xx} N_x + k_{yy} N_y + 2k_{xy} N_{xy} = \frac{\rho h}{g} \frac{\partial^2 w}{\partial t^2}, \]

where
\[ Q_x = M'_x - M'_{xy}, \]
\[ Q_y = M'_y - M'_{xy}. \]

From equation 5.4 in conjunction with equations 5.1, 5.2 and 5.3, the following equations of motion are obtained. It is assumed that the motions are simple harmonic in nature.
\[
\begin{align*}
&\left[ u'' + \frac{1 - \nu}{2} u' + (\frac{h'}{h}) u' + \frac{1 - \nu(h')}{2} u' + \Omega u'' \right] + \\
&\left[ \frac{1 + \nu}{2} v'' + \frac{1 - \nu(h')}{2} v' + \nu(h') v' \right] + \\
&\left[ -(k_x + \nu k_y) w' - (1 - \nu) k_{xy} w' \right] + \\
&\left[ (k_x' + \nu k_y') + (\frac{h'}{h})(k_x + \nu k_y) + (1 - \nu)(\frac{h'}{h} k_{xy} + k_{xy}^2) \right] v = 0 ,
\end{align*}
\]

\[
\begin{align*}
&\left[ \frac{1 + \nu}{2} u'' + \nu(h') u' + \frac{1 - \nu(h')}{2} u' \right] + \\
&\left[ \frac{1 - \nu}{2} v'' + v' + (\frac{h'}{h}) v' + \frac{1 - \nu}{2} (\frac{h'}{h}) v' + \Omega v' \right] - \\
&\left[ (\nu k_x + k_y) w' + (1 - \nu) k_{xy} w' \right] + \\
&\left[ (\nu k_x' + k_y') + (\frac{h'}{h})(\nu k_x + k_y) + (1 - \nu)(k_{xy}^2 + \frac{h'}{h} k_{xy}) \right] w = 0 ,
\end{align*}
\]

\[
\begin{align*}
&\left[ (k_x + \nu k_y) u' + (1 - \nu) k_{xy} u' \right] + \left[ (\nu k_x + k_y) v' + (1 - \nu) k_{xy} v' \right] - \\
&\left[ (k_x^2 + 2\nu k_x k_y + k_y^2 + 2(1 - \nu) k_{xy}^2) v - \Omega v \right] \\
&+ \frac{h'}{12} \left( w'''' + 2w'' + w' \right) + \frac{1}{12h} \left( 2(h^3)'w''' + 2(h^3)'w'' + 2(h^3)'w' \right) \\
&+ 2(h^3)'w'' \right) + \frac{1}{12h} \left( (h^3)'' + \nu(h^3)'' \right) w'' + \\
&\frac{1}{12h} \left( (h^3)'' + \nu(h^3)'' \right) w' \right] = 0 .
\end{align*}
\]

5.5(a-c)

In these equations

\[ \Omega = \rho w^2(1 - \nu^2)/E_0 . \]
5.3 The Equations of the Middle Surface and of its Curvatures

The equation of the middle surface is often given in Monge's form as
\[ z = f(x,y) \]  \hspace{1cm} 5.6

If \( x \) and \( y \) are taken as parameters of the surface then the curvatures of the surface may be expressed accurately in terms of the partial derivatives of \( z \) with respect to \( x \) and \( y \). Since \( x \) and \( y \) are assumed as the Cartesian co-ordinates, the curvatures may be approximated by the following:

\[
\begin{align*}
    k_x &= \frac{\partial^2 z}{\partial x^2}, \\
    k_y &= \frac{\partial^2 z}{\partial y^2}, \\
    k_{xy} &= \frac{\partial^2 z}{\partial x \partial y}.
\end{align*}
\]  \hspace{1cm} 5.7(a-c)

5.4 The Variation of Thickness

The thickness \( h \) of a plate may also be defined by an equation of the form 5.6. Let it be
\[ h = h_0 + h_1(x,y) \]  \hspace{1cm} 5.8

In equation 5.8, \( h \) is the thickness at any point \((x,y)\) of the plate and \( h_0 \) is the thickness at a suitably defined point, preferably the origin. \( h_1(x,y) \) defines the variation of \( h \) from point to point. For constant thickness
\[ h = h_0. \]
5.5 Free Vibrations of Uniform Plates of Constant Curvatures with \( k_{xy} = 0 \).

If \( f(x,y) \) of equation 5.6 is of order higher than the second, then at least one of the curvatures of the plate will be variable. This will render some of the coefficients of equation 5.5(a-c) variable. Moreover, if \( h \) is variable, then most of the coefficients of equation 5.5(a-c) are also variable. It may be verified that there are no closed form solutions of the system of equations 5.5(a-c), even for simply supported boundary conditions, if any one of the curvatures, or the thickness, is variable. This is also true, if \( k_{xy} \) is also non-zero, no matter whether it is constant or variable. The problem may then be solved only numerically.

However, there is a class of double curvature plates of considerable practical interest, for which \( k_x, k_y \) and \( h \) are constant and \( k_{xy} = 0 \). The equation of the middle surface of such a plate is a quadratic one and in general may be written as

\[
z = \tilde{a}_0 + \tilde{a}_1 x + \frac{1}{2} \tilde{a}_2 x^2 + \tilde{b}_0 + \tilde{b}_1 y + \frac{1}{2} \tilde{b}_2 y^2 + \tilde{c}_{xy} .
\]

5.9

Here \( \tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{b}_0, \tilde{b}_1, \tilde{b}_2 \) and \( \tilde{c}_o \) are constants. The condition \( k_{xy} = 0 \) requires that \( \tilde{c}_o = 0 \). The equation 5.9 represents some of the familiar surfaces which are given below.

(i) Circular cylindrical surface when \( \tilde{a}_2 = \tilde{b}_1 = \tilde{c}_o = 0; \tilde{a}_0, \tilde{a}_1, \tilde{b}_0 \) arbitrary, with \( \tilde{b}_2 = k_y \);

(ii) Elliptic paraboloid surface when \( \tilde{a}_0 = \tilde{a}_1 = \tilde{b}_0 = \tilde{b}_1 = \tilde{c}_o = 0, \tilde{a}_2 \) and \( \tilde{b}_2 \) positive and \( \tilde{a}_2 \neq \tilde{b}_2 \) with \( k_x = \tilde{a}_2 \) and \( k_y = \tilde{b}_2 \);
(iii) A spherical surface when \( \tilde{a}_0 = \tilde{a}_1 = \tilde{b}_0 = \tilde{b}_1 = \tilde{c}_0 = 0 \) and
\( \tilde{a}_2 = \tilde{b}_2 \) (both +ve or -ve) with \( k_x = k_y = \tilde{a}_2 = \tilde{b}_2 \).
(iv) An hyperbolic paraboloid when \( \tilde{a}_0 = \tilde{a}_1 = \tilde{b}_0 = \tilde{b}_1 = \tilde{c}_0 = 0 \)
with \( \tilde{a}_2 \) positive and \( \tilde{b}_2 \) negative with \( k_x = \tilde{a}_2 \) and \( k_y = \tilde{b}_2 \).

Out of the above four surfaces, only the first one is developable although in this theory they are all assumed to be developable.

With \( k_x \), \( k_y \) and \( h \) constant and \( k_{xy} = 0 \), the system of equations 5.5(a-c) simplifies to a great extent giving that below.

\[
\begin{align*}
    u'' + \frac{1 - v}{2} u'' + \Omega u + \frac{1 + v}{2} v'' - (k_x + vk_y) w' &= 0, \\
    \frac{1 + v}{2} u' + \frac{1 - v}{2} v'' + v' + \Omega v - (vk_x + k_y) w' &= 0, \\
    -(k_x + vk_y) u' - (vk_x + k_y) v' + [(k_x^2 + 2vk_xk_y + k_y^2 - \Omega) w + \frac{h^2}{12} (w'''' + 2w'''' + w''')] &= 0.
\end{align*}
\]

5.10(a-c)

For \( k_x = 0 \), the set of equations 5.10(a-c) reduces to the simple set of equations given by Flügge [6], Vlasov [3,4], Donell-Jenkins and others, for constant thickness circular cylindrical plates. Therefore, the order of accuracy of this system for a double curvature plate is similar to that of Flügge's simple set of equations for the circular cylindrical plate. The non-developable character of the doubly curved plates is reflected in the equation 5.10(c) by the term \( 2vk_xk_y w \).
5.6 Solutions for All Edges Simply Supported

Let the origin of co-ordinates be at one of the corners of the plate and let $L_x$ and $L_y$ represent the length of the plate along the $x$ and $y$ directions respectively. The simply supported edge conditions are as follows.

At $x = 0$ and at $x = L_x$,

\[ v = 0, \]
\[ w = 0, \]

\[ u' + vv' - (k_x + v,k_y)w = 0, \quad \text{(from $N_x = 0$)} \]
\[ w'' + vw'' = 0, \quad \text{(from $M_x = 0$)}. \]

At $y = 0$ and at $y = L_y$,

\[ u = 0, \]
\[ w = 0, \]

\[ vu' + v - (vk_x + k_y)w = 0, \quad \text{(from $N_y = 0$)} \]
\[ vw'' + w'' = 0, \quad \text{(from $M_y = 0$)}. \]

The equations 5.10 and the boundary conditions 5.11 are exactly satisfied by the following functions for $u$, $v$ and $w$.

\[ u = q_1 \cos \frac{\max}{L_x} \sin \frac{ny}{L_y}, \]
\[ v = q_2 \sin \frac{\max}{L_x} \cos \frac{ny}{L_y}, \]
\[ w = q_3 \sin \frac{\max}{L_x} \sin \frac{ny}{L_y}, \]

where $q_1$, $q_2$ and $q_3$ are amplitudes of vibrations (to be determined).
Substituting for \( u, v \) and \( w \) from 5.12 into 5.10, gives the following equations for the determination of the amplitudes

\[
\begin{bmatrix}
a_{11} + \Omega & a_{12} & a_{13} \\
a_{12} & a_{22} + \Omega & a_{23} \\
a_{13} & a_{23} & a_{33} + \Omega
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\tag{5.13}
\]

The coefficients \( a_{ij} \) of 5.13 are as follows

\[
a_{11} = -\left(\frac{(m\pi)^2}{L_x} + \frac{1 - \nu}{2} \frac{(n\pi)^2}{L_y}\right)
\]

\[
a_{12} = -\frac{1 + \nu}{2} \frac{(m\pi)}{L_x} \frac{(n\pi)}{L_y}
\]

\[
a_{13} = -(k_x + \nu k_y) \frac{(m\pi)}{L_x}
\]

\[
a_{22} = -\left(\frac{1 - \nu}{2} \frac{(m\pi)^2}{L_x} + \frac{(n\pi)^2}{L_y}\right)
\]

\[
a_{23} = -(\nu k_x + k_y) \frac{(m\pi)}{L_y}
\]

\[
a_{33} = -\left[(k_x^2 + 2\nu k_x k_y + k_y^2) + \frac{1}{12} \left(\frac{(m\pi)^2}{L_x} + \frac{(n\pi)^2}{L_y}\right)^2\right]
\]

For \( \{q_i\} \neq \{0\}, D(\Omega) = 0, \) is the frequency equation.

Neglecting the effects of the inplane inertia on the out of plane motion, the solution for the lowest \( \Omega \) may be written as
\[ \Omega = \frac{h^2 \pi^4}{12} \left[ \left( \frac{m}{L_x} \right)^2 + \left( \frac{n}{L_y} \right)^2 \right]^2 + (1 - \nu^2) \frac{\left[ k_y \left( \frac{m}{L_x} \right)^2 + k_x \left( \frac{n}{L_y} \right)^2 \right]^2}{\left( \frac{m}{L_x} \right)^2 + \left( \frac{n}{L_y} \right)^2} \]  

5.14(a)

Since \( \Omega = \frac{\rho(1 - \nu^2)\omega^2}{Eg} \),

\[ \omega^2 = \frac{Egh^2 \pi^4}{12\rho(1-\nu^2)} \left[ \left( \frac{m}{L_x} \right)^2 + \left( \frac{n}{L_y} \right)^2 \right]^2 + \frac{Eg}{\rho} \frac{\left[ k_y \left( \frac{m}{L_x} \right)^2 + k_x \left( \frac{n}{L_y} \right)^2 \right]^2}{\left( \frac{m}{L_x} \right)^2 + \left( \frac{n}{L_y} \right)^2} \]  

5.14(b)

The second part of the solution may be regarded as the correction to the flat plate frequency due to the curvatures \( k_x \) and \( k_y \). The expression 5.14(b) is the same as the expression derived by Vlasov [3] using equations of shallow shells, in curvilinear co-ordinates, neglecting the effects of the inplane inertias on the out of plane motions. A similar solution is also given by Reissner [21], using Marguerre's equations of shallow shells, in Cartesian co-ordinates, neglecting the inplane inertias.

5.7 Any Two Opposite Edges Simply Supported

When two opposite edges are simply supported, the equations 5.10(a-c) consistent with the boundary conditions at the other two edges may be solved for their true solutions. The substitution of the known simply supported solutions into equations 5.10(a-c) leads to a set of ordinary differential equations with constant coefficients, which may then be solved. The following functions satisfy the simply supported edge conditions.
(i) Edges $x = 0$ and $x = L_x$ are simply supported.

\[
\begin{align*}
  f_1 &= q_1 \cos \frac{m \pi x}{L_x} , \\
  f_2 &= q_2 \sin \frac{m \pi x}{L_x} , \\
  f_3 &= q_3 \sin \frac{m \pi x}{L_x} .
\end{align*}
\]

5.15(a-c)

(ii) Edges $y = 0$ and $y = L_y$ simply supported.

\[
\begin{align*}
  g_1 &= q_1 \sin \frac{n \pi y}{L_y} , \\
  g_2 &= q_2 \cos \frac{n \pi y}{L_y} , \\
  g_3 &= q_3 \sin \frac{n \pi y}{L_y} .
\end{align*}
\]

5.16(a-c)

If the functions 5.15(a-c) are substituted into equation 5.10(a-c) a system of ordinary differential equations for the solution of $g_1$, $g_2$ and $g_3$ is obtained. Similarly, if functions 5.16(a-c) are substituted, then a system of ordinary equations for the determination of $f_1$, $f_2$ and $f_3$ is obtained. These ordinary differential equations give solutions, consistent with the given boundary conditions, without introducing any simplifying assumptions.

5.8 Boundary Conditions other than a Pair of Opposite Edges Simply Supported

One of the methods that can be readily applied to the solution of this set of equations is the method of Kantorovich, provided reasonable functional approximations in either of the directions is available. The resulting ordinary differential equations may be solved applying the
modified matrix progression method discussed in Chapter III.

In order to determine the approximate function for \( w \) in the \( x \) direction, \( k_y, v \) and all the derivatives with respect to \( y \), in the equations 5.10(a-c) are set equal to zero. Thus, three equations are obtained which are given below.

\[
\begin{align*}
\dddot{u} + \Omega u &= 0, \\
\dddot{v} &= 0, \quad 5.17(a-c) \\
\dddot{w} - \frac{12}{h^2} (\Omega - k_x^2) w &= 0,
\end{align*}
\]

where \( \Omega = \rho \omega^2 / Eg \).

Equation 5.17(b) gives \( v = 0 \). The equations 5.17(a) and 5.17(c) constitute the equations of motion of an arch element along the direction of \( x \). These two equations are uncoupled due to the assumption that the arch element is very flat. Since these two equations are uncoupled, the transverse vibrations of the arch are determined by the equation 5.17(c) alone. Let

\[
\frac{12}{h^2} (\Omega - k_x^2) = \Omega_b
\]

in equation 5.17(c). In general \( k_x \) is a small quantity and therefore \( k_x^2 < \Omega \). If the inequality \( k_x^2 < \Omega \) holds, then equation 5.17 represents the transverse vibrations of a straight uniform beam whose eigenvalues of vibrations are \( \Omega_b \).

For given boundary conditions at the ends \( x = 0 \), and \( x = L_x \), the eigenfunction \( w \) of 5.17(c) is allowed to represent \( f_3 \) approximately.
It may be seen from the equations 5.10(a-c) that the nature of $f_2$ is the same as that of $f_3$, and of $f_1$, the same as that of $f'_3$. Therefore, it may be assumed that

$$f_2 = f_3,$$  \hspace{1cm} 5.19(a-b)

$$f_1 = f'_3.$$

Similarly, for given conditions at the edges $y = 0$ and $y = L_y$, the mode shape of $\varepsilon_3$ may be approximated by the eigenfunctions of the equation

$$\dddot{w} - 12(\Omega - k_y^2)w/h^2 = 0$$  \hspace{1cm} 5.20

and

$$\varepsilon_2 = \varepsilon_3,$$  \hspace{1cm} 5.21(a-b)

$$\varepsilon_1 = \varepsilon_3.$$

The equations 5.10(a-c) are reduced with respect to the direction of $x$. The reduced equations and their coefficients are given in Appendix 9. The required beam integrals may be found in Appendix 4.

The reduction in the direction of $y$ is exactly similar to the reduction in the direction of $x$. It is preferable that the reduction be carried out in the shallower direction. If the curvatures are equal, as in the case of a spherical cap, either of the directions may be used.

Since the equations 5.10(a-c) contain only the curvatures $k_x$ and $k_y$ of the surface, which are constant, no other properties of the surface other than the magnitude and direction of the curvatures need be known for computational purposes.
5.9 Doubly Curved Surfaces formed by Intersecting Straight Lines

This class of surface is as important as the doubly curved surfaces discussed earlier. The form of this class is very simple and is generated by intersecting straight lines. The equation of the middle surface may be written as

\[ z = \bar{c}_o \, xy \]  

where \( \bar{c}_o \) is a constant. It may be seen that for this class of surfaces

\[
\begin{align*}
k_x &= 0 , \\
k_y &= 0 , \\
k_{xy} &= \bar{c}_o .
\end{align*}
\]

Only \( h \) = constant shall be considered.

The equations 5.5(a-c) reduced for \( k_x = k_y = 0 \) and \( h \) = constant are the following.

\[
\begin{align*}
&u'' + \frac{1 - v}{2} u'' + \frac{\Omega}{h} u + \frac{1 + v}{2} v''' - (1 - v) k_{xy} v' = 0 , \\
&\frac{1 + v}{2} u'' + \frac{1 - v}{2} v''' + v'' + \frac{\Omega}{h} v - (1 - v) k_{xy} v' = 0 , \\
&(1 - v) k_{xy} u' + (1 - v) k_{xy} v' - 2(1 - v) k_{xy}^2 v + \frac{\Omega}{h} w - \frac{h^2}{12} \{w''' + 2w'''} = 0 .
\end{align*}
\]

It is apparent from these equations and the simply-supported boundary conditions stated in section 5.6 that this class of plates has no exact solutions even for the simply supported boundary conditions. However, this class of plates has exact solutions for pinned edges.
(simple support preventing axial movement). These conditions may be stated as

(a) At \( x = 0 \) and at \( x = L_x \)

\[ u = 0, \quad \text{5.25(a-c)} \]
\[ w = 0, \]
\[ u' + v' - 2k_{xy} w = 0, \quad \text{(from } N_{xy} = 0 \text{)} \]
\[ w'' + vw''' = 0. \quad \text{(from } M_x = 0 \text{)} \]

(b) At \( y = 0 \) and at \( y = L_y \)

\[ v = 0, \quad \text{5.26(a-c)} \]
\[ w = 0, \]
\[ u' + v' - 2k_{xy} w = 0, \quad \text{(from } N_{xy} = 0 \text{)} \]
\[ vw'' + w''' = 0. \quad \text{(from } M_y = 0 \text{)} \]

The eigenfunctions that satisfy these boundary conditions and the differential equations 5.24(a-c) are

\[ u = q_1 \sin \frac{m \pi x}{L_x} \cos \frac{n \pi y}{L_y}, \]
\[ v = q_2 \cos \frac{m \pi x}{L_x} \sin \frac{n \pi y}{L_y}, \quad \text{5.27(a-c)} \]
\[ w = q_3 \sin \frac{m \pi x}{L_x} \sin \frac{n \pi y}{L_y}; \]

where \( q_1, q_2 \) and \( q_3 \) are amplitudes of vibrations which are yet to be determined.

Substitution of the functions for \( u, v \) and \( w \) from equation 5.27 into 5.24(a-c), the following frequency determinant is obtained,
\[
\begin{bmatrix}
\lambda_m^2 + \frac{1 - \nu}{2} \lambda_n^2 - \frac{\Omega}{h} & \frac{1 + \nu}{2} \lambda_m \lambda_n & (1 - \nu) k_{xy} \lambda_n \\
\frac{1 + \nu}{2} \lambda_m \lambda_n & \frac{1 - \nu}{2} \lambda_m^2 + \lambda_n^2 - \frac{\Omega}{h} & (1 - \nu) k_{xy} \lambda_m \\
(1 - \nu) k_{xy} \lambda_n & (1 - \nu) k_{xy} \lambda_m & 2(1 - \nu) k_{xy}^2 + \frac{h^2}{12} (\lambda_m^2 + \lambda_n^2)^2 - \frac{\Omega}{h}
\end{bmatrix} = 0;
\]

where \( \lambda_m = \frac{m\pi}{L_x} \) and \( \lambda_n = \frac{n\pi}{L_y} \).

This determinantal equation is a cubic one in \( \Omega \). The lowest value of \( \Omega \) for a given pair of \( m, n \) is the frequency parameter of the transverse vibration for this mode.

The role of this equation for the class of plate under consideration is the same as the role of the simply supported solutions of plate with constant \( h \), constant \( k_x \) and \( k_y \) with \( k_{xy} = 0 \).

Also the equation 5.28 may form the basis for comparison and evaluation of solutions by approximate methods for this class of surfaces. A simple expression for \( \omega \) may be obtained by deleting \( \Omega/h \) from the first two rows of 5.28 (neglecting the effects of inplane inertias). For other boundary conditions the Kantorovich method may be applied. Though the general application of the method may be followed as was done before, the relation between the functional approximations will be different. Taking \( f_3 \) and \( g_3 \) as the basis, the functional relations will be as follows.
\( f_1 \) same in shape as \( f_3 \),  
\( f_2 \) same in shape as \( f'_3 \),  
\( g_1 \) same in shape as \( g'_3 \),  
\( g_2 \) same in shape as \( g_3 \).

It should be noted that, for plates with \( k_{\text{xy}} = 0 \) and \( h = \text{constant} \), these functional relations only hold when \( f_1 \) is interchanged with \( f_2 \) and \( g_1 \) is interchanged with \( g_2 \).

We shall not continue this topic further, since all these will be covered in the next Chapter. However, it may be realised that the knowledge of the differential equations is necessary for a better understanding of the problem.

5.10 Results and Discussion

(i) Simply Supported Plates

The equation 5.14(b) gives the square of the frequency of the simply supported plate approximately. This expression is the same as the expression derived by Vlasov [3] and is similar to the one given by Reissner [21], under similar approximations. The first part of this expression is the same as the expression for a flat plate and the second part may be regarded as the correction to the flat plate frequency due to the small curvatures. This correction is always positive and therefore the curved plate frequencies should always be greater than the corresponding flat plate frequencies. For small curvatures, and also for higher modes, this correction will be insignificant. For a square anticlastic plate
(k_x and k_y having opposite signs) of equal curvatures (k_x = -k_y) the correction vanishes for all modes with m=n.

It may be seen from this expression that for the same aspect ratio and f_0, the frequencies of a singly curved plate will always be greater than the corresponding frequencies of an anticlastic plate, and lower than the corresponding frequencies of a synclastic plate. The expression 5.14(b) is sufficient to obtain all necessary data for free vibrations of simply supported shallow plates.

(ii) Clamped Plates

It has been stated in section 5.5 that when k_x = 0, the set of equations 5.10(a-c) reduces for singly curved plates to the simple set of equations given by Flügge [6].

This simple set of equations is usually regarded as of doubtful accuracy. If it can be established that this set of equations is accurate enough for a shallow, singly curved plate, then it will be reasonable to assume that the set of equations 5.10(a-c) will yield dependable solutions for shallow doubly curved plates.

For this, a plate which was analysed before, using the exact set of equations of Flügge, in Chapter III is selected. The dimensions of the plate are

\[ L_x = L_y = 17 \text{ in.} \]
\[ k_x = 0, \quad k_y = \frac{1}{R} = 0.01042 \text{ in}^{-1}, \]
\[ \nu = 0.33. \]

For all edges clamped, the results are compared in Table 7. It may be seen that the results obtained by using the simple set are always
fractionally higher than those obtained by using the so-called exact set of equations. For the lowest eigenvalue the difference is about 0.5% and in the 4th eigenvalue it is about 1.7%. In terms of frequencies, these differences are negligible.

Moreover, the mode shapes for \( u, v \) and \( w \) obtained by using the simple set of equations are identical to those obtained by using the exact set of equations.

A similar comparison for this plate is presented in Table 8, when its curved edges are clamped and the straight edges are simply supported. The effects of the inplane inertias on the out of plane motions are neglected in using both sets of equations. The difference in magnitudes of the eigenvalues is again very small.

From the above it may be concluded that for the solution of the problem of free vibrations of shallow singly curved plates, the simple set of equations may be used with reasonably good accuracy. Since a singly curved plate is a special form of a doubly curved plate, it may be reasonable to expect accurate solutions for all uniform shallow plates with constant curvatures and \( k_{xy} = 0 \) by using equations 5.10(a-c).

So far, only singly curved plates are considered. Since spherical plates are the simplest form of doubly curved plates, it is intended to begin the analysis of the solutions with spherical plates. No previous solutions of clamped spherical plates with rectangular base plane projections are noticed. Therefore, it is not possible to compare the solutions obtained here directly with solutions
obtained elsewhere. The previous solutions are mainly confined to hemi-
spherical shells and spherical caps with either one or two boundaries.
Such a particular set of solutions for shallow clamped spherical caps
with one edge is due to Reissner [52]. It is possible to obtain some
idea about the order of magnitude and the trend of variations of the
lowest eigenvalues that might be expected for rectangular spherical
plates from Reissner's set of solutions. In Table 9, some of Reissner's
solutions for spherical caps and some solutions for spherical plates
obtained by the method of Kantorovich are given. The maximum rise to
thickness ratio \( f_h/h \) 16, 12 and 8 are taken from the tables given by
Reissner. The sides of the plates are taken to be equal to the diameter
of the base plane circles of the caps. The value of \( \nu = 0.3 \) is used by
Reissner.

For all the plates

\[
h = 0.02 \text{ in},
\]

and \( R = 96 \text{ in} \) are assumed.

From Table 9 it may be seen that the order of magnitude of the
solutions is very much the same. The trend of variation with the variation
of \( f_h/h \) is similar.

From this it seems that the solutions obtained here have
reasonable magnitude and correct trend of variation.

In Table 10, a few more values of \( \Omega \) including the values of \( m, n \)
for the above-mentioned plates are given. It may be seen that the
calculated mode shapes of \( u \) and \( v \) are compatible with the mode shapes
of \( w \) for all the \( \Omega \) values quoted.
The values of $\Omega$ for three square plates of equal base plane sides but of different curvatures are given in Table 11. The following are common for all the three plates.

\[ L_x = L_y = 17 \text{ in. } , \]
\[ h = 0.02 \text{ in. } , \]
\[ \nu = 0.33. \]

The first plate is a singly curved one with $k_x = 0$ and $k_y = 0.01042 \text{ in}^{-1}$.

The second one is an anticlastic plate with $k_x = 0.01042 \text{ in}^{-1}$ and $k_y = -k_x$.

The third one is a synclastic one with $k_x = 0.01042 \text{ in}^{-1}$ and $k_y = k_x$.

From this table it may be seen that the value of $\Omega$ of the singly curved plate is lower than the $\Omega$ of the anticlastic plate. This is contrary to what might be expected in the light of equation 5.14 for a simply supported plate. Since the behaviour of a clamped plate is different from the behaviour of a simply supported plate, in the light of Table 11, it may be concluded that a clamped singly curved plate will have lower frequencies than a doubly curved plate. As well, an anticlastic plate will have lower frequencies than a synclastic plate when all the edges are clamped.

In Table 12, the lowest values of $\Omega$ of a rectangular plate of $L_x = 11 \text{ in.}$, $L_y = 9 \text{ in.}$, and $h = 0.028 \text{ in.}$ are given. The $k_x = 0.001 \text{ in}^{-1}$ is kept constant and the $k_y$ is varied. In this case also, the $\Omega$ of the synclastic plates is higher than the $\Omega$ of the anticlastic plates.
5.11 Conclusions

The method of Kantorovich matrix progression may be used to study the problems of free vibrations of second order surfaces of uniform thickness. For shallow plates, the equations of Vlasov in the Cartesian co-ordinate system may be relied upon to yield reasonably accurate solutions. Even for square plates, the Kantorovich method may be relied upon, at least, to yield the lowest frequencies with an acceptable accuracy. However, the present analysis will need modifications in case of degenerate modes.
CHAPTER VI

FINITE ELEMENT METHOD

6.1 Introduction

The success of the method used in Chapter V depends mainly on the accuracy of the assumed mode shapes. For plates of variable thickness and curvatures virtually no simple analytical expressions for the mode shapes are available. Therefore, it is necessary to look for other methods for the solution of the general problem. One such method is the finite, or discrete, element analysis. The application of the method to non-structural problems is beyond the scope of this thesis and therefore will not be discussed.

In the finite element method of analysis, the structure is assumed to be divided in some manner into a number of elements which are connected to one another and also to the external boundaries, if any, at some discrete points known as the 'nodal points'. So far as structural analysis is concerned, this is not an entirely new concept. For example, the members of a frame structure may be visualised as finite elements. Usually the finite element method is used for solution of complicated problems. Therefore, the assemblage contains a large number of unknowns. The formulation and the subsequent solution of the equations in these large numbers of unknowns are organised in matrix notations suitable for handling with high speed digital computers. Once the finite element formulation is carried out in matrix notations, there
is hardly any difference between the matrix method of structural analysis and the finite element method of analysis. However, the formal introduction of the finite element method for the analysis of structures (distinct from the matrix method) has greatly assisted in solving some of the structural problems which are otherwise analytically intractable. Because of this, at present, the method is very popular with structural engineers dealing with unusual or complicated structures.

After discretisation of the structure suitable variation of displacements or of forces or of both are prescribed within an element in terms of the nodal unknowns. Since displacements and forces are related to strains and stresses, the prescribed variations of the displacements and the forces reflect the variation of the stresses and the strains in each of the elements. The prescription of these variations should conform to certain conditions known as the conditions of conformity which will be discussed later.

If the displacements at the nodal points are the basic unknowns of the problem the method is known as the 'displacement method' which is formulated on the basis of the variational principle of minimum potential energy. In that method, stress-displacement relations are satisfied, but the equilibrium is satisfied only approximately [96]. The element's nodal loads \( \{F\} \) are expressed in terms of the element's so-called 'stiffness matrix' \([K]\) and the element's nodal displacements \( \{u\} \) as \( \{F\} = [K]\{u\} \). Since stiffness is involved, the method is sometimes known as 'stiffness method' also. The element's stiffness matrix is a function of the material and geometric properties of the element and
the prescribed displacement patterns. The condition of compatibility of displacements requires that the elements meeting at a common node must have the same displacements at that node. Satisfaction of this compatibility condition together with equilibrium at each node leads to the nodal load-displacement or equilibrium equations of the whole structure. Therefore, the displacement method is known as 'the equilibrium method' also \[ 41 \], though local equilibrium is only approximately satisfied. The formation of the system equilibrium equations from the elements' equilibrium equations is known as the 'assembly' which yields the system stiffness matrix known as the 'overall stiffness matrix' and the system load matrix known as the 'overall load matrix'. The imposition of conditions on some nodal displacements is known as 'constraining'. The 'constraining' of the system is necessary to eliminate possible 'rigid body displacements' or the 'non-straining' displacements. The solution of the constrained system of equilibrium equations yields the necessary displacements at the nodes.

For dynamical system, the so called 'element mass matrix' and the 'overall mass matrix' are also required. The element mass matrix depends upon the element's material and geometrical properties and also the sum of the squares of the velocities of the element in three mutually orthogonal directions. Since velocity is equal to the 'first time derivative' of the displacements the mass matrix is again derivable in terms of the nodal displacements. The overall mass matrix is formed from the elements' mass matrices exactly in the same
way as the overall stiffness matrix is formed from the elements' stiffness matrices. The overall load matrix is replaced by the overall mass matrix, in the case of free vibrations, yielding the so-called eigensystem, solution of which yields eigenvalues and eigenvectors.

There are two other alternative approaches, one known as the 'force method' where all the basic nodal unknowns are forces or stresses and the other is known as the 'mixed method' where the basic nodal unknowns may be both displacements and forces. Reference may be made to Livesley [41] and Rubinstein [68] for the displacement and the force method. Also reference may be made to Reissner [96] for the variational basis of the mixed method.

For our purpose, the displacement method is the simplest and the easiest to formulate, particularly for the formulation of the mass matrix. Therefore the displacement method is preferable and subsequently any reference to finite element method will mean the 'displacement method of structural analysis using matrices'.

In recent years, the method has been extensively used for the solution of plate and shell problems. Usually the structure is idealised as an assembly of flat triangular or rectangular elements. Recent advances in the solution of shell type structures include the use of rectangular curved elements of zero Gaussian curvature. Some elements with non-zero Gaussian curvatures are also developed. A brief description of some of these elements is given later.

Some very good elements are available for the analysis of shells of revolution and singly curved shells. In the case of uniform
singly curved plates, the coefficients of the first fundamental form of the surface are constant. Therefore, there is no need for numerical integrations while deriving the mass and stiffness matrices using accurate shell equations in curvilinear co-ordinate system. For doubly curved shells or plates the coefficients of the first fundamental form of the surface are always variable requiring numerical integrations in deriving the element stiffness and mass matrices, if the equations are derived in curvilinear co-ordinates. The elements available at present are all claimed to be in the curvilinear co-ordinate system. The approximations under which they are derived are not clear. Sometimes only the strain displacement relations are given together with the assumed displacement function. The actual matrices have to be derived by the user. Since the finite element method is to make analysis easier, rather than make it complicated, we set forth to derive a simple finite element for the analysis of doubly curved rectangular plates, under clearly defined approximations. Since we are interested only in rectangular plates the simple rectangular configuration of the element will be used. The element is expected to handle a wide variety of plates including flat and the curved ones. Since we are interested in vibrations of the plates in presence of static membrane stresses, the effects of applied inplane shear and direct tensile and compressive forces on the stiffness of the element are taken into account.
6.2 General Discussion of the Strain Energy Expression of a Doubly Curved Plate

The finite element formulation of the problem of vibrations of curved plates may be accomplished directly from the expressions of the strain and kinetic energies of the plate. The stiffness matrix and the consistent mass matrix of each element may be formed from consideration of the expressions for the element's strain and kinetic energies. The strain energy $S$ may be written as

$$S = S_m + S_b + S_{mb} + S_{cmb} + S_I,$$

$$= S_a + S_I.$$

The components of $S_a$ given by equation 6.1(a) are derived in Appendix 10. The physical interpretation of the components of $S$ are as follows.

The part $S_m$ is due to the membrane stretching of the plate and is similar to the corresponding energy of a flat plate. $S_b$ is an energy due to the bending of the plate and is contributed entirely by the changes of curvatures and is similar to the corresponding flat plate energy. In the case of a flat plate $S_b$ represents the total bending energy. But for a curved plate $S_b$ is not the complete bending energy.

The additional bending energy $S_{mb}$ and $S_{cmb}$ are generated by the special shapes of the curved plates. The energy $S_{mb}$ is associated with the Gaussian curvature $K_G = k_x k_y$ of the plate. In the functional form (equation A10.11)

$$S_{mb} = K_G \cdot f(h, \varepsilon_x, \varepsilon_y, \varepsilon_{xy}).$$
and is not a function of the changes of curvatures, when $K_G = 0$, $S_{mb} = 0$. Physically, a surface may be unrolled or developed into a plane, purely by continuous bending, without stretching or tearing, only when its Gaussian curvature $K_G = 0$. A familiar example of this kind of surface is a circular cylinder. On the other hand if $K_G \neq 0$, the surface cannot be developed into a plane by bending alone, but needs stretching. A spherical cap is a familiar example of this kind of surface. It may be concluded that $S_{mb}$ is due to the forces of stretching, which resist development of the plate into a plane by continuous bending, acting with the midplane strains. If $S_{mb}$ is neglected, then effectively the surface is assumed to be developable ($K_G = 0$). Clearly, for a flat plate, $S_{mb}$ does not exist.

The part of the energy denoted by $S_{cmb}$ is due to the warping of the faces, and therefore will exist for both developables and non-developables, provided the surface is not already flat. Due to the existence of the curvatures, the lengths of curved line elements of the plate across the thickness are different. Consequently, the same amount of change in these lengths produces different strains, and therefore different stresses across the thickness of the plate. The resultant force due to these stresses is eccentric and therefore gives rise to an additional bending energy which is $S_{cmb}$.

The part of the energy $E$ given by $S_I$ is due to the applied membrane stresses. In aircraft type of structures $S_I$ is usually induced due to pressurisation of the cabins.

In order to account for all the physically admissible components of the strain energy of a thin doubly curved plate, the variations of the
strains across the thickness should be assumed to be at least quadratic in order. In the computation of the elementary volume of the plate, the varying lengths of the line elements across the thickness should also be accounted for.

Although the expression for the strain energy derived on the basis of quadratic variation of strains and the warping of the faces (as derived in Appendix 10) is accurate, it is too complicated to be used in its general form. Therefore, it is necessary to simplify the strain energy expression, if possible. The two major assumptions in this direction are indicated in Appendix 10.

6.3 Simplification of the Strain Energy Expression

It is possibly to simplify the strain energy expression under certain conditions. One of these conditions concerns the ratio of the maximum rise to the minimum base width of the plate. If this ratio is less than 1/5 (Vlasov [3], p.343) the plate is assumed to be shallow. If the plate is shallow, the strain energy expression may be considerably simplified. Probably, all shallow surfaces may be assumed to be developables without appreciable error. This assumption allows $S_{mb}$ to be taken as zero. This is also justified in the sense that the product of the curvatures $K_0$ is negligibly small. The assumption that $S_{mb} = 0$ seems to be particularly reasonable in the finite element approximation, because the individual elements tend to be very shallow.

If the thickness and the curvatures are small, $S_{cmb}$ may be neglected. The warping of the faces of a small finite element will
be small. This assumption is equivalent to the assumption that the elements of a curved plate are, individually, not much different from a flat plate. The elementary volume will be given by the product of the differential lengths in the three mutually orthogonal directions.

6.4 The Strain Energy of a Shallow Plate Element

(i) Without applied inplane stresses

If the simplifications mentioned in section 6.3 are carried out, the strain energy expression A10.l4 will be obtained under the simplified assumptions of A10.13(a-d). These assumptions are shown to be consistent with the geometry of the finite elements which are essentially not much different from flat plate elements. This being so, there seems to be little reason to have the added complications of using the curvilinear co-ordinate systems. Instead, the more straightforward Cartesian co-ordinate system in (x,y,z) may be used. Also it may be noticed that under the simplifying assumptions, the curvilinear and the Cartesian co-ordinate systems are equivalent to each other (In Appendix 10, A→B→l, α→x, β→y, γ→z). Therefore, it is reasonable to use the equations 5.1(a-f) which are in Cartesian co-ordinates. This set has also the added advantage that the curvature $k_{xy}$ is included. Moreover, it is shown in the previous Chapter that, although approximate in nature, this set of equations yields dependable solutions for shallow plates.

Since the strain-displacement relations and the assumptions used here are different from those used in Appendix 10, the new strain
energy expression will be derived here.

Assuming linear variation of strain across the thickness of the plate, the strains at a point \((x,y,z)\) inside the plate may be written as

\[
\begin{align*}
\varepsilon_x &= \varepsilon_x + z \chi_x , \\
\varepsilon_y &= \varepsilon_y + z \chi_y , \\
\varepsilon_{xy} &= \varepsilon_{xy} + z \chi_{xy} .
\end{align*}
\]

6.3(a-c)

This assumption is valid only for thin shells. Some authors, notably Vlasov \[3\] and \[4\] object to the use of this approximation even in thin shells. However, authors like Love \[2\] , Langhaar \[7\] and \[8\] and Warburton \[22\] have used it when deriving strain energy expressions for thin shells.

The stress at a point \((x,y,z)\) may be written as

\[
\begin{align*}
\sigma_x &= E(e_x + v e_y)/(1 - \nu^2) , \\
\sigma_y &= E(e_y + v e_x)/(1 - \nu^2) , \\
\sigma_{xy} &= E e_{xy}/(2(1 + \nu)) .
\end{align*}
\]

6.4(a-c)

The strain energy \(S_o\) may be written as

\[
S_o = \frac{1}{2} \int \int \int_{z=\pm h/2} (\sigma_{xx} e_x + \sigma_{yy} e_y + \sigma_{xy} e_{xy}) \, dz \, dx \, dy ,
\]

6.5

where \(S_o = S_m + S_b\).

It may be noticed that in equation 6.5 the elementary volume is taken as \((dz \, dx \, dy)\). The warping of the faces of the element is not considered.
From equation 6.5, in conjunction with equation 6.4 and 6.3, and carrying out the integrations with respect to z, the following strain energy expression is obtained.

\[
S_o = \frac{1}{2} \cdot \frac{E}{1 - \nu^2} \int \int \left[ h (\varepsilon_x^2 + \varepsilon_y^2 + 2\nu \varepsilon_x \varepsilon_y + \frac{1 - \nu}{2} \varepsilon_{xy}^2) \\
+ \frac{h^3}{12} (\chi_x^2 + \chi_y^2 + 2\nu \chi_x \chi_y + \frac{1 - \nu}{2} \chi_{xy}^2) \right] \, dx \, dy.
\]

6.7

The part of the energy associated with h is \( S_m \) (membrane) and the part associated with \( h^3 \) is \( S_b \) (bending).

For ease of matrix manipulations the equation 6.7 is derived in a different way. For this the stress resultants and the stress couples given by equations 5.2 and 5.3 respectively are redefined as the following.

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
D & \nu D & 0 & 0 & 0 & 0 \\
\nu D & D & 0 & 0 & 0 & 0 \\
0 & 0 & D(1-\nu)/2 & 0 & 0 & 0 \\
0 & 0 & 0 & K & \nu K & 0 \\
0 & 0 & 0 & \nu K & K & 0 \\
0 & 0 & 0 & 0 & 0 & K(1-\nu)/2
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_{xy} \\
\chi_x \\
\chi_y \\
\chi_{xy}
\end{bmatrix}
\]

In matrix notation \( \{N\} = [E^*] \{\varepsilon\} \).

6.8
The vector \( \{ \varepsilon \} \) is defined as the following.

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_{xy} \\
\chi_x \\
\chi_y \\
\chi_{xy}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 & -k_x \\
0 & \frac{\partial}{\partial y} & -k_y \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & -2k_{xy} \\
0 & 0 & \frac{\partial^2}{\partial x \partial y} \\
0 & 0 & \frac{\partial^2}{\partial y \partial y}
\end{bmatrix}
\begin{bmatrix}
u(x,y) \\
v(x,y) \\
w(x,y)
\end{bmatrix}
\]

In matrix notation \( \{ \varepsilon \} = [B] \{ u \} \).

With equations 6.8 and 6.9 the strain energy expression 6.7 may be rewritten as

\[ S_o = \frac{1}{2} \int \int \{N\}^T \{ \varepsilon \} \ dx \ dy \] 6.10

(ii) The strain energy due to applied membrane stresses

Let \( s_x \), \( s_y \) and \( s_{xy} \) be the applied membrane stresses. \( s_x \) and \( s_y \) are direct stresses applied in the direction of \( x \) and \( y \) respectively. \( s_x \) and \( s_y \) are positive and tensile when their sense of direction is as shown in (b) of Figure 18. \( s_{xy} \) is the applied membrane shear stress and is positive if its sense of direction is as shown in (b) of Figure 18. The contribution to the total strain energy \( S \) by the applied membrane stresses is denoted by \( S_I \).
In order to obtain an expression for $S_I$, it is necessary to define the strain-displacement relations, given by equations 5.1(a-c) more completely by including some non-linear terms. Such a set of relations, consistent with the approximations with which the relations 5.1(a-c) are derived, may be found in reference [53]. For the sake of completeness they are given below.

\[ \varepsilon_x = u' - k_x w + \frac{1}{2}(w')^2 , \]
\[ \varepsilon_y = v' - k_y w + \frac{1}{2}(w')^2 , \]
\[ \varepsilon_{xy} = u' + v' - 2k_{xy} w + w'w' . \]

The expression for $S_I$ may be written as

\[ S_I = \int \int h[s_x \varepsilon_x + s_y \varepsilon_y + s_{xy} \varepsilon_{xy}] \, dx \, dy \]
\[ = \int \int h[s_x (u' - k_x w + \frac{1}{2}(w')^2) + s_y (v' - k_y w + \frac{1}{2}(w')^2) \]
\[ + s_{xy} (u' + v' - 2k_{xy} w + w'w')] \, dx \, dy . \]

6.11(a)

Usually the contributions of the linear terms are neglected in the expression 6.11(d) yielding

\[ S_I = \frac{1}{2} \int \int h[s_x (w')^2 + s_y (w')^2 + 2s_{xy} (w'w')] \, dx \, dy . \]

6.11(e)

The expression 6.11(e) is the same as that given by Timoshenko and Gere [98, pp.340-351] for flat plates. A consequence of this expression for $S_I$ is that the effects of the membrane stresses $s_x$, $s_y$ and $s_{xy}$ are equivalent to a transverse distributed surface load of magnitude

\[ (s_x w'' + s_y w'' + 2s_{xy} w'') \] (see Vlasov [3, pp.532-535]).
The expression 6.11(e) may be written in a symmetrical matrix form as follows

\[ S_I = \frac{1}{2} \int \int h[w', w'] \begin{bmatrix} s_x & s_{xy} \\ s_{xy} & s_y \end{bmatrix} \begin{bmatrix} w' \\ w'' \end{bmatrix} \, dx \, dy \]

\[ = \frac{1}{2} \int \int h(w^*)^T \begin{bmatrix} I_p \end{bmatrix} (w^*) \, dx \, dy \]

where \( (w^*)^T = \begin{bmatrix} w' & w'' \end{bmatrix} \)

and \( \begin{bmatrix} I_p \end{bmatrix} \) is the 2 x 2 matrix of the inplane stresses.

6.5 The Kinetic Energy

The consistent mass matrix of the individual elements may be derived from their expressions of kinetic energy. The expression for the kinetic energy is as follows

\[ T = \frac{\rho}{2} \int \int \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right) \, dz \, dx \, dy . \]

Integrating with respect to \( z \)

\[ T = \frac{\rho}{2} \int \int h \left( \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right) \, dx \, dy . \]

For free vibrations expression 6.14(a) reduces to

\[ T = \frac{\rho \omega^2}{2} \, e^{i \omega t} \int \int h(u^2 + v^2 + w^2) \, dx \, dy . \]
In matrix notation

\[ T = \frac{\rho \omega^2}{2} e^{i\omega t} \int \int h \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \, dx \, dy \]

\[ = \frac{\rho \omega^2}{2} e^{i\omega t} \int \int h(u)T(u) \, dx \, dy . \quad 6.14(c) \]

For free vibrations the constant \( e^{i\omega t} \) is a non zero factor of the kinetic potential of the system and therefore may be dropped from equation 6.14(c). Then equation 6.14(c) becomes

\[ T = \frac{\rho \omega^2}{2} \int \int h(u)^T(u) \, dx \, dy . \quad 6.15 \]

6.6 Element Properties

The next step in the determination of the element properties is to assume suitable displacement fields for \( \{u\} \) of the element. These displacement fields should be such as to maintain continuity of displacements and normal slopes along the element boundaries, including the nodal points. These displacement fields are in general described by parametric expansions of \( \{u\} \). The total number of parameters should be equal to the total number of degrees of freedom of the element. Thus the displacements may be written as

\[ \{u\} = [P(x,y)] \{a\} , \quad 6.16 \]

where \([P(x,y)]\) is the matrix of the functions defining \( \{u\} \) and \( \{a\} \) is
the column matrix of the parameters.

From equation 6.9 and 6.16

\[
\{ \epsilon \} = [ \mathbf{B} ] [ \mathbf{P}(x,y) ] \{ \mathbf{a} \}
= [ \mathbf{P}^* ] \{ \mathbf{a} \}.
\]  \hspace{1cm} 6.17

From equation 6.8 and 6.17

\[
\{ N \} = [ \mathbf{E}^* ] [ \mathbf{P}^* ] \{ \mathbf{a} \}.
\]  \hspace{1cm} 6.18

In the absence of the applied inplane stresses, the strain energy of an element may be written as

\[
S_o = \frac{1}{2} \int_{\mathcal{E}} \int_{\mathcal{O}} (a)^T [ \mathbf{P}^* ]^T [ \mathbf{E}^* ] [ \mathbf{P}^* ] \{ \mathbf{a} \} \, dx \, dy
= \frac{1}{2} \{ \mathbf{a} \}^T \int_{\mathcal{O}} \int_{\mathcal{O}} [ \mathbf{P}^* ]^T [ \mathbf{E}^* ] [ \mathbf{P}^* ] \, dx \, dy \{ \mathbf{a} \}
= \frac{1}{2} \{ \mathbf{a} \}^T [ \mathbf{K}_o ] \{ \mathbf{a} \}
\]  \hspace{1cm} 6.19

where \([ \mathbf{K}_o ] = \int_{\mathcal{O}} \int_{\mathcal{O}} [ \mathbf{P}^* ]^T [ \mathbf{E}^* ] [ \mathbf{P}^* ] \, dx \, dy\) is the element stiffness matrix when no inplane stresses are applied.

The next stage is to determine the contribution of the inplane stresses to the stiffness of the element.

\{ \mathbf{w}^* \} may be written as

\[
\{ \mathbf{w}^* \} = [ \mathbf{P}(x,y) ] \{ \mathbf{a} \}
\]  \hspace{1cm} 6.20

The first and the second rows of \([ \mathbf{P}(x,y) ]\) may be obtained by differentiating the third row of \([ \mathbf{P}(x,y) ]\) with respect to \(x\) and \(y\) respectively.
From equation 6.12 in conjunction with equation 6.20

\[ S_I = \frac{1}{2} (a)^T \int_0^b h [P(x,y)]^T [I_p] [P(x,y)] \, dx \, dy \, (a) \]

\[ = \frac{1}{2} (a)^T [K_I] \, (a) \quad , \quad 6.21 \]

where \([K_I] = \int_0^b h \, [P(x,y)]^T [I_p] \, [P(x,y)] \, dx \, dy\) is the element stiffness matrix due to the inplane stresses.

The stiffness matrix due to combined bending and stretching in the presence of applied membrane stresses is given by

\[ [\bar{K}] = [\bar{K}_o] + [\bar{K}_I] \quad 6.22 \]

Similarly from equation 6.15, the kinetic energy of the element may be written as

\[ T = \frac{\rho \omega^2}{2} (a)^T \int_0^b h [P(x,y)]^T [P(x,y)] \, dx \, dy \, (a) \]

\[ = \frac{\rho \omega^2}{2} (a)^T [M] \, (a) \quad , \quad 6.23 \]

where \([M] = \int_0^b h \, [P(x,y)]^T [P(x,y)] \, dx \, dy\) is the element mass matrix.

The next stage is to determine \(\{a\}\). It may be determined by introducing the nodal co-ordinates successively into the expression 6.16. Thus the nodal displacements \(\{u\}\) are given by

\[ \{u\} = [C] \, \{a\} \quad \cdot \quad 6.24 \]

The matrix \([C]\) is square, and if non-singular,

\[ \{a\} = [C]^{-1} \, \{u\} \quad . \quad 6.25 \]
Introducing \( \{a\} \) from equation 6.25 into equation 6.19, 6.22 and 6.23 successively the following expressions are obtained.

\[
S_o = \frac{1}{2} \{u\}^T [\{c\}^{-1}]^T [\bar{K}_o] \{c\}^{-1} \{u\} , \quad 6.26
\]

\[
S_I = \frac{1}{2} \{u\}^T [\{c\}^{-1}]^T [\bar{K}_I] \{c\}^{-1} \{u\} , \quad 6.27
\]

\[
T = \frac{\rho \omega^2}{2} \{u\}^T [\{c\}^{-1}]^T [\bar{K}] [\{c\}^{-1}] \{u\} . \quad 6.28
\]

From equation 6.26 and 6.27 in view of equation 6.22, the total strain energy of the element may be written as

\[
S = S_o + S_I = \frac{1}{2} \{u\}^T [\{c\}^{-1}]^T [\bar{K}] [\{c\}^{-1}] \{u\} \quad 6.29
\]

The total kinetic and strain energies of the element are given by equation 6.28 and 6.29 respectively.

6.7 Determination of the Natural Frequencies and Mode Shapes

The overall stiffness matrix \([\bar{A}]\) and the overall mass matrix \([\bar{M}]\) of the complete plate may be obtained by assembling the stiffness and mass matrices, respectively, of all the constituent elements by standard matrix methods of structural analysis (see for example Rubinstein |68, ch.VI & VII| and Livesley |41,Ch.IV| ). The corresponding assembly of the nodal unknowns \(\{u\}\) of all the constituents elements will give an overall vector \(\{d\}\) of all the unknowns.

The approximate expression for the kinetic-potential \(\Lambda\) for the whole plate is given by

\[
\Lambda = \frac{1}{2} \{d\}^T [\{A\} - \lambda \{M\}] \{d\} , \quad 6.30(a)
\]
where \[ \lambda = \rho \omega^2. \] Equation 6.30(a) is subsequently constrained, if required, by deleting those elements of \( \{d\} \) which are known to be zero. In general the constraining of elements of \( \{d\} \) is necessary to reflect edge conditions (edge constraints) or intermediate conditions of symmetry or anti-symmetry (intermediate constraints). Correspondingly, \( [\bar{A}] \) is constrained to \( [\bar{A}_1] \) and \( [\bar{M}] \) is constrained to \( [\bar{M}_1] \) by deleting the rows and columns of \( [\bar{A}] \) and \( [\bar{M}] \) corresponding to the vanishing elements of \( \{d\} \). The equation 6.30(a) is then reduced to

\[
\lambda = \frac{1}{2} \{d_1\}^T [\bar{A}_1] - \lambda [\bar{M}_1] \{d_1\}, \tag{6.30(b)}
\]

where \( \{d_1\} \) is the constrained vector of unknowns.

Sometimes the conditions of constraints may be more complicated. In such cases the constraining has to be done in an entirely different way. We shall discuss it briefly with a specific example. Let \( \{d\} \) of 6.30(a) be constituted with only 8 elements. Let there be two equations of constraints such as \( (b_{ij} \text{ are coefficients}) \)

\[
\begin{bmatrix}
  b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\
  b_{21} & b_{22} & b_{23} & b_{24} & b_{25}
\end{bmatrix}
\begin{bmatrix}
  d_2 \\
  d_4 \\
  d_5 \\
  d_7 \\
  d_8
\end{bmatrix} = \{0\}, \tag{6.30(c)}
\]

In general it will be incorrect to assume the trivial solution for the above equation 6.30(c). The correct procedure is to express any two of the unknowns in terms of the other three as shown below.
\[
\begin{align*}
\begin{bmatrix}
    d_2 \\
d_4 \\
    d_5 \\
d_7 \\
d_8 \\
\end{bmatrix} &= -\begin{bmatrix}
    b_{11} & b_{12} \\
    b_{21} & b_{22} \\
\end{bmatrix}^{-1} \begin{bmatrix}
    b_{13} & b_{14} & b_{15} \\
    b_{23} & b_{24} & b_{25} \\
\end{bmatrix} \begin{bmatrix}
    d_5 \\
d_7 \\
d_8 \\
\end{bmatrix}, \\
&= \begin{bmatrix}
    c_{11} & c_{12} & c_{13} \\
    c_{21} & c_{22} & c_{23} \\
\end{bmatrix} \begin{bmatrix}
    d_5 \\
d_7 \\
d_8 \\
\end{bmatrix}.
\end{align*}
\]

Now re-arrange the vector \( \{d\} \) as follows.

\[
\begin{align*}
\begin{bmatrix}
    d_2 \\
d_4 \\
d_5 \\
d_7 \\
d_8 \\
d_1 \\
d_3 \\
d_6 \\
\end{bmatrix} &= \begin{bmatrix}
    c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
    c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\
    1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix} \begin{bmatrix}
    d_5 \\
d_7 \\
d_8 \\
d_1 \\
d_3 \\
d_6 \\
\end{bmatrix}.
\end{align*}
\]

In matrix notation

\[
\{d\} = [B_0] \{d_1\}. \tag{6.30(e)}
\]

It is necessary to make row and column changes in equation 6.30(a) corresponding to the arrangement of 6.30(e). For the general case also, the form of the equation 6.30(e) will be the same. With equation
6.30(e) equation 6.30(a) may be written as

\[ \Lambda = \frac{1}{2} \{ d_1 \}^T [B_0]^T [[\bar{A}] - \lambda [\bar{M}]] [B_0] \{ d_1 \}. \quad 6.30(f) \]

It may be noted that when relations such as 6.30(c) exist or some of the elements of \{d\} are zero, then it is not possible to make \( \Lambda \) stationary with respect to the elements of \{d\} without application of the proper constraints.

After the removal of the zero elements of \{d\}, if any, or the application of the proper constraints, the elements of the vector \( \{d_1\} \) are all independent and may be treated as generalised co-ordinates.

The kinetic potential of equation 6.30(b) is then made stationary by forming the differentials of \( \Lambda \) with respect to each of the generalised co-ordinates in turn, and equating to zero. It may be observed that the coefficient matrix of the system of equations thus obtained will involve both \([\bar{A}_1]\) and \([\bar{M}_1]\) without altering them. Therefore, the formal differentiations need not be performed. The set of linear homogeneous algebraic equations may be directly written down from 6.30(b) as follows.

\[ [[\bar{A}_1] - \lambda [\bar{M}_1]] \{ d_1 \} = \{ 0 \}. \quad 6.30(g) \]

(In the case of equation 6.30(f) \[ [B_0]^T [[\bar{A}] - \lambda [\bar{M}]] [B_0] \{ d_1 \} = \{ 0 \}).

The system 6.30(g) represents an eigen system with \( \lambda \) representing the eigenvalues and \( \{ d_1 \} \) the non-zero components of the eigenvector \{d\}.

If the constraining of the system 6.30(g) is such that all possible rigid body motions are prevented, then all the eigenvalues of the system 6.30(g) will be non-zero.

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Another eigen system may be obtained from 6.30(a) directly if the system does not require any constraining. The system of equations may be written as follows.

\[
[[\mathbf{A}] - \lambda \mathbf{[M]}] \{d\} = \{0\}.
\]  

If the system representation is ideal, the six lowest eigenvalues of the system should be zero corresponding to 3 rigid body displacements and 3 rigid body rotations. This will ensure that under rigid body displacements and rotations there will be no straining anywhere in the plate.

If all the displacements at a node (generally at a boundary) are known to vanish, that node does not contribute to the nodal equilibrium of the system and therefore may be excluded while deriving the overall stiffness and mass matrices of the system.

There are standard methods and computer procedures available for the solution of the eigen systems. In general these procedures are readily applicable to both the systems given by 6.30(g) and 6.31.

6.8 Considerations for the Selection of Displacement Fields

The description of the displacement fields is one of the most crucial phases of the finite element analysis. Broadly, an assumed displacement field should satisfy two conditions which are discussed below.

(i) Conditions of admissibility.

An admissible function will describe the deformations of a region which was continuous before the deformations in such a way that the region
remains continuous after the deformations. In order to satisfy this condition, it is necessary that

(a) both the inplane and the out of plane displacements across element interfaces be continuous,

(b) transverse slopes across element interfaces be equal.

If conditions (a) and (b) are satisfied the displacement field is geometrically admissible. The condition (b) is seldom satisfied.

Moreover, the displacement fields should be such that each individual displacement be represented independently of all other displacements.

(ii) Criteria of convergence

Ideally the convergence of the solutions towards true solutions should be rapid and monotonic with increase of number of elements in a given domain. In order to accomplish this, it is suggested that in addition to the satisfaction of conditions (a) and (b) of section 6.8(i), the displacement fields should be capable of representing the following two sets of deformations.

(a) Rigid body displacements

Under rigid body displacements, there should be no straining anywhere in the region. This includes direct strains and changes of curvatures. It appears that, though this condition is important, it should not be enforced upon, if the strain-displacement relations themselves are not inherently strain free under rigid body displacements. Most of the simplified versions of the strain-displacement relations of shell theory
are not free of strains under rigid body displacements. If, for such a set of relations, the rigid body modes are satisfied by constraining of the assumed displacement fields, the effect is to increase the constraints of the system artificially. Since the strain displacement relations used here are not entirely free of strain under rigid body displacements, such strain may be expected.

It is shown (|35| and |39|) that even for a set of strain displacement relations which are inherently free of strains under rigid body displacements, it is not possible to account for all the components of rigid body displacements by polynomial expansions of the displacement fields for a curved plate.

(b) Constant strain condition

The assumed displacement fields should account for any constant strains that might develop in parts of the region. However, it may be verified that a curved plate cannot admit a constant change of curvature without accompanying change of direct strains, and vice versa. Since this is so, the constant strain condition is not attainable, on both physical and mathematical grounds, for a curved plate, except the trivial ones which are the rigid body conditions. Over and above these conditions the assumed functions should be symmetrical in x and y if the problem is geometrically symmetrical.

If the displacements and the required derivatives across the element boundaries are continuous, then the finite element method becomes a special case of the method of Ritz in which the continuum is defined in a piecewise manner (see Oliveira |58|). The conditions under which the
solutions in the method of Ritz converge to the true solutions have been mathematically investigated (for a proof see Kantorovich [1, pp.347-357]). Kantorovich [1, pp.258-262] has shown that an arbitrarily close approximation to the actual minimum of the energy integral may be obtained by Ritz's method. Using a number of families of functions a sequence of successively more exact approximations can be obtained. The sufficient condition that the sequence of integrals tends to the true minimum is that the system of families of approximating functions be relatively complete. By relative completeness it is meant that "any admissible function, together with its partial derivatives, may be approximated as closely as one pleases by means of functions of the given families".

The convergence of the finite element method in the theory of elasticity has been studied by relatively few authors. Melosh [55] has given some convergence criteria on which the formulation of the finite element should be based. He claims that satisfaction of the criteria will result in monotonic convergence of the solution (not necessarily to the true solution). Tocher and Kapur [56] have shown that monotonic convergence is not always maintained even when these conditions are satisfied. Melosh has explicitly stated that the derivatives of the displacement need not be continuous and the inclusion of the rigid body modes though desirable does not ensure convergence to exact solution.

Bazeley et al [54] contend that the satisfaction of the constant strain conditions more than anything else will ensure convergence. Also, it is shown by them that the convergence properties of the so-called non-conforming elements are better than the conforming elements for practical
size elements. It is not difficult to see why this should be so. The discontinuities in the assemblage due to the use of the non-conforming elements reduce the stiffness of the overall structure. This will tend to result in smaller frequencies and higher displacements in case of vibrations and statics respectively than with the same number of conforming elements, provided the solutions are not already very close to the true ones. There is no guarantee that the convergence of solutions with non-conforming elements will always be monotonic. Cowper et al [19, pp. 41-42] have shown that the solutions obtained with the non-conforming element of Bazeley et al [54] oscillate with increase of number of elements. Also from [19, pp. 43-44] it may be seen that the solutions with the non-conforming elements are rapidly converging towards some solutions not necessarily the true one (they have crossed the true value once).

Tong and Pian [95] have presented a theoretical development to show the sufficient conditions that will ensure a finite element displacement analysis to converge to the exact displacement solutions when the size of the elements is progressively reduced. The convergence criteria given by them are not any different from those already mentioned for the convergence in the method of Ritz which in the finite element language is conformity. It is interesting to note that Tong and Pian have not mentioned the so-called condition of constant strain postulated by Bazeley et al [54].

In order that the total strain energy (sum of the strain energy of all the elements) be finite, it is required that the highest derivatives of the displacements appearing in the expression for the strain energy be finite. For this, it is required that a displacement be continuous together
with its derivatives up to one less than the highest derivative appearing in the energy integral. In the classical shell theory the energy integral contains \( u', u'', v', v', w', w'' \) and \( w' \). Therefore it is required that \( u, v, w, w' \) and \( w' \) are to be continuous. In the non-classical shell theories where the shearing deformation is included, \( u', u'', v', v', w', w', \psi_x', \psi_x', \psi_y', \psi_y' \) (\( \psi_x \) and \( \psi_y \) are the rotations of the middle surface and are not functions of \( w \)) are contained in the energy integral requiring the continuity of \( u, v, w, \psi_x \) and \( \psi_y \) only.

Another paper on the study of convergence of finite element solutions is due to Johnson and McLay [57]. The authors have claimed to have investigated the foundation of the theory of the finite element method as it applies to elasticity theory. The problem considered was an inplane one which was not typical of the more usual bending problems. A counter example is presented in which a set of functions, admissible to the variational principle, is shown not to converge.

A paper by Oliveira [58] (reference [97] is more mathematical than [58]) seems to be one of the first papers which deals with the question of completeness and convergence of the finite element method. The conclusions are claimed to be 'valid for all the different linear continuous structural theories, i.e. for two and three-dimensional elasticity, as well as for beams, shells and plates'. The criterion for completeness is essentially the same as the one which is presented by Bazeley et al [54], namely the condition of constant strain [58, p.14].

If the second derivatives of the displacements are continuous and finite within an element, then according to Oliveira they are complete provided they fulfil certain conditions. The essence of these conditions is the following.
The expansions of the displacement functions within an element are polynomials. The total number of parameters allowed in these expansions are equal to the total number of degrees of freedom of the element. In these polynomial expansions, the free parameters and the parameters associated with the linear parts of the expansions are completely arbitrary. The non-linear terms of the expansions can vanish independent of the free parameters and the parameters associated with the linear parts of the expansions.

Assuming that these conditions are satisfied, then Oliveira contends that completeness will ensure convergence to the exact solution if the displacements are compatible, that is the element is a conforming one. Then the sufficient condition for convergence is conformity and is similar to the one given by Tong and Pian [95].

The completeness criterion is shown by Oliveira to ensure convergence even when the element is non-conforming, provided the density of the body forces like gravitational forces, magnetic forces, and inertia forces within the elements remain finite irrespective of their size.

The remarks and the findings of Oliveira except in the case of the conforming elements, appear to contradict some of the findings of other investigators. First, the polynomial expansions are accepted practice in finite element analysis, so also the number of parameters allowed in the expansion are generally equal to the degrees of freedom of the element. As such they conform to present practice. However, Cantin and Clough [39] have shown that polynomial expansions alone cannot account for all the
rigid body modes of a curved plate. Again in the case of curved plates, there exist certain relations between the parameters associated with the linear parts of the expansions of the displacements, which are responsible for defining the rigid body modes. Therefore, they cannot be completely arbitrary except in their relative magnitudes.

Again the constant strain conditions, other than the rigid body ones, cannot be generated in a general curved plate because of the nature of the strain-displacement relations, whether the effects of the shearing deformations are included or not. This may be verified from any general linear theory. When shearing deformations are included reference may be made to the strain-displacement relations given by Naghdi [18].

Even though the results using non-conforming elements converge, satisfying the criteria postulated by Oliveira, the results of reference [19] show that their destination is not easily predictable even in the case of flat plates.

From the above discussion it appears that conformity is highly desirable whatever the views of Bazeley et al [54] and Oliveira [58] might be. Also, at least in the case of curved plates, there is no point in trying to incorporate the constant strain conditions except possibly the rigid body ones.

6.9 Assumed Displacement Fields

Utku [59] seems to have made the first attempts to develop a doubly curved triangular finite element. He took the rotations of the middle surface as independent variables (not functions of u, v and w),
together with \( u, v \) and \( w \). In such a formulation the potential functional of the plate contains only first derivatives of \( u, v, w \) and the two rotations. Because of this, continuity of \( u, v, w \) and the two rotations across element boundaries guarantees both the continuity of displacements and normal slopes. With this advantage in hand, Utku prescribed the displacement fields and the rotations with simple linear functions. Since the linear distribution of \( w \) is not compatible with the linear distribution of the rotations, the element would have considerable strain under rigid body displacements. Subsequently these strains were eliminated by inducing fictitious nodal forces. This elimination process complicated the formulation. The curvature \( k_{xy} \) was not considered.

Later, Connor and Brebbia \[40\] used an orthogonal curvilinear co-ordinate system in their doubly curved rectangular element formulation. The third curvature \( k_{xy} \) is also taken into account. The formulation is based on the strain-displacement relations given by Reissner \[60\]. Five degrees of freedom at each of the corner nodes are considered.

Rectangular elements are also derived by Schmit et al \[61\] and Gallagher and Yang \[62\]. Both these elements are designed to study the problems of elastic instability. Consequently, the strain-displacement relations contain nonlinear terms. These elements are derived in the curvilinear co-ordinate systems. None of these elements takes into account the curvature \( k_{xy} \). The element derived by Schmit et al is \( 48 \times 48 \) in size corresponding to the nodal displacements \( u, u', w, u'' \), \( v, v', v'' \), \( w', w'' \). They have used the same interpolation functions to describe the displacements \( u, v \) and \( w \). The interpolation functions were
derived by Bogner et al. [63] and were also given by Butlin and Leckie [65].

Gallagher and Yang [62] have used the same interpolation functions to
describe \( w \) for their 24 x 24 stiffness matrix. This matrix corresponds
to the nodal displacements \( u, v, w, w', w'' \) and \( w''' \). The derivatives of
\( u, v \) and \( w \), in the case of these matrices, are with respect to the curvi-
linear co-ordinates of the surface.

For the inplane displacements \( u \) and \( v \) the following two functions
are usually used for flat plates (see Zienkiewicz [37 p.66]).

\[
\begin{align*}
u &= a_1 + a_2 x + a_3 y + a_4 xy, \\
v &= a_5 + a_6 x + a_7 y + a_8 xy. \\
\end{align*}
\]

These two functions are used by Connor and Brebbia and also by Gallagher
and Yang in the formulation of their curved elements. These functions
maintain displacement compatibility across element boundaries.

The equivalent to the interpolation function mentioned above
for the displacement \( w \) may be written as the following.

\[
\begin{array}{c|cccc}
\text{} & 1 & x & x^2 & x^3 \\
\hline
1 & a_9 & a_{10} & a_{12} & a_{15} \\
y & a_{11} & a_{13} & a_{16} & a_{19} \\
y^2 & a_{14} & a_{17} & a_{21} & a_{22} \\
y^3 & a_{18} & a_{20} & a_{23} & a_{24} \\
\end{array}
\]

In the expansion of \( w \), \( a_j (j = 9 \text{ to } 24) \) is a coefficient of the polynomial
term \( x^p y^q \) (\( p = 0, 1, 2, 3 \) and \( q = 0, 1, 2, 3 \)) formed by the product of row
and column elements at \( a_j \).
The expansion of \( w \) corresponding to the first twelve terms (\( a_9 \) to \( a_{20} \)) of 6.32(c) was originally used by Adini and Clough \( |64| \) for the analysis of flat plates. Connor and Brebbia \( |40| \) have used it for the development of their curved plate element. Though this function maintains displacement continuity, it does not maintain continuity of normal slopes.

It is not always possible to satisfy all the desirable requirements—discussed in the previous section—by the assumed displacement functions. It may be possible to obtain better displacement field descriptions by assuming more and more terms in the expansion of \( \{u\} \). This is severely limited by available computer capacities. Therefore it is intended to use a six degree of freedom per node system, with a rectangular element of four nodes. Thus, only 24 parameters are allowed in \( \{a\} \) in the expansion of \( \{u\} \). The displacements \( u, v \) and \( w \) are prescribed according to equations 6.32 (a-c).

The nodal displacements considered by Connor and Brebbia for their five degree of freedom system are

\[
\{ u \ v \ w \ w' \ w'' \}. \tag{6.33}
\]

For the six degree of freedom system, the nodal displacements are assumed to be

\[
\{ u \ v \ w \ w' \ w'' \ w''' \}. \tag{6.34}
\]

The selection of \( w''' \) as a nodal displacement preserves the symmetry of the derivatives of \( w \) that are considered as nodal displacements. The necessary derivatives of \( w \) are the following.
\[ w' = \begin{array}{c|ccc} 1 & 2x & 3x^2 \\ \hline 1 & a_{10} & a_{12} & a_{15} \\ y & a_{13} & a_{16} & a_{19} \\ y^2 & a_{17} & a_{21} & a_{22} \\ y^3 & a_{20} & a_{23} & a_{24} \end{array}, \quad 6.35(a) \]

\[ w^* = \begin{array}{c|cccc} 1 & 1 & x & x^2 & x^3 \\ \hline 1 & a_{11} & a_{13} & a_{16} & a_{19} \\ 2y & a_{14} & a_{17} & a_{21} & a_{22} \\ 3y^2 & a_{18} & a_{20} & a_{23} & a_{24} \end{array}, \quad 6.35(b) \]

\[ w'' = \begin{array}{c|cc} 2 & 6x \\ \hline 1 & a_{12} & a_{15} \\ y & a_{16} & a_{19} \\ y^2 & a_{21} & a_{22} \\ y^3 & a_{23} & a_{24} \end{array}, \quad 6.35(c) \]

\[ w''' = \begin{array}{c|cccc} 1 & 1 & x & x^2 & x^3 \\ \hline 2 & a_{14} & a_{17} & a_{21} & a_{22} \\ 6y & a_{18} & a_{20} & a_{23} & a_{24} \end{array}, \quad 6.35(d) \]

\[ w'''' = \begin{array}{c|ccc} 1 & 2x & 3x^2 \\ \hline 1 & a_{13} & a_{16} & a_{19} \\ 2y & a_{17} & a_{21} & a_{22} \\ 3y^2 & a_{20} & a_{23} & a_{24} \end{array}, \quad 6.35(e) \]
The properties of the functions 6.32(a-c) will be discussed later. Using these functions and their derivatives the following required matrices are derived.

The stiffness matrix \([K_0(24x24)]\) is derived in Appendix 11.
The stiffness matrix \([K_1(24x24)]\) is derived in Appendix 12.
The mass matrix \([M(24x24)]\) is derived in Appendix 13.
The \([C(24x24)]^{-1}\) matrix is given in Appendix 14.

The co-ordinate system and the node numbering for which these matrices are derived is shown in Figure 19. The order of node numbering of a full plate, when the nodes on the boundaries do not contribute to the nodal equilibrium of the system, is shown in (a) of Figure 20. The node numbering of a quarter of the plate, when advantage of symmetries is taken, is shown in (b) of Figure 20.

6.10 The Properties of the Assumed Displacement Fields

In this section, it is proposed to study the continuity of the assumed displacements \(u, v, w, w', w''\) across element boundaries. For this the functions \(u, v, w, w', w''\) are expanded with \(\{u\}\) as the parameters of expansions along the faces of an element. It is only necessary to select any pair of orthogonal faces. Referring to Figure 19(a) the faces joining the nodes 1-2 and 1-3 are selected. Along the fact 1-3 where \(y=0\), the following expressions for the nodal displacements are obtained.
\[ u = u_1(1 - x/a) + u_3 x/a , \]
\[ v = v_1(1 - x/a) + v_3 x/a , \]
\[ w = w_1(1 - 3x^2/a^2 + 2x^3/a^3) + w_3(3x^2/a^2 - 2x^3/a^3) \]
\[ + w_1'(x - 2x^2/a + x^3/a^2) + w_3'(-x^2/a + x^3/a^2) , \]
\[ w' = w_1'(1 - 3x^2/a^2 + 2x^3/a^3) + w_3'(3x^2/a^2 - 2x^3/a^3) \]
\[ + w_1'(x - 2x^2/a + x^3/a^2) + w_3'(-x^2/a + x^3/a^2) , \]
\[ w' = 6x/a^2 \{w_1(-1 + x/a) + w_3(1 - x/a)\} \]
\[ + w_1'(1 - 4x/a + 3x^2/a^2) + w_3'(-2x/a + 3x^2/a^2) , \]
\[ w'' = 6x/a^2 \{w_1'(-1 + x/a) + w_3'(1 - x/a)\} \]
\[ + w_1'(1 - 4x/a + 3x^2/a^2) + w_3'(-2x/a + 3x^2/a^2) . \]

The expressions for the above functions at the face \( y = b \) may be obtained by replacing the subscript 1 by 2 and 3 by 4 in the expressions
6.36(a-f).

Along the face 1-2 where \( x=0 \), the following expressions for the nodal displacements are obtained.

\[ u = u_1(1 - y/b) + u_2 y/b , \]
\[ v = v_1(1 - y/b) + v_2 y/b , \]
\[ w = w_1(1 - 3y^2/b^2 + 2y^3/b^3) + w_2(3 - 2y/b) y^2/b^2 + \]
\[ w_1'(y - 2y^2/b + y^3/b^2) + w_2'(-y^2/b + y^3/b^3) , \]
\[ w' = w_1'(-6y/b^2 + 6y^2/b^3) + w_2'(6y/b^2 - 6y^2/b^3) \]
\[ + w_1'(1 - 4y/b + 3y^2/b^2) + w_2'(-2y/b + 3y^2/b^2) , \]
\[ w' = w_1'(1 - 3y^2/b^2 + 2y^3/b^3) + w_2'(3y^2/b^2 - 2y^3/b^3) \]
\[ + w_1'(y - 2y^2/b + y^3/b^2) + w_2'(-y^2/b + y^3/b^2) , \]
\[ w'' = w_1'(-6y/b^2 + 6y^2/b^2) + w_2'(6y/b^2 - 6y^2/b^3) \]
\[ + w_1'(1 - 4y/b + 3y^2/b^2) + w_2'(-2y/b + 3y^2/b^2) . \]
Similar expressions for the face \( x = a \) may be obtained by replacing the subscript 1 by 3 and 2 by 4 in the above expressions.

From equations 6.36 and 6.37 it may be seen that the chosen displacements \( u, v, w \) and the slopes \( w', w' \) and \( w'' \) are continuous across the element boundaries. This is because they are functions of \( w \) and its derivatives at nodes lying on the same line.

It may be noticed that for shallow plates the assumed Cartesian co-ordinate system is the same as the curvilinear co-ordinate system used under the same approximations. Therefore, in this system both the local and the global co-ordinate systems are the same and no co-ordinate transformations are required.

6.11 Deduction of Associate Conditions for Constraints

Suppose that \( w = 0 \) along a line \( y = 0 \) is one of the given conditions under which it is required to solve a particular problem. From equation 6.36(c) it may be seen that if \( w \) is set equal to zero at the nodes along the line \( (w_1 = w_3 = 0) \), then \( w \neq 0 \) everywhere along the line. For \( w \) to vanish everywhere an associate condition that \( w' = 0 \) along the same line is required. For the correct solution of the problem the associate conditions, if any, are also to be imposed along with the given conditions. In this section the associate conditions that may arise in this analysis are investigated.

(i) Associate conditions along a line \( y = \text{constant} \).
   (a) \( u = 0 \) and \( v = 0 \) need no associate conditions.
   (b) \( w = 0 \) needs the associate condition \( w' = 0 \).
(c) \( w' = 0 \) needs the associate condition \( w = 0 \).
(d) \( w' = 0 \) needs the associate condition \( w'' = 0 \).
(e) \( w'' = 0 \) needs the associate condition \( w' = 0 \).

(ii) Associate conditions along a line \( x = \) constant.

(a) \( u = 0 \) and \( v = 0 \) need no associate conditions.
(b) \( w = 0 \) needs the associate condition that \( w' = 0 \).
(c) \( w' = 0 \) needs the associate condition that \( w = 0 \).
(d) \( w' = 0 \) needs the associate condition that \( w'' = 0 \).
(e) \( w'' = 0 \) needs the associate condition that \( w' = 0 \).

From the conditions stated above in (i) and (ii) the clamped boundary conditions may be deduced as the following.

(iii) \( x = \) constant clamped.

The necessary boundary conditions are \( u = v = w = w' = 0 \). But these conditions are not sufficient in view of (ii). Therefore, the necessary and sufficient conditions that \( x = \) constant is clamped are

\[ u = v = w = w' = w'' = 0. \]

(iv) \( y = \) constant clamped.

The necessary boundary conditions are \( u = v = w = w' = 0 \). But they are not sufficient. The necessary and sufficient conditions that the edge is clamped are again

\[ u = v = w = w' = w'' = 0 . \]
From (iii) and (iv) it is clear that in this formulation all the nodal displacements on a clamped edge vanish. Therefore, the nodes on a clamped boundary do not contribute to the nodal equilibrium of the whole system. Consequently, the nodes on a clamped boundary need not be considered while assembling the overall stiffness and mass matrices with considerable saving of computer storage. The node numbering when all the edges are clamped is shown in (a) of Figure 20 for sixteen elements.

(v) Intermediate constraints (for \( k_{xy} = 0 \) and \( h = \text{constant} \))

(a) symmetrical modes of vibrations along \( x \).

The necessary conditions are \( u = w' = 0 \) which are satisfied when \( u = w' = w'' = 0 \).

(b) antisymmetrical modes of vibrations along \( x \).

The necessary conditions are \( v = w = 0 \) which are satisfied when \( v = w = w' = 0 \).

(c) symmetrical modes of vibrations along \( y \).

The necessary conditions are \( v = w' = 0 \) which are satisfied when \( v = w' = w'' = 0 \).

(d) antisymmetrical modes of vibrations along \( y \).

The necessary conditions are \( u = w = 0 \) which are satisfied when \( u = w = w' = 0 \).

The node numbering for quarter of a plate when the advantage of symmetry is available is shown in (b) of Figure 20. The clamped boundary conditions and the conditions of intermediate constraints are summarised in Figure 21 for \( k_{xy} = 0 \) and \( h = \text{constant} \).
(vi) The intermediate constraints derived in (v) are true only when \( k_{xy} = 0 \) and \( h = \text{constant} \). For a surface like a hyper whose surface equation is given by \( z = c_0 \cdot xy \) where \( c_0 \) is a constant, the intermediate constraints derived above are not applicable. Doubly curved surfaces like the hyper have \( k_x = k_y = 0 \) and \( k_{xy} = c_0 \). For such surfaces the intermediate constraints are as follows, when \( h = \text{constant} \).

(a) Symmetrical modes of vibrations along \( x \).

The necessary conditions are \( v = w' = 0 \), which are satisfied when \( v = w' = w'' = 0 \).

(b) Antisymmetrical modes of vibrations along \( x \).

The necessary conditions are \( u = w = 0 \) which are satisfied when \( u = w = w'' = 0 \).

(c) Symmetrical modes of vibrations along \( y \).

The necessary conditions are \( u = w' = 0 \) which are satisfied when \( u = w' = w'' = 0 \).

(d) Antisymmetrical modes of vibrations along \( y \).

The necessary conditions are \( v = w = 0 \) which are satisfied when \( v = w = w' = 0 \).

For static problems with symmetry, only conditions (a) or (c) or both (a) and (c) are needed. A surface for which \( k_x \neq 0 \), \( k_y \neq 0 \) and \( k_{xy} \neq 0 \) and also \( h \) is not a constant, the functional relationships between \( u, v \) and \( w \) are too complicated for a simple formulation of the conditions of the intermediate constraints. These conditions are summarised in Figure 22.
6.12 Results and Discussions

The properties of the element under rigid body displacements may be investigated by solving the unconstrained system equations. A plate with different curvatures is analysed, a 2 x 2 mesh being used for better definition of the modes. The results are presented in Table 13. For the flat plate, four of the rigid body modes are non-straining (zero eigenvalues) and the other two are very nearly so (the eigenvalues for these two frequencies are very small). The curved plates without twist have only three non-straining rigid body displacements and the fourth is also almost non-straining. The other two rigid body displacements are definitely straining. The doubly curved plate, with twist, has four completely non-straining displacements, whereas the other two are definitely straining. Therefore, it may be concluded that the element is nearly free of strains under rigid body displacements only when all curvatures are zero. Otherwise, there may be straining under two of the rigid body displacements. These two straining displacements are the rotations about the x and y co-ordinate axes.

The intermediate constraints and the associate conditions deduced in section 6.11 were verified by computing the frequencies of a plate with and without the application of the intermediate constraints. The results are presented in Table 14 and they are found to be identical.

In Table 15, the frequencies of symmetrical modes of vibration of a flat plate are compared with those obtained by Warburton's \( |28| \) formula. The frequencies of the lower modes are smaller than those of Warburton. Since the solutions of Warburton are fractionally higher than the exact ones, it may be concluded that the finite element solutions are more accurate.
It may be observed, from the table, that a larger number of elements is needed for more accurate solutions for the higher modes. The two lowest frequencies obtained with 9 elements are lower than the corresponding Warburton solutions. In Table 16 the lowest frequencies of the same clamped plate with 9 elements (3 x 3 mesh in quarter of the plate) are compared with the corresponding solutions of Warburton. The difference between the solutions is very small. The same plate is also analysed by Mason [66]. The results, given by Mason for the clamped plate, are for six degrees of freedom per nodal point. The transverse displacement $w$ and its derivatives are considered as nodal unknowns. The system developed in this section is effectively a four degrees of freedom system per nodal point for a flat plate. A direct comparison of the two sets of results, however, shows excellent agreement.

From Table 17 and Table 18 it may be seen that the frequencies of a clamped singly curved rectangular plate monotonically decrease with increase of the number of elements. The mesh patterns are shown in Figure 23. The lowest values for the first symmetrical mode frequency, and for the frequency of the first symmetric-antisymmetric mode obtained by the method of Kantorovich for the same plate, are 889.76 Hz and 965.73 Hz respectively. It may be seen from the tables that the solutions with the finite element method are gradually converging towards similar solutions with increase of number of elements.

The first 32 frequencies of the plate are given in Table 19. Mode shapes of some of the higher frequencies are found to be degenerate, though they were not expected since the aspect ratio of the plate is $3/4$. Also
they may not be expected in view of Webster's [32] results. Positive proof that degenerate modes do exist even in curved rectangular plates will be provided in the ninth chapter. The mode shapes of these degenerate modes are shown in Figure 24 and 25. However, there is a possibility that all of the degenerate modes shown in Figure 24 and 25 are not degenerate but some of them just require more definition.

In Figure 24, the mode of frequency 3103.02 Hz may not be a degenerate mode but may be (4,3) mode. Similarly, in Figure 25, the modes of frequencies 5099.42 Hz, 6193.36 Hz and 6384.39 Hz may be regular modes with nodal patterns (2,7),(6,1) and (2,8) respectively. Since they are of high frequency modes, doubts may be cast on the accuracy of the frequencies. Admittedly, these frequencies together with their mode shapes may be more refined by using more numbers of elements. However, the degeneracy of most of the modes will be there. Even now there are modes of higher order frequency than these degenerate modes with regular mode shapes. Another point worth observing in Table 19 is that the degenerate modes tend to appear in groups in between the regular ones.

Also it may be observed that the second and the third frequency have the same m and n (1,3) of w. This confirms our findings in Chapter III using an entirely different approach. However, it is worth mentioning that the analysis presented in Chapter II and III is much more convenient to explain their existence than the finite element method. Probably except a physical explanation, the finite element method will not be able to explain them. Also, it may be noted that only two of them could be found. It may be mentioned that for the
lower frequency the two interior nodal lines are much nearer to the edges of the plate, whereas for the higher frequency they are nearer to the centre of the plate. Their existence and their mode shapes will also be confirmed in the ninth chapter. It may also be noted that the mode shapes of u and v are different in both the cases. For the lower one n of u is greater than the n of u of the higher one whereas for the n of v the opposite is happening. The mode shapes of u, v and w satisfy the boundary conditions together with the conditions of shape compatibility discussed in Chapter III.

In Table 20 and Table 21 some frequencies of the same clamped singly curved rectangular plate under the action of biaxial tensile forces are given. In these cases also, the frequencies show the solutions to be rapidly converging with increase in the number of elements. The lowest symmetric-antisymmetric mode frequency obtained for the same plate under the same system of bi-axial tensile forces was 1365.57 Hz with the method of Kantorovich. The lowest symmetrical mode frequency obtained by the method of Kantorovich was 1231.79 Hz. Comparing these solutions with the finite element solutions of 1363.80 Hz. and 1231.89 Hz respectively, with only 20 elements in a quarter of the plate, the following may be concluded. The present element may be relied upon to study vibrations of curved plates in the presence of applied membrane stresses. Also, due to the increase of stiffness in the presence of these membrane stresses, it seems that the solutions become slightly less sensitive to the size of the elements, than in the absence of these stresses.
In Table 22 the first two frequencies of the plate together with their mode shapes are given when various bi-axial tensile and compressive stresses are applied. Plenty of degenerate modes even at the lower range of frequencies are observed. Even one of them is the lowest frequency mode at 1500 lb/in$^2$ tension. This is an unexpected result. The lowest frequencies are converted into a non-dimensional frequency parameter $\rho \omega^2 (1 - \nu^2)/(Egk_y^2)$ which is the same as $\Delta_\omega$ used up to Chapter III. These frequency parameters are plotted against the hoop stresses in Figure 26. The frequency parameters for the simply supported plate using an exact solution given in Chapter IV are also plotted in the same figure. The variation of the frequency parameter with the variation of the stresses is essentially a series of straight lines both for the compressions and the tensions and also for both the boundary conditions. However, for the clamped case, there seems to be slight non-linearity near the critical stress. Incidentally, in this bi-axial stress system the critical stress for the clamped plate (nearly 2000lb/in$^2$) is just about twice the critical stress (about 1000lb/in$^2$) of the simply supported plate. Since we are not interested here either in buckling or in post buckling behaviour the study is not oriented towards them. However, it is being shown that the present formulation may be used to find whether a plate will be able to withstand a given system of stresses without buckling. The test is to find the eigenvalues. If the lowest eigenvalue is negative or the frequency is imaginary, then the plate will buckle, otherwise not.
In Table 23, some frequencies of a clamped square spherical plate are presented. The \( m, n \) of \( w \) are given in the parentheses. Those without the \( m, n \) quoted are the degenerate modes. The nodal patterns of the symmetrical and the antisymmetrical modes of vibrations are given in Figure 27 and Figure 28 respectively. It may be observed that the solutions maintain absolute reciprocity; that is, the frequencies for \( m,n \) mode are the same as the frequencies of the \( n,m \) mode (column 3 of the table). The 6th mode in Figure 27 might need some more definition.

Table 24 is a table of comparisons between some of the frequencies of the above-mentioned spherical plate computed by the method of Kantorovich, and computed by the method of finite elements. It is proper to compare only the like modes. Since the Kantorovich method is not formulated to account for the modes whose nodal lines are not parallel to the edges, it is not possible to compare most of the frequencies. The differences between the frequencies are reasonably small. It may be observed that the lowest symmetric mode is actually a degenerate one. For the same mode, the frequency obtained by the finite element method is 860.33 Hz, compared to the frequency obtained by the Kantorovich method of 872.67 Hz. The difference between the two frequencies is 1.43 per cent. Also, it may be observed that the 3,1 mode frequency is not the same as the 1,3 mode frequency, by the method of Kantorovich, which in fact should be the same. The reason for this discrepancy is that the prescribed mode shapes are not the exact ones. The 3,1 mode frequency 1092.02 Hz by the method of Kantorovich should not be assumed to be the second symmetric mode frequency and should not be compared with the second degenerate mode frequency of 952.08 Hz.
Even when the frequencies are comparable, the frequencies obtained by the method of Kantorovich are fractionally higher than the corresponding frequencies by the finite element method. This is due to the fact that the apex of the plate is a singular point and the nodal line pattern around it is very complicated, and not easily prescribable in the method of Kantorovich.

Changing the sign of $k_y$ an anticlastic plate may be obtained. Its frequencies and modes are given in Table 25. The modes of the first eight frequencies are shown in Figure 29 and Figure 30 for this anticlastic plate. They are in general quite different from those of the spherical plate. The complete reciprocity of the frequencies and their mode shapes are maintained. The 6th mode shown in Figure 29 might need more definition.

Some of these frequencies obtained by the methods of finite element are compared with the corresponding frequencies obtained by the method of Kantorovich (Table 26). The order of the percentage difference is similar to the spherical plate. The frequencies obtained by the method of Kantorovich are in general higher than those obtained by the method of finite element.

In Table 27 the frequencies of the plate when $k_x = k_{xy} = 0$ are given. Since $k_x = 0$ and $k_y \neq 0$, the plate though square in plan has lost its absolute symmetry and the definition of being geometrically square. An immediate consequence is the disappearance of most of the degenerate modes. Only 5 out of the first 32 frequencies are found to be degenerate and their nodal patterns are shown in Figure 31. Compared with the rectangular plate frequencies given in Table 19 this number appears to
be strangely low. The first mode of Figure 31 seems to lack some definition. Another consequence is the loss of the reciprocity of the frequencies and the mode shapes. That is why there are four columns in Table 27 instead of three.

The frequencies and their mode shapes for this case are compared with the Kantorovich solutions in Table 28. Since for this case, the Kantorovich solution is quite reliable, it is possible to have some conclusive idea about the relative accuracy of the finite element analysis. It may be seen that the percentage difference is reasonably small. Unlike the other two cases, the frequencies by the method of Kantorovich are smaller than those by the finite element method. This is expected, because the straight beam vibration functions are adequate representation of the solutions in the straight direction. However, in the case of the doubly curved plates the straight beam vibration functions are fairly inadequate representatives of the modes in any of the directions, in the Kantorovich method. On the other hand, the level of accuracy of the finite element method is more or less constant for a certain number of degrees of freedom.

6.13 Conclusions

The finite element analysis presented here may be relied upon for the analysis of all thin plates with rectangular boundaries in the presence of applied membrane stresses or in their absence. It is shown that though the element cannot account for all the rigid body displacements, the results converge to known solutions. Also, it is shown that convergence
to the known solutions is monotonic. The results obtained with the element confirm some findings using the method of Kantorovich like the possibility of existence of more than one frequency for a given m and n of w. The element is found to be convenient to predict modes of vibrations of both singly and doubly curved plates even when the nodal lines are not parallel to the edges of the plates. Also, the present formulation may be used to predict critical stresses over which buckling may occur in plates. The element may be used to study vibration characteristics of plates in presence of applied inplane shear stress also.
CHAPTER VII
APPROXIMATE ANALYSIS OF RANDOM VIBRATIONS
OF CURVED PLATES - AND STATICS OF CURVED PLATES

7.1 Introduction

A structure may experience various types of loading during its lifetime. These loads may be divided into two broad classes. They are (a) static and (b) dynamic. In general, the criteria for design of a structure which is expected to carry only static loading are different from the one which is expected to carry dynamic loading. Also, it is true that more or less all structures in their lifetime experience both static and dynamic loadings.

The direction and the magnitude of the static loads on a proposed structure may be estimated fairly accurately. Therefore, it becomes relatively easier to design the structure to carry the static load safely. On the other hand, the nature and magnitude of the dynamic loads on a structure are difficult to estimate.

Jet noise, turbulence in air and the uneven surfaces of airfields are major sources of random excitation in aerospace engineering. In civil engineering, the tall buildings and chimneys experience random excitations from gusty winds. Dams and other massive structures situated in earthquake zones have to be designed to withstand severe earthquake shocks. The cumulative effect of random excitation on a structure is to cause fatigue leading to serious structural damage and sometimes to failure.
The problems of random vibrations are studied by a combination of structural and statistical methods. The theory of probability is widely used for systematic presentation and analysis of random data such as the time history of pressure, velocity, acceleration and other related quantities, which may be called the generalised forces. To the action of these random forces the structure responds randomly. The random response of the structure is required to be predicted in statistical sense. Due to this random response the structure fatigues. The ultimate aim of random analysis of a structure is to predict probability of failure of the structure due to fatigue.

7.2 Analysis of Response to Random Excitation

The random response of a two dimensional spatial system to random excitation may be analysed by a normal mode approach given by Powell [102 and 103]. A review of the method as applied to actual dynamical systems is given by Clarkson [51 and 101]. Matrix formulation of the problem using the normal mode approach may be found in Kouskoulas and Hurty [104]. The computation of response using the equation developed by Powell, for all the modes having frequencies in the bandwidth of excitation, is prohibitively lengthy. Therefore, an alternative method, avoiding the use of the normal modes, is given. The method is more or less similar to the one given by Olson [105] where he separates the final solutions to real and imaginary parts. Later Lindberg et al [106] have used the method given by Olson to study the random response of a multi-bay panel system using finite elements.
Theoretically, an infinite number of independent co-ordinates are necessary to describe the motion of the continuous system. However, using the finite element method of analysis the continuous system is reduced to a discrete system, requiring only a finite number of independent co-ordinates to describe the motion of the system. Let these independent co-ordinates be represented by \( \{Q(t)\} \), where the \((t)\) emphasises the dependence of these co-ordinates on time. The elements of \( \{Q(t)\} \) represent the basic nodal unknowns of the whole system like the displacements and the rotations. If \( [\bar{A}] \) and \( [\bar{M}] \) are the overall stiffness and the overall mass matrix of the system respectively, then the elastic and the inertia forces of the system are given by \( [\bar{A}] \{Q(t)\} \) and \( [\bar{M}]\ddot{\{Q(t)\}} \) respectively \((\dot{} = \frac{d}{dt})\).

The vibrating systems usually contain frictional forces which are very complex in nature. In a great many practical problems, the frictional forces are assumed to be proportional to the velocity \( \dot{Q}(t) \) of the system. Frictional forces of this type are known as viscous damping forces. The retarding forces due to the viscous damping are then given by \( [\bar{C}]\dot{Q}(t) \). In practice, it is extremely difficult to find the coefficients of the matrix \( [\bar{C}] \). It is generally assumed to be symmetric and its size is proportional to that of \( [\bar{A}] \) and/or \( [\bar{M}] \). Equating the sum of the internal forces to the external forces \( \{L(t)\} \) the following system equation is obtained.

\[
[\bar{M}]\ddot{\{Q(t)\}} + [\bar{C}]\dot{\{Q(t)\}} + [\bar{A}]\{Q(t)\} = \{L(t)\} \quad \text{(7.1)}
\]
The elements of \( \{ L(t) \} \) will depend upon the basic nodal unknowns of the system. In this case, they will be forces and moments. The equation 7.1 represents the general system equation. If \( \{ L(t) \} = \{ \mathbf{0} \} \), then the system represents free vibrations with damping; if both \( \{ \mathbf{C} \} = [0] \) and \( \{ L(t) \} = \{ \mathbf{0} \} \), it represents free vibrations; if \( \{ L(t) \} \) is static \( \{ Q(t) \} \) is static; if \( \{ L(t) \} \) is transient \( \{ Q(t) \} \) is transient, and if \( \{ L(t) \} \) is random \( \{ Q(t) \} \) is random. The case when \( \{ L(t) \} \) is random is of immediate interest to us and accordingly \( \{ L(t) \} \) in equation 7.1 is assumed random.

Since \( \{ L(t) \} \) and \( \{ Q(t) \} \) are random, there are no periodic components in them, and therefore direct harmonic analysis is not possible. Again, since the random process, say \( \{ L(t) \} \) spans between \( t = -\infty \) and \( t = \infty \) its Fourier transform \( \{ \mathcal{L}(if) \} \) is defined mathematically as \( \{ \mathcal{L}(if) \} = \int_{-\infty}^{\infty} \{ L(t) \} e^{-2\pi i ft} \, dt \) where \( f \) is frequency in Hz and \( i = \sqrt{-1} \). This integral cannot converge to a definite value and therefore cannot be defined sensibly in a physical way. This difficulty is avoided by assuming the record to extend between \( t = -T_1/2 \) and \( t = T_1/2 \) only. Outside this interval the process is assumed to be zero. This new process \( \{ L(t) \}_{T_1} \) is identical to the original process \( \{ L(t) \} \) within the interval of definition and its Fourier transform is definable. This new function \( \{ L(t) \}_{T_1} \) is then used with the limits of integration between \(-T_1/2 \) and \( T_1/2 \). Subsequently the subscript \( T_1 \) will be dropped. Also, since the integral between \(-T_1/2 \) and \( T_1/2 \) is the same as the integral between \( -\infty \) and \( \infty \), the limits of integration with respect to \( t \) will be retained as \( -\infty \) and \( \infty \). However, the integral with respect to \( f \) will still be between \( -\infty \) and \( \infty \) whenever necessary.
The Fourier transform \{\lambda(if)\} of \{L(t)\} may be written as
\[
\{\lambda(if)\} = \int_{-\infty}^{\infty} \{L(t)\} e^{-2i\pi ft} \, dt ,
\]
and its inverse transform may be written as
\[
\{L(t)\} = \int_{-\infty}^{\infty} \{\lambda(if)\} e^{2i\pi ft} \, df .
\]
\{\lambda(if)\} is known as the Fourier spectrum of the impressed forces.

Similarly the Fourier transform of the output \{Q(t)\} and its inverse may be written as follows
\[
\{q(if)\} = \int_{-\infty}^{\infty} \{Q(t)\} e^{-2i\pi ft} \, dt ,
\]
\[
\{Q(t)\} = \int_{-\infty}^{\infty} \{q(if)\} e^{2i\pi ft} \, df .
\]

Differentiating equation 7.5 with respect to \(t\)
\[
\{\dot{Q}(t)\} = 2i\pi \int_{-\infty}^{\infty} \dot{f}\{q(if)\} e^{2i\pi ft} \, df ,
\]
\[
\{\ddot{Q}(t)\} = -4\pi^2 \int_{-\infty}^{\infty} f^2\{q(if)\} e^{2i\pi ft} \, df .
\]

Substituting from 7.3, 7.5, 7.6 and 7.7 into equation 7.1 the following equation is obtained, which is in the frequency domain.
\[
\int_{-\infty}^{\infty} \left[ -4\pi^2 f^2[I]\right] + 2i\pi f[C] + [A]\{q(if)\} e^{2i\pi ft} \, df = \int_{-\infty}^{\infty} \{\lambda(if)\} e^{2i\pi ft} \, df .
\]

If equation 7.8 is to hold for all time \(t\) then the integrands on both sides of this equation must be equal, yielding
\[
\left[ -4\pi^2 f^2[I] + 2i\pi f[C] + [A]\right]\{q(if)\} = \{\lambda(if)\} ,
\]
or
\[
\{q(if)\} = [Z(if)]^{-1} \{\lambda(if)\} ,
\]
where \([Z(if)] = [-\omega^2[M] + i\omega[C] + [A]]\) is the complex impedance of the system. Equation 7.10 establishes the relationship between the Fourier spectra \(\{q(if)\}\) of the output and the Fourier spectra \(\{\lambda(if)\}\) of the input. Knowing \(\{q(if)\}\), the displacements \(\{Q(t)\}\) at the nodes may be determined from equation 7.5 which is

\[
\{Q(t)\} = \int_{-\infty}^{\infty} [Z(if)]^{-1} \{\lambda(if)\} e^{2i\pi ft} df . \tag{7.11}
\]

At any other points the displacements may be found from their prescribed variation.

The coefficients of the viscous damping matrix \([C]\) are usually extremely difficult to find. The usual practice is to use a damping matrix which is proportional to the stiffness matrix (Thompson [49], p.72) so that

\[
[Z(if)] = [-\omega^2[M] + i\eta[A] + [A]] \tag{7.12}
\]

where \(\eta\) is known as a loss factor. The form 7.12 of \([Z(if)]\) is used here.

The output spectral density \([S_o(if)]\) is given by [91]

\[
[S_o(if)] = \lim_{T_1 \to \infty} \frac{2}{T_1} [\{q(if)\} \{q*(if)\}^T] , \tag{7.13}
\]

where \(\{\cdot\}\) and \([\cdot]\) denote complex conjugates and \(\cdot^T\) and \([\cdot]^T\) denote transpose. Then \(\{q*(if)\}\) is

\[
\{q*(if)\} = \int_{-\infty}^{\infty} \{Q(t)\} e^{2i\pi ft} dt . \tag{7.14}
\]

Substituting from 7.10 into 7.13 the following equation is obtained

\[
[S_o(if)] = \lim_{T_1 \to \infty} \frac{2}{T_1} [Z(if)]^{-1} \{\lambda(if)\} \{\lambda*(if)\}^T \left[[Z*(if)]^{-1}\right]^T . \tag{7.15}
\]
In this equation \([Z(\text{if})]\) is not random and so that the time average is not applicable to \([Z(\text{if})]\) and \([[[Z(\text{if})]^{-1}]^T]\). Thus, applying the time average to the rest of the equation

\[
\lim_{T_1 \to \infty} \frac{2}{T_1} \{\lambda(\text{if})\}{\lambda^*(\text{if})}^T = [S_\lambda(\text{if})] , \quad 7.16
\]

where \([S_\lambda(\text{if})]\) is the spectra of the input forces. Introducing \([S_\lambda(\text{if})]\) from equation 7.16 into equation 7.15 the following equation is obtained

\[
[S_\phi(\text{if})] = [Z(\text{if})]^{-1} [S_\lambda(\text{if})] [[Z^*(\text{if})]^{-1}]^T . \quad 7.17
\]

Equation 7.17 gives the complex output spectra in terms of the complex impedance of the system and the complex input spectra. The diagonal elements of \([S_\phi(\text{if})]\) are all real and they represent the output power spectral densities, whereas the off-diagonal elements are all complex and they represent the cross-spectra of the output. Moreover, it may be noted that the elements \(s_{ij}\) of \([S_\phi(\text{if})]\) are complex conjugates of \(s_{ij}(\text{if} \neq j)\). Similarly the diagonal elements of \([S_\lambda(\text{if})]\) are all real and they represent the power spectral densities of the input, whereas the off-diagonal elements are all complex and they represent the cross-spectra of the exciting forces. The success of the predictions by the analysis developed above will greatly depend upon the accuracy with which \([S_\lambda(\text{if})]\) is determined. Two extreme cases of the input spectra are as follows.
(a) When the exciting forces are uncorrelated, then the off-diagonal elements of $[S_1(\text{if})]$ are all zeros, whereas the diagonal elements are all real and constant, with magnitudes equal to the power of the exciting forces at the frequency. When the spectral densities of the exciting forces are identical and equal to $S$, then in this uncorrelated case $[S_1(\text{if})] = S[I]$ where $[I]$ is a unit matrix.

(b) When the exciting forces are fully correlated, then $[S_1(\text{if})]$ is a matrix of constant real elements. When the spectral densities of the exciting forces are identical and equal to $S$, then in this fully correlated case $[S_1(\text{if})] = S[I]$ where $[I]$ is a matrix of unit elements.

But in the most general case $[S_1(\text{if})]$ depends not only on the frequency $f$ but also on space and on time difference $\tau$. Therefore, the determination of $[S_1(\text{if})]$ for the most general case is quite difficult. Sometimes, it is determined experimentally and sometimes by prescribing suitable variations of the input in time and space.

Since $u$, $v$, $w$, $w'$, $w'$ and $w''$ are assumed as basic nodal unknowns, the diagonal elements of $[S_0(\text{if})]$ will give the power spectra of them at each node. The off-diagonal elements will give the cross-spectra of these displacements. The spectra of the stresses will have to be obtained by further differentiating these displacements. It will be shown later in this chapter that stresses obtained in this manner are in general discontinuous across element boundaries.

The direct method developed for the complete description of the response is not quite economical from the computational time point of
view, since the inversion of the complex $[Z(i\omega)]$ requires a large amount of computing time. However, if the eigenvalues of the system are known, then the method may be applied advantageously. Computing the responses at a few of the lowest eigenvalues, it may be determined whether the major part of the response results from the contribution of a few of the lowest frequency modes (the eigenfunctions are not necessary). If not, the use of the method will be expensive. Under such a situation, a much more simplified design method is required. In what follows, we discuss such a simple design method.

7.3 Design Method (Single Mode Response)

Clarkson [51 and 101] has given the formulas for the mean square displacements $y^2(t)$ and the mean square stress $\sigma^2(t)$ of a two-dimensional continuum by greatly simplifying the general theory given by Powell. Formulas identical to the ones given by Clarkson, were derived by Miles [48] from the consideration of single degree of freedom system.

The major simplifying assumptions used by Clarkson are

(a) the major part of the response results from the contribution of one predominant mode;

(b) the pressure field is random in time but uniform in space.

According to Lin [50], apparently Miles also considered a pressure field which was random in time but nearly uniformly distributed over the panel. With these limitations, Miles gave the following expressions for the mean square displacement and the mean square stresses respectively.
\[ y^2(t) = \frac{\pi}{2\eta} \omega \: S_I(\omega) \: y_o^2, \]
\[ \sigma^2(t) = \frac{\pi}{2\eta} \: f \: S_I(f) \: \sigma_o^2. \]

In these equations \( S_I(\omega) \) and \( S_I(f) \) = the power spectral densities of the input pressure at the \( \omega \) and \( f \) respectively,
\( \eta \) = loss factor.
\( y_o \) and \( \sigma_o \) are the displacement and stresses at the point of interest due to a unit uniformly distributed static load. From equation 7.18(b) the r.m.s. value of the stress \( \sigma(t) \) is

\[ \sqrt{\sigma^2(t)} = \sqrt{\left(\frac{\pi}{2\eta} \: f \: S_I(f)\right) \: \sigma_o}. \]

The dimension of \( S_I(f) \) in equation 7.18(b) is \((\text{lb/} \text{in}^2)^2/\text{Hz}\) if the stress \( \sigma(t) \) is in \( \text{lb/} \text{in}^2 \). In aeronautical engineering the pressure is usually given in dB but in structural designs it is given in \( \text{lb/} \text{ft}^2 \). Therefore, when the pressure is given in \( x_p \) dB, then the pressure \( p_s \) in \( \text{lb/} \text{ft}^2 \) that is to be used in equation 7.18(c) to give \( \sigma(t) \) in \( \text{lb/} \text{in}^2 \) is as follows. The base pressure level is given by 0.0002 dyne/cm² = 4.19 \times 10^{-7}\text{lb/ft}^2.

Then
\[ x_p = 20 \log_{10} \left( \frac{p_s}{4.19 \times 10^{-7}} \right) \text{ dB} \]
\[ p_s = 4.19 \times 10^{(0.05x_p - 7)} \text{ lb/ft}^2. \]

Then
\[ \sqrt{\sigma^2(t)} = \sqrt{\frac{\pi}{2\eta} \: f^2 \: p_s \: \sigma_o / 144} \text{ lb/in}^2. \]

The equation 7.18(d) is the usual equation which is used to compute the r.m.s. stress when the pressure is given in \( \text{lb/} \text{ft}^2 \).
Thus, in order to use the simple design method, it is necessary to know the static deflections and stresses. Therefore, the next stage will be to discuss the static analysis of curved plates.

7.4 Static Analysis of Curved Plates

Since the stiffness matrix is already developed in Chapter VI, it is convenient to use the finite element method for the analysis of curved plates under static loading also. Since we are interested only in the uniformly distributed normal load, the load vector corresponding to the basic displacements is as follows

\[
\begin{bmatrix}
  u \\
  v \\
  w \\
  w' \\
  w''
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  0 \\
  0 \\
  p_z \\
  0 \\
  0
\end{bmatrix}
\]

where \( p_z \) is the intensity of the normal surface load.

There are two different ways in which the load may be assumed to be applied to the plate. One of them is called the consistent load approach and the other is called the discrete load approach.

The consistent load approach is the more accurate of the two. The load is assumed to be distributed in some way over the surface. Then the consistent load matrix may be visualised as the load matrix that transfers the surface loads to the nodes in a consistent way. The following is the derivation of the consistent load matrix.
7.4(a) The Consistent Load Matrix

The work done by the surface loads in producing the displacements is

\[
\int_0^a \int_0^b \{u \ v \ w \ w' \ w''\} \begin{bmatrix} 0 \\ 0 \\ p_z \\ 0 \\ 0 \end{bmatrix} \, dx \, dy
\]

\[
= \int_0^a \int_0^b \{v\}^T \{L\} \, dx \, dy.
\]

\[
= \int_0^a \int_0^b \{a\}^T [V(x,y)]^T \{L\} \, dx \, dy
\]

\[
= \{u\}^T [\{C\}^{-1}]^T \int_0^a \int_0^b [V(x,y)]^T \{L\} \, dx \, dy. \quad 7.20
\]

The matrix \(\{v\}\) defines the nodal displacements in terms of the variable displacement field description matrix \([V(x,y)]\) which in turn may be derived from matrix \([P(x,y)]\) of Chapter VI. \(\{L\}\) is the load matrix which is assumed to be constant here, but may be given any variation; for example, instead of \(p_z\) one can write \((1 + x + y + xy + \ldots) p_z\). Note the difference between \(\{v\}\) and \(\{u\}\). \(\{v\}\) defines the nodal displacements assumed to be unknowns whereas \(\{u\}\) defines all the nodal unknowns of an element.

The matrix

\[
\{L\} = \int_0^a \int_0^b [V(x,y)]^T \{L\} \, dx \, dy \quad 7.21
\]

may be called the consistent load matrix. This matrix is given in Appendix 15. Like the overall stiffness and the overall mass matrix,
in this case the overall load matrix is to be obtained by assembling the load matrices of all the elements.

7.4(b) Discrete Load Matrix

Instead of deriving the consistent load matrix, the load corresponding to each nodal displacement may be prescribed. Suppose a uniformly distributed load $p_z$ over an element is given. If $(a,b)$ is the area of the element then the equivalent discrete load at each node corresponding to displacement $w$ is $p_z ab/4$. Similarly, the corresponding contribution of all the elements (maximum 4 and minimum 1 in this case) meeting at that node are added arithmetically giving the total load. When only concentrated loads are applied, and the nodal points are so chosen that the loads act on them, then the discrete load matrix is preferable. Also for loadings of widely varying magnitudes which cannot be defined mathematically conveniently, the discrete loading is again preferable.

7.5 Intermediate Constraints

In the case of vibrations, wherever applicable, the dimensions of the problem could be reduced by applying intermediate constraints. The problem of free vibrations could be solved in such cases by breaking it up into four problems. In the case of statics one set of loading will uniquely define a set of displacements, if rigid body motions are prevented. Therefore, the symmetry of the loading is an additional consideration. Except for simple type of loading, in general it is not possible to take
advantage of symmetry. For uniformly distributed loading, if the other conditions are suitable, then the plate may be assumed to be symmetrically deformed.

7.6 Calculation of Displacements and the Stresses

The load displacement equation may be written as

\[ [\bar{A}] \{d\} = \{L_S\} \]  \hspace{1cm} 7.22

where \([\bar{A}]\) is the overall stiffness matrix and \(\{d\}\) is the column of all the unknowns. These two matrices are defined adequately in Chapter VI. \(\{L_S\}\) may be defined as the overall load matrix which may be obtained either the consistent or the discrete way. However, the dimension of \([\bar{A}]\) and the lengths of \(\{d\}\) and \(\{L_S\}\) are equal. The system 7.22 is necessarily singular reflecting the fact that \(\{L_S\}\) will be unaltered if small arbitrary rigid body displacements are given to the system. When the system is sufficiently constrained to prevent all possible rigid body displacements, the system 7.22 is constrained to the following system which is necessarily non-singular.

\[ [\bar{A}_1] \{d_1\} = \{L_{S1}\} \]  \hspace{1cm} 7.23

Since \([\bar{A}_1]\) is non-singular

\[ \{d_1\} = [\bar{A}_1]^{-1} \{L_{S1}\} \]  \hspace{1cm} 7.24

where \(\{d_1\}\) contains the required solutions.

It may be seen that for the calculation of the stresses the derivatives \(u', u'', v', v'', w', w''\) and \(w''\) are also required. Since, in
this formulation, they are not used as nodal displacements the vector \( \{d_1\} \) does not contain them. Therefore, they have to be obtained by further differentiation of \( u, v \) and \( w \). Usually, these higher derivatives are not continuous across element boundaries. In general, for this reason, the stresses are not continuous across element boundaries. Sometimes this difficulty is avoided by computing the stresses at the centre of the elements. This obscures the real state of the stresses and their discontinuities. Also, like clamped boundaries where the stresses are of considerable interest, may not be obtained if they are calculated at the centre of elements.

The properties of the elements are obtained on the basis of the co-ordinates of the centre of the elements. However, the thickness and the curvatures at the nodes have to be used when the stresses are computed at the nodes.

Though the strain matrix is used in deriving the element stiffness matrix, it is not written down explicitly. At any point \( x, y \) on the surface, preferably a node or the centre of an element, the equation 6.17 of Chapter VI is

\[
\{\varepsilon\} = [\mathbf{D}] \cdot [\mathbf{P}(x,y)] \cdot \{u\} \\
= [\mathbf{P}^*] \cdot \{u\} , \quad \text{7.25}
\]

and from equation 6.8 of Chapter VI

\[
\{N\} = [\mathbf{E}^*] \cdot \{\varepsilon\} \\
= [\mathbf{E}^*] \cdot [\mathbf{P}^*][\mathbf{C}]^{-1} \cdot \{u\} . \quad \text{7.26}
\]

The strain matrix \([\mathbf{P}^*]\) is given in Appendix 16.
Then the stresses due to the inplane stress resultants are given by

\[
\begin{bmatrix}
\sigma_{nx} \\
\sigma_{ny} \\
\tau_{nxy}
\end{bmatrix} = \frac{1}{h} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} \quad 7.27
\]

The extreme fibre stresses due to the moments are

\[
\begin{bmatrix}
\sigma_{mx} \\
\sigma_{my} \\
\tau_{mxy}
\end{bmatrix} = \frac{6}{h^2} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} \quad 7.28
\]

The maximum and the minimum stresses are given by

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \begin{bmatrix}
\sigma_{nx} \\
\sigma_{ny} \\
\tau_{nxy}
\end{bmatrix} \pm \begin{bmatrix}
\sigma_{mx} \\
\sigma_{my} \\
\tau_{mxy}
\end{bmatrix} \quad 7.29
\]

7.7 Results and Discussions

The computations in this chapter are confined to the following singly curved plate.

- \( L_x = 3 \text{ in} \),
- \( L_y = 4 \text{ in} \),
- \( k_x = 0, \ k_y = 1/30 \text{ in}^{-1}, \ k_{xy} = 0 \),
- \( \nu = 0.33 \),
- \( E = 10^7 \text{ lb/in}^2 \),
- \( \rho = 0.096 \text{ lb/in}^3 \)
Static load - normal to the surface, uniformly distributed with unit magnitude.

(a) Static results

Since the boundary conditions are the same at all four edges and the loading is symmetrical, it is necessary to consider only a quarter of the plate. For the study of the convergence of the solutions 12, 20 and 30 elements respectively were used in a quarter of the plate. The mesh pattern for the 12 and 20 elements are shown in Figure 23 and for that of 30 elements is shown in (a) of Figure 32.

With 12 elements in a quarter of the plate, the nodal displacements computed by assuming consistent loads were found to be identical with those computed on the basis of discrete loading. This is probably due to the fact that the assumed mesh pattern is so fine that even the discrete loading is adequate to represent the distributed loading. As expected, the convergence of the solutions is from below and is monotonic. With 12, 20 and 30 elements the transverse displacement computed at the centre of the plate are $9.931 \times 10^{-3}$ in, $9.994 \times 10^{-3}$ in, and $10.030 \times 10^{-3}$ in respectively. From this it appears that even with 12 elements, the solutions are acceptably accurate. The same displacement obtained with an approximate Rayleigh-Ritz analysis (not presented here) is $0.0124$ in.

Except for the stresses due to the twisting moments, in general all other stresses will show a certain amount of discontinuity across element boundaries in this formulation. An idea of the general order of discontinuity of the stresses could be obtained from (b), (c) and (d) of Figure 32, where
the maximum stresses around the nodes 18 and 25 (refer to (a) of Figure 32) of the plate are shown. When the stresses are discontinuous as shown, then the usual practice is to accept the arithmetic average of the stresses contributed by the elements meeting at the node as the actual stress at the node. Thus, in general, the stress at a node will mean the average stress.

The results obtained with 30 elements on a quarter of the plate are presented in the form of figures. First, the discretized form of quarter of the plate with 30 elements is drawn, where the nodal points are not numbered. If the nodal numbers are required they may be obtained by comparing with (a) of Figure 32. Then the values of the quantities (displacement, rotation or stress) are given at the nodes. When no values at the boundary nodes are given, it may be understood that they are zero. The displacements u, v and w are given in Figure 33. The slopes w', w' and w'' are given in Figure 34. The maximum and minimum stresses are given in Figure 35 and Figure 36 respectively. The actual quantities may be obtained by dividing the quantities given at the nodes by the numerical factor given with each figure.

From Figure 33, it may be observed that the computed functions u, v and w satisfy the clamped boundary conditions and the displacement functions u, v are compatible with the symmetric deformation of w. Also, it may be observed that the magnitudes of u and v are much smaller than the magnitudes of w. Thus, in this particular case, for all practical purposes w may be assumed to be equal to the total displacement. Therefore, maximum value of w may be used to define \( y_0 \) when the mean square displacement

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is calculated using the design method for curved plates.

The slopes \( w' \) and \( w'' \) of Figure 34 are of no practical use here. However, \( w''' \) is useful, since the twisting moment is directly proportional to it, in this formulation.

Since all the nodal displacements at a clamped boundary vanish identically in this formulation, the stresses at any two nodes of an element which are on a clamped boundary, will depend only on the nodal displacements of the other two nodes. Consequently, the stresses at the clamped boundaries will maintain continuity. Again, due to symmetry, the stresses at the centre of the plate also have to be continuous. The stresses will be discontinuous at all other nodal points. An examination of Figure 35 and Figure 36 will show that the intensity of stresses is higher (except the shear stress) in the centre and in the edges of the plate than in the rest of it. Thus, the maximum stresses could be predicted without any uncertainty in this case. This is very useful since the maximum stresses are required in the design method. Therefore, referring to (a) of Figure 32, the node 1, which is situated at the centre of the plate, and the nodes 7 and 36, which are situated at the centre of the straight edge and the curved edge respectively, are of special interest. It may be observed that the shearing stresses which are not of any interest here are zero at these three nodes.

The combined stresses together with their direct and bending components at these three nodes are given in Table 29. (If desired, the direct and the bending stresses at different nodal points may easily be found from Figure 35 and Figure 36). From Table 29, it may be observed
that except at the centre of the plate, the direct stresses are smaller than the bending stresses. At node 36, where the combined stresses are the maximum, the direct stress is 57% of the bending stress. This is also a useful result, because the maximum stress used in deriving the mean square stress in the design method was also due to bending. Thus, in this case, there will be little error if we use the maximum combined stress instead of the maximum bending stress to evaluate the mean square stress of the plate, using the design method.

The variations of the combined stresses along the central straight line generator and along the central arch are shown in (a) and (b) respectively of Figure 37. It may be seen that the maximum variation occurs near the edges in both the cases.

(b) Results with the design method

Since the displacements and the stresses due to the unit uniformly distributed static loading are known at various points of the plate, it is possible to use the design method to predict r.m.s. stresses and displacements at these points. As the magnitudes of \( u \) and \( v \) are small compared to that of \( w \), in calculating the r.m.s. displacements \( w \) may be used in place of \( y_0 \) in equation 7.18(a). However, at points where the membrane stresses are significantly large (comparable or greater than the bending stresses), the method may not be used to predict the r.m.s. stress. In this case, such a critical point is the centre of the plate. It is not a big difficulty, as the maximum stress appears at a boundary where the membrane stress is insignificant.
Again the design method is based on uniform pressure
distribution over the plate. It may be argued that the curved plate
will not respond to such a pressure field at its lowest resonant
frequency, if the lowest frequency mode is an antisymmetrical one.
Under such a situation, the logical step will be to use the lowest
symmetrical mode frequency. The approximations under which the method
is being applied are so severe it appears to be immaterial which
frequency one chooses to use if it is near about the lowest one.
However, we shall use the frequency of the lowest symmetric mode.

Since the plates under consideration are very thin, usually
most of them may buckle under a unit uniformly distributed static load.
Alternatively, the stresses and the displacements predicted by the
linear small deflection theory may be too big. For example, assume
the 3 in x 4 in plate mentioned earlier to be flat. The displacement
calculated at the centre of the plate under unit static pressure
computed on the basis of the table provided by Timoshenko and Woinowsky-
Krieger [99, p.202] is approximately 0.0788 in. compared to its thickness
of 0.013 in. The estimated maximum bending stress computed on the basis
of the above-mentioned table, at the centre of an edge is approximately
22400 lb/in². The lowest natural frequency of the plate is 406.15 Hz.
Apparently the central displacement is too big to be predicted by the
small displacement theory. But, considering that the actual load
intensity coming over the plate is very small compared to unity, we
may still use these large values, assuming them to be just parameters
obtained by extending a linear load deflection plot.
Now, this method will be applied to compute the r.m.s. stress and displacement. A pressure level of 120 dB ($p_s = 0.419 \text{ lb/ft}^2$) is assumed. The maximum r.m.s. stress is at node 36 ((a) of Figure 32). To compute maximum $\sqrt{\sigma^2(t)}$ for the curved plate at node 36 we use $f = 972.5$ Hz (lowest symmetric mode frequency), $p_s = 0.419 \text{ lb/ft}^2$ and $\sigma_0 = 4738 \text{ lb/in}^2$, $\eta = 0.02$ in equation 7.18(d), yielding $\sqrt{\sigma^2(t)} \approx 3800 \text{ lb/in}^2$. From random fatigue data given in reference [101], it may be seen that the plate will fail due to fatigue when the number of stress reversals are well over $10^7$ times, providing the plate is made of either of the materials specified in the said reference. When the plate is flat, the $\sqrt{\sigma^2(t)}$ estimated at the same point under the same pressure is about 11600 lb/in$^2$, in which case the plate will fail due to fatigue in less than $10^6$ number of stress reversals.

The r.m.s. displacement at the centre of the plate is estimated to be 0.0081 in. when the plate is assumed to be curved and is 0.041 in. when it is assumed to be flat. Thus for the curved plate the estimated maximum r.m.s. displacement is about half the thickness of the plate, whereas for the flat plate it is about three times the thickness of the plate. The overall effects of curvature on behaviour of plates may be seen very well from this exercise.

(c) Computation with the direct method

The equation 7.17 is used to obtain the output power spectra $[S_o(\text{if})]$ with a unit matrix used as the input power spectra $[S_0(\text{if})]$, and $\eta = 0.01$. Trial solutions using only 12 elements in quarter of the
plate, found to require excessive amount of computing time. Most of the
time is required in inverting the complex $[Z(\omega)]$ matrix. The power
spectrum of $\omega$ for this type of input is shown in Figure 38 (Figure 38
is drawn exactly in the same way as Figure 33), for input frequencies
of 890 Hz and 912 Hz respectively. Since the used mesh sizes are not
sufficiently fine these results are not the correct ones. However, they
are presented here to show the applicability of the method only.

7.8 Conclusions

The finite element analysis developed in the previous section
for the analysis of eigenvalue problems of curved plates is generalised
to include both static and random analysis. For the static case, though
the stresses in general are discontinuous, it is shown that the maximum
stresses can still be predicted without difficulty for clamped plates.
Based on the static deflections and stresses, the r.m.s. stresses and
displacements of a panel under random loading may be roughly estimated.
However, the direct application of the finite element method to the analysis
of random vibrations, though expedient from the point of view of
formulation, is found to be too time consuming for practical computation.
CHAPTER VIII

COMPUTATIONAL ASPECTS OF THE PROBLEM

8.1 Introduction

The methods of solutions discussed above require a large amount of computing. Therefore, in this chapter the computational aspect of the problem will be discussed briefly. First the method of Kantorovich and then the finite element method will be considered. Originally, programming efforts were made in FORTRAN V language for the method of Kantorovich which did not require any sophisticated eigenvalue subroutines. However, in the finite element method, for the solution of the large eigen systems, sophisticated eigenvalue procedures are required. Since such procedures may be found in ALGOL 60 language in the relevant literature, it was found more convenient to write the programme in that language.

8.2 Roots of Cubic Polynomials

Whenever the functions $u$, $v$ and $w$ are explicitly prescribed, the equation for the determination of the eigenvalues reduces to a cubic polynomial. In general, the roots of the polynomial are all positive and real. The problem is to find the roots of the cubic which are the eigenvalues and corresponding to each of the eigenvalues the amplitude ratios.
It is possible to obtain the roots of the equation precisely and correctly in terms of the coefficients of the polynomial. However, in the actual computation quite a few square and cube roots are involved, which introduces appreciable error. Therefore, an iterative method due to Bairstow is preferable. The method may be found in Kuo [67, Chapter VI]. A simple subroutine to evaluate the roots of a cubic polynomial in complex form is written. The flow chart of a complete program is given in Figure 39. When no membrane stresses are applied it is required only to remove loop 5 from the flow chart. Since these programs are written for singly curved plates only, the symbols of chapter II, III and IV are to be used. The following set of identifiers are necessary.

\[ mc \rightarrow \text{total number of plates to be analysed} \]
\[ mb, nb \rightarrow \text{compute for } m=1 \text{ to } mb \text{ and } n=1 \text{ to } nb \]
\[ nc \rightarrow \text{a counter} \]
\[ \varepsilon \rightarrow \text{a small number defining accuracy; about } 10^{-12} \]
\[ md \rightarrow \text{defines the number of times a plate is to be analysed for different combinations of inplane stress, } E, \text{ etc.} \]

No special problems were encountered and the program could produce an average of 3 eigenvalues and eigenvectors per second. The time gain without inplane stresses is not noticeable.

8.3 General Method of Solution

It has been shown in Chapter III that the general method of solution may be formulated either in real form or in complex form. It is logical to question the necessity of the complex formulation, when the
formulation may be done in real form for which standard methods for
finding the solutions are available. First, it is extremely simple
to formulate with complex arithmetic which eliminates a fairly large
amount of algebra involved in the real formulation. Secondly, it is
easier to form the higher derivatives of the exponential functions.
Moreover, the complex formulation may very effectively be used as an
overall check on the relative performance of the formulation with real
numbers. Also, most of the modern high speed digital computers can
handle complex arithmetic as easily as real arithmetic in the FORTRAN
language. Even in the formulation with real arithmetic, there are
areas where the use of complex arithmetic is much more expedient. Such
areas are like the computation of the roots of the eighth order character-
istic equation and the constants $\alpha_j$ and $\beta_j$ which relate the arbitrary
constants $A_j$ and $B_j$ respectively to the arbitrary constants $C_j$.

On the other hand, the computations with the complex
formulation require more computing time and storage. In the case of
the real formulation, the accuracy of the solutions may sometimes be
improved by using double precision arithmetic, if desired. The same
facility is not available with complex numbers. Advantage due to
symmetry of a problem cannot be taken easily in the complex formulation.
Here, the use of the complex formulation may be regarded as a check on
the real formulation.

8.3(a) Computation with Real Formulation

The general criterion for the existence of a zero of $D(\Delta)$
between two values of $\Delta$ is that $D(\Delta)$ must change sign. Once the interval
in which a zero of the determinant lies is found, two criteria for the evaluation of the proper $\Delta$ must be specified. These criteria are as follows.

(i) If the absolute value of the determinant subject to rounding error is less than a prescribed small number $\epsilon_1$, where $|\epsilon_1| \geq 0$, the $\Delta$ corresponding to that $D(\Delta)$ is an eigenvalue.

(ii) Sometimes the value of the determinant may be too large to satisfy the condition for $\epsilon_1$ or it may require a very long time before doing that. The extra cost involved may not justify the improved accuracy. Therefore, another small number $\epsilon_2$ should be specified. When the existence of a zero of $D(\Delta)$ is known between two intervals $\Delta_a$ and $\Delta_b$ and if $|\Delta_a - \Delta_b| \leq \epsilon_2$, then $\Delta_a$ is the eigenvalue; if $|D(\Delta_a)| < |D(\Delta_b)|$, else, $\Delta_b$ is the eigenvalue. The following example taken from an actual computation will illustrate the point.

\[
\begin{align*}
\Delta_c &= 0.10244446 & \Delta_b &= 0.10244438 & \Delta_a &= 0.10244431 \\
D(\Delta_c) &= 96.093370 & D(\Delta_b) &= 24.977136 & D(\Delta_a) &= -45.259283
\end{align*}
\]

It may be seen that the $D(\Delta)$ has changed sign between $\Delta_b$ and $\Delta_a$. Since $|D(\Delta_b)| < |D(\Delta_a)|$ and $|\Delta_a - \Delta_b| < \epsilon_2$ it may be assumed that $\Delta_b$ is the eigenvalue.

If within one interval there are two roots, then there will be no change of sign of $D(\Delta)$ at its ends $\Delta_a$ and $\Delta_b$. Then the existence of the eigenvalues will be indicated by a general depression of the curve of $D(\Delta)$ against $\Delta$. If the eigenvalues are discrete, both of them may be obtained by gradually narrowing down the interval between $\Delta_a$ and $\Delta_b$. 

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Ultimately they will be obtained by the criterion of change of sign.

However, if there is only one root in the disprssion then it becomes extremely difficult to obtain it. The best way to tackle this problem seems to be to calculate the value of $\Delta$ which gives the lowest value of $D(\Delta)$ in the depression.

The eigenvectors corresponding to the eigenvalues computed on the basis as stated above, must satisfy the given boundary conditions and conditions of shape compatibility. The $\Delta$ which do not satisfy these conditions, though apparently gives $D(\Delta) = 0$, is not a real eigenvalue of the problem.

Sometimes there may be exponent overflow. This happens when the product of a positive real part of a root of the eighth order characteristic equation and the central angle is very large. Normalisation with respect to this is also not possible. Because the negative real part corresponding to the large positive real part will then underflow. This exponent overflow and underflow seems to increase when $L/b$ decreases from unity and also the value of $m$ increases from 1. No normalisation is adopted in this case. The argument and the exponents may be stored in separate words. By this, not all the troubles may be eliminated.

8.3(b) Computation with Complex Formulation

In this case most of the elements of $D(\Delta)$ are complex and therefore, for every real $\Delta$, the value of $D(\Delta)$ will be complex. When $D(\Delta) = 0$, both the real and the imaginary parts of $D(\Delta)$ must be zero.
simultaneously. But in practice this seldom happens. Also it does not seem to be worthwhile to try to locate the value of $\Delta$ at which $D(\Delta)$ is identically zero. Again, the criterion of zero crossing is more confusing than helpful, since the real and the imaginary parts of $D(\Delta)$ may change sign independent of each other except where $D(\Delta) = 0$. Therefore, it is better to avoid treating two parts of $D(\Delta)$ separately and also to avoid the criterion of zero crossing altogether. In this case, it is found to be very convenient to work with the modulus of the complex value of $D(\Delta)$, which is defined as the square root of the sum of the squares of the real and the imaginary parts of $D(\Delta)$. As this is always positive there is no question of zero crossing. The obvious criterion is to look for the minimum of the modulus in the interval of interest. Again, the eigenfunctions of the roots obtained this way must satisfy the conditions at the edges together with the conditions of shape compatibility. Otherwise, they may not be treated as true eigenvalues.

8.3(c) The Computer Programs

The flow chart for both the real and the complex formulation are given in Figure 3. The roots of the characteristic equation of order eight are evaluated by using the iterative method of Bairstow. The subroutines for the evaluation of the determinants are available in the magnetic tape library of the computer of the University of Southampton.

As the frequencies are proportional to the square of the eigenvalues, the error in the frequencies will be much less than the error in
eigenvalues. But a small error in an eigenvalue may produce a big error in the mode shape where the eigenvalues enter directly into the computation.

Once an eigenvalue is known, the constants $C_j = 1 \ldots 8$ may be evaluated to a ratio by solving a set of ordinary algebraic equations. Once the $C_j$ are known, the $A_j$ and the $B_j$ may be calculated without difficulty. Then $A_j$, $B_j$ and $C_j$ are normalised with respect to the largest among them. Knowing the normalised eigenvectors the eigenfunctions may be computed. These eigenfunctions may again be normalised so that the largest value is $+1$.

### 8.4 Computation with the Modified Matrix Method

The essential features of the computations with this method are given in Figure 4. The major steps in the computation are (a) evaluation of $e^{[A]x}$ where $[A]$ is a square matrix of constant elements and $x$ is a scalar parameter, (b) evaluation of a determinant and (c) inversion of a matrix.

The series $e^{[A]x} = I + [A]x + \frac{[A]^2x^2}{2!} + \ldots \ldots$ (where $I$ is a unit matrix) is absolutely and uniformly convergent for all values of $x$, (Chapter III). Therefore, the series may be evaluated to any desired degree of accuracy. The evaluation involves only matrix additions and multiplications requiring no special computing efforts. However, the series is rapidly convergent if $x < 1$. A subroutine is written for this purpose.

The evaluation of a determinant is done by triangulation, using a Fortran translation of the C.A.C.M. Algorithm No.41. Since the system is small and well conditioned the inverse of a matrix is found from its adjoint.
The subroutine for the evaluation of a determinant is used in the inversion subroutine.

Subsequently the eigenvalues are found by bisections. The eigenvector corresponding to each eigenvalue is obtained by using stored information and back substitution according to the theory given in Chapter III.

In these programs, if the initial value of iteration could be estimated properly, then they are very efficient. The most reliable and self generating initial value is obtained by using an approximate simply supported solution along the direction in which the final solutions are sought.

Time required is 170 seconds for 40 $\Delta$ steps with all the eigenvalues and eigenvectors therein calculated. Maximum number of eigenvalues and eigenvectors computed in 170 seconds are 6. This estimate is for the ATLAS Laboratory at Harwell. The time estimate for the ICT 1907 computer at the University of Southampton is 45 minutes. The storage requirement is 14300 words.

When the modified matrix method is used for doubly curved plates, in Figure 4, the input will be $L_x$ = length along $x$, $L_y$ = length along $y$, $k_x$ = curvature along a curve $y = constant$, $k_y$ = curvature along a curve $x = constant$, $h$ = thickness and $\nu$ = Poisson's ratio.

8.5 Computations with the Finite Element Method

All the matrix methods of analysing structures are computer oriented methods. The finite element method being only a particularization of the general matrix method of structural analysis is therefore also
computer oriented. Since the method is being used here for the dynamic (free and random) and static analysis of curved plates, it is necessary to discuss briefly the computational procedures adopted here for each of the cases. The flow charts for the computer programs for the free and random vibration analysis are given in Figure 40(a) and in Figure 40(b) respectively, whereas for the static analysis it is given in Figure 41. The identifiers used in these flow charts have the following significance.

NEL, NNO, NBC and CBR are the controlling integers of the analysis where

NEL → number of elements,
NNO → number of nodes,

NBC → total number of degrees of freedom that are constrained to be zero. This includes boundary conditions, intermediate constraints and any other constraints that might be applied. Since higher derivatives of u, v and w are not included in the formulation, no provision for the general type of constraint is made.

CBR → stands for clamped boundary restraint. If the displacements at a node vanish identically particularly at a clamped boundary, then CBR = (total nodes on the boundary) x (degrees of freedom at each node). If any of the nodal displacements at the boundary nodes is not zero, then CBR = 0. Then all the boundary conditions have to be included in NBC. Note that CBR = NBC - all other constraints, other than the boundary ones.

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The co-ordinates of the nodes of each of the elements are required to find the dimensions, the curvatures and the thickness of each of the elements. Referring to (a) of Figure 19, the global coordinates \((x, y)\) of node 1 and those of node 4 of each element are required. For the formation of the overall system matrices the node numbers (connectivity) of each element (from 1 to NEL) are to be given. Referring to (b) of Figure 19 for elements I and II the data are

\[
\begin{array}{cccccc}
  x_6 & y_6 & x_2 & y_2 & 6 & 7 \\
  x_7 & y_7 & x_3 & y_3 & 7 & 8 \\
\end{array}
\]

The subscripts of x and y indicate the node at which they are specified.

The equation of the surface in polynomial form is to be given. From it, the curvatures \(k_x, k_y\) and \(k_{xy}\) at the centre of each element are calculated. The constants of the equation are the \(SG[I]\).

The thickness \(h\) may also be given in polynomial form. However, in this formulation only the average thickness of each element requires to be prescribed.

When there are inplane stresses present, it is convenient to prescribe them along with \(h\) for each element from 1 to NEL as follows \((s_x, s_y, s_{xy}\) are the inplane stresses as defined in Chapter VI).

\[
\begin{array}{ccc}
  h & s_x & s_y \\
  & s_{xy} & .
\end{array}
\]

Similarly, in the case of statically applied distributed normal loading, the loads for each element from 1 to NEL may be prescribed together with \(h\) conveniently as follows \((p_z = \text{the intensity of the normal load})\).
It is necessary to input $E$ and $\nu$ for static; $E$, $\nu$, $\rho/g$ for free vibrations (notations as in Chapter VI) with or without applied membrane stresses; $E$, $\nu$, $\rho/g$ and $\eta$ (loss factor) for random vibrations.

Moreover, in the static analysis when the loads are assumed to be discrete, then they have to be specified separately. For this case there will be another controlling integer with the first four, which will give the total number of nodal loads to be imposed. Referring to (b) of Figure 19, let there be a concentrated normal load $P_z$ at node 7. Then the specification will be

$$7 \quad 3 \quad P_z,$$

where 7 is the node number, 3 specifies that the given load $P_z$ corresponds to the 3rd nodal unknown ($w$ in this case). Similarly, in the case of the random vibrations the input information has to be provided.

It is always better to print out all the input data, to see if the correct data is provided.

From these input data, the program will then calculate the geometry of each element and form the stiffness, mass and load matrices for each of the elements. The next stage is to form the overall stiffness, mass and load matrices by assembling the individual element matrices. Then these overall matrices are constrained in the case of static and also constrained in the case of dynamics if required.
From this point the methods of solutions of problems of free vibrations, random vibrations and static are different which will be discussed next.

8.5(a) Free Vibrations

The system equations (equation 6.30(g)) may be solved by a variety of methods. One of the methods that can be applied is the iterative method described for the method of Kantorovich. Find the values of \( D(\lambda) = |[\bar{A}] - \lambda[\bar{M}]| \) (\( D(\lambda) \) is frequency determinant) for a range of values of \( \lambda \), and from there find the values of \( \lambda \) that give \( D(\lambda) = 0 \). The obvious difficulty with this method is that it cannot find repeated zero roots which are necessary to investigate free plates or rigid body modes.

The method of calculation of the eigenvalues and the corresponding eigenvectors adopted here uses a series of procedures written in ALGOL 60 which appeared in the Handbook Series of Linear Algebra in the Numerische Mathematik. A little explanation of each step together with the reference of each specific procedure used are given below.

The eigenvalue problem is equivalent to a homogeneous polynomial equation in \( \lambda \) of degree \( N \), where \( N \) is equal to the order of the system. The aim is to factorise the equation to obtain the roots. As the system is in matrix form, it is more convenient to obtain the eigenvalues direct from the matrices rather than converting the system into polynomial form. Moreover, the roots of polynomials are extremely sensitive to the accuracy with which the coefficients are determined. To effect this factorization, various properties of matrices are used to transform one complicated
system to another equivalent simpler system. However, the mathematics involved in these transformations are not always simple.

The first step towards this simplification is to reduce the pair of square matrices (the overall stiffness and the overall mass matrices) to a single matrix. For our convenience in this chapter capital letters will represent square matrices and small letters will represent vectors. Then let the system equation be written as

$$Ad = \lambda M d \quad 8.1$$

where $A$ is the overall symmetric stiffness matrix

$M$ is the overall symmetric positive definite mass matrix

$d$ is the vector of the unknowns

$\lambda$ eigenvalue.

Since $M$ is symmetric positive definite there exists a real nonsingular lower triangular matrix $H$ such that

$$HH^T = M \quad 8.2$$

Substituting for $M$ from 8.2 into 8.1

$$Ad = \lambda HH^T d$$

or

$$(H^{-1}AH^{-T})(H^Td) = \lambda(H^Td)$$

or

$$Bx = \lambda x, \quad 8.3$$

where $B = H^{-1}AH^{-T}$ is the reduced equation which is symmetric and $x = H^Td$ is the new eigenvector. Thus the system 8.1 is reduced to the system 8.3. With this transformation the $\lambda$ is unchanged but the eigenvector $d$ is changed to

$$x = H^Td \quad 8.4$$
The application of the method of decomposition together with ALGOL 60 procedure are given by Martin et al \cite{69}. Procedures in ALGOL 60 to reduce the system equation 8.1 to the form of equation 8.3 are given by Martin et al \cite{70}, one of which is being used here.

The reduction to a single matrix of the system of equations is only the first step towards factorization. The symmetric matrix $B$ of equation 8.3 is fully populated and therefore is not of convenient form to obtain the eigenvalues from it. The usual procedure is to obtain a matrix $C$ which is similar to matrix $B$ but with fewer non-zero elements about the main diagonal. The matrices $B$ and $C$ are called similar over a field, if there exists a non-singular matrix $Q$ over that field such that

$$C = Q^{-1} B Q .$$

Moreover, if $Q$ is orthogonal, that is $Q^T Q = Q Q^T = I$ ($I$ is unit matrix), the $B$ and $C$ are said to be orthogonally similar. Now the eigenvalues of the similar matrix $C$ are

$$C - \lambda I = Q^{-1} B Q - \lambda I = Q^{-1} B Q - Q^{-1} \lambda I Q$$

$$= Q^{-1} (B - \lambda I) Q .$$

Again from equation 8.5

$$B = QCQ^{-1} .$$

The system equation from equation 8.3 is

$$(B - \lambda I)x = (QCQ^{-1} - Q\lambda IQ^{-1})x$$

$$= Q(C - \lambda I)(Q^{-1}x) .$$
From equation 8.6 and equation 8.8 it may be seen that (since Q is non-zero) under similarity the eigenvalues remain invariant where the eigenvectors are changed to \( Q^{-1}x \).

If by the transformation under similarity C is reduced to a diagonal matrix, then all the eigenvalues of the system are given by the diagonal elements of C. In Jacobi type methods the goal is to obtain C as a diagonal matrix. To achieve this, theoretically an infinite number of iterations are required. In practice, the iterations are terminated when the off diagonal elements are smaller than a small prescribed number. A short description of the method and an Algol procedure on Jacobi type method may be found in Reference [77].

On the other hand, the direct method of Householder aims at partial reduction of the problem to tridiagonal form. A description of the Householder's method may be found in Wendorff [78]. Wilkinson [72] has given the theory and an algorithm on Householder's method. The algorithm reduces any symmetric matrix to symmetric tridiagonal form by orthogonal similarity transformations. Later on, Martin et al [71] have given improved versions of that procedure. One of the improved versions is used here. Next we shall consider the solution of the system. Reference is made to Fox and Mayers [79, pp.104-105].

Assuming that C is reduced to a symmetric tridiagonal form and denoting the diagonal elements of C with c and the off diagonal elements with b the following matrix is obtained.
\[
C - \lambda I = \begin{bmatrix}
  c_1 - \lambda & b_2 \\
  b_2 & c_2 - \lambda & b_3 \\
  & \cdots & \cdots \\
  b_{n-1} & c_{n-1} - \lambda & b_n \\
  b_n & c_n - \lambda
\end{bmatrix}
\]

The determinant of \( C - \lambda I \) may be computed from the Sturm sequence

\[
f_0(\lambda) = 1, \quad f_1(\lambda) = c_1 - \lambda \tag{8.10}
\]

\[
f_{k+1}(\lambda) = (c_{k+1} - \lambda) f_k(\lambda) - b_{k+1} \frac{b_k}{b_{k+1}} f_{k-1}(\lambda),
\]

where \( f_k(\lambda) \) is the determinant of the leading submatrix of order \( k \) of \( C - \lambda I \). When \( k = N \), the order of the matrix, then \( f_N(\lambda) = |C - \lambda I| \). The computed \( f_N(\lambda) \) is the exact value of the determinant. In this case only the sign of \( f_k(\lambda) \) is necessary, since the number of equalities in sign of successive members of the sequence \( f_0(\lambda), f_1(\lambda), \ldots \), \( f_N(\lambda) \) is equal to the number of eigenvalues greater than \( \lambda \). By using successive values of \( \lambda \) and using a method of bisection any desired eigenvalues may be obtained.

Wilkinson [75] has given an Algol procedure to find the eigenvalues of a symmetric tridiagonal matrix using Sturm sequence and the so-called method of bisection. Later on, Barth et al. [73] have given an improved version of the same procedure which is being used here. The procedure can find very close or coincident eigenvalues. Moreover, it is not necessary to find all the eigenvalues of the system. In fact, this is the only known procedure that can find all the eigenvalues between, say, the \( m_1 \)th and the \( m_2 \)th eigenvalues.
Another procedure for the determination of the eigenvalues of a symmetric tridiagonal matrix is given by Reinich and Bauer \cite{74}. The method is based on the so-called QR Transformation with Newton shift. Details of the method may be found in Francis (\cite{80}, \cite{81}) and Ortega and Kaiser \cite{82}. It may be used to find either the lowest or the highest \textit{ml} eigenvalues. In practice, it was observed that this procedure and the bisection procedure of Barth et al \cite{73} yield identical eigenvalues.

However, Reinich and Bauer \cite{74} recommend the use of the bisection procedure of \cite{73} if clusters of eigenvalues are expected.

Once the eigenvalues are known, it is required to find the corresponding eigenvectors. The first step is to find the eigenvectors of the tridiagonal matrix $C - \lambda I$ whose eigenvector corresponding to $\lambda$ is $z = Q^{-1}x$ according to equation 8.8. Wilkinson \cite{76} has given the so-called method of inverse iteration and an Algol procedure for the calculation of the eigenvectors of such a matrix when the eigenvalues are given. Initially a vector $v$ is assumed such that

\begin{equation}
(C - \lambda I)z = v \quad \text{8.11}
\end{equation}

The solution of this equation should be such that for the given eigenvalue, the vector $v$ should be zero, when at least one of the elements of $z$ is not identically zero. Solution of 8.11 will yield a vector $z$ which is used as the initial value for the solution of

\begin{equation}
(C - \lambda I)y = z \quad \text{8.12}
\end{equation}

where $y$ is the second iterated value of the vector. By a proper choice of $v$ the number of iterations required is cut to a minimum. The initial
value \( v \) is found as follows. Let

\[
C - \lambda I = LU
\]

where \( L \) is a unit lower triangular matrix and \( U \) is an upper triangular matrix. With this equation 8.11 and 8.12 become

\[
LUz = v \quad \text{and} \quad LUy = z
\]

respectively.

Now assuming \( v = Le \) where \( e \) is a vector of unit elements, one obtains

\[
Uz = e
\]

Knowing \( z = Q^{-1}x \), the vector \( x \) of the reduced equation 8.3 is computed by backtransformation using an Algol procedure given by Wilkinson [72]. Then the eigenvector \( d \) of equation 8.1 is computed from the relation \( H^T d = x \).

8.5(b) Random Vibration

The solution of the problem of discrete frequency response is equivalent to solving the complex linear system

\[
Ax = b
\]

where \( A, x \) and \( b \) are complex. This complex system may easily be solved by any standard method like Gauss elimination or Crout factorization. Based on Crout factorization, Boulder et al [83] have given Algol procedures that can solve the complex system 8.15 adequately. When the inverse of \( A \) is required the same procedure of reference [83] may be used. In that case it is only necessary to use in the right hand side of the equation 8.15 a unit matrix of the same order as the system matrix. The time required for
inversion of complex matrices appears to be very high. No time gain in computing is observed by converting the complex system to real systems.

8.5(c) Static Problem

The problem reduces to solving a linear real algebraic system similar to equation 8.15 in form. The Algol procedures of reference [83] are used to solve them. Once the vector \( x \) is obtained, it is rearranged by putting in the elements which were constrained to be zero. Then at each node of each of the elements the stresses are computed.
CHAPTER IX

EXPERIMENTAL STUDY OF FREE VIBRATION CHARACTERISTICS
OF A CLAMPED SINGLE CURVED RECTANGULAR PLATE

9.1 Introduction

The experimental determination of structural behavior is an integral part of the analysis of any structure, particularly if it is a complex one. In complex structures numerous parameters can enter into and influence the analysis. It is virtually impossible to account for all of them without making some simplifying assumptions and idealizations. The validity of the assumptions and the idealizations cannot be verified or substantiated easily without taking recourse to some experimental techniques.

However, the experiments also suffer from different types of shortcomings, like imperfections in the test specimens, residual stresses and others. In fact, the tendency is to treat the experimental and the analytical techniques of investigations as complementary to one another instead of rivals. For fair comparison of results, the test specimens should be as similar to the ones used in the analysis as possible.

9.2 The Test Specimen

Usually, for curved plates, the reproduction of the boundary conditions, particularly the clamped ones, is quite difficult. This has led to difficulties in comparing experimental and theoretical solutions. Naturally, if the boundary conditions are different, there will be no agreement between the theoretical and the experimental results.
In order to remove the uncertainties from the reproduction of the boundary conditions as much as practicable, the Lockheed-Georgia Company, U.S.A., has machined a thin singly curved plate from a block of aluminium. The size of the block is 10 in x 9 in. The actual panel is 3 in along the straight edges and 4 in along the curved edges. The radius of curvature is 30 in and the thickness is 0.013 in. The geometry of the plate is shown in Figure 42. The picture of the plate is shown in (a) of Plate 1. This plate was made available to us for testing.

The plate was machined out of the block by an electrical diffusion process. The possible residual stresses induced by the machining are relieved by heat treating the panel.

9.3 Methods of Excitation

The steady state vector response method given by Kennedy and Pancu [84] is one of the most popular methods for the measurement of natural frequencies and damping ratios of structures. The method is very time consuming and tedious and not easy to apply outside laboratories. Part of these troubles are alleviated by the slow frequency sweep method of Hok [85]. In this method, the frequency at which the maximum response occurs is shifted in the direction in which the frequency is changing. Moreover, the response at resonance is less than the steady state maximum. These introduce errors in the experimental results.

Transient methods of measuring frequency response with rapid frequency sweep require extremely short test times. Therefore, the method has advantage over the steady and quasi-steady state methods. The
transient method is not widely employed because of the difficulty of analysing the test data. Since a computer is available at Southampton to digitise the test data and then analyse it, the transient method of testing with rapid frequency sweep is preferred.

The success of the transient test method will depend upon the availability of a sweep generator to generate the forcing function. Such a portable electronic generator has been designed by White [86]. This generator can rapidly generate a rectangular wave, N-wave or sinewave between 0 and 20 KHz of frequency. The sweep time and the lower and upper limits of frequency of interest and the type of wave may be set up in the generator.

The pulse excitation is not advantageous for multi-degree of freedom systems. Some of the resonances may not be excited due to the zeros in the modulus spectrum. Also, since the spectrum is of infinite extent some frequencies outside the range of interest may be excited. Therefore, the swept sinewave is preferable as a transient forcing function. The forcing function should have a 'flat' spectrum between the two frequency limits. The importance of this consideration will be seen later. The following is the summary of a theoretical investigation of Thrall et al [87] into the properties of swept sinewaves.

The equation of the swept sinewave is

\[ X(t) = \sin(at^2 + bt), \quad 0 < t < T \]

where \( t = \) time, \( T = \) duration
\( a = \pi(f_2 - f_1)/T, \quad b = 2\pi f_1, \)
\( f_1 = \) initial frequency, \( f_2 = \) final frequency.
The power spectrum of the swept sinewave has the following properties

(a) the mean value of the spectrum is \( \frac{\pi}{4a} \),

(b) the spectrum is not flat but has two peaks, approximately 1.4 times the mean spectrum level in height, occurring at \( f_1 + 1.2 \sqrt{\frac{a}{2\pi}} \) and \( f_2 - 1.2 \sqrt{\frac{a}{2\pi}} \) Hz.

(c) the amplitude of the ripple superimposed on the mean spectrum level is proportional to \( \frac{1}{\sqrt{T}} \)

(d) the cut off rate of the spectrum at \( f_1 \) and \( f_2 \) is high.

The main disadvantage (b) may be overcome by slightly extending the sweep outside the range of interest. Because of (d) unwanted resonance may be excluded by limiting the range of interest.

White [86] has shown that his sweep oscillator is capable of generating almost ideal swept sinewaves.

9.4 Vibration Measuring Probe

The vibrations of a structure may be measured in several different ways. Some of these require the attachment of strain gauges or other measuring transducers to the vibrating structure. Since the plate under test is very thin, the stiffness and the mass of the plate may vary considerably if such measuring devices are attached to the surface, even locally. This may change the vibration characteristics of the plate to a considerable extent. This suggests the use of non-contacting measuring devices. One of the simplest devices is a non-contacting capacitance probe,
which is being used here. The encased probe may be seen in (b) of Plate 1. A probe with a maximum amplitude limit of $5 \times 10^{-3}$ inch was used. This ensures that the vibrations will always be in the small amplitude range and hence linear.

The probe consists of a flat, circular inner electrode, separated from an outer guard ring by an insulating sleeve. The outer radius of the guard ring is 0.125 inch, whereas the radius of the electrode is 0.0707 inch. The output voltage amplitude is proportional to the distance between the probe and the structure. The mean amplitude determines the mean distance between the probe and the structure.

The distance between the probe and the plate surface is adjusted by a pair of differential co-axial threaded cylinders. The probe attached to the cylinders is mounted to a rigid fork as shown in (b) of Plate 1. The fork is held in position by the two steel plates which are held to the block by screwing (b) of Plate 1). Evidently, the arrangement is extremely simple. Movement of the probe near the edges of the plate is restricted. This is unimportant for vibration measurements. The output of the probe is recorded on magnetic tape. The position of the fork with the plate when measurements are taken may be seen in (a) of Plate 2.

9.5 Method of Excitation of the Plate

Since the plate is situated in a cavity, to excite the plate with grazing incidence is impracticable. The apparatus used in exciting the plate is shown in Figure 43. In the case of discrete frequency test the sweep oscillator is replaced by a decade oscillator. The sweep
oscillator described by White [86] is used.

The output of the oscillator is passed through an amplifier (25 watts), and then to the loudspeaker (100 watts, 8 ohm impedance, 55 Hz resonance, 18 in diameter). The sound from the loudspeaker then excites the plate. The vibration of the plate is measured by the probe and is transmitted to the vibration meter. The output from the vibration meter is then recorded on magnetic tape. The tape recording of the output is played back in the oscilloscope for visual inspection. The whole set-up is shown in Figure 43, also it is shown in (b) of Plate 2.

As shown in (a) of Plate 2, the loudspeaker is about 9 inches away from the plate. There is no physical connection between the plate and the loudspeaker. The loudspeaker is simply placed on rubber foam. The block is placed on thick foam which is placed on a wooden plank. A 5 in x 6 in hole is made over the loudspeaker through the plank and the foam for the transmission of sound to the plate. The foam cuts off any possible response of the plank to the sound being transmitted to the block of aluminium containing the test plate. It was found unnecessary to fasten the block of aluminium to the test rig.

9.6 Boundary Conditions of the Test Plate

At a glance, it appears that the test plate is clamped. But the material into which the edges of the plate are embedded also has the same material properties as the plate. Therefore, the plate is actually elastically supported. The effect of this may be very small. However, there is an approximate way in which this elastic support condition may be
included in the theory. This approximation, based on the so-called Vogt's artifice (see Ref. [92, pp.441-452]), is widely used in the design of arch dams in civil engineering. The arch dams are just special curved plates. The artifice simply extends the point of actual support into the foundation. This extension in our case becomes (see Ref. [93]) equal to $3Ah^3$ where

$$A = \frac{(1 - v^2)}{12} \cdot \frac{18}{h^2 \pi} \cdot \frac{1}{1 + 0.25 \left( \frac{L_y}{L_x} \right)}.$$  

The total extension at each edge is approximately of the order of the thickness $h$. The length $L_x$ and $L_y$ are to be increased by an amount approximately equal to $2h$.

9.7 Theoretical Background of the Test

The vector diagram method given by Kennedy and Pancu [84] may be said to be the classical method of experimental analysis. The frequency response function $H(i\omega)$ is given by

$$H(i\omega) = \frac{Y(i\omega)}{X(i\omega)},$$

where $X(i\omega)$ and $Y(i\omega)$ are the Fourier transforms of the transient excitation $X(t)$ and the system response $Y(t)$ respectively. Then the vector diagram is a plot between the Real $[H(i\omega)]$ as the ordinate and the Imaginary $[H(i\omega)]$ as the abscissa. It may be noted that in this case records of both the input and the output, together with their Fourier transforms, are needed.

White [88] has discussed the limits of resolution of close natural frequencies for frequency response analysis by direct Fourier
transform methods. Also some possible sources of error in the transient testing of structures with close natural frequencies are discussed.

An alternative method, known as the Characteristic Response Function Method (CRFM), is to be used here. The method is a new one whose full exposition may be found in Kandianis [89]. Here we shall only give the salient features of the method.

The method rests on the assumption that the input spectrum $G_{pp}(f)$ is flat. Therefore, information on the absolute magnitudes of the input is not necessary.

The working of the classical method and the CRFM is shown in Figure 44. From a comparison of the working of the methods it may be seen that the CRFM is the quicker of the two since it dispenses with the necessity of recording and processing the input data. Moreover, it eliminates the effects of noise on the measurement of the frequency response.

In the classical method the information on phase is retained. Therefore, it is necessary to know both the input and the system output and their Fourier transforms even though the input power spectrum may be flat. The modulus of the Fourier transform of the input may be flat; its phase, however, may vary with frequency as in the case of the swept sinewave excitation.

In the CRFM the input power spectral density used here is a swept sinewave which is shown to satisfy the condition that its spectrum should be 'flat'. Then the autocorrelation function of the output $R_{yy}(\tau)$
(where $\tau$ is time delay) and the power spectral density function $S_y(f)$ are related by the following relations (Robson [91, p.50]; Bendat and Piersol [90, p.22]).

$$S_y(f) = 2 \int_{-\infty}^{\infty} R_{yy}(\tau) e^{-2\pi ft} d\tau,$$

$$R_{yy}(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} S_y(f) e^{2\pi ft} df.$$  \hspace{1cm} 9.3(a-b)

The equation 9.3(a) gives the Fourier transform of the autocorrelation function $R_{yy}(\tau)$ of the output.

Kandianis [89] has shown that the Fourier transform $f(i\omega)$ of the autocorrelation function $R_{yy}(\tau)$ of the impulse resonance yields an expression in terms of the frequency response function $H(i\omega)$, which for a single degree of freedom system is given by

$$f(i\omega) = K \left\{ i\omega - \frac{1}{\omega_n} H(i\omega) + 2\eta H(i\omega) \right\}$$

$$= K F(i\omega),$$  \hspace{1cm} 9.4

where $K$ is a constant, $F(i\omega)$ is the characteristic response function, and $H(i\omega)$ is the frequency response function given by

$$H(i\omega) = \frac{1}{M \omega_n^2 \left[ 1 - \left( \frac{\omega}{\omega_n} \right)^2 + 2i\eta \frac{\omega}{\omega_n} \right]}.$$  \hspace{1cm} 9.5

where $\eta$ is the damping ratio, $\omega_n$ is the circular frequency at resonance and $\omega$ is any circular frequency, $M$ is the mass. Also it is shown by Kandianis that the function $f(i\omega)$ has the same properties as the frequency
response function $H(i\omega)$ and hence may be utilized in the measurement of the response represented conveniently by plots of $\text{Re}[F(i\omega)]$ vs $\text{Im}[F(i\omega)]$ which are proportional to the real and the imaginary part of $F(i\omega)$ respectively. From the same plots the damping and the natural frequencies may be measured. While drawing the vector plots we shall denote them as $\text{Re}[F(i\omega)]$ and $\text{Im}[F(i\omega)]$. From the transient method one can obtain only the natural frequencies and damping. The mode shapes may be found using a discrete frequency response test. The response of the plate to discrete frequency excitation is measured by a capacitance probe and is transmitted to a vibration meter, as shown in Figure 43. The vibrations are monitored and observed on an oscilloscope and the frequencies of the peak responses are noted. These are assumed to be the natural frequencies. Then the probe is removed and fine dry sand is sprinkled over the surface. The sand setting on the nodal lines indicates the mode shape of the vibration. Since the plate could be moved over the plank freely, it is possible to excite limited and selected areas of the plate even with the big loudspeaker whenever necessary (particularly when antisymmetrical modes are required to be excited).

9.8 Results and Discussions

The results of the CRFM are presented first. Since the results are plotted by a computer, the values of the quantities are not given along the axes. However, with each of the figures there is a note defining the origin and the scales along the axes. We have shown the direction of the
axes and whenever necessary also the quantity defined along the axes. For example, in the response curves the frequency $f$ is plotted along the direction of $x$. Therefore, in that case we have given both $f$ and $x$ along that axis. Sometimes it is not necessary to know whether the response is in mechanical or electrical units (like inches or volts) since one is proportional to the other. In such cases we merely indicate the direction of $x$ or $y$. Similarly, in the vector plots, the frequency resolutions are taken in such a way that the values of the frequencies given at the points are the exact values.

Figure 45 is the input power spectral density $G_{pp}(f)$ which is expected to be 'flat'. In fact, this assumption was made while deriving the theory of the CRFM. The $G_{pp}(f)$ of Figure 45 was obtained by testing the excitation system alone as shown in Figure 46. In place of the specimen a microphone was placed and the sound pressure level was tape recorded. The power spectral density of this was computed on a digital computer after analogue to digital conversion of the signals. Referring to Figure 46, the excitation is generated by the constrained tandem combination of the sweep oscillator [power spectrum $G_s(f)$], the power amplifier [power spectrum $G_a(f)$] and the loudspeaker [power spectrum $G_l(f)$]. The tandem system is constrained because for all times the input to one is always the output of the one immediately preceding it. Also, the tandem combination is a product one where the total power spectrum $G_{pp}(f)$ of the system is the product of $G_s(f)$, $G_a(f)$ and $G_l(f)$. Since the sweep oscillator and the amplifier have suitable characteristics, it is found that the loudspeaker is not quite suitable. Also the
characteristics of the independent constrained tandem combination of
the microphone, the sound pressure level meter and the tape recorder
(Figure 46) are supposed to be suitable from the data supplied by the
manufacturers. From this it may be seen that the acoustic excitation
that is used is not the ideal one. However, if the structure under test
is a lightly damped one (the resonant peaks are sharp) and the excitation
spectrum across each resonant peak is approximately constant or only
slowly varying, then the excitation spectrum outside this frequency
region is unimportant and does not affect the resonant response.

Figure 47(a-b) shows the time history and the autocorrelation
function respectively of the displacement response at an arbitrarily
selected point at about the centre of a quadrant of the plate. The
autocorrelation function is a one sided one and is zero for all values
of time lag \( \tau \) less than zero. Since it is a very rapidly decaying
function only about 500 time lags are sufficient. The rest of the data
may be supplied as zeros (or zero extensions). One beneficial aspect
of this is that the small noisy signals beyond 500 time lags are
completely removed.

Figure 48, Figure 49 and Figure 50 are the plots of the
modulus of the Fourier transform of the output (written as frequency
response function \( |H(\omega)| \) in the plots) at 10 Hz, 5 Hz and 1 Hz resolution
respectively, together with the phase against frequency. It may be seen
that the plots are not quite smooth (noisy) and the phase information
cannot be utilised, since the phase information of the input is not
available. Though the frequencies for the peak responses may be extracted
from these figures, we prefer not to do so, since we cannot find them accurately.

The next figure (Figure 51) is the plot of the modulus of the Fourier Transform of the one-sided autocorrelation function of the response of Figure 47(b) (the characteristic response function \( |F(i\omega)| \)) together with its phase against frequency at 10 Hz resolution. A comparison between Figure 51 and Figure 50 shows that Figure 51 is smoother than Figure 50 (not noisy) and all its phase information is available.

Again we prefer not to pick up the frequencies of the peak responses from Figure 51 since it cannot be done from this figure accurately. But it is possible to pick up areas of interest from this figure. Such an area of interest is picked up from Figure 51 and is plotted in a magnified scale and at 1 Hz resolution in Figure 52. The next three plots (Figure 53 to Figure 55) are the vector plots of selected frequency bands of the response of Figure 51. The extra constructions of Figure 53 are for the determination of damping of the plate. Figure 56 shows the input power spectral density between 1180 and 1260 Hz. Due to the spiky nature of the input spectrum shown in that figure, the vector plot of Figure 55 shows too many lobes all of which are not necessarily due to the presence of natural frequencies.

Since, compared to the lowest frequency, the higher frequencies have smaller responses, it is preferable that the plate should be tested separately in smaller frequency ranges. Therefore separate tests are carried out between frequency ranges of 1250-1650 Hz, 1650-2050 Hz and 2050-2450 Hz respectively. The records for these cases also are taken at
almost the same position at which the previous record was taken. The amplitude and the phase plots between 1250-1650 is given in Figure 57 followed by vector plots of selected frequency bands in Figure 58 and Figure 59. Similarly, the amplitude and phase plots between 1650-2050 Hz are given in Figure 60, followed by vector plots of selected frequency bands in Figure 61 and Figure 62. Again, the amplitude and phase plots between 2050-2450 Hz are shown in Figure 63 followed by vector plots of selected frequency bands in Figure 64 to Figure 67.

Beyond 2500 Hz the rapid frequency test is not continued because of the limitation on the frequency response of the tape recorder. However, the discrete frequency response test and the determination of the mode shapes with sand patterns are continued up to 5000 Hz.

A comparison between the calculated and the measured frequencies of the clamped singly curved plate shown in Figure 42 is presented in Table 30. For convenience, \( f_c \) will denote calculated resonant frequency (the first 32 of them may be found in Table 19), \( f_r \) and \( f_d \) measured resonant frequencies by the rapid and discrete frequency test respectively. Reference to 'observed mode shape' will mean observed sand pattern at a peak response frequency which is assumed to be a resonant frequency. Obvious difficulty with this assumption is that some close natural frequencies may lie in apparent depressions instead of peaks (an example of such a case is given in reference |84|). If two close peaks are observed then it may be worthwhile to check the mode shapes in the intermediate depression also.
It may be observed from Table 30 that the \( f_r \) and \( f_d \) agree reasonably well. This confirms that the CRFM may be used even though the input spectrum may not be absolutely 'flat' for the whole frequency range of interest, provided only smaller frequency bands within the range are analysed.

The lowest frequency mode is the 1,2 mode predicted by the theory and confirmed by the experiment. The difference between the \( f_c \) and the \( f_r \) for this mode is less than 10% (incidentally this is the biggest difference that may be observed in the table). From Figure 53 the half power points may be estimated to be around 802 and 827 Hz. The damping ratio estimated from this is about 0.015 (loss factor \( \approx 0.03 \)). Therefore, the plate is lightly damped.

Now consider the second and the third frequencies of Table 30. They have the same \( m, n \) (1,3) of \( w \). The mode shapes observed experimentally were exactly similar to the ones predicted theoretically. The nodal lines corresponding to the smaller frequency lie nearer to the edges whereas those corresponding to the bigger one lie nearer to the centre. Thus the experiment confirms that more than one mode with \( w \) having the same \( m \) and \( n \) is an analytically and experimentally obtainable fact in curved plates. For these two modes the magnitudes of \( f_c \) and \( f_r \) agree within 5%.

The fourth and fifth frequencies need no comment. It may be observed that for \( f_r \) we have bracketed two frequencies for the sixth mode. The two peaks corresponding to these two frequencies may clearly be seen in the amplitude plot of Figure 60 and its vector plot in Figure 62. However, the phase plot in Figure 60 indicates only one resonance there.
The discrete frequency test also shows only one resonance. Similarly, for the seventh frequency we have bracketed two frequencies. The discrete frequency response test also shows two \((1,4)\) modes. Also it appears as if the theoretical prediction of the sixth and the seventh modes are not correct in magnitude when compared with the corresponding \(f_d\)'s. Also it was observed that in the discrete frequency response test, the responses at frequencies 1735 Hz and 1802 Hz were very strong. Moreover, the peaks at these resonances were found to be quite broad. Because of this, probably one resonance might have been missed in between these two frequencies. The relative strength of the responses within this zone may be seen from the vector plot of Figure 62.

The next three frequencies hardly need any comment. However, from Figure 63 it may be seen that the resonance at 2225 Hz is in a depression rather than in a peak. The phase plot shows a change of sign. The vector plot of Figure 65 also shows only a phase change and a slight increase of response. However, the discrete frequency test clearly shows the \((3,2)\) mode there.

Now considering all the vector plots and the amplitude plots one may easily see that there are many more bulges and spikes suggesting the existence of more resonant frequencies than we would like to admit. However, it may be suggested that most of the small bulges and spikes are due to the nature of the excitation. In this experiment we have conveniently used the rapid frequency test as the overall guide to the possibility of existence of natural frequencies, which are subsequently verified by the discrete frequency tests.
Comparison between the calculated and the experimental frequencies beyond those of Table 30 are provided in Table 31. In this table, the measured frequencies are all obtained by discrete frequency response test. But from Figure 67, the two $f_c's$ corresponding to the first two bracketed $f_d's$ may be seen to be 2313 and 2339 Hz respectively. From this table it may be seen that whenever the modes agree the $f_c's$ and the $f_d's$ compare reasonably well (within 5\% of each other). Also it may be observed that there are quite a few degenerate modes. Though the magnitudes of some of the calculated and the experimental frequencies compare reasonably well, not a single degenerate mode shape agrees entirely (comparing the sand patterns of Table 31 with Figure 24 and Figure 25). The theoretically obtained mode shape for the second one is more or less the same as the sand pattern if either of the figures is rotated by 90°. The third one tends to agree. The degenerate mode of frequency 3064.94 Hz shown in Figure 24 is possibly not a degenerate mode but is the (2,5) mode (6th mode of Table 31). Similarly, most likely the degenerate mode of 3103.02 Hz of Figure 24 is the 4,1 mode (compare with the 8th mode of Table 31). The degenerate mode of 4478.56 Hz of Figure 24 looks somewhat like the 20th mode of Table 31. The degenerate mode of 4698.54 Hz of Figure 24 appears to be the (3,6) mode (number 21 of Table 31). Similarly, the degenerate mode of frequency 5099.42 Hz shown in Figure 25 may not be degenerate but appears to be the (2,7) mode (number 23 in Table 31). Thus it may be seen that most of the degenerate modes shown in Figure 24 and Figure 25 are not degenerate but are regular modes which need more definition. However,
there is no doubt that some of them are really degenerate. The experiment and the theory show that even in rectangular plates, there may be degenerate modes, although one to one correspondence between the theoretical and the experimental degenerate mode shapes could not be obtained here.

9.9 Conclusions

The experimental results show that the theoretical results are of reasonable order of magnitude and the predicted mode shapes are correct but are doubtful in the case of the degenerate ones. The experimental results have helped immensely in establishing the correctness of some results obtained theoretically.

The theory predicts and the experiment shows that \((1,2)\) is the lowest frequency mode. The \((1,1)\) mode does not seem to exist for the plate. The theory and the experiment both agree that there are two modes with \(w\) having the same mode numbers \((1,3)\). The experiment and the theory both agree that the nodal lines corresponding to the lower frequency are nearer to the edges than those corresponding to the higher one. Also, the magnitudes of these frequencies predicted by the theory and obtained experimentally compare reasonably well.

The experiment and the theory both show that there are degenerate resonant modes though the plate is a rectangular one. Usually degenerate modes are not expected in rectangular plates. However, these frequencies are of very high order.

Consideration of elastic support has little effect on the frequencies. Assuming the plate to be 3.034 in x 4.034 in instead of
3 in x 4 in, the two lowest frequencies have decreased to 879 Hz and 957.35 Hz from 890 Hz and 965.73 Hz respectively. These two results were obtained by the method of Kantorovich.
CHAPTER X
GENERAL DISCUSSION AND CONCLUSIONS

Two different approaches are employed for the study of free vibration characteristics of curved plates with or without the presence of static membrane stresses. One of them is the differential equation approach and the other is the energy approach.

In the differential equation approach, the differential equations of motion are solved. Exact solutions of these equations can be obtained only for some special boundary conditions. Otherwise, the solutions will only be approximate. An approximate method due to Kantorovich \(|1|\) is used to reduce the partial differential equations of motion to ordinary differential equations by prescribing suitable variation of displacements in one of the directions. The accuracy of these ordinary differential equations and the accuracy of the final solutions depend largely on the accuracy with which these variations are prescribed. In general the ordinary differential equations are approximate and therefore the final solutions are also approximate.

For clamped boundaries, it was found that several standard methods of solution of ordinary differential equations are not suitable for this system of equations. The general method of solution, both with real and complex numbers, is found to produce pseudo eigenvalues. The single step matrix integration and the multi-step matrix progression technique are found to yield no solutions at all. The difficulties
experienced with these methods are attributed to the presence of large off-diagonal elements in the matrix equations of the systems and the diminished influence of the boundary conditions of one end at the other. It is shown that all these difficulties may be overcome by using the modified matrix progression technique. Apart from the pseudo eigenvalues the modified matrix progression technique and the general method of solution produced more or less identical solutions for singly curved plates. Therefore, preference may be given to the modified matrix progression technique.

A comparison of these solutions with a truncated power series solution in conjunction with the Rayleigh-Ritz method given by Webster [32] shows very close agreement. However, a comparison with a one term Rayleigh-Ritz solution shows that the one term Rayleigh-Ritz solution is applicable only for a very limited range of panel geometry. Moreover, it was found that the omission of the effects of inplane inertias on the out of plane motion might not be justifiable even for some shallow plates. Due to the omission of the effects of the inplane inertias, the vibration frequencies may sometimes be higher and sometimes be lower than the actual frequencies. Therefore, in general, the inplane inertias should not be omitted.

It is observed that the mode yielding the lowest frequency need not necessarily be the $l, l$ mode of $w$. This is known to be true in complete cylinders also. However, the lowest frequency mode is always associated with $m=1$. In the case when the lowest frequency mode is not the $(1,1)$ mode of $w$, but a higher one, say the $(1,3)$ one, it is not known what happens to the lower ones (in this case the $1, l$ and the $1,2$ modes).
There is a possibility that they may be modes of very high frequencies, but evidence suggests that they do not exist. If these modes really do not exist, then Rayleigh-Ritz type of solutions using completely defined displacement functions might produce frequencies for non-existent modes. Therefore, sufficient amount of care should be taken in evaluating solutions of this type (such solutions may be found in reference 14).

Also, it is found that there may be more than one (though not more than 2 are detected) mode with \( w \) having the same \( m \) and \( n \). That this can happen is explained both on the basis of theory and physical reasoning. Briefly, the basis of the argument is that the mode shape of the plate in a particular mode of deformation cannot be defined uniquely by the mode shape of \( w \) alone. The mode shapes of \( u \), \( v \), together with that of \( w \) are also necessary. However, the mode shapes of \( u \), \( v \) and \( w \) all cannot be independent. They must be compatible with one another in a particular mode of deformation. This has been discussed in detail in Chapter III.

The comparison between the computed clamped plate frequencies and the experimental frequencies given in reference 14, 15, and 47 shows that the computed frequencies are always higher than the experimental frequencies. In general, the experimental frequencies are bounded between the simply supported and the clamped plate frequencies. If allowances are made for deficiencies in the experiments, it may be concluded that the experimental and the theoretical frequencies compare reasonably well.

The Kantorovich-Matrix-Progression technique is successfully extended to include the effects of inplane tensile static stresses on the free vibration characteristics of uniform singly curved plates. It is
shown that the frequencies increase with the increase of the membrane stresses. The variation of the frequency parameter with the stresses is linear over the range where the mode number is constant. If the lowest frequency of free vibrations without the inplane stresses appears at one of the higher order modes, then with the increase of the inplane stresses the mode number decreases until it becomes unity.

It is demonstrated that the Kantorovich-Matrix-Progression technique of analysis may be used in the analysis of doubly curved plates also, provided reasonable functional approximations could be made.

Therefore, it may be concluded that the Kantorovich-Matrix-Progression technique as developed here may be used successfully to predict free vibration characteristics of curved plates. It is possible that polynomial expansions may be used to prescribe the mode shapes of u, v and w in one of the directions, so that the method may be used to predict degenerate modes also. Moreover, higher approximations may be used to refine the solutions further. The fourth order Runge-Kutta technique in conjunction with the Kantorovich reduction may also be tried. The recursive technique of improving the solutions as proposed by Kerr [34] may also be developed to predict more refined solutions.

The energy method of analysis is applied here for the study of dynamic and static behaviour of curved plates in conjunction with the finite element technique. The displacement method of analysis is used. A doubly curved rectangular element is developed. It is shown that all the rigid body modes, particularly the rotations, are not included. Comparison of solutions for free vibrations of clamped flat plates shows
excellent agreement with solutions obtained by other methods. For clamped singly curved plates, the finite element solutions are shown to converge to the Kantorovich solutions. For doubly curved plates, the finite element solutions are shown to be better than the Kantorovich solution. It is because the mode shapes prescribed for the doubly curved plates in the method of Kantorovich were straight beam vibration functions which were not sufficiently representative of the true mode shapes.

The effects of inplane stress (both tensile and compressive) on the stiffness of the element is also taken into account. It is shown that in the presence of the membrane stresses also the results obtained by the finite element method converge monotonically to the Kantorovich solutions with the increase of the number of elements. It is shown that the element is capable of predicting free vibration characteristics for the range of stress system for which the eigenvalues are zeros or positive. For a tensile stress system the conclusions drawn from the Kantorovich method are entirely applicable to the finite element method also. For a compressive stress system the behaviour of the plate is more or less similar to the behaviour under the tensile stress system.

The results obtained with the element confirm some findings using the method of Kantorovich like the possibility of existence of more than one frequency for a given m and n of w. Moreover, unlike the Kantorovich method, the element is found to be convenient to predict modes of vibrations of both singly and doubly curved plates even when the nodal lines are not parallel to the edges of the plates. Also
computations with the element show that there may be degenerate modes of vibrations even for rectangular plates.

Subsequently, the finite element method developed for the analysis of free vibrations was extended to include random and static analysis also. A method for the analysis of random vibrations which does not require the knowledge of the eigenvalues and the eigenvectors is advanced. Though theoretically attractive, it is shown to require too much computing time. Therefore, an approximate design method is used to predict the r.m.s. stresses and displacements. This method is based on the approximation that major part of the response is due to the response of a single mode. For the application of this design method the static stresses and displacements are required. It is shown that though the static stresses in general are discontinuous, the maximum stresses of interest may be predicted without difficulty in this case.

The extension to the finite element technique developed here may be carried out in several directions. The first one will be to assume quadratic variation of stresses across the thickness of the plate, together with improved displacement functions for $u$ and $v$. Attempts may be made to represent all the rigid body modes in the displacement functions. The development of a conforming element in curvilinear coordinate systems utilizing the general energy expression given in Appendix 10 may be a major achievement. Also attention may be paid to utilize the force and the mixed method. Elements of different shapes may also be considered.
The theoretically obtained results for the free vibrations are verified experimentally. The agreements between the theoretical and the experimental results are reasonably good. Also, the experiment has substantiated the fact that there could be more than one frequency for a given pair of \( m \) and \( n \) of \( w \), the fact which was predicted both by the method of Kantorovich and the finite element analysis. Also, it has confirmed that at the higher modes of vibrations there could be degenerate modes even in rectangular plates.

In order to improve the experiment with the Characteristic Response Function Method, efforts may be directed towards generating the required acoustic input of 'flat' spectrum over the whole frequency range of interest.

It is hoped that the findings of this thesis will help a little towards the understanding of the problem.
APPENDIX 1

The reduced equation 2.9(a-c) and its coefficients are given in this Appendix.

The reduced equations 2.9(a-c) may be written as follows.

\[ p_{11} \ddot{g}_1 + a_{12} \dddot{g}_1 + a_{13} \dddot{g}_2 + p_{12} \ddot{g}_3 + a_{16} \dddot{g}_3 = 0, \]

\[ a_{21} \ddot{g}_1 + a_{22} \dddot{g}_2 + p_{21} \ddot{g}_2 + p_{22} \ddot{g}_3 = 0, \quad \text{Al.1(a-c)} \]

\[ p_{31} \ddot{g}_1 + a_{34} \dddot{g}_1 + p_{33} \dddot{g}_2 + p_{32} \dddot{g}_2 + p_{34} \dddot{g}_3 + a_{39} \dddot{g}_3 = 0. \]

The coefficients of these equations are given below.

\[ a_{11} = \int_{0}^{\bar{x}} f_1'' f_1 \, dx \]

\[ a_{12} = 0.5(1 - \nu)(1 + k) \int_{0}^{\bar{x}} f_1 f_1 \, dx \]

\[ a_{13} = 0.5(1 + \nu) \int_{0}^{\bar{x}} f_2' f_1 \, dx \]

\[ a_{14} = \nu \int_{0}^{\bar{x}} f_3' f_1 \, dx \]

\[ a_{15} = k \int_{0}^{\bar{x}} f_3''' f_1 \, dx \]

\[ a_{16} = 0.5k(1 - \nu) \int_{0}^{\bar{x}} f_3'' f_1 \, dx \]

\[ a_{17} = \Delta \int_{0}^{\bar{x}} f_1 f_1 \, dx \]

\[ a_{21} = 0.5(1 + \nu) \int_{0}^{\bar{x}} f_1' f_2 \, dx \]

\[ a_{22} = \int_{0}^{\bar{x}} f_2' f_2 \, dx \]
$$a_{23} = 0.5(1 - \nu)(1 + 3k) \int_{0}^{\bar{x}} f_2'' f_2 \, dx$$

$$a_{24} = \int_{0}^{\bar{x}} f_3 f_2 \, dx$$

$$a_{25} = 0.5k(3 - \nu) \int_{0}^{\bar{x}} f_3' f_2 \, dx$$

$$a_{26} = \Delta \int_{0}^{\bar{x}} f_2 f_2 \, dx$$

$$a_{31} = \nu \int_{0}^{\bar{x}} f_1 f_3 \, dx$$

$$a_{32} = \int_{0}^{\bar{x}} f_2 f_3 \, dx$$

$$a_{33} = (1 + k) \int_{0}^{\bar{x}} f_3 f_3 \, dx$$

$$a_{34} = 0.5k(1 - \nu) \int_{0}^{\bar{x}} f_1' f_3 \, dx$$

$$a_{35} = k \int_{0}^{\bar{x}} f_1'' f_3 \, dx$$

$$a_{36} = 0.5k(3 - \nu) \int_{0}^{\bar{x}} f_2' f_3 \, dx$$

$$a_{37} = k \int_{0}^{\bar{x}} f_3''' f_3 \, dx$$

$$a_{38} = 2k \int_{0}^{\bar{x}} f_3'' f_3 \, dx$$

$$a_{39} = k \int_{0}^{\bar{x}} f_3 f_3 \, dx$$

$$a_{40} = 2k \int_{0}^{\bar{x}} f_3 f_3 \, dx$$

$$a_{41} = \Delta \int_{0}^{\bar{x}} f_3 f_3 \, dx$$

$$p_{11} = a_{11} + a_{17} \quad p_{12} = a_{14} - a_{15}$$

$$p_{21} = a_{23} + a_{26} \quad p_{22} = a_{24} - a_{25}$$

$$p_{31} = a_{31} - a_{35} \quad p_{32} = a_{33} + a_{37} - a_{41}$$

$$p_{33} = a_{32} - a_{36} \quad p_{34} = a_{38} + a_{40}$$
APPENDIX 2

The reduced equation 2.10(a-c) and its coefficients are given in this Appendix.

The reduced equations 2.10(a-c) may be written as follows

\[ a_{11} f''_1 + F f'_1 + a_{13} f'_2 + q f'_3 - a_{16} f'''_3 = 0 \]
\[ a_{23} f''_2 + S f'_2 + a_{24} f'_3 + a_{21} f'_1 - a_{25} f''_3 = 0 \]
\[ a_{37} f'''_3 + a_{32} f'_2 + Q f'_3 + T f'_1 - a_{36} f''_2 + a_{38} f'''_3 - a_{35} f'_1 = 0 \]

The coefficients of equation 2.10(a-c)

\[ a_{11} = \int_{0}^{\theta} g_1 g_1 \, d\phi \]
\[ a_{12} = 0.5(1 - v)(1 + k) \int_{0}^{\theta} g_1 \, d\phi \]
\[ a_{13} = 0.5(1 + v) \int_{0}^{\theta} g_2 g_1 \, d\phi \]
\[ a_{14} = v \int_{0}^{\theta} g_3 g_1 \, d\phi \]
\[ a_{15} = 0.5 k(1 - v) \int_{0}^{\theta} g_3 \, d\phi \]
\[ a_{16} = k \int_{0}^{\theta} g_3 g_1 \, d\phi \]
\[ a_{17} = \Delta \int_{0}^{\theta} g_1 g_1 \, d\phi \]
\[ a_{21} = 0.5(1 + v) \int_{0}^{\theta} g_1 g_2 \, d\phi \]
\[ a_{22} = \int_{0}^{\theta} g_2 g_2 \, d\phi \]
\[ a_{23} = 0.5(1 - \nu)(1 + 3k) \int_0^\theta g_2 g_2 \, d\phi \]
\[ a_{24} = \int_0^\theta g_3^* g_2 \, d\phi \]
\[ a_{25} = 0.5k(3 - \nu) \int_0^\theta g_3^* g_2 \, d\phi \]
\[ a_{26} = A \int_0^\theta g_2 g_2 \, d\phi \]
\[ a_{31} = \nu \int_0^\theta g_1 g_3 \, d\phi \]
\[ a_{32} = \int_0^\theta g_2^* g_3 \, d\phi \]
\[ a_{33} = (1 + k) \int_0^\theta g_3 g_3 \, d\phi \]
\[ a_{34} = 0.5k(1 - \nu) \int_0^\theta g_1^* g_3 \, d\phi \]
\[ a_{35} = k \int_0^\theta g_1 g_3 \, d\phi \]
\[ a_{36} = 0.5k(3 - \nu) \int_0^\theta g_2^* g_3 \, d\phi \]
\[ a_{37} = k \int_0^\theta g_3 g_3 \, d\phi \]
\[ a_{38} = 2k \int_0^\theta g_3^* g_3 \, d\phi \]
\[ a_{39} = k \int_0^\theta g_3^* g_3 \, d\phi \]
\[ a_{40} = A \int_0^\theta g_3 g_3 \, d\phi \]

\[ P = a_{12} + a_{17} \quad q = a_{14} + a_{15} \quad S = a_{22} + a_{26} \]
\[ Q = a_{33} + a_{39} + a_{38} - a_{40} \]
\[ T = a_{31} + a_{34} \]
This appendix shows how the ordinary differential equations 2.9(a-c) or A1.1(a-c) are reduced to a set of first order differential equations, the notation being the same as in Appendix 1. In addition, the formulation of the straight edge boundary equations is given for the clamped case.

The equations A1.1(a-c) are rewritten in the following form:

\[ g_1^{**} = -(p_{11}/a_{12})g_1 - (p_{12}/a_{12})g_2 - (a_{13}/a_{12})g_3 - (a_{16}/a_{12})g_3^{**} , \]

\[ g_2^{**} = -(p_{21}/a_{22})g_2 - (a_{21}/a_{22})g_1 - (p_{22}/a_{22})g_3^{**} , \]

\[ g_3^{***} = ((p_{11}a_{34}/a_{12} - p_{31})/a_{39})g_1 + ((p_{12}a_{34}/a_{12} - p_{32})/a_{39})g_3 + ((a_{13}a_{34}/a_{12} - p_{33})/a_{39})g_3^{**} + ((a_{16}a_{34}/a_{12} - p_{34})/a_{39})g_3^{**} . \]

Let the following variables be defined:

\[ g_4 = g_3^{*} , \]
\[ g_5 = g_4^{*} = g_3^{**} , \]
\[ g_6 = g_5^{*} = g_4^{**} = g_3^{***} , \]
\[ g_7 = g_1^{*} , \]
\[ g_8 = g_2^{*} . \]

From equation A3.2(a-e)

\[ g_6^{*} = g_3^{***} ; \]
\[ g_7^{*} = g_1^{**} , \]
\[ g_8^{*} = g_2^{**} . \]

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Combining the equations A 3.1, A 3.2 and A 3.3 the following system is obtained.

\[
\frac{d}{d\phi} \begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6 \\
\varepsilon_7 \\
\varepsilon_8
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
z_1 & 0 & z_3 & 0 & z_5 & 0 & 0 & z_8 \\
\frac{-p_{11}}{a_{21}} & 0 & \frac{-p_{12}}{a_{12}} & 0 & \frac{-a_{16}}{a_{12}} & 0 & 0 & \frac{-a_{13}}{a_{12}} \\
0 & -\frac{p_{21}}{a_{22}} & 0 & -\frac{p_{22}}{a_{22}} & 0 & 0 & -\frac{a_{21}}{a_{22}} & 0
\end{bmatrix} \begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6 \\
\varepsilon_7 \\
\varepsilon_8
\end{bmatrix} \tag{A 3.4}
\]

where

\[
z_1 = \frac{1}{a_{39}} \left( p_{11} \frac{a_{34}}{a_{12}} - p_{31} \right),
\]

\[
z_3 = \frac{1}{a_{39}} \left( p_{12} \frac{a_{34}}{a_{12}} - p_{32} \right),
\]

\[
z_5 = \frac{1}{a_{39}} \left( a_{16} \frac{a_{34}}{a_{12}} - p_{34} \right),
\]

\[
z_8 = \frac{1}{a_{39}} \left( a_{13} \frac{a_{34}}{a_{12}} - p_{33} \right).
\]

Equation A 3.4 represents a set of first order differential equations and can be written in matrix notation in the form

\[
\frac{dG}{d\phi} = [A] [G]. \tag{A 3.5}
\]

The boundary conditions

(a) Edge \( \phi = 0 \) clamped.

The conditions are \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0 \).
In matrix form the boundary conditions are

\[ \{ G_o \} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \\ g_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} g_5 \\ g_6 \\ g_7 \\ g_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \]  

A 3.6(a)

In matrix notation

\[ \{ G_o \} = [J_o]\{ G_{o^\bullet\bullet} \}. \]

A 3.6(b)

(b) Edge $\phi = 0$ clamped.

The conditions are again $g_1 = g_2 = g_3 = g_4 = 0$. In matrix form

\[ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \\ g_8 \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \]

A 3.7(a)

In matrix notation

\[ [K_o]\{ G_o \} = \{ 0 \}. \]

A 3.7(b)
APPENDIX 4

Beam Vibration Functions and the Required Integrals

In subsequent paragraphs $F_m$ and $F_n$ will denote $F_{b \chi}$ and $F_{b \phi}$ respectively. The functions and integrals with respect to $x$ will only be stated. The corresponding functions along $\phi$ and the integrals may be obtained by replacing $m$ by $n$, $\bar{x}$ by $\theta$, $\sigma_m$ by $\sigma_n$, $I_m$ by $I_n$, and $\lambda_m$ by $\lambda_n$. The meaning of $\sigma$, $I$ and $\lambda$ will be apparent from the following sections.

Reference for all functions and tables [31].

(a) Two opposite edges $x = 0$ and $x = \bar{x}$ are clamped.

$$ F_m = \cosh \frac{\lambda_m x}{\bar{x}} - \cos \frac{\lambda_m x}{\bar{x}} - \sigma_m (\sinh \frac{\lambda_m x}{\bar{x}} - \sin \frac{\lambda_m x}{\bar{x}}) . $$

The boundary values are

$$ F_m(0) = F_m(\bar{x}) = 0 , $$
$$ F'_m(0) = F'_m(\bar{x}) = 0 . $$

$\lambda_m$ are the roots of the equation

$$ \cos \lambda_m \cosh \lambda_m - 1 = 0 . $$

$$ \sigma_m = \frac{\cosh \lambda_m - \cos \lambda_m}{\sinh \lambda_m - \sin \lambda_m} . $$

$$ \sigma^2_m = \frac{\sinh \lambda_m + \sin \lambda_m}{\sinh \lambda_m - \sin \lambda_m} . $$

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The following.

\[ I_m = \frac{2}{\sigma_m} - \frac{2\sigma_m}{\lambda_m} \]

\[ = \frac{2}{\sigma_m} - \frac{2\tanh \lambda_m \sinh \lambda_m}{\lambda_m \sinh \lambda_m - \sin \lambda_m} \]

is needed.

The values of \( \lambda_m \) and \( \sigma_m \) are given in the table below.

<table>
<thead>
<tr>
<th>m</th>
<th>( \lambda_m )</th>
<th>( \sigma_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.730 04</td>
<td>0.982 502 2</td>
</tr>
<tr>
<td>2</td>
<td>7.853 20</td>
<td>1.000 777 3</td>
</tr>
<tr>
<td>3</td>
<td>10.995 6</td>
<td>0.999 966 5</td>
</tr>
<tr>
<td>4</td>
<td>14.137 2</td>
<td>1.000 001 5</td>
</tr>
<tr>
<td>5</td>
<td>17.278 8</td>
<td>0.999 999 9</td>
</tr>
<tr>
<td>( m&gt;5 )</td>
<td>((2m+1)\pi/2)</td>
<td>1.0</td>
</tr>
</tbody>
</table>

(b) Edge \( x = 0 \) clamped and \( x = \bar{x} \) simply supported

\[ F_m = \cosh \frac{\lambda_m x}{\bar{x}} - \cos \frac{\lambda_m x}{\bar{x}} - \sigma_m (\sinh \frac{\lambda_m x}{\bar{x}} - \sin \frac{\lambda_m x}{\bar{x}}) \]

The boundary values are

\[ F_m(0) = F'_m(0) = 0, \]
\[ F_m(\bar{x}) = F''_m(\bar{x}) = 0. \]
\( \lambda_m \) are roots of the equation

\[
\tan \lambda_m = \tanh \lambda_m.
\]

\( \sigma_m = \cot \lambda_m = \coth \lambda_m. \)

The following

\[
\tau_m = \sigma_m \left( \sigma_m - \frac{1}{\lambda_m} \right)
\]

is needed.

The values of \( \lambda_m \) and \( \sigma_m \) are given in the table below.

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \lambda_m )</th>
<th>( \sigma_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.926 60</td>
<td>1.000 777 3</td>
</tr>
<tr>
<td>2</td>
<td>7.068 58</td>
<td>1.000 001 4</td>
</tr>
<tr>
<td>3</td>
<td>10.210 2</td>
<td>1.000 000 0</td>
</tr>
<tr>
<td>4</td>
<td>13.351 8</td>
<td>1.000 000 0</td>
</tr>
<tr>
<td>5</td>
<td>16.493 4</td>
<td>1.000 000 0</td>
</tr>
<tr>
<td>( m&gt;5 )</td>
<td>(2m+1)\pi/4</td>
<td>1.0</td>
</tr>
</tbody>
</table>

(c) \underline{Edges \( x = 0 \) and \( x = \bar{x} \) are simply supported}

\[
F_m = \sin \frac{\lambda_m x}{\bar{x}}.
\]

The boundary values are

\[
F_m(0) = F_m(\bar{x}) = 0,
\]

\[
F_m''(0) = F_m''(\bar{x}) = 0.
\]
\[ \lambda_m = m \pi . \]

\[ I_m = 1 . \]

Also

\[ \int_0^{\bar{x}} \sin^2 \frac{\lambda m}{x} \, dx = \int_0^{\bar{x}} \cos^2 \frac{\lambda m}{x} \, dx = \frac{\bar{x}}{2} , \]

and

\[ \int_0^{\bar{x}} \sin \frac{\lambda m}{x} \cos \frac{\lambda m}{x} \, dx = 0 . \]

In actual substitution of the value of the first integral only \( \bar{x} \) is needed to be used as \( 1/2 \) drops out as a common factor.

With these functions and notations, all the required integrals may be written down.

\[ \int_0^{\bar{x}} F_m F_m \, dx = \bar{x} \]

\[ \int_0^{\bar{x}} F_m' F_m' \, dx = \frac{\lambda m}{x} \bar{x} \]

\[ \int_0^{\bar{x}} F_m F_m' \, dx = \frac{(\lambda m)^2}{x} I_m \bar{x} \]

\[ \int_0^{\bar{x}} F_m' F_m' \, dx = -\frac{(\lambda m)^2}{x} I_m \bar{x} \] \hspace{1cm} A4.1(a-f)

\[ \int_0^{\bar{x}} F_m'' F_m' \, dx = \frac{\lambda m}{x} \bar{x} \]

\[ \int_0^{\bar{x}} F_m''' F_m' \, dx = \frac{(\lambda m)^4}{x} \bar{x} \]

These integrals are true for the three types of boundary conditions under discussion except when two opposite edges are simply supported a factor of \( 1/2 \) is needed. As pointed out before the total omission of the \( 1/2 \) does not make any difference to the final results.
Now \( f_1, f_2, f_3 \) and \( g_1, g_2, g_3 \) may be formally expressed according to equation 1.18(a-c) as the following.

\[
\begin{align*}
f_1 &= \left( \frac{\pi}{\lambda_m} \right) F'_m \\
g_1 &= \frac{1}{\lambda_n} F'_n \\
f_2 &= F_m \\
g_2 &= \left( \frac{\theta}{\lambda_n} \right) F'_n \\
f_3 &= F_m \\
g_3 &= \frac{1}{\lambda_n} F'_n
\end{align*}
\]

With these notations and the above integrals, the coefficients of the reduced equations are given as follows.

(a) The coefficients of the reduced equation 2.9(a-c) given in Appendix 1 are written in terms of the beam integrals as follows.

\[
\begin{align*}
a_{11} &= -\left( \frac{\lambda}{\lambda_n} \right)^2 \bar{x} \\
a_{12} &= 0.5(1 - \nu)(1 + k) \bar{I}_m \bar{x} \\
a_{13} &= 0.5(1 + \nu) \left( \frac{\lambda}{\lambda_n} \right) \bar{I}_m \bar{x} \\
a_{14} &= \nu \left( \frac{\lambda}{\lambda_n} \right) \bar{I}_m \bar{x} \\
a_{15} &= -k \left( \frac{\lambda}{\lambda_n} \right)^3 \bar{x} \\
a_{16} &= 0.5k(1 - \nu) \left( \frac{\lambda}{\lambda_n} \right) \bar{I}_m \bar{x} \\
a_{17} &= \Delta \bar{I}_m \bar{x} \\
a_{21} &= -0.5(1 + \nu) \left( \frac{\lambda}{\lambda_n} \right) \bar{I}_m \bar{x} \\
a_{22} &= \bar{x}
\end{align*}
\]
\[ a_{23} = -0.5(1 - \nu)(1 + 3k) \left( \frac{\lambda}{x} \right)^2 \Im\bar{x} \]

\[ a_{24} = \bar{x} \]

\[ a_{25} = -0.5k(3 - \nu) \left( \frac{\lambda}{x} \right)^2 \Im\bar{x} \]

\[ a_{26} = \Delta\bar{x} \]

\[ a_{31} = -\nu \left( \frac{\lambda}{x} \right) \Im\bar{x} \]

\[ a_{32} = \bar{x} \]

\[ a_{33} = (1 + k) \bar{x} \]

\[ a_{34} = -0.5k(1 - \nu) \left( \frac{\lambda}{x} \right) \Im\bar{x} \]

\[ a_{35} = k \left( \frac{\lambda}{x} \right)^3 \bar{x} \]

\[ a_{36} = -0.5k(3 - \nu) \left( \frac{\lambda}{x} \right)^2 \Im\bar{x} \]

\[ a_{37} = k \left( \frac{\lambda}{x} \right)^4 \bar{x} \]

\[ a_{38} = -2k \left( \frac{\lambda}{x} \right)^2 \Im\bar{x} \]

\[ a_{39} = k \bar{x} \]

\[ a_{40} = 2k \bar{x} \]

\[ a_{41} = \Delta\bar{x} \]
(b) The coefficients of the reduced equation 2.10(a-c) given in Appendix 2 are written in terms of the beam integrals below.

\[ a_{11} = \theta \]

\[ a_{12} = -0.5(1 - \nu)(1 + k) \left( \frac{\lambda n}{\delta} \right)^2 I_n \theta \]

\[ a_{13} = -0.5(1 + \nu) \left( \frac{\lambda n}{\delta} \right) I_n \theta \]

\[ a_{14} = \nu \theta \]

\[ a_{15} = -0.5k(1 - \nu) \left( \frac{\lambda n}{\delta} \right)^2 I_n \theta \]

\[ a_{16} = k \theta \]

\[ a_{17} = \Delta \theta \]

\[ a_{21} = 0.5(1 + \nu) \left( \frac{\lambda n}{\delta} \right) I_n \theta \]

\[ a_{22} = -\left( \frac{\lambda n}{\delta} \right)^2 \theta \]

\[ a_{23} = 0.5(1 - \nu)(1 + 3k) I_n \theta \]

\[ a_{24} = \left( \frac{\lambda n}{\delta} \right) I_n \theta \]

\[ a_{25} = 0.5k(3 - \nu) \left( \frac{\lambda n}{\delta} \right) I_n \theta \]

\[ a_{26} = \Delta I_n \theta \]

\[ a_{31} = \nu \theta \]
\[ a_{32} = -\left(\frac{n}{\theta}\right) I_n \theta \]

\[ a_{33} = (1 + k) \theta \]

\[ a_{34} = -0.5k (1 - \nu) \left(\frac{n}{\theta}\right) I_n \theta \]

\[ a_{35} = k \theta \]

\[ a_{36} = -0.5k (3 - \nu) \left(\frac{n}{\theta}\right) I_n \theta \]

\[ a_{37} = k \theta \]

\[ a_{38} = -2k \left(\frac{n}{\theta}\right)^2 I_n \theta \]

\[ a_{39} = k \left(\frac{n}{\theta}\right)^4 \theta \]

\[ a_{40} = \Delta \theta \]
APPENDIX 5

In this appendix the coefficients $b_{ij}$ of equation 3.68 are given.

The symbols which are not defined in this appendix are defined in Appendix 2. Let

$$d_{11} = \int_{0}^{\theta} g_1 \delta_1 \, d\phi,$$

$$d_{22} = \int_{0}^{\theta} g_2 \delta_2 \, d\phi,$$

$$d_{33} = \int_{0}^{\theta} g_3 \delta_3 \, d\phi,$$

$$M = \frac{m\pi}{X}.$$

The coefficients $b_{ij}$ of equation 3.6 are as follows.

$$b_{11} = \frac{(a_{12} - a_{11}M^2)}{d_{11}},$$

$$b_{12} = \frac{a_{13}M}{d_{11}}.$$

$$b_{13} = \frac{(a_{16}M^3)}{d_{11}}.$$

$$b_{21} = \frac{-a_{21}M}{d_{22}}.$$

$$b_{22} = \frac{(a_{22} - a_{23}M^2)}{d_{22}}.$$

$$b_{23} = \frac{(a_{24} + a_{25}M^2)}{d_{22}}.$$

$$b_{31} = \frac{-(TM + a_{35}M^2)}{d_{33}}.$$

$$b_{32} = \frac{(a_{32} + a_{36}M^2)}{d_{33}}.$$

$$b_{33} = \frac{(a_{33} + a_{38} + a_{39} + a_{37}M^4 - a_{38}M^2)}{d_{33}}.$$
APPENDIX 6

\[ d_1 = \bar{x} f_1 \frac{dx}{x}, \quad d_2 = \bar{x} f_2 \frac{dx}{x}, \quad d_3 = \bar{x} f_3 \frac{dx}{x} \]

\[ M = \frac{mn}{dx}, \quad N = \frac{nm}{dx} \]

(a) Coefficients of equation 3.73.

The coefficients \( a_{ij} \) and \( p_{ij} \) are defined in Appendix 1.

\[ b_{11} = (a_{11} - a_{12}N^2)/d_1 \]
\[ b_{12} = -a_{13}N/d_1 \]
\[ b_{13} = (p_{12} - a_{16}N^2)/d_1 \]
\[ b_{21} = a_{21}N/d_2 \]
\[ b_{22} = (a_{23} - a_{22}N^2)/d_2 \]
\[ b_{23} = p_{22}N/d_2 \]
\[ b_{31} = (p_{31} - a_{34}N^2)/d_3 \]
\[ b_{32} = -p_{33}N/d_3 \]
\[ b_{33} = (a_{33} + a_{37} - p_{34}N^2 + a_{39}N^4)/d_3 \]

(b) Coefficients of equation 3.75

\[ c_{11} = - \left[ \frac{N^2 + \frac{1-v}{2} (1 + \nu) N^2}{MN} \right] \]
\[ c_{12} = - \frac{1 + \nu}{2} MN \]
\[ c_{13} = M \left[ \nu - \frac{1-v}{2} MN^2 + \frac{MN^2}{2} \right] \]
\[ c_{22} = - \left[ \frac{1 - \nu}{2} (1 + 3k) M^2 + N^2 \right] \]
\[ c_{23} = N \left[ 1 + \frac{3 - \nu}{2} kM^2 \right] \]
\[ c_{33} = - \left[ 1 + k(M^2 + N^2)^2 - 2N^2 + 1 \right] \]
The eigenvalues and eigenfunctions are derived for a beam which is subjected to a unit non-dimensional tensile load and has both ends clamped. Subsequently, the eigenfunctions of the beam are used to derive the constant coefficients of the reduced equations for a plate vibrating in the presence of tensile membrane stresses.

The equation of transverse vibrations of motion of the beam described above is

\[ \frac{d^4 \bar{w}}{dn^4} - \frac{d^2 \bar{w}}{dn^2} - \Delta_b \bar{w} = 0. \]  \hspace{1cm} A 7.1

In this equation \( n \) is the non-dimensional axial coordinate, \( \bar{w} \) is the transverse displacement, and \( \Delta_b \) is the frequency parameter.

The four characteristic roots of \( A 7.1 \) are \( \chi, -\chi, i\mu, -i\mu \), where

\[ \chi = \sqrt{\frac{1}{2}(1 + \sqrt{1 + 4\Delta_b})}, \]
\[ \mu = \sqrt{\frac{1}{2}(-1 + \sqrt{1 + 4\Delta_b})}. \]

The frequency equation for \( \bar{w} = \frac{d\bar{w}}{dn} = 0 \) at both ends of the beam is

\[ \frac{1}{2\sqrt{\Delta_b}} \sinh \chi \sin \mu - \cosh \chi \cos \mu + 1 = 0. \]  \hspace{1cm} A 7.2

The eigenfunction is given by

\[ \bar{w} = C_1 \cosh \chi n + C_2 \sinh \chi n + C_3 \cos \mu n + C_4 \sin \mu n. \]  \hspace{1cm} A 7.3

The constants \( C_1, C_2, C_3, C_4 \) have the following values,

\[ C_1 = (0.5 + \mu^2)\sigma_m/\chi, \]
\[ C_2 = -\mu/\chi, \]
\[ C_3 = -(\chi^2 - 0.5)\sigma_m/\chi, \]
\( C_4 = 1.0.\)

\( \sigma_m \) is defined as the following

\[
\sigma_m = \frac{\tan \chi \tan \mu}{\mu \tan \mu + \chi \tanh \chi}.
\]

The values of \( A_b, \chi, \mu, C_1, C_2, C_3 \) and \( C_4 \) are listed below for the first three modes.

<table>
<thead>
<tr>
<th>Mode</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_b )</td>
<td>512.85934</td>
<td>3849.5758</td>
<td>14716.531</td>
</tr>
<tr>
<td>( \chi )</td>
<td>4.8116439</td>
<td>7.9086643</td>
<td>11.036881</td>
</tr>
<tr>
<td>( \mu )</td>
<td>4.7065823</td>
<td>7.8451877</td>
<td>10.991485</td>
</tr>
<tr>
<td>( C_1 )</td>
<td>0.99421013</td>
<td>0.99124455</td>
<td>0.99591889</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>-0.97816514</td>
<td>-0.99197380</td>
<td>-0.99588688</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>-0.99421013</td>
<td>-0.99124455</td>
<td>-0.99591889</td>
</tr>
<tr>
<td>( C_4 )</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

It is assumed here that

\[
\begin{align*}
    f_1(x) &= \bar{w}', \\
    f_2(x) &= \bar{w}, \\
    f_3(x) &= \bar{w}.
\end{align*}
\]

Integrals similar to those given in equation A 4.1(a-f) of Appendix 4 are also needed. However, in this case, it is found easier to compute these integrals by the computer. Assuming these integrals to be known, the reduced equations take the following form.

\[
\begin{align*}
P_{11} \bar{e}_1 + a_{12} \bar{e}_2 + a_{13} \bar{e}_3 + P_{12} \bar{g}_3 + a_{16} \bar{g}_3 &= 0, \\
a_{21} \bar{e}_1 + a_{22} \bar{e}_2 + P_{21} \bar{g}_2 + P_{22} \bar{g}_3 &= 0, \\
P_{31} \bar{e}_1 + a_{34} \bar{e}_2 + P_{33} \bar{g}_2 + P_{32} \bar{g}_3 + P_{34} \bar{g}_3 + a_{39} \bar{g}_3 &= 0.
\end{align*}
\]

The coefficients of these equations are given below.
\[ a_{11} = (1 + q_x) \int o \ x \ f_1'' f_1 \ dx \]

\[ a_{12} = \{0.5(1 - v)(1 + k) + q_x\} \int o \ x \ f_1 f_1 \ dx \]

\[ a_{13} = 0.5(1 + v) \int o \ x \ f_2' f_1 \ dx \]

\[ a_{14} = (v - q_x) \int o \ x \ f_3' f_1 \ dx \]

\[ a_{15} = k \int o \ x \ f_3''' f_1 \ dx \]

\[ a_{16} = 0.5k(1 - v) \int o \ x \ f_3' f_1 \ dx \]

\[ a_{17} = \Delta \int o \ x \ f_1 f_1 \ dx \]

\[ a_{21} = 0.5(1 + v) \int o \ x \ f_1' f_2 \ dx \]

\[ a_{22} = (1 + q_x) \int o \ x \ f_2 f_2 \ dx \]

\[ a_{23} = \{0.5(1 - v)(1 + 3k) + q_x\} \int o \ x \ f_2'' f_2 \ dx \]

\[ a_{24} = (1 + q_x) \int o \ x \ f_3 f_2 \ dx \]

\[ a_{25} = 0.5k(3 - v) \int o \ x \ f_3'' f_2 \ dx \]

\[ a_{26} = \Delta \int o \ x \ f_2 f_2 \ dx \]

\[ a_{31} = (v - q_x) \int o \ x \ f_1' f_3 \ dx \]

\[ a_{32} = (1 + q_x) \int o \ x \ f_2 f_3 \ dx \]

\[ a_{33} = (1 + k) \int o \ x \ f_3' f_3 \ dx \]

\[ a_{34} = 0.5(1 - v)k \int o \ x \ f_1 f_3 \ dx \]

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\[ a_{35} = k \int_{0}^{\bar{x}} f_1'' f_3 \, dx \]
\[ a_{36} = 0.5(1 - \nu)k \int_{0}^{\bar{x}} f_2'' f_3 \, dx \]
\[ a_{37} = k \int_{0}^{\bar{x}} f_3''' f_3 \, dx \]
\[ a_{38} = 2k \int_{0}^{\bar{x}} f_3'' f_3 \, dx \]
\[ a_{39} = k \int_{0}^{\bar{x}} f_3 f_3 \, dx \]
\[ a_{40} = (2k - q_\phi) \int_{0}^{\bar{x}} f_3 f_3 \, dx \]
\[ a_{41} = \Delta \int_{0}^{\bar{x}} f_3 f_3 \, dx \]
\[ a_{42} = q_x \int_{0}^{\bar{x}} f_3'' f_3 \, dx \]

\[ p_{11} = a_{11} + a_{17} \]
\[ p_{12} = a_{14} - a_{15} \]
\[ p_{21} = a_{23} + a_{26} \]
\[ p_{22} = a_{24} - a_{25} \]
\[ p_{31} = a_{31} - a_{35} \]
\[ p_{32} = a_{33} + a_{37} - a_{41} - a_{42} \]
\[ p_{33} = a_{32} - a_{36} \]
\[ p_{34} = a_{38} + a_{40} \]
APPENDIX 8

Let \( d_1 = \frac{\bar{x}}{0} f_1 f_1 \, dx \), \( d_2 = \frac{\bar{x}}{0} f_2 f_2 \, dx \), \( d_3 = \frac{\bar{x}}{0} f_3 f_3 \, dx \),

\[
M = \frac{m \pi}{x}, \quad N = \frac{n \pi}{6}.
\]

(a) Coefficients of equation 3.73 with inplane stresses.

The coefficients \( a_{ij} \) and \( p_{ij} \) are defined in Appendix 7.

\[
\begin{align*}
b_{11} &= (a_{11} - a_{12}N^2)/d_1 \\
b_{12} &= -a_{13}N/d_1 \\
b_{13} &= (p_{12} - a_{16}N^2)/d_1 \\
b_{21} &= a_{21}N/d_2 \\
b_{22} &= (-a_{22}N^2 + a_{23})/d_2 \\
b_{23} &= p_{22}N/d_2 \\
b_{31} &= (p_{31} - a_{34}N^2)/d_3 \\
b_{32} &= -p_{33}N/d_3 \\
b_{33} &= (a_{33} + a_{37} - a_{42} - p_{34}N^2 + a_{39}N^4)/d_3.
\end{align*}
\]

(b) Coefficients of equation 3.75 with inplane stresses.

\[
\begin{align*}
c_{11} &= - \left[ (1 + q_x)M^2 - \frac{1 - \nu}{2} (1 + k) + q_\phi \right] N^2 \\
c_{12} &= - \frac{1 + \nu}{2} MN \\
c_{13} &= M \left[ (\nu - q_\phi) + k(M^2 - \frac{1 - \nu}{2} N^2) \right]
\end{align*}
\]
\[ c_{22} = -(1 + q_\phi)N^2 - \left( \frac{1 - \nu}{2} (1 + 3k) + q_x \right)M^2 \]
\[ c_{23} = -N[(1 + q_\phi) + k \left( \frac{3 - \nu}{2} \right)M^2] \]
\[ c_{33} = -[1 + k + q_x M^2 \left( 2k - q_\phi \right)N^2 + k(M^2 + N^2)^2] \].
APPENDIX 9

Reduction of the system of equations 5.10(a-c) along the direction of x.

The reduced equations are as follows

\[(\Delta t_1 + t_2)g_1 + \frac{1 - \nu}{2} t_1 g''_1 + \frac{1 + \nu}{2} t_3 g''_2 - (k_x + \nu k_y) t_4 g_3 = 0,\]

\[\frac{1 + \nu}{2} t_5 g''_1 + (\frac{1 - \nu}{2} t_6 + \Delta t_7) g_2 + t_7 g''_2 - (\nu k_x + k_y) t_8 g_3 = 0, \quad \Delta 9.1(a-c)\]

\[-(k_x + \nu k_y) t_9 g_1 - (\nu k_x + k_y) t_8 g''_2 +\]

\[\{(k_x^2 + 2\nu k_x k_y + k_y^2 - \Delta) t_{10} + \frac{h^2}{12} t_{11}\} g_3 + \frac{h^2}{6} t_{12} g''_3 + \frac{h^2}{12} t_{10} g''_3 = 0.\]

The coefficients \(t_{ij}\) are as follows

\[t_1 = \int_0^L f_1 f_1' \, dx \quad t_2 = \int_0^L f_1' f_1 \, dx \quad t_3 = \int_0^L f_2 f_1' \, dx\]

\[t_4 = \int_0^L f_3 f_1' \, dx \quad t_5 = \int_0^L f_1' f_2 \, dx \quad t_6 = \int_0^L f_2' f_2 \, dx\]

\[t_7 = \int_0^L f_2 f_2 \, dx \quad t_8 = \int_0^L f_3 f_2 \, dx \quad t_9 = \int_0^L f_1' f_3 \, dx\]

\[t_{10} = \int_0^L f_3 f_3 \, dx \quad t_{11} = \int_0^L f_3' f_3 \, dx \quad t_{12} = \int_0^L f_3'' f_3 \, dx\]

\(f_3\) is represented by the characteristic function of a beam consistent with the conditions at the ends \(x = 0\) and \(x = L_x\). Then \(f_2 = f_3\) and \(f_1 = f_3'\) according to 5.19(a-b). The above integrals may be found in Appendix 4.

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APPENDIX 10

The Strain Energy of a Doubly Curved Plate

(Most of the symbols used in this Appendix are local)

In this Appendix the expressions for strain energy of a doubly curved plate are derived. The strain displacement relations due to Vlasov [3], in the curvilinear orthogonal co-ordinate system are used.

Rectangular co-ordinates \((x, y, z)\) of a surface may be given in a certain closed interval by two independent parameters \(\alpha\) and \(\beta\) such that

\[
\begin{align*}
  x &= f_x(\alpha, \beta), \\
  y &= f_y(\alpha, \beta), \\
  z &= f_z(\alpha, \beta). 
\end{align*}
\]

\[A10.1(a-c)\]

The curves \(\alpha = \text{constant}\) and \(\beta = \text{constant}\) are called the parametric or co-ordinate curves.

The square of the length of a line element on the surface is given by

\[(dL)^2 = A^2 \, d\alpha^2 + 2C \, d\alpha d\beta + B^2 d\beta^2 ,\]

where

\[
\begin{align*}
  A^2 &= (\frac{\partial x}{\partial \alpha})^2 + (\frac{\partial y}{\partial \alpha})^2 + (\frac{\partial z}{\partial \alpha})^2 , \\
  C &= \frac{\partial x}{\partial \alpha} \frac{\partial x}{\partial \beta} + \frac{\partial y}{\partial \alpha} \frac{\partial y}{\partial \beta} + \frac{\partial z}{\partial \alpha} \frac{\partial z}{\partial \beta} , \\
  B^2 &= (\frac{\partial x}{\partial \beta})^2 + (\frac{\partial y}{\partial \beta})^2 + (\frac{\partial z}{\partial \beta})^2 . 
\end{align*}
\]

\[A10.2(a-d)\]

\(A^2, B^2\) and \(C\) are called the coefficients of first fundamental form. When the parametric lines are orthogonal, then \(C = 0\).
The coefficients of second fundamental form \( a, b \) and \( c \) are defined as follows.

\[
\begin{align*}
    a &= \frac{1}{\sqrt{A^2 B^2 - C^2}} \left| \begin{array}{ccc}
    \frac{\partial^2 x}{\partial \alpha^2} & \frac{\partial^2 y}{\partial \alpha^2} & \frac{\partial^2 z}{\partial \alpha^2} \\
    \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} & \frac{\partial z}{\partial \alpha} \\
    \frac{\partial x}{\partial \beta} & \frac{\partial y}{\partial \beta} & \frac{\partial z}{\partial \beta} \\
\end{array} \right| \quad \text{A10.3(a)} \\

    c &= \frac{1}{\sqrt{A^2 B^2 - C^2}} \left| \begin{array}{ccc}
    \frac{\partial^2 x}{\partial \alpha \partial \beta} & \frac{\partial^2 y}{\partial \alpha \partial \beta} & \frac{\partial^2 z}{\partial \alpha \partial \beta} \\
    \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} & \frac{\partial z}{\partial \alpha} \\
    \frac{\partial x}{\partial \beta} & \frac{\partial y}{\partial \beta} & \frac{\partial z}{\partial \beta} \\
\end{array} \right| \quad \text{A10.3(b)} \\

    b &= \frac{1}{\sqrt{A^2 B^2 - C^2}} \left| \begin{array}{ccc}
    \frac{\partial^2 x}{\partial \beta^2} & \frac{\partial^2 y}{\partial \beta^2} & \frac{\partial^2 z}{\partial \beta^2} \\
    \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} & \frac{\partial z}{\partial \alpha} \\
    \frac{\partial x}{\partial \beta} & \frac{\partial y}{\partial \beta} & \frac{\partial z}{\partial \beta} \\
\end{array} \right| \quad \text{A10.3(c)}
\end{align*}
\]

When the parametric lines are the lines of curvature, then \( C = c = 0 \). In the orthogonal case the principal curvatures are given by

\[
\begin{align*}
    k_\alpha &= \frac{a}{A} \\
    k_\beta &= \frac{b}{B} \\
\end{align*}
\]

A10.4(a-b)

The total or Gaussian curvature \( K_G = k_\alpha k_\beta \).
In order to derive the expression for the strain energy, a triply orthogonal system of curvilinear co-ordinates in \((a, \beta, \gamma)\) are considered. The middle surface of the three dimensional finite body lies at \(\gamma=0\), so that the thickness \(h(a, \beta)\) of the body is bounded at each point \((a, \beta)\) by \(\gamma = \pm h(a, \beta)/2\). In such a co-ordinate system the following quantities are defined. The subscripts denote the direction of action or simply the direction of the quantities.

\[
\begin{align*}
\sigma_a, \sigma_\beta, \sigma_\gamma & \quad \text{direct stresses} \\
\sigma_{a\beta}, \sigma_{\gamma a}, \sigma_{\beta \gamma} & \quad \text{shear stresses} \\
e_a, e_\beta, e_\gamma & \quad \text{direct strain} \\
e_{a\beta}, e_{\gamma a}, e_{\beta \gamma} & \quad \text{shear strain} \\
V & \quad \text{total volume} \\
dV & \quad \text{elementary volume} \\
S_a & \quad \text{total strain energy}
\end{align*}
\]

The total strain energy stored in a finite body is given by

\[
S_a = \frac{1}{2} \iiint_V \left[ \sigma_a e_a + \sigma_\beta e_\beta + \sigma_\gamma e_\gamma + \sigma_{a\beta} e_{a\beta} + \sigma_{\gamma a} e_{\gamma a} + \sigma_{\beta \gamma} e_{\beta \gamma} \right] dV \tag{A10.5}
\]

The effects of the normal stresses (the stresses with subscript \(\gamma\)) are usually small and are neglected. This leads to the plane stress condition and equation A10.5 becomes

\[
S_a = \frac{1}{2} \iiint_V \left[ \sigma_a e_a + \sigma_\beta e_\beta + \sigma_{a\beta} e_{a\beta} \right] dV \tag{A10.6}
\]
The stress strain relations are as follows:

\[ \sigma_\alpha = \frac{E}{1 - \nu^2} (e_\alpha + \nu e_\beta), \]
\[ \sigma_\beta = \frac{E}{1 - \nu^2} (e_\beta + \nu e_\alpha), \]
\[ \sigma_{\alpha\beta} = \frac{E}{2(1 + \nu)} e_{\alpha\beta}. \]

With these relations the expression A10.6 becomes

\[ S_\alpha = \frac{1}{2} \cdot \frac{E}{(1 - \nu^2)} \iiint \left[ e_\alpha^2 + e_\beta^2 + 2\nu e_\alpha e_\beta + \frac{1 - \nu}{2} e_{\alpha\beta}^2 \right] dv \]

A10.8

The square of the line element dL and the elementary volume dV in the triply orthogonal system are given by the following equations.

\[ (dL)^2 = A^2 (1 + k_\alpha \gamma)^2 \, da^2 + B^2 (1 + k_\beta \gamma)^2 \, db^2 + d\gamma^2, \]
\[ dV = (1 + k_\alpha \gamma)(1 + k_\beta \gamma) \text{AB} \, da \, db \, d\gamma. \]

A10.9(a-b)

The following quantities are defined at a point (\(\alpha, \beta, 0\)), that is at the middle surface of the body. The subscripts define their direction of action or simply direction, or their association with a particular co-ordinate line.

- \( \varepsilon_\alpha, \varepsilon_\beta \) direct strain
- \( \varepsilon_{\alpha\beta} \) shear strain
- \( \chi_\alpha, \chi_\beta \) change of curvature
- \( \chi_{\alpha\beta} \) change of twist curvature
\[ \phi_\alpha = -k_\alpha \chi_\alpha \]
\[ \phi_\beta = -k_\beta \chi_\beta \]
\[ \psi = \frac{1}{2}(k_\alpha - k_\beta)^2 \varepsilon_{\alpha\beta} - \frac{1}{2}(k_\alpha + k_\beta)\chi_{\alpha\beta} \]

These quantities vary with \( \alpha \) and \( \beta \) but are independent of \( \gamma \). For a thin doubly curved plate, the expressions for \( e_\alpha \), \( e_\beta \) and \( e_{\alpha\beta} \) given by Vlasov [3, p.249] may be used in equation A10.6 and A10.7. For thin plates, usually the first three terms involving up to \( \gamma^2 \), in the expansion of \( e_\alpha \), \( e_\beta \) and \( e_{\alpha\beta} \) are retained. The expansions of \( e_\alpha \), \( e_\beta \), \( e_{\alpha\beta} \), \( \sigma_\alpha \), \( \sigma_\beta \) and \( \sigma_{\alpha\beta} \) truncated at the third term are as follows.

\[ e_\alpha = \varepsilon_\alpha + \chi_\alpha \gamma + \phi_\alpha \gamma^2, \]
\[ e_\beta = \varepsilon_\beta + \chi_\beta \gamma + \phi_\beta \gamma^2, \]
\[ e_{\alpha\beta} = \varepsilon_{\alpha\beta} + \chi_{\alpha\beta} \gamma + \psi \gamma^2, \]
\[ \sigma_\alpha = \frac{E}{1 - \nu^2} \left[ \varepsilon_\alpha + \nu \varepsilon_\beta + (\chi_\alpha + \nu \chi_\beta) \gamma + (\phi_\alpha + \nu \phi_\beta) \gamma^2 \right], \]
\[ \sigma_\beta = \frac{E}{1 - \nu^2} \left[ \varepsilon_\beta + \nu \varepsilon_\alpha + (\chi_\beta + \nu \chi_\alpha) \gamma + (\phi_\beta + \nu \phi_\alpha) \gamma^2 \right], \]
\[ \sigma_{\alpha\beta} = \frac{E}{2(1 + \nu)} \left[ \varepsilon_{\alpha\beta} + \chi_{\alpha\beta} \gamma + \psi \gamma^2 \right]. \]

A10.10(a-f)

The expressions for the strains \( e_\alpha \), \( e_\beta \) and \( e_{\alpha\beta} \) from equations A10.10(a-c) and the expression for the elementary volume \( dV \) from equation A10.9(b) are substituted into the expression for \( S_a \) given by equation A10.8. In the resulting expression all terms involving powers of \( \gamma \) higher than the 2nd are neglected. The remainder of the expression is then integrated with respect to \( \gamma \) between the limits of \( \pm h/2 \) (the \( \alpha \) and \( \beta \) in the parentheses are
dropped). This resulted in the expression of $S_a$ of the following form.

$$S_a = S_m + S_b + S_{mb} + S_{cmb},$$  \hspace{1cm} A10.11

where

$$S_m = \frac{1}{2} \frac{E}{1 - \nu} \int \int h(\epsilon^2_{\alpha} + \epsilon^2_{\beta} + 2 \nu \epsilon_{\alpha} \epsilon_{\beta} + \frac{1 - \nu}{2} \epsilon^2_{\alpha\beta})AB \, \text{d}a \, \text{d}b,$$

$$S_{mb} = \frac{1}{24} \frac{E}{1 - \nu} \int \int k^3_{\alpha\beta} h(\epsilon^2_{\alpha} + \epsilon^2_{\beta} + 2 \nu \epsilon_{\alpha} \epsilon_{\beta} + \frac{1 - \nu}{2} \epsilon_{\alpha\beta})AB \, \text{d}a \, \text{d}b,$$

$$S_b = \frac{1}{24} \frac{E}{1 - \nu} \int \int h^3(\chi_{\alpha}^2 + \chi_{\beta}^2 + 2 \nu \chi_{\alpha} \chi_{\beta} + \frac{1 - \nu}{2} \chi_{\alpha\beta}^2)AB \, \text{d}a \, \text{d}b,$$

$$S_{cmb} = \frac{1}{24} \frac{E}{1 - \nu} \int \int \left[ h^3(2k_{\alpha} \chi_{\beta}(\epsilon_{\beta} + \nu \epsilon_{\alpha})) + 2k_{\alpha} \chi_{\alpha}(\epsilon_{\alpha} + \nu \epsilon_{\beta}) + \frac{1 - \nu}{2}(k_{\alpha} - k_{\beta})^2 \epsilon_{\alpha\beta}^2 + \frac{1 - \nu}{2}(k_{\alpha} + k_{\beta})\epsilon_{\alpha\beta}\chi_{\alpha\beta}^2 \right] AB \, \text{d}a \, \text{d}b.$$

For the solution of a problem the strain displacement relations are necessary. The following are the strain displacement relations given by Vlasov.

$$\epsilon_{\alpha} = \frac{1}{A} \frac{\partial u}{\partial \alpha} + \frac{1}{AB} \frac{\partial A}{\partial \beta} \nu + k_{\alpha} w,$$

$$\epsilon_{\beta} = \frac{1}{AB} \frac{\partial u}{\partial \alpha} + \frac{1}{B} \frac{\partial v}{\partial \beta} + k_{\beta} w,$$

$$\epsilon_{\alpha\beta} = \frac{A}{B} \frac{\partial}{\partial \beta} \left( \frac{u}{A} \right) + \frac{B}{A} \frac{\partial}{\partial \alpha} \left( \frac{v}{B} \right),$$

$$\chi_{\alpha} = \frac{3k_{\alpha}}{3a} \frac{u}{A} + \frac{3k_{\alpha}}{3A} \frac{v}{B} - k_{\alpha} w - \frac{1}{A} \frac{\partial}{\partial \alpha} \left( \frac{1}{A} \frac{\partial w}{\partial \alpha} \right) - \frac{1}{AB} \frac{\partial A}{\partial \beta} \frac{\partial w}{\partial \beta},$$

$$\chi_{\beta} = \frac{3k_{\beta}}{3a} \frac{u}{A} + \frac{3k_{\beta}}{3A} \frac{v}{B} - k_{\beta} w - \frac{1}{B} \frac{\partial}{\partial \beta} \left( \frac{1}{B} \frac{\partial w}{\partial \beta} \right) - \frac{1}{A} \frac{\partial A}{\partial \alpha} \frac{\partial w}{\partial \alpha},$$

$$\chi_{\alpha\beta} = (k_{\alpha} - k_{\beta}) \left[ \frac{A}{B} \frac{\partial}{\partial \beta} \left( \frac{u}{A} \right) - \frac{B}{A} \frac{\partial}{\partial \alpha} \left( \frac{v}{B} \right) \right] - \frac{1}{AB} \left( \frac{\partial w}{\partial \alpha} \right)^2 \left( \frac{\partial w}{\partial \beta} \right).$$  \hspace{1cm} A10.12(a-f)
Substitution of these relations into the strain energy expression of A10.11 will yield the strain energy expression in terms of the displacements. Use of this expression to solve a general problem will apparently need a tremendous amount of effort. However, for specific problems of thin plates, it will yield accurate expressions for the strain energy. For example, for a circular cylindrical plate of uniform thickness with $x = Ra$, $y = R \cos \beta$ and $z = R \sin \beta$, it will yield exactly the same strain energy expression given by Bleich and Di Maggio [11] and Warburton [20] using Flügge's [6] strain-displacement relations. The following is a list of the parametric equations of some surfaces which are often encountered in practice.

<table>
<thead>
<tr>
<th>Surface</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circular cylindrical</td>
<td>$Ra$</td>
<td>$R \cos \beta$</td>
<td>$R \sin \beta$</td>
</tr>
<tr>
<td>Spherical</td>
<td>$R \sin \alpha$</td>
<td>$R \cos \alpha \sin \beta$</td>
<td>$R \cos \alpha \cos \beta$</td>
</tr>
<tr>
<td>Hyperboloid of one sheet</td>
<td>$\frac{\alpha - \beta}{\alpha + \beta}$</td>
<td>$\frac{1 + \alpha \beta}{\alpha + \beta}$</td>
<td>$- \frac{\alpha \beta - 1}{\alpha + \beta}$</td>
</tr>
<tr>
<td>Hyperboloid of two sheets</td>
<td>$\frac{a \sinh \alpha}{\cos \beta}$</td>
<td>$\frac{5 \sinh \alpha}{\sin \beta}$</td>
<td>$- \frac{c \cosh \alpha}{\sin \beta}$</td>
</tr>
<tr>
<td>Circular cone</td>
<td>$R \sin \alpha \cos \beta$</td>
<td>$R \sin \alpha \sin \beta$</td>
<td>$R \cos \alpha$</td>
</tr>
<tr>
<td>Elliptic paraboloid</td>
<td>$\frac{\alpha}{a} \cos \beta$</td>
<td>$\frac{5 \alpha \sin \beta}{a}$</td>
<td>$\alpha^2$</td>
</tr>
<tr>
<td>Hyperbolic paraboloid</td>
<td>$\frac{\alpha}{a} \cosh \beta$</td>
<td>$\frac{5 \alpha \sinh \beta}{a}$</td>
<td>$\alpha^2$</td>
</tr>
<tr>
<td>&quot;</td>
<td>$\frac{a(a + \beta)}{a}$</td>
<td>$\frac{5(a - \beta)}{a}$</td>
<td>$\alpha \beta$</td>
</tr>
<tr>
<td>&quot;</td>
<td>$a$</td>
<td>$\beta$</td>
<td>$\alpha \beta$</td>
</tr>
</tbody>
</table>

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R, ̅a, ̅b and ̅c are constant. It may be seen that if A = B = 1, 
dx = dα, dy = dβ and k_α = k_β = 0 are substituted in equation A10.11 and
A10.12, the strain energy S_a reduces to S_m + S_b. This is the complete
flat plate energy due to inplane stretching and combined bending.

Since the strain energy expression A10.11 and the strain-
displacement relations A10.12 are quite difficult to use in practice,
sometimes they are simplified as the following.

\[ e_α = ε_α + X_αγ \]
\[ e_β = ε_β + X_βγ \]
\[ e_αβ = ε_αβ + X_αβγ \]
\[ dV = ABdαdβdγ \]

The assumptions inherent in this simplification are:
(i) the distribution of the strains across the thickness is
linear,
(ii) the variation of the lengths of the line elements across
the thickness is neglected.

Consequently, \( S_a \) reduces to

\[ S_a = S_m + S_b \]

It has been discussed in Chapter I that some authors, notably
Vlasov [3 and 4] and Langhaar and Carver [33] object to the use of the first
assumption. The reason for this is that some terms arising out of the
presence of the curvatures could not be accounted for, which are of the
same order of magnitude as some of the moment terms. Sometimes, "during the mathematical handling of equations, it may happen that the large terms cancel and just the small ones become decisive" - Flügge [6, p.217].
APPENDIX I

The element stiffness matrix $[\bar{K}_o]$

The element stiffness matrix $[\bar{K}_o]$ is given by the double integral (equation 6.19)

$$[\bar{K}_o] = \int_{x=0}^{a} \int_{y=0}^{b} [P^*]^T [E^*] [P^*] \, dx \, dy, \quad A\ 11.1$$

where $[P^*] = [D] [P(x,y)]$.

The matrix $[P^*]$ contains the expansions of the displacement fields $\{u(x,y)\}$ given by equation 6.32(a-c). The quantities $a$ and $b$ are the lengths of the sides of an element along the direction of $x$ and $y$ respectively in the $x$-$y$ plane as shown in (a) of Fig.19.

The matrix $[\bar{K}_o]$ given explicitly in this Appendix is correct for second order surfaces with constant thickness. The same matrix may be used for higher order surfaces and variable thickness provided the elements are so small that the variation of curvatures and thickness over an element is small. Otherwise, the elements of $[\bar{K}_o]$ have to be numerically evaluated.

The matrix $[\bar{K}_o]$ is partitioned into sixteen 6th order square submatrices. This facilitates in assembling the overall stiffness matrix $[\bar{A}]$ and also the presentation of $[\bar{K}_o]$.

The following notations are used here

$$A = k_x + \nu k_y,$$

$$B = \nu k_x + k_y,$$

$$C = (1 - \nu)k_{xy},$$

$$H = k_x^2 + k_y^2 + 2\nu k_x k_y + 2(1 - \nu)k_{xy},$$

$$D = Eh/(1 - \nu^2),$$

$$K = Dh^2/12.$$
The stiffness matrix $[\tilde{K}_o]$ is partitioned into submatrices in the following form.

$$
[\tilde{K}_o] = 
\begin{bmatrix}
S_{11} & S_{21} & \text{symmetrical} \\
S_{21} & S_{22} & \text{} \\
S_{31} & S_{32} & S_{33} \\
S_{41} & S_{42} & S_{43} & S_{44}
\end{bmatrix}
$$

Since $[\tilde{K}_o]$ is symmetrical, only its lower triangle is given here.
$$[S_{11}] = \begin{bmatrix}
0 & 0 & \frac{(1-v)Dab}{2} & \text{Symmetrical} \\
0 & Dab & 0 & 0 \\
0 & 0 & \frac{Dab^2}{2} & \frac{(1-v)Dab^2}{4} \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{(1-v)Dab}{2} & \frac{1-v}{4} \text{Dab}^2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$$

$$[S_{21}] = \begin{bmatrix}
0 & vDab & 0 & \frac{vDab^2}{2} & 0 & 0 \\
0 & \frac{vDab^2}{2} & \frac{1-v}{4} \text{Dab}^2 & \frac{1+v}{8} \text{Dab}^2 & 0 & \frac{1-v}{4} \text{Dab}^2 \\
0 & -DAab & -DCab & -\frac{Dab(Ab+Ca)}{2} & 0 & -DCab \\
0 & \frac{-DAab^2}{2} & \frac{-DCa^2b}{2} & \frac{-Dab^2}{3} + \frac{Ca}{4} & 0 & \frac{-DCab^2}{2} \\
0 & \frac{-DAa^2b}{2} & \frac{-Dc^2b}{2} & \frac{-Dab^2}{3} + \frac{Ca}{2} & 0 & \frac{-DCab^2}{3} \\
0 & \frac{-Daa^2b}{3} & \frac{-Dca^2b}{3} & \frac{-Dab^2}{2} + \frac{Ca}{3} & 0 & \frac{-DCab^2}{3} \\
\end{bmatrix}$$

$$[S_{22}] = \begin{bmatrix}
Dab & \frac{Dab^2}{3}(a^2 + \frac{1-v}{2} b^2) & \text{Symmetrical} \\
\frac{-DBab}{2} & -\frac{Dab}{2}(Ba + Cb) & DHab \\
\frac{-DBa^2b}{2} & -\frac{Da^2b}{3}(Ba + Cb) & \frac{DHa^2b}{2} & \frac{DHa^3b}{3} \\
\frac{-DBab}{2} & -\frac{Da^2b}{4} + \frac{Cb}{3} & \frac{DHa^2b}{4} & \frac{DHa^2b^2}{3} & \frac{DHa^3b}{3} \\
\frac{-DBa^2b}{2} & -\frac{Da^2b}{2} + \frac{Cb}{3} & \frac{DHa^2b}{3} & \frac{DHa^4b}{6} & \frac{DHa^5b}{5} + 4Kab \\
\end{bmatrix}$$
\[
\begin{bmatrix}
S_{31} &=& 
\begin{bmatrix}
0 & - \frac{D_{2a}^2 b^2}{4} & - \frac{D_{c} c^2}{4} & - \frac{D_{a} a^2 b}{6} (Ab + Ca) & 0 & - \frac{D_{ca}^2 b}{4} \\
0 & - \frac{D_{a} a^3 b}{3} & - \frac{D_{c} c^3}{3} & - \frac{D_{a} a b^3 (Ab)}{6} (Ab + Ca) & 0 & - \frac{D_{ca}^2 b}{3} \\
0 & - \frac{D_{2a}^2 b^4}{4} & - \frac{D_{c} c^2}{4} & - \frac{D_{a} a^4 b^2 (Ab)}{5} + \frac{Ca}{6} & 0 & - \frac{D_{ca}^2 b}{4} \\
0 & - \frac{D_{2a}^2 b^2}{6} & - \frac{D_{c} c^2}{6} & - \frac{D_{a} a^3 b^2 (Ab)}{9} + \frac{Ca}{6} & 0 & - \frac{D_{ca}^2 b}{6} \\
0 & - \frac{D_{2a}^2 b^3}{6} & - \frac{D_{c} c^3}{6} & - \frac{D_{a} a^2 b^3 (Ab)}{8} + \frac{Ca}{9} & 0 & - \frac{D_{ca}^2 b}{6} \\
0 & - \frac{D_{a} a^4 b}{4} & - \frac{D_{c} c^4}{4} & - \frac{D_{a} a b^4 (Ab)}{5} + \frac{Ca}{8} & 0 & - \frac{D_{ca}^2 b}{4}
\end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
S_{32} &=& 
\begin{bmatrix}
- \frac{D_{ba}^2 b^6}{4} & - \frac{D_{ba}^2 b^2 (Ba + Cb)}{6} & D_{Ha} a^2 b^2 & D_{Ha} a^2 b^2 & D_{Ha} a^2 b^3 & D_{Ha} a^2 b^3 \\
- \frac{D_{ba}^3}{3} & - \frac{D_{ba}^3 (Ba + Cb)}{2} & \frac{D_{ba} a^3 b^2}{3} & \frac{D_{ba} a^3 b^2}{3} & \frac{D_{ba} a^4 b}{4} & \frac{D_{ba} a^4 b}{4} \\
- \frac{D_{ba} a^2 b}{4} & - \frac{D_{ba} a^2 b (Ba + Cb)}{6} & \frac{D_{ba} a^3 b}{4} & \frac{D_{ba} a^3 b}{4} & \frac{D_{ba} a^4 b^2}{8} & \frac{D_{ba} a^4 b^2}{8} \\
- \frac{D_{ba} a b^2}{6} & - \frac{D_{ba} a b^2 (Ba + Cb)}{9} & \frac{D_{ba} a b^3}{6} & \frac{D_{ba} a b^3}{6} & \frac{D_{ba} a^4 b^3}{10} & \frac{D_{ba} a^4 b^3}{10} \\
- \frac{D_{ba} a^2 b^3}{6} & - \frac{D_{ba} a^2 b^3 (Ba + Cb)}{9} & \frac{D_{ba} a b^4}{6} & \frac{D_{ba} a b^4}{6} & \frac{D_{ba} a^4 b^4}{12} & \frac{D_{ba} a^4 b^4}{12} \\
- \frac{D_{ba} a^4 b}{4} & - \frac{D_{ba} a^4 b (Ba + Cb)}{5} & \frac{D_{ba} a^2 b^4}{4} & \frac{D_{ba} a^2 b^4}{4} & \frac{D_{ba} a^4 b^5}{5} & \frac{D_{ba} a^4 b^5}{5} + 4vKab
\end{bmatrix}
\end{bmatrix}
\]
\[
[S_{33}] = \begin{align*}
\text{DHa}^{3b^3/9} + 2K(1-\nu)ab \\
\text{DHa}^{2b^4/8} + \text{DHa}^{5b^3/5} + 4Kab \\
\text{DHa}^{b^2/10} + 6\sqrt{\text{Ka}^2b} + 12\text{Ka}^3b \\
\text{DHa}^{b^3/12} + 2\nu\text{Kab}^2 + 3\text{Ka}^2b^2 + 4\text{Kab}^3/3 \\
\text{DHa}^{b^3/12} + 2K(1-\nu)a^2b + 8K(1-\nu)a^3b/3 \\
\text{DHa}^{b^4/12} + 2K(1-\nu)ab^2 + 4\nu\text{Ka}^2b^2 + 8K(1-\nu)ab^3/3 \\
\text{DHa}^{b^5/10} + 6\text{Kab}^2 + 9\nu\text{Ka}^2b^2 + 3\text{Ka}^2b^2 + 12\text{Kab}^3 \\
\text{DHa}^{b^4/16} + 4\nu\text{Kab}^3 + 2\nu\text{Ka}^2b^2 + 4\nu\text{Kab}^3 + 12\text{Kab}^3
\end{align*}
\] Symmetrical
\[
[S_{41}] = \\
\begin{bmatrix}
0 & -\frac{DAb^2}{8} & -\frac{DAb^2}{8} & -\frac{A^b}{2}(\frac{Ab}{5} + \frac{Ca}{5}) & 0 & -\frac{DAb^2}{8} \\
0 & -\frac{DAb^4}{8} & -\frac{DAb^4}{8} & -\frac{DAb^4}{2}(\frac{Ab}{5} + \frac{Ca}{6}) & 0 & -\frac{DAb^4}{8} \\
0 & -\frac{DAb^3}{9} & -\frac{DAb^3}{9} & -\frac{DAb^3}{12}(Ab + Ca) & 0 & -\frac{DAb^3}{9} \\
0 & -\frac{DAb^3}{12} & -\frac{DAb^3}{12} & -\frac{DAb^3}{16}(Ab + Ca) & 0 & -\frac{DAb^3}{12} \\
0 & -\frac{DAb^4}{12} & -\frac{DAb^4}{12} & -\frac{DAb^4}{15}(Ab + Ca) & 0 & -\frac{DAb^4}{12} \\
0 & -\frac{DAb^4}{16} & -\frac{DAb^4}{16} & -\frac{DAb^4}{20}(Ab + Ca) & 0 & -\frac{DAb^4}{16}
\end{bmatrix}
\]
\[
\begin{bmatrix}
\mathbf{[S_{42}^{-}]} = \\
- \frac{DBa_4b^2}{8} - \frac{DHa_4b^2}{8} + \frac{DHa_4b^3}{10} + \frac{DHa_4b^4}{12} + \frac{DHa_4b^5}{12} + \\
- \frac{DBa_2b^4}{8} - \frac{DHa_2b^4}{8} + \frac{DHa_2b^5}{10} + \frac{DHa_2b^6}{16} + \\
- \frac{DBa_3b^3}{9} - \frac{DHa_3b^3}{9} + \frac{DHa_3b^4}{12} + \frac{DHa_3b^5}{15} + \\
- \frac{DBa_4b^3}{12} - \frac{DHa_4b^3}{12} + \frac{DHa_4b^4}{15} + \frac{DHa_4b^5}{16} + \\
- \frac{DBa_3b^4}{12} - \frac{DHa_3b^4}{12} + \frac{DHa_3b^5}{15} + \frac{DHa_3b^6}{20} + \\
- \frac{DBa_4b^4}{16} - \frac{DHa_4b^4}{16} + \frac{DHa_4b^5}{20} + \frac{DHa_4b^6}{24} + \\
3 \frac{KBa_2b^2}{2} + 3v \frac{KBa_2b^2}{2} + \\
4 \frac{Kab_2(b^2 + va^2)}{3} + 2 \frac{Kab_2(b^2 + va^2)}{2} + \\
2 \frac{Kab_2(b^2 + va^2)}{2} + \\
3 \frac{KBa_2b^2(b^2 + va^2)}{2}
\end{bmatrix}
\]
\[
S_{43} = \begin{bmatrix}
+2K(1-v)a^3 b & +3vK_2^2 & +6K_2^3 b^2 & +K_2 b(2b^2 + 3(1-v)a^2) & +2K_2 b^3 & +6vK_2 b^3 \\
DHA^5 b^5/15 & DHA^2 b^6/12 & DHA^5 b^4/20 & DHA^4 b^5/20 & DHA^3 b^6/18+ & DHA^2 b^7/14 \\
+2K(1-v)a^2 b^2 & +3K_2 b^2 & +6vK_2 b^2 & +2K_2 b^3 & Kab^2(2a^2 + & +6K_2 b^3 \\
DHA^4 b^4/16 & DHA^3 b^5/15+ & DHA^6 b^3/18 & DHA^5 b^4 & +2Kab^2 \left(\frac{y^2}{2} + \frac{a^2}{3}\right) + \frac{8K(1-v)a^3 b^2}{3} & +2Kab^2 b^2 & DHA^3 b^6 \\
+2K(1-v)a^2 b^2 & \frac{1}{3}Kab^2 (vb^2 + a^2) & +Kab^2 (2b^2/3 + va^2) & +3va^2/5 & +2Kab^2 (vb^2/2 + 3va^2/5) & +6Kab^2 (a^2/3 + vb^2/2) & +3va^2/5 \\
+2K(1-v)a^3 b^2 & 2Kab^2 (vb^2 + a^2) & 4Kab^2 (b^2 + va^2) & +3va^2/5 & +4Kab^2 (vb^2/2 + 3va^2/5) & +3K(1-v)a^2 b^4 & +4Kab^2 (a^2 + 3vb^2/5) \\
DHA^4 b^5/20 & DHA^3 b^6/18+ & DHA^6 b^4/24+ & DHA^5 b^5 & +4Kab^2 (b^2/5) & DHA^4 b^6/24 & +DHA^3 b^7/21 \\
+2K(1-v)a^2 b^3 & 2Kab^2 (vb^2/2 + a^2) & 3Kab^2 (b^2/3 + va^2) & +3va^2/5 & +4Kab^2 (vb^2/2 + 3va^2/5) & +3K(1-v)a^2 b^4 & +4Kab^2 (a^2 + 3vb^2/5) \\
DHA^5 b^5/25 & DHA^4 b^6/24+ & DHA^7 b^4/28+ & DHA^6 b^5 & +6Kab^3 (b^2/5) & DHA^5 b^6/30 & +DHA^4 b^7/28+ \\
+2K(1-v)a^3 b^3 & \frac{3}{2}Kab^2 (a^2 + vb^2) & 18Kab^3 (b^2/6 + va^2/5) & +3K(1-v) & +6Kab^3 (a^2/5 + 3K(1-v) & +18Kab^3 (a^2/6 + 3K(1-v) & +18Kab^3 (a^2/6 + 3K(1-v) \\
\end{bmatrix}
\]
<table>
<thead>
<tr>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2K a^4 b^2 \left{ 2b^2 \left( \frac{9}{5} (1-v) a^2 \right) \right}$</td>
</tr>
<tr>
<td>$D H a^4 b^2 / 24$</td>
</tr>
<tr>
<td>$3 a^2 b^2 \left( b^2 + (a^2 + v b^2) + 3 K (1-v) \right)$</td>
</tr>
<tr>
<td>$- \frac{4 K (1-v) a^3 b^3}{9}$</td>
</tr>
<tr>
<td>$D H a^4 b^2 / 28$</td>
</tr>
<tr>
<td>$3 a^2 b^2 \left( b^2 + (a^2 + v b^2) + 3 K (1-v) \right)$</td>
</tr>
<tr>
<td>$- \frac{4 K (1-v) a^3 b^3}{9}$</td>
</tr>
<tr>
<td>$D H a^4 b^2 / 30$</td>
</tr>
<tr>
<td>$3 a^2 b^2 \left( b^2 + (a^2 + v b^2) + 3 K (1-v) \right)$</td>
</tr>
<tr>
<td>$- \frac{4 K (1-v) a^3 b^3}{9}$</td>
</tr>
</tbody>
</table>

*Symmetrical*
APPENDIX 12

The stiffness Matrix \( [\bar{K}_I] \)

According to equation 6.21 the element stiffness matrix \( [\bar{K}_I] \)
due to the applied inplane stresses is given by

\[
[\bar{K}_I] = \int_0^b \int_0^a \left[ \begin{array}{c} \{p(x,y)\} \\ \{p(x,y)\}^T \end{array} \right] \left[ \begin{array}{c} I_p \\ I_p \end{array} \right] \left[ \begin{array}{c} \{p(x,y)\} \\ \{p(x,y)\}^T \end{array} \right] \, dx \, dy.
\]

\[ A \, 12.1. \]

The matrix \( \left[ \{p(x,y)\} \right] \) contains the expansions of the vector \( \{w^*\} \) which
may be obtained from equations 6.35(a-b). The integration limits \( a \) and \( b \)
are defined in (a) of Fig.19.

The matrix \( [\bar{K}_I] \), given explicitly in the Appendix, is correct for
any element with constant thickness, provided the applied stresses \( s_x, s_y \)
and \( s_{xy} \) are constant. However, it is possible to use the same matrix for
variable \( h, s_x, s_y \) and \( s_{xy} \) provided they do not vary appreciably within an
element. Otherwise, the elements of \( [\bar{K}_I] \) have to be evaluated numerically.

The matrix \( [\bar{K}_I] \) is partitioned into sixteen 6th order square sub-
matri ces, so that each of these submatrices may be added to the corresponding
submatrices of \( [\bar{K}_O] \) in order to form \( [\bar{K}] \) according to equation 6.22.

Since \( [\bar{K}_I] \) is symmetrical only its lower triangle is given. The partitioning
is done as the following.

\[
[\bar{K}_I] = \begin{bmatrix}
\bar{s}_{11} \\
\bar{s}_{21} & \bar{s}_{22} \\
\bar{s}_{31} & \bar{s}_{32} & \bar{s}_{33} \\
\bar{s}_{41} & \bar{s}_{42} & \bar{s}_{43} & \bar{s}_{44}
\end{bmatrix}
\]

\[ A \, 12.2 \]

For convenience \( s_x \) is represented by \( x \),

\( s_y \) is represented by \( y \),

and \( s_{xy} \) is represented by \( z \).

in the submatrices \( [\bar{s}_{i,j}] \).
\[
\begin{bmatrix}
11 & = & [0] \\
21 & = & [0] \\
22 & = & hab \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 & x \\
0 & 0 & 0 & z & y \\
0 & 0 & 0 & ax & az & 4a^2x/3 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
31 & = & [0] \\
32 & = & hab \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & \frac{1}{2}(bx+az) & \frac{1}{2}(ay+bz) & 2a\left(\frac{bx}{4} + \frac{az}{3}\right) \\
0 & 0 & 0 & bz & by & abz \\
0 & 0 & 0 & a^2x & a^2z & 3a^3x/2 \\
0 & 0 & 0 & a\left(\frac{bx}{2} + \frac{az}{3}\right) & a\left(\frac{ay}{3} + \frac{bz}{2}\right) & 2a^2\left(\frac{bx}{3} + \frac{az}{4}\right) \\
0 & 0 & 0 & b\left(\frac{bx}{3} + \frac{az}{2}\right) & b\left(\frac{ay}{2} + \frac{bz}{3}\right) & 2ab\left(\frac{bx}{6} + \frac{az}{3}\right) \\
0 & 0 & 0 & b^2z & b^2y & ab^2z \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
S_{33}
\end{bmatrix}
= \begin{bmatrix}
\frac{b^2x}{3} + \frac{a^2y}{3} + \frac{abz}{2} \\
2b(\frac{ay}{4} + \frac{bz}{3}) \\
\frac{4b^2y}{3} \\
3a^2(\frac{bx}{6} + \frac{az}{4}) \\
a(b^2 \frac{x}{3} + \frac{a^2y}{4} + \frac{abz}{2}) \\
b(\frac{b^2x}{4} + \frac{a^2y}{3} + \frac{abz}{2}) \\
3b^2(\frac{ay}{6} + \frac{bz}{4})
\end{bmatrix}
\begin{bmatrix}
a^2bz \\
a^2bz \\
a^2b^2z \\
3a^2(\frac{ay}{4} + \frac{3bz}{4}) \\
a^2b^2(\frac{ay}{4} + \frac{3bz}{4}) \\
a^2b^2(\frac{ay}{4} + \frac{3bz}{4}) \\
a^2b^2(\frac{ay}{4} + \frac{3bz}{4})
\end{bmatrix}
\begin{bmatrix}
a^2bz \\
a^2b^2z \\
a^2b^2z \\
a^2b^2z \\
a^2b^2z \\
a^2b^2z \\
a^2b^2z
\end{bmatrix}
\]

[Symmetrical]

\[
\begin{bmatrix}
S_{41}
\end{bmatrix}
= \begin{bmatrix} 0 \end{bmatrix}
\]

\[
\begin{bmatrix}
S_{42}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a^2(\frac{bx}{2} + \frac{az}{2}) \\
a^2(\frac{ay}{4} + \frac{bz}{2}) \\
a^2b(\frac{bx}{3} + \frac{az}{4}) \\
a^2b(\frac{ay}{4} + \frac{bz}{3}) \\
a^2b(\frac{bx}{3} + \frac{2az}{5}) \\
a^2b(\frac{ay}{3} + \frac{bz}{4}) \\
a^2b(\frac{bx}{3} + \frac{2az}{5})
\end{bmatrix}
\begin{bmatrix}
a^2b^2(\frac{bx}{4} + \frac{az}{3}) \\
ab^2(\frac{bx}{4} + \frac{az}{3}) \\
ab^2(\frac{bx}{4} + \frac{az}{3}) \\
ab^2(\frac{bx}{4} + \frac{az}{3}) \\
ab^2(\frac{bx}{4} + \frac{az}{3}) \\
ab^2(\frac{bx}{4} + \frac{az}{3}) \\
ab^2(\frac{bx}{4} + \frac{az}{3})
\end{bmatrix}
\]
\[
\begin{bmatrix}
3b^2 x & 2a^2 y \\
5 & 7 & 2
\end{bmatrix}
= \text{Symmetrical}
\]

\[
\begin{align*}
&\quad a^4 \left( \frac{3b^2 x}{5} + \frac{2a^2 y}{7} + \frac{abz}{2} \right) \\
&= a^2 b^2 \left( \frac{b^2 x}{5} + \frac{a^2 y}{5} + \frac{5abz}{8} \right) \quad b^4 \left( \frac{b^2 x}{7} + \frac{3a^2 y}{5} + \frac{abz}{2} \right) \\
&= a^2 b \left( \frac{3b^2 x}{8} + \frac{a^2 y}{6} + \frac{8abz}{15} \right) \quad ab^3 \left( \frac{b^2 x}{6} + \frac{3a^2 y}{8} + \frac{8abz}{15} \right) \quad 4a^2 b^2 \left( \frac{b^2 x}{15} + \frac{a^2 y}{15} + \frac{abz}{8} \right) \\
&= a^4 b \left( \frac{9b^2 x}{20} + \frac{a^2 y}{7} + \frac{a^2 y}{2} \right) \quad a^2 b^3 \left( \frac{b^2 x}{6} + \frac{3a^2 y}{10} + \frac{11abz}{20} \right) \quad a^3 b^2 \left( \frac{3b^2 x}{10} + \frac{2a^2 y}{9} + \frac{abz}{2} \right) \\
&= a^3 b^2 \left( \frac{3b^2 x}{10} + \frac{a^2 y}{6} + \frac{11abz}{20} \right) \quad ab^4 \left( \frac{b^2 x}{7} + \frac{9a^2 y}{20} + \frac{abz}{2} \right) \quad 2b^3 \left( \frac{2b^2 x}{9} + \frac{3a^2 y}{10} + \frac{abz}{2} \right) \\
&= a^2 b^4 \left( \frac{b^2 x}{7} + \frac{9a^2 y}{25} + \frac{abz}{2} \right) \quad a^2 b^4 \left( \frac{b^2 x}{7} + \frac{9a^2 y}{25} + \frac{abz}{2} \right) \quad a^3 b^3 \left( \frac{b^2 x}{4} + \frac{a^2 y}{4} + \frac{12abz}{25} \right)
\end{align*}
\]
APPENDIX 13

The element mass matrix \([M]\)

The element mass matrix \([M]\) is given in equation 6.23 as

\[
[M] = \int_{x=0}^{a} \int_{y=0}^{b} h \left[\begin{array}{c}
P(x,y) \\
N^T(P(x,y))
\end{array}\right] \ dx \ dy.
\]

\[A \ 13.1\]

The matrix \([P(x,y)]\) contains the displacement field \(u(x,y)\) given by equation 6.32(a-c). The quantities \(a\) and \(b\) are the lengths of the sides of an element along the direction of \(x\) and \(y\) respectively in the \(x\)-\(y\) plane as shown in (a) of Fig.19.

Since \([M]\) is independent of the curvatures the matrix \([M]\), which is given explicitly in this Appendix, is correct for all plates with constant thickness. The same matrix may be used for variable thickness without appreciable error provided the thickness varies slowly over an element. Otherwise, the elements of \([M]\) for rapidly varying thickness will have to be evaluated numerically.

The matrix \([M]\) is partitioned into sixteen 6th order square submatrices. The partitioning facilitates the assembly of the overall mass matrix and also the presentation of the matrix \([M]\).

Since \([M]\) is symmetrical, only the lower triangle is given here.

The partitioned form of \([M]\) is the following.

\[
[M] = \begin{bmatrix}
M_{11} & M_{12} & \text{symmetrical} \\
M_{21} & M_{22} & \\
M_{31} & M_{32} & M_{33} \\
M_{41} & M_{42} & M_{43} & M_{44}
\end{bmatrix}
\]

\[A \ 13.2\]

The submatrices \([M_{ij}]\) \((i = 1,4; j = 1,4)\) are the following.
\[ \begin{bmatrix} M_{11} \end{bmatrix} = h \begin{bmatrix} \text{ab} \\
\text{Symmetrical} \\
\end{bmatrix} \]
\[
\begin{bmatrix}
\frac{a^2 b}{2} & \frac{a^3 b}{3} & \\
\frac{a b^2}{2} & \frac{a^2 b^2}{4} & \frac{a b^3}{3} & \\
\frac{a^2 b^2}{4} & \frac{a^3 b^2}{6} & \frac{a^2 b^3}{6} & \frac{a^3 b^3}{9} & \\
0 & 0 & 0 & 0 & \text{ab} \\
0 & 0 & 0 & 0 & \frac{a^2 b}{2} & \frac{a^3 b}{3} & \\
\end{bmatrix}
\]

\[ \begin{bmatrix} M_{21} \end{bmatrix} = h \begin{bmatrix} \text{ab} \\
\text{Symmetrical} \\
\end{bmatrix} \]
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & \frac{a b^2}{2} & \frac{a^2 b^2}{4} & \\
0 & 0 & 0 & 0 & 0 & \frac{a b^2}{4} & \frac{a^2 b^2}{6} & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\end{bmatrix}
\]

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\[
\begin{bmatrix}
M_{22}
\end{bmatrix} = h
\begin{bmatrix}
\text{Symmetrical} \\
\frac{a^2 b^3}{6} & \frac{a^3 b^3}{9} \\
0 & 0 & ab \\
0 & 0 & \frac{a^2 b}{2} & \frac{a^3 b}{3} \\
0 & 0 & \frac{a b^2}{2} & \frac{a^2 b^2}{4} & \frac{a^3 b}{3} \\
0 & 0 & \frac{a^3 b}{3} & \frac{a^4 b}{4} & \frac{a^3 b^2}{6} & \frac{a^5 b}{5}
\end{bmatrix}
\]

\[
\begin{bmatrix}
M_{31}
\end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}
\]

\[
\begin{bmatrix}
M_{32}
\end{bmatrix} = h
\begin{bmatrix}
0 & 0 & \frac{a^2 b^2}{4} & \frac{a^3 b^2}{6} & \frac{a^2 b^3}{6} & \frac{a^4 b^2}{8} \\
0 & 0 & \frac{a b^3}{3} & \frac{a^2 b^3}{6} & \frac{a b^4}{4} & \frac{a^3 b^3}{9} \\
0 & 0 & \frac{a^4 b}{4} & \frac{a^5 b}{5} & \frac{a^4 b^2}{8} & \frac{a b^6}{6} \\
0 & 0 & \frac{a^3 b^2}{6} & \frac{a^4 b^2}{8} & \frac{a^3 b^3}{9} & \frac{a^5 b^2}{10} \\
0 & 0 & \frac{a^2 b^3}{6} & \frac{a^3 b^3}{9} & \frac{a b^4}{8} & \frac{a^4 b^3}{12} \\
0 & 0 & \frac{a^4 b^4}{4} & \frac{a^5 b^4}{8} & \frac{a^5 b^5}{10} & \frac{a^3 b^4}{12}
\end{bmatrix}
\]
\[
\begin{bmatrix}
M_{33} &=& h \\
\begin{bmatrix}
a^3 b^3 / 9 \\
2 b^4 / 8 & ab^5 / 5 & \text{Symmetrical} \\
a^5 b^2 / 10 & a^4 b^3 / 12 & a^7 b / 7 \\
a^4 b^3 / 12 & a^3 b^4 / 12 & a^6 b^2 / 12 & a^5 b^3 / 15 \\
a^3 b^4 / 12 & a^2 b^5 / 10 & a^5 b^3 / 15 & a^4 b^4 / 16 & a^3 b^5 / 15 \\
a^2 b^5 / 10 & ab^6 / 6 & a^4 b^4 / 16 & a^3 b^5 / 15 & a^2 b^6 / 12 & ab^7 / 17
\end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
M_{41} &=& \begin{bmatrix} 0 \end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
M_{42} &=& h \\
\begin{bmatrix}
0 & 0 & a^4 b^2 / 8 & a^5 b^2 / 10 & a^4 b^3 / 12 & a^6 b^2 / 12 \\
0 & 0 & a^2 b^4 / 8 & a^3 b^4 / 12 & a^2 b^5 / 10 & a^4 b^4 / 16 \\
0 & 0 & a^3 b^3 / 9 & a^4 b^3 / 12 & a^3 b^4 / 12 & a^5 b^3 / 15 \\
0 & 0 & a^4 b^3 / 12 & a^5 b^3 / 15 & a^4 b^4 / 16 & a^6 b^3 / 18 \\
0 & 0 & a^3 b^4 / 12 & a^4 b^4 / 16 & a^3 b^5 / 15 & a^5 b^4 / 20 \\
0 & 0 & a^4 b^4 / 16 & a^5 b^4 / 20 & a^4 b^5 / 20 & a^6 b^4 / 24
\end{bmatrix}
\end{bmatrix}
\]
\[
\begin{bmatrix}
M_{43} = h & \\
a^5b^3/15 & a^4b^4/16 & a^7b^2/14 & a^6b^3/18 & a^5b^4/20 & a^4b^5/20 \\
a^3b^5/15 & a^2b^6/12 & a^5b^4/20 & a^4b^5/20 & a^3b^6/18 & a^2b^7/14 \\
a^4b^4/16 & a^3b^5/15 & a^6b^3/18 & a^5b^4/20 & a^4b^5/20 & a^3b^6/18 \\
a^5b^4/20 & a^4b^5/20 & a^7b^3/21 & a^6b^4/24 & a^5b^5/25 & a^4b^6/24 \\
a^4b^5/20 & a^3b^6/18 & a^6b^4/24 & a^5b^5/25 & a^4b^6/24 & a^3b^7/21 \\
a^5b^5/25 & a^4b^6/24 & a^7b^4/28 & a^6b^5/30 & a^5b^6/30 & a^4b^7/28
\end{bmatrix}
\]

\[
\begin{bmatrix}
M_{44} = h & \\
a^7b^3/21 & \\
a^5b^5/25 & a^3b^7/21 & \text{Symmetrical} \\
a^6b^4/24 & a^4b^6/24 & a^5b^5/25 \\
a^7b^4/28 & a^5b^6/30 & a^6b^5/30 & a^7b^5/35 \\
a^6b^5/30 & a^4b^7/28 & a^5b^6/30 & a^6b^6/36 & a^5b^7/35 \\
a^7b^5/35 & a^5b^7/35 & a^6b^6/36 & a^7b^6/42 & a^6b^7/42 & a^7b^7/49
\end{bmatrix}
\]
APPENDIX 14

The $[C]$ and the $[C]^{-1}$ matrices

The parameters $\{a\}$ for each element may be expressed in terms of the nodal unknowns $\{u\}$ as shown in equation 6.25. The displacement fields $\{u(x,y)\}$ and their derivatives may then be written formally with $\{u\}$ as the parameters instead of $\{a\}$. This facilitates the study of the continuity of the displacements and slopes across element boundaries. Moreover, this helps in formulating the boundary conditions correctly.

The inversion of $[C]$ is therefore necessary in this formulation. However, the inversion of $[C]$ may be avoided if $\{u\}$ instead of $\{a\}$ provides the parameters for describing $\{u(x,y)\}$ in equation 6.16.

One of the best methods for inverting a large matrix is by partitioning. Sometimes the amount of work in inverting a matrix may be reduced by judiciously rearranging the matrix. The inverse of the rearranged matrix may again be rearranged to obtain the inverse of the original matrix. The inverse of any matrix is correct if the product of the matrix and its inverse be a unit matrix, in the partitioned form, this check may be applied at each stage of the inversion.

The proper order of the components of the vector $\{u\}$ is the following

$$\begin{bmatrix}
  u_1 & v_1 & w_1 & w_1' & w_1'' & u_2 & v_2 \\
  w_2 & w_2' & w_2'' & u_3 & v_3 & w_3 & w_3' \\
  w_3' & w_3'' & u_4 & v_4 & w_4 & w_4' & w_4''
\end{bmatrix}^T$$

In order to simplify the inversion of $[C]$, the components
of \{u\} are rearranged in the following order

\[
\begin{bmatrix}
  u_1 & u_2 & u_3 & u_4 & v_1 & v_2 & v_3 & v_4 \\
  w_1 & w_1' & w_1 & w_2 & w_3 & w_4 & w_2' & w_3'
\end{bmatrix}^{T}
\]

Let the modified form of \{u\} be denoted by \{\bar{u}\} and the modified form of \[C\] be denoted by \[C^*\]. Then

\[
\{\bar{u}\} = [C^*] \{a\}
\]

Now, A14.2(a) is partitioned in the following way.

\[
\begin{bmatrix}
  u^* \\
  u^{**}
\end{bmatrix}
= \begin{bmatrix}
  C_1(8x8) & 0 \\
  0 & C_2(16x16)
\end{bmatrix}
\begin{bmatrix}
  a^* \\
  a^{**}
\end{bmatrix}
\]

\{u^*\} and \{a^*\} contain the first eight components of \{\bar{u}\} and \{a\} respectively. \{u^{**}\} and \{a^{**}\} contain the remaining 16 components of \{\bar{u}\} and \{a\} respectively. \[C_1\] and \[C_2\] are given below.

\[
[C_1] = \begin{bmatrix}
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  1 & 0 & b & 0 & 0 & 0 & 0 & 0 \\
  1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
  1 & a & b & ab & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 & b & 0 \\
  0 & 0 & 0 & 0 & 1 & a & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & a & b & ab
\end{bmatrix}
\]

A14.4
From equation A.14.3 it may be seen that the inverse of $[C^*]$ is given by the inverse of $[C_1]$ and the inverse of $[C_2]$ in situ. Or written formally

$$[C^*]^{-1} = \begin{bmatrix} [C_1]^{-1} & 0 \\ 0 & [C_2]^{-1} \end{bmatrix}$$

The matrices $[C_1]^{-1}$ and $[C_2]^{-1}$ are given below:

$$[C_1]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/a & 0 & 1/a & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/b & 1/b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/ab & -1/ab & -1/ab & 1/ab & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/a & 0 & 1/a & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/b & 1/b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/ab & -1/ab & -1/ab & 1/ab & 0 \end{bmatrix}$$

Let

$$[C_2]^{-1} = \begin{bmatrix} Y_1 & Y_2 \\ (16\times8) & (16\times8) \end{bmatrix}$$

Where,
$$\begin{bmatrix} y_1 \end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3/a^2 & -2/a & 0 & 0 & 3/a^2 & 0 & 0 & -1/a & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3/b^2 & 0 & -2/b & 3/b^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2/a^3 & 1/a^2 & 0 & 0 & -2/a^3 & 0 & 0 & 1/a^2 & 0 & 0 \\
0 & 0 & -3/a^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3/b^2 & 0 & 0 & 0 & 0 & 0 & 3/b^2 & 0 & 0 \\
2/b^3 & 0 & 1/b^2 & -2/b^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2/a^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2/b^3 & 0 & 0 & 0 & 0 & 0 & -2/b^3 & 0 & 0 \\
9/(a^2b^2) & 6/(ab^2) & 6/(a^2b) & -9/(a^2b^2) & -9/(a^2b^2) & 9/(a^2b^2) & -6/(ab^2) & 3/(ab^2) & 0 & 0 \\
-6/(a^3b^2) & -3/(a^2b^2) & -4/(a^3b) & 6/(a^3b^2) & 6/(a^3b^2) & -6/(a^3b^2) & 3/(a^2b^2) & -3/(a^2b^2) & 0 & 0 \\
-6/(a^2b^3) & -4/(ab^3) & -3/(a^2b^2) & 6/(a^2b^3) & 6/(a^2b^3) & -6/(a^2b^3) & 4/(ab^3) & -2/(ab^3) & 0 & 0 \\
4/(a^3b^3) & 2/(a^2b^3) & 2/(a^3b^2) & -4/(a^3b^3) & -4/(a^3b^3) & 4/(a^3b^3) & -2/(a^2b^3) & 2/(a^2b^3) & 0 & 0
\end{bmatrix}$$
$$\begin{bmatrix} y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1/b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3/(ab^2) & 3/(a_1^2b) & -6/(a_1^2b) & -3/(a_1^2b) & 4/(ab) & 2/(ab) & 2/(ab) & 1/(ab) \\ 3/(a_1^2b) & -2/(a_1^3b) & 4/(a_1^3b) & 2/(a_1^3b) & -2/(a_1^2b) & -2/(a_1^2b) & -1/(a_1^2b) & -1/(a_1^2b) \\ 2/(ab^3) & -3/(a_1^2b^2) & 3/(a_1^2b^2) & 3/(a_1^2b^2) & -2/(ab^2) & -1/(ab^2) & -2/(ab^2) & -1/(ab^2) \\ -2/(a_1^2b^3) & 2/(a_1^3b^2) & -2/(a_1^3b^2) & -2/(a_1^3b^2) & 1/(a_1^2b^2) & 1/(a_1^2b^2) & 1/(a_1^2b^2) & 1/(a_1^2b^2) \\ \end{bmatrix}$$
Knowing \( [c_1]^{-1} \) and \( [c_2]^{-1} \) from A 14.7 and A 14.8 respectively, the \( [C^*]^{-1} \) is obtained. The \( [C]^{-1} \) may be obtained from \( [C^*]^{-1} \) by the necessary row and column exchanges. However, it is not necessary to obtain \( [C]^{-1} \) in its proper form. It may be recalled from equations 6.29 and 6.28 that both \( [K] \) and \( [M] \) are to be premultiplied by \( ([C]^{-1})^T \) and postmultiplied by \( [C]^{-1} \). Instead of using \( [C]^{-1} \) and its transpose, \( [C^*]^{-1} \) and its transpose are used to postmultiply and premultiply respectively, both \( [K] \) and \( [M] \). The resulting matrices are then rearranged by performing the necessary row and column exchanges.
APPENDIX 15

Derivation of the Consistent Load Matrix

Equation 7.21 is \( (L) = \int_{0}^{b} \int_{0}^{a} [V(x,y)] \{L\} \, dx \, dy \). The matrix \([V(x,y)]\) is directly derivable from the matrix \([P(x,y)]\). The elements of \((L)\) are independent of the physical and geometrical properties of the plate and therefore are applicable to all plates, where the finite elements are rectangular. The matrix \((L)\) is partitioned into four submatrices which is advantageous in the assembly of the overall load matrix.

\[
\begin{bmatrix}
L_{11} \\
L_{12} \\
L_{13} \\
L_{14}
\end{bmatrix}
\]

\((L)\) is \(24 \times 1\), and each of the submatrices are \(6 \times 1\). The elements of the submatrices are as follows.

\[
\begin{align*}
(L_{11}) &= \{0\} \\
(L_{12}) &= p_{z}ab 
\begin{bmatrix}
0 \\
0 \\
1 \\
a/2 \\
b/2 \\
a^2/3
\end{bmatrix} \\
(L_{13}) &= p_{z}ab 
\begin{bmatrix}
ab/4 \\
b^2/3 \\
a^3/4 \\
a^2b/6 \\
ab^2/6 \\
b^3/4
\end{bmatrix} \\
(L_{14}) &= p_{z}a^2b^2 
\begin{bmatrix}
a^2/8 \\
b^2/8 \\
ab/9 \\
a^2b/12 \\
ab^2/12 \\
a^2b^2/16
\end{bmatrix}
\end{align*}
\]
APPENDIX 16

The Strain Matrix \([P^*(x,y)]\)

The strain matrix \([P^*(x,y)]\) defines the strains \(\{\varepsilon\}\) at any point \((x,y)\) on the surface (note that \(x,y\) with reference to the global co-ordinates). The values of \(k_x, k_y, k_{xy}\) and \(h\) must be calculated at the current point \((x,y)\). For convenience of presentation, \([P^*(x,y)]\) is partitioned as follows.

\[
\begin{bmatrix}
P^*(x,y) \\
(6\times24)
\end{bmatrix}
= 
\begin{bmatrix}
P^*_1(x,y) & P^*_2(x,y) \\
(6\times12) & (6\times12)
\end{bmatrix}
\]

Also for convenience \(a, b\) and \(c\) will represent \(-k_x, -k_y\) and \(-2k_{xy}\) respectively in this appendix.

\[
[P^*_1(x,y)] = 
\begin{bmatrix}
0 & 1 & 0 & y & 0 & 0 & 0 & 0 & a & ax & ay & ax^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & b & bx & bx^2 \\
0 & 0 & 1 & x & 0 & 1 & 0 & y & c & cx & cy & cx^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

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\[
[P_2(x,y)] = \begin{bmatrix}
axy & ay^2 & ax^3 & ax^2y & ax^2 & ay^3 & ax^3y & ax^2y^2 & ax^3y^2 & ax^2y^3 & ax^3y^3 \\
by & by^2 & bx^3 & bx^2y & by^3 & bx^3y & bxy & bx^3y & bx^3y^2 & bx^3y^3 & bx^3y^3 \\
cxy & cy^2 & cx^3 & cx^2y & cxy^2 & cy^3 & cx^3y & cxy^3 & cx^3y^2 & cx^3y^3 & cx^3y^3 \\
0 & 0 & 6x & 2y & 0 & 0 & 6xy & 0 & 2y^2 & 6xy^2 & 2y^3 & 6xy^3 \\
0 & 2 & 0 & 0 & 2x & 6y & 0 & 6xy & 2x^2 & 2x^3 & 6x^2y & 6yx^3 \\
2 & 0 & 0 & 4x & 4y & 0 & 6x^2 & 6y^2 & 8xy & 12x^2y & 12xy^2 & 18x^2y^2
\end{bmatrix}
\]
REFERENCES


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TABLE NO. 1

Comparison of $\Delta_w$ of a clamped square plate computed by various methods.

<table>
<thead>
<tr>
<th>General solution (complex)</th>
<th>General solution (real)</th>
<th>Modified matrix method</th>
<th>$u$</th>
<th>$v$</th>
<th>$w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.063043</td>
<td>0.063043</td>
<td>+</td>
<td>2, 1</td>
<td>1, 1*</td>
<td>1, 3</td>
</tr>
<tr>
<td>0.102444</td>
<td>0.102444</td>
<td>0.102445</td>
<td>2, 3</td>
<td>1, 2</td>
<td>1, 3</td>
</tr>
<tr>
<td>0.128125</td>
<td>0.126829</td>
<td>0.126830</td>
<td>2, 4</td>
<td>1, 3</td>
<td>1, 4</td>
</tr>
<tr>
<td>0.136250</td>
<td>0.133742</td>
<td>+</td>
<td>2, 5</td>
<td>1, 3*</td>
<td>1, 5</td>
</tr>
<tr>
<td>0.271731</td>
<td>0.271731</td>
<td>0.271732</td>
<td>2, 4</td>
<td>1, 1</td>
<td>1, 4</td>
</tr>
<tr>
<td>0.336062</td>
<td>0.336094</td>
<td>0.336062</td>
<td>2, 5</td>
<td>1, 4</td>
<td>1, 5</td>
</tr>
<tr>
<td>0.697512</td>
<td>0.697500</td>
<td>0.697512</td>
<td>2, 6</td>
<td>1, 1</td>
<td>1, 6</td>
</tr>
<tr>
<td>0.835618</td>
<td>0.835000</td>
<td>0.835618</td>
<td><em>:</em></td>
<td>**</td>
<td>**</td>
</tr>
</tbody>
</table>

+ eigenvalue not detected  
* mode shape not compatible.  
** computed mode shapes need more definition.

Plate dimensions:  
$L = 17$ in.  
$b = 17$ in.  
$h = 0.02$ in.  
$R = 96$ in.  
$v = 0.33$.  

The eigenvalues and mode shapes of rectangular plates. The curved edges of the plates are simply supported and the straight edges are clamped. Exact method of solution is used. $v = 0.33$.

<table>
<thead>
<tr>
<th>L-in.</th>
<th>b-in.</th>
<th>h-in.</th>
<th>R-in.</th>
<th>$\Delta_w$</th>
<th>$m, n$</th>
<th>u</th>
<th>v</th>
<th>w</th>
</tr>
</thead>
<tbody>
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<td>20.0</td>
<td>17.0</td>
<td>0.02</td>
<td>96.0</td>
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<td>1,2</td>
<td>1,3</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td>0.06288</td>
<td>2,2</td>
<td>1,3</td>
<td>1,4</td>
<td></td>
</tr>
<tr>
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<td></td>
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<td></td>
<td>0.20436</td>
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<td>1,3</td>
<td>1,4</td>
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<td></td>
<td></td>
<td></td>
<td>0.31290</td>
<td>1,3</td>
<td>1,4</td>
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<td></td>
</tr>
<tr>
<td>10.2</td>
<td>17.0</td>
<td>0.02</td>
<td>96.0</td>
<td>0.15605</td>
<td>2,3</td>
<td>1,4</td>
<td>1,3</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.17160</td>
<td>2,4</td>
<td>1,3</td>
<td>1,4</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.38580</td>
<td>2,5</td>
<td>1,4</td>
<td>1,5</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.39091</td>
<td>2,6</td>
<td>1,3</td>
<td>1,4</td>
<td></td>
</tr>
</tbody>
</table>
### TABLE NO. 3

Comparison of $A_w$ of a clamped square plate for different radii and thicknesses computed by the general and the modified matrix methods.

<table>
<thead>
<tr>
<th>$R$ in.</th>
<th>$h$ in.</th>
<th>General solution (real)</th>
<th>Modified Matrix solution</th>
<th>$m,n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>96</td>
<td>0.032</td>
<td>0.186172</td>
<td>0.186151</td>
<td>2,3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.200168</td>
<td>0.200175</td>
<td>2,2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.487734</td>
<td>0.487736</td>
<td>2,4</td>
</tr>
<tr>
<td>96</td>
<td>0.04</td>
<td>0.237968</td>
<td>0.237975</td>
<td>2,2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.259094</td>
<td>0.259095</td>
<td>2,3</td>
</tr>
<tr>
<td>48</td>
<td>0.02</td>
<td>0.050396</td>
<td>0.050398</td>
<td>2,4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.054478</td>
<td>0.054475</td>
<td>2,3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.105430</td>
<td>0.105435</td>
<td>2,3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.141250</td>
<td>0.141409</td>
<td>2,6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.260000</td>
<td>0.259335</td>
<td>2,2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.290000</td>
<td>0.287138</td>
<td>2,7</td>
</tr>
<tr>
<td>48</td>
<td>0.032</td>
<td>0.081626</td>
<td>0.081628</td>
<td>2,3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.093506</td>
<td>0.094456</td>
<td>2,4</td>
</tr>
<tr>
<td>48</td>
<td>0.04</td>
<td>0.102219</td>
<td>0.102222</td>
<td>2,3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.126602</td>
<td>0.126572</td>
<td>2,4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.270872</td>
<td>0.270873</td>
<td>2,4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.335625</td>
<td>0.335641</td>
<td>1,5</td>
</tr>
</tbody>
</table>

Plate dimensions:

$L = 17$ in.,

$b = 17$ in.,

$\nu = 0.33$. 
### TABLE NO. 4

Comparison between calculated and measured frequencies of a square plate.

<table>
<thead>
<tr>
<th>$m, n$ of w</th>
<th>Rayleigh-Ritz solution*</th>
<th>Webster solution**</th>
<th>Kantorovich solution</th>
<th>Experiment***</th>
<th>Simply supported</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 3</td>
<td>226.0</td>
<td>112.2</td>
<td>0.1024</td>
<td>112.8</td>
<td>85</td>
</tr>
<tr>
<td>1, 4</td>
<td>232.5</td>
<td>-</td>
<td>0.1268</td>
<td>126.0</td>
<td>129</td>
</tr>
<tr>
<td>1, 4</td>
<td>-</td>
<td>-</td>
<td>0.2717</td>
<td>183.7</td>
<td>-</td>
</tr>
<tr>
<td>1, 5</td>
<td>267.2</td>
<td>-</td>
<td>0.3361</td>
<td>204.3</td>
<td>190</td>
</tr>
<tr>
<td>1, 6</td>
<td>327.0</td>
<td>-</td>
<td>0.6975</td>
<td>294.3</td>
<td>-</td>
</tr>
</tbody>
</table>

*Rayleigh-Ritz: one term beam function approximation.*

**Webster [32]: only the lowest frequency is available.

***Experiment by Rucker [15], also presented by Sewall [14].

Plate dimensions:

$L = b = 17 \text{ in}$,
$h = 0.02 \text{ in}$,
$R = 96 \text{ in}$,
$\nu = 0.33$,
$E = 10^7 \text{ lb/in}^2$,
$\rho = 0.096 \text{ lb/in}^3$. 
## TABLE NO. 5

Comparison between the calculated and the measured frequencies of a square plate for different radii and thicknesses.

<table>
<thead>
<tr>
<th>n of $R_1$, $R_2$ and $R_3$ respectively*</th>
<th>Frequency c.p.s.</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Rayleigh-Ritz solution</td>
<td>Webster's solution</td>
<td>Kantorovich modified matrix sol.</td>
<td>Experiment by Rucker</td>
<td>Simply supported plate</td>
</tr>
<tr>
<td>------------------------------------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td>---------------------------</td>
<td>------------------------</td>
<td>-------------------------</td>
</tr>
<tr>
<td></td>
<td>$R = 96$ in., $h = 0.032$ in.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2, 3, 2</td>
<td>254</td>
<td>150</td>
<td>157.7</td>
<td>117</td>
<td>86</td>
</tr>
<tr>
<td>3, 2, 3</td>
<td>252</td>
<td>150</td>
<td>152</td>
<td>125</td>
<td>112</td>
</tr>
<tr>
<td>4, 1, 4</td>
<td>293</td>
<td></td>
<td>246.1</td>
<td>229</td>
<td>182</td>
</tr>
<tr>
<td></td>
<td>$R = 96$ in., $h = 0.04$ in.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2, 3, 2</td>
<td>260.5</td>
<td>165</td>
<td>171.9</td>
<td>123</td>
<td>94</td>
</tr>
<tr>
<td>3, 2, 3</td>
<td>273.5</td>
<td></td>
<td>179.3</td>
<td>197</td>
<td>137</td>
</tr>
<tr>
<td></td>
<td>$R = 48$ in., $h = 0.02$ in.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3, 4, 3</td>
<td>423.8</td>
<td>158</td>
<td>164.5</td>
<td>86</td>
<td>94.2</td>
</tr>
<tr>
<td>4, 3, 4</td>
<td>393.8</td>
<td></td>
<td>158.2</td>
<td>148</td>
<td>119.6</td>
</tr>
<tr>
<td>3, 4, 5</td>
<td>393</td>
<td></td>
<td>228.7</td>
<td>241</td>
<td>175</td>
</tr>
<tr>
<td>6, 3, 6</td>
<td>421</td>
<td></td>
<td>264.9</td>
<td>387</td>
<td>247</td>
</tr>
<tr>
<td>7, 6, 7</td>
<td>477</td>
<td></td>
<td>379.5</td>
<td>439</td>
<td>332.8</td>
</tr>
<tr>
<td></td>
<td>$R = 48$ in., $h = 0.032$ in.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3, 2, 3</td>
<td>434</td>
<td>203</td>
<td>201.4</td>
<td>144</td>
<td>126</td>
</tr>
<tr>
<td>4, 3, 4</td>
<td>432</td>
<td></td>
<td>215.5</td>
<td>270</td>
<td>185</td>
</tr>
<tr>
<td></td>
<td>$R = 48$ in., $h = 0.04$ in.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3, 2, 3</td>
<td>451</td>
<td>225</td>
<td>225.3</td>
<td>180</td>
<td>149</td>
</tr>
<tr>
<td>4, 3, 4</td>
<td>465</td>
<td></td>
<td>250.7</td>
<td>289</td>
<td>229</td>
</tr>
<tr>
<td>4, 1, 4</td>
<td>366.8</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5, 4, 5</td>
<td>534</td>
<td></td>
<td>408.2</td>
<td>398</td>
<td>347</td>
</tr>
</tbody>
</table>

* $m = 1$ for $f_2$ and $f_3$, $m = 2$ for $f_1$.

Plate dimensions: $L = 17$ in.,
$b = 17$ in.,
$\nu = 0.33$
$E = 10^7$ lb/in$^2$. 
TABLE NO. 6

Calculated and measured lowest frequencies of a rectangular plate for different radii and thicknesses.

<table>
<thead>
<tr>
<th>R ins</th>
<th>h ins</th>
<th>f c.p.s.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Rayleigh Ritz</td>
</tr>
<tr>
<td></td>
<td></td>
<td>all edges clamped</td>
</tr>
<tr>
<td>96</td>
<td>0.008</td>
<td>314.5 (1,1)</td>
</tr>
<tr>
<td>96</td>
<td>0.048</td>
<td>345.6 (1,1)</td>
</tr>
<tr>
<td>72</td>
<td>0.028</td>
<td>394.9 (1,2)</td>
</tr>
<tr>
<td>72</td>
<td>0.048</td>
<td>433.6 (1,1)</td>
</tr>
<tr>
<td>48</td>
<td>0.028</td>
<td>531.9 (1,2)</td>
</tr>
<tr>
<td>48</td>
<td>0.048</td>
<td>619.8 (1,1)</td>
</tr>
</tbody>
</table>

*xccyss: curved edges clamped, straight edges simply supported
**Sevall 14.

The numbers in parentheses indicate m and n for w.

Plate dimensions: L = 11 in., b = 9 in., v = 0.33 in., E = 10^7 lb/in^2
Comparison of eigenvalues of a clamped singly curved square plate computed by using the Flugge's exact and the approximate set of equations.

<table>
<thead>
<tr>
<th>$\Delta_w^*$</th>
<th>$\Delta_e$</th>
<th>$\Delta_s$</th>
<th>$u$</th>
<th>$v$</th>
<th>$w$</th>
<th>$\Delta_s/\Delta_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.102445</td>
<td>0.102975</td>
<td>2, 3</td>
<td>1, 2</td>
<td>1, 3</td>
<td>1.005</td>
</tr>
<tr>
<td></td>
<td>0.126830</td>
<td>0.127825</td>
<td>2, 4</td>
<td>1, 3</td>
<td>1, 4</td>
<td>1.008</td>
</tr>
<tr>
<td></td>
<td>0.271732</td>
<td>0.272942</td>
<td>2, 4</td>
<td>1, 1</td>
<td>1, 4</td>
<td>1.005</td>
</tr>
<tr>
<td></td>
<td>0.336062</td>
<td>0.341893</td>
<td>1, 5</td>
<td>1, 4</td>
<td>1, 5</td>
<td>1.017</td>
</tr>
</tbody>
</table>

$\Delta_w = \Omega/k^2_y$

$\Delta_e$ and $\Delta_s$ are $\Delta_w$ obtained by using the Flugge's exact and the simple set of equations respectively.

Plate dimensions:

$L_x = L_y = 17$ in.

$k_x = 0$, $k_y = \frac{1}{R} = 0.01042$ in$^{-1}$

$v = 0.33$
TABLE No. 8

Comparison between the eigenvalues of a singly curved plate using the Flugge's exact and the simple set of equations.

<table>
<thead>
<tr>
<th>m, n of w</th>
<th>( \Delta_w )</th>
<th>( \Delta_e )</th>
<th>( \Delta_s )</th>
<th>( \Delta_s/\Delta_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1</td>
<td>0.293985</td>
<td>0.291923</td>
<td>0.993</td>
<td></td>
</tr>
<tr>
<td>1, 2</td>
<td>0.087827</td>
<td>0.087123</td>
<td>0.992</td>
<td></td>
</tr>
<tr>
<td>1, 3</td>
<td>0.065333</td>
<td>0.065659</td>
<td>1.005</td>
<td></td>
</tr>
<tr>
<td>1, 4</td>
<td>0.117312</td>
<td>0.119196</td>
<td>1.016</td>
<td></td>
</tr>
</tbody>
</table>

\* \( \Delta_w, \Delta_e \) and \( \Delta_s \) are defined in Table 7.

The dimensions of the plate are given in Table 7.
The boundary conditions:
The curved edges clamped and the straight edges simply supported.
Approximations:
Inplane inertias are neglected.
### TABLE No. 9

(a) $\Omega$ values of clamped spherical cap with one boundary obtained by Reissner and (b) $\Omega$ for spherical plates on rectangular base plane obtained here.

(a) Reissner's lowest values of $\Omega$ for spherical caps

<table>
<thead>
<tr>
<th>$f_n/h$</th>
<th>$\Omega \times 1000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.1097</td>
</tr>
<tr>
<td>12</td>
<td>0.1174</td>
</tr>
<tr>
<td>8</td>
<td>0.1377</td>
</tr>
</tbody>
</table>

(b) Lowest values of $\Omega$ obtained by the method of Kantorovich for rectangular spherical plates

<table>
<thead>
<tr>
<th>$f_n/h$</th>
<th>$\Omega \times 1000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.1137</td>
</tr>
<tr>
<td>12</td>
<td>0.1201</td>
</tr>
<tr>
<td>8</td>
<td>0.1332</td>
</tr>
</tbody>
</table>

$v = 0.3,$
$h = 0.02 \text{ in.},$
$R = 96 \text{ in.},$

$f_n = L_0^2/(8R).$
### TABLE No: 10

Values of $\Omega$ of clamped square spherical plates.

<table>
<thead>
<tr>
<th>1000 $\Omega$</th>
<th>$m, n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_x = L_y = 15.677$</td>
<td>$L_x = L_y = 13.577$</td>
</tr>
<tr>
<td>$f_n/h = 16$</td>
<td>$f_n/h = 12$</td>
</tr>
<tr>
<td>$0.113735$</td>
<td>$0.120105$</td>
</tr>
<tr>
<td>$0.117109$</td>
<td>$0.123478$</td>
</tr>
<tr>
<td>$0.139150$</td>
<td></td>
</tr>
<tr>
<td>$0.149178$</td>
<td></td>
</tr>
</tbody>
</table>

Other dimensions:

- $h = 0.02$ in.,
- $k_x = k_y = 0.01042$ in$^{-1}$,
- $f_n = L_x^2 k_x / 8$ in$^{-1}$,
- $v = 0.3$. 
### TABLE No. 11

Comparison of the lower values of $\Omega$ for three different clamped square plates.

<table>
<thead>
<tr>
<th>$1000 \Omega$</th>
<th>anticlastic plate</th>
<th>synclastic plate</th>
<th>$m, n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Singly curved plate</td>
<td>0.079270</td>
<td>0.109441</td>
<td>2, 2</td>
</tr>
<tr>
<td>0.011173</td>
<td>0.080688</td>
<td>0.111780</td>
<td>2, 3</td>
</tr>
<tr>
<td>0.013870</td>
<td>0.100780</td>
<td>0.112374</td>
<td>2, 4</td>
</tr>
<tr>
<td>0.029616</td>
<td>0.112354</td>
<td>0.134704</td>
<td>2, 3</td>
</tr>
<tr>
<td>0.037098</td>
<td></td>
<td></td>
<td>2, 5</td>
</tr>
</tbody>
</table>

Common dimensions of the plates:

\[
L_x = L_y = 17 \text{ in.} \\
h = 0.02 \text{ in.} \\
v = 0.33
\]

Curvatures:

- singly curved plate $k_x = 0$, $k_y = 0.01042 \text{ in}^{-1}$.
- anticlastic plate $k_x = 0.01042 \text{ in}^{-1}$, $k_y = -k_x$.
- synclastic plate $k_x = k_y = 0.01042 \text{ in}^{-1}$. 
TABLE No. 12

The lower values of $\Omega$ of clamped rectangular plates.

<table>
<thead>
<tr>
<th>$k_y = 0.02$</th>
<th>$k_y = -0.02$</th>
<th>$u$</th>
<th>$v$</th>
<th>$w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.9855$</td>
<td>$0.9190$</td>
<td>$2, 2$</td>
<td>$1, 3$</td>
<td>$1, 2$</td>
</tr>
<tr>
<td>$1.4295$</td>
<td>$1.3781$</td>
<td>$2, 3$</td>
<td>$1, 2$</td>
<td>$1, 3$</td>
</tr>
</tbody>
</table>

$\Omega_{x} = 0.01$, $\Omega_{y} = 0.028$ in., $k_x = 0.001$ in$^{-1}$, $v = 0.33$. 

$L_x = 11$ in., $L_y = 9$ in.
TABLE No. 13

First eight frequencies of free plates with four elements on the complete plates.

<table>
<thead>
<tr>
<th>Flat plate $k_x = k_y = k_{xy} = 0$</th>
<th>Singly curved $k_x = k_{xy} = 0$</th>
<th>Doubly curved with $k_{xy} = 0$</th>
<th>Doubly curved with $k_{xy} \neq 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.3031101</td>
<td>0.3996966</td>
<td>0.0</td>
</tr>
<tr>
<td>0.5915085</td>
<td>103.6089</td>
<td>212.6196</td>
<td>202.4926</td>
</tr>
<tr>
<td>0.9345491</td>
<td>181.0350</td>
<td>212.6198</td>
<td>224.2031</td>
</tr>
<tr>
<td>1435.326</td>
<td>1448.721</td>
<td>1464.183</td>
<td>1464.410</td>
</tr>
<tr>
<td>2076.465</td>
<td>2097.731</td>
<td>2121.046</td>
<td>2121.445</td>
</tr>
</tbody>
</table>

plate dimensions:

- $L_x = L_y = 1$ in.,
- $h = 0.01$ in.,
- $k_x, k_y = 0.01$ in$^{-1}$,
- $k_{xy} = 0.002$ in$^{-1}$,
- $E = 33.7 \times 10^6$ lb/in$^2$, $\nu = 0.3$, $\rho = 0.27$ lb/cu.in.
TABLE No. 14

Comparison of frequencies of a clamped singly curved plate with 16 elements in the full plate and 4 elements in quarter of the plate. The modes of vibrations are symmetrical along x and antisymmetrical along y.

<table>
<thead>
<tr>
<th>4 elements in quarter plate</th>
<th>16 elements in full plate</th>
</tr>
</thead>
<tbody>
<tr>
<td>185.02</td>
<td>185.02</td>
</tr>
<tr>
<td>284.25</td>
<td>284.25</td>
</tr>
<tr>
<td>285.34</td>
<td>285.34</td>
</tr>
<tr>
<td>369.27</td>
<td>369.27</td>
</tr>
<tr>
<td>450.73</td>
<td>450.73</td>
</tr>
<tr>
<td>519.02</td>
<td>519.02</td>
</tr>
</tbody>
</table>

Dimensions of the plate:

\[ L_x = 17 \text{ in.}, \quad L_y = 17 \text{ in.}, \quad h = 0.02 \text{ in.}, \quad k_x = 0, \quad k_y = 0.01042 \text{ in}^{-1}. \]

Properties of material:

\[ \nu = 0.33, \quad E = 10^7 \text{ lb/in}^2, \quad \rho = 0.096 \text{ lb/cu.in}. \]
TABLE No. 15

Comparison of the frequencies of symmetrical modes of vibrations of a clamped flat plate with 9 and 16 elements (in quarter plate) with Warburton's solutions. The mesh patterns are 3x3 and 4x4 respectively.

<table>
<thead>
<tr>
<th>9 elements</th>
<th>16 elements</th>
<th>Warburton's sol.</th>
<th>m, n</th>
</tr>
</thead>
<tbody>
<tr>
<td>574.92</td>
<td>574.75</td>
<td>576.63</td>
<td>1, 1</td>
</tr>
<tr>
<td>1471.79</td>
<td>1466.14</td>
<td>1471.00</td>
<td>3, 1</td>
</tr>
<tr>
<td>2653.33</td>
<td>2640.87</td>
<td>2643.67</td>
<td>1, 3</td>
</tr>
<tr>
<td>3268.08</td>
<td>3236.90</td>
<td>3214.94</td>
<td>5, 1</td>
</tr>
<tr>
<td>3469.39</td>
<td>3450.92</td>
<td>3455.53</td>
<td>3, 3</td>
</tr>
<tr>
<td>5187.22</td>
<td>5126.93</td>
<td>5105.01</td>
<td>5, 3</td>
</tr>
</tbody>
</table>

Dimension of plate:

\[
\begin{align*}
L &= 4 \text{ in.}, L_y = 2.75 \text{ in.}, h = 0.015 \text{ in.}, \ k_x = k_y = k_{xy} = 0.\nonumber
\end{align*}
\]

Properties of material:

\[
E = 33.7 \times 10^6 \text{ lb/in}^2, \ \nu = 0.3, \ \rho = 0.27 \text{ lb/cu.in.}
\]
**TABLE No. 16**

Comparison of a few of the lower frequencies of a clamped flat plate with 9 elements (in quarter plate) with Warburton's solutions. The mesh pattern is 3x3.

<table>
<thead>
<tr>
<th>m, n</th>
<th>(f_f = \text{frequency with finite element})</th>
<th>(f_w = \text{frequency by Warburton})</th>
<th>(f_w/f_f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1</td>
<td>574.92</td>
<td>576.63</td>
<td>1.003*</td>
</tr>
<tr>
<td>2, 1</td>
<td>908.01</td>
<td>911.04</td>
<td>1.0033</td>
</tr>
<tr>
<td>1, 2</td>
<td>1396.80</td>
<td>1399.62</td>
<td>1.002</td>
</tr>
<tr>
<td>3, 1</td>
<td>1471.79</td>
<td>1471.00</td>
<td>0.9995**</td>
</tr>
<tr>
<td>2, 2</td>
<td>1706.42</td>
<td>1711.81</td>
<td>1.0032</td>
</tr>
<tr>
<td>3, 2</td>
<td>2236.58</td>
<td>2237.46</td>
<td>1.0004</td>
</tr>
</tbody>
</table>

* with 16 elements \(f_w/f_f = 1.0033\)

** with 16 elements \(f_w/f_f = 1.0033\)

Dimensions of plate:

\(L_x = 4\ \text{in.}, \ L_y = 2.75\ \text{in.}, \ h = 0.015\ \text{in.}, \ k_x = k_y = k_{xy} = 0.\)

Properties of material:

\(E = 33.7 \times 10^6\ \text{lb/in.}^2, \ \nu = 0.3, \ \rho = 0.27\ \text{lb/cu.in.}\)
TABLE No.17

Change of frequencies of symmetrical modes of vibrations of a clamped singly curved plate, with the increase of number of elements (in quarter of the plate). Mesh patterns are shown in Fig.23.

<table>
<thead>
<tr>
<th>Frequency Hz.</th>
<th>12 elements</th>
<th>20 elements</th>
<th>24 elements (m,n) of W.</th>
</tr>
</thead>
<tbody>
<tr>
<td>987.06</td>
<td>977.39</td>
<td>972.51 (1, 3)*</td>
<td></td>
</tr>
<tr>
<td>1337.11</td>
<td>1319.87</td>
<td>1310.89 (1, 3)**</td>
<td></td>
</tr>
<tr>
<td>2078.63</td>
<td>2067.71</td>
<td>2067.51 (3, 1)**</td>
<td></td>
</tr>
<tr>
<td>2600.97</td>
<td>2581.85</td>
<td>2577.74 (degenerate)</td>
<td></td>
</tr>
<tr>
<td>2656.40</td>
<td>2612.59</td>
<td>2592.27 (degenerate)</td>
<td></td>
</tr>
</tbody>
</table>

*(m,n) of v is (1,2) and of u (2,3). Frequency obtained by method of Kantorovich 965.73 Hz for this mode.
** (m,n) of v is (1,4) and that of u (2,1).
*** Corresponding frequency by the method of Kantorovich is 2063.29 Hz.

Dimensions of the plate: \( L_x = 3 \) in., \( L_y = 4 \) in., \( h = 0.013 \) in., \( k_x = 0, k_y = 1/30 \) in\(^{-1}\), \( k_{xy} = 0 \).

Properties of material: \( E = 10^7 \) lb/in\(^2\), \( v = 0.33 \), \( \rho = 0.096 \) lb/in\(^3\).
TABLE No.18

Change of frequencies of a singly curved clamped plate with the increase of number of elements (in quarter of the plate). Modes of vibrations are symmetrical along the straight line generators and antisymmetrical along the curved direction.

<table>
<thead>
<tr>
<th>Frequency Hz.</th>
<th>12 elements</th>
<th>20 elements</th>
<th>24 elements (m,n) of W.</th>
</tr>
</thead>
<tbody>
<tr>
<td>911.53</td>
<td>897.11</td>
<td>890.00 (1,2)*</td>
<td></td>
</tr>
<tr>
<td>1861.39</td>
<td>1831.07</td>
<td>1815.59 (1,4)</td>
<td></td>
</tr>
<tr>
<td>2248.56</td>
<td>2234.69</td>
<td>2233.50 (3,2)</td>
<td></td>
</tr>
<tr>
<td>3153.54</td>
<td>3126.56</td>
<td>3119.19 (3,4)</td>
<td></td>
</tr>
<tr>
<td>3629.70</td>
<td>3559.33</td>
<td>3527.11 (1,6)</td>
<td></td>
</tr>
</tbody>
</table>

*The corresponding frequency obtained by the method of Kantorovich and matrix progression is 889.76 Hz.

The dimensions and material properties of the plate are the same as those of Table No. 17. The mesh patterns are shown in Figure 23.
<table>
<thead>
<tr>
<th>No.</th>
<th>Frequency</th>
<th>( w(m,n) )</th>
<th>No.</th>
<th>Frequency</th>
<th>( w(m,n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>890.00</td>
<td>1, 2</td>
<td>17</td>
<td>3527.11</td>
<td>1, 6</td>
</tr>
<tr>
<td>2</td>
<td>972.51</td>
<td>1, 3 ( v(1,2) u(2,3) )</td>
<td>18</td>
<td>3674.84</td>
<td>4, 3</td>
</tr>
<tr>
<td>3</td>
<td>1310.89</td>
<td>1, 3 ( v(1,4) u(2,1) )</td>
<td>19</td>
<td>3850.74</td>
<td>3, 5</td>
</tr>
<tr>
<td>4</td>
<td>1371.46</td>
<td>2, 1</td>
<td>20</td>
<td>3993.66</td>
<td>2, 6</td>
</tr>
<tr>
<td>5</td>
<td>1454.19</td>
<td>2, 2</td>
<td>21</td>
<td>4217.92</td>
<td>4, 4</td>
</tr>
<tr>
<td>6</td>
<td>1775.10</td>
<td>2, 3</td>
<td>22</td>
<td>4478.56</td>
<td>degenerate</td>
</tr>
<tr>
<td>7</td>
<td>1815.59</td>
<td>1, 4</td>
<td>23</td>
<td>4643.79</td>
<td>1, 7</td>
</tr>
<tr>
<td>8</td>
<td>2067.51</td>
<td>3, 1</td>
<td>24</td>
<td>4698.54</td>
<td>degenerate</td>
</tr>
<tr>
<td>9</td>
<td>2233.50</td>
<td>3, 2</td>
<td>25</td>
<td>4765.30</td>
<td>degenerate</td>
</tr>
<tr>
<td>10</td>
<td>2318.90</td>
<td>2, 4</td>
<td>26</td>
<td>4940.53</td>
<td>degenerate</td>
</tr>
<tr>
<td>11</td>
<td>2577.74</td>
<td>degenerate*</td>
<td>27</td>
<td>5099.42</td>
<td>degenerate</td>
</tr>
<tr>
<td>12</td>
<td>2592.27</td>
<td>degenerate</td>
<td>28</td>
<td>5623.74</td>
<td>degenerate</td>
</tr>
<tr>
<td>13</td>
<td>3064.94</td>
<td>degenerate</td>
<td>29</td>
<td>5842.91</td>
<td>4, 6</td>
</tr>
<tr>
<td>14</td>
<td>3103.02</td>
<td>degenerate</td>
<td>30</td>
<td>6193.36</td>
<td>degenerate</td>
</tr>
<tr>
<td>15</td>
<td>3119.19</td>
<td>3, 4</td>
<td>31</td>
<td>6384.89</td>
<td>degenerate</td>
</tr>
<tr>
<td>16</td>
<td>3307.74</td>
<td>4, 2</td>
<td>32</td>
<td>6420.98</td>
<td>6, 2</td>
</tr>
</tbody>
</table>

*The degenerate mode shapes are shown in Figure 24 and 25.*
TABLE No. 20

Change of frequencies of a clamped plate, under bi-axial tension with the increase of number of elements (in quarter of the plate). Symmetrical modes of vibrations. Mesh patterns are shown in Fig. 23.

<table>
<thead>
<tr>
<th>Frequency Hz.</th>
<th>20 elements with ( (m,n) ) of ( w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12 elements</td>
<td></td>
</tr>
<tr>
<td>1235.27</td>
<td>1231.89 ((1, 1)^*)</td>
</tr>
<tr>
<td>1933.10</td>
<td>1917.27 ((1, 3))</td>
</tr>
<tr>
<td>2493.93</td>
<td>2483.10 ((3, 1)^{**})</td>
</tr>
<tr>
<td>3208.31</td>
<td>3189.60 ((3, 3))</td>
</tr>
<tr>
<td>3509.69</td>
<td>3471.13 ((1, 5))</td>
</tr>
</tbody>
</table>

* Corresponding frequency by Kantorovich matrix progression is 1231.79 Hz.

** Corresponding frequency by Kantorovich matrix progression is 2479.7 Hz.

The dimensions and material properties of the plate are the same as those given in Table 17.

Inplane stresses:

\[ s_x = 1500 \text{ lb/in}^2, \quad s_y = 3000 \text{ lb/in}^2, \quad s_{xy} = 0. \]
Change of frequencies of a singly curved clamped plate under bi-axial tension with the increase of number of elements (in quarter of the plate). The vibrations are symmetrical parallel to the straight edges and antisymmetrical parallel to the curved edges. The mesh patterns are shown in Fig.23.

<table>
<thead>
<tr>
<th>Frequency Hz.</th>
<th>12 elements</th>
<th>20 elements with ((m,n)) of (w)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1373.61</td>
<td>1363.80 (1,2)*</td>
<td></td>
</tr>
<tr>
<td>2631.93</td>
<td>2607.44 (1,4)</td>
<td></td>
</tr>
<tr>
<td>2752.17</td>
<td>2738.47 (3,2)</td>
<td></td>
</tr>
<tr>
<td>3857.36</td>
<td>3830.58 (3,4)</td>
<td></td>
</tr>
<tr>
<td>4546.49</td>
<td>4481.31 (1,6)</td>
<td></td>
</tr>
</tbody>
</table>

*Frequency corresponding to the (1, 2) mode by the method of Kantorovich is 1365.569 Hz.

The dimensions and the material properties of the plate are the same as those given in Table 17.

The inplane stresses are the same as those given in Table 20.
Frequencies of a clamped singly curved plate under uniform bi-axial membrane stresses. The axial stress is half of the hoop stress. Dimensions and the properties of the material of the plate are the same as those given in Table 17.

<table>
<thead>
<tr>
<th>applied hoop stress $\text{lb/in}^2$</th>
<th>FREQUENCY $\text{Hz.}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tension</td>
</tr>
<tr>
<td>100</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>992.89 (1,2)</td>
</tr>
<tr>
<td></td>
<td>1039.11 (deg)**</td>
</tr>
<tr>
<td>1000</td>
<td>1078.90 (1,2)</td>
</tr>
<tr>
<td></td>
<td>1087.86 (1,3)</td>
</tr>
<tr>
<td>1500</td>
<td>1129.31 (deg)</td>
</tr>
<tr>
<td></td>
<td>1157.64 (1,2)</td>
</tr>
<tr>
<td>1750</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>1166.25 (1,1)</td>
</tr>
<tr>
<td></td>
<td>1230.70 (1,2)</td>
</tr>
<tr>
<td>3000</td>
<td>1231.89 (1,1)</td>
</tr>
<tr>
<td></td>
<td>1363.80 (1,2)</td>
</tr>
</tbody>
</table>

* The stress system is beyond critical. Ignore the imaginary frequency.

**(deg) stands for degenerate mode.

*** $m,n$ of $w$. 
TABLE No. 23

Frequencies of a clamped shallow square spherical plate with 25 elements (in quarter plate). A 5x5 mesh pattern is used.

<table>
<thead>
<tr>
<th>Symmetrical modes of vibrations</th>
<th>Antisymmetrical modes of vibrations</th>
<th>Symmetric-anti Symmetric modes of vibrations</th>
</tr>
</thead>
<tbody>
<tr>
<td>860.33</td>
<td>909.02 (2,2)</td>
<td>835.58 (1,2)</td>
</tr>
<tr>
<td>952.08</td>
<td>1295.04</td>
<td>1053.68 (3,2)</td>
</tr>
<tr>
<td>1110.00</td>
<td>1312.19</td>
<td>1197.70 (1,4)</td>
</tr>
<tr>
<td>1223.07</td>
<td>1765.23 (4,4)</td>
<td>1486.02 (3,4)</td>
</tr>
<tr>
<td>1552.97</td>
<td>2137.09</td>
<td>1668.00 (5,2)</td>
</tr>
<tr>
<td>1579.46</td>
<td>2143.49</td>
<td>2022.91 (7,6)</td>
</tr>
<tr>
<td>1855.42</td>
<td>2623.64</td>
<td>2145.27 (5,8)</td>
</tr>
<tr>
<td>1865.51</td>
<td>2630.41</td>
<td>2338.71 (3,6)</td>
</tr>
</tbody>
</table>

* The frequencies for the m,n modes of the x-symmetric y-antisymmetric modes are same as the frequencies of the x-antisymmetric y-symmetric modes with m and n interchanged. The numbers in the brackets indicate mode numbers. The modes of the other frequencies are degenerate.

Dimensions of the plate:

\[ L = L = 5 \text{ in}, h = 0.01 \text{ in}, k_x = k_y = 0.02 \text{ in}^{-1}, k_{xy} = 0. \]

Properties of material:

\[ E = 33.7 \times 10^6 \text{ lb/} \text{in}^2, \quad v = 0.3, \quad \rho = 0.27 \text{ lb/} \text{in}^3. \]
comparison of frequencies of a clamped square spherical plate obtained by the method of kantorovich and the finite element method (with 25 elements in quarter plate). mesh pattern used is 5x5.

<table>
<thead>
<tr>
<th>Frequency Hz.</th>
<th>m, n of w</th>
<th>Percentage difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kantorovich-Matrix Progression</td>
<td>Finite element</td>
<td></td>
</tr>
<tr>
<td>837.03</td>
<td>835.58</td>
<td>1, 2</td>
</tr>
<tr>
<td>872.67</td>
<td>860.33</td>
<td>1, 3</td>
</tr>
<tr>
<td></td>
<td>degenerate (symmetrical)</td>
<td></td>
</tr>
<tr>
<td>934.18</td>
<td>909.02</td>
<td>2, 2</td>
</tr>
<tr>
<td>1029.02</td>
<td>952.08</td>
<td>3, 1</td>
</tr>
<tr>
<td></td>
<td>degenerate (symmetrical)</td>
<td></td>
</tr>
<tr>
<td>1077.26</td>
<td>1053.68</td>
<td>3, 2</td>
</tr>
<tr>
<td>1224.60</td>
<td>1197.70</td>
<td>1, 4</td>
</tr>
</tbody>
</table>

dimensions of the plate are given in table 23.
TABLE 25

Frequencies of a clamped shallow anticlastic plate.

<table>
<thead>
<tr>
<th>Symmetrical modes</th>
<th>Antisymmetrical modes</th>
<th>Symmetric-antisymmetric modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>724.54</td>
<td>702.96 (2,2)</td>
<td>691.35 (1,2)</td>
</tr>
<tr>
<td>883.89</td>
<td>1197.53</td>
<td>917.42 (3,2)</td>
</tr>
<tr>
<td>903.11</td>
<td>1201.09</td>
<td>1126.77 (1,4)</td>
</tr>
<tr>
<td>1087.34</td>
<td>1673.55 (4,4)</td>
<td>1379.57 (3,4)</td>
</tr>
<tr>
<td>1499.93</td>
<td>2080.90</td>
<td>1594.45 (5,2)</td>
</tr>
<tr>
<td>1499.98</td>
<td>2082.10</td>
<td>1970.13 (7,6)</td>
</tr>
<tr>
<td>1781.02</td>
<td>2569.66</td>
<td>2073.58 (5,8)</td>
</tr>
<tr>
<td>1783.20</td>
<td>2573.20</td>
<td>2279.18 (3,6)</td>
</tr>
</tbody>
</table>

*Reciprocity of the vibrations is completely maintained.
The numbers in the brackets indicates the modes of \( w \).
The frequencies without mode numbers are degenerate modes. frequencies.
plate dimensions:
\[ L_x = L_y = 5 \text{ in}, \]
\[ h = 0.01 \text{ in}, \]
\[ k_x = 0.02 \text{ in}^{-1}, k_y = -k_x, \]
\[ k_{xy} = 0. \]

Properties of material:
\[ E = 33.7 \times 10^6 \text{ lb/in}^2, \]
\[ v = 0.3, \rho = 0.27 \text{ lb/in}^3. \]

A 5x5 mesh pattern is used in quarter of the plate.
TABLE No. 26

Comparison of frequencies of a clamped square antilastic plate obtained by the method of Kantorovich and the finite element method.

<table>
<thead>
<tr>
<th>m, n of w</th>
<th>Frequency Hz</th>
<th>Percentage difference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Kantorovich-matrix Progression</td>
<td>Finite element</td>
</tr>
<tr>
<td>1, 2</td>
<td>699.51</td>
<td>691.35</td>
</tr>
<tr>
<td>2, 2</td>
<td>735.24</td>
<td>702.96</td>
</tr>
<tr>
<td>3, 2</td>
<td>908.45</td>
<td>917.42</td>
</tr>
<tr>
<td>4, 1</td>
<td>1133.13</td>
<td>1126.77</td>
</tr>
<tr>
<td>1, 3</td>
<td>741.75</td>
<td>724.54 (first symmetrical mode)</td>
</tr>
</tbody>
</table>

Mesh pattern and dimensions of the plate are the same as those given in Table 25.
**TABLE 27.**

Frequencies of a clamped shallow singly curved plate.

<table>
<thead>
<tr>
<th>Symmetrical</th>
<th>Antisymmetrical</th>
<th>Symmetric along straight line generators and antisymmetric along curved direction</th>
<th>Antisymmetric along straight line generators and symmetric along the curved direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>530.52 (1,3)</td>
<td>656.52 (2,2)</td>
<td>445.36 (1,2)</td>
<td>674.28 (2,3)</td>
</tr>
<tr>
<td>745.82 (dg)*</td>
<td>1099.05 (2,4)</td>
<td>891.24 (3,2)</td>
<td>834.55 (dg)</td>
</tr>
<tr>
<td>859.01 (3,1)</td>
<td>1192.45 (4,2)</td>
<td>947.69 (1,4)</td>
<td>1118.41 (4,1)</td>
</tr>
<tr>
<td>1050.04 (3,3)</td>
<td>1649.12 (4,4)</td>
<td>1330.96 (3,4)</td>
<td>1365.05 (4,3)</td>
</tr>
<tr>
<td>1370.48 (1,5)</td>
<td>2006.16 (2,6)</td>
<td>1580.95 (5,2)</td>
<td>1503.54 (2,5)</td>
</tr>
<tr>
<td>1483.02 (5,1)</td>
<td>2065.49 (6,2)</td>
<td>1871.93 (1,6)</td>
<td>1954.16 (dg)</td>
</tr>
<tr>
<td>1727.53 (3,5)</td>
<td>2535.35 (4,6)</td>
<td>2055.62 (5,4)</td>
<td>2041.19 (4,5)</td>
</tr>
<tr>
<td>1768.45 (dg)</td>
<td>2553.65 (6,4)</td>
<td>2226.69 (3,6)</td>
<td>2262.62 (dg)</td>
</tr>
</tbody>
</table>

*dg - frequency of a degenerate mode shown in Fig. 31.*

The numbers in the bracket indicate the mode of \( w \).

**plate dimensions:**

\[
\begin{align*}
L_x & = L_y = 5 \text{ in.} \\
h & = 0.01 \text{ in.} \\
k_y & = 0.02 \text{ in}^{-1} \\
k_x & = k_{xy} = 0.
\end{align*}
\]

**Properties of material:**

\[
E = 33.7 \times 10^6 \text{ lb/in}^2, \quad v = 0.3, \quad \rho = 0.27 \text{ lb/in}^3.
\]

A 5x5 mesh pattern is used in quarter of the plate.
TABLE No. 28

Comparison of the frequencies of a clamped square shallow singly curved plate, obtained by the method of Kantorovich and the method of finite element.

<table>
<thead>
<tr>
<th>m, n of w</th>
<th>Frequency Hz.</th>
<th>Percentage difference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Kantorovich-matrix Progression</td>
<td>Finite element</td>
</tr>
<tr>
<td>1, 2</td>
<td>437.70</td>
<td>445.36</td>
</tr>
<tr>
<td>1, 3</td>
<td>509.06</td>
<td>530.52</td>
</tr>
<tr>
<td>2, 2</td>
<td>650.15</td>
<td>656.52</td>
</tr>
<tr>
<td>3, 1</td>
<td>858.26</td>
<td>859.01</td>
</tr>
<tr>
<td>3, 2</td>
<td>884.92</td>
<td>891.24</td>
</tr>
<tr>
<td>4, 1</td>
<td>1116.43</td>
<td>1118.41</td>
</tr>
<tr>
<td>4, 2</td>
<td>1179.03</td>
<td>1192.45</td>
</tr>
</tbody>
</table>

The mesh pattern and dimensions of the plate are the same as those given in Table 27.
TABLE NO. 29

Maximum stresses in a singly curved plate due to unit static uniformly distributed loading.

<table>
<thead>
<tr>
<th>Nodal point</th>
<th>Direction of stress</th>
<th>Combined stresses**</th>
<th></th>
<th>Direct stress</th>
<th></th>
<th>Bending stress</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1*</td>
<td>x</td>
<td>162.04</td>
<td>-1482.41</td>
<td>-4660.19</td>
<td>822.23</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>y</td>
<td>-1847.04</td>
<td>-3052.03</td>
<td>-2449.53</td>
<td>602.50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>x</td>
<td>-1476.79</td>
<td>299.96</td>
<td>-588.42</td>
<td>-888.38</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>y</td>
<td>-4475.13</td>
<td>908.96</td>
<td>-1783.00</td>
<td>-2692.05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>x</td>
<td>-5006.25</td>
<td>4469.79</td>
<td>-268.23</td>
<td>-4738.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>y</td>
<td>-1652.06</td>
<td>1475.03</td>
<td>-78.14</td>
<td>-1553.17</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Location of the nodal points are given in (a) of Figure 32.

** The stresses are in lb/in².
TABLE NO. 30

Comparison between calculated and measured frequencies (Hz) of a clamped singly curved rectangular plate.

<table>
<thead>
<tr>
<th>Serial no.</th>
<th>Calculated frequency</th>
<th>Experiment</th>
<th>m,n of w</th>
<th>Sand pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Rapid frequency</td>
<td>Discrete frequency</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>890</td>
<td>812</td>
<td>814</td>
<td>1,2</td>
</tr>
<tr>
<td>2</td>
<td>973</td>
<td>926</td>
<td>940</td>
<td>1,3</td>
</tr>
<tr>
<td>3</td>
<td>1311</td>
<td>1258</td>
<td>1260</td>
<td>1,3</td>
</tr>
<tr>
<td>4</td>
<td>1371</td>
<td>1335</td>
<td>1306</td>
<td>2,1</td>
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<tr>
<td>5</td>
<td>1454</td>
<td>1480</td>
<td>1452</td>
<td>2,2</td>
</tr>
<tr>
<td>6</td>
<td>1775</td>
<td>[1778]</td>
<td>1802</td>
<td>2,3</td>
</tr>
<tr>
<td>7</td>
<td>1816</td>
<td>[1738]</td>
<td>[1735]</td>
<td>1,4</td>
</tr>
<tr>
<td>8</td>
<td>2068</td>
<td>2129</td>
<td>2100</td>
<td>3,1</td>
</tr>
<tr>
<td>9</td>
<td>2234</td>
<td>2225</td>
<td>2225</td>
<td>3,2</td>
</tr>
<tr>
<td>10</td>
<td>2319</td>
<td>2288</td>
<td>2280</td>
<td>2,4</td>
</tr>
</tbody>
</table>

* The longer side is curved.
Comparison between calculated and measured frequencies (Hz) of a clamped singly curved rectangular plate.

<table>
<thead>
<tr>
<th>Serial No.</th>
<th>Frequency Calculated</th>
<th>Frequency Measured</th>
<th>$m,n$ of $w$</th>
<th>Sand pattern of the degenerate modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-</td>
<td>2329</td>
<td>dg*</td>
<td>[Diagram 1]</td>
</tr>
<tr>
<td>2</td>
<td>2577</td>
<td>2518</td>
<td>dg</td>
<td>[Diagram 2]</td>
</tr>
<tr>
<td>3</td>
<td>2592</td>
<td>2622</td>
<td>dg</td>
<td>[Diagram 3]</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>2649</td>
<td>dg</td>
<td>[Diagram 4]</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>2713</td>
<td>dg</td>
<td>[Diagram 5]</td>
</tr>
<tr>
<td>6</td>
<td>-</td>
<td>3016</td>
<td>2,5</td>
<td>[Diagram 6]</td>
</tr>
<tr>
<td>7</td>
<td>3119</td>
<td>3113</td>
<td>3,4</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-</td>
<td>3188</td>
<td>4,1</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>3308</td>
<td>3332</td>
<td>4,2</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-</td>
<td>3348</td>
<td>4,2</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>3527</td>
<td>3403</td>
<td>1,6</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>-</td>
<td>3445</td>
<td>1,6</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>-</td>
<td>3534</td>
<td>no**</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>3675</td>
<td>3699</td>
<td>4,3</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>3851</td>
<td>3812</td>
<td>3,5</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>3994</td>
<td>3930</td>
<td>2,6</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>4218</td>
<td>4195</td>
<td>4,4</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>-</td>
<td>4331</td>
<td>5,6</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>4644</td>
<td>4408</td>
<td>1,7</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>-</td>
<td>4544</td>
<td>dg</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>-</td>
<td>4670</td>
<td>3,6</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>-</td>
<td>4723</td>
<td>5,2</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>-</td>
<td>4984</td>
<td>2,7</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>-</td>
<td>5150</td>
<td>5,3</td>
<td></td>
</tr>
</tbody>
</table>

* dg : Degenerate mode.
**no : Sand pattern not clear.
*** The longer side is curved.
Fig. 1  Geometry of a singly curved rectangular plate.
Forces in a shell element

Moments in a shell element

Fig. 2  Forces and moments in an element of circular cylindrical shell.
\[ \Lambda_I = \text{Initial value of } \Lambda. \]
\[ \Lambda_F = \text{Final value of } \Lambda. \]
\[ \epsilon = \text{Increment of } \Lambda \text{ in each step}. \]

The interval between \( \Lambda_I \) and \( \Lambda_F \) is scanned for \( \Lambda_w \). The scanning may be performed from \( \Lambda_I \) to \( \Lambda_F \) or from \( \Lambda_F \) to \( \Lambda_I \). The following flow diagram is shown for scanning from \( \Lambda_I \).

Input data \( L, R, h, \theta, v, \Lambda_F, \epsilon, m \)

Form the coefficients of ordinary differential equations which are independent of \( \Lambda \).

Input \( \Lambda_I \) and form the coefficients which are dependent on \( \Lambda \).

Find roots \( \lambda_j \) of the characteristic equation and the constants \( a_j, b_j \).

Form frequency determinant \( D(\Lambda) \) either in real or in complex form.

\[
\begin{align*}
\text{YES} & \quad \text{PRINT } \Lambda \\
\text{NO} & \quad \Lambda_I = \Lambda_I + \epsilon
\end{align*}
\]

Compute \( a_j, b_j, c_j \), compute \( u, v, w \); also the stress resultants if needed.

Print out results.

\[
\begin{align*}
\text{YES} & \quad \Lambda_I = \Lambda_I + \epsilon \\
\text{NO} & \quad \Lambda_I > \Lambda_F
\end{align*}
\]

If no eigenvalue is obtained redefine \( \Lambda_F \); otherwise 'EXIT'.

Exit

Fig. 3 Flow diagram for iteration.
\( \Delta_I \) = Initial value of \( \Delta \).

\( \Delta_F \) = Final value of \( \Delta \).

\( \epsilon \) = Increment of \( \Delta \) in each step.

The interval between \( \Delta_I \) and \( \Delta_F \) is scanned for \( \Delta_w \). The scanning is performed from \( \Delta_I \) to \( \Delta_F \) in the following flow chart.

- **Input data** \( L, R, h, \theta, \nu, \Delta, \epsilon, m, n \)
  - (additional inplane load data \( p_f \) and \( p_x \) for the inplane load case)
- Coefficients of ordinary differential equations which are independent of \( \Delta \).
- Form the boundary condition matrices \( J, K \).
- **Input \( \Delta_I \) and form the coefficients which are dependent on \( \Delta \). Form the matrix \( [A] \) and \( e^{[A] \theta_1} \)
- **Apply modified matrix progression and form \( D(\Delta) \).**
- **YES** \( D(\Delta) = 0? \) **NO** \( \Delta_I = \Delta_I + \epsilon \)
- **YES** \( \Delta_I > \Delta_F \) **NO**
  - \( \Delta_I = \Delta_I + \epsilon \)
  - If necessary redefine \( \Delta_F \), else 'exit'.
- **Exit**
- **Exit**

Fig. 4 Flow diagram for iteration; modified matrix method.
Fig. 5  $D(\Delta) \text{vs} \Delta$ of a clamped plate. General method of solution with real co-efficients.
Fig. 6  \(D(\Delta)\) vs \(\Delta\) of a clamped plate. Modified matrix progression method.

\[L = 17\,\text{in},\ b = 17\,\text{in},\ h = 0.02\,\text{in},\ R = 96\,\text{in},\ m = 1,\ \nu = 0.33\]
<table>
<thead>
<tr>
<th>Simply supported</th>
<th>$g_2$</th>
<th>$g_2=0$ At the edges</th>
<th>$g_3$ Unchanged</th>
<th>$n$ of $g_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_3$</td>
<td>$g_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sin n\pi\phi/\theta$</td>
<td>$\cos n\pi\phi/\theta$</td>
<td>Alternative shapes of $g_2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: The figures are not to scale

Fig. 7 Possible mode shapes of $g_2$ when $g_2=0$ at the edges and $g_3$ is a sine function.
Webster's solutions for clamped boundaries[32]
(K) Kantorovich solution for clamped boundaries
(RR) Raleigh-Ritz solution for clamped boundaries using beam functions.
(S) Simply supported boundaries.

Fig. 8 Variation of lowest $\Delta_w$ for a square plate with central angle.
Fig. 9  Variation of lowest $\Delta_w$ with aspect ratio $L/b$. 

$h/b = 0.0012$

$b/R = 0.177$

$m = 1$

$\nu = 0.33$

*indicates direction of Kantorovich reduction
Fig. 10 Variation of lowest $\Delta_w$ with the variation of $h/b$ of a clamped square singly curved plate.
Fig. 11 Variation of eigenvalues with variation of inplane forces of a simply supported square plate.
Fig. 12 Variation of eigenvalues with the variation of inplane forces of a simply supported rectangular plate.
The figures at the points indicate the value of \( n \).

**Fig. 13** Lowest eigenvalues of a simply supported square plate.
L/b = 1.22
h/b = 0.0031
b/R = 0.0936
\nu = 0.33

The figures at the points indicate value of \( n \).

Fig. 14 Lowest eigenvalues of a simply supported rectangular plate.
$L/b = 1.0$
$h/b = 0.0012$
$b/R = 0.177$
$m = 1$
$\nu = 0.33$

---

**Fig. 15** Variation of lowest eigenvalues of a square plate with cylindrical pressure.

- $S$ — all edges simply supported
- $C$ — all edges clamped
- $XCC$-$YSS$ — curved edges clamped and straight edges simply supported.
C all edges clamped
XCC-YSS curved edges clamped; straight edges simply supported.
S all edges simply supported.

Fig. 16 Variation of lowest eigenvalues of a square plate with cylindrical pressure.
Fig. 17 Variation of lowest eigenvalues of a rectangular plate with cylindrical pressure.
(a) A shallow shell \( f_h < \frac{1}{5} \) of the smaller of \( L_x \) and \( L_y \)

(b) Stress resultants and applied inplane stresses.

c) Stress couples and transverse shears.

Fig. 18 A shallow shell.
(a) Co-ordinates and order of node numbering of an element.

(b) Discretization and node numbering of a plate. The equation of the surface is defined in the X-Y system. The integers and the Roman numerals indicate the node and element numbers respectively.

Fig. 19
(a) Node numbering when the nodes on the boundaries do not contribute to the nodal equilibrium of the system.

(b) Node numbering of a quarter plate when the nodes at the boundaries do not contribute to the nodal equilibrium of the system.

Fig. 20
(a) Symmetrical modes of vibrations (quarter plate)

(b) Antisymmetrical modes of vibrations (quarter plate)

Fig. 21 Clamped boundary conditions and intermediate constraints ($h=\text{constant}, K_{xy}=0$).
(a) Symmetrical modes of vibrations (quarter plate shown)

(b) Anti-symmetrical modes of vibrations (quarter plate shown)

Fig. 22 Clamped boundary conditions and intermediate constraints when $k_x = k_y = 0$ and $k_{xy} \neq 0$; $h =$ constant.
Fig 23 Discretization of quarter of a clamped singly curved plate.
Fig. 24 Degenerate modes of a clamped singly curved rectangular plate of aspect ratio 0.75.
Fig. 25  Degenerate modes of a clamped singly curved rectangular plate of aspect ratio 0.75.
Fig. 26 Frequency parameter vs applied hoop stress $s_y (s_x = s_y / 2$ assumed)
Fig 27 Nodal patterns of symmetrical modes of a clamped square spherical plate.

\(a = b = 5\text{ in.}, \ h = 0.01\text{ in.}, \ k_x = k_y = 0.02\text{ in}^{-1}, k_{xy} = 0.\ E = 33.7 \times 10^6\text{ lb/in}^2, \ \nu = 0.3, \ \rho = 0.27\text{ lb/in}^3.\)
Fig. 28 Nodal patterns of antisymmetrical modes of vibrations of a clamped square spherical plate.
Fig. 29 Nodal patterns of symmetrical modes of a clamped square anticlastic plate ($k_y = -k_x, k_{xy} = 0$).
Fig 30  Nodal patterns of antisymmetrical modes of vibrations of a clamped square anti-clastic plate \((k_y=-k_x, k_{xy}=0)\)
(a) Symmetrical modes of vibration.

(b) Modes of vibration are antisymmetric along the straight-line generators and symmetrical along the curved direction.

\( x \) - direction of straight line generator
\( y \) - curved direction

Fig. 31 Degenerate modes of a clamped square singly curved plate \( k_x = k_{xy} = 0, k_y = 1/R \).
The dimensions are in inches

(a) Discretization and nodal reference of quarter of a plate.

(b) Maximum $\sigma_x$

(c) Maximum $\sigma_y$

(d) Maximum $\tau_{xy}$

Plate stresses due to uniform static load of unit magnitude.

Fig. 32.
<table>
<thead>
<tr>
<th></th>
<th>0.00</th>
<th>0.00</th>
<th>0.00</th>
<th>0.00</th>
<th>0.00</th>
<th>0.00</th>
<th>0.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.45</td>
<td>-0.47</td>
<td>-0.56</td>
<td>-0.70</td>
<td>-0.81</td>
<td>-0.69</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>-0.83</td>
<td>-0.87</td>
<td>-1.01</td>
<td>-1.23</td>
<td>-1.44</td>
<td>-1.25</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>-1.00</td>
<td>-1.04</td>
<td>-1.17</td>
<td>-1.40</td>
<td>-1.63</td>
<td>-1.48</td>
<td>0.00</td>
<td>0.00</td>
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<tr>
<td>-0.72</td>
<td>-0.75</td>
<td>-0.83</td>
<td>-0.99</td>
<td>-1.16</td>
<td>-1.14</td>
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<td>0.00</td>
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<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
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</tbody>
</table>

\[ u \times 10^5 \text{ in.} \]

<table>
<thead>
<tr>
<th></th>
<th>0.00</th>
<th>3.71</th>
<th>6.86</th>
<th>8.72</th>
<th>8.41</th>
<th>5.29</th>
<th>0.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>3.45</td>
<td>6.45</td>
<td>8.24</td>
<td>7.99</td>
<td>5.07</td>
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<tr>
<td>0.00</td>
<td>2.83</td>
<td>5.28</td>
<td>6.82</td>
<td>6.73</td>
<td>4.37</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>1.90</td>
<td>3.56</td>
<td>4.65</td>
<td>4.71</td>
<td>3.19</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.89</td>
<td>1.66</td>
<td>2.18</td>
<td>2.27</td>
<td>1.68</td>
<td>0.00</td>
<td>0.00</td>
</tr>
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<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

\[ v \times 10^5 \text{ in.} \]

<table>
<thead>
<tr>
<th></th>
<th>10.03</th>
<th>9.73</th>
<th>8.77</th>
<th>7.00</th>
<th>4.41</th>
<th>1.55</th>
<th>0.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.58</td>
<td>9.30</td>
<td>8.41</td>
<td>6.75</td>
<td>4.29</td>
<td>1.53</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>8.11</td>
<td>7.90</td>
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\[ w \times 10^3 \text{ in.} \]

Fig. 33. u, v and w due to uniformly distributed unit static load.
### (a) Normal slope along y. \((w^* x 10^3)\).

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### (b) Normal slope along x. \((w^* x 10^3)\).

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### (c) The twist. \((w'' x 10^3)\).

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**Fig. 34.** Slopes and twist due to uniformly distributed unit static load.
### (a) $\sigma_{x\text{-max}} \times 10^{-2}$ lb/in$^2$.

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### (b) $\sigma_{y\text{-max}} \times 10^{-3}$ lb/in$^2$.

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### (c) $\tau_{xy\text{-max}} \times 10^{-1}$ lb/in$^2$.

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Fig. 35. Maximum stresses due to unit uniformly distributed static load.
(a) $\sigma_{x-min} \times 10^{-3}$ lb/in$^2$.

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(b) $\sigma_{y-min} \times 10^{-3}$ lb/in$^2$.

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(c) $\tau_{xy-min} \times 10^{-2}$ lb/in$^2$.

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Fig. 36. Minimum stresses due to uniformly distributed unit static load.
(a) Combined stress along the central generator

(b) Combined stress along the central arch

Fig. 37 Variation of combined stresses.
Fig. 38. Spectral density of the transverse displacements of a singly curved plate.
Input mc, mb, nb, md, ε -
nc = 0
nc = nc + 1
Input L, R, h, φ, ν
Output L, R, h, φ, ν
nd = 0
nd = nd + 1
Input q_x, q_φ
Output q_x, q_φ
m = n = 0
m = m + 1
n = n + 1
Form co-efficients of cubic
Bairstow's iteration, comp. Δ_u, Δ_v, Δ_w
Output Δ_u, Δ_v, Δ_w
nf = 0
nf = nf + 1
Compute A, B, C for 4(nf)

nf = 3?
no
yes
n = nb?
no
yes
m = mb?
no
yes
nd = md?
no
yes
nc = nc?
no
yes
EXIT

Fig. 39. Flow chart for solving cubic frequency equation.
Fig. 40(a). Flow diagram of solution of eigensystem using finite elements.
Fig. 40(b) Flow diagram of random analysis using finite element method.
Fig. 41 Flow diagram for static analysis using finite element method.
Fig 42 Details of a singly curved clamped plate.
Fig. 43. Block diagram of instrumentation for rapid frequency test.
(For discrete frequency response test replace the sweep oscillator by a decade oscillator and remove the filter and the tape recorder).

\( V_{1-6} \) are terminal supply.
Transient excitation $x(t)$ \hspace{1cm} System \hspace{1cm} Response $y(t)$

$x(t) \rightarrow x(i\omega)$ \hspace{1cm} $y(i\omega) = \frac{y(i\omega)}{x(i\omega)}$

$H(i\omega) =$ Frequency response function

(a) Classical method

Excitation $x(t)$ \hspace{1cm} System \hspace{1cm} Response $y(t)$

$y(t) \rightarrow R_{yy}(\tau)$ \hspace{1cm} $F(i\omega)$

$F(i\omega) = \left\{ i\left(\frac{\omega}{\omega_n}\right)H(i\omega) + 2\eta H(i\omega) \right\}$

Assumption:
Excitation of constant spectral density function over frequency regime of interest.

(b) Characteristic response function method (ref. 89)

Fig. 44 The classical and characteristic response function method.
Fig. 45 Power spectral density $G_{pp}(f)$ vs frequency $f$. 
(a) Measurement of force input.

(b) Power spectrum of the components of the force generating tandem of (a).

Fig. 46 Effects of the component input power spectra on the overall system input power spectrum $G_{pp}(f)$. 
(a) Time history of the output (displacement response).

(b) Autocorrelation of the output (1000 time lags).

Fig. 47 Time history and autocorrelation of the output.

$t =$ time
\( \tau =$ time lag
\( w(t) =$ transverse displacement
\( R_{ww}(\tau) =$ autocorrelation of \( w(t) \)
Fig. 48 amplitude and phase of frequency response function $H(i\omega)$ with 1 Hz resolution.
Amplitude and phase of frequency response function $H(\omega)$ with 5 Hz resolution.
Fig. 50 Amplitude and phase of frequency response function $H(i\omega)$ with 10 Hz resolution.
Fig. 51 Amplitude and phase of Characteristic Response function $F(i\omega)$ with 10 Hz resolution.
Fig. 52 Magnification of the amplitude and phase of Fig. 51 between 750 and 990 Hz at 1 Hz resolution.
Fig. 53 Vector plot.
(The figures at the points are frequencies in Hz).
Fig. 55 Vector plot (the figures at the points indicate the frequencies in Hz).
(a) $G_{pp}(f)$ vs frequency $f$

(b) Phase vs frequency

Fig. 56 Magnification of the power spectral density of Fig. 45 between 1180 and 1260 Hz and its phase.
Fig. 57 Amplitude and phase of the Characteristic Response function $F(i\omega)$ between 1250 and 1650 Hz at 1 Hz resolution. (This response curve is of a separate excitation from that of Fig. 51.)
Fig. 58 Vector plot (the figures at the points indicate the frequencies in Hz).
Fig. 59 Vector plot (the figure at the points indicate the frequencies in Hz).
Fig. 60 Amplitude and phase of the Characteristic Response function $F(i\omega)$ between 1650 and 2050 Hz at one Hz resolution. (This response curve is of a separate excitation from that of Fig. 51.)
Fig. 61 Vector plot (the figures at the points indicate the frequencies in Hz).
Fig. 62 Vector plot (the figures at the points indicate the frequencies in Hz; the vector plot of Fig. 61 may be seen at the left hand corner of the first quadrant.)
Fig. 63 Amplitude and phase of the Characteristic Response function $F(i\omega)$ between 2050 and 2450 Hz at one Hz resolution. (This response curve is of a separate excitation from that of Fig. 51.)
Fig. 64 Vector plot (the figures at the points indicate the frequencies in Hz.)
Fig. 65 Vector plot (the figures at the points indicate the frequencies in Hz; also note that the scale of the plot is very distorted).
Fig. 66 Vector plot (the figures at the points indicate the frequencies in Hz).
Fig. 67 Vector plot (the figures at the points indicate the frequencies in Hz).
Plate 1. The test plate and the capacitance probe with its attachments.
(a) The loud speaker and the test plate in test position.

(b) The set up of the test.

Plate 2. The test fixture.