"A STUDY OF THE RANK FORCE METHOD FOR STRUCTURAL MATRIX VIBRATION ANALYSIS."

by

John Robinson.

D.C.Ae.,C.Eng.,A.F.R.Ae.S.,A.M.I.Mech.E.

Thesis submitted for the Degree of Doctor
of Philosophy, October 1967.

Institute of Sound and Vibration Research,
University of Southampton, England.

ABSTRACT OF THESIS.

In the field of structural vibration analysis using finite element techniques and a distributed structural mass representation the displacement method has virtually received complete priority over the force method. To date, no published work has been found on a force approach which adopts an automatic selection of redundancy technique and a distributed structural mass representation. This thesis formulates and investigates such a force method ("The Rank Force Method") and the rank technique is used for automatic selection of redundancies. Distinction is made between static and dynamic redundancy. Procedures for deriving element dynamic flexibility matrices are presented and then applied to give particular dynamic matrices. It is shown that such matrices can be separated into an element static flexibility matrix and an element inverse mass matrix. Endload, beam and rectangular plate elements are considered. Using these elements the rank force method is applied to a number of structural configurations to evaluate their eigenvalues. These results are compared with thoses obtained using alternative procedures. Element loads and structural reactions for a given frequency and applied loading are also given. A general discussion of the rank force method for vibration analysis is given.

When adopting this force approach for eigenvalue evaluation a highly reduced structural dynamic flexibilty matrix can be used, a method is presented and investigated. It is also shown that in vibration analyses the structural reactions need not be imposed until the final stages of the force formulation. This is ideal when analysing large practical structures.

All results are obtained by writing the rank force method as a computerized structural analysis research system, such a system is presented and descibed.

CONTENTS.

INTRODUCTION.

The analysis of complex structural configurations under static and dynamic loading conditions has been revolutionized by the introduction of finite element techniques which incorporate matrix formulation of the problem. These techniques are ideally suited for solution by high speed electronic digital computers. The structure to be analysed is replaced by an equivalent mathematical model which is established by idealizing the structure under consideration into a finite number of structural elements. The static and dynamic properties of each structural element is expressable in matrix form. The finite structural elements are assembled in accordance with basic structural rules to give the overall structural behaviour. Finite element techniques are classified into two main approaches, namely, the displacement approach and the force approach. In the former the displacements are considered as the unknowns and the equations of equilibrium are enforced to give the correct displacement system. In the latter, the internal element loads and external structural reactions are considered as the unknowns, the correct system of loads being that which satisfies the energy equations.

The advantages and disadvantages of the matrix force and displacement approaches for computerized structural matrix analysis have been a well discussed topic for many years, particularly in static analysis and lumped mass vibration analysis. A correlation study of these methods was carried out by Gallagher[19]. He found that no impartial evidence existed upon which to base the view that either approach is "best".

Any comparison of the two approaches should be made when the amount of computer programme input data, to be prepared by the engineer, is the same in both cases. This is the case when adopting a matrix force approach which incorporates an automatic selection of redundancy technique, such as "The Rank Technique" .
The main points for comparison for a given problem are ;

1. Computer storage required.
2. Total computer running time for a solution.
3. Accuracy of solution.

Comparison should also be made for the same programme generality, programming language and using the same computer. Simplicity of theoretical derivation and apprehension should not be overlooked. These points are very important, since, for practical applications, the programme user is only interested in how much it will cost to solve a problem in the quickest time possible. He also requires that the amount of input data to be prepared by him is a minimum, that the results be well presented, meaningful and accurate. The users confidence in the results has to be established. The user is not very interested in the programming and theoretical gimmickry which the programmers and theorists enjoy and appreciate. However, the user must be made aware of the limitations in the results.

The displacement and force approaches have received much attention in the field of static analysis with the displacement formulation being the most popular. The lack of popularity in the force approach was initially due to the excessive amount of computer programme input data which had to be prepared by the engineer. This impediment was caused by the fact that structural redundancies had to be manually selected and the basic and redundant load systems generated by hand. Such a force approach is referred to as a semi-automated force approach. To emphasize this problem further explanation of these load systems will be given. To carry out a static analysis of a structure by a semi-automated force approach requires first of all the realization of redundancy and then the selection of redundancies. This is a difficult task when considering a complex structural configuration and it is even more difficult to select a satisfactory set of redundancies. When a structure is redundant it means that the loads in the structural elements cannot be found by equilibrium considerations only. In mathematical terms, it means that the system of equilibrium equations has an infinite number of solutions. The correct system of redundants is that which satisfies the energy equations which are derived by minimizing the total complementary potential in the structure. When a structure is determinate, non-redundant, the structural element properties need not be known in order to evaluate the internal load system. When selecting redundancies for an applied load condition the actual redundant loads should be as small as possible, in which case a determinate solution from equilibrium considerations can be obtained with little possibility of error. This demands that the determinate structure, that is , an indeterminate structure with redundancies removed, behaves [illegible] as possible like the actual structure. This [illegible]ssitates consideration of the applied load [illegible]ewing the simple plane frame structure [illegible] that for the applied load P the best [illegible] shown in figure 1(b). If the applied lo[illegible]gure 2(a) the best determinate stru[illegible]her member, see figure 2(b). A more [illegible]ing can be established by considering the e[illegible]ions obtained by minimizing the total complementary [illegible]tential in the structure. These are given by, for a semi-automated force approach,

SEE:
Robinson, John,
"A General Note on the Selection of Redundancies in Structural Analysis,"
The Aeronautical Journal, Technical Note, July 1968.

$$[F_{k\lambda}]\{P_\lambda\} + [F_{kk}]\{q^k\} = \{0\}$$

Therefore, the static redundancies $\{q^k\}$ are given by,

$$\{q^k\} = -[F_{kk}]^{-1}[F_{k\lambda}]\{P_\lambda\}$$

It can be seen that the inverse of matrix $[F_{kk}]$ is required. For well conditioned equations this matrix should have predominant terms on the diagonal. If matrix $[F_{kk}]$ was a simple diagonal matrix its inverse would be a diagonal matrix whose nonzero elements are the reciprocal of the corresponding elements in the original matrix. In this case the equations are well conditioned.

From these few points it can be seen that the manual selection of redundancies such that ideal requirements exist is impossible. Initially one could assume a set of redundants, run the computer programme, and investigate the results to see if the redundants had been well chosen, if not the process would be continued. However, this would be very time consuming.

The basic and redundant load systems will now be explained. The generalized element boundary loads can be expressed in terms of the generalized applied loads and selected redundancies by equilibrium considerations. Therefore,

$$\{ q_\alpha \} = [\gamma_{\alpha\lambda}]\{ P_\lambda \} + [\bar{\gamma}_{\alpha k}]\{ q^k \}$$

where,

$[\gamma_{\alpha\lambda}]\{P_\lambda\} =$ basic load system.

$[\bar{\gamma}_{\alpha k}]\{q^k\} =$ redundant load system.

To generate the basic and redundant load systems the following procedure is adopted :

<u>Basic Load System.</u>

First, all the selected redundant loads are set equal to zero, that is, $\{q^k\} = \{0\}$, the structure now becomes statically determinate. The first applied load P_1 is then set equal to unity and all other applied loads are set to zero. The first column of the $[\gamma_{\alpha\lambda}]$ matrix can then be generated by considering static equilibrium of the structural joints and elements. Therefore,

$$\{q_\alpha\} = \begin{bmatrix} q_1 \\ \vdots \\ q_\alpha \\ \vdots \\ q_L \end{bmatrix} = \begin{bmatrix} \gamma_{11} \\ \vdots \\ \gamma_{\alpha 1} \\ \vdots \\ \gamma_{L1} \end{bmatrix} = \{\gamma_{\alpha 1}\}$$

The second applied load P_2 is then set equal to unity and all other applied loads are set to zero, still keeping all the redundant loads equal to zero. The second column of the $[\gamma_{\alpha\lambda}]$ matrix can then be generated, again by static reasoning. Therefore,

$$\{q_\alpha\} = \begin{bmatrix} q_1 \\ \vdots \\ q_\alpha \\ \vdots \\ q_L \end{bmatrix} = \begin{bmatrix} \gamma_{12} \\ \vdots \\ \gamma_{\alpha 2} \\ \vdots \\ \gamma_{L2} \end{bmatrix} = \{\gamma_{\alpha 2}\}$$

This procedure is repeated for all the applied loads until the complete $[\gamma_{\alpha\lambda}]$ matrix has been generated.

Redundant Load System.

First, all the applied loads are set equal to zero, that is, $\{P_\lambda\} = \{0\}$. The first redundant q' is then set equal to unity and all other redundants are set to zero. The first column of the $[\bar{\gamma}_{\alpha k}]$ matrix can then be generated. Therefore,

$$\{q_\alpha\} = \begin{bmatrix} q_1 \\ \vdots \\ q_\alpha \\ \vdots \\ q_L \end{bmatrix} = \begin{bmatrix} \bar{\gamma}_{11} \\ \vdots \\ \bar{\gamma}_{\alpha 1} \\ \vdots \\ \bar{\gamma}_{L1} \end{bmatrix} = \{\bar{\gamma}_{\alpha 1}\}$$

This procedure is repeated for all the redundant loads until the complete $[\bar{\gamma}_{\alpha k}]$ matrix has been generated.

By now the lack of popularity in the semi-automated force approach should be appreciated. For large, complex problems the task of manually generating the basic and redundant load transformation matrices ($[\gamma_{\alpha\lambda}]$ and $[\bar{\gamma}_{\alpha k}]$,respectively) is very time consuming, laborious, and prone to error. This effort is completely impractical.

In the field of structural vibration analysis, using a distributed structural mass representation, the displacement approach has virtually received complete priority over the force approach. The only published work that has been found which adopts the force approach using a distributed structural mass representation is that by Levien and Hartz[23,24]. The initial part of their work discusses the semi-automated force approach for static structural analysis. They then continue to discuss the basic and redundant load systems such that inertia loads are considered. They quickly conclude that the semi-automated force method of analysis is of no practical use for the dynamic analysis of general frame structures. Levien and Hartz then present a hybrid system of analysis which removes the need to generate the basic and redundant load systems and therefore the need to select redundancies. This hybrid system is presented for the analysis of rigid joint frames and the unknowns are taken as particular generalized element loads and displacements. The eigenvalues for two plane frame configurations are given and these results will be used for comparison purposes. The element dynamic representation adopted was based on transcendental functions.

It can be seen that the same impediments in the semi-automated force approach exist in both static and vibration analysis. However, in recent years these impediments have been investigated for static analysis and one method of automatically selecting redundancies, "The Rank Technique", is given in references 27,28,29 and 31. In reference 31 the author describes how the rank technique can be used to automatically generate the basic and redundant load systems but points out that there is no need to generate these systems directly if the rank technique is incorporated into a force method. A force approach which incorporates the rank technique for automatic selection of redundancies is referred to as "The Rank Force Method"[31].

The work carried out in this thesis is a study of the rank force method for structural matrix vibration analysis using a distributed structural mass representation. This is a pure force method where the unknowns are the generalized element boundary loads and generalized structural reactions. Since this is a research project only simple structural configurations are analysed, such as collinear beam structures, general plane frames and two dimensional rectangular plate structures.

The first chapter discusses structural dynamic redundancy and differentiates between static redundancy and dynamic redundancy. Also presented in this chapter is the rank technique for vibration analysis and as a computer programme subroutine, examples of redundancy sets are also given. Chapter 2 formulates the rank force method for vibration analysis but the work has been restricted to internal stress distributions and structural response for a given frequency due to harmonic applied loads. The work also covers the eigenvalue problem. All loads and displacements are assumed to vary harmonically with time and in phase. The structure is assumed undamped and steady state conditions are assumed to exist.

In Chapter 3 approximate procedures for the derivation of element flexibility matrices are presented. These derivations have been specifically given for endload, plane beam and rectangular plate elements but the general procedures would be the same when considering other element types. Three derivation procedures have been investigated, one of a more exact nature and the others being more approximate. The more exact derivation adopts transcendental functions whereas the other derivations adopt series functions (polynomials). These functions represent the respective element internal loadings. Having established derivation procedures the next step is to obtain particular element dynamic flexibility matrices by considering specific types of elements and internal loading functions. This work is contained in Chapter 4.

To carry out vibration analysis of structures, even for the very simple configurations, it is essential to write a computerized system. Such a system is presented in Appendix 4 using A.S.A. Fortran as the basic programming language and written for an I.C.T. 1900 Computer.

To analyse a given configuration, using the rank force method, a set of standard subroutines, a force-subroutine and a master programme are required. The force-subroutine generates the system of joint equilibrium equations for the particular configuration, and by following the force formulation and calling the standard subroutines generates the internal loading system, structural response and the structural dynamic stiffness matrix for a given frequency. The master programme controls the analysis, for example, reading input data and selecting output. In the case of eigenvalue evaluation an iterative procedure is contained within the master programme. The master programme and force-subroutine have been written for various types of structural configurations but the general concepts are the same for any configuration. The structural types considered are ; collinear beam structures, general plane frames and two dimensional rectangular plate structures. It should be noted that the standard subroutines apply to any configuration. The programmes have been written and restricted to analyse problems within the computer core storage mainly because this research project is concerned with investigating the force formulation as a vibration analysis tool. The various programme and subroutine listings are given and their arguments, limitations and applications are described.

Before an analysis can be carried out the actual structure has to be transformed into a mathematical model. This transformation is referred to as "structural idealization". When undertaking a structural research project it is desirable to keep in mind future practical application of the work, therefore, in Appendix 5 the practical application of a computerized structural analysis system is discussed.

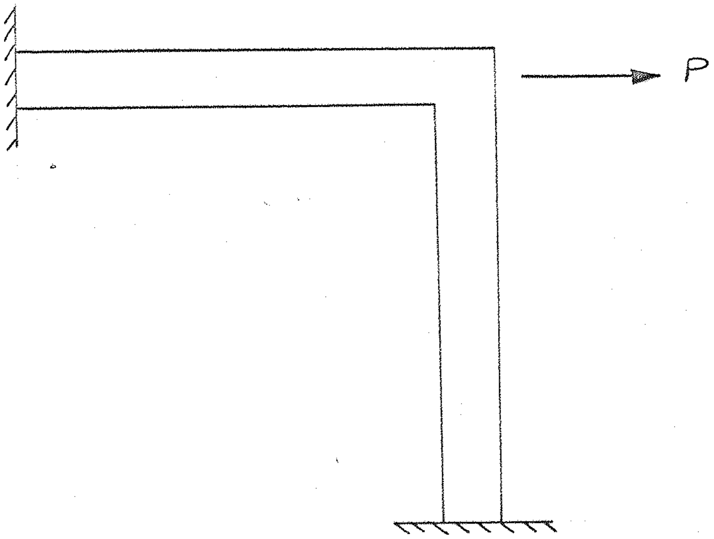
Chapter 5 presents results obtained using these computerized systems for various structural configurations. These results are discussed and compared with results obtained by alternative analysis procedures. Conclusions, recommendations and areas for future research are given.

During the latter stages of this research work a number of important items appeared, these are discussed in appendices. In the rank force method for vibration analyses the inversion of a large matrix has to be carried out in order to evaluate eigenvalues. In Appendix 1 a method is proposed which only requires the inversion of a very small submatrix which is contained in the initial large matrix. This reduced matrix is immediately extractable.

Some results are given to help substantiate the proposed method and they do encourage further investigations.

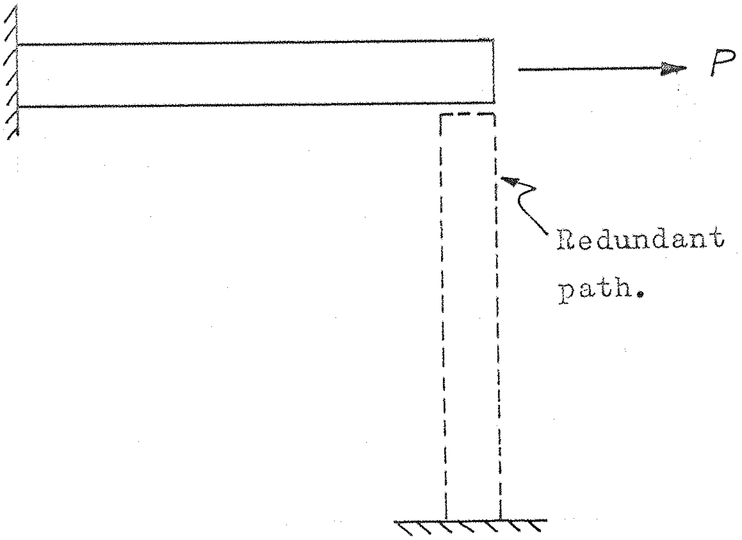
A procedure is presented in Appendix 2 for the delayed imposition of the generalized structural reactions in the rank force method for vibration analyses. This enables unconstrained structural dynamic flexibility matrices to be generated which are ideal for the vibration analysis of large structural configurations using "block elements". A block element being itself an assembly of finite elements, a substructure.

Appendix 3 suggests a further approach for deriving a dynamic flexibility matrix for a rectangular plate element using an s-system of element generalized loads.



Actual structure and applied loading.

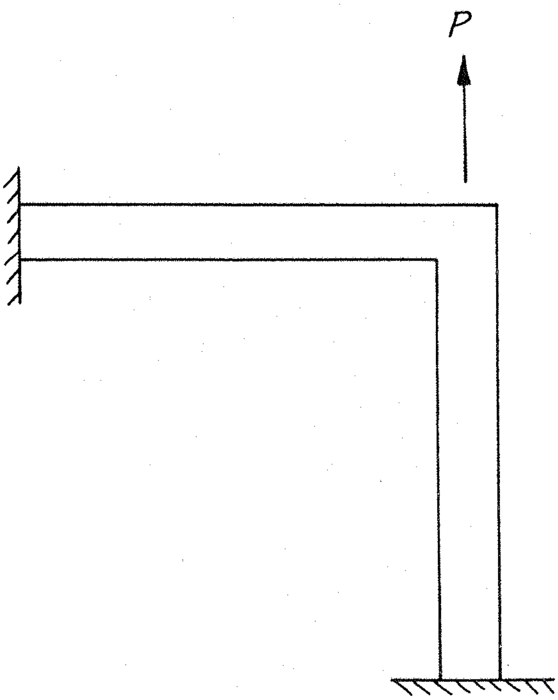
(a)



Determinate structure for applied load.

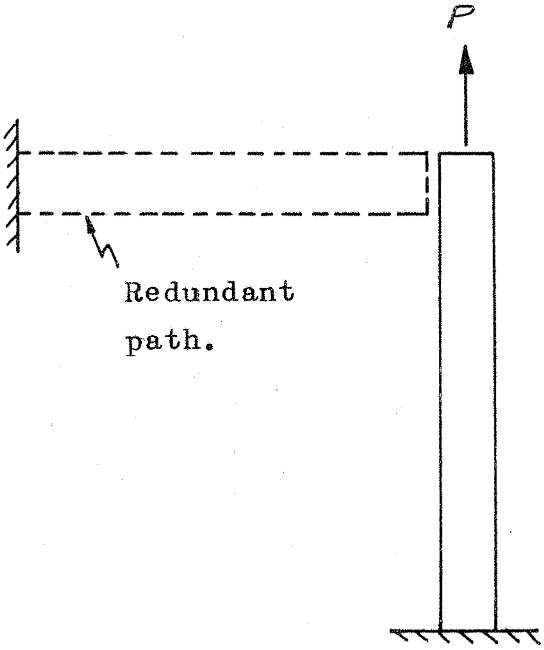
(b)

Fig. 1 .



Actual structure and applied loading.

(a)



Determinate structure for applied load.

(b)

Fig. 2 .

Acknowledgements

The author would like to express his sincere gratitude to his supervisor, Professor B.L. Clarkson, for his continual encouragement and invaluable suggestions. I would also like to acknowledge my fellow sufferers, Tony Craggs and Neil Popplewell, with whom many a problem has been discussed whilst visiting the Swan and other such relaxing abodes. My thanks are also due to Dr. M. Petyt, Dr. C.A. Mercer, V. Mason and E. Wilby who have on many an occasion helped in a dilemma.

My final acknowledgements go to the Science Research Council, who sponsored my entire research project, and the University Computing Staff, the computer has been my biggest problem.

A computer is like a woman, when it functions correctly it is sheer delight but otherwise it is a pain in the ----.

Nomenclature.

$\{P_\lambda\}$ = generalized applied discrete loads.

$\{q^k\}$ = column matrix of redundants.

$[F_{kk}]$

$[F_{k\lambda}]$

$[\gamma_{\alpha\lambda}]$

$[\bar{\gamma}_{\alpha k}]$

Transformation matrices.

These are use in a structural static analysis see reference 31.

$[\Omega_\alpha]$ = submatrix of $\{_t q_\alpha\}$ coefficients in the system of joint equilibrium equations.

$[\Omega_e]$ = submatrix of $\{_t R_e\}$ coefficients in the system of joint equilibrium equations.

$[\psi_\lambda]$ = submatrix of $\{_t P_\lambda\}$ coefficients in the system of joint equilibrium equations.

$\{_t\delta_\alpha\}$ = vector of time dependent generalized element boundary displacements. $\{_t\delta_\alpha\} = \{\delta_\alpha\} \sin \omega t$.

$\{_t q_\alpha\}$ = vector of time dependent generalized element boundary loads. $\{_t q_\alpha\} = \{q_\alpha\} \sin \omega t$.

$\{_t R_e\}$ = vector of time dependent generalized structural reactions. $\{_t R_e\} = \{R_e\} \sin \omega t$.

$\{_t P_\lambda\}$ = vector of time dependent generalized applied loads. $\{_t P_\lambda\} = \{P_\lambda\} \sin \omega t$.

$\{_t\Delta_\lambda\}$ = vector of time dependent generalized structural displacements. $\{_t\Delta_\lambda\} = \{\Delta_\lambda\} \sin \omega t$.

$[\gamma_\alpha]$ = submatrix of $\{_t q_\alpha\}$ coefficients in the system of joint equilibrium equations after applying the rank technique.

$[\gamma_e]$ = submatrix of $\{_tR_e\}$ coefficients in the system of joint equilibrium equations after applying the rank technique.

$[\gamma_\lambda]$ = submatrix of $\{_tR_\lambda\}$ coefficients in the system of joint equilibrium equations after applying the rank technique.

π^* = total complementary potential in the structure.

NE = total number of finite elements in the structure.

$\{_tq^k\}$ = vector of automatically selected dynamic redundancies, time dependent. $\{_tq^k\} = \{q^k\} \sin \omega t$.

$[f_{md}]$ = element dynamic flexibility matrix.

$\{_tq_{m\alpha}\}$ = vector of time dependent generalized element boundary loads for element m. $\{_tq_{m\alpha}\} = \{q_{m\alpha}\} \sin \omega t$.

$\{_t\delta_{m\alpha}\}$ = vector of corresponding generalized element boundary displacements for element m.

$\{_t\delta_{m\alpha}\} = \{\delta_{m\alpha}\} \sin \omega t$.

$[F_d]$ = assembled element dynamic flexibility matrix, band matrix.

$\{0\}$ = null vector.

$[_\alpha\partial_k]$ = matrix of partial derivatives, $_\alpha\partial_k$ is the partial derivative of $_tq_\alpha$ with respect to $_tq^k$. This matrix is contained in matrix $[\gamma_\alpha \vdots \gamma_e]$.

$[_e\partial_k]$ = matrix of partial derivatives, $_e\partial_k$ is the partial derivative of $_tR_e$ with respect to $_tq^k$. This matrix is contained in matrix $[\gamma_\alpha \vdots \gamma_e]$.

$[\vartheta]$ = $[_\alpha\partial_k][F_d]$, used in energy equations and generated for a given frequency.

$[\Delta_{\alpha\lambda}]$ = matrix which relates the generalized element boundary loads, $\{{}_tq_\alpha\}$, and the generalized applied loads, $\{{}_tP_\lambda\}$, for a given frequency.

$[\Delta_{e\lambda}]$ = matrix which relates the generalized structural reactions, $\{{}_tR_e\}$, and the generalized applied loads, $\{{}_tP_\lambda\}$, for a given frequency.

$[{}_\alpha\partial_\lambda]$ = matrix of partial derivatives, ${}_\alpha\partial_\lambda$ is the partial derivative of ${}_tq_\alpha$ with respect to ${}_tP_\lambda$ for a given frequency. This matrix is the transpose of matrix $[\Delta_{\alpha\lambda}]$.

$[{}_e\partial_\lambda]$ = matrix of partial derivatives, ${}_e\partial_\lambda$ is the partial derivative of ${}_tR_e$ with respect to ${}_tP_\lambda$ for a given frequency. This matrix is the transpose of matrix $[\Delta_{e\lambda}]$.

$[\mathcal{F}_d]$ = structural dynamic flexibility matrix for a given frequency.

$[\mathcal{K}_d]$ = $[\mathcal{F}_d]^{-1}$ = structural dynamic stiffness matrix for a given frequency.

$det[\mathcal{K}_d]$ = determinant of the structural dynamic stiffness matrix for a given frequency.

$\tilde{W}_1^*$ = complementary virtual work done by the virtual generalized element boundary loads.

$\tilde{W}_2^*$ = complementary virtual work done by the element inertia loading.

$\tilde{U}^*$ = complementary virtual work done by the virtual internal element loading.

$P(x,t)$ = internal element endload distribution, function of position (x) and time (t). $P(x,t) = P(x) \sin \omega t$.

$\omega_x(x,t)$ = element distributed loading in the x-direction, function of position (x) and time (t).

$u_x(x,t)$ = element displacement in the x-direction, function of position (x) and time (t). $u_x(x,t) = u_x(x) \sin \omega t$.

$\left.\begin{matrix} \lfloor T_{Pq} \rfloor \\ \lfloor T_{uq} \rfloor \\ \lfloor T_{\omega q} \rfloor \\ \lfloor T_1 \rfloor \\ \lfloor T_2 \rfloor \end{matrix}\right]$ row transformation matrices for an endload element.

$[f_m]$ = element static flexibility matrix.

$[m_{mf}]$ = element inverse mass matrix.

$[k_{md}]$ = element dynamic stiffness matrix.

$[k_m]$ = element static stiffness matrix.

$[m_{mk}]$ = element mass matrix.

$M(x,t)$ = internal element bending moment distribution, function of position (x) and time (t). $M(x,t) = M(x) \sin \omega t$.

$Q(x,t)$ = internal element shear distribution, function of position (x) and time (t). $Q(x,t) = Q(x) \sin \omega t$.

$\omega_z(x,t)$ = element distributed loading in the z-direction, function of position (x) and time (t).

$u_z(x,t)$ = element displacement in the z-direction, function of position (x) and time (t). $u_z(x,t) = u_z(x) \sin \omega t$.

$\left.\begin{matrix} \lfloor T_{Mq} \rfloor \\ \lfloor T_{uq} \rfloor \end{matrix}\right\}$ row transformation matrices for a plane beam element.

$\theta(x,t)$ = element rotation in the xz-plane for a plane beam element, function of position (x) and time (t). $\theta(x,t) = \theta(x) \sin \omega t$.

i, j = element specifying nodes, endload and beam elements.

u_y = amplitude of element displacement in the y-direction. $u_y(x,z,t) = u_y(x,z)\,\sin\omega t = u_y \sin\omega t$.

ω_y = element distributed loading in the y-direction.

$$\left.\begin{matrix} M_x \\ M_z \\ M_{xz} \end{matrix}\right\}$$ rectangular plate moments.

$$\left.\begin{matrix} Q_x \\ Q_z \end{matrix}\right\}$$ rectangular plate shears.

$$\left.\begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix}\right]$$ rectangular plate equivalent shears.

$$\left.\begin{matrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{matrix}\right]$$ rectangular plate nodal concentrated loads.

$$\left.\begin{matrix} \lfloor T_{xuq} \rfloor \\ \lfloor T_{zuq} \rfloor \\ \lfloor T_{xzuq} \rfloor \\ \lfloor T_{xMq} \rfloor \\ \lfloor T_{zMq} \rfloor \\ \lfloor T_{xzMq} \rfloor \\ \lfloor T_{uq} \rfloor \end{matrix}\right]$$ row transformation matrices for a rectangular plate element.

$[\Omega_{m\alpha}]$ = submatrix of $\{{}_t q_{m\alpha}\}$ coefficients in the system of overall equilibrium equations when using an s-system for the generalized element boundary loads, rectangular plate element.

$[\Psi_{m\beta}]$ = submatrix of $\{{}_t s_{m\beta}\}$ coefficients in the system of overall equilibrium equations, using an s-system.

$\{{}_t s_{m\beta}\}$ = vector of time dependent generalized equivalent discrete nodal loads, rectangular plate element, using an s-system. $\{{}_t s_{m\beta}\} = \{s_{m\beta}\} \sin \omega t$.

$[\gamma_{m\alpha}]$ = submatrix of $\{{}_t q_{m\alpha}\}$ coefficients in the system of overall equilibrium equations after applying the rank technique, using an s-system.

$[\gamma_{m\beta}]$ = submatrix of $\{{}_t s_{m\beta}\}$ coefficients in the system of overall equilibrium equations after applying the rank technique, using an s-system.

$[{}_\alpha\partial_k]_m$ = matrix of partial derivatives when using an s-system, rectangular plate element.

$[\phi_m] = [{}_\alpha\partial_k]_m [f_{md}]$, used in energy equations and generated for a given frequency, using an s-system.

$[\Delta_{m\alpha\beta}]$ = matrix which relates the q-system and the s-system for a given frequency, rectangular plate element.

$[\varphi_{md}]$ = element dynamic flexibility matrix for a rectangular plate element corresponding to an s-system.

$A_1, A_2, A_3, \ldots$ = distribution constants.

$(\bar{x},\bar{y},\bar{z})$ = global axes and coordinates (in).

(x,y,z) = local axes and coordinates (in).

$\{{}_tq^*_{m\alpha}\}$ = generalized element boundary loads relative to the local axes.

$$\{{}_tq^*_{m\alpha}\} = \{q^*_{m\alpha}\}\,\mathrm{Sin}\,\omega t.$$

$\{{}_tq_{m\alpha}\}$ = generalized element boundary loads relative to the global axes.

$\{{}_t\delta^*_{m\alpha}\}$ = corresponding generalized element boundary displacements relative to the local axes.

$[C_m]$ = transformation matrix, rotation of axes.

$[f^*_{md}]$ = element dynamic flexibility matrix relative to the local axes.

$[f_{md}]$ = element dynamic flexibility matrix relative to the global axes.

$\bar{x}_i\,,\bar{z}_i$ = $\bar{x}$ and $\bar{z}$ ordinates of node i.

$\bar{x}_j\,,\bar{z}_j$ = $\bar{x}$ and $\bar{z}$ ordinates of node j.

Inclined plane beam element.

$\left.\begin{matrix}M_\xi\\ M_\zeta\\ M_{\xi\zeta}\end{matrix}\right\}$ rectangular plate moments (non-dimensional).

$\left.\begin{matrix}Q_\xi\\ Q_\zeta\end{matrix}\right\}$ rectangular plate shears (non-dimensional).

$\left.\begin{matrix}F_1\,,F_3\\ F_5\,,F_6\\ F_7\,,F_8\\ F_{10}\,,F_{11}\\ F_{12}\end{matrix}\right]$ transcendental functions used in the endload and beam type elements using procedure 3.

E = Young's modulus of elasticity for the element material, lb per in^2.

ω = angular frequency, radians per second.

μ = density of element material, lb per in^3.

ν = Poisson's ratio.

t = time, seconds.

m = element number.

$386 \cdot 4$ = g × 12, in per sec^2 ($g = 32.2$ ft per sec^2).

x, z = local axes and coordinates (in).

I = second moment of area of element cross section, in^4.

A = cross sectional area of element, in^2.

ℓ = length of element, in.

i, j = element specifying nodes.

$\lambda_1^2 = \dfrac{\omega^2 \rho}{AE} = \dfrac{\omega^2 \mu}{386 \cdot 4 E}$

= endload element frequency parameter when using transcendental functions.

$\lambda_2^4 = \dfrac{\omega^2 \rho}{EI} = \omega^2 \dfrac{\mu A}{386 \cdot 4 EI}$

= plane beam element frequency parameter when using transcendental functions.

$\lambda = \dfrac{\rho \omega^2 \ell^4}{840 EI}$

= plane beam element frequency parameter when using polynomials.

$\rho = \dfrac{\mu A}{386 \cdot 4}$

= mass per unit length, lb sec^2 per in^2.

Endload and plane beam elements.

x, y, z = local axes and coordinates (in).
t_p = plate thickness, in.
a, b = rectangular plate element dimensions, in.
$\xi = \frac{x}{a}$
$\zeta = \frac{z}{b}$
$D = \frac{E t_p^3}{12(1-\nu^2)}$ = plate stiffness factor.
$\rho = \frac{\mu t_p}{386 \cdot 4}$ = mass per unit area,
lb $\sec^2$ per in^3.

Rectangular plate element.

A.S.A. = American Standards Association.
I.C.T. = International Computers and Tabulators Ltd.

CHAPTER 1.

STRUCTURAL DYNAMIC REDUNDANCY.

Synopsis.

This chapter differentiates between static and dynamic redundancy and the rank technique for vibration analysis is described. This technique is a method for automatically selecting redundancies in the matrix force approach and was initially developed for static analysis. The basic concept of the rank technique is to investigate a system of linear equations by applying the fundamental theorem for linear equations which compares the rank of the coefficient and augmented matrices. The joint equilibrium equations for a given frequency constitute such a system in structural vibration analysis. Evaluation of the rank of a matrix and the general investigation of the system of equations is carried out using the Jordan elimination procedure. The rank technique has other capabilities which are also described. Also contained in this chapter is the rank technique as a computer programme subroutine and examples of automatically selected redundancy sets are given. In chapter 2 this technique is incorporated into a matrix force approach thus presenting a fully automated force method (The Rank Force Method). The computer programme input data for the rank force method is the same as in the matrix displacement method.

1.1 The Question of Redundancy.

In the static analysis of elastic structures the matrix force formulation initially consists of assembling the system of equilibrium and release equations. The equilibrium equations consist of ;

1. Joint equilibrium equations.
2. Element equilibrium equations.
3. External equilibrium equations.

This system of equations is investigated by applying the rank technique, which, among other things, automatically isolates a consistent set of redundancies.

In the vibration formulation only the joint equilibrium equations are assembled. There are no resulting element equilibrium equations since these equations are a function of the inertia loading and the element boundary loads, and the inertia loading is expressed in terms of the boundary loads, initially from equilibrium considerations. For simplicity the external equilibrium equations will be replaced in the vibration analysis by compatibility equations which are obtained by considering the added redundancy.

In vibration analysis the degree of indeterminacy is much higher than in the static analysis although the number of unknowns, element boundary loads and structural reactions, is the same. Since the number of unknowns is constant for the two forms of analysis, the maximum order of the system of equations required for a unique solution of the element boundary loads and structural reactions in terms of the applied load system is the same. The difference being in the percentage of energy equations contained in the total system. In the vibration analysis the redundancies will be referred to as "dynamic redundancies" and in the static analysis as "static redundancies". Because of the higher degree of indeterminacy in the vibration analysis the coefficient matrix corresponding to the unknowns is more populated thus increasing the numerical problem. However, it should be pointed out that in frame type structures the joint equilibrium equations contain Boolean matrices and hence the automatic selection of redundancies becomes numerically sound. The joint equilibrium equations for plate structures also contain Boolean matrices depending on the system of element loads adopted.

1.2 "The Rank Technique" in Vibration Analysis.

The rank technique is a method for automatically selecting the redundancies, static or dynamic, in the matrix force approach and was initially developed for static analysis. This technique, among other things, can automatically generate the basic and redundant load systems if required. If the rank force method is adopted these two systems are not required directly.

To solve the eigenvalue and harmonic forcing function problems it is required to investigate a system of joint equilibrium equations. These equations can be investigated automatically and in a systematic manner. Considering the prescribed load conditions, the technique automatically determines whether the structure is unstable, determinate or redundant by using the notion of rank of a matrix. If the structure is redundant, the technique not only determines the degree of indeterminacy but also automatically isolates a consistent set of redundants. The joint equilibrium equations constitute a set of statements which relate the internal element boundary loads and structural reactions to the applied loading system. These equations are either consistent or inconsistent. If the equations are inconsistent, this implies that equilibrium cannot be formulated on the structure, or the structure is unstable for the particular applied loading system. If the equations are consistent, they may be either sufficient or insufficient to determine the unknowns. In the former case, the structure is determinate; in the latter, redundant. Consider the system of joint equilibrium equations: any set of values of the unknowns which simultaneously satisfy these equations is called a solution. When such a system has one or more solutions, it is said to be consistent. The matrix of coefficients corresponding to the unknowns (element loads and structural reactions) is defined as the coefficient matrix of the system and the matrix obtained by connecting the coefficient matrix and the matrix of coefficients corresponding to the knowns (applied loads) is defined as the augmented matrix of the system.

To determine whether a system is consistent or inconsistent use the fundamental theorem for linear equations:

> "A system of linear equations is consistent if, and only if, the coefficient matrix and the augmented matrix have the same rank."

The rank of a matrix is defined as the dimension of the largest square submatrix of the original matrix having a non-zero determinant.

However, there is no need to calculate the rank of the coefficient matrix directly. The system of equations can be investigated by applying the Jordan elimination procedure to the augmented matrix only. A systematic way of determining the rank of a matrix, in particular the augmented matrix, will now be described:

> Locate the largest absolute value in the first row and divide all elements in this row by the actual value corresponding to this. Now this row is multiplied by the coefficient of the corresponding element in the second row and subtracted from the second row. This is continued for each of the remaining rows. The column corresponding to that element has now a one in the first row and zeros in all other rows. The same process is performed in turn for the remaining rows until either all of the rows are exhausted or all of the remaining rows have all zeros as elements. If, after exhausting all the rows, the largest absolute value in a row has always been located in the submatrix which corresponds to the unknowns in the augmented matrix, then the rank of the coefficient matrix and the augmented matrix are equal and the system is consistent. If the largest absolute value in any row is located in the submatrix which corresponds to the knowns in the augmented matrix, then the ranks are unequal and the system is inconsistent.

After generating a system of linear equations, it may not be immediately apparent which, if any, are dependent equations. On completion of the Jordan elimination procedure, the dependent equations will appear as null rows; the remaining equations will constitute the system of independent equations. The redundancies will be isolated in the process of rank determination and are identified by noting the columns in the coefficient matrix, in the independent system, which were not selected in the elimination procedure, that is, none of the elements in the columns were isolated as maximum absolute values.

The rank technique is shown as a flow chart in figure 3. This technique is described in detail for static analysis in references 27, 28, and 31 with the added sophistication of selecting the largest absolute value in a row instead of the first non-zero value. The method of locating redundancies is also different.

1.3 "The Rank Technique" as a Computer Programme Subroutine.

To study the rank force method for structural vibration analysis a computerized structural analysis has been written in A.S.A. Fortran for an I.C.T. 1900 computer. The rank technique is used as a subroutine and its listing is given in table 1.

This subroutine investigates a rectangular array using the Jordan elimination procedure. Typically, this array is an augmented matrix consisting of a submatrix (coefficient matrix, which need not be square) corresponding to the unknowns and a submatrix corresponding to the knowns. Initially two vectors are formed, one null vector, denoted by IDEP, and a vector IQ. The latter vector contains integer numbers starting at 1 and increasing in sequence to the actual number of unknowns (N8). The next step is to locate the maximum absolute value in the first row, scanning only the coefficient matrix. The actual maximum value is then used to normalize the corresponding row in the augmented matrix. The Jordan elimination procedure is then applied to reduce the element in the other rows which has the same column location as the maximum value(used for normalizing) to zero. This procedure is repeated until all rows have been normalized. At each stage the actual maximum value in a row is stored in a vector which can be used to evaluate a determinant. This vector is referred to as the normvector, XMAX.

If the maximum absolute value in a row is zero the remaining elements in that row, contained in the submatrix corresponding to the known quantities, are investigated. If they are all zero a dependent equation has been found, the corresponding row number of such an equation is stored in vector IDEP. If any of the elements are nonzero it means there is no solution to the system of equations, that is, the coefficient and augmented matrices have unequal rank. A statement NO SOLUTION is printed out. In the case of a rectangular coefficient matrix the redundant column numbers are stored in vector IQ. Each time a maximum value in a row is located its column location is noted and the corresponding row in the IQ vector has its value set equal to zero. The redundancies are given by the remaining nonzero elements.

This subroutine can be used to evaluate a determinant and to invert a matrix. In a matrix inversion the augmented matrix would consist of the matrix to be inverted and a unit matrix. The inverse would appear in submatrix corresponding to the unit matrix. It should be noted that this subroutine has not been optimized.

From the subroutine listing it can be seen that the subroutine has been given the name RANTEC.

The subroutine arguments will now be described;

XKD = rectangular (or square) array.

N9 = actual number of rows.

N8 = number of columns in coefficient matrix.

N7 = total number of columns, that is, number of columns in the augmented matrix.

N9MAX
N8MAX
N7MAX } corresponding maximum values, dynamic dimensioning.

IDEP = vector of dependent equations, row numbers.

XMAX = normvector, vector of normalized row elements.

IQ = vector of redundant load numbers. This vector initially consists of the element load numbers, element loads which are not isolated as redundants have their corresponding load number set to zero.

Figure 4 gives added definition for some of the arguments. As an example, consider the joint equilibrium equations given by,

$$[\Omega_\alpha \vdots \Omega_e]\{{}_t q_\alpha \vdots {}_t R_e\} + [\Psi_\lambda]\{{}_t P_\lambda\} = \{0\}$$

The coefficient matrix is given by $[\Omega_\alpha \vdots \Omega_e]$ of order, say, N1 × MC and the augmented matrix by $[\Omega_\alpha \vdots \Omega_e, \Psi_\lambda]$ of order, say, N1 × LM. Let the maximum number of unknowns, element boundary loads and structural reactions, be 84, maximum number of joint equations be 42 and the maximum number of applied loads be 38. The augmented matrix will be denoted by OM. The program call statement to investigate matrix OM using subroutine RANTEC would be,

```
CALL RANTEC(OM,N1,MC,LM,42,84,122,IDEP,XMAX,IQ)
```

Figure 5 gives added definition for the example.

1.4 Redundancy Sets.

To illustrate the rank technique for dynamic redundancy selection two simple structural configurations will be investigated.

As a first example consider the simply supported beam shown in figure 6.

The system of joint equilibrium equations is given by,

$$\left[\begin{array}{cccccccc|cc} 1 & & & & & & & & -1 & \\ & 1 & & & & & & & & \\ & & 1 & & 1 & & & & & \\ & & & 1 & & 1 & & & & \\ & & & & & & 1 & & & -1 \\ & & & & & & & 1 & & \end{array}\right] \{ {}_t q_\alpha \,\vdots\, {}_t R_e \} + \left[\begin{array}{cccc} & & & \\ -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & \\ & & & -1 \end{array}\right] \{ {}_t R_\lambda \} = \{ 0 \}$$

or in contracted matrix form as,

$$[\Omega_\alpha \,\vdots\, \Omega_e] \{ {}_t q_\alpha \,\vdots\, {}_t R_e \} + [\Psi_\lambda] \{ {}_t R_\lambda \} = \{ 0 \}$$

The augmented matrix is therefore,

$$[\Omega_\alpha \,\vdots\, \Omega_e , \Psi_\lambda] = \left[\begin{array}{cccccccc|cc|cccc} 1 & & & & & & & & -1 & & & & & \\ & 1 & & & & & & & & & -1 & & & \\ & & 1 & & 1 & & & & & & & -1 & & \\ & & & 1 & & 1 & & & & & & & -1 & \\ & & & & & & 1 & & & -1 & & & & \\ & & & & & & & 1 & & & & & & -1 \end{array}\right]$$

Applying the rank technique results in an equivalent augmented matrix given by,

$$\left[\begin{array}{cccccccc:cc:cccc}
1 & & & & & & & & -1 & & & & & \\
 & 1 & & & & & & & & & -1 & & & \\
 & & 1 & & 1 & & & & & & & -1 & & \\
 & & & 1 & & 1 & & & & & & & -1 & \\
 & & & & & & 1 & & & -1 & & & & \\
 & & & & & & & 1 & & & & & & -1
\end{array}\right]$$

The isolated redundancies are therefore q_5, q_6, R_1 and R_2, considering amplitudes only. In the static analysis this beam would be determinate.

As a second example consider the plane frame shown in figures 7 and 8.

The augmented matrix for this plane frame is given by,

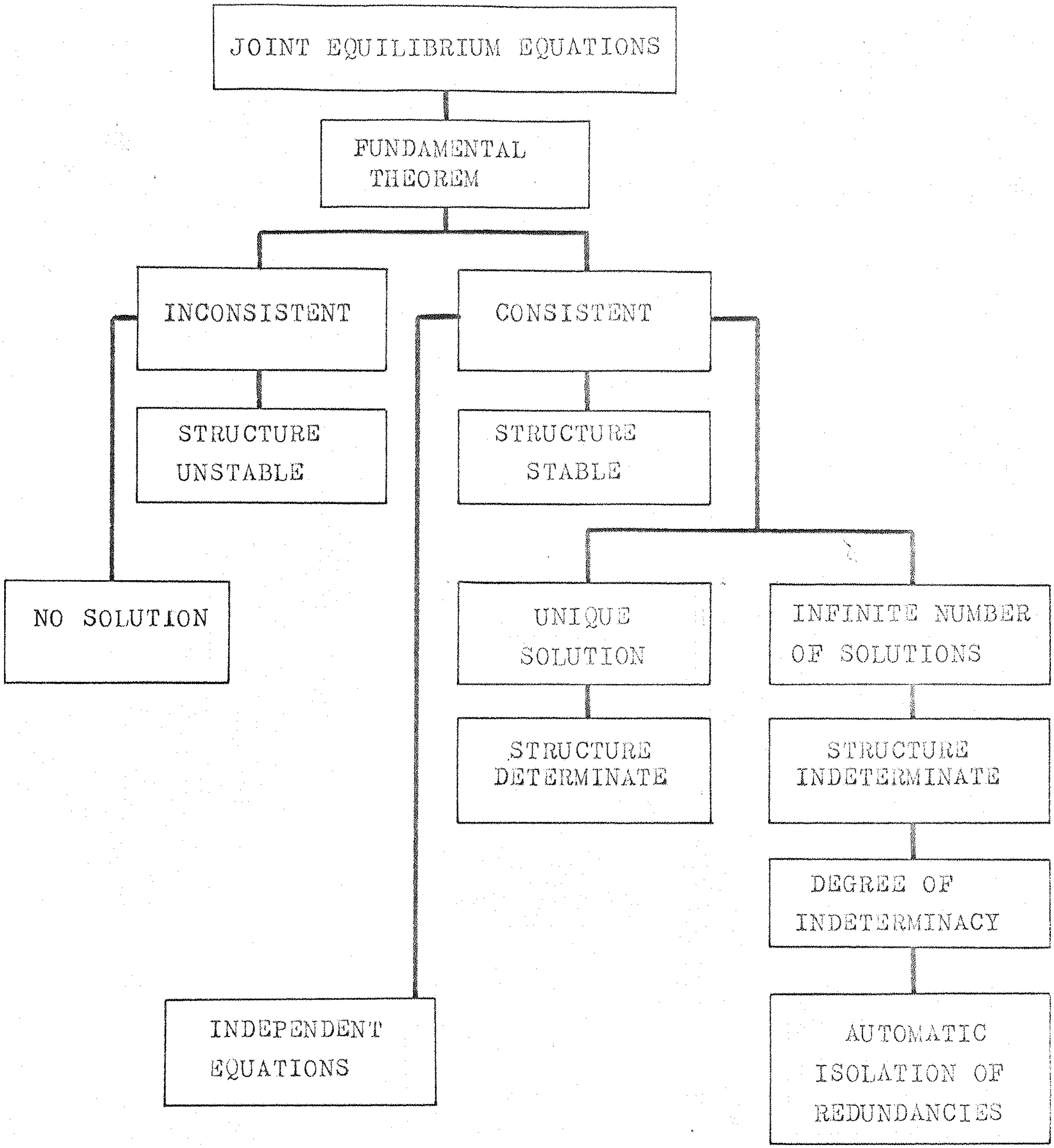
$$\left[\begin{array}{*{30}{c}|*{3}{c}|*{9}{c}}
1&&&&&&1&&&&&&1&&&&&&&&&&&&&&&&&&-1&&&&&&&&&&&\\
&1&&&&&&1&&&&&&1&&&&&&&&&&&&&&&&&-1&&&&&&&&&&\\
&&1&&&&&&1&&&&&&1&&&&&&&&&&&&&&&&&-1&&&&&&&&\\
&&&1&&&&&&&&&&&&&&&1&&&&&&&&&&&&&&&&-1&&&&&&&\\
&&&&1&&&&&&&&&&&&&&&1&&&&&&&&&&&&&&&&-1&&&&&&\\
&&&&&1&&&&&&&&&&&&&&&1&&&&&&&&&&&&&&&&-1&&&&&\\
&&&&&&&&&1&&&&&&&&&&&&1&&&1&&&&&&&&&&&&&-1&&&&\\
&&&&&&&&&&1&&&&&&&&&&&&1&&&1&&&&&&&&&&&&&-1&&&\\
&&&&&&&&&&&1&&&&&&&&&&&&1&&&1&&&&&&&&&&&&&-1&&\\
&&&&&&&&&&&&&&&1&&&&&&&&&&&1&&&&&&&&&&&&&-1&\\
&&&&&&&&&&&&&&&&1&&&&&&&&&&&&1&&&&-1&&&&&&&&&\\
&&&&&&&&&&&&&&&&&1&&&&&&&&&&&&1&&&&&&&&&&&&-1
\end{array}\right]$$

Applying the rank technique results in an equivalent augmented matrix given by,

$$\left[\begin{array}{cccccccccccccccccccccccccccccc|ccc|ccccccccc}
1 & & & & & & 1 & & & & & & 1 & & & & & & & & & & & & & & & & & & -1 & & & & & & & & & & & \\
& 1 & & & & & & 1 & & & & & & 1 & & & & & & & & & & & & & & & & & & -1 & & & & & & & & & & \\
& & 1 & & & & & & 1 & & & & & & 1 & & & & & & & & & & & & & & & & & & & -1 & & & & & & & & \\
& & & 1 & & & & & & & & & & & & & & & 1 & & & & & & & & & & & & & & & & -1 & & & & & & & \\
& & & & 1 & & & & & & & & & & & & & & & 1 & & & & & & & & & & & & & & & & -1 & & & & & & \\
& & & & & 1 & & & & & & & & & & & & & & & 1 & & & & & & & & & & & & & & & & -1 & & & & & \\
& & & & & & & & & 1 & & & & & & & & & & & & 1 & & & 1 & & & & & & & & & & & & & -1 & & & & \\
& & & & & & & & & & 1 & & & & & & & & & & & & 1 & & & 1 & & & & & & & & & & & & & -1 & & & \\
& & & & & & & & & & & 1 & & & & & & & & & & & & 1 & & & 1 & & & & & & & & & & & & & -1 & & \\
& & & & & & & & & & & & & & & 1 & & & & & & & & & & & & 1 & & & & & & & & & & & & & -1 & \\
& & & & & & & & & & & & & & & & 1 & & & & & & & & & & & & 1 & & & & -1 & & & & & & & & & \\
& & & & & & & & & & & & & & & & & 1 & & & & & & & & & & & & 1 & & & & & & & & & & & & -1
\end{array}\right]$$

The isolated redundancies are therefore $q_7, q_8, q_9, q_{13}, q_{14}, q_{15}, q_{19} \ldots\ldots\ldots\ldots\ldots\ldots q_{30}, R_1, R_2$ and R_3, considering amplitudes only. In the static analysis this plane frame would have six degrees of indeterminacy.

It can be seen that in both these examples the augmented matrix is the same before and after applying the rank technique. It should be noted that this would not always be the case. It should also be noted that the choice of redundants is influenced by the method of numbering the nodes and element loads.



"The Rank Technique."

Fig. 3 .

```
      SUBROUTINE RANTEC(XKD,N9,N8,N7,N9MAX,N8MAX,N7MAX,IDEP.XMAX.IQ)
C     JOHN ROBINSON.  I,S.V.R.
C     AUTOMATIC SELECTION OF REDUNDANCIES IN THE MATRIX FORCE METHOD.
C     THE RANK TECHNIQUE.
      DIMENSION XKD(N9MAX,N7MAX),IDEP(N9MAX),XMAX(N9MAX),IQ(N8MAX)
      DO9 L=1,N9
    9 IDEP(L)=0
      DO1 M=1,N8
    1 IQ(M)=M
      DO30 I=1,N9
C     MAX ABS VALUE IN A ROW AND ITS COLUMN LOCATION
C     INITIAL ABS VALUE
      ZMAX=ABS(XKD(I,1))
      K=1
      DO6 J=2,N8
      RABS=ABS(XKD(I.J))
      IF(RABS-ZMAX)6,6,5
C     MAX ABS VALUE IF DIFFERENT FROM INITIAL VALUE
    5 ZMAX=RABS
C     LOCATION OF MAX ABS VALUE
      K=J
    6 CONTINUE
C     ARRAY OF NORMALIZED ELEMENTS FOR DETERMINANT SOLUTION
      XMAX(I)=XKD(I,K)
      IQ(K)=0
C     IDENTIFICATION OF DEPENDENT EQUATIONS
      IF(ABS(XMAX(I)).GT.1.0E-08)GO TO 14
      IF(N8-N7)3,8,8
    3 DO2 J=N8+1,N7
      IF(ABS(XKD(I,J)).GT.1.0E-08)GO TO 4
    2 CONTINUE
      GO TO 8
    4 WRITE(6,160)
  160 FORMAT(12H NO SOLUTION)
      GO TO 60
    8 IDEP(I)=I
      GO TO 30
C     XKD ROW NORMALIZING
   14 DO16 J=1,N7
   16 XKD(I,J)=XKD(I,J)/XMAX(I)
      DO28 M=1,N9
      IF(M-I)20,28,20
   20 AXKD=XKD(M,K)
      DO22 J=1,N7
   22 XKD(M,J)=XKD(M,J)-AXKD*XKD(I,J)
   28 CONTINUE
   30 CONTINUE
   60 RETURN
      END
```

A.S.A. Fortran listing of subroutine RANTEC.

Table 1 .

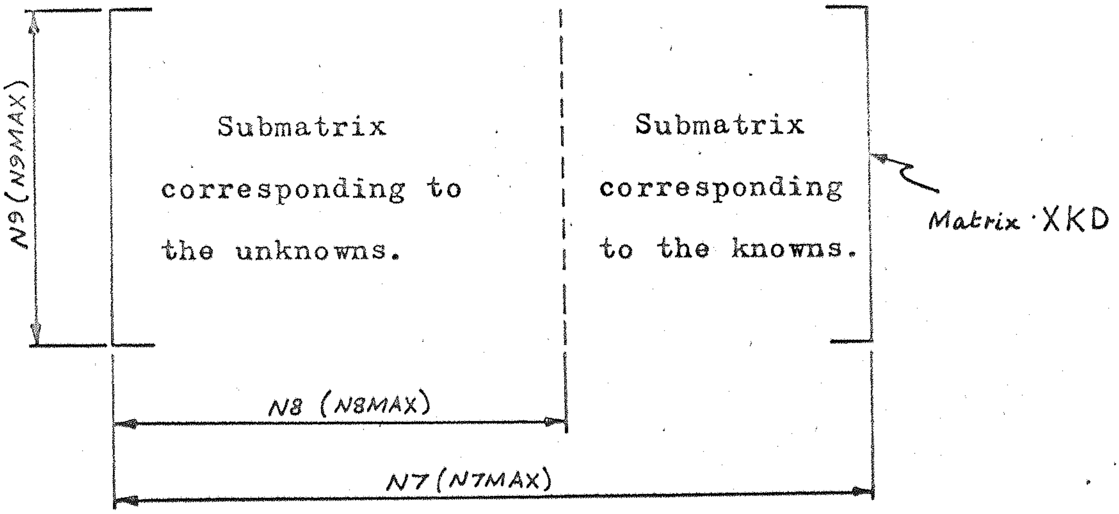

N9 (N9MAX)
Submatrix corresponding to the unknowns.
Submatrix corresponding to the knowns.
Matrix XKD
N8 (N8MAX)
N7 (N7MAX)

Fig. 4.

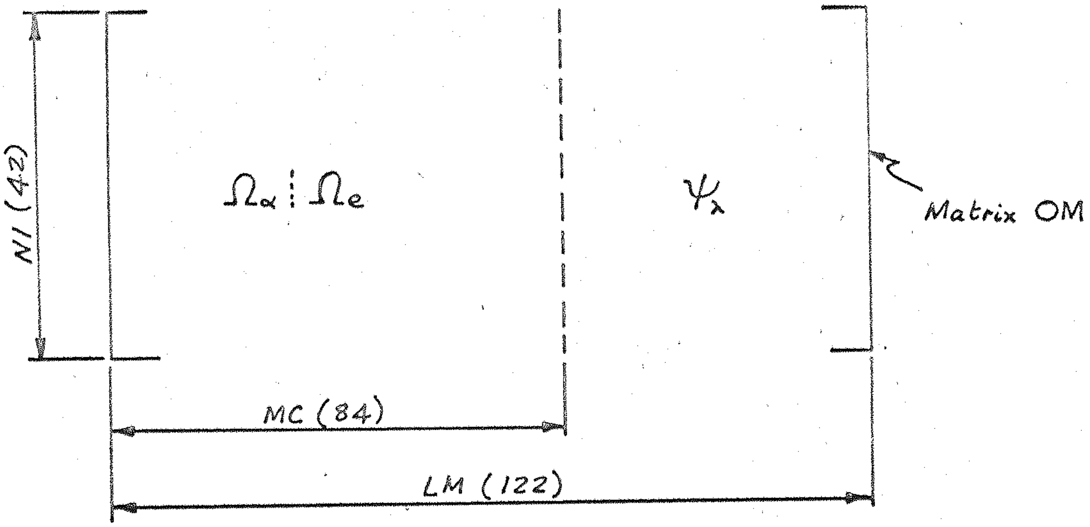
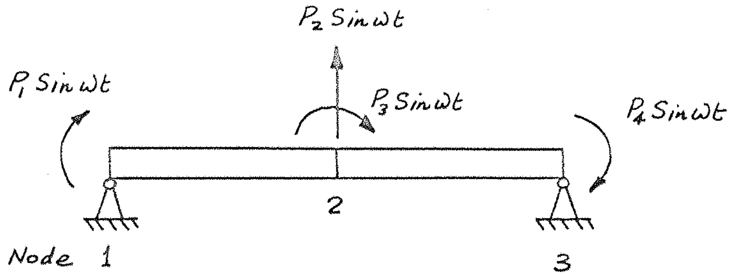
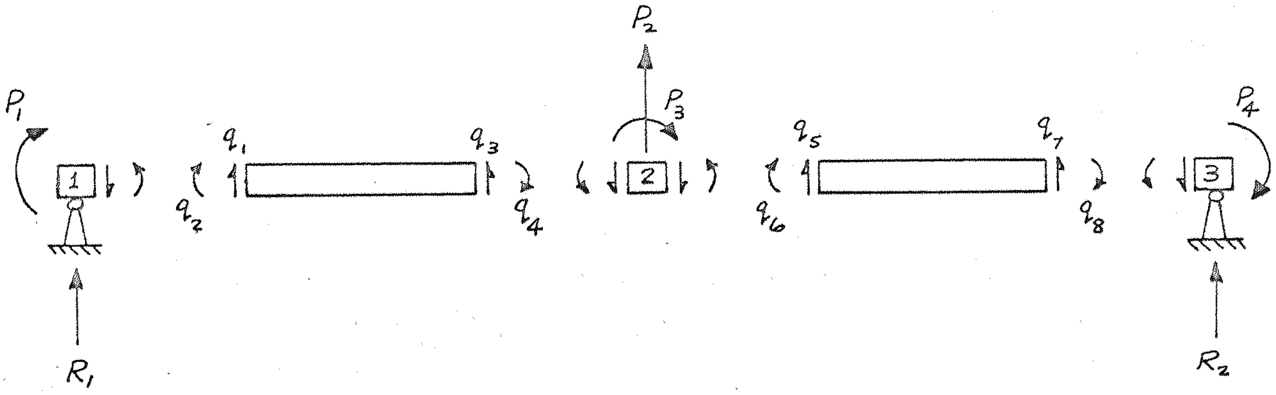

N1 (42)
$\Omega_\alpha \vdots \Omega_e$
Ψ_λ
Matrix OM
MC (84)
LM (122)

Fig. 5.



Applied harmonic forcing system.

(a)

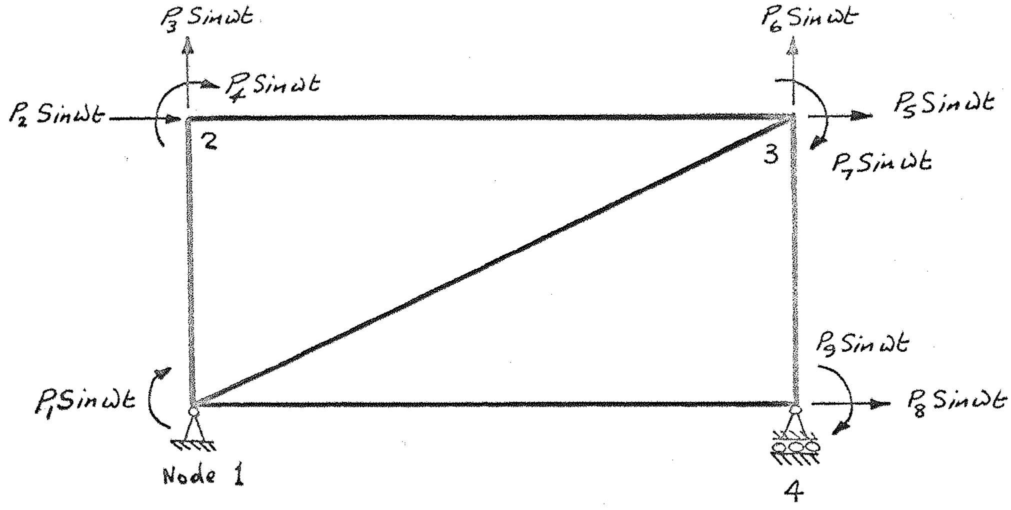


Freebody diagram showing complete generalized load system, amplitudes only. All loads are shown positive.

(b)

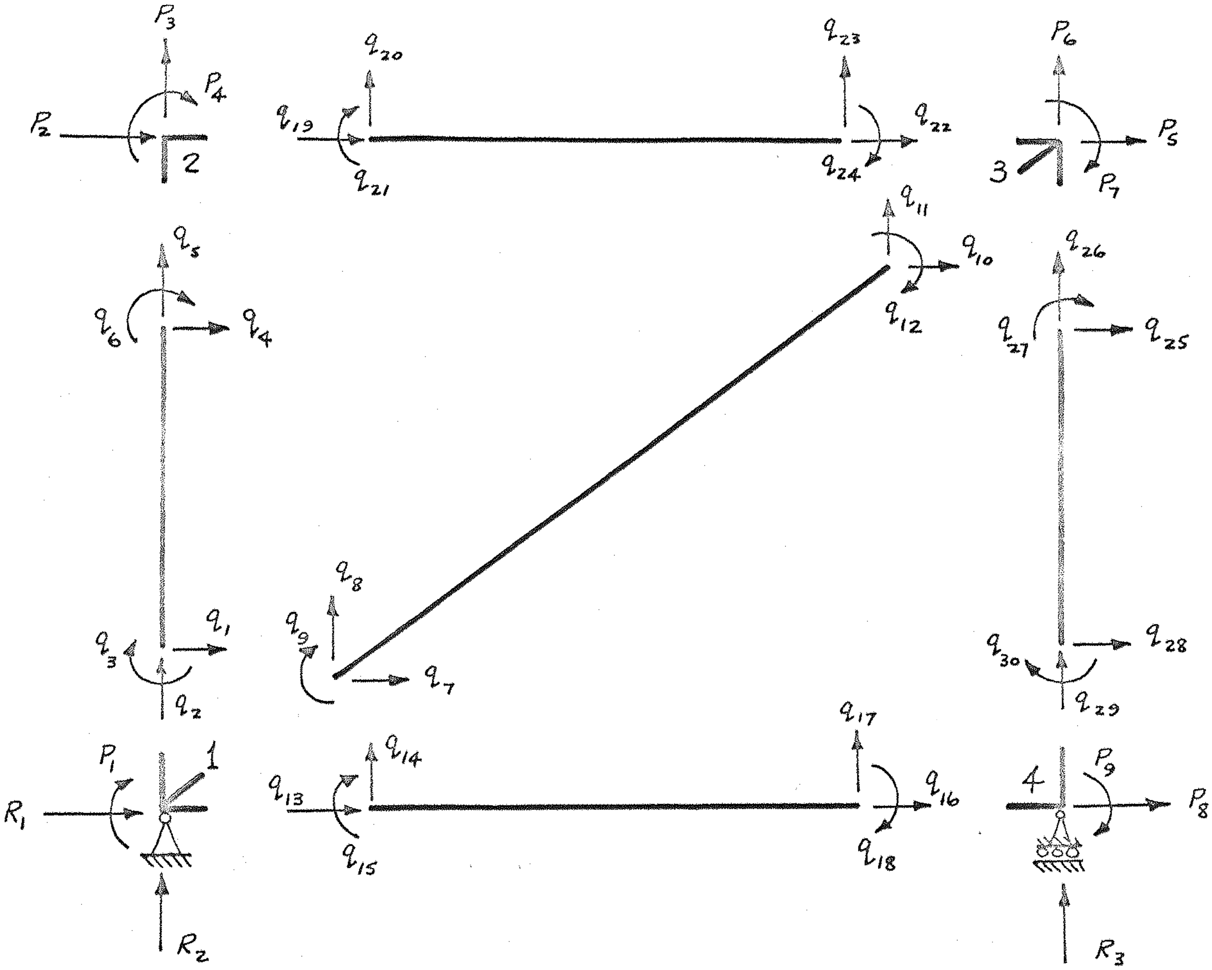
Simply supported beam idealized as two finite elements.

Fig. 6 .



Applied harmonic forcing system.

Fig. 7.



Freebody diagram showing complete generalized load system, amplitudes only. All loads are shown positive.

Fig. 8 .

CHAPTER 2.

"THE RANK FORCE METHOD."

Synopsis.

A matrix force approach which incorporates the rank technique for automatic selection of redundancies is referred to as the rank force method. This method is presented for the vibration analysis of redundant elastic structures and uses a distributed mass representation of the structure. Only undamped structures are considered and it is assumed that steady state conditions exist. All generalized loads and displacements are assumed to vary harmonically and in phase. The first step in the rank force method is to generate a system of joint equilibrium equations for a given idealization. Such equations relate the generalized element boundary loads and structural reactions to the generalized applied load system. The joint equilibrium equations form a rectangular system, that is, there are less equations than unknowns. This means that the element loads and structural reactions can't be evaluated from equilibrium considerations alone and thus the structure is dynamically redundant. The rank technique is applied to the system of joint equilibrium equations and a consistent set of redundancies is automatically selected and hence the degree of redundancy. Therefore, in order to obtain a unique solution for the unknowns it is required to generate a further set of independent linear equations. The number of required equations being equal to the degree of redundancy. The additional equations are obtained by minimizing the total complementary potential in the structure with respect to each selected redundancy. The total energy is obtained by summation of the individual element energies . The element energy is described by its dynamic flexibility matrix and procedures for deriving such matrices are given in Chapter 3 and particular matrices are derived in Chapter 4. The system of energy equations is assembled with the investigated system of joint equilibrium equations thus forming a system of independent equations with the same number of equations as unknowns. Applying the rank technique to this system of equations results in a unique solution for the element loads and structural reactions in terms of the applied load system for a given frequency. The structural response for a given frequency can then be obtained by differentiating the total complementary potential with respect to each of the applied discrete loads. This also gives the structural dynamic flexibility matrix. The eigenvalue formulation is achieved by inverting this matrix.

In order to investigate the vibration characteristics of typical structures the rank force method has been written in the form of a computerized system, this is described in Appendix 4. Using this system collinear beam structures, general plane frames and two dimensional plate structures are investigated and the results are given and discussed in Chapter 5.

2.1 The Rank Force Method.

It is assumed that the structure to be analysed can be idealized into a system of discrete structural elements, finite elements. The force approach is described for an undamped structure and steady state conditions are assumed. The generalized load system, element boundary loads, structural reactions and applied loads, and their respective corresponding displacements are assumed to vary harmonically and in phase. In this presentation it is assumed that the structural reactions have no corresponding displacements. The generalized applied loads are assumed to be point generalized loads and act at the selected nodes. Therefore, for a given idealization, a system of joint equilibrium equations can be assembled which relates the generalized element boundary loads and the generalized applied nodal loads. In this formulation all applied loads are assumed time dependent, however, it should be noted that static applied loads can be included in the vector of applied loads if required.

In the case of a constrained structure the respective applied loads are replaced by unknown reactions. Therefore, considering a constrained structure, the system of joint equilibrium equations can be written in contracted matrix notation as,

$$[\,\Omega_\alpha \,\vdots\, \Omega_e\,]\{\,{}_t q_\alpha \,\vdots\, {}_t R_e\,\} + [\,\Psi_\lambda\,]\{\,{}_t P_\lambda\,\} = \{0\} \qquad 2.1.1$$

Although no direct releases are considered in this presentation it will be pointed out that they can be accounted for by nulling the respective columns in the submatrix $[\Omega_\alpha]$.

Example, one direct release for a plane beam type element is a pinned end, therefore, the column corresponding to the moment at the respective boundary of the element would be nulled.

The system of equations given in equation 2.1.1 can now be investigated by applying the rank technique. This technique will automatically isolate a consistent set of redundancies and hence the degree of redundancy. This gives the number of additional equations which are required for a unique solution of the generalized element boundary loads and structural reactions in terms of the applied load system for a given frequency. After applying the rank technique the resulting equations are given by,

$$[\gamma_\alpha \vdots \gamma_e]\{{}_tq_\alpha \vdots {}_tR_e\} + [\gamma_\lambda]\{{}_tP_\lambda\} = \{0\} \qquad 2.1.2$$

The coefficient matrices corresponding to the unknowns and knowns, as given in equations 2.1.1 and 2.1.2 need only be generated once for a given structural configuration since their matric coefficients are independent of frequency. However, this means that the coefficient matrices in equation 2.1.2 have to be stored separately in the computer and made available for all values of assumed frequency before combining with the energy equations. This has the disadvantage of taking up considerable additional storage space but would given significant time saving because of the iterative nature of the eigenvalue problem.

The additional equations, which are energy equations, are generated by minimizing the total complementary potential in the structure with respect to the automatically selected dynamic redundancies. It is assumed that the total complementary potential, π^* , is given by,

$$\pi^* = \frac{1}{2} \sum_{m=1}^{NE} \lfloor {}_t q_{m\alpha} \rfloor [f_{m\alpha}] \{ {}_t q_{m\alpha} \} \qquad 2.1.3$$

where,

$[f_{m\alpha}]$ = element dynamic flexibility matrix.

This can be written as,

$$\pi^* = \frac{1}{2} \lfloor {}_t q_\alpha \rfloor [F_\alpha] \{ {}_t q_\alpha \} \qquad 2.1.4$$

It should be noted that the vector of isolated dynamic redundancies, $\{ {}_t q^k \}$, is contained in the vector of unknowns, $\{ {}_t q_\alpha \,\vdots\, {}_t R_e \}$. Therefore,

$$\left\{ \frac{\partial \pi^*}{\partial {}_t q^k} \right\} = \{ 0 \} \qquad 2.1.5$$

or

$$\begin{matrix} \frac{\partial \pi^*}{\partial q^1} = \frac{\partial \pi^*}{\partial q_1}\frac{\partial q_1}{\partial q^1} + \cdots + \frac{\partial \pi^*}{\partial q_\alpha}\frac{\partial q_\alpha}{\partial q^1} + \cdots + \frac{\partial \pi^*}{\partial R_1}\frac{\partial R_1}{\partial q^1} + \cdots + \frac{\partial \pi^*}{\partial R_e}\frac{\partial R_e}{\partial q^1} + \cdots = 0 \\ \vdots \\ \frac{\partial \pi^*}{\partial q^k} = \frac{\partial \pi^*}{\partial q_1}\frac{\partial q_1}{\partial q^k} + \cdots + \frac{\partial \pi^*}{\partial q_\alpha}\frac{\partial q_\alpha}{\partial q^k} + \cdots + \frac{\partial \pi^*}{\partial R_1}\frac{\partial R_1}{\partial q^k} + \cdots + \frac{\partial \pi^*}{\partial R_e}\frac{\partial R_e}{\partial q^k} + \cdots = 0 \\ \vdots \end{matrix} \qquad 2.1.6$$

Note ; when the equations are expanded the t-subscript is dropped.

Hence, in contracted matrix form,

$$[{}_{\alpha}\partial_k \,\vdots\, {}_{e}\partial_k]\left\{ \frac{\partial \pi^*}{\partial_t q_\alpha} \,\vdots\, \frac{\partial \pi^*}{\partial_t R_e} \right\} = \{0\} \qquad 2.1.7$$

Assuming that no work is done by the reactions and no lack of fit of elements the energy equations can be written as,

$$[{}_{\alpha}\partial_k][F_d]\{{}_t q_\alpha\} = \{0\} \qquad 2.1.8$$

In the actual computer programme the matrix product $[{}_{\alpha}\partial_k][F_d]$ is performed without assembling $[F_d]$. The $[\gamma_\alpha \,\vdots\, \gamma_e]$ matrix will give the relationship between the element loads and the dynamic redundancies, that is, the $\frac{\partial q_\alpha}{\partial q^k}$ terms. These terms form the array $[{}_{\alpha}\partial_k]$. Therefore, the $[{}_{\alpha}\partial_k]$ matrix can be automatically extracted from the $[\gamma_\alpha \,\vdots\, \gamma_e]$ matrix and it should be noted that this is constant for a given structural idealization but the same remarks apply as for the $[\gamma_\alpha \,\vdots\, \gamma_e]$ matrix. The $\frac{\partial \pi^*}{\partial q_\alpha}$ terms are obtained by assembling the individual element energies and differentiating to give,

$$\left\{ \frac{\partial \pi^*}{\partial_t q_\alpha} \right\} = [F_d]\{{}_t q_\alpha\} \qquad 2.1.9$$

Combining equations 2.1.2 and 2.1.8 gives,

$$\left[\begin{array}{c:c} \gamma_\alpha & \gamma_e \\ \hdashline \varphi & 0 \end{array}\right]\{{}_t q_\alpha \,\vdots\, {}_t R_e\} + \left[\begin{array}{c} \gamma_\lambda \\ \hdashline 0 \end{array}\right]\{{}_t R_\lambda\} = \{0\} \qquad 2.1.10$$

where,

$$[\varphi] = [{}_{\alpha}\partial_k][F_d]$$

Continuing the elimination procedure, as used in the rank technique, to this system of equations gives a unique solution for the element boundary loads and structural reactions in terms of the applied loads for a given frequency. To obtain the direct relationships some rearranging is generally necessary to give a unit coefficient matrix for the unknowns. Therefore, in partitioned form,

$$\begin{bmatrix} {}_t q_\alpha \\ \hline {}_t R_e \end{bmatrix} = \begin{bmatrix} \Delta_{\alpha\lambda} \\ \hline \Delta_{e\lambda} \end{bmatrix} \{ {}_t P_\lambda \} \qquad 2.1.11$$

This equation gives the generalized element boundary loads and structural reactions due to a system of harmonic forcing functions, that is,

$$\begin{bmatrix} {}_t q_\alpha \\ \hline {}_t R_e \end{bmatrix} = \begin{bmatrix} q_\alpha \\ \hline R_e \end{bmatrix} \mathrm{Sin}\,\omega t \; ; \quad \{ {}_t P_\lambda \} = \{ P_\lambda \} \, \mathrm{Sin}\,\omega t \qquad 2.1.12$$

The generalized structural displacements, structural response, corresponding to the applied load system are obtained using the matrix equation,

$$\left\{ \frac{\partial \pi^*}{\partial {}_t P_\lambda} \right\} = \{ {}_t \Delta_\lambda \} \qquad 2.1.13$$

or

$$\begin{array}{lllll}
\dfrac{\partial \pi^*}{\partial P_1} = \dfrac{\partial \pi^*}{\partial q_1}\dfrac{\partial q_1}{\partial P_1} + & + \dfrac{\partial \pi^*}{\partial q_\alpha}\dfrac{\partial q_\alpha}{\partial P_1} + & + \dfrac{\partial \pi^*}{\partial R_1}\dfrac{\partial R_1}{\partial P_1} + & + \dfrac{\partial \pi^*}{\partial R_e}\dfrac{\partial R_e}{\partial P_1} + & = \Delta_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\dfrac{\partial \pi^*}{\partial P_\lambda} = \dfrac{\partial \pi^*}{\partial q_1}\dfrac{\partial q_1}{\partial P_\lambda} + & + \dfrac{\partial \pi^*}{\partial q_\alpha}\dfrac{\partial q_\alpha}{\partial P_\lambda} + & + \dfrac{\partial \pi^*}{\partial R_1}\dfrac{\partial R_1}{\partial P_\lambda} + & + \dfrac{\partial \pi^*}{\partial R_e}\dfrac{\partial R_e}{\partial P_\lambda} + & = \Delta_\lambda \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

2.1.14

In contracted matrix form,

$$\{{}_t\Delta_\lambda\} = [\,{}_\alpha\partial_\lambda \,\vdots\, {}_e\partial_\lambda\,]\left\{ \frac{\partial\pi^*}{\partial_t q_\alpha} \,\vdots\, \frac{\partial\pi^*}{\partial_t R_e} \right\} \tag{2.1.15}$$

Therefore,

$$\{{}_t\Delta_\lambda\} = [\,{}_\alpha\partial_\lambda\,][\,F_d\,]\{\,{}_t q_\alpha\,\} \tag{2.1.16}$$

The $\frac{\partial q_\alpha}{\partial P_\lambda}$ terms are contained in matrix $[{}_\alpha\partial_\lambda]$ and the $\frac{\partial\pi^*}{\partial q_\alpha}$ terms have previously been defined. As before $\left\{\frac{\partial\pi^*}{\partial_t R_e}\right\}=\{0\}$. It should be noted that,

$$[\,{}_\alpha\partial_\lambda\,] = [\,\Delta_{\alpha\lambda}\,]^T \tag{2.1.17}$$

Substituting from equations 2.1.11 and 2.1.17 into equation 2.1.16 gives,

$$\{{}_t\Delta_\lambda\} = [\,\Delta_{\alpha\lambda}\,]^T[\,F_d\,][\,\Delta_{\alpha\lambda}\,]\{\,{}_tP_\lambda\,\} \tag{2.1.18}$$

or

$$\{\,{}_t\Delta_\lambda\,\} = [\,\mathcal{F}_d\,]\{\,{}_tP_\lambda\,\} \tag{2.1.19}$$

where,

$[\mathcal{F}_d]$ = structural dynamic flexibility matrix.

$$= [\,\Delta_{\alpha\lambda}\,]^T[\,F_d\,][\,\Delta_{\alpha\lambda}\,] \tag{2.1.20}$$

In the actual computer programme the assembled element dynamic flexibility matrix is in fact not generated. The structural dynamic flexibility matrix can be readily assembled by taking advantage of the band form of $[F_d]$ and using a submatrix multiplication and summation technique. Equation 2.1.19 gives the structural response due to a system of harmonic forcing functions, that is,

$$\{{}_t\Delta_\lambda\} = \{\Delta_\lambda\} Sin\,\omega t \quad ; \quad \{{}_tR_\lambda\} = \{R_\lambda\} Sin\,\omega t \qquad 2.1.21$$

The eigenvalue problem can now be formulated. Rearranging equation 2.1.19 gives,

$$[\mathcal{K}_d]\{{}_t\Delta_\lambda\} = \{{}_tR_\lambda\} \qquad 2.1.22$$

where,

$[\mathcal{K}_d] = [\mathcal{F}_d]^{-1} =$ structural dynamic stiffness matrix.

The eigenvalue evaluation requires the solution of the system of homogeneous equations,

$$[\mathcal{K}_d]\{{}_t\Delta_\lambda\} = \{0\} \qquad 2.1.23$$

In this equation the frequency is contained in matrix $[\mathcal{K}_d]$. The structural dynamic stiffness matrix has to be generated for every assumed value of frequency. The frequency values which give a zero determinant, that is,

$$det\,[\mathcal{K}_d] = 0 \qquad 2.1.24$$

are the eigenvalues which have corresponding eigenvectors.

When the structural dynamic stiffness matrix has to be generated for every assumed value of the frequency parameter, displacement or force approach, the formulation will be referred to as a "continuous generation process" otherwise it will be referred to as a "singular generation process".

The rank force method has been written as a computerized system for the analysis of collinear beam structures, plane frames and two dimensional rectangular plate structures. In Appendix 4 this system is presented and further detailed explanation of the various steps in the rank force method are given. It should be noted that any theoretical presentation does not necessarily appear in the same manner when it is written as a computer programme or subroutine.

CHAPTER 3.

APPROXIMATE PROCEDURES FOR THE DERIVATION OF ELEMENT DYNAMIC FLEXIBILITY MATRICES.

Synopsis.

The total complementary potential in a structure is obtained by summation of the individual element energies. This is expressable in a quadratic form, that is,

$$\pi^* = \frac{1}{2}\sum_{m=1}^{NE} \lfloor {}_t q_{m\alpha} \rfloor [f_{m\alpha}] \{ {}_t q_{m\alpha} \}$$

The $[f_{m\alpha}]$ matrix is the element dynamic flexibility matrix which represents the approximate vibration behaviour of the respective structural element. This matrix is derived using the principle of virtual forces, that is,

$$\tilde{W}_1^* + \tilde{W}_2^* - \tilde{U}^* = 0$$

where, $\tilde{W}_1^*$ is the complementary virtual work done by the virtual generalized element boundary loads, $\tilde{W}_2^*$ is the complementary virtual work done by the element inertia loading and $\tilde{U}^*$ is the complementary virtual work done by the virtual internal element loads.

The element dynamic flexibility matrix can be derived in a number of ways and various procedures are presented in this chapter. However, each procedure adopts the principle of virtual forces.

In actual fact three procedures have been formulated and presented for the following element types,

1. Endload element, used for inclined plane beam element.
2. Plane beam element, shear and bending.
3. Inclined plane beam element, endload, shear and bending.
4. Rectangular plate element.

In order to assess the various methods of derivation particular element dynamic flexibility matrices have been derived and are given in Chapter 4. Each of these particular matrices have been investigated by analysing simple structural configurations, the results and their discussion are given in Chapter 5.

Introduction.

In the theoretical formulation of the rank force method the assembled element dynamic flexibility matrix is a band matrix formed simply by using the individual element dynamic flexibility matrices. A minimum band width is obtained by numbering the generalized element boundary loads consecutively on an element and continuing the sequence on subsequent elements. No element numbering is required since this is carried out automatically within the computer programme. Each element is recognized by its specifying nodes. The elements are numbered in the order in which they are given in the computer programme input data and the element load numbering is automatically established in conjunction with this.

Each element dynamic flexibility matrix represents the approximate vibration behaviour of the respective structural element and alternative approximate procedures for deriving this matrix will now be presented.

3.1 Derivation Procedure 1.

3.1.1 Endload Element.

The generalized element boundary loads for the endload element shown in figure 10 are denoted by q_α and $q_{\alpha+3}$. The endload vibration equation for this element is derived by considering equilibrium of the element increment shown in figure 9.

Therefore,

$$\frac{\partial P(x,t)}{\partial x} dx + \omega_x(x,t) dx = 0$$

and hence,

$$\omega_x(x,t) = -\frac{\partial P(x,t)}{\partial x} \qquad 3.1.1.1$$

When $\omega_x(x,t)$ is an inertia loading,

$$\omega_x(x,t) = -\rho \frac{\partial^2 u_x(x,t)}{\partial t^2} \qquad 3.1.1.2$$

Therefore, the general endload vibration equation is,

$$\frac{\partial P(x,t)}{\partial x} = \rho \frac{\partial^2 u_x(x,t)}{\partial t^2} \qquad 3.1.1.3$$

where,

$$P(x,t) = AE \frac{\partial u_x(x,t)}{\partial x} \qquad 3.1.1.4$$

When the time function of the endload and displacement distributions is assumed harmonic, that is,

$$P(x,t) = P(x) \operatorname{Sin} \omega t \qquad 3.1.1.5$$

$$u_x(x,t) = u_x(x) \operatorname{Sin} \omega t \qquad 3.1.1.6$$

and hence,

$$\frac{\partial^2 u_x(x,t)}{\partial t^2} = -\omega^2 \sin\omega t \; u_x(x) \qquad 3.1.1.7$$

Equation 3.1.1.3 reduces to,

$$\frac{dP(x)}{dx} = -\rho\,\omega^2 u_x(x) \qquad 3.1.1.8$$

The procedure for deriving the element dynamic flexibility matrix for an endload element is first of all to assume an endload distribution in the form of a polynomial and then evaluate the constant terms by consideration of the element boundary load conditions.

This results in the contracted matrix equation,

$$P(x) = \lfloor T_{pq} \rfloor \{ q_{m\alpha} \} \qquad 3.1.1.9$$

This equation relates the amplitude of the internal endload at station x in the element to the amplitudes of the generalized element boundary loads. Considering amplitudes only the element dynamic flexibility matrix can now be derived using the principle of virtual forces, that is,

$$\tilde{W}_1^* + \tilde{W}_2^* - \tilde{U}^* = 0 \qquad 3.1.1.10$$

The complementary virtual work done by the virtual generalized element boundary loads, $\tilde{W}_1^*$, is given by,

$$\tilde{W}_1^* = \lfloor \tilde{q}_{m\alpha} \rfloor \{ \delta_{m\alpha} \} \qquad 3.1.1.11$$

where,

$\lfloor \tilde{q}_{m\alpha} \rfloor$ = virtual generalized element boundary loads.

and

$\{\delta_{m\alpha}\}$ = generalized element boundary displacements corresponding to the actual generalized element boundary loads, $\{q_{m\alpha}\}$, (amplitudes).

The complementary virtual work done by the inertia loading, $\tilde{W}_2^*$, is given by,

$$\tilde{W}_2^* = \int_0^\ell \tilde{\omega}_x(x)\, u_x(x)\, dx \qquad 3.1.1.12$$

It will be assumed that the displacement function $u_x(x)$ in equation 3.1.1.12 is obtained by rearranging equation 3.1.1.8, that is,

$$u_x(x) = -\frac{1}{\rho\omega^2}\,\frac{d\,P(x)}{dx} \qquad 3.1.1.13$$

Differentiating equation 3.1.1.9 gives,

$$\frac{d\,P(x)}{dx} = \lfloor T_{uq} \rfloor \{ q_{m\alpha} \} \qquad 3.1.1.14$$

Therefore,

$$u_x(x) = -\frac{1}{\rho\omega^2} \lfloor T_{uq} \rfloor \{ q_{m\alpha} \} \qquad 3.1.1.15$$

and

$$\omega_x(x) = -\lfloor T_{uq} \rfloor \{ q_{m\alpha} \} = -\lfloor q_{m\alpha} \rfloor \{ T_{uq} \} \qquad 3.1.1.16$$

Substituting from equations 3.1.1.15 and 3.1.1.16 into equation 3.1.1.12 results in,

$$\tilde{W}_2^* = \frac{1}{\rho\omega^2} \lfloor \tilde{q}_{m\alpha} \rfloor \left(\int_0^\ell \{T_{uq}\} \lfloor T_{uq} \rfloor \, dx \right) \{ q_{m\alpha} \} \qquad 3.1.1.17$$

The complementary virtual work done by the virtual internal endload is given by,

$$\tilde{U}^* = \int_0^\ell \tilde{P}(x)\, \frac{d\,u_x(x)}{dx}\, dx \qquad 3.1.1.18$$

From equation 3.1.1.4,

$$\frac{d u_x(x)}{dx} = \frac{P(x)}{AE} = \frac{1}{AE} \lfloor T_{pq} \rfloor \{ q_{m\alpha} \} \qquad 3.1.1.19$$

It can be seen at this stage that the first derivative of the displacement as given by equation 3.1.1.19 is not the same as if equation 3.1.1.13 were differentiated. The assumption regarding the displacement, $u_x(x)$ (equation 3.1.1.19), will be appreciated when considering the plate element derivation. It is felt that such an assumption is acceptable in view of the approximate nature of finite element techniques and particularly if good results are obtained. It should be noted that for a linear endload distribution the first derivative of the displacement would be zero if equation 3.1.1.13 were used which is of course unrealistic.

Substituting from equation 3.1.1.19 results in,

$$\tilde{U}^* = \frac{1}{AE} \lfloor \tilde{q}_{m\alpha} \rfloor \left(\int_0^{\ell} \{ T_{pq} \} \lfloor T_{pq} \rfloor \, dx \right) \{ q_{m\alpha} \} \qquad 3.1.1.20$$

Therefore by substituting the respective terms in equation 3.1.1.10 the element dynamic flexibility matrix for an endload element is obtained, that is,

$$[f_{md}] = \frac{1}{AE} \int_0^{\ell} \{ T_{pq} \} \lfloor T_{pq} \rfloor \, dx - \frac{1}{\rho \omega^2} \int_0^{\ell} \{ T_{uq} \} \lfloor T_{uq} \rfloor \, dx \qquad 3.1.1.21$$

In contracted matrix form,

$$[f_{md}] = [f_m] - \frac{1}{\omega^2} [m_{mf}] \qquad 3.1.1.22$$

where,

$[f_m]$ = element static flexibility matrix.

$[m_{mf}]$ = element inverse mass matrix.

The sign convention for the generalized element boundary displacements corresponding to the generalized element boundary loads is given in figure 10 .

In the displacement method the element dynamic stiffness matrix is separated into a static stiffness matrix and a mass matrix. Before progressing to further derivation procedures a clarification of definition will be presented. In the displacement method, considering amplitudes only,

$$\begin{aligned} \{q_{m\alpha}\} &= [k_{md}]\{\delta_{m\alpha}\} \\ &= [k_m]\{\delta_{m\alpha}\} - \omega^2[m_{mk}]\{\delta_{m\alpha}\} \\ &= \{q_{m\alpha}\}_1 - \{q_{m\alpha}\}_2 \end{aligned}$$

Now,

$$\{q_{m\alpha}\}_2 = \omega^2[m_{mk}]\{\delta_{m\alpha}\} = [m_{mk}]\{\ddot{\delta}_{m\alpha}\}$$

where,

$\{\ddot{\delta}_{m\alpha}\} = \omega^2\{\delta_{m\alpha}\} =$ acceleration amplitudes.

Therefore, considering the relationship force equals mass times acceleration the definition of mass matrix is established.

In the force approach, again considering amplitudes only,

$$\begin{aligned} \{\delta_{m\alpha}\} &= [f_{md}]\{q_{m\alpha}\} \\ &= [f_m]\{q_{m\alpha}\} - \frac{1}{\omega^2}[m_{mf}]\{q_{m\alpha}\} \\ &= \{\delta_{m\alpha}\}_1 - \{\delta_{m\alpha}\}_2 \end{aligned}$$

Therefore, following a similar procedure as in the displacement approach the definition of inverse mass matrix becomes obvious.

3.1.2 Plane Beam Element.

The beam element to be considered is shown in figure 12. The generalized element boundary loads are denoted by $q_{\alpha+1}$, $q_{\alpha+2}$, $q_{\alpha+4}$ and $q_{\alpha+5}$

The relationships between the incremental loads and the displacement in the z-direction, see figure 11, are,

$$M(x,t) = EI \frac{\partial^2 u_z(x,t)}{\partial x^2} \qquad 3.1.2.1$$

$$Q(x,t) = -\frac{\partial M(x,t)}{\partial x} \qquad 3.1.2.2$$

$$\omega_z(x,t) = -\frac{\partial Q(x,t)}{\partial x} = \frac{\partial^2 M(x,t)}{\partial x^2} \qquad 3.1.2.3$$

When $\omega_z(x,t)$ is an inertia loading the general bending vibration equation is given by,

$$\frac{\partial^2 M(x,t)}{\partial x^2} = -\rho \frac{\partial^2 u_z(x,t)}{\partial t^2} \qquad 3.1.2.4$$

If a harmonic time function is assumed equation 3.1.2.4 reduces to,

$$\frac{d^2 M(x)}{dx} = -\rho \omega^2 u_z(x) \qquad 3.1.2.5$$

A bending moment distribution is now assumed in the form of a polynomial and the constant terms evaluated by consideration of the element boundary load conditions. The following contracted matrix expression is then obtained for the internal bending moment at station x in the element to the amplitudes of the generalized element boundary loads.

$$M(x) = \lfloor T_{Mq} \rfloor \{ q_{m\alpha} \} \qquad 3.1.2.6$$

where,

$$\{ q_{m\alpha} \} = \{ q_{\alpha+1} \quad q_{\alpha+2} \quad q_{\alpha+4} \quad q_{\alpha+5} \} \qquad 3.1.2.7$$

The complementary work done by the virtual generalized element boundary loads is given by,

$$\tilde{W}_1^* = \lfloor \tilde{q}_{m\alpha} \rfloor \{ \delta_{m\alpha} \} \qquad 3.1.2.8$$

The complementary virtual work done by the inertia loading is,

$$\tilde{W}_2^* = \int_0^{\ell} \tilde{\omega}_3(x) \; u_3(x) \; dx \qquad 3.1.2.9$$

Now, from equation 3.1.2.5,

$$u_3(x) = \frac{1}{\rho\omega^2} \; \frac{d^2M(x)}{dx^2} \qquad 3.1.2.10$$

Differentiating equation 3.1.2.6 twice gives,

$$\frac{d^2M(x)}{dx^2} = \lfloor T_{uq} \rfloor \{ q_{m\alpha} \} \qquad 3.1.2.11$$

Therefore,

$$\tilde{W}_2^* = \frac{1}{\rho\omega^2} \lfloor \tilde{q}_{m\alpha} \rfloor \left(\int_0^{\ell} \{ T_{uq} \} \lfloor T_{uq} \rfloor \, dx \right) \{ q_{m\alpha} \} \qquad 3.1.2.12$$

If a third degree polynomial is assumed for the bending moment distribution the second derivative of the moment expression gives a linear function for the displacement $u_3(x)$. This says that the inertia loading is linear along the length of the element. Considering lumped mass representation this linear variation seems reasonable as an initial assumption. It should be realized that in the matrix displacement approach when a third degree polynomial is assumed for the beam displacement function that this gives a linear bending moment distribution which again is not quite realistic but it is an approximation.

The complementary work done by the virtual internal moments is given by,

$$\tilde{U}^* = \int_0^\ell \tilde{M}(x) \frac{d^2 u_z(x)}{dx^2} dx \qquad 3.1.2.13$$

To be realistic in representing the internal complementary energy the second derivative of the displacement in equation 3.1.2.13 will be that given by equation 3.1.2.1, therefore,

$$\frac{d^2 u_z(x)}{dx^2} = \frac{M(x)}{EI} = \frac{1}{EI} \lfloor T_{Mq} \rfloor \{ q_{m\alpha} \} \qquad 3.1.2.14$$

Hence,

$$\tilde{U}^* = \frac{1}{EI} \lfloor \tilde{q}_{m\alpha} \rfloor \left(\int_0^\ell \{T_{Mq}\} \lfloor T_{Mq} \rfloor dx \right) \{ q_{m\alpha} \} \qquad 3.1.2.15$$

Note ; if the second derivative of the displacement was obtained using equation 3.1.2.10 it would give zero curvature, that is, $\frac{d^2 u_z(x)}{dx^2} = 0$. Again this is unrealistic but the displacement function given by equation 3.1.2.10 is only used for inertia calculations.

Therefore, by applying the principle of virtual forces and substituting the respective terms the element dynamic flexibility matrix for the beam element shown in figure 12 is given by,

$$[f_{md}] = \frac{1}{EI}\int_0^{\ell} \{T_{Mq}\}\lfloor T_{Mq} \rfloor \, dx - \frac{1}{\rho\omega^2}\int_0^{\ell} \{T_{uq}\}\lfloor T_{uq} \rfloor \, dx \qquad 3.1.2.16$$

In contracted matrix form,

$$[f_{md}] = [f_m] - \frac{1}{\omega^2}[m_{mf}]$$

The sign convention for the generalized element boundary loads and corresponding displacements is given in figure 12.

3.1.3 Rectangular Plate Element.

3.1.3(a) q-system of boundary loads.

The generalized element boundary loads for this derivation are distributed moments and distributed equivalent shears along the respective boundaries and four concentrated nodal loads. This system of loads will be referred to as a q-system. One such system is shown in figure 13. All generalized loads are assumed to vary harmonically with time.

The relationship between the incremental loads, as shown in figure 14, and the displacement in the y-direction are,

$$\begin{bmatrix} \frac{\partial^2 u_y}{\partial x^2} \\ \frac{\partial^2 u_y}{\partial z^2} \\ \frac{\partial^2 u_y}{\partial x \partial z} \end{bmatrix} = \frac{1}{D(1-\nu^2)} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & -(1+\nu) \end{bmatrix} \begin{bmatrix} M_x \\ M_z \\ M_{xz} \end{bmatrix} \quad 3.1.3.1$$

where,

$$D = \frac{E t_p^3}{12(1-\nu^2)}$$

The shears and moments are related by the equations,

$$\left.\begin{aligned} Q_x &= -\frac{\partial M_x}{\partial x} + \frac{\partial M_{xz}}{\partial z} \\ Q_z &= \frac{\partial M_{xz}}{\partial x} - \frac{\partial M_z}{\partial z} \end{aligned}\right] \quad 3.1.3.2$$

In deriving the element dynamic flexibility matrix an equivalent plate loading system will be adopted, see figure 15. This equivalent system is given by,

$$\left.\begin{aligned}
V_1 &= \left(Q_z + \frac{\partial M_{xz}}{\partial x} \right)_{z=0} \\
V_2 &= \left(Q_x + \frac{\partial M_{xz}}{\partial z} \right)_{x=a} \\
V_3 &= \left(Q_z + \frac{\partial M_{xz}}{\partial x} \right)_{z=b} \\
V_4 &= \left(Q_x + \frac{\partial M_{xz}}{\partial z} \right)_{x=0} \\
W_1 &= 2\left(M_{xz}\right)_{x=0,\, z=0} \\
W_2 &= 2\left(M_{xz}\right)_{x=a,\, z=0} \\
W_3 &= 2\left(M_{xz}\right)_{x=a,\, z=b} \\
W_4 &= 2\left(M_{xz}\right)_{x=0,\, z=b}
\end{aligned}\right] \qquad 3.1.3.3$$

The M_x and M_z moments are unchanged.

The relationship between the moments, bending and twisting, and the distributed load applied perpendicular to the plane of the plate is,

$$\frac{\partial^2 M_x}{\partial x^2} - 2\,\frac{\partial^2 M_{xz}}{\partial x \partial z} + \frac{\partial^2 M_z}{\partial z^2} = w_y \qquad 3.1.3.4$$

Therefore, when w_y is an inertia load,

$$\frac{\partial^2 M_x}{\partial x^2} - 2\,\frac{\partial^2 M_{xz}}{\partial x \partial z} + \frac{\partial^2 M_z}{\partial z^2} = \rho\omega^2 u_y \qquad 3.1.3.5$$

The next step in the procedure is to assume distributions for the moments M_x, M_z and M_{xz}. The constant terms in the assumed polynomials are evaluated using equations 3.1.3.2 and 3.1.3.3 and the boundary load conditions as given by the generalized element load system.

The complementary virtual work done by the virtual inertia loading is given by,

$$\tilde{W}_z^* = \int_0^b \int_0^a \tilde{w}_y \, u_y \; dx \, dz \qquad 3.1.3.6$$

Substituting from equation 3.1.3.5 gives,

$$\tilde{W}_z^* = \frac{1}{\rho\omega^2} \int_0^b \int_0^a \left(\frac{\partial^2 \tilde{M}_x}{\partial x^2} - 2\,\frac{\partial^2 \tilde{M}_{xz}}{\partial x \partial z} + \frac{\partial^2 \tilde{M}_z}{\partial z^2} \right) \left(\frac{\partial^2 M_x}{\partial x^2} - 2\,\frac{\partial M_{xz}}{\partial x \partial z} + \frac{\partial^2 M_z}{\partial z^2} \right) dx \, dz$$

3.1.3.7

where, from the final moment expressions,

$$\left.\begin{aligned} \frac{\partial^2 M_x}{\partial x^2} &= \lfloor T_{xuq} \rfloor \{ q_{m\alpha} \} \\ \frac{\partial^2 M_z}{\partial z^2} &= \lfloor T_{zuq} \rfloor \{ q_{m\alpha} \} \\ \frac{\partial^2 M_{xz}}{\partial x \partial z} &= \lfloor T_{xzuq} \rfloor \{ q_{m\alpha} \} \end{aligned} \right] \qquad 3.1.3.8$$

The complementary work done by the virtual internal moments is given by,

$$\tilde{U}^* = \int_0^b\int_0^a \left(\tilde{M}_x \frac{\partial^2 u_y}{\partial x^2} - 2\tilde{M}_{xz}\frac{\partial^2 u_y}{\partial x \partial z} + \tilde{M}_z \frac{\partial^2 u_y}{\partial z^2}\right) dx\, dz \qquad 3.1.3.9$$

Substituting from equation 3.1.3.1 gives,

$$\tilde{U}^* = \frac{1}{D(1-\nu^2)}\int_0^b\int_0^a \left(\tilde{M}_x (M_x - \nu M_z) + 2(1+\nu)\tilde{M}_{xz} M_{xz} + \tilde{M}_z(-\nu M_x + M_z)\right) dx\, dz$$

Therefore,

$$\tilde{U}^* = \frac{1}{D(1-\nu^2)}\int_0^b\int_0^a \left(\tilde{M}_x M_x + \tilde{M}_z M_z + 2(1+\nu)\tilde{M}_{xz} M_{xz} - \nu(\tilde{M}_x M_z + \tilde{M}_z M_x)\right) dx\, dz$$

$$3.1.3.10$$

The assumed final moment distributions can be written as,

$$\left.\begin{aligned} M_x &= \lfloor T_{xMq} \rfloor \{ q_{m\alpha} \} \\ M_z &= \lfloor T_{zMq} \rfloor \{ q_{m\alpha} \} \\ M_{xz} &= \lfloor T_{xzMq} \rfloor \{ q_{m\alpha} \} \end{aligned}\right] \qquad 3.1.3.11$$

The complementary work done by the virtual generalized element boundary loads is given by,

$$\tilde{W}_1^* = \lfloor \tilde{q}_{m\alpha} \rfloor \{ \delta_{m\alpha} \} \qquad 3.1.3.12$$

Hence, applying the principle of virtual forces the element dynamic flexibility matrix for a rectangular plate element is given by,

$$[f_{md}] = [f_m] - \frac{1}{\omega^2}[m_{mf}]$$

where,

$[f_m]$ = element static flexibility matrix

$$= \left(\frac{12}{E t_p^3}\right) \int_0^b \int_0^a \Big(\{T_{xMq}\} \lfloor T_{xMq} \rfloor + \{T_{zMq}\} \lfloor T_{zMq} \rfloor$$

$$+ 2(1+\nu) \{T_{xzMq}\} \lfloor T_{xzMq} \rfloor$$

$$- \nu \left(\{T_{xMq}\} \lfloor T_{\underset{z}{x}Mq} \rfloor + \{T_{zMq}\} \lfloor T_{\underset{x}{z}Mq} \rfloor \right) \Big) \, dx dz$$

3.1.3.13

$[m_{mf}]$ = element inverse mass matrix

$$= \frac{1}{\rho} \int_0^b \int_0^a \{T_{uq}\} \lfloor T_{uq} \rfloor \, dx dz$$ 3.1.3.14

and

$$\lfloor T_{uq} \rfloor = \left(\lfloor T_{xuq} \rfloor - z \lfloor T_{xzuq} \rfloor + \lfloor T_{zuq} \rfloor \right)$$ 3.1.3.15

3.1.3(b) s-system of boundary loads.

The q-system of generalized element boundary loads is not a convenient system for use in the rank force method. It would be more convenient to have equivalent discrete generalized nodal loads, these will be referred to as an s-system. The s-system for plate bending is shown in figure 16 . Such a system would then be consistent with the element nodal displacements as adopted in the displacement approach. Now, how can a q-system be replaced by an s-system so that the two systems are equivalent. The two element boundary loading systems have a dependency which is established from equilibrium considerations. In other words, the two systems must have the same generalized load resultants.

To derive the element dynamic flexibility matrix corresponding to the s-system the q-loads are considered as the unknowns and the s-loads as the applied loads.

Therefore, for overall equilibrium between the two systems,

$$[\Omega_{m\alpha}]\{{}_tq_{m\alpha}\} + [\Psi_{m\beta}]\{{}_tS_{m\beta}\} = \{0\} \qquad 3.1.3.16$$

Applying the rank technique to these equations results in the following system of independent equations,

$$[\gamma_{m\alpha}]\{{}_tq_{m\alpha}\} + [\gamma_{m\beta}]\{{}_tS_{m\beta}\} = \{0\} \qquad 3.1.3.17$$

This now presents a redundant problem with a set of redundancies, contained in $\{{}_t q_{m\alpha}\}$, being isolated by the rank technique. Therefore, to obtain a unique solution for the q-system in terms of the s-system the total complementary potential in the plate is minimized with respect to the isolated q-redundancies. The resulting energy equations are given by,

$$[{}_\alpha\partial_k]_m \, [f_{md}] \, \{{}_t q_{m\alpha}\} = [\vartheta_m] \, \{{}_t q_{m\alpha}\} = \{0\} \qquad 3.1.3.18$$

Assembling these equations with the system of independent equations and again applying the rank technique results in the relationship,

$$\{{}_t q_{m\alpha}\} = [\Delta_{m\alpha\beta}] \, \{{}_t S_{m\beta}\} \qquad 3.1.3.19$$

Therefore, for the same plate potential, the element dynamic flexibility matrix corresponding to the s-system is given by,

$$[\not{p}_{md}] = [\Delta_{m\alpha\beta}]^T [f_{md}] [\Delta_{m\alpha\beta}] \qquad 3.1.3.20$$

Having found the equivalent s-loads in an analysis the more meaningful q-loads can be calculated using equation 3.1.3.19.

It would appear that to derive more complex element representations the q-system can be increased to include prescribed values of the distributed loads at other points on the boundaries. The s-sytem will still contain the same number of terms for this element but this enables more complex loading distributions to be considered.

3.2 Derivation Procedure 2.

This procedure derives a displacement function by integration which is adopted when calculating the complementary work done by the virtual inertia loading.

3.2.1 Endload Element.

Considering a harmonic time function equation 3.1.1.4 reduces to,

$$P(x) = AE \frac{d\,u_x(x)}{dx} \qquad 3.2.1.1$$

Rearranging this equation gives,

$$\frac{d\,u_x(x)}{dx} = \frac{1}{AE} P(x) = \frac{1}{AE} \lfloor T_{pq} \rfloor \{ q_{m\alpha} \} \qquad 3.2.1.2$$

Therefore, a displacement function can be obtained by integrating equation 3.2.1.2, that is,

$$u_x(x) = \frac{1}{AE}\left(\left(\int \lfloor T_{pq} \rfloor \, dx\right) \{ q_{m\alpha} \} + C\right) \qquad 3.2.1.3$$

The constant of integration will be evaluated by applying d'Alembert's principle to the overall element, that is,

$$\int_0^{\ell} \omega_x(x)\, dx + q_\alpha + q_{\alpha+3} = 0 \qquad 3.2.1.4$$

where,

$$\omega_x(x) = \rho\omega^2 u_x(x)$$

3.2.2 Plane Beam Element.

Considering a harmonic time function,

equation 3.1.2.1 can be rearranged and integrated twice to give,

$$u_z(x) = \frac{1}{EI}\left(\left(\iint M(x)\,dx\,dx\right) + C_1 x + C_2\right) \qquad 3.2.2.1$$

or

$$u_z(x) = \frac{1}{EI}\left(\left(\iint \lfloor T_{Mq} \rfloor\,dx\,dx\right)\{q_{M\alpha}\} + C_1 x + C_2\right) \qquad 3.2.2.2$$

The constants of integration are evaluated by considering overall equilibrium of the beam element, for equilibrium in the z-direction,

$$\int_0^{\ell} \omega_z(x)\,dx + q_{\alpha+1} + q_{\alpha+4} = 0 \qquad 3.2.2.3$$

and for moment equilibrium about node i,

$$-\int_0^{\ell} \omega_z(x)\,x\,dx + q_{\alpha+2} - \ell\, q_{\alpha+4} + q_{\alpha+5} = 0 \qquad 3.2.2.4$$

where,

$$\omega_z(x) = \rho\omega^2 u_z(x)$$

3.2.3 Rectangular Plate Element.

Incremental considerations give the expression,

$$d^2u_y = \frac{\partial^2 u_y}{dx^2}\,dx\,dx + 2\frac{\partial^2 u_y}{\partial x\,\partial z}\,dx\,dz + \frac{\partial^2 u_y}{\partial z^2}\,dz\,dz \qquad 3.2.3.1$$

The second partial derivatives contained in equation 3.2.3.1 are replaced by the moments as given by equation 3.1.3.1. If equation 3.2.3.1 is integrated twice the resulting displacement function is given by,

$$u_y = u_1 + u_2 + u_3 \qquad 3.2.3.2$$

where,

$$\left.\begin{aligned} u_1 &= \iint \frac{\partial^2 u_y}{\partial x^2}\, dx\, dx + f_1(z)x + f_2(z) \\ u_2 &= \iint 2\frac{\partial^2 u_y}{\partial x \partial z}\, dx\, dz + f_3(z)z + f_4(x) \\ u_3 &= \iint \frac{\partial^2 u_y}{\partial z^2}\, dz\, dz + f_5(x)z + f_6(x) \end{aligned}\right] \qquad 3.2.3.3$$

This approach gives the general displacement function if the f-functions can be evaluated. One solution for u_y , not the general solution, is,

$$u_y = \iint \frac{\partial^2 u_y}{\partial x^2}\, dx\, dx + \iint 2\frac{\partial^2 u_y}{\partial x \partial z}\, dx\, dz + \iint \frac{\partial^2 u_y}{\partial z^2}\, dz\, dz$$

$$+ C_1 x + C_2 z + C_3 \qquad 3.2.3.4$$

The constants of integration are evaluated using the equilibrium equations,

$$\left.\begin{aligned} &\int_0^b\int_0^a \rho\omega^2 u_y\, dx\, dz + F(Y) = 0 \\ &\int_0^b\int_0^a \rho\omega^2 u_y z\, dx\, dz + M(X) = 0 \\ &\int_0^b\int_0^a \rho\omega^2 u_y x\, dx\, dz + M(Z) = 0 \end{aligned}\right] \qquad 3.2.3.5$$

where,

F(Y), M(X) and M(Z) are the element boundary load resultants.

3.3 Derivation Procedure 3.

3.3.1 Endload Element.

This procedure adopts two previously given equations, they are,

$$w_x(x,t) = -\frac{\partial P(x,t)}{\partial x} \qquad 3.1.1.1$$

and

$$P(x,t) = AE\frac{\partial u_x(x,t)}{\partial x} \qquad 3.1.1.4$$

Differentiating equation 3.1.1.4 gives,

$$\frac{\partial P(x,t)}{\partial x} = AE\frac{\partial^2 u_x(x,t)}{\partial x^2} \qquad 3.3.1.1$$

and hence,

$$AE\frac{\partial^2 u_x(x,t)}{\partial x^2} = -w_x(x,t) \qquad 3.3.1.2$$

When $w_x(x,t)$ is an inertia load,

$$w_x(x,t) = -\rho\frac{\partial^2 u_x(x,t)}{\partial t^2}$$

Therefore,

$$AE\frac{\partial^2 u_x(x,t)}{\partial x^2} = \rho\frac{\partial^2 u_x(x,t)}{\partial t^2}$$

or

$$AE\frac{\partial^2 u_x(x,t)}{\partial x^2} - \rho\frac{\partial^2 u_x(x,t)}{\partial t^2} = 0 \qquad 3.3.1.3$$

Assuming a harmonic time function, that is,

$$u_x(x,t) = u_x(x)\,\mathrm{Sin}\,\omega t$$

equation 3.3.1.3 becomes,

$$AE \frac{d^2 u_x(x)}{dx^2} + \rho\omega^2 u_x(x) = 0$$

or

$$\frac{d^2 u_x(x)}{dx^2} + \lambda_1^2 u_x(x) = 0 \qquad 3.3.1.4$$

where,

$$\lambda_1^2 = \omega^2 \frac{\rho}{AE} = \omega^2 \frac{\mu}{386{\cdot}4 E} \qquad 3.3.1.5$$

Solving equation 3.3.1.5 gives the general form,

$$u_x(x) = A_1 \operatorname{Cos} \lambda_1 x + A_2 \operatorname{Sin} \lambda_1 x \qquad 3.3.1.6$$

The constant terms are evaluated using equation 3.1.1.4 , considering a harmonic time function, that is,

$$P(x) = AE \frac{d u_x(x)}{dx} \qquad 3.3.1.7$$

and the element boundary load conditions.

3.3.2 Plane Beam Element.

This procedure adopts three previously given equations, they are,

$$M(x,t) = EI \frac{\partial^2 u_z(x,t)}{\partial x^2} \qquad 3.1.2.1$$

$$Q(x,t) = -\frac{\partial M(x,t)}{\partial x} = -EI \frac{\partial^3 u_z(x,t)}{\partial x^3} \qquad 3.1.2.2$$

and

$$w_z(x,t) = \frac{\partial^2 M(x,t)}{\partial x^2} \qquad 3.1.2.3$$

Differentiating equation 3.1.2.1 twice gives,

$$\frac{\partial^2 M(x,t)}{\partial x^2} = EI \frac{\partial^4 u_z(x,t)}{\partial x^4} \qquad 3.3.2.1$$

and hence,

$$EI \frac{\partial^4 u_z(x,t)}{\partial x^4} = \omega_z(x,t) \qquad 3.3.2.2$$

When $\omega_z(x,t)$ is an inertia load,

$$\omega_z(x,t) = -\rho \frac{\partial^2 u_z(x,t)}{\partial x^2}$$

Therefore,

$$EI \frac{\partial^4 u_z(x,t)}{\partial x^4} = -\rho \frac{\partial^2 u_z(x,t)}{\partial x^2}$$

or

$$EI \frac{\partial^4 u_z(x,t)}{\partial x^4} + \rho \frac{\partial^2 u_z(x,t)}{\partial t^2} = 0 \qquad 3.3.2.3$$

Assuming a harmonic time function, that is,

$$u_z(x,t) = u_z(x) \, \mathrm{Sin}\, \omega t$$

equation 3.3.2.3 becomes,

$$EI \frac{d^4 u_z(x)}{dx^4} - \rho \omega^2 u_z(x) = 0$$

or

$$\frac{d^4 u_z(x)}{dx^4} - \lambda_z^4 \, u_z(x) = 0 \qquad 3.3.2.4$$

where,

$$\lambda_z^4 = \omega^2 \frac{\rho}{EI} = \omega^2 \frac{\mu A}{386{\cdot}4\, EI} \qquad 3.3.2.5$$

Solving equation 3.3.2.4 gives the general form,

$$u_z(x) = A_1 \cos\lambda_2 x + A_2 \sin\lambda_2 x + A_3 \cosh\lambda_2 x + A_4 \sinh\lambda_2 x \qquad 3.3.2.6$$

The constant terms are evaluated using equations 3.1.2.1 and 3.1.2.2, considering a harmonic time function, that is,

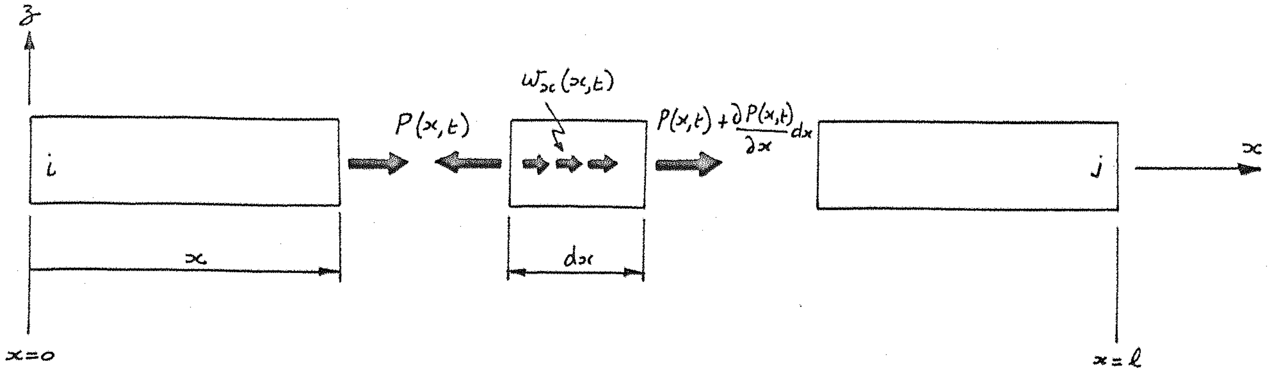
$$M(x) = EI \frac{d^2 u_z(x)}{dx^2}$$

and

$$Q(x) = -EI \frac{d^3 u_z(x)}{dx^3}$$

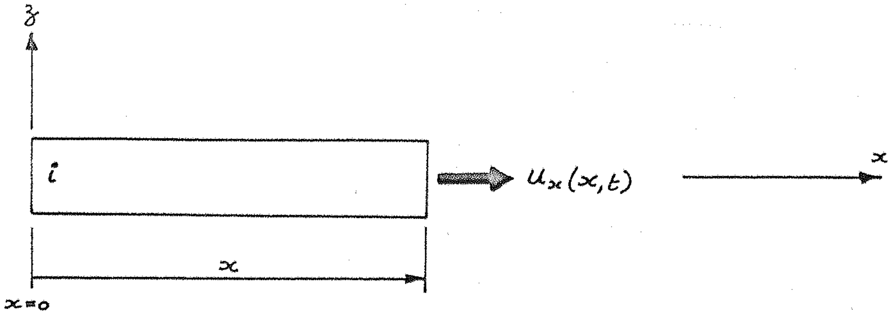
and the element boundary load conditions.

This chapter has presented derivation procedures for evaluation of the dynamic flexibility matrix for various types of elements. Basically, the general formulation can be applied to any type of element and in the next chapter particular element dynamic flexibility matrices are derived.



Positive sign convention for incremental loading.

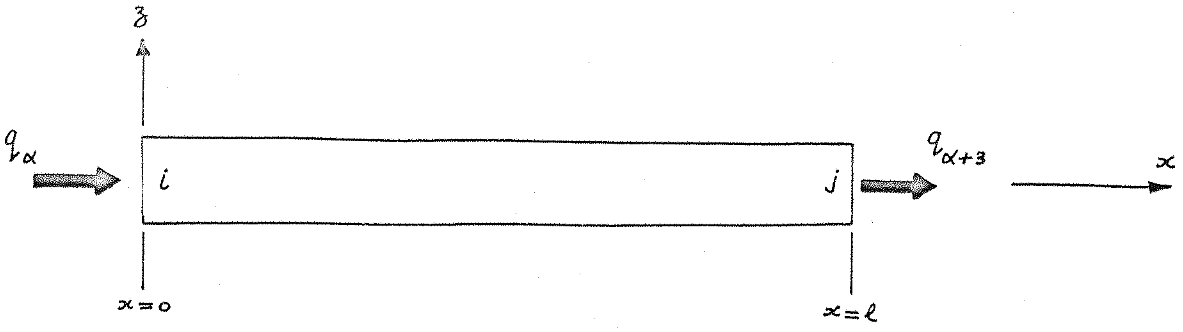
(a)



Positive sign convention for displacement in the x-direction.

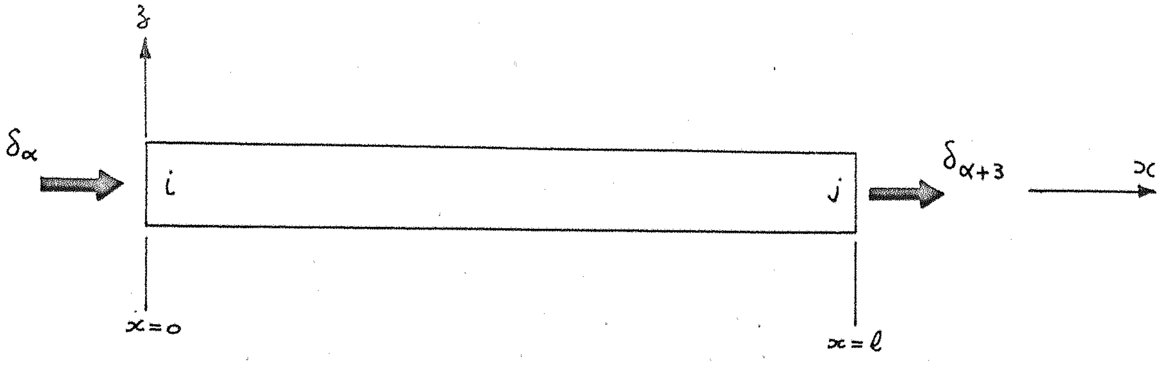
(b)

Fig. 9 .



Positive sign convention for the generalized element boundary loads.

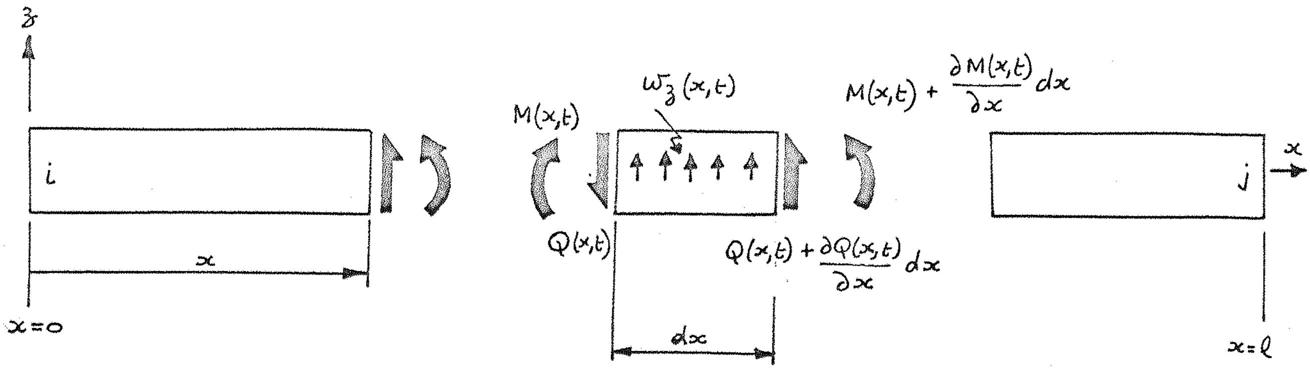
(a)



Positive sign convention for the generalized element boundary displacements.

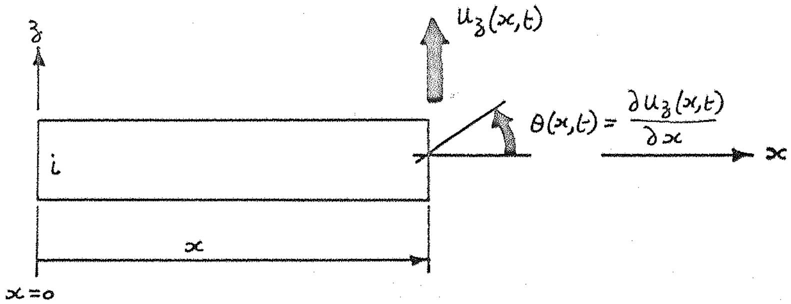
(b)

Fig. 10 .



Positive sign convention for incremental loading.

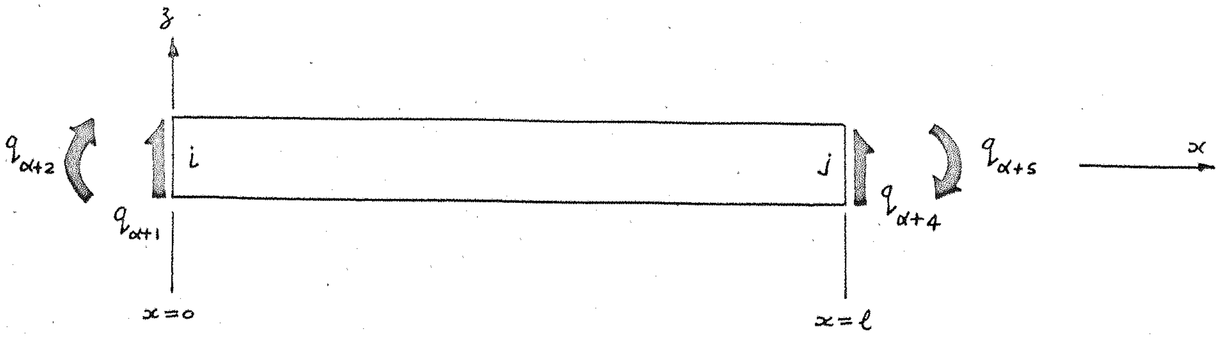
(a)



Positive sign convention for displacement in the z-direction and rotation.

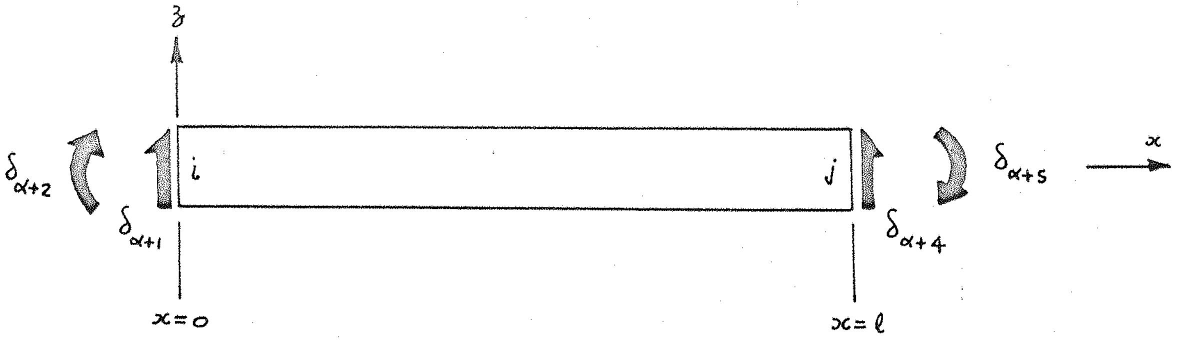
(b)

Fig. 11 .



Positive sign convention for the generalized element boundary loads.

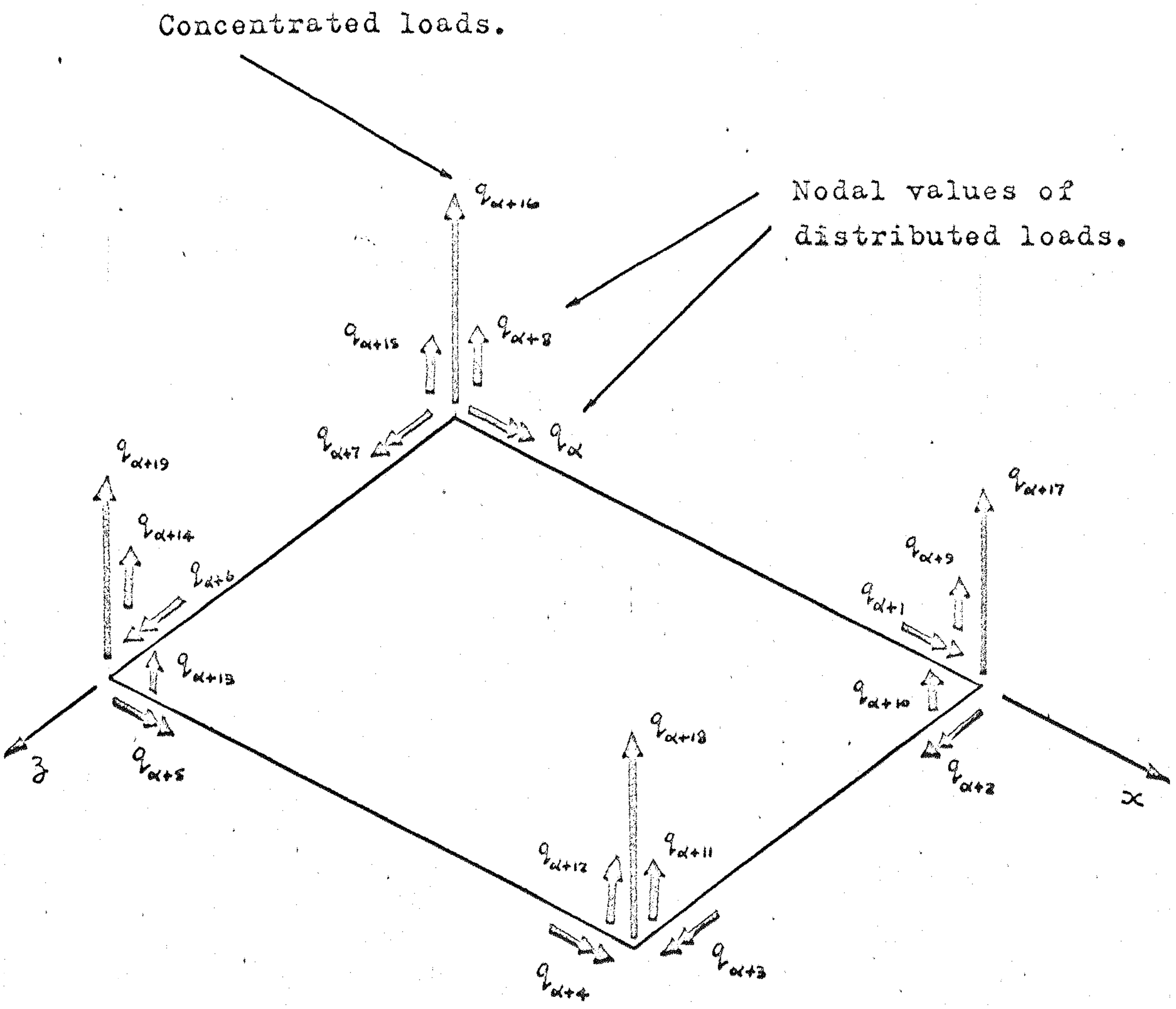
(a)



Positive sign convention for the generalized element boundary displacements.

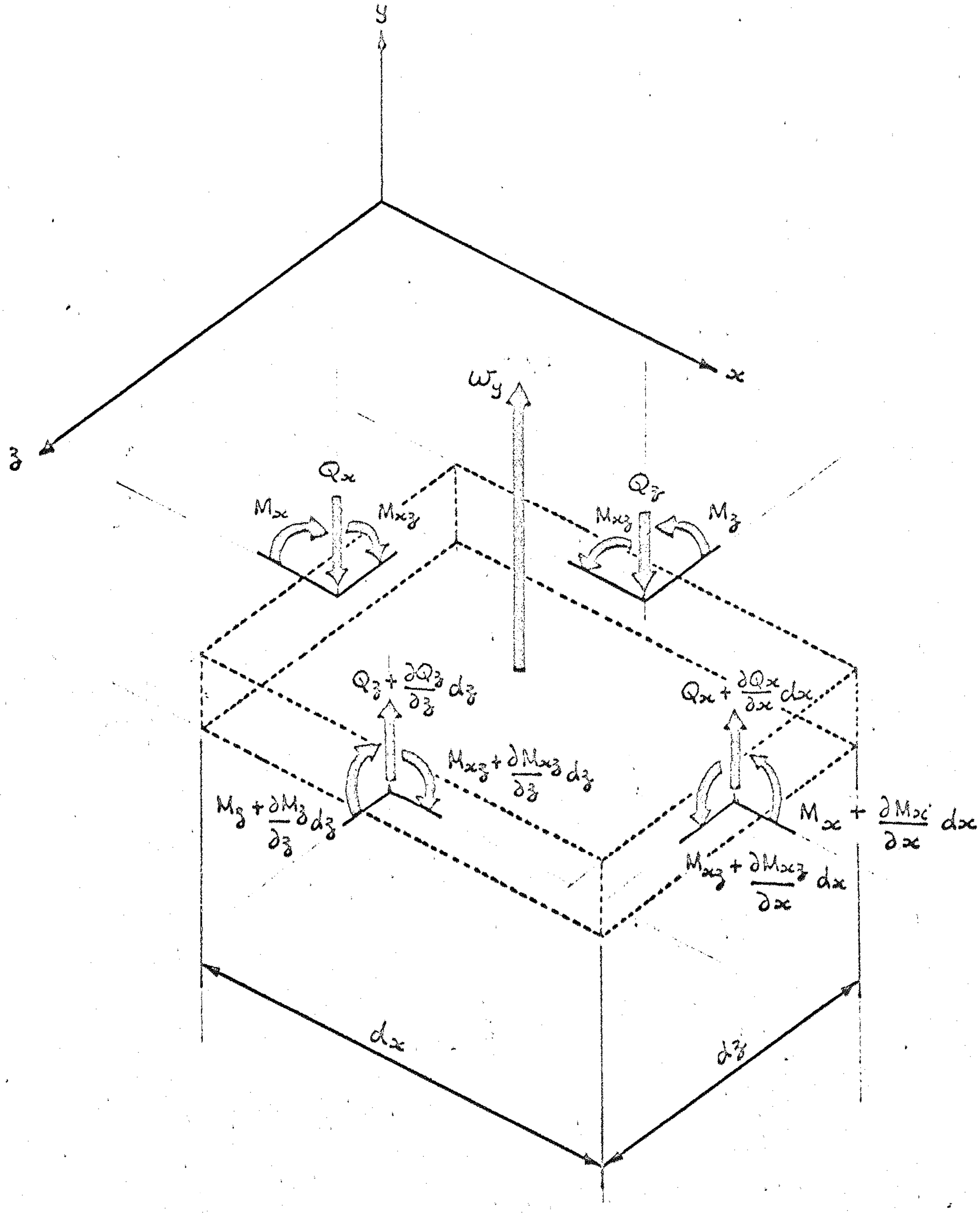
(b)

Fig. 12 .



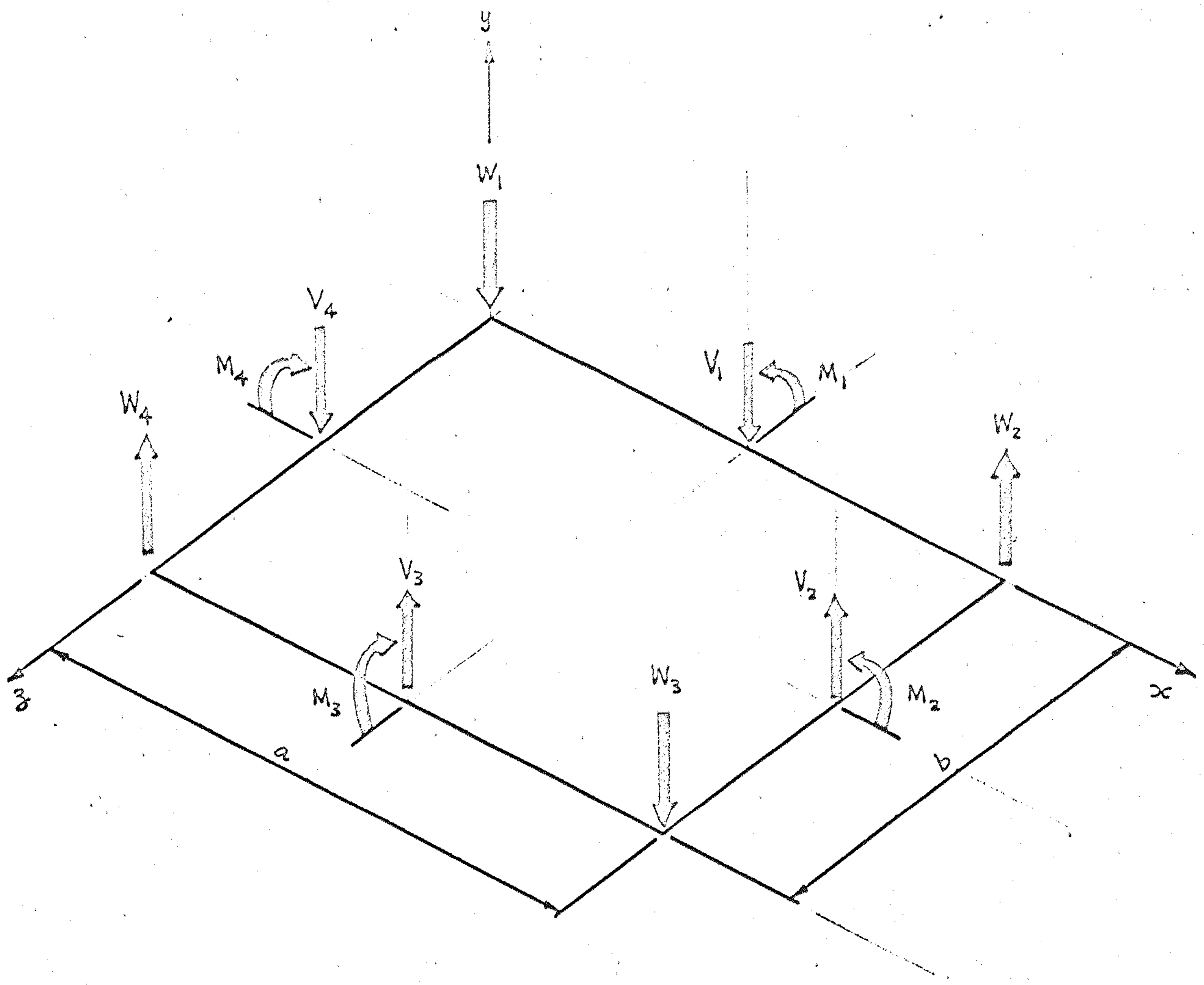
One q-system of generalized element boundary loads. These are shown positive and the corresponding displacements will have the same convention.

Fig. 13 .



Positive sign convention for incremental loading.

Fig. 14 .



V's are equivalent shears (lb/in)

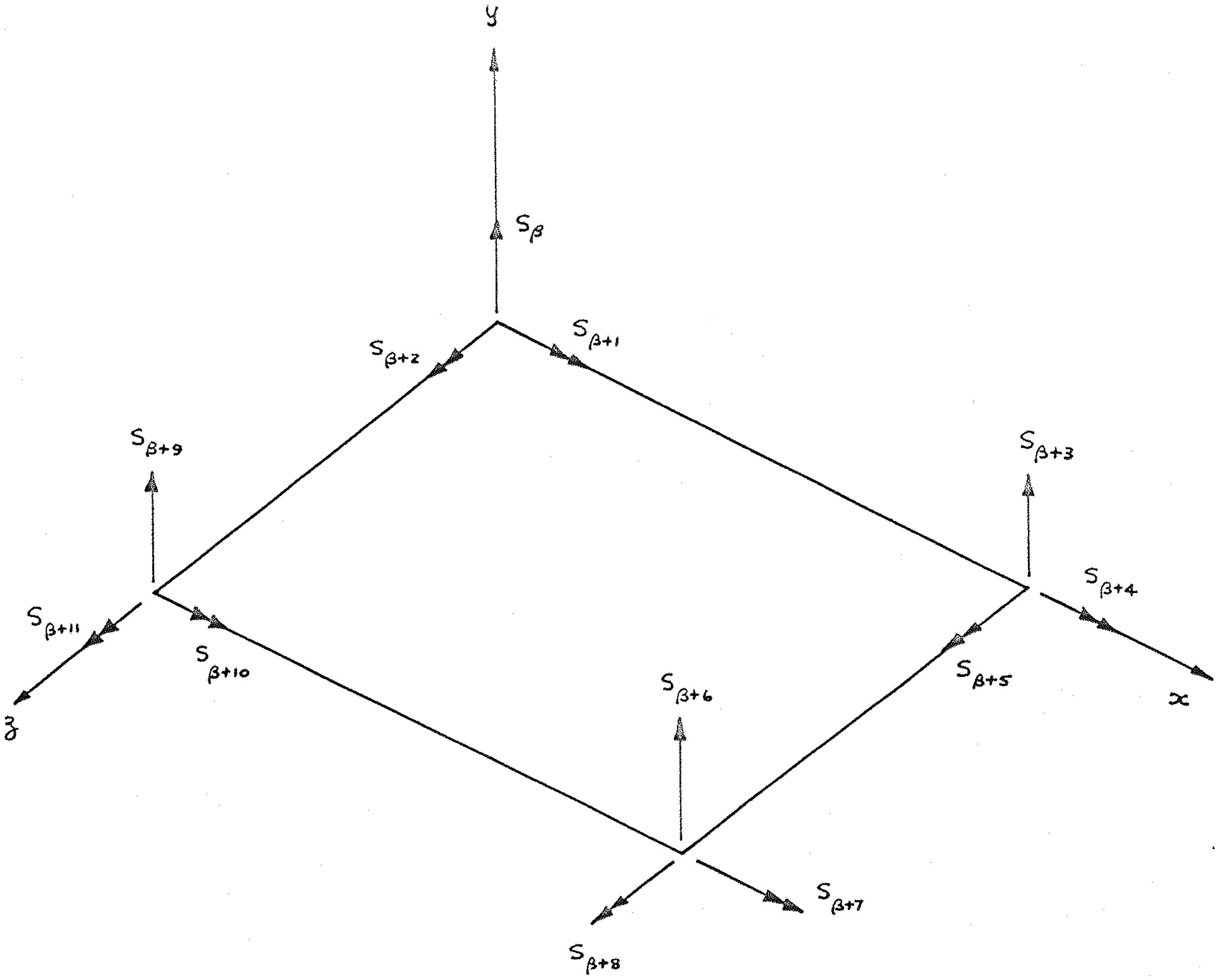
M's are moments (lb.in/in)

W's are concentrated nodal loads (lbs.)

V's and W's are equivalent to the Q_x , Q_z and M_{xz} systems at the plate boundaries.

Equivalent plate loading system.

Fig. 15.



s-system of generalized element boundary loads for plate bending. These are shown positive and the corresponding displacements will have the same convention.

Fig. 16 .

CHAPTER 4.

PARTICULAR ELEMENT DYNAMIC FLEXIBILITY MATRICES.

Synopsis.

Particular element dynamic flexibility matrices are derived in this chapter and a type designation system has been established. Three derivation procedures were presented in Chapter 3 but all the procedures are not applied to the various element types. All three procedures have been applied to a plane beam element, shear and bending, which is used in the analysis of collinear beam structures. Based on the results for such structural configurations the second procedure, see Chapter 3, was disregarded for other types of elements. Procedure 3 is a so called exact solution and is only used for beam type elements, that is, the plane beam element and the inclined plane beam element. Procedure 1 was applied to a rectangular plate element whose generalized element boundary load vector contains twelve terms. In this plate element derivation the bending moments and equivalent shears are assumed constant along the respective boundaries. This element has been used to analyse two dimensional plate structures and the results have been compared with those obtained by alternative methods. The inclined beam element has been used for the analysis of general plane frames and again a comparison of the results have been made. See Chapter 5 for results and their discussion.

Introduction.

In this chapter particular element dynamic flexibility matrices will be derived using the procedures presented in Chapter 3. In each case the section and elastic properties are assumed constant throughout the element. In all derivations only amplitude values are considered.

An element type designation system will now be established for ease of reference. This is given by,

Element Type PI/NFD

where,

PI = procedure 'I' adopted to derive the element dynamic flexibility matrix. In the work contained in this chapter 'I' will take on values from 1 to 3.

N = number given to an element type derived using procedure 'I'.

FD = dynamic flexibility matrix.

4.1 Derivation Procedure 1.

4.1.1 Endload Element (Element Type P1/1FD).

The endload distribution for the endload element shown in figure 10 will be assumed linear, that is,

$$P(x) = A_1 + A_2(x) \qquad 4.1.1.1$$

The boundary conditions for evaluation of the constant terms are,

$$\left.\begin{aligned} P(x) &= -q_\alpha \quad \text{at } x=0 \\ \text{and} \\ P(x) &= q_{\alpha+3} \quad \text{at } x=\ell \end{aligned}\right] \qquad 4.1.1.2$$

The resulting endload distribution is given by,

$$P(x) = \lfloor \; (-1+\tfrac{x}{\ell}) \quad (\tfrac{x}{\ell}) \; \rfloor \{ q_\alpha \quad q_{\alpha+3} \} \qquad 4.1.1.3$$

or,in contracted matrix form,

$$P(x) = \lfloor T_{pq} \rfloor \{ q_{m\alpha} \}$$

where,

$$\lfloor T_{pq} \rfloor = \lfloor \; (-1+\tfrac{x}{\ell}) \quad (\tfrac{x}{\ell}) \; \rfloor \qquad 4.1.1.4$$

The displacement function required for the complementary work done by the virtual inertia loading is obtained from the relationship,

$$u_x(x) = -\frac{1}{\rho\omega^2}\frac{dP(x)}{dx}$$

From equation 4.1.1.3,

$$\frac{dP(x)}{dx} = \lfloor \frac{1}{\ell} \quad \frac{1}{\ell} \rfloor \{ q_\alpha \quad q_{\alpha+3} \} \qquad 4.1.1.5$$

or, in contracted form,

$$\frac{dP(x)}{dx} = \lfloor T_{uq} \rfloor \{ q_{m\alpha} \}$$

where,

$$\lfloor T_{uq} \rfloor = \lfloor \frac{1}{\ell} \quad \frac{1}{\ell} \rfloor \qquad 4.1.1.6$$

Therefore, using equation 3.1.1.21, the element dynamic flexibility matrix is given by,

$$[f_{md}] = \frac{1}{AE}\int_0^\ell \{T_{pq}\}\lfloor T_{pq}\rfloor\, dx - \frac{1}{\rho\omega^2}\int_0^\ell \{T_{uq}\}\lfloor T_{uq}\rfloor\, dx$$

$$= \frac{1}{AE}\int_0^\ell \begin{bmatrix} (-1+\frac{x}{\ell}) \\ (\frac{x}{\ell}) \end{bmatrix} \begin{bmatrix} (-1+\frac{x}{\ell}) & (\frac{x}{\ell}) \end{bmatrix} dx - \frac{1}{\rho\omega^2}\int_0^\ell \begin{bmatrix} (\frac{1}{\ell}) \\ (\frac{1}{\ell}) \end{bmatrix} \begin{bmatrix} (\frac{1}{\ell}) & (\frac{1}{\ell}) \end{bmatrix} dx$$

Therefore,

$$[f_{md}] = \frac{\ell}{6AE} \left[\begin{array}{c|c} 2 & -1 \\ \hline -1 & 2 \end{array}\right] - \frac{1}{\ell\rho\omega^2} \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & 1 \end{array}\right]$$

4.1.1.7

4.1.2 Plane Beam Element (Element Type P1/2FD).

It will be assumed that the internal bending moment varies as a 3rd degree polynomial, that is,

$$M(x) = A_1 + A_2 x + A_3 x^2 + A_4 x^3 \qquad 4.1.2.1$$

Therefore,

$$Q(x) = -(A_2 + 2A_3 x + 3A_4 x^2) \qquad 4.1.2.2$$

Using the boundary conditions,

$$\left.\begin{array}{l} \left.\begin{array}{l} M(x) = q_{\alpha+2} \\ Q(x) = -q_{\alpha+1} \end{array}\right] \text{ at } x=0 \\ \left.\begin{array}{l} M(x) = -q_{\alpha+5} \\ Q(x) = q_{\alpha+4} \end{array}\right] \text{ at } x=\ell \end{array}\right] \qquad 4.1.2.3$$

to evaluate the constant terms results in the moment distribution,

$$M(x) = \lfloor T_{Mq} \rfloor \{ q_{m\alpha} \} \qquad 4.1.2.4$$

where,

$$\lfloor T_{Mq} \rfloor = \lfloor \left(x - 2\frac{x^2}{\ell} + \frac{x^3}{\ell^2}\right) \left(1 - 3\frac{x^2}{\ell^2} + 2\frac{x^3}{\ell^3}\right) \left(\frac{x^2}{\ell} - \frac{x^3}{\ell^2}\right) \left(-3\frac{x^2}{\ell^2} + 2\frac{x^3}{\ell^3}\right) \rfloor$$

4.1.2.5

Differentiating equation 4.1.2.4 twice with respect to x gives,

$$\frac{d^2 M(x)}{dx^2} = \lfloor T_{uq} \rfloor \{ q_{m\alpha} \}$$

where,

$$\lfloor T_{uq} \rfloor = \lfloor \left(-\frac{4}{\ell} + \frac{6x}{\ell^2}\right) \left(-\frac{6}{\ell^2} + \frac{12x}{\ell^3}\right) \left(\frac{2}{\ell} - \frac{6x}{\ell^2}\right) \left(-\frac{6}{\ell^2} + \frac{12x}{\ell^3}\right) \rfloor$$

4.1.2.6

The element static flexibility matrix is given by,

$$[f_m] = \frac{1}{EI} \int_0^{\ell} \{T_{Mq}\} \lfloor T_{Mq} \rfloor \, dx$$

and the element inverse mass matrix by,

$$[m_{mf}] = \frac{1}{\rho} \int_0^{\ell} \{T_{uq}\} \lfloor T_{uq} \rfloor \, dx$$

The element dynamic flexibility matrix is found using the relationship,

$$[f_{md}] = [f_m] - \frac{1}{\omega^2} [m_{mf}]$$

Therefore, making the respective substitutions and integrating results in,

$$[f_{md}] = \frac{\ell}{420\,EI} \begin{bmatrix} 4\ell^2 & 22\ell & 3\ell^2 & -13\ell \\ 22\ell & 156 & 13\ell & -54 \\ 3\ell^2 & 13\ell & 4\ell^2 & -22\ell \\ -13\ell & -54 & -22\ell & 156 \end{bmatrix} - \frac{1}{\rho\omega^2} \cdot \frac{2}{\ell^3} \begin{bmatrix} 2\ell^2 & 3\ell & -\ell^2 & 3\ell \\ 3\ell & 6 & -3\ell & 6 \\ -\ell^2 & -3\ell & 2\ell^2 & -3\ell \\ 3\ell & 6 & -3\ell & 6 \end{bmatrix}$$

4.1.2.7

This can be written as,

$$[f_{md}] = \frac{\ell}{420EI} \cdot \frac{1}{\lambda} \begin{bmatrix} \ell^2(4\lambda-2) & \ell(22\lambda-3) & \ell^2(3\lambda+1) & \ell(-13\lambda-3) \\ \ell(22\lambda-3) & (156\lambda-6) & \ell(13\lambda+3) & (-54\lambda-6) \\ \ell^2(3\lambda+1) & \ell(13\lambda+3) & \ell^2(4\lambda-2) & \ell(-22\lambda+3) \\ \ell(-13\lambda-3) & (-54\lambda-6) & \ell(-22\lambda+3) & (156\lambda-6) \end{bmatrix}$$

4.1.2.8

where,

$$\lambda = \frac{\rho \omega^2 \ell^4}{840EI} \quad \text{and} \quad \rho = \frac{\mu A}{386 \cdot 4} \qquad 4.1.2.9$$

4.1.3 Inclined Plane Beam Element (Element Type P1/3FD).

In all derivations the generalized element boundary loads and corresponding displacements are relative to the global axes $(\bar{x},\bar{y},\bar{z})$. For element types P1/1FD and P1/2FD the local axes (x,y,z) and the global axes have been assumed the same. In order to develop an inclined beam element for general plane frame analysis the element dynamic flexibility matrix will be derived relative to the local axes and then transformed relative to the global axes. The generalized element boundary loads and corresponding displacements relative to the local axes will be denoted by $\{q^{*}_{M\alpha}\}$ and $\{\delta^{*}_{M\alpha}\}$, amplitudes only. Figure 17 shows the two systems. An inclined beam element showing local and global axes is given in figure 18. The dynamic flexibility matrix for an inclined beam element, relative to the local axes, is derived by assembling the dynamic flexibility matrices for element types P1/1FD and P1/2FD. Therefore,

$$[f^*_{md}] = \begin{bmatrix} \frac{\ell}{3AE} - \frac{1}{\ell\rho\omega^2} & 0 & 0 & -\frac{\ell}{6AE} - \frac{1}{\ell\rho\omega^2} & 0 & 0 \\ 0 & \frac{\ell^3}{420EI}\cdot\frac{(4\lambda-2)}{\lambda} & \frac{\ell^2}{420EI}\cdot\frac{(22\lambda-3)}{\lambda} & 0 & \frac{\ell^3}{420EI}\cdot\frac{(3\lambda+1)}{\lambda} & \frac{\ell^2}{420EI}\cdot\frac{(-13\lambda-3)}{\lambda} \\ 0 & \frac{\ell^2}{420EI}\cdot\frac{(22\lambda-3)}{\lambda} & \frac{\ell}{420EI}\frac{(156\lambda-6)}{\lambda} & 0 & \frac{\ell^2}{420EI}\cdot\frac{(13\lambda+3)}{\lambda} & \frac{\ell}{420EI}\cdot\frac{(-54\lambda-6)}{\lambda} \\ -\frac{\ell}{6AE} - \frac{1}{\ell\rho\omega^2} & 0 & 0 & \frac{\ell}{3AE} - \frac{1}{\ell\rho\omega^2} & 0 & 0 \\ 0 & \frac{\ell^3}{420EI}\cdot\frac{(3\lambda+1)}{\lambda} & \frac{\ell^2}{420EI}\cdot\frac{(13\lambda+3)}{\lambda} & 0 & \frac{\ell^3}{420EI}\cdot\frac{(4\lambda-2)}{\lambda} & \frac{\ell^2}{420EI}\cdot\frac{(-22\lambda+3)}{\lambda} \\ 0 & \frac{\ell^2}{420EI}\cdot\frac{(-13\lambda-3)}{\lambda} & \frac{\ell}{420EI}\cdot\frac{(-54\lambda-6)}{\lambda} & 0 & \frac{\ell^2}{420EI}\cdot\frac{(-22\lambda+3)}{\lambda} & \frac{\ell}{420EI}\cdot\frac{(156\lambda-6)}{\lambda} \end{bmatrix}$$

4.1.3.1

The generalized element boundary load vector relative to the local axes is,

$$\{ q^*_{m\alpha} \} = \{ q^*_{\alpha} \;\; q^*_{\alpha+1} \;\; q^*_{\alpha+2} \;\; q^*_{\alpha+3} \;\; q^*_{\alpha+4} \;\; q^*_{\alpha+5} \} \qquad 4.1.3.2$$

The generalized element boundary load vector relative to the global axes is,

$$\{ q_{m\alpha} \} = \{ q_{\alpha} \;\; q_{\alpha+1} \;\; q_{\alpha+2} \;\; q_{\alpha+3} \;\; q_{\alpha+4} \;\; q_{\alpha+5} \} \qquad 4.1.3.3$$

The two element load systems are related by a simple transformation matrix, rotation of axes, that is,

$$\{ q^{*}_{m\alpha} \} = [C_m] \{ q_{m\alpha} \} \tag{4.1.3.4}$$

where,

$$[C_m] = \begin{bmatrix} \frac{(\bar{x}_j - \bar{x}_i)}{l} & \frac{(\bar{z}_j - \bar{z}_i)}{l} & 0 & 0 & 0 & 0 \\ -\frac{(\bar{z}_j - \bar{z}_i)}{l} & \frac{(\bar{x}_j - \bar{x}_i)}{l} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(\bar{x}_j - \bar{x}_i)}{l} & \frac{(\bar{z}_j - \bar{z}_i)}{l} & 0 \\ 0 & 0 & 0 & -\frac{(\bar{z}_j - \bar{z}_i)}{l} & \frac{(\bar{x}_j - \bar{x}_i)}{l} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.1.3.5}$$

The element dynamic flexibility relative to the global axes is found by equating the element energy formulated in terms of the respective element load systems.

Therefore, the element dynamic flexibility matrix for an inclined plane beam element is given by,

$$[f_{md}] = [C_m]^T [f_{md}^*][C_m] \qquad 4.1.3.6$$

Therefore,

$$[f_{md}] = \begin{bmatrix} f(1,1) & f(1,2) & f(1,3) & f(1,4) & f(1,5) & f(1,6) \\ & f(2,2) & f(2,3) & f(1,5) & f(2,5) & f(2,6) \\ & & f(3,3) & -f(1,6) & -f(2,6) & f(3,6) \\ & & & f(1,1) & f(1,2) & -f(1,3) \\ & \text{SYM} & & & f(2,2) & -f(2,3) \\ & & & & & f(3,3) \end{bmatrix}$$

4.1.3.7

where,

$$f(1,1) = \left(\frac{\bar{x}_j - \bar{x}_i}{\ell}\right)^2 \left(\frac{\ell}{3AE} - \frac{1}{\ell\rho\omega^2}\right) + \left(\frac{\bar{z}_j - \bar{z}_i}{\ell}\right)^2 \frac{\ell^3}{420EI}\,\frac{(4\lambda-2)}{\lambda}$$

$$f(2,2) = \left(\frac{\bar{z}_j - \bar{z}_i}{\ell}\right)^2 \left(\frac{\ell}{3AE} - \frac{1}{\ell\rho\omega^2}\right) + \left(\frac{\bar{x}_j - \bar{x}_i}{\ell}\right)^2 \frac{\ell^3}{420EI}\,\frac{(4\lambda-2)}{\lambda}$$

$$f(3,3) = \frac{\ell}{420EI}\,\frac{(156\lambda-6)}{\lambda}$$

$$f(1,2) = \frac{(\bar{x}_j-\bar{x}_i)}{\ell}\,\frac{(\bar{z}_j-\bar{z}_i)}{\ell}\left(\frac{\ell}{3AE} - \frac{1}{\ell\rho\omega^2} - \frac{\ell^3}{420EI}\,\frac{(4\lambda-2)}{\lambda}\right)$$

$$f(1,3) = -\frac{(\bar{z}_j-\bar{z}_i)}{\ell}\,\frac{\ell^2}{420EI}\,\frac{(22\lambda-3)}{\lambda}$$

$$f(1,4) = \left(\frac{\bar{x}_j-\bar{x}_i}{\ell}\right)^2\left(-\frac{\ell}{6AE} - \frac{1}{\ell\rho\omega^2}\right) + \left(\frac{\bar{z}_j-\bar{z}_i}{\ell}\right)^2\frac{\ell^3}{420EI}\,\frac{(3\lambda+1)}{\lambda}$$

$$f(1,5) = \frac{(\bar{x}_j-\bar{x}_i)}{\ell}\,\frac{(\bar{z}_j-\bar{z}_i)}{\ell}\left(-\frac{\ell}{6AE} - \frac{1}{\ell\rho\omega^2} - \frac{\ell^3}{420EI}\,\frac{(3\lambda+1)}{\lambda}\right)$$

$$f(1,6) = \frac{(\bar{z}_j-\bar{z}_i)}{\ell}\,\frac{\ell^2}{420EI}\,\frac{(13\lambda+3)}{\lambda}$$

$$f(2,3) = \frac{(\bar{x}_j-\bar{x}_i)}{\ell}\,\frac{\ell^2}{420EI}\,\frac{(22\lambda-3)}{\lambda}$$

$$f(2,5) = \left(\frac{\bar{z}_j-\bar{z}_i}{\ell}\right)^2\left(-\frac{\ell}{6AE} - \frac{1}{\ell\rho\omega^2}\right) + \left(\frac{\bar{x}_j-\bar{x}_i}{\ell}\right)^2\frac{\ell^3}{420EI}\,\frac{(3\lambda+1)}{\lambda}$$

$$f(2,6) = -\frac{(\bar{x}_j-\bar{x}_i)}{\ell}\,\frac{\ell^2}{420EI}\,\frac{(13\lambda+3)}{\lambda}$$

$$f(3,6) = \frac{\ell}{420EI}\,\frac{(-54\lambda-6)}{\lambda}$$

and

$$\ell = \left((\bar{x}_j-\bar{x}_i)^2 + (\bar{z}_j-\bar{z}_i)^2\right)^{\frac{1}{2}}, \quad \lambda = \frac{\rho\omega^2\ell^4}{840EI}, \quad \rho = \frac{\mu A}{386.4}$$

4.1.4 Rectangular Plate Element (Element Type P1/4FD).

The q-system of generalized element boundary loads will be adopted for this element type. It is assumed for this particular plate element that the distributed boundary loadings are uniform along each respective boundary. The generalized element boundary load system is shown in figure 19. In the derivation of the element dynamic flexibility matrix non-dimensional forms of the relevant elasticity equations will be used. Therefore,

$$\left.\begin{aligned}
Q_\xi &= -\frac{1}{a}\frac{\partial M_\xi}{\partial \xi} + \frac{1}{b}\frac{\partial M_{\xi\eta}}{\partial \eta} \\
Q_\eta &= \frac{1}{a}\frac{\partial M_{\xi\eta}}{\partial \xi} - \frac{1}{b}\frac{\partial M_\eta}{\partial \eta} \\
V_1 &= \left(Q_\eta + \frac{1}{a}\frac{\partial M_{\xi\eta}}{\partial \xi} \right)_{\eta=0} \\
V_2 &= \left(Q_\xi + \frac{1}{b}\frac{\partial M_{\xi\eta}}{\partial \eta} \right)_{\xi=1} \\
V_3 &= \left(Q_\eta + \frac{1}{a}\frac{\partial M_{\xi\eta}}{\partial \xi} \right)_{\eta=1} \\
V_4 &= \left(Q_\xi + \frac{1}{b}\frac{\partial M_{\xi\eta}}{\partial \eta} \right)_{\xi=0} \\
W_1 &= 2\left(M_{\xi\eta} \right)_{\xi=0,\,\eta=0} \\
W_2 &= 2\left(M_{\xi\eta} \right)_{\xi=1,\,\eta=0} \\
W_3 &= 2\left(M_{\xi\eta} \right)_{\xi=1,\,\eta=1} \\
W_4 &= 2\left(M_{\xi\eta} \right)_{\xi=0,\,\eta=1}
\end{aligned}\right] \qquad 4.1.4.1$$

The assumed moment distributions are,

$$\left.\begin{aligned} M_\xi &= A_1 + A_2\xi + A_3\xi^2 + A_4\xi^3 \\ M_\eta &= A_5 + A_6\eta + A_7\eta^2 + A_8\eta^3 \\ M_{\xi\eta} &= A_9 + A_{10}\xi + A_{11}\eta + A_{12}\xi\eta \end{aligned}\right] \quad 4.1.4.2$$

Therefore,

$$\frac{\partial M_\xi}{\partial \xi} = A_2 + 2A_3\xi + 3A_4\xi^2$$

$$\frac{\partial M_\eta}{\partial \eta} = A_6 + 2A_7\eta + 3A_8\eta^2$$

$$\frac{\partial M_{\xi\eta}}{\partial \xi} = A_{10} + A_{12}\eta$$

$$\frac{\partial M_{\xi\eta}}{\partial \eta} = A_{11} + A_{12}\xi$$

Hence,

$$\left.\begin{aligned} Q_\xi &= -\frac{1}{a}\left(A_2 + 2A_3\xi + 3A_4\xi^2\right) + \frac{1}{b}\left(A_{11} + A_{12}\xi\right) \\ Q_\eta &= -\frac{1}{b}\left(A_6 + 2A_7\eta + 3A_8\eta^2\right) + \frac{1}{a}\left(A_{10} + A_{12}\eta\right) \\ V_1 &= \left(-\frac{A_6}{b} + \frac{2A_{10}}{a}\right) \\ V_2 &= \left(-\frac{A_2}{a} - \frac{2}{a}A_3 - \frac{3}{a}A_4\right) + \frac{2}{b}\left(A_{11} + A_{12}\right) \end{aligned}\right]$$

$$V_3 = \left(\frac{-A_6}{b} - \frac{2}{b} A_7 - \frac{3}{b} A_8 \right) + \frac{2}{a} \left(A_{10} + A_{12} \right)$$

$$V_4 = \left(\frac{-A_2}{a} + \frac{2}{b} A_{11} \right)$$

$$W_1 = 2A_9$$

$$W_2 = 2 \left(A_9 + A_{10} \right)$$

$$W_3 = 2 \left(A_9 + A_{10} + A_{11} + A_{12} \right)$$

$$W_4 = 2 \left(A_9 + A_{11} \right)$$

4.1.4.3

The constant terms are evaluated using equations 4.1.4.2 and 4.1.4.3 with the following boundary load conditions,

1. $M_\xi = -q_{\alpha+3} \quad at \quad \xi = 0$
2. $M_\xi = q_{\alpha+1} \quad at \quad \xi = 1$
3. $M_\zeta = q_\alpha \quad at \quad \zeta = 0$
4. $M_\zeta = -q_{\alpha+2} \quad at \quad \zeta = 1$
5. $V_1 = -q_{\alpha+4} \quad at \quad \xi = 0$
6. $V_2 = q_{\alpha+5} \quad at \quad \xi = 1$

7. $V_3 = q_{\alpha+6}$ at $\eta = 1$

8. $V_4 = -q_{\alpha+7}$ at $\xi = 0$ 4.1.4.4

9. $W_1 = -q_{\alpha+8}$ at $\xi = 0, \eta = 0$

10. $W_2 = q_{\alpha+9}$ at $\xi = 1, \eta = 0$

11. $W_3 = -q_{\alpha+10}$ at $\xi = 1, \eta = 1$

12. $W_4 = q_{\alpha+11}$ at $\xi = 0, \eta = 1$

Therefore,

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \\ A_{10} \\ A_{11} \\ A_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & \frac{a}{b} & 0 & 0 & \frac{a}{b} \\ 0 & 3 & 0 & 3 & 0 & a & 0 & -2a & -2\frac{a}{b} & \frac{a}{b} & \frac{a}{b} & -2\frac{a}{b} \\ 0 & -2 & 0 & -2 & 0 & -a & 0 & a & \frac{a}{b} & -\frac{a}{b} & -\frac{a}{b} & \frac{a}{b} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & \frac{b}{a} & \frac{b}{a} & 0 & 0 \\ -3 & 0 & -3 & 0 & -2b & 0 & b & 0 & -2\frac{b}{a} & -2\frac{b}{a} & \frac{b}{a} & \frac{b}{a} \\ 2 & 0 & 2 & 0 & b & 0 & -b & 0 & \frac{b}{a} & \frac{b}{a} & -\frac{b}{a} & -\frac{b}{a} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} q_{\alpha} \\ q_{\alpha+1} \\ q_{\alpha+2} \\ q_{\alpha+3} \\ q_{\alpha+4} \\ q_{\alpha+5} \\ q_{\alpha+6} \\ q_{\alpha+7} \\ q_{\alpha+8} \\ q_{\alpha+9} \\ q_{\alpha+10} \\ q_{\alpha+11} \end{bmatrix}$$

4.1.4.5

Substituting for the constant terms in equation 4.1.4.2 gives,

$$M_\xi = \begin{bmatrix} 0 \\ 3\xi^2-2\xi^3 \\ 0 \\ -1+3\xi^2-2\xi^3 \\ 0 \\ a(\xi^2-\xi^3) \\ 0 \\ a(\xi-2\xi^2+\xi^3) \\ \frac{a}{b}(\xi-2\xi^2+\xi^3) \\ \frac{a}{b}(\xi^2-\xi^3) \\ \frac{a}{b}(\xi^2-\xi^3) \\ \frac{a}{b}(\xi-2\xi^2+\xi^3) \end{bmatrix}^T \{q_{m\alpha}\} \; ; \; \frac{\partial^2 M_x}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 M_\xi}{\partial \xi^2} = \begin{bmatrix} 0 \\ \frac{1}{a^2}(6-12\xi) \\ 0 \\ \frac{1}{a^2}(6-12\xi) \\ 0 \\ \frac{1}{a}(2-6\xi) \\ 0 \\ \frac{1}{a}(-4+6\xi) \\ \frac{1}{ab}(-4+6\xi) \\ \frac{1}{ab}(2-6\xi) \\ \frac{1}{ab}(2-6\xi) \\ \frac{1}{ab}(-4+6\xi) \end{bmatrix}^T \{q_{m\alpha}\}$$

$$\xleftarrow{\quad 4.1.4.6 \quad} \qquad \xrightarrow{\quad 4.1.4.7 \quad}$$

$$M_\eta = \begin{bmatrix} 1-3\eta^2+2\eta^3 \\ 0 \\ -3\eta^2+2\eta^3 \\ 0 \\ b(\eta-2\eta^2+\eta^3) \\ 0 \\ b(\eta^2-\eta^3) \\ 0 \\ \frac{b}{a}(\eta-2\eta^2+\eta^3) \\ \frac{b}{a}(\eta-2\eta^2+\eta^3) \\ \frac{b}{a}(\eta^2-\eta^3) \\ \frac{b}{a}(\eta^2-\eta^3) \end{bmatrix}^T \{q_{m\alpha}\} \; ; \; \frac{\partial^2 M_y}{\partial y^2} = \frac{1}{b^2}\frac{\partial^2 M_\eta}{\partial \eta^2} = \begin{bmatrix} \frac{1}{b^2}(-6+12\eta) \\ 0 \\ \frac{1}{b^2}(-6+12\eta) \\ 0 \\ \frac{1}{b}(-4+6\eta) \\ 0 \\ \frac{1}{b}(2-6\eta) \\ 0 \\ \frac{1}{ab}(-4+6\eta) \\ \frac{1}{ab}(-4+6\eta) \\ \frac{1}{ab}(2-6\eta) \\ \frac{1}{ab}(2-6\eta) \end{bmatrix}^T \{q_{m\alpha}\}$$

$$\xleftarrow{\quad 4.1.4.8 \quad} \qquad \xrightarrow{\quad 4.1.4.9 \quad}$$

and

$$M_{\xi\eta} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2}(1-\xi-\eta+\xi\eta) \\ \frac{1}{2}(\xi-\xi\eta) \\ -\frac{1}{2}\xi\eta \\ \frac{1}{2}(\eta-\xi\eta) \end{bmatrix}^T \{q_{m\alpha}\} \qquad 4.1.4.10$$

$$2\frac{\partial^2 M_{xz}}{\partial x \partial z} = \frac{2}{ab}\frac{\partial^2 M_{\xi\eta}}{\partial\xi\partial\eta} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{ab} \\ -\frac{1}{ab} \\ -\frac{1}{ab} \\ -\frac{1}{ab} \end{bmatrix}^T \{q_{m\alpha}\} \qquad 4.1.4.11$$

Therefore,

$$\{T_{uq}\} = \begin{bmatrix} \frac{1}{b^2}(-6+12\eta) \\ \frac{1}{a^2}(6-12\xi) \\ \frac{1}{b^2}(-6+12\eta) \\ \frac{1}{a^2}(6-12\xi) \\ \frac{1}{b}(-4+6\eta) \\ \frac{1}{a}(2-6\xi) \\ \frac{1}{b}(2-6\eta) \\ \frac{1}{a}(-4+6\xi) \\ \frac{1}{ab}(6\xi+6\eta-7) \\ \frac{1}{ab}(-6\xi+6\eta-1) \\ \frac{1}{ab}(-6\xi-6\eta+5) \\ \frac{1}{ab}(6\xi-6\eta-1) \end{bmatrix} \qquad 4.1.4.12$$

The element inverse mass matrix is given by,

$$[m_{m\xi}] = \frac{ab}{\rho}\int_0^1\!\!\int_0^1 \left(\frac{1}{a^2}\frac{\partial^2 \tilde{M}_\xi}{\partial \xi^2} - \frac{2}{ab}\frac{\partial^2 \tilde{M}_{\xi\varsigma}}{\partial \xi \partial \varsigma} + \frac{1}{b^2}\frac{\partial^2 \tilde{M}_\varsigma}{\partial \varsigma^2}\right)\left(\frac{1}{a^2}\frac{\partial^2 M_\xi}{\partial \xi^2} - \frac{2}{ab}\frac{\partial^2 M_{\xi\varsigma}}{\partial \xi \partial \varsigma} + \frac{1}{b^2}\frac{\partial^2 M_\varsigma}{\partial \varsigma^2}\right) d\xi\, d\varsigma$$

$$= \frac{ab}{\rho}\int_0^1\!\!\int_0^1 \{T_{uq}\}\lfloor T_{uq}\rfloor\, d\xi\, d\varsigma$$

4.1.4.13

Substituting from equation 4.1.4.12 and integrating results in the element inverse mass matrix,

$$[m_{m\xi}] = \frac{1}{\rho}\begin{bmatrix}
12\frac{a}{b^3} & 0 & 12\frac{a}{b^3} & 0 & 6\frac{a}{b^2} & 0 & -6\frac{a}{b^2} & 0 & \frac{6}{b^2} & \frac{6}{b^2} & -\frac{6}{b^2} & -\frac{6}{b^2} \\
 & 12\frac{b}{a^3} & 0 & 12\frac{b}{a^3} & 0 & 6\frac{b}{a^2} & 0 & -6\frac{b}{a^2} & -\frac{6}{a^2} & \frac{6}{a^2} & \frac{6}{a^2} & -\frac{6}{a^2} \\
 & & 12\frac{a}{b^3} & 0 & 6\frac{a}{b^2} & 0 & -6\frac{a}{b^2} & 0 & \frac{6}{b^2} & \frac{6}{b^2} & -\frac{6}{b^2} & -\frac{6}{b^2} \\
 & & & 12\frac{b}{a^3} & 0 & 6\frac{b}{a^2} & 0 & -6\frac{b}{a^2} & -\frac{6}{a^2} & \frac{6}{a^2} & \frac{6}{a^2} & -\frac{6}{a^2} \\
 & & & & 4\frac{a}{b} & 1 & -2\frac{a}{b} & 1 & \frac{4}{b} & \frac{4}{b} & -\frac{2}{b} & -\frac{2}{b} \\
 & & & & & 4\frac{b}{a} & 1 & -2\frac{b}{a} & -\frac{2}{a} & \frac{4}{a} & \frac{4}{a} & -\frac{2}{a} \\
 & & & & & & 4\frac{a}{b} & 1 & -\frac{2}{b} & -\frac{2}{b} & \frac{4}{b} & \frac{4}{b} \\
 & \text{SYM}^{L} & & & & & & 4\frac{b}{a} & \frac{4}{a} & -\frac{2}{a} & -\frac{2}{a} & \frac{4}{a} \\
 & & & & & & & & \frac{7}{ab} & \frac{1}{ab} & -\frac{5}{ab} & \frac{1}{ab} \\
 & & & & & & & & & \frac{7}{ab} & \frac{1}{ab} & -\frac{5}{ab} \\
 & & & & & & & & & & \frac{7}{ab} & \frac{1}{ab} \\
 & & & & & & & & & & & \frac{7}{ab}
\end{bmatrix}$$

4.1.4.14

The element static flexibility matrix is given by,

$$\lfloor f_m \rfloor = \left(\frac{12}{E t_p^3}\right) ab \int_0^1\int_0^1 \Big(\{T_{\xi Mq}\}\lfloor T_{\xi Mq}\rfloor + \{T_{\eta Mq}\}\lfloor T_{\eta Mq}\rfloor$$

$$+ 2(1+\nu)\{T_{\xi\eta Mq}\}\lfloor T_{\xi\eta Mq}\rfloor - \nu\big(\{T_{\xi Mq}\}\lfloor \underset{\eta}{T_{\eta Mq}}\rfloor + \{T_{\eta Mq}\}\lfloor \underset{\xi}{T_{\xi Mq}}\rfloor\big)\Big)\, d\xi\, d\eta$$

4.1.4.15

where,

$\lfloor T_{\xi Mq}\rfloor$ is given by equation 4.1.4.6

$\lfloor T_{\eta Mq}\rfloor$ is given by equation 4.1.4.8

$\lfloor T_{\xi\eta Mq}\rfloor$ is given by equation 4.1.4.10

Therefore,

$$[f_m] = \left(\frac{12}{E t_p^3}\right) \frac{ab}{420} \times$$

$$\left[\begin{array}{cccccccccccc}
156 & -105\nu & -54 & 105\nu & 22b & -\frac{35}{2}a\nu & 13b & -\frac{35}{2}a\nu & 22\frac{b}{a}-\frac{35}{2}\frac{a}{b}\nu & 22\frac{b}{a}-\frac{35}{2}\frac{a}{b}\nu & 13\frac{b}{a}-\frac{35}{2}\frac{a}{b}\nu & 13\frac{b}{a}-\frac{35}{2}\frac{a}{b}\nu \\
 & 156 & 105\nu & -54 & -\frac{35}{2}b\nu & 22a & -\frac{35}{2}b\nu & 13a & 13\frac{a}{b}-\frac{35}{2}\frac{b}{a}\nu & 22\frac{a}{b}-\frac{35}{2}\frac{b}{a}\nu & 22\frac{a}{b}-\frac{35}{2}\frac{b}{a}\nu & 13\frac{a}{b}-\frac{35}{2}\frac{b}{a}\nu \\
 & & 156 & -105\nu & -13b & \frac{35}{2}a\nu & -22b & \frac{35}{2}a\nu & -13\frac{b}{a}+\frac{35}{2}\frac{a}{b}\nu & -13\frac{b}{a}+\frac{35}{2}\frac{a}{b}\nu & -22\frac{b}{a}+\frac{35}{2}\frac{a}{b}\nu & -22\frac{b}{a}+\frac{35}{2}\frac{a}{b}\nu \\
 & & & 156 & \frac{35}{2}b\nu & -13a & \frac{35}{2}b\nu & -22a & -22\frac{a}{b}+\frac{35}{2}\frac{b}{a}\nu & -13\frac{a}{b}+\frac{35}{2}\frac{b}{a}\nu & -13\frac{a}{b}+\frac{35}{2}\frac{b}{a}\nu & -22\frac{a}{b}+\frac{35}{2}\frac{b}{a}\nu \\
 & & & & 4b^2 & -\frac{35}{12}ab\nu & 3b^2 & -\frac{35}{12}ab\nu & 4\frac{b^2}{a}-\frac{35}{12}a\nu & 4\frac{b^2}{a}-\frac{35}{12}a\nu & 3\frac{b^2}{a}-\frac{35}{12}a\nu & 3\frac{b^2}{a}-\frac{35}{12}a\nu \\
 & & & & & 4a^2 & -\frac{35}{12}ab\nu & 3a^2 & 3\frac{a^2}{b}-\frac{35}{12}b\nu & 4\frac{a^2}{b}-\frac{35}{12}b\nu & 4\frac{a^2}{b}-\frac{35}{12}b\nu & 3\frac{a^2}{b}-\frac{35}{12}b\nu \\
 & & & & & & 4b^2 & -\frac{35}{12}ab\nu & 3\frac{b^2}{a}-\frac{35}{12}a\nu & 3\frac{b^2}{a}-\frac{35}{12}a\nu & 4\frac{b^2}{a}-\frac{35}{12}a\nu & 4\frac{b^2}{a}-\frac{35}{12}a\nu \\
 & & & & & & & 4a^2 & 4\frac{a^2}{b}-\frac{35}{12}b\nu & 3\frac{a^2}{b}-\frac{35}{12}b\nu & 3\frac{a^2}{b}-\frac{35}{12}b\nu & 4\frac{a^2}{b}-\frac{35}{12}b\nu \\
 & \text{SYM}^{\underset{\sim}{L}} & & & & & & & \begin{array}{c}4\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\\+\\\frac{70}{3}\left(1+\frac{3}{4}\nu\right)\end{array} & \begin{array}{c}3\frac{a^2}{b^2}+4\frac{b^2}{a^2}\\-\frac{35}{3}\left(1+\frac{3}{2}\nu\right)\end{array} & \begin{array}{c}3\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\\+\frac{35}{6}\end{array} & \begin{array}{c}4\frac{a^2}{b^2}+3\frac{b^2}{a^2}\\-\frac{35}{3}\left(1+\frac{3}{2}\nu\right)\end{array} \\
 & & & & & & & & & \begin{array}{c}4\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\\+\\\frac{70}{3}\left(1+\frac{3}{4}\nu\right)\end{array} & \begin{array}{c}4\frac{a^2}{b^2}+3\frac{b^2}{a^2}\\-\frac{35}{3}\left(1+\frac{3}{2}\nu\right)\end{array} & \begin{array}{c}3\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\\+\frac{35}{6}\end{array} \\
 & & & & & & & & & & \begin{array}{c}4\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\\+\\\frac{70}{3}\left(1+\frac{3}{4}\nu\right)\end{array} & \begin{array}{c}3\frac{a^2}{b^2}+4\frac{b^2}{a^2}\\-\frac{35}{3}\left(1+\frac{3}{2}\nu\right)\end{array} \\
 & & & & & & & & & & & \begin{array}{c}4\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\\+\\\frac{70}{3}\left(1+\frac{3}{4}\nu\right)\end{array}
\end{array}\right]$$

4.1.4.16

The element dynamic flexibility matrix can now be obtained using the relationship,

$$[f_{md}] = [f_m] - \frac{1}{\omega^2}[m_{mf}]$$

4.1.5 Rectangular Plate Element (Element Type P1/5FD).

This element type will adopt the s-system of generalized element boundary loads as shown in figure 16. The element dynamic flexibility matrix corresponding to this system will be derived using the q-system and corresponding dynamic flexibility matrix as adopted and derived for Element Type P1/4FD.

Figure 20 shows the q-resultants which must be in overall equilibrium with the s-system.

Therefore, the equilibrium equations

$$[\,\Omega_{m\alpha}\,]\{\,_t q_{m\alpha}\,\} + [\,\Psi_{m\beta}\,]\{\,_t S_{m\beta}\,\} = \{0\}$$

are given by,

q-loads												s-loads											
α	$\alpha+1$	$\alpha+2$	$\alpha+3$	$\alpha+4$	$\alpha+5$	$\alpha+6$	$\alpha+7$	$\alpha+8$	$\alpha+9$	$\alpha+10$	$\alpha+11$	β	$\beta+1$	$\beta+2$	$\beta+3$	$\beta+4$	$\beta+5$	$\beta+6$	$\beta+7$	$\beta+8$	$\beta+9$	$\beta+10$	$\beta+11$
				$-a$	$-b$	$-a$	$-b$	-1	-1	-1	-1	1			1			1			1		
$-a$		$-a$			$\frac{b^2}{2}$	ab	$\frac{b^2}{2}$			b	b		1			1		$-b$	1		$-b$	1	
	$-b$		$-b$	$-\frac{a^2}{2}$	$-ab$	$-\frac{a^2}{2}$			$-a$	$-a$				1	a		1	a		1			1

$$= \{0\} \qquad 4.1.5.1$$

The remaining procedure for the derivation of the element dynamic flexibility matrix corresponding to an s-system, Chapter 3 (3.1.3(b)), becomes too involved for manual generation. The derivation is completed by writing the procedure as a computer programme subroutine, the element dynamic flexibility matrix being evaluated within the computer. Equation 4.1.5.1 is written into the subroutine. This subroutine (FMD50) is described in Appendix 4.

4.2 Derivation Procedure 2.

4.2.1 Endload Element (Element Type P2/1FD).

In this procedure the required displacement function is given by equation 3.2.1.3, that is,

$$u_x(x) = \frac{1}{AE}\left(\left(\int \lfloor T_{pq} \rfloor \, dx\right)\{q_{m\alpha}\} + C\right)$$

Substituting from equation 4.1.1.4 gives,

$$u_x(x) = \frac{1}{AE}\left(\lfloor \left(-x+\frac{x^2}{2\ell}\right) \quad \left(\frac{x^2}{2\ell}\right) \rfloor \{q_{m\alpha}\} + C\right) \qquad 4.2.1.1$$

The constant of integration, C , is evaluated using the equilibrium equation,

$$\int_0^\ell \rho\omega^2 u_x(x)\,dx + q_\alpha + q_{\alpha+3} = 0$$

Therefore,

$$C = -\lfloor \left(-\frac{\ell}{3}+\frac{AE}{\ell\rho\omega^2}\right) \quad \left(\frac{\ell}{6}+\frac{AE}{\ell\rho\omega^2}\right) \rfloor \{q_{m\alpha}\} \qquad 4.2.1.2$$

and

$$u_x(x) = \lfloor T_{\omega q} \rfloor \{q_{m\alpha}\} \qquad 4.2.1.3$$

where,

$$\lfloor T_{\omega q} \rfloor = \frac{1}{AE}\left(\lfloor \left(\left(-x+\frac{x^2}{2\ell}+\frac{\ell}{3}\right)-\frac{AE}{\ell\rho\omega^2}\right) \quad \left(\left(\frac{x^2}{2\ell}-\frac{\ell}{6}\right)-\frac{AE}{\ell\rho\omega^2}\right) \rfloor\right) \qquad 4.2.1.4$$

The element inverse mass matrix is derived from the expression for the work done by the virtual inertia loading, that is,

$$\tilde{W}_2^* = \int_0^\ell \left(\rho\widetilde{\omega^2 u_x(x)}\right) u_x(x)\,dx$$

In order to retain the usual form for the element dynamic flexibility matrix, that is,

$$[f_{md}] = [f_m] - \frac{1}{\omega^2}[m_{mf}]$$

the inverse mass matrix is given by,

$$[m_{mf}] = \rho\omega^4 \int_0^{\ell} \{T_{\omega q}\} \lfloor T_{\omega q} \rfloor \, dx \qquad 4.2.1.5$$

Equation 4.2.1.4 can be written as,

$$\lfloor T_{\omega q} \rfloor = \frac{1}{AE}\left(\lfloor T_1 \rfloor + \frac{AE}{\rho\omega^2} \lfloor T_2 \rfloor \right) \qquad 4.2.1.6$$

where,

$$\lfloor T_1 \rfloor = \left\lfloor \left(-x + \frac{x^2}{2\ell} + \frac{\ell}{3}\right) \quad \left(\frac{x^2}{2\ell} - \frac{\ell}{6}\right) \right\rfloor \qquad 4.2.1.7$$

and

$$\lfloor T_2 \rfloor = \left\lfloor \left(-\frac{1}{\ell}\right) \quad \left(-\frac{1}{\ell}\right) \right\rfloor \qquad 4.2.1.8$$

Therefore, the element inverse mass matrix is given by,

$$[m_{mf}] = \frac{\rho\omega^4}{(AE)^2} \int_0^{\ell} \left(\{T_1\} + \frac{AE}{\rho\omega^2}\{T_2\} \right)\left(\lfloor T_1 \rfloor + \frac{AE}{\rho\omega^2} \lfloor T_2 \rfloor \right) dx$$

$$= \frac{\rho\omega^4}{(AE)^2} \int_0^{\ell} \Big(\{T_1\}\lfloor T_1 \rfloor + \frac{AE}{\rho\omega^2}\{T_1\}\lfloor T_2 \rfloor + \frac{AE}{\rho\omega^2}\{T_2\}\lfloor T_1 \rfloor$$

$$+ \frac{(AE)^2}{(\rho\omega^2)^2}\{T_2\}\lfloor T_2 \rfloor \Big) \, dx$$

Hence,

$$[m_{mf}] = \frac{\rho\omega^4}{(AE)^2}\int_0^{\ell} \{T_1\}\lfloor T_1 \rfloor dx + \frac{\omega^2}{AE}\int_0^{\ell}\{T_1\}\lfloor T_2 \rfloor dx$$

$$+ \frac{\omega^2}{AE}\int_0^{\ell}\{T_2\}\lfloor T_1 \rfloor dx + \frac{1}{\rho}\int_0^{\ell}\{T_2\}\lfloor T_2 \rfloor dx$$

Evaluating the integrals results in,

$$\int_0^{\ell}\{T_1\}\lfloor T_2 \rfloor dx = [0]$$

$$\int_0^{\ell}\{T_2\}\lfloor T_1 \rfloor dx = [0]$$

$$\frac{\rho\omega^4}{(AE)^2}\int_0^{\ell}\{T_1\}\lfloor T_1 \rfloor dx = \frac{\rho\ell^3\omega^4}{360\,(AE)^2}\begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix}$$

$$\frac{1}{\rho}\int_0^{\ell}\{T_2\}\lfloor T_2 \rfloor dx = \frac{1}{\rho\ell}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Now, the element static flexibility matrix is the same as element type P1/1FD. Therefore, the element dynamic flexibility matrix is given by,

$$[f_{md}] = [f_m] - \frac{1}{\omega^2}[m_{mf}]$$

$$= [f_m] - \frac{1}{\omega^2}\left(\frac{1}{\rho}\int_0^\ell \{T_2\}\lfloor T_2 \rfloor dx + \frac{\rho\omega^4}{(AE)^2}\int_0^\ell \{T_1\}\lfloor T_1 \rfloor dx \right)$$

Hence,

$$[f_{md}] = \frac{\ell}{6AE}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \frac{1}{\ell\rho\omega^2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$- \frac{\rho\omega^2}{(AE)^2}\frac{\ell^3}{360}\begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix} \qquad 4.2.1.9$$

4.2.2 Plane Beam Element (Element Type P2/2FD).

The required displacement function is obtained using equation 3.2.2.2, that is,

$$u_z(x) = \frac{1}{EI}\left(\left(\iint \lfloor T_{Mq} \rfloor \, dx\, dx\right)\{q_{m\alpha}\} + C_1 x + C_2\right)$$

Substituting from equation 4.1.2.5 gives,

$$u_z(x) = \frac{1}{EI}\Bigg(\Big\lfloor \left(\frac{x^3}{6} - \frac{x^4}{6\ell} + \frac{x^5}{20\ell^2}\right) \left(\frac{x^2}{2} - \frac{x^4}{4\ell^2} + \frac{x^5}{10\ell^3}\right) \left(\frac{x^4}{12\ell} - \frac{x^5}{20\ell^2}\right) \left(-\frac{x^4}{4\ell^2} + \frac{x^5}{10\ell^3}\right)\Big\rfloor \{q_{M\alpha}\}$$

$$+ C_1 x + C_2 \Bigg) \qquad 4.2.2.1$$

The constant terms of integration are evaluated using the equilibrium equations,

$$\int_0^\ell \rho\omega^2 u_z(x)\, dx + q_{\alpha+1} + q_{\alpha+4} = 0$$

and

$$-\int_0^\ell \rho\omega^2 u_z(x)\, x\, dx + q_{\alpha+2} - \ell\, q_{\alpha+4} + q_{\alpha+5} = 0$$

Therefore,

$$C_1 = \Big\lfloor \left(\frac{-11}{210}\ell^2 + \frac{6}{\ell^2}\frac{EI}{\rho\omega^2}\right) \left(-\frac{13}{35}\ell + \frac{12}{\ell^3}\frac{EI}{\rho\omega^2}\right) \left(-\frac{13}{420}\ell^2 - \frac{6}{\ell^2}\frac{EI}{\rho\omega^2}\right)$$

$$\left(\frac{9}{70}\ell + \frac{12}{\ell^3}\frac{EI}{\rho\omega^2}\right)\Big\rfloor \{q_{m\alpha}\} \qquad 4.2.2.2$$

and

$$C_2 = \Big\lfloor \left(\frac{\ell^3}{105} - \frac{4}{\ell}\frac{EI}{\rho\omega^2}\right) \left(\frac{11}{210}\ell^2 - \frac{6}{\ell^2}\frac{EI}{\rho\omega^2}\right) \left(\frac{\ell^3}{140} + \frac{2}{\ell}\frac{EI}{\rho\omega^2}\right)$$

$$\left(-\frac{13}{420}\ell^2 - \frac{6}{\ell^2}\frac{EI}{\rho\omega^2}\right)\Big\rfloor \{q_{m\alpha}\} \qquad 4.2.2.3$$

Hence,

$$u_z(x) = \lfloor T_{wq} \rfloor \{ q_{mx} \} \tag{4.2.2.4}$$

where,

$$\begin{aligned}\lfloor T_{wq} \rfloor = \frac{1}{EI} \Big\lfloor & \left(\left(\frac{x^3}{6} - \frac{x^4}{6\ell} + \frac{x^5}{20\ell^2} \right) + \left(-\frac{11\ell^2 x}{210} + \frac{6}{\ell^2} \frac{EIx}{\rho\omega^2} \right) + \left(\frac{\ell^3}{105} - \frac{4}{\ell} \frac{EI}{\rho\omega^2} \right) \right) \\ & \left(\left(\frac{x^2}{2} - \frac{x^4}{4\ell^2} + \frac{x^5}{10\ell^3} \right) + \left(-\frac{13}{35} \ell x + \frac{12}{\ell^3} \frac{EIx}{\rho\omega^2} \right) + \left(\frac{11}{210} \ell^2 - \frac{6}{\ell^2} \frac{EI}{\rho\omega^2} \right) \right) \\ & \left(\left(\frac{x^4}{12\ell} - \frac{x^5}{20\ell^2} \right) + \left(-\frac{13}{420} \ell^2 x - \frac{6}{\ell^2} \frac{EIx}{\rho\omega^2} \right) + \left(\frac{\ell^3}{140} + \frac{2}{\ell} \frac{EI}{\rho\omega^2} \right) \right) \\ & \left(\left(-\frac{x^4}{4\ell^2} + \frac{x^5}{10\ell^3} \right) + \left(\frac{9}{70} \ell x + \frac{12}{\ell^3} \frac{EIx}{\rho\omega^2} \right) + \left(-\frac{13}{420} \ell^2 - \frac{6}{\ell^2} \frac{EI}{\rho\omega^2} \right) \right) \Big\rfloor \end{aligned} \tag{4.2.2.5}$$

The element inverse mass matrix is given by,

$$[m_{mf}] = \rho\omega^4 \int_0^\ell \{T_{wq}\} \lfloor T_{wq} \rfloor \, dx$$

and the element static flexibility matrix is the same as element type P1/2FD. Therefore the element dynamic flexibility matrix is given by the relationship,

$$[f_{md}] = [f_m] - \frac{1}{\omega^2} [m_{mf}]$$

Hence,

$$[f_{md}] = \frac{\ell}{420EI}\begin{bmatrix} 4\ell^2 & 22\ell & 3\ell^2 & -13\ell \\ 22\ell & 156 & 13\ell & -54 \\ 3\ell^2 & 13\ell & 4\ell^2 & -22\ell \\ -13\ell & -54 & -22\ell & 156 \end{bmatrix} - \frac{1}{\rho\omega^2}\cdot\frac{2}{\ell^3}\begin{bmatrix} 2\ell^2 & 3\ell & -\ell^2 & 3\ell \\ 3\ell & 6 & -3\ell & 6 \\ -\ell^2 & -3\ell & 2\ell^2 & -3\ell \\ 3\ell & 6 & -3\ell & 6 \end{bmatrix}$$

$$-\frac{\rho\omega^2}{(EI)^2}\cdot\frac{\ell^5}{420}\begin{bmatrix} \frac{71}{10395}\ell^2 & \frac{223}{6930}\ell & \frac{1097}{166320}\ell^2 & -\frac{1681}{55440}\ell \\ \frac{223}{6930}\ell & \frac{118}{770} & \frac{1681}{55440}\ell & -\frac{2951}{21560} \\ \frac{1097}{166320}\ell^2 & \frac{1681}{55440}\ell & \frac{71}{10395}\ell^2 & -\frac{223}{6930}\ell \\ -\frac{1681}{55440}\ell & -\frac{2951}{21560} & -\frac{223}{6930}\ell & \frac{118}{770} \end{bmatrix}$$

4.2.2.6

This can be written as,

$$[f_{md}] = \frac{\ell}{420EI}\cdot\frac{1}{\lambda}\begin{bmatrix} \ell^2\left(4\lambda-2-\frac{71\times840}{10395}\lambda^2\right) & \ell\left(22\lambda-3-\frac{223\times840}{6930}\lambda^2\right) & \ell^2\left(3\lambda+1-\frac{1097\times840}{166320}\lambda^2\right) & \ell\left(-13\lambda-3+\frac{1681\times840}{55440}\lambda^2\right) \\ \ell\left(22\lambda-3-\frac{223\times840}{6930}\lambda^2\right) & \left(156\lambda-6-\frac{118\times840}{770}\lambda^2\right) & \ell\left(13\lambda+3-\frac{1681\times840}{55440}\lambda^2\right) & \left(-54\lambda-6+\frac{2951\times840}{21560}\lambda^2\right) \\ \ell^2\left(3\lambda+1-\frac{1097\times840}{166320}\lambda^2\right) & \ell\left(13\lambda+3-\frac{1681\times840}{55440}\lambda^2\right) & \ell^2\left(4\lambda-2-\frac{71\times840}{10395}\lambda^2\right) & \ell\left(-22\lambda+3+\frac{223\times840}{6930}\lambda^2\right) \\ \ell\left(-13\lambda-3+\frac{1681\times840}{55440}\lambda^2\right) & \left(-54\lambda-6+\frac{2951\times840}{21560}\lambda^2\right) & \ell\left(-22\lambda+3+\frac{223\times840}{6930}\lambda^2\right) & \left(156\lambda-6-\frac{118\times840}{770}\lambda^2\right) \end{bmatrix}$$

4.2.2.7

where,

$$\lambda = \frac{\rho\omega^2\ell^4}{840EI}, \qquad \rho = \frac{\mu A}{386.4}$$

4.3 Derivation Procedure 3.

4.3.1 Endload Element (Element Type P3/1FD).

To derive the element dynamic flexibility matrix using this procedure requires the solution of equation 3.3.1.6, that is,

$$u_x(x) = A_1 \cos \lambda_1 x + A_2 \sin \lambda_1 x$$

The constant terms are evaluated using this equation in conjunction with the relationship,

$$P(x) = AE \frac{d\,u_x(x)}{dx}$$

where,

$$\frac{d\,u_x(x)}{dx} = -A_1 \lambda_1 \sin \lambda_1 x + A_2 \lambda_1 \cos \lambda_1 x \qquad 4.3.1.1$$

and the element boundary load conditions,

$$\left.\begin{array}{lll} P(x) = -q_\alpha & \text{at} & x = 0 \\ P(x) = q_{\alpha+3} & \text{at} & x = l \end{array}\right] \qquad 4.3.1.2$$

The endload distribution is given by,

$$P(x) = AE\left(-A_1 \lambda_1 \sin \lambda_1 x + A_2 \lambda_1 \cos \lambda_1 x\right) \qquad 4.3.1.3$$

Therefore, the constant terms are given by,

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \frac{1}{AE\lambda_1} \begin{bmatrix} -\dfrac{\cos \lambda_1 l}{\sin \lambda_1 l} & -\dfrac{1}{\sin \lambda_1 l} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} q_\alpha \\ q_{\alpha+3} \end{bmatrix} \qquad 4.3.1.4$$

Hence, the general displacement function is,

$$u_x(x) = \frac{1}{AE\lambda_1}\left[\begin{array}{c|c} \cos\lambda_1 x & \sin\lambda_1 x \end{array}\right]\left[\begin{array}{c|c} -\frac{\cos\lambda_1 \ell}{\sin\lambda_1 \ell} & -\frac{1}{\sin\lambda_1 \ell} \\ \hline -1 & 0 \end{array}\right]\left[\begin{array}{c} q_\alpha \\ \hline q_{\alpha+3} \end{array}\right]$$

4.3.1.5

The generalized element boundary displacements corresponding to the generalized element boundary loads are obtained using equation 4.3.1.5 by evaluating the displacement at $x = o$ and $x = \ell$.

Therefore,

$$\left[\begin{array}{c} \delta_\alpha \\ \hline \delta_{\alpha+3} \end{array}\right] = \frac{1}{AE\lambda_1}\left[\begin{array}{c|c} -\frac{\cos\lambda_1 \ell}{\sin\lambda_1 \ell} & -\frac{1}{\sin\lambda_1 \ell} \\ \hline -\frac{1}{\sin\lambda_1 \ell} & -\frac{\cos\lambda_1 \ell}{\sin\lambda_1 \ell} \end{array}\right]\left[\begin{array}{c} q_\alpha \\ \hline q_{\alpha+3} \end{array}\right]$$

4.3.1.6

The element dynamic flexibility matrix is therefore,

$$[f_{md}] = \frac{1}{AE\lambda_1}\left[\begin{array}{c|c} -\frac{\cos\lambda_1 \ell}{\sin\lambda_1 \ell} & -\frac{1}{\sin\lambda_1 \ell} \\ \hline -\frac{1}{\sin\lambda_1 \ell} & -\frac{\cos\lambda_1 \ell}{\sin\lambda_1 \ell} \end{array}\right]$$

4.3.1.7

Using the notation given by Bishop and Johnson,[3]

$$[f_{md}] = \frac{1}{AE\lambda_1}\left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{12} & F_{11} \end{array}\right] \tag{4.3.1.8}$$

where,

$$\left.\begin{aligned} F_{11} &= -\frac{\cos\lambda_1 \ell}{\sin\lambda_1 \ell} \\ F_{12} &= -\frac{1}{\sin\lambda_1 \ell} \end{aligned}\right] \tag{4.3.1.9}$$

and

$$\lambda_1 = \omega\left(\frac{\mu}{386 \cdot 4\, E}\right)^{\frac{1}{2}}$$

4.3.2 Plane Beam Element (Element Type P3/2FD).

The element dynamic flexibility matrix for this element is derived using equation 3.3.2.6, that is,

$$u_3(x) = A_1 \mathrm{Cos}\,\lambda_2 x + A_2 \mathrm{Sin}\,\lambda_2 x + A_3 \mathrm{Cosh}\,\lambda_2 x + A_4 \mathrm{Sinh}\,\lambda_2 x$$

The constant terms are evaluated using this equation in conjunction with the relationships,

$$M(x) = EI \frac{d^2 u_3(x)}{dx^2}$$

and

$$Q(x) = -EI \frac{d^3 u_3(x)}{dx^3}$$

where,

$$\frac{d^2 u_3(x)}{dx^2} = -A_1 \lambda_2^2 \mathrm{Cos}\,\lambda_2 x - A_2 \lambda_2^2 \mathrm{Sin}\,\lambda_2 x + A_3 \lambda_2^2 \mathrm{Cosh}\,\lambda_2 x + A_4 \lambda_2^2 \mathrm{Sinh}\,\lambda_2 x \qquad 4.3.2.1$$

$$\frac{d^3 u_3(x)}{dx^3} = A_1 \lambda_2^3 \mathrm{Sin}\,\lambda_2 x - A_2 \lambda_2^3 \mathrm{Cos}\,\lambda_2 x + A_3 \lambda_2^3 \mathrm{Sinh}\,\lambda_2 x + A_4 \lambda_2^3 \mathrm{Cosh}\,\lambda_2 x \qquad 4.3.2.2$$

and the element boundary load conditions,

$$\left.\begin{array}{l} \left.\begin{array}{l} M(x) = q_{\alpha+2} \\ Q(x) = -q_{\alpha+1} \end{array}\right] \text{ at } x = 0 \\ \\ \left.\begin{array}{l} M(x) = -q_{\alpha+5} \\ Q(x) = q_{\alpha+4} \end{array}\right] \text{ at } x = l \end{array}\right] \qquad 4.3.2.3$$

The resulting general displacement function, displacement in the z-direction, and the general rotation function, given by $\theta(x) = \dfrac{du_3(x)}{dx}$, are,

$$\begin{bmatrix} u_3(x) \\ \theta(x) \end{bmatrix} = \left[\begin{array}{c|c} \dfrac{F_5(\operatorname{Cos}\lambda_2 x + \operatorname{Cosh}\lambda_2 x) + (F_1+F_3)\operatorname{Sin}\lambda_2 x + (F_1-F_3)\operatorname{Sinh}\lambda_2 x}{-2EI\lambda_2^3 F_3} & \dfrac{(F_1-F_3)\operatorname{Cos}\lambda_2 x + (F_1+F_3)\operatorname{Cosh}\lambda_2 x - F_6(\operatorname{Sin}\lambda_2 x + \operatorname{Sinh}\lambda_2 x)}{2EI\lambda_2^2 F_3} \\ \hline \dfrac{(F_1+F_3)\operatorname{Cos}\lambda_2 x + (F_1-F_2)\operatorname{Cosh}\lambda_2 x - F_5(\operatorname{Sin}\lambda_2 x - \operatorname{Sinh}\lambda_2 x)}{-2EI\lambda_2^2 F_3} & \dfrac{F_6(\operatorname{Cos}\lambda_2 x + \operatorname{Cosh}\lambda_2 x) + (F_1-F_3)\operatorname{Sin}\lambda_2 x - (F_1+F_3)\operatorname{Sinh}\lambda_2 x}{-2EI\lambda_2 F_3} \end{array}\right.$$

$$\left.\begin{array}{c|c} \dfrac{F_8(\operatorname{Cos}\lambda_2 x + \operatorname{Cosh}\lambda_2 x) - F_{10}(\operatorname{Sin}\lambda_2 x + \operatorname{Sinh}\lambda_2 x)}{2EI\lambda_2^3 F_3} & \dfrac{F_{10}(\operatorname{Cos}\lambda_2 x + \operatorname{Cosh}\lambda_2 x) + F_7(\operatorname{Sin}\lambda_2 x + \operatorname{Sinh}\lambda_2 x)}{-2EI\lambda_2^2 F_3} \\ \hline \dfrac{F_{10}(\operatorname{Cos}\lambda_2 x + \operatorname{Cosh}\lambda_2 x) + F_8(\operatorname{Sin}\lambda_2 x - \operatorname{Sinh}\lambda_2 x)}{-2EI\lambda_2^2 F_3} & \dfrac{F_7(\operatorname{Cos}\lambda_2 x + \operatorname{Cosh}\lambda_2 x) - F_{10}(\operatorname{Sin}\lambda_2 x - \operatorname{Sinh}\lambda_2 x)}{-2EI\lambda_2 F_3} \end{array}\right]$$

$$\times . \left\{ q_{\alpha+1} \quad q_{\alpha+2} \quad q_{\alpha+4} \quad q_{\alpha+5} \right\}$$

4.3.2.4

The generalized element boundary displacements corresponding to the generalized element boundary loads are obtained using equation 4.3.2.4 by evaluating the displacement in the z-direction and the rotation at $x=0$ and $x=\ell$. Therefore,

$$\begin{bmatrix} \delta_{\alpha+1} \\ \delta_{\alpha+2} \\ \delta_{\alpha+4} \\ \delta_{\alpha+5} \end{bmatrix} = \frac{1}{EI\lambda_2^3 F_3} \begin{bmatrix} -F_5 & F_1\lambda_2 & F_8 & -F_{10}\lambda_2 \\ F_1\lambda_2 & F_6\lambda_2^2 & F_{10}\lambda_2 & F_7\lambda_2^2 \\ F_8 & F_{10}\lambda_2 & -F_5 & -F_1\lambda_2 \\ -F_{10}\lambda_2 & F_7\lambda_2^2 & -F_1\lambda_2 & F_6\lambda_2^2 \end{bmatrix} \begin{bmatrix} q_{\alpha+1} \\ q_{\alpha+2} \\ q_{\alpha+4} \\ q_{\alpha+5} \end{bmatrix}$$

4.3.2.5

When evaluating these generalized displacements it should be remembered that the sign convention for the displacement in the z-direction is the same in both equation 4.3.2.4 (theory of elasticity, see figure 11) and for the corresponding boundary displacement. However, in the case of rotations the two sign conventions are opposite, see figures 11 and 12 . For example, from equation 4.3.2.4 the rotation at x = 0 due to $q_{\alpha+1} = 1.0$ is given by,

$$\left(\frac{du_3(x)}{dx} \right)_{x=0} = - \frac{F_1}{EI\lambda_2^2 F_3}$$

From equation 4.3.2.5 the rotation at x = 0 due to $q_{\alpha+1} = 1.0$ is given by

$$\left(\delta_{\alpha+2} \right)_{q_{\alpha+1}=1.0} = \frac{F_1}{EI\lambda_2^2 F_3}$$

See figures for further clarification. Therefore, when evaluating the generalized element boundary rotations using equation 4.3.2.4 the sign of the resulting quantity must be reversed.

The element dynamic flexibility matrix is therefore,

$$[f_{md}] = \frac{1}{EI\lambda_2^3 F_3}\begin{bmatrix} -F_5 & F_1\lambda_2 & F_8 & -F_{10}\lambda_2 \\ F_1\lambda_2 & F_6\lambda_2^2 & F_{10}\lambda_2 & F_7\lambda_2^2 \\ F_8 & F_{10}\lambda_2 & -F_5 & -F_1\lambda_2 \\ -F_{10}\lambda_2 & F_7\lambda_2^2 & -F_1\lambda_2 & F_6\lambda_2^2 \end{bmatrix} \qquad 4.3.2.6$$

where, again using the same notation as Bishop and Johnson,

$$\left.\begin{aligned} F_1 &= \mathrm{Sin}\,\lambda_2\ell\;\mathrm{Sinh}\,\lambda_2\ell \\ F_3 &= \mathrm{Cos}\,\lambda_2\ell\;\mathrm{Cosh}\,\lambda_2\ell - 1 \\ F_5 &= \mathrm{Cos}\,\lambda_2\ell\;\mathrm{Sinh}\,\lambda_2\ell - \mathrm{Sin}\,\lambda_2\ell\;\mathrm{Cosh}\,\lambda_2\ell \\ F_6 &= \mathrm{Cos}\,\lambda_2\ell\;\mathrm{Sinh}\,\lambda_2\ell + \mathrm{Sin}\,\lambda_2\ell\;\mathrm{Cosh}\,\lambda_2\ell \\ F_7 &= \mathrm{Sin}\,\lambda_2\ell + \mathrm{Sinh}\,\lambda_2\ell \\ F_8 &= \mathrm{Sin}\,\lambda_2\ell - \mathrm{Sinh}\,\lambda_2\ell \\ F_{10} &= \mathrm{Cos}\,\lambda_2\ell - \mathrm{Cosh}\,\lambda_2\ell \end{aligned}\right] \qquad 4.3.2.7$$

and

$$\lambda_2 = \left(\frac{\omega^2\mu A}{386{\cdot}4\,EI}\right)^{\frac{1}{4}}$$

4.3.3 Inclined Plane Beam Element (Element Type P3/3FD).

The general derivation of the element dynamic flexibility matrix for this element type is the same as element type P1/3FD. However, in this case the element dynamic flexibility matrix relative to the local axes is derived by assembling the dynamic flexibility matrices for element types P3/1FD and P3/2FD. Therefore,

$$[f_{md}^{*}] = \begin{bmatrix} \frac{F_{11}}{AE\lambda_1} & 0 & 0 & \frac{F_{12}}{AE\lambda_1} & 0 & 0 \\ 0 & \frac{-F_5}{EI\lambda_2^3 F_3} & \frac{F_1}{EI\lambda_2^2 F_3} & 0 & \frac{F_8}{EI\lambda_2^3 F_3} & \frac{-F_{10}}{EI\lambda_2^2 F_3} \\ 0 & \frac{F_1}{EI\lambda_2^2 F_3} & \frac{F_6}{EI\lambda_2 F_3} & 0 & \frac{F_{10}}{EI\lambda_2^2 F_3} & \frac{F_7}{EI\lambda_2 F_3} \\ \frac{F_{12}}{AE\lambda_1} & 0 & 0 & \frac{F_{11}}{AE\lambda_1} & 0 & 0 \\ 0 & \frac{F_8}{EI\lambda_2^3 F_3} & \frac{F_{10}}{EI\lambda_2^2 F_3} & 0 & \frac{-F_5}{EI\lambda_2^3 F_3} & \frac{-F_1}{EI\lambda_2^2 F_3} \\ 0 & \frac{-F_{10}}{EI\lambda_2^2 F_3} & \frac{F_7}{EI\lambda_2 F_3} & 0 & \frac{-F_1}{EI\lambda_2^2 F_3} & \frac{F_6}{EI\lambda_2 F_3} \end{bmatrix}$$

4.3.3.1

The element dynamic flexibility matrix relative to the global axes is given by,

$$[f_{md}] = [C_m]^T [f_{md}^{*}] [C_m]$$

where $[C_m]$ is given by equation 4.1.3.5. Therefore,

$$[f_{md}] = \begin{bmatrix} f(1,1) & f(1,2) & f(1,3) & f(1,4) & f(1,5) & f(1,6) \\ & f(2,2) & f(2,3) & f(1,5) & f(2,5) & f(2,6) \\ & & f(3,3) & -f(1,6) & -f(2,6) & f(3,6) \\ & & & f(1,1) & f(1,2) & -f(1,3) \\ & \text{SYM}^{\underline{L}} & & & f(2,2) & -f(2,3) \\ & & & & & f(3,3) \end{bmatrix}$$

where,

$$f(1,1) = \left(\frac{\bar{x}_j - \bar{x}_i}{l}\right)^2 \frac{F_{11}}{AE\lambda_1} - \left(\frac{\bar{z}_j - \bar{z}_i}{l}\right)^2 \frac{F_5}{EI\lambda_2^3 F_3}$$

$$f(2,2) = \left(\frac{\bar{z}_j - \bar{z}_i}{l}\right)^2 \frac{F_{11}}{AE\lambda_1} - \left(\frac{\bar{x}_j - \bar{x}_i}{l}\right)^2 \frac{F_5}{EI\lambda_2^3 F_3}$$

$$f(3,3) = \frac{F_6}{EI\lambda_2 F_3}$$

$$f(1,2) = \frac{(\bar{x}_j-\bar{x}_i)}{\ell}\,\frac{(\bar{z}_j-\bar{z}_i)}{\ell}\left(\frac{F_{11}}{AE\lambda_1}+\frac{F_5}{EI\lambda_2^3F_3}\right)$$

$$f(1,3) = -\frac{(\bar{z}_j-\bar{z}_i)}{\ell}\,\frac{F_1}{EI\lambda_2^2F_3}$$

$$f(1,4) = \left(\frac{\bar{x}_j-\bar{x}_i}{\ell}\right)^2\frac{F_{12}}{AE\lambda_1} + \left(\frac{\bar{z}_j-\bar{z}_i}{\ell}\right)^2\frac{F_8}{EI\lambda_2^3F_3}$$

$$f(1,5) = \frac{(\bar{x}_j-\bar{x}_i)}{\ell}\,\frac{(\bar{z}_j-\bar{z}_i)}{\ell}\left(\frac{F_{12}}{AE\lambda_1}-\frac{F_8}{EI\lambda_2^3F_3}\right)$$

$$f(1,6) = \frac{(\bar{z}_j-\bar{z}_i)}{\ell}\,\frac{F_{10}}{EI\lambda_2^2F_3}$$

$$f(2,3) = \frac{(\bar{x}_j-\bar{x}_i)}{\ell}\,\frac{F_1}{EI\lambda_2^2F_3}$$

$$f(2,5) = \left(\frac{\bar{z}_j-\bar{z}_i}{\ell}\right)^2\frac{F_{12}}{AE\lambda_1} + \left(\frac{\bar{x}_j-\bar{x}_i}{\ell}\right)^2\frac{F_8}{EI\lambda_2^3F_3}$$

$$f(2,6) = -\frac{(\bar{x}_j-\bar{x}_i)}{\ell}\,\frac{F_{10}}{EI\lambda_2^2F_3}$$

$$f(3,6) = \frac{F_7}{EI\lambda_2F_3}$$

$$\ell = \left((\bar{x}_j-\bar{x}_i)^2+(\bar{z}_j-\bar{z}_i)^2\right)^{\frac{1}{2}}$$

$$\rho = \frac{\mu A}{386\cdot 4}$$

$$F_1 = \sin\lambda_2 \ell \sinh\lambda_2 \ell$$

$$F_3 = \cos\lambda_2 \ell \cosh\lambda_2 \ell - 1$$

$$F_5 = \cos\lambda_2 \ell \sinh\lambda_2 \ell - \sin\lambda_2 \ell \cosh\lambda_2 \ell$$

$$F_6 = \cos\lambda_2 \ell \sinh\lambda_2 \ell + \sin\lambda_2 \ell \cosh\lambda_2 \ell$$

$$F_7 = \sin\lambda_2 \ell + \sinh\lambda_2 \ell$$

$$F_8 = \sin\lambda_2 \ell - \sinh\lambda_2 \ell$$

$$F_{10} = \cos\lambda_2 \ell - \cosh\lambda_2 \ell$$

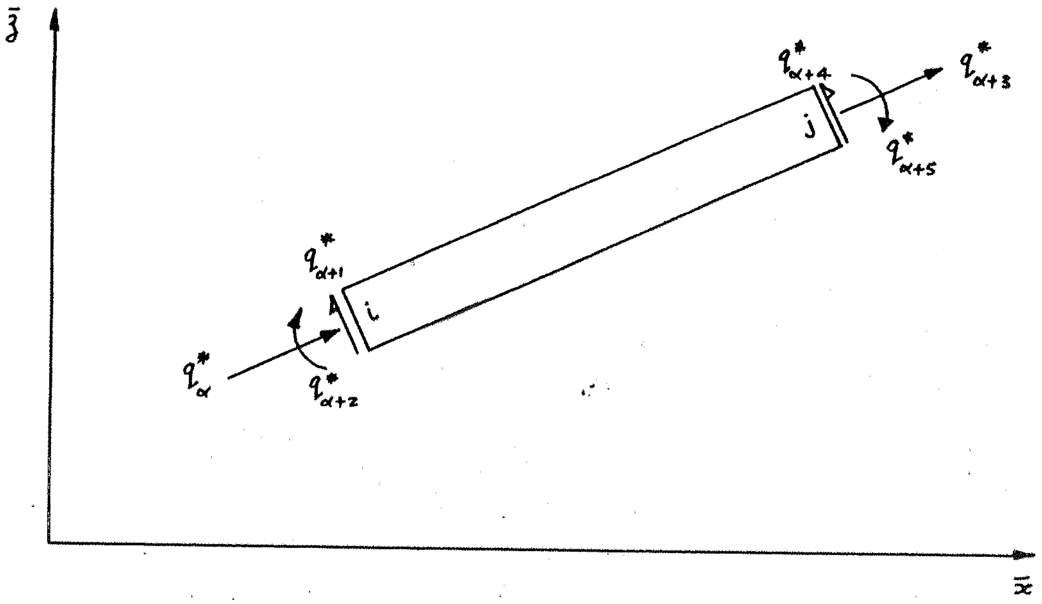
$$F_{11} = -\frac{\cos\lambda_1 \ell}{\sin\lambda_1 \ell}$$

$$F_{12} = -\frac{1}{\sin\lambda_1 \ell}$$

$$\lambda_1 = \omega\left(\frac{\mu}{386 \cdot 4 E}\right)^{\frac{1}{2}}$$

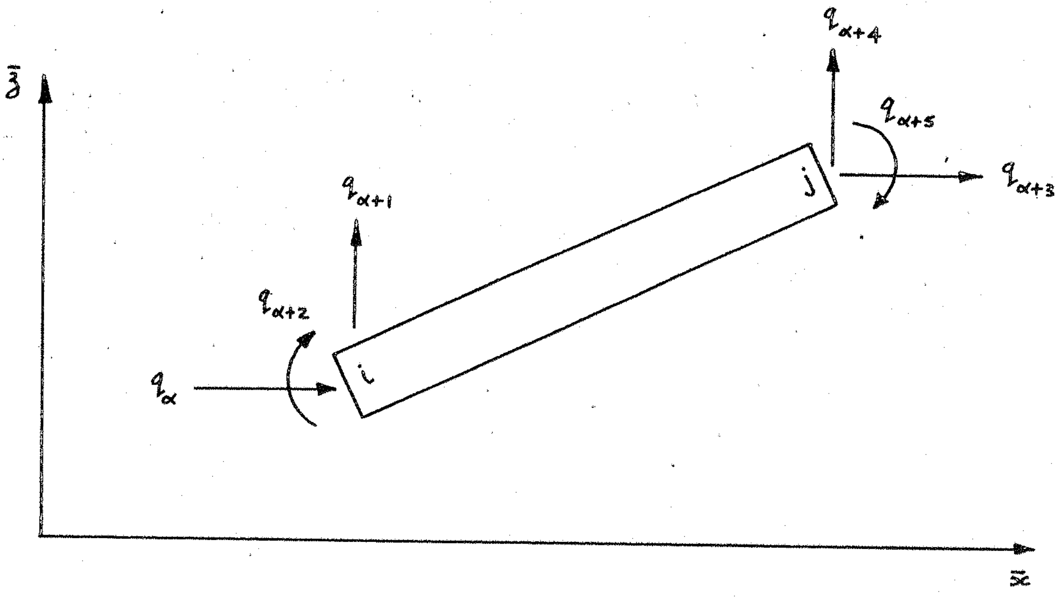
$$\lambda_2 = \left(\frac{\omega^2 \mu A}{386 \cdot 4 EI}\right)^{\frac{1}{4}}$$

In the initial research study of the rank force method for vibration analysis only certain types of structural elements have been considered. These were felt sufficient for the initial investigation. In future work the element loading systems should be considered in a more complex form, particularly for the rectangular plate element. Also, elements of irregular shape should be investigated and consistent beam and plate element matrices should be derived for use in structural configurations consisting of beam/plate combinations.



Generalized element boundary loads relative to the local axes.

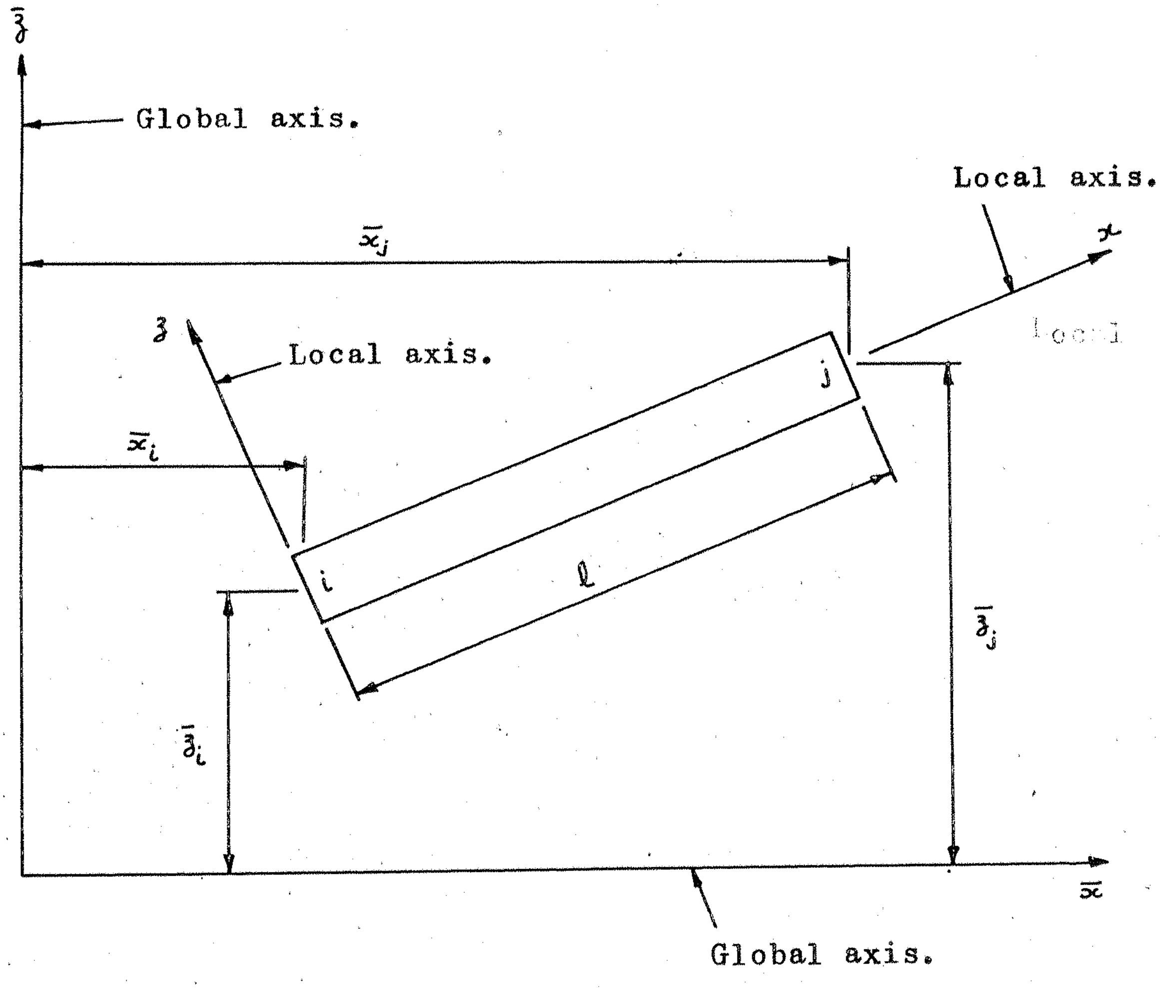
(a)



Generalized element boundary loads relative to the global axes.

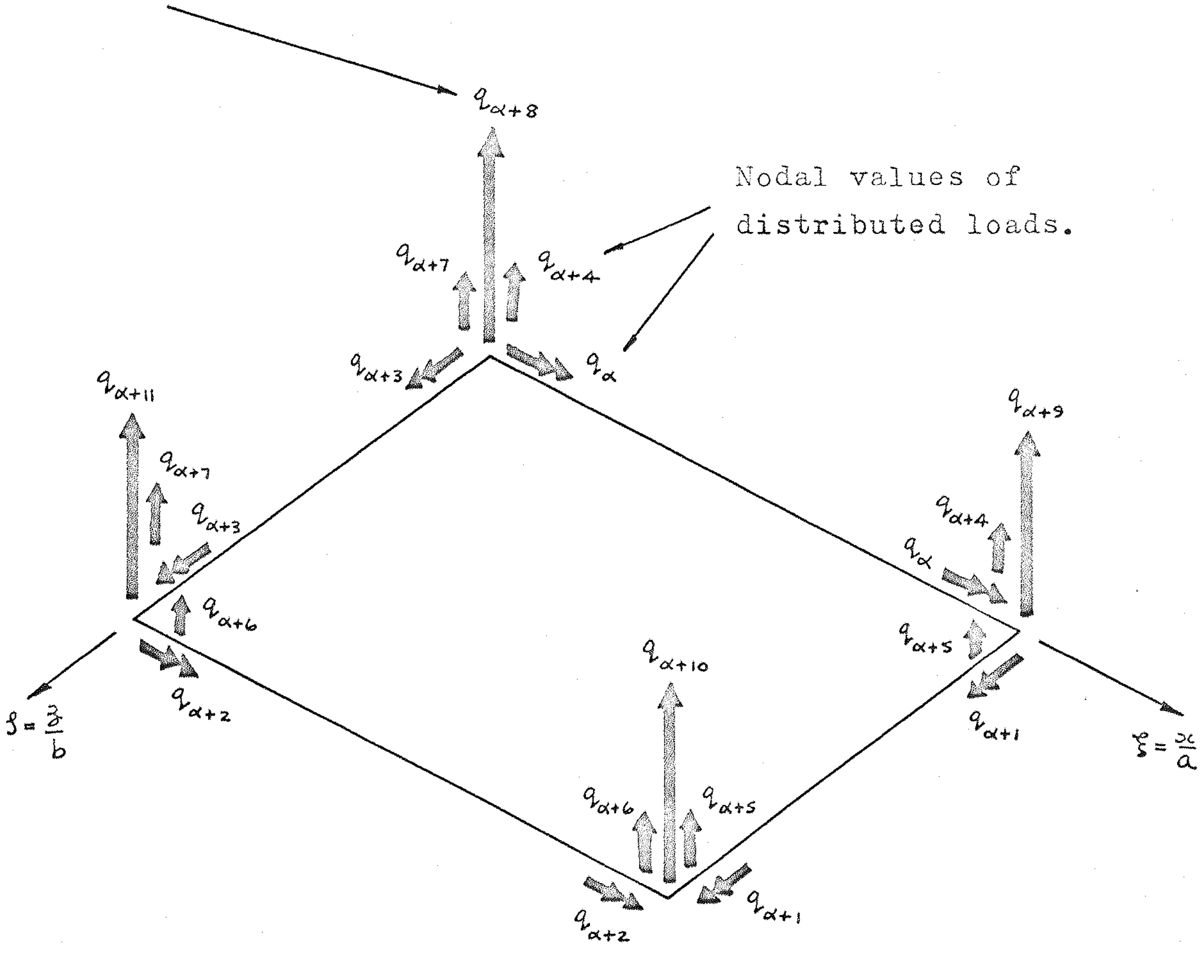
(b)

Fig. 17.



Inclined plane beam element showing local and global axes.

Fig. 18 .

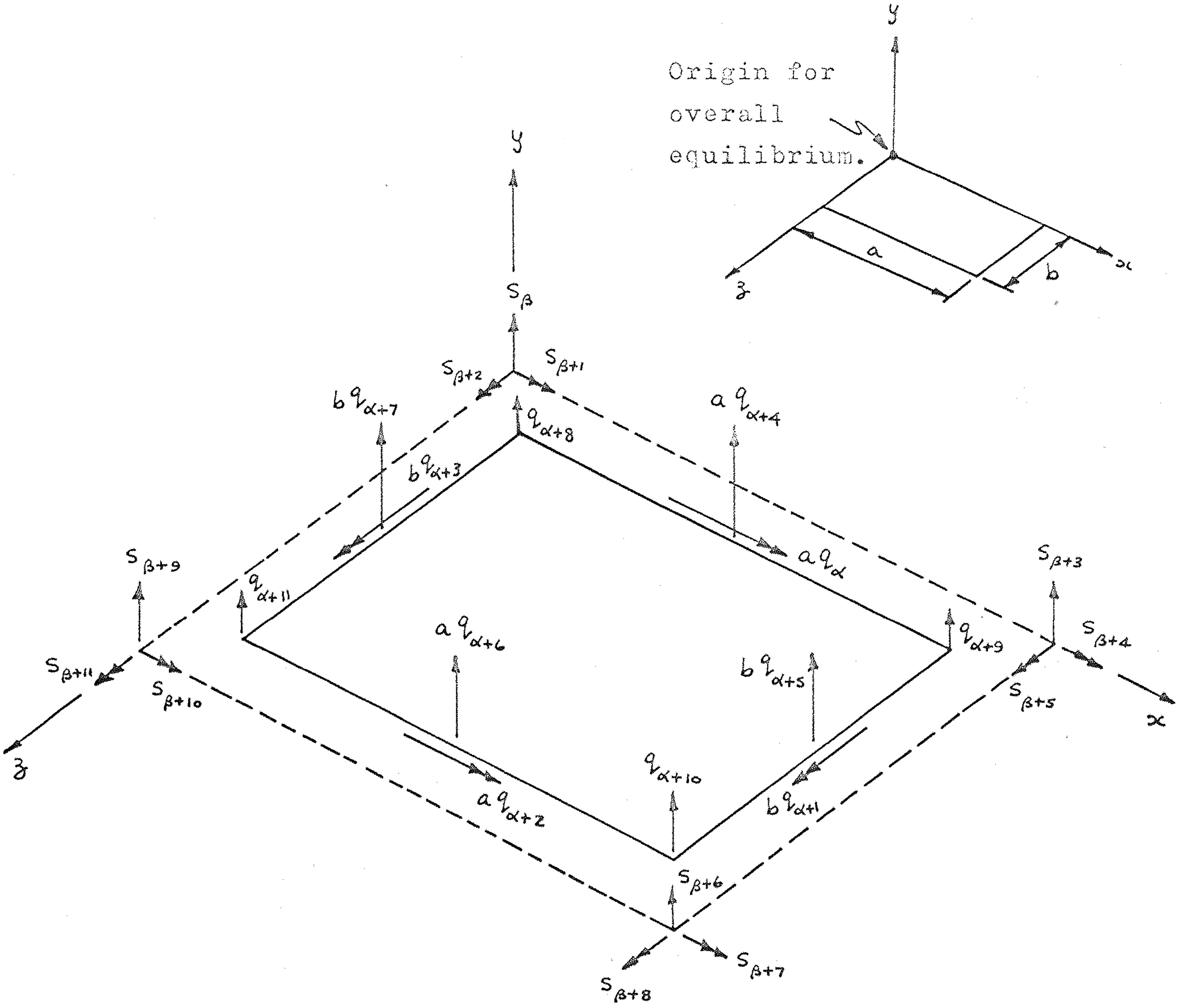


Concentrated nodal loads.

Nodal values of distributed loads.

Assumed q-system of generalized element boundary loads. Positive as shown.

Fig. 19 .



q-system and s-system for the derivation of Element Type P1/5FD.

Fig. 20 .

CHAPTER 5.

RESULTS AND DISCUSSION.

5.1 Collinear beam structures.

The rank force method was applied to a series of collinear beam structures and the corresponding eigenvalues evaluated. These are given in tables 2 and 3 . The numbers in parentheses give the percentage error of eigenvalues, that is,

$$\frac{\omega(\text{calculated}) - \omega(\text{reference})}{\omega(\text{reference})} \times 100\,\%$$

where,

ω (calculated) = calculated eigenvalue using the respective representation.

ω (reference) = reference eigenvalue. This is taken as that evaluated using Element Type P3/2FD, see table 4 .

The structural model, a simply supported beam, was idealized into various arrangements of finite elements of equal length up to a maximum of six elements. The beam properties are given in table 5 . The accuracy of the results depends on the structural element representation and the number of discrete elements used in the structural idealization.

In Element Type P1/2FD the assumption of a 3rd degree polynomial to represent the bending moment results in a linear displacement function for the element inertia loading. This is of course not realistic but is an approximation which is better than a lumped mass representation for a continuous structural system. In reference 8 Archer derives a dynamic stiffness matrix for a plane beam element and adopts a 3rd degree polynomial to represent the distorted shape of the element. This results in a linear bending moment distribution which again is not realistic. As a consequence of this the dynamic flexibility matrix for a plane beam element (Element Type P1/2FD) is somewhat similar to the dynamic stiffness matrix derived by Archer. The eigenvalues of a simply supported beam evaluated using Element Type P1/2FD are given in table 2. For the lower modes the eigenvalues, as shown in table 2, are identical to those obtained by Archer using the displacement approach but for higher modes differences in the two sets of results would be expected. An explanation can be obtained by formulating the determinantal equation for a simply supported beam idealized as one finite element ($NE = 1$) using the displacement and rank force methods. The determinantal equation given by the displacement method would be a 2nd degree polynomial in the frequency parameter, λ, which has two eigenvalues. The determinantal equation given by the rank force method would be a higher degree polynomial which would give a higher number of eigenvalues, two of which would be equal to those given by the displacement method. This equality of eigenvalues is due to the similarity in the dynamic flexibility and stiffness matrices for the plane beam element.

When the structural dynamic stiffness matrix, using the displacement method, can be separated into the structural static stiffness matrix and mass matrix the number of eigenvalues that can be calculated is given by the order of these matrices, or in other words, the number of unconstrained degrees of freedom for the structure. In the rank force method this separation is not possible and the number of eigenvalues which can be computed is difficult to assess. The force determinantal equation cannot be formed simply even for the most trivial configurations. When trying to formulate this equation it is essential not to make simplifications by cancellation and rearrangement of terms otherwise artificial or spurious eigenvalues (single and double) will be present. To evaluate higher modes than those given in table 2 it is necessary to divide the beam into a higher number of structural finite elements.

Element Type P2/2FD was then investigated because it doesn't use a linear displacement function for the element inertia loading. However, it should be remembered that the incremental loading equation is not satisfied. Initially it was felt that this representation would give better results than Element Type P1/2FD but this is not the case as can be seen from table 3.

To investigate the variation of particular generalized element boundary loads with frequency in a collinear beam structure a simply sypported beam idealized as two finite elements was again taken as the structural model. The applied and internal loading systems for this model are shown in figure 21 and table 5 gives the beam properties.

The variations of shear load q_1 and bending moment q_4 with frequency for the separate unit applied loads are shown in tables 8 and 9 and figures 27 to 31 using Element Type P1/2FD and in tables 6 and 7 and figures 22 to 26 using Element Type P3/2FD (reference element). Table 10 gives the percentage error in q_1 using Element Type P1/2FD. The percentage error is given by,

$$\frac{q_1(P3/2FD) - q_1\,(P1/2FD)}{q_1\,(P3/2FD)} \times 100\ \%$$

It can be seen that the error changes with frequency and applied load, close to the first eigenvalue the error in q_1 is very large for P_1 ,P_2 and P_4 . The error for P_3 is small simply because q_1 does not resonate at the first eigenvalue as can be seen from figure 28 . Away from the first eigenvalue the error is small but generally speaking this error is higher than that in the corresponding eigenvalues. It should also be noted that when q_1 resonates the error is lower bound below the first eigenvalue and upper bound above the first eigenvalue. Further eigenvalues cannot be investigated with this structural model because of the error in the corresponding second eigenvalues.

Table 11 gives the percentage error in q_4 using Element Type P1/2FD. The percentage error is given by,

$$\frac{q_4\,(P3/2FD) - q_4\,(P1/2FD)}{q_4\,(P3/2FD)} \times 100\ \%$$

The same comments as for q_1 apply to the bending moment q_4.

5.2 General plane frames.

The first four eigenvalues for the frame structures shown in figures 32 and 33 and with element properties as given in table 14 have been computed using the rank force method. These were evaluated using Element Types P1/3FD and P3/3FD and the respective eigenvalues and those obtained by a hybrid method (Levien[23,24]) and the displacement method (Burch[40]) are given in tables 12 and 13. There is good agreement between all the results when using Element Type P3/3FD in the rank force method. However, this was expected since all three sets of results adopt transcendental functions for the element dynamic representation. The hybrid method uses particular generalized element boundary loads and displacements as the unknowns, the displacement method uses displacements as unknowns and the rank force method uses generalized element boundary loads as unknowns.

Element Type P1/3FD gives good results for the lower modes but the frames must be idealized into a larger number of discrete structural elements in order to obtain the higher eigenvalues more accurately. This was also shown by the collinear beam results when adopting polynomials to represent the element internal loading distribution.

The eigenvalues given by Burch are in the form of bounds, this is because he uses a graphical approach for the evaluation.

In figure 34 a bent cantilever is shown with element properties as given in table 15. The first two eigenvalues of this simple frame structure were evaluated using Element Type P3/3FD and are compared with those given by Bishop and Johnson[3] in table 16. The element loads and structural reactions for this frame were also evaluated at a frequency of 150 radians per second and are given in table 17 and figures 36 and 37. The applied loading system is shown in figure 35 (a) and the numbering system for the element loads and structural reactions in figure 35 (b).

5.8 Rectangular plate structures.

Using element types Pl/4FD(q-system) and Pl/5FD(s-system, 8 equations) the first two eigenvalues of a rectangular cantilever plate were computed and are shown in table 18 . The plate idealization and properties are shown in figure 38 . These results are compared with those of Barton[38] (Ritz Method) and the experimental results of Zienkiewicz[20] . To give some comparison with the displacement method the results given by Zienkiewicz[20] have been quoted but his plate was idealized as twelve finite elements, three generalized displacements were assumed per node (no slope continuity between nodes) and the assumed displacement function is given in reference 41. The lumped mass results of Zienkiewicz[20] are also quoted.

In the rectangular plate analyses a great deal of trouble was encountered in keeping the analyses in core (6 element limit) and also numerical difficulties. Although some results have been obtained, which show good agreement with results calculated by alternative procedures, the author feels that the derivation procedures are sound but the results and other configurations should be further investigated. Before carrying out other investigations the programmes should be rewritten to enable more finite elements to be considered.

The plate boundary loadings for a finite element were assumed uniform along the respective boundaries, more complex distributions could be investigated.

5.4 General discussion.

In the field of finite element techniques for structural vibration analysis the displacement approach is by far the most popular. No published work on the vibration application of a force method which adopts an automatic selection of redundancy technique and uses a distributed structural mass representation has been found to date. The author has only found one paper[23,24] which starts out to apply a force approach and finally develops a hybrid method. The reason for this abandonment was that Levien[23,24] had no automatic selection of redundancy technique available to him and it was this impediment which led to his development of a hybrid approach. This impediment existed for many years for static analysis which accounts a great deal for the popularity in the displacement approach. It was therefore felt that a force approach which adopts an automatic selection of redundancy technique, in particular the rank force method, and uses a distributed structural mass representation should be studied in an effort to give a more realistic assessment of the force approach based on an actual investigation. The research work has been limited to the eigenvalue problem, element loads and structural reactions and structural displacements due to a system of harmonic forcing functions. All loads and displacements have been assumed to vary harmonically with time and in phase, the structure is assumed undamped and steady state conditions are assumed to exist. Only simple structural configurations have been considered, mainly because particular solutions were available. It can be seen from the collinear beam, plane frame and rectangular plate results that the rank force method gives good results and therefore this method is not deficient from this point of view. The comparison of the displacement and force approaches now becomes a matter of comparing the manner in which the results are obtained.

5.4.1 The eigenvalue problem.

In the displacement approach the following system of equations can be immediately assembled for a given frequency parameter,

$$[\mathcal{K}_d]\{{}_t\Delta_\lambda\} = \{{}_tP_\lambda\}$$

The eigenvalue problem is given by,

$$[\mathcal{K}_d]\{{}_t\Delta_\lambda\} = \{0\}$$

The eigenvalues are those values of the frequency parameter which give a zero determinant value for the matrix $[\mathcal{K}_d]$, that is,

$$\det[\mathcal{K}_d] = 0$$

When using an element dynamic stiffness matrix which contains transcendental functions or a condensed element dynamic stiffness matrix as developed by Pestel[18], using polynomials for the distorted shape of the element and using higher order derivatives as generalized element boundary displacements, the structural dynamic stiffness matrix, $[\mathcal{K}_d]$, has to be generated for every assumed value of the frequency parameter. Under these circumstances the procedure is a continuous generation process, as previously defined. However, by using polynomials for the distorted shape of the elements and adopting the uncondensed form of the element dynamic stiffness matrix, even when condensation is possible, the structural dynamic stiffness matrix can be expanded into the familiar eigenvalue iteration form,

$$\left([\mathcal{K}] - \lambda [\mathcal{M}] \right) \{ {}_{\epsilon}\Delta_{\lambda} \} = \{ 0 \}$$

where $[\mathcal{K}]$ is the structural static stiffness matrix and $[\mathcal{M}]$ is the structural mass matrix. Both these matrices are independent of frequency and therefore constant for a given structural idealization, the procedure then becomes a singular generation process. Having found an eigenvalue it is then a relatively simple matter to calculate the corresponding eigenvector.

At an eigenvalue the structural dynamic stiffness matrix is singular and therefore it has no inverse, that is , no structural dynamic flexibility matrix. Now, at a true eigenvalue,

$$[\mathcal{K}_d]^{-1} = \frac{1}{|\mathcal{K}_d|}\, adj\, [\mathcal{K}_d] = \frac{1}{0}\, adj\, [\mathcal{K}_d] = (\infty)\, adj\, [\mathcal{K}_d]$$

Therefore, the terms in the inverse matrix can be zero and infinity. This now brings up the point, what is the eigenvalue criteria in the rank force method ?

In this method the following system of equations are generated for a given frequency,

$$[\mathcal{F}_{\alpha}]\{{}_{t}P_{\lambda}\} = \{{}_{t}\Delta_{\lambda}\}$$

The frequency is contained in the structural dynamic flexibility matrix, $[\mathcal{F}_{\alpha}]$, and this matrix has to be generated for every assumed value of frequency irrespective of the method of deriving the element dynamic flexibility matrices. The individual element dynamic flexibility matrices can themselves be expanded into a static flexibility matrix and an inverse mass matrix. The rank force method is always a continuous generation process. In the displacement approach the eigenvalue formulation is obtained by nulling the applied load vector which presents a system of homogeneous linear equations and the eigenvalue criteria becomes obvious. Following the same reasoning with the rank force method gives,

$$[\mathcal{F}_{\alpha}]\{0\} = \{{}_{t}\Delta_{\lambda}\}$$

In the eigenvalue problem the eigenvector contains finite values, that is, it is not a null vector. Therefore, the interpretation of this latter equation is :

> What transformation matrix exists such that when it is multiplied by a null vector it gives and finite vector ?

This literal interpretation is therefore, at a true eigenvalue there is no structural dynamic flexibility matrix.

This is consistent with the fact that in the displacement approach the structural dynamic stiffness matrix has no inverse at a true eigenvalue. The force criteria also indicates that no eigenvector can be found. The question that now remains is, how can eigenvalues be found in the rank force method ?

Three methods which can be adopted are;

1. Plot each generalized element boundary load against frequency for each unit generalized applied load. See collinear beam results. This method will then give the individual eigenvalue bounds and adopts the equations,

$$\{ {}_t q_\alpha \} = [\Delta_{\alpha\lambda}] \{ {}_t P_\lambda \}$$

2. Plot each generalized structural displacement against frequency for each unit generalized applied load. Again this method gives the individual eigenvalue bounds and adopts the equations,

$$\{ {}_t \Delta_\lambda \} = [\mathcal{F}_\alpha] \{ {}_t P_\lambda \}$$

This method was used by Burch[39].

3. Invert the structural dynamic flexibility matrix to give,

$$[\mathcal{F}_\alpha]^{-1} \{ {}_t \Delta_\lambda \} = \{ {}_t P_\lambda \}$$

The inverse can be obtained providing the frequency is not an eigenvalue. This formulation now becomes that of the displacement approach and the eigenvalues are found by investigating the sign of the determinant. In practice the structural dynamic flexibility matrix can be inverted simply because the matrix will never be singular but as the frequency approaches a true eigenvalue this method becomes numerically difficult and in fact the eigenvectors become unreliable.

The eigenvalues obtained for the collinear beam, plane frame and plate structures were computed using method 3 and are in good agreement with the exact values. However, as expected the eigenvectors were generally unreliable. Which eigenvalues had been computed was established by covering a full range of frequencies and plotting the sign change. Method three was chosen because it could be computerized and subroutines for determinant, eigenvalue and eigenvector evaluation had already been written and proven. Methods 1 and 2 are very time consuming for the user with method 2 being the better of these graphical approaches. In these two methods the eigenvalues are located by the variables tending to infinity.

One further comment on the eigenvalue evaluation. When in the displacement approach the structural dynamic stiffness matrix can be expanded into the static stiffness and mass matrices the number of eigenvalues which can be evaluated is equal to the order of the structural dynamic stiffness matrix. Also it is known that the first eigenvalue obtained is either the lowest or the highest depending on the iteration form adopted. In such a displacement method the eigenvalues can be placed in order without consideration of the corresponding eigenvector. This is a distinct advantage when investigating an unfamiliar structure. In a continuous generation process, in particular the rank force method, it is not known how many eigenvalues can be computed. When an eigenvalue is located it is not readily known which one it corresponds to.

Finally, for future eigenvalue evaluation using the rank force method the procedure proposed in Appendix 1 should be considered. In fact, if this procedure can be proven conclusively it becomes a very powerful process in a force formulation.

5.4.2 Element loads and structural response.

The main objectives of a structural vibration analysis is to evaluate the generalized element loads and structural displacements (response) for a given frequency and a given system of applied forcing functions, harmonic in this work. In the displacement approach these can be obtained in two ways ;

1. By evaluating the displacements for a given frequency using the equations,

$$[\mathcal{K}_d]\{{}_t\Delta_\lambda\} = \{{}_tP_\lambda\}$$

   which give,

$$\{{}_t\Delta_\lambda\} = [\mathcal{K}_d]^{-1}\{{}_tP_\lambda\}$$

   The structural displacements correspond to element displacements, the equality of displacements being decided from compatibility considerations. Therefore, using the individual element dynamic stiffness matrices the respective element loads can be calculated. The disadvantage of this method is having to invert the structural dynamic stiffness matrix, this becomes very difficult close to an eigenvalue.

2. Using the normal mode approach in which the structural displacements are expressed in terms of eigenvectors. This method can become difficult when the eigenvalues are clustered together.

Because of the numerical difficulties in the displacement approach for evaluating the element loads and structural displacements it has been a natural feeling that this is where the force approach could be at an advantage. This is because the element loads and structural displacements can be evaluated without using normal modes or having to invert a structural dynamic stiffness matrix. Therefore, the unreliabilty of the eigenvectors in the rank force method presents no problem. However, this is a misconception due to the fact that in a structural static analysis the matrix to be inverted in the force approach, order equal to the degree of static redundancy, is in many cases smaller than in the displacement approach. In vibration analyses the dynamic redundancy is much higher than the static redundancy for a given structure. Therefore, the matrix to be inverted in the rank force method for vibration analysis is, in general, larger than in the displacement approach. In the rank force method the inversion of concern is in the form of an elimination procedure and occurs when the rank technique is extended to the energy equations. The number of energy equations being equal to the degree of dynamic redundancy. To emphasize this point consider a simply supported beam idealized as six finite elements, see figure 39 . The degree of static redundancy is zero but the degree of dynamic redundancy is 12. If the same beam was analysed by the displacement approach the matrix to be inverted would be of order (12 × 12) for both static and dynamic analyses. As a second example consider the plane frame shown in figure 40 . The degree of static redundancy is 6 but the degree of dynamic redundancy is 30. Using the displacement approach the matrix to be inverted would be of order (6 × 6) for both static and dynamic analyses.

5.4.3 Computer storage and running time.

The displacement approach (Direct Stiffness Method) enables a much larger problem to be analysed than the rank force method for a given computer storage. The computer programmes for the vibration analysis of the various structural configurations, adopting the rank force method, were written so that the analyses were carried out within the computer core storage. Table 19 shows the respective structural limitations. These programmes use all the core storage efficiently at the expense of computer running time. In the eigenvalue evaluation no advantage can be taken of sparse matrix techniques , however, these techniques would save storage for element load evaluation only since the assembled independent and energy equations are sparsely populated. A realistic comparison of the two approaches from the point of view of computer storage and running time could not be made since apart from the collinear beam work no displacement programme was available. In any case it is very difficult to realistically assess the two methods in this respect unless the programmes for both approaches are written for the same generality, storage limitations, computer,

computer language, accuracy of results, programming and derivation effort and in each case the programmes should be optimized.

One programme for the eigenvalue evaluation of collinear beam structures using the displacement approach (Direct Stiffness Method) was available. Taking this programme and accepting it as it stood showed that this programme could analyse a collinear beam structure with approximately twice the number of finite elements than possible with the rank force method for a given storage. Although no actual comparison of running times has been made for these respective programmes it is felt that the displacement approach would take less time. In this particular case the displacement formulation was a singular generation process. Again this type of comparison is difficult for the same reasons as stated for the storage problem. Any time comparison which is undertaken for eigenvalue evaluation should be made for the same method of evaluation, initial value and step size. Accuracy should also be considered.

5.4.4 Other comments.

1. It will be noticed that procedure 2 for the derivation of element dynamic flexibility matrices was only applied to a plane beam element (Element Type P2/2FD). The reason for this is that based on the collinear beam results procedure 1, which also adopts polynomials to represent the element internal loading, gave better results and is simpler to apply.

2. In the displacement approach the structural element properties can be amended within an analysis far more readily than in the rank force method.

3. The displacement derivation procedure for element dynamic matrices is simpler and more flexible in its application. It lends itself more easily to irregular structural elements. The use of a triangular element in the displacement approach is a distinct advantage, particularly for structures containing curved surfaces.

4. The joint equilibrium equations for collinear beam structures, plane frames and rectangular plate structures(using an s-system) are all Boolean matrices. This means that the selection of redundancies is numerically sound although not necessarily sound from a structural point of view.

5. In the displacement approach the element distorted shape is generally expressed in terms of polynomials and as a result the structural dynamic stiffness matrix can be separated as previously described. The formulation in this state is a singular generation process. If transcendental functions were adopted this matrix could not be expanded and the formulation would be a continuous generation process. However, using transcendental functions would require less elements in a structural idealization and therefore a much larger problem could be analysed for a given computer storage. Do transcendental functions present more numerical difficulties or is a

singular generation process the aim ? This area is worthy of future investigations. In the restricted problems solved in this thesis using transcendental functions no numerical difficulties were encountered.

5.4.5 New developments.

Although the general conclusion is that the displacement approach is the more practical of the two main finite element techniques for vibration analyses the following new developments have occured in the force approach ;

1. A matrix force approach has been formulated for vibration analyses using an automatic selection of redundancy technique and a distributed structural mass representation.

2. Differentiation has been made between static and dynamic redundancy.

3. Procedures have been established for derivation of element dynamic flexibility matrices. Such matrices can be separated into a static flexibility matrix and an inverse mass matrix.

4. A method has been established for transforming the element dynamic flexibility matrix for a rectangular plate element corresponding to the distributed element boundary loading to a matrix corresponding to equivalent discrete nodal loads. This enables more complex

distributed loadings to be considered for the same number of equivalent loads. In other words, although the number of generalized distributed loads may increase, depending on on the complexity of the assumed distributions, the number of equivalent nodal loads will be the same.

5. A technique is suggested for the eigenvalue evaluation by the rank force method using a highly reduced structural dynamic flexibility matrix. This will save computer storage and running time.

6. In structural vibration analyses using the rank force method a delayed imposition of the structural reactions can be made. This is ideal for the analysis of large practical structures using block elements.

5.4.6 Possible areas for continued research.

The amount of computer input data required by the displacement method and the rank force method is the same. Although it is presently felt that the displacement approach is a more practical tool this work on the rank force method does keep the force approach abreast with its counterpart in the advancement of the state-of-the-art, if only as an educational tool. However, impediments that now exist in the force approach may in future years be removed but only if further development and research in this area is continued. An automatic selection of redundancy technique was only developed because of continual research and at the time of development this appeared to be an impossible task. Therefore, suggested research areas are ;

1. Development of improved element dynamic flexibility matrices with consideration towards irregular shapes.
2. Methods of reducing computer storage and running time.
3. Further investigations of the leading submatrix technique for eigenvalue calculations.
4. Investigate possible numerical problems when using an assemblage of finite elements with a wide variation of properties between the elements.
5. Develop programmes which use combinations of element types, such as, plates and beams.
6. Considerations to general loading and damping.
7. Develop programmes for analysis by block elements in order to analyse large practical structures.

Mode	Number of finite elements = NE					
	NE = 1	2	3	4	5	6
1	166·62 (11.0)	150·71 (0.395)	150·24 (0·081)	150·16 (0·0259)	150·14 (0·0107)	150·13 (0·0053)
2		666·51 (11.0)	607·60 (1·18)	602·87 (0·395)	601·50 (0·165)	600·99 (0·081)
3			1499·65 (11.0)	1375·82 (1·83)	1361·87 (0·794)	1356·47 (0·394)
4				2666·05 (11.0)	2457·36 (2·3)	2430·41 (1·18)
5					4165·70 (11.0)	3852·79 (2.66)
6						5998·61 (11.0)

Eigenvalues (ω radians per second) for a simply supported beam using Element Type P1/2FD. The numbers in parentheses give the percentage error of eigenvalues.

Table 2.

Mode	Number of finite elements = NE				
	NE = 1	2	3	4	5
1	208·14 (38·6)	151·32 (0·796)	150·37 (0·162)	150·20 (0·049)	150·16 (0·0224)
2		832·58 (38·6)	615·05 (2·42)	605·28 (0·796)	602·50 (0·332)
3			1873·31 (38·6)	1402·28 (3·78)	1372·96 (1·61)
4				3330·33 (38·6)	2517·21 (4·81)
5					5203·64 (38·6)
6					

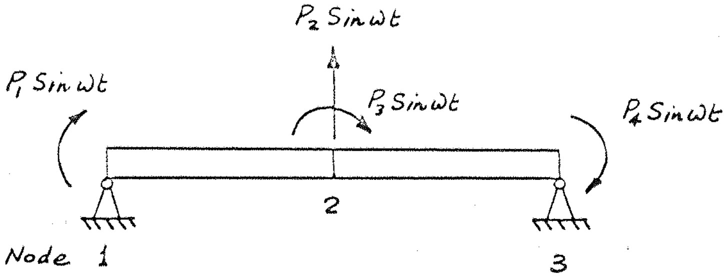
Eigenvalues (ω radians per second) for a simply supported beam using Element Type P2/2FD. The numbers in parentheses give the percentage error of eigenvalues.

Table 3 .

Mode	Eigenvalue
1	150·1264
2	600·5057
3	1351·1378
4	2402·0227
5	3753·1605
6	5404·5511

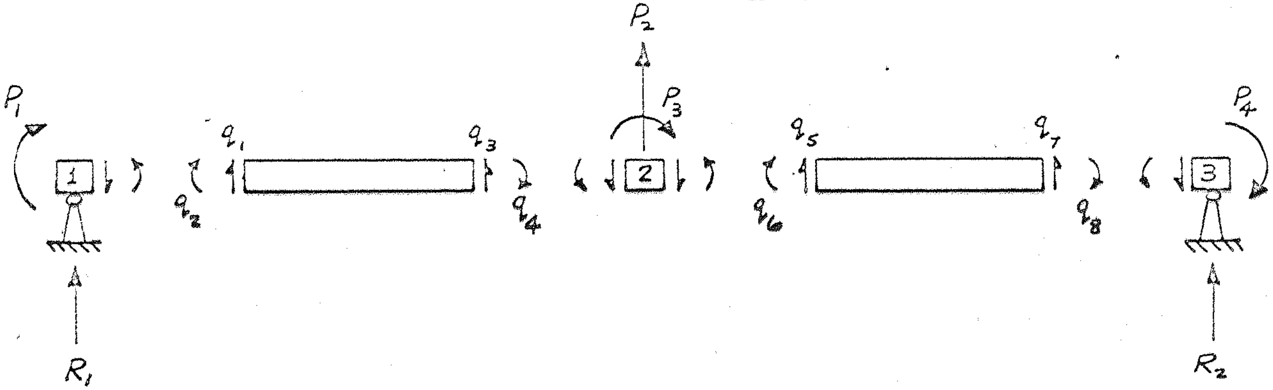
Eigenvalues (ω radians per second) for a simply supported beam using Element Type P3/2FD.

Table 4.



Applied harmonic forcing system.

(a)



Freebody diagram showing complete generalized load system, amplitudes only. All loads are shown positive.

(b)

Simply supported beam idealized as two finite elements.

Fig. 21 .

Total beam length.	60.0 in.
Cross sectional area.	1.366 in^2.
Second moment of area.	0.1 in^4.
Material density.	0.283 lb per in^3.
Young's modulus.	30.0×10^6 lb per in^2.

Beam properties.

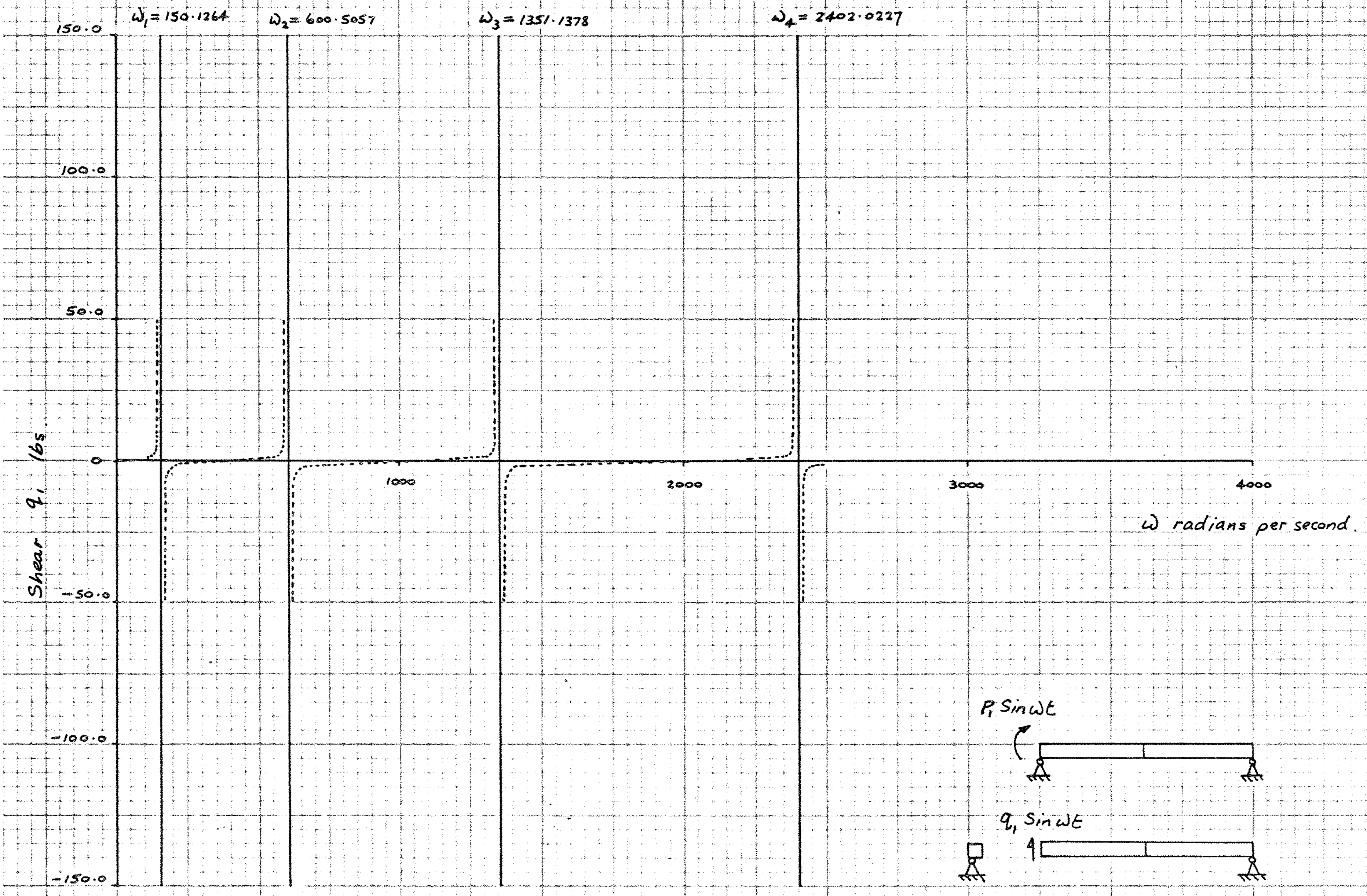
Table 5.

ω radians/sec.	Shear q_1 lbs			
	$P_1=1{\cdot}0$ lb.in.	$P_2=1{\cdot}0$ lb.	$P_3=1{\cdot}0$ lb.in.	$P_4=1{\cdot}0$ lb.in
100·0	0·01116	−1·00665	−0·01756	−0·04244
140·0	0·20820	−4·74484	−0·01848	−0·23740
149·0	2·18200	−42·43473	−0·01874	−2·21062
150·0	19·71357	−377·26227	−0·01877	−19·74212
150·1	93·76422	−1791·52547	−0·01877	−93·79277
150·2	−34·16195	651·68462	−0·01878	34·13341
150·3	−14·47465	275·68499	−0·01878	14·44612
151·0	−2·90380	54·69861	−0·01880	2·87531
152·0	−1·37440	25·49006	−0·01883	1·34598
200·0	−0·08767	0·96274	−0·02062	0·06326
300·0	−0·04731	0·35960	−0·02730	0·03754
400·0	−0·02387	0·26013	−0·04244	0·04620
500·0	0·03017	0·23140	−0·09072	0·09111
598·0	3·92769	0·22829	−3·98248	3·98009
600·0	19·68746	0·22840	−19·74212	19·73967
600·2	32·56818	0·22841	−32·62283	32·62037
600·4	93·73811	0·22842	−93·79274	93·79028
600·6	−107·33251	0·22843	107·27789	−107·28036
600·8	−34·18801	0·22844	34·13340	−34·13587
601·0	−20·35345	0·22846	20·29885	−20·30133
700·0	−0·15987	0·24123	0·11221	−0·11792
900·0	−0·07457	0·31741	0·04812	−0·06500
1200·0	0·04869	0·92672	0·03754	−0·14332
1345·0	3·57612	23·31769	0·03847	−3·65533
1350·0	19·66123	125·71637	0·03854	−19·73990
1350·5	35·09110	223·94574	0·03855	−35·16972
1351·0	160·68056	1023·47289	0·03856	−160·75913
1351·5	−62·65562	−398·32971	0·03856	62·57711
1352·0	−26·27794	−166·74241	0·03857	26·19948
1355·0	−5·92483	−37·17191	0·03862	5·84670
1400·0	−0·54878	−2·96804	0·03945	0·47559
1600·0	−0·16342	−0·62211	0·04620	0·11567
1800·0	−0·10134	−0·37844	0·06025	0·09127
2000·0	−0·04688	−0·29484	0·09111	0·10723
2200·0	0·06694	−0·26062	0·18759	0·19490
2390·0	3·21065	−0·25097	3·31615	3·31676
2400·0	19·63498	−0·25093	19·73967	19·73993
2401·0	38·87211	−0·25093	38·97672	38·97695
2402·0	1494·46368	−0·25093	1494·56820	1494·56840
2403·0	−41·25278	−0·25093	−41·14834	−41·14818
2404·0	−20·40569	−0·25093	−20·30133	−20·30120
2408·0	−6·81901	−0·25093	−6·71497	−6·71498

Unit variations of shear load q_1 with frequency ω.

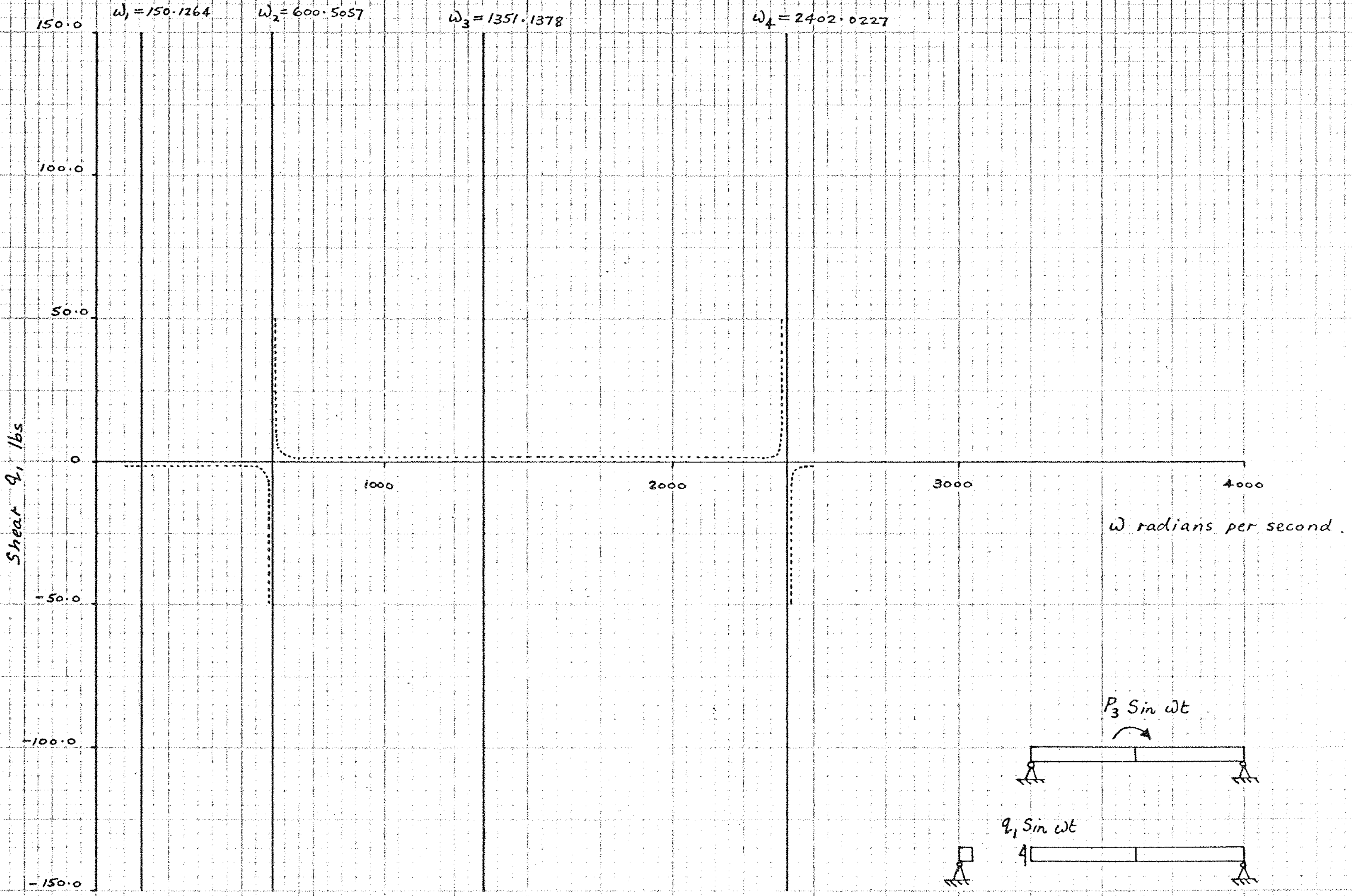
Element Type P3/2FD.

Table 6.



Amplitude distribution of shear load q_1 for $P_1 = 1 \cdot 0$ lb.in.
Element Type P3/2FD.

Fig. 22.



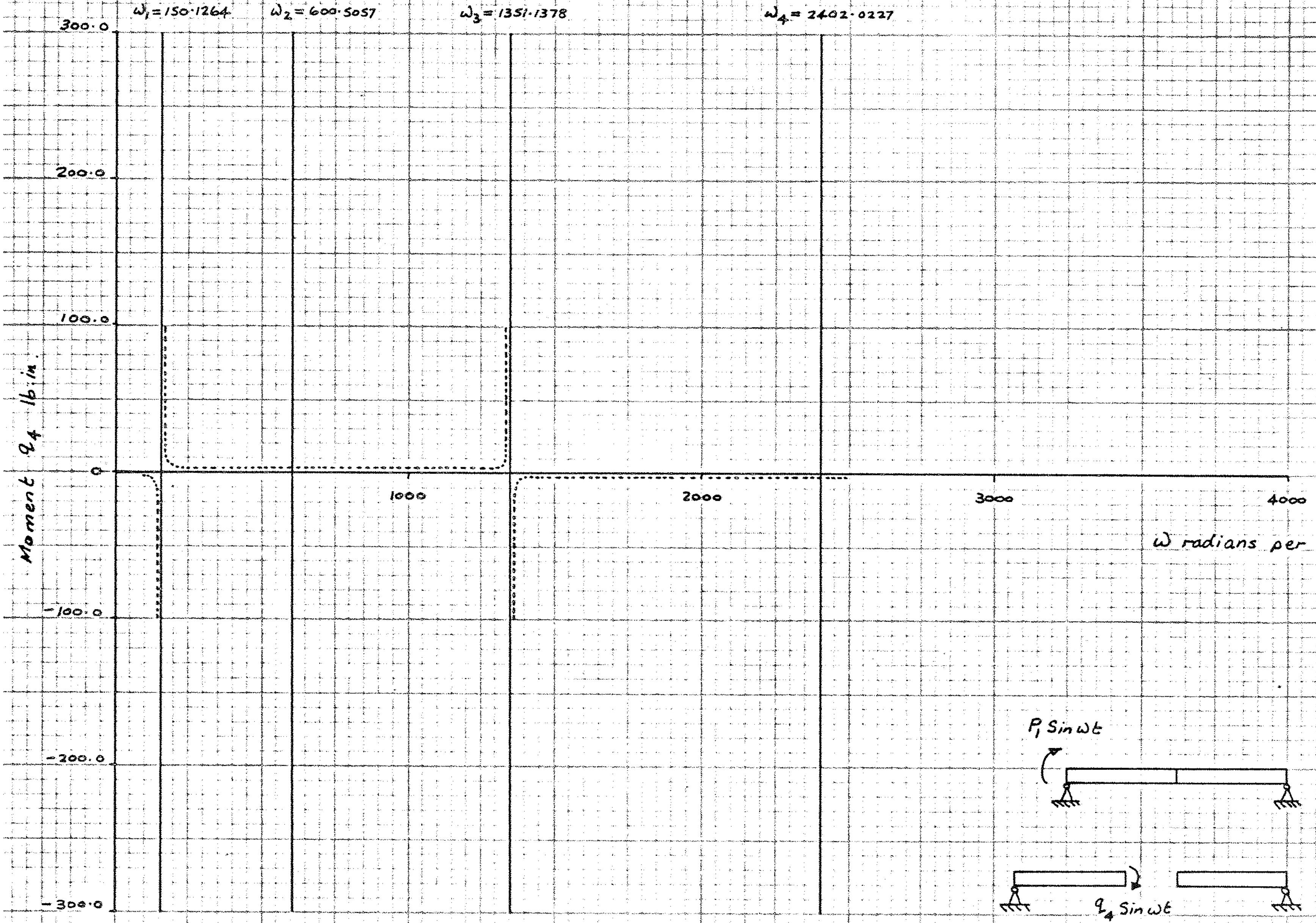
Amplitude distribution of shear load q_1 for $P_3 = 1.0$ lb.in
Element Type P3/2FD.

Fig. 23.

ω radians/sec	Moment q_4 lb.in.			
	$P_1 = 1.0$ lb.in.	$P_2 = 1.0$ lb.	$P_3 = 1.0$ lb.in.	$P_4 = 1.0$ lb.in.
100.0	-1.00665	24.70518	0.50000	1.00665
140.0	-4.74484	96.12707	0.50000	4.74484
149.0	-42.43473	815.95865	0.50000	42.43473
150.0	-377.26227	7210.69450	0.50000	377.26227
150.1	-1791.52547	34221.13192	0.50000	1791.52547
150.2	651.68462	-12440.74361	0.50000	-651.68462
150.3	275.68499	-5259.67956	0.50000	-275.68499
151.0	54.69861	-1039.15001	0.50000	-54.69861
152.0	25.49006	-481.30688	0.50000	-25.49006
200.0	0.96274	-12.81956	0.50000	-0.96274
300.0	0.35960	-1.14679	0.50000	-0.35960
400.0	0.26013	0.98430	0.50000	-0.26013
500.0	0.23140	1.86141	0.50000	-0.23140
598.0	0.22829	2.36758	0.50000	-0.22829
600.0	0.22840	2.37623	0.50000	-0.22840
600.2	0.22841	2.37710	0.50000	-0.22841
600.4	0.22842	2.37796	0.50000	-0.22842
600.6	0.22843	2.37882	0.50000	-0.22843
600.8	0.22844	2.37968	0.50000	-0.22844
601.0	0.22846	2.38054	0.50000	-0.22846
700.0	0.24123	2.77143	0.50000	-0.24123
900.0	0.31741	3.60563	0.50000	-0.31741
1200.0	0.92672	7.75782	0.50000	-0.92672
1345.0	23.31769	150.39882	0.50000	-23.31769
1350.0	125.71637	802.29222	0.50000	-125.71637
1350.5	223.94574	1427.64012	0.50000	-223.94574
1351.0	1023.47289	6517.58836	0.50000	-1023.47289
1351.5	-398.32971	-2533.88790	0.50000	398.32971
1352.0	-166.74242	-1059.5570	0.50000	166.74242
1355.0	-37.17191	-234.68370	0.50000	37.17191
1400.0	-2.96804	-16.90735	0.50000	2.96804
1600.0	-0.62211	-1.85128	0.50000	0.62211
1800.0	-0.37844	-0.17219	0.50000	0.37844
2000.0	-0.29484	0.50632	0.50000	0.29484
2200.0	-0.26062	0.90201	0.50000	0.26062
2390.0	-0.25097	1.17780	0.50000	0.25097
2400.0	-0.25093	1.19099	0.50000	0.25093
2401.0	-0.25093	1.19230	0.50000	0.25093
2402.0	-0.25093	1.19361	0.50000	0.25093
2403.0	-0.25093	1.19493	0.50000	0.25093
2404.0	-0.25093	1.19624	0.50000	0.25093
2408.0	-0.25093	1.20148	0.50000	0.25093

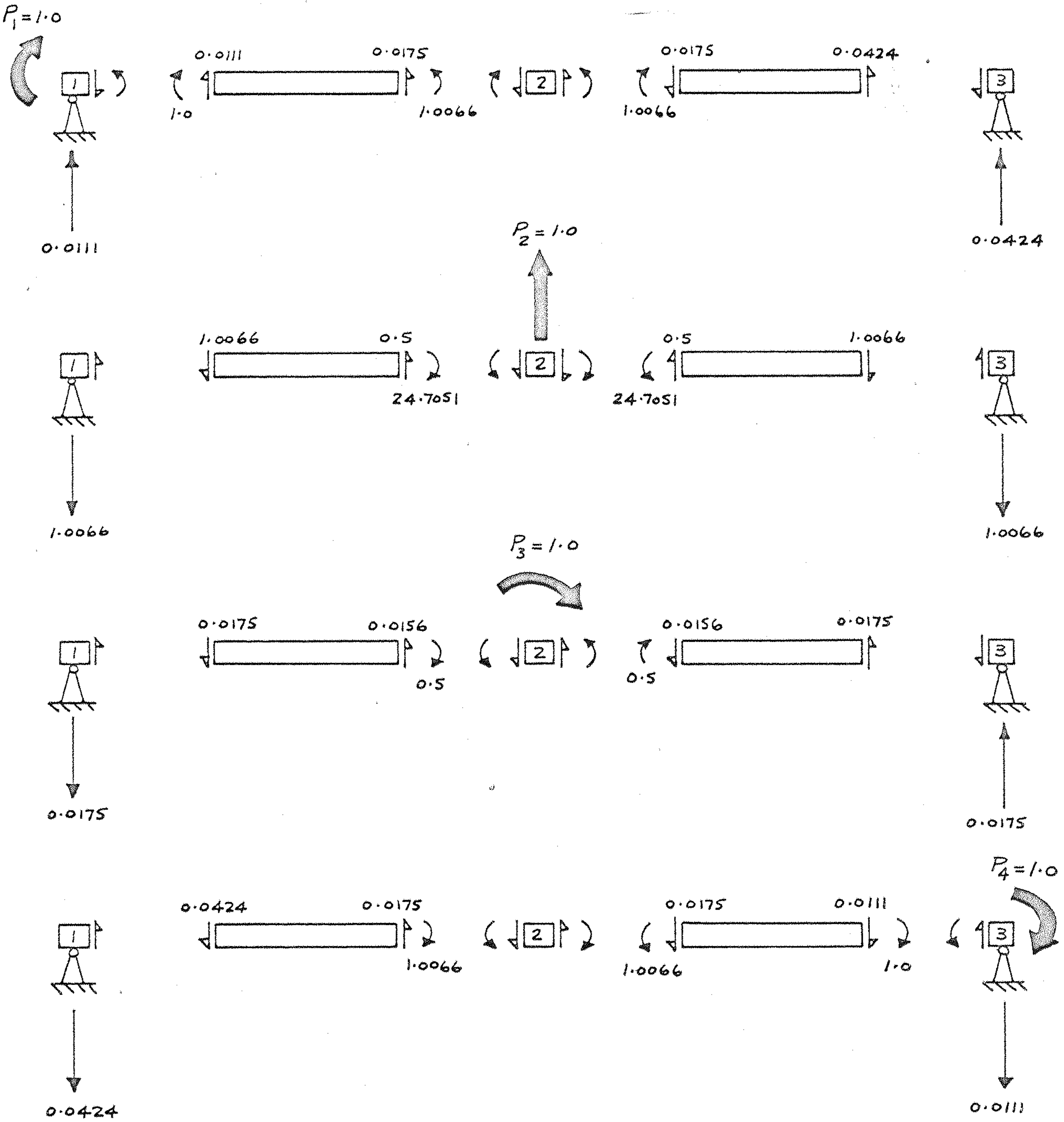
Unit variations of bending moment q_4 with frequency ω. Element Type P3/2FD.

Table 7.



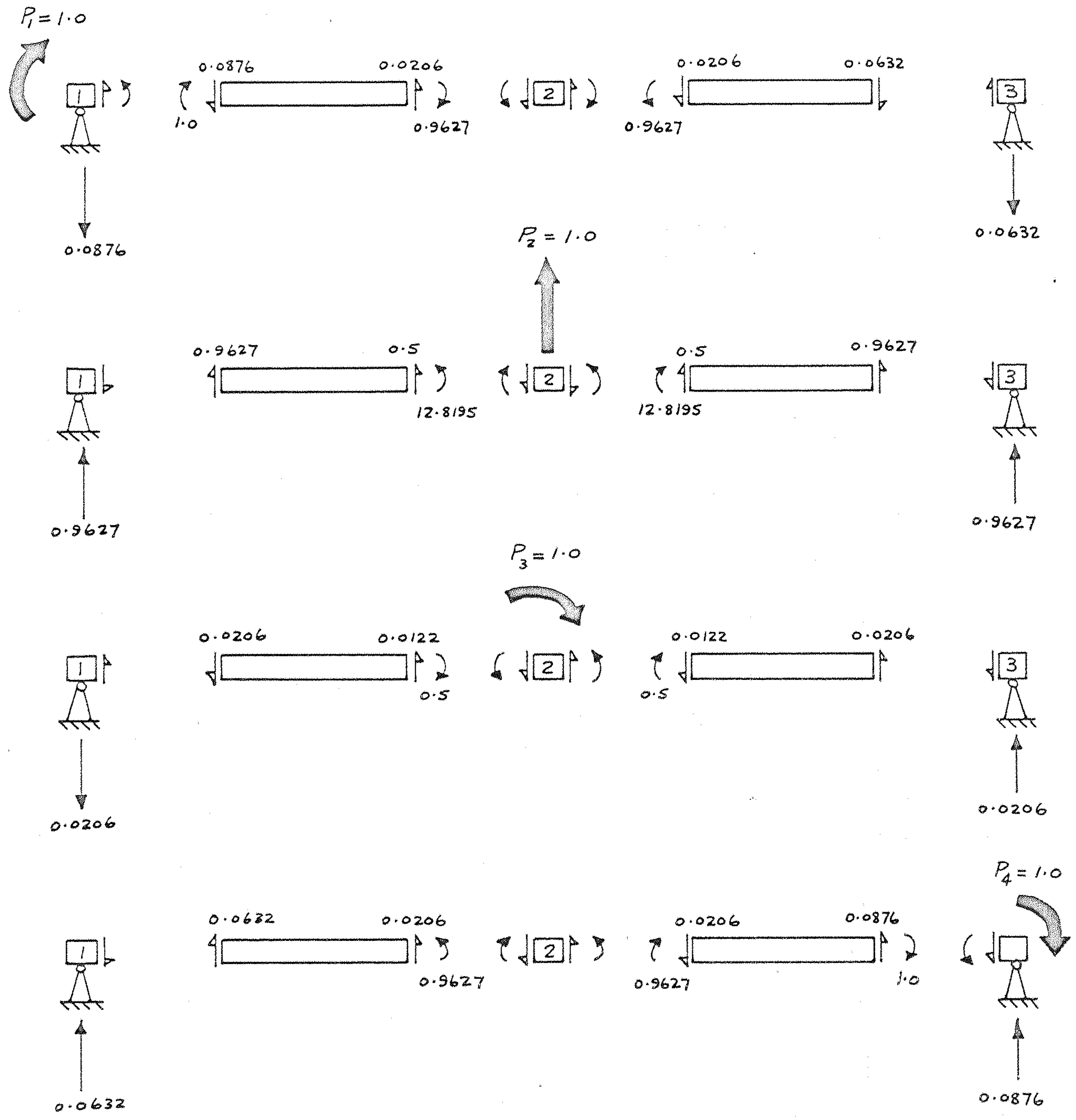
Amplitude distribution of bending moment q_4 for $P_1 = 1.0$ lb.in.
Element Type P3/2FD.

Fig. 24.



Unit load distributions for $\omega = 100.0$ radians per second, amplitudes only. Element Type P3/2FD.

Fig. 25 .



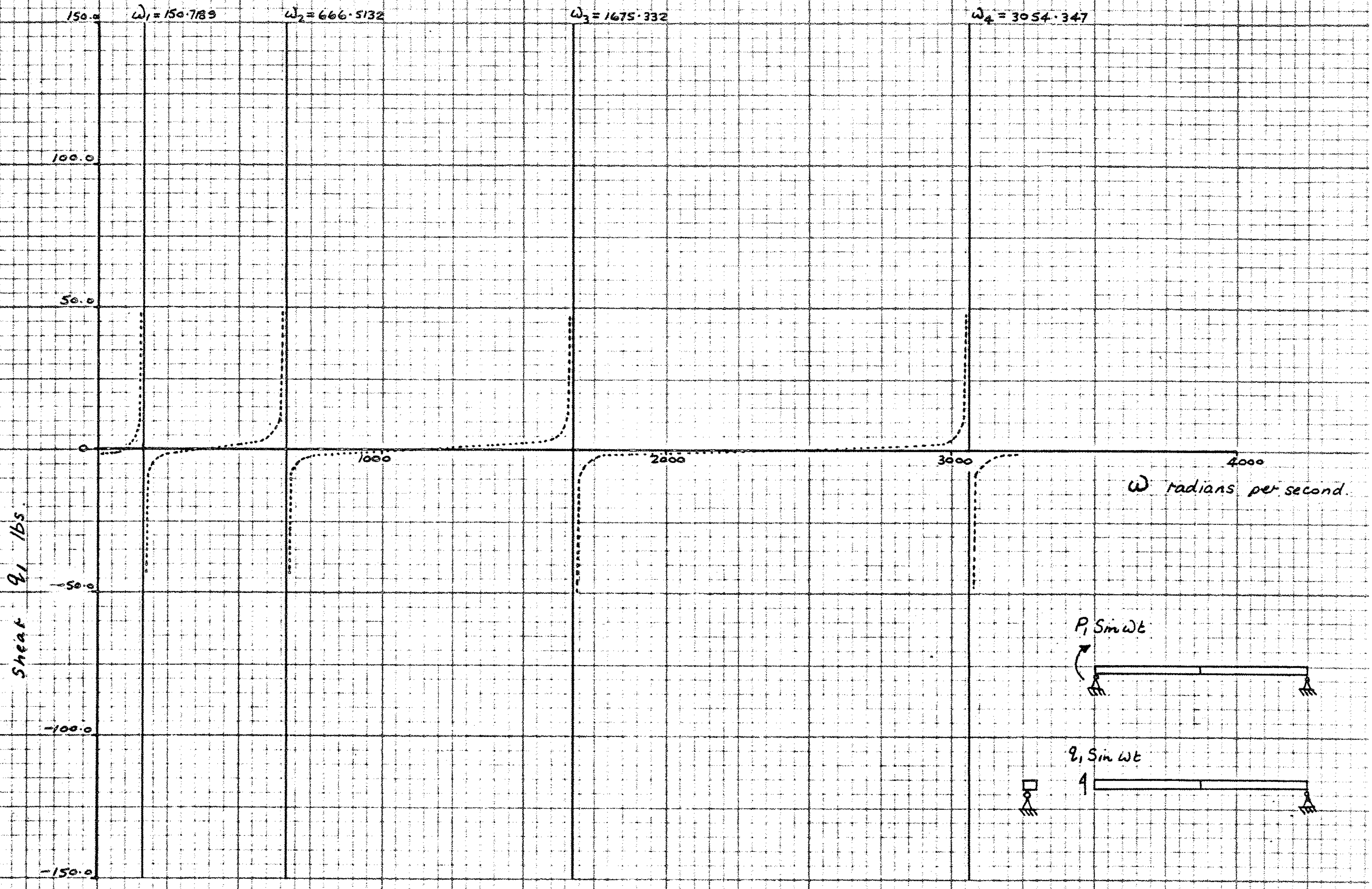
Unit load distributions for $\omega = 200.0$ radians per second, amplitudes only. Element Type P3/2FD.

Fig. 26 .

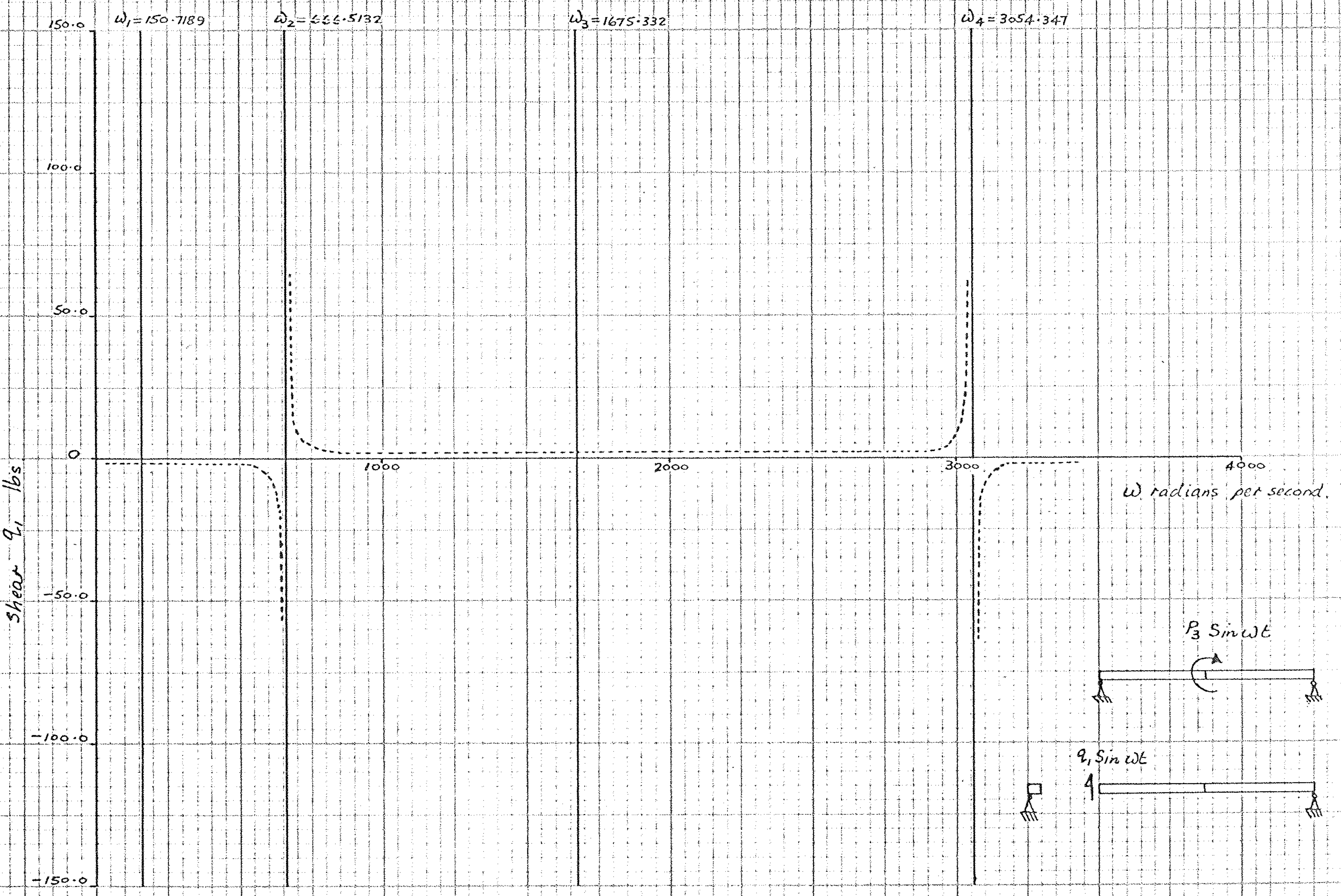
ω radians/sec.	Shear q_1 lbs.			
	$P_1 = 1.0$ lb.in.	$P_2 = 1.0$ lb.	$P_3 = 1.0$ lb.in.	$P_4 = 1.0$ lb.in.
20	−0·01601	−0·51145	−0·01670	−0·01723
60	−0·00989	−0·62045	−0·01698	−0·02270
100	0·01097	−1·00360	−0·01756	−0·04226
140	0·19687	−4·53521	−0·01846	−0·22611
145	0·40313	−8·47785	−0·01860	−0·43206
150	3·47674	−67·28126	−0·01874	−3·50535
155	−0·62950	11·28776	−0·01889	0·60123
160	−0·31120	5·20207	−0·01905	0·28328
200	−0·08898	0·99063	−0·02053	0·06439
240	−0·06410	0·56716	−0·02251	0·04391
280	−0·05257	0·41367	−0·02510	0·03808
500	0·00815	0·23955	−0·06868	0·06897
540	0·03613	0·23583	−0·09438	0·09349
580	0·08855	0·23501	−0·14446	0·14242
620	0·22838	0·23654	−0·28181	0·27861
660	2·06374	0·24009	−2·11455	2·11012
665	9·10864	0·24067	−9·15910	9·15452
667	−28·59024	0·24091	28·53991	−28·54456
670	−4·05000	0·24127	3·99987	−4·00462
700	−0·48072	0·24549	0·43273	−0·43846
740	−0·25267	0·25262	0·20771	−0·21488
1400	0·06483	0·94102	0·05272	−0·17489
1440	0·10099	1·11624	0·05303	−0·20703
1500	0·18459	1·53046	0·05379	−0·28468
1580	0·45459	2·89850	0·05533	−0·54667
1620	0·87848	5·06906	0·05632	−0·96646
1640	1·44876	7·99831	0·05687	−1·53466
1670	10·30694	53·57637	0·05776	−10·38964
1680	−12·05778	−61·52125	0·05808	11·97615
1700	−2·38023	−11·72383	0·05874	2·30079
1740	−0·98219	−4·54102	0·06019	0·90725
2800	0·10914	−0·38882	0·33463	0·38344
2840	0·17607	−0·38069	0·39982	0·44718
2900	0·33976	−0·36955	0·56112	0·60652
2960	0·70874	−0·35950	0·92793	0·97155
3000	1·40487	−0·35334	1·62273	1·66527
3050	20·26036	−0·34618	20·47666	20·51794
3055	−136·66404	−0·34549	−136·44789	−136·40672
3100	−2·18305	−0·33955	−1·96819	−1·92807
3160	−1·07351	−0·33223	−0·86024	−0·82140

Unit variations of shear load q_1 with frequency ω .
Element Type P1/2FD.

Table 8 .



Amplitude distribution of shear load q_1 for $P_1 = 1.0$ lb.in.



Amplitude distribution of shear load q_1 for $P_3 = 1.0$ lb.in.

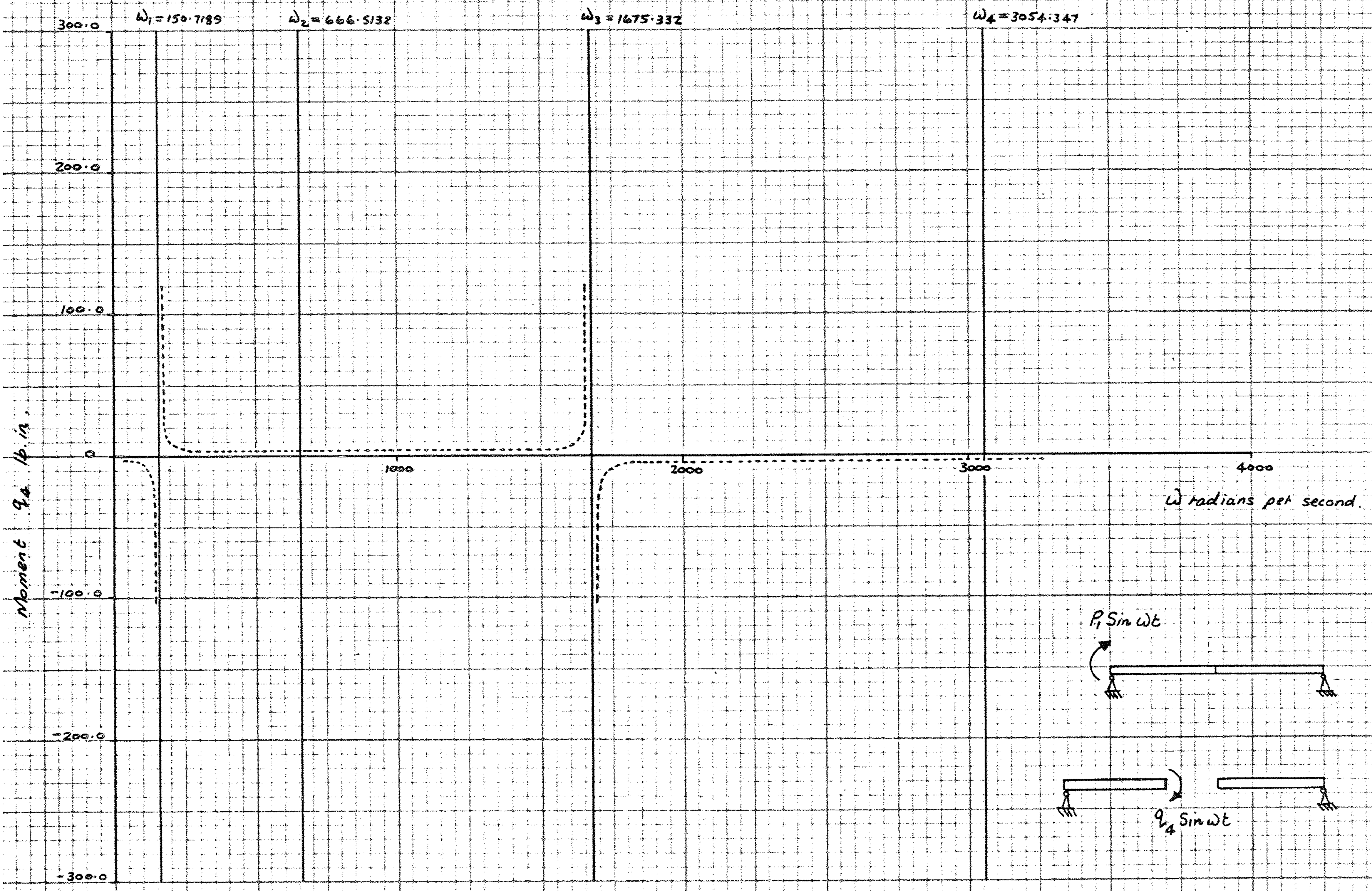
Element Type PI/2FD.

Fig. 28.

ω radians/sec.	Moment q_4 lb.in			
	$P_1 = 1.0$ lb.in.	$P_2 = 1.0$ lb.	$P_3 = 1.0$ lb.in.	$P_4 = 1.0$ lb.in.
20	−0·51145	15·21997	0·5	0·51145
60	−0·62042	17·31089	0·5	0·62042
100	−1·00326	24·64784	0·5	1·00326
140	−4·52977	92·13895	0·5	4·52977
145	−8·46630	167·45945	0·5	8·46630
150	−67·17779	1290·79024	0·5	67·17779
155	11·26825	−210·11667	0·5	−11·26825
160	5·19201	−93·85676	0·5	−5·19201
200	0·98651	−13·36510	0·5	−0·98651
240	0·56290	−5·22578	0·5	−0·56290
280	0·40869	−2·23401	0·5	−0·40869
500	0·22631	1·62058	0·5	−0·22631
540	0·22036	1·83057	0·5	−0·22036
580	0·21703	2·00123	0·5	−0·21703
620	0·21578	2·14433	0·5	−0·21578
660	0·21624	2·26790	0·5	−0·21624
665	0·21640	2·28227	0·5	−0·21640
667	0·21648	2·28796	0·5	−0·21648
670	0·21659	2·29644	0·5	−0·21659
700	0·21819	2·37761	0·5	−0·21819
740	0·22150	2·47767	0·5	−0·22150
1400	0·65279	5·18822	0·5	−0·65279
1440	0·76428	5·77207	0·5	−0·76428
1500	1·02799	7·14287	0·5	−1·02799
1580	1·89939	11·64315	0·5	−1·89939
1620	3·28225	18·76671	0·5	−3·28225
1640	5·14856	28·37456	0·5	−5·14856
1670	34·18829	177·82543	0·5	−34·18829
1680	−39·14552	−199·56825	0·5	39·14552
1700	−7·41743	−36·28411	0·5	7·41743
1740	−2·84102	−12·72706	0·5	2·84102
2800	−0·19655	0·95449	0·5	0·19655
2840	−0·19139	0·98231	0·5	0·19139
2900	−0·18432	1·02052	0·5	0·18432
2960	−0·17795	1·05501	0·5	0·17795
3000	−0·17404	1·07620	0·5	0·17404
3050	−0·16949	1·10088	0·5	0·16949
3055	−0·16906	1·10324	0·5	0·16906
3100	−0·16529	1·12374	0·5	0·16529
3160	−0·16064	1·14904	0·5	0·16064

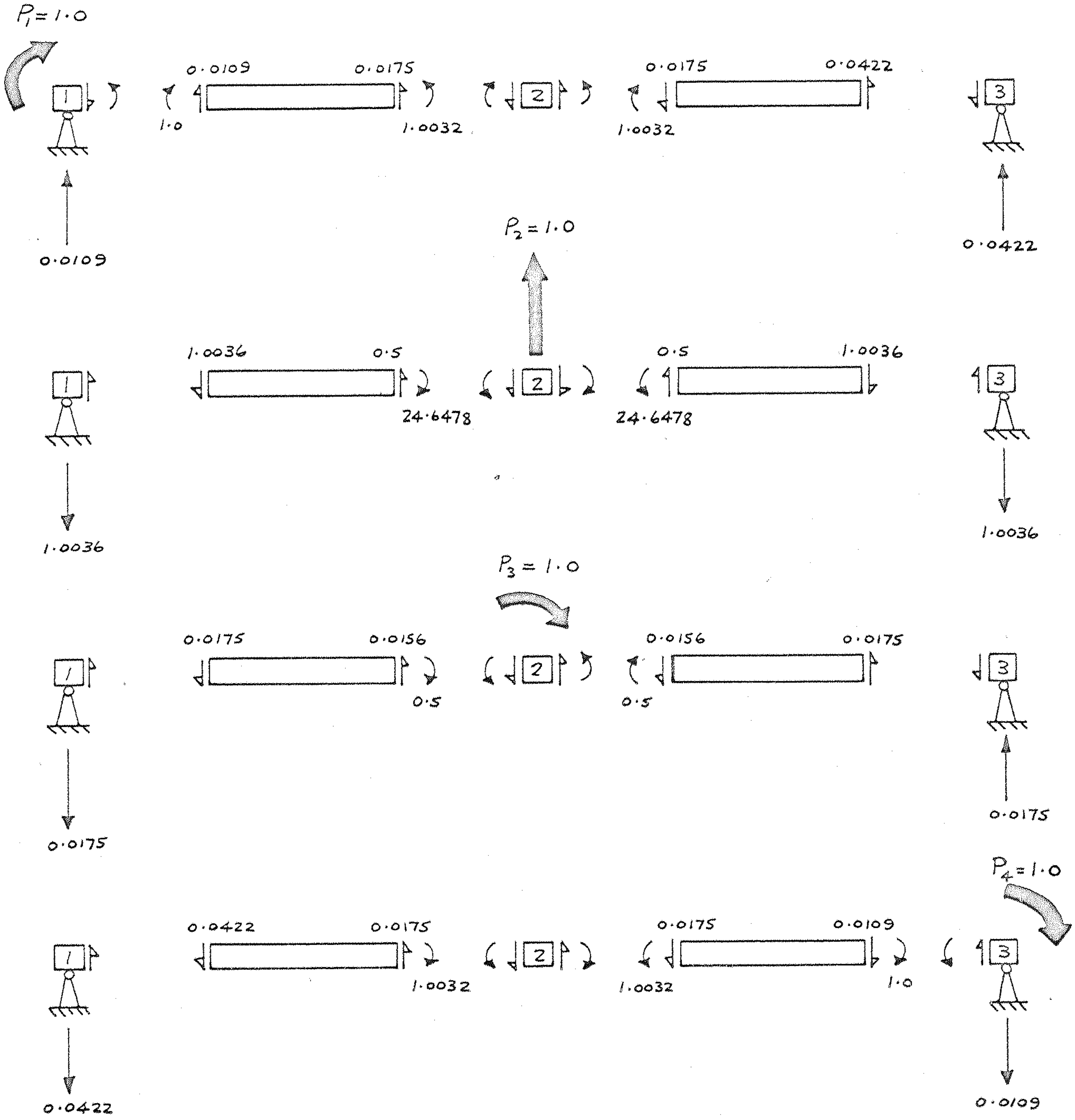
Unit variations of bending moment q_4 with frequency ω.
Element Type P1/2FD.

Table 9.



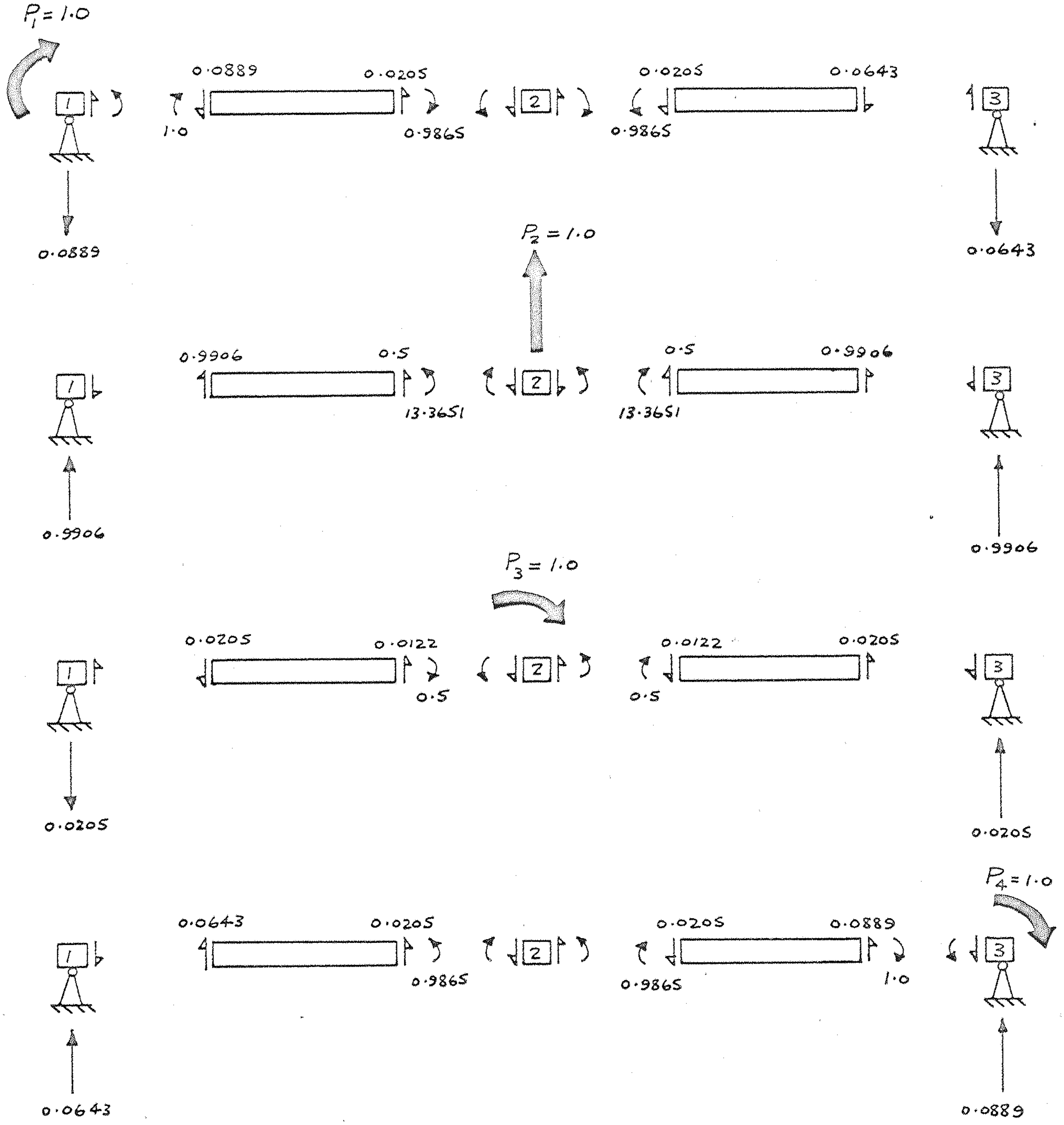
Amplitude distribution of bending moment q_4 for $P_1 = 1{\cdot}0$ lb.in.
Element Type P1/2FD.

Fig. 29.



Unit load distributions for $\omega = 100.0$ radians per second, amplitudes only. Element Type P1/2FD.

Fig. 30 .



Unit load distributions for $\omega = 200.0$ radians per second, amplitudes only. Element Type P1/2FD.

Fig. 31.

Frequency ω radians/second		Applied load			
		$P_1 = 1.0$ lb.in.	$P_2 = 1.0$ lb.	$P_3 = 1.0$ lb.in.	$P_4 = 1.0$ lb.in.
100.0	P3/2FD	$q_1 = 0.01116$	−1.00665	−0.01756	−0.04244
	P1/2FD	$q_2 = 0.01097$	−1.00360	−0.01756	−0.04226
	Difference	0.00019	−0.00305	0.0	−0.00018
	Percentage error.	1.70	0.316	0.0	0.425
150.0	P3/2FD	$q_1 = 19.71357$	−377.26227	−0.01877	−19.74212
	P1/2FD	$q_1 = 3.47674$	−67.28126	−0.01874	−3.50535
	Difference	16.23683	−309.98101	−0.00003	−16.23677
	Percentage error.	82.4	82.3	0.167	82.4
200.0	P3/2FD	$q_1 = -0.08767$	0.96274	−0.02062	0.06326
	P1/2FD	$q_1 = -0.08898$	0.99063	−0.02053	0.06439
	Difference	0.00131	−0.02789	−0.00009	−0.00113
	Percentage error.	−1.49	−2.9	0.436	−1.78

Percentage error in shear load q_1 when using Element Type P1/2FD.

Table 10 .

Frequency ω radians/second		Applied load			
		$P_1 = 1{\cdot}0$ lb.in.	$P_2 = 1{\cdot}0$ lb.	$P_3 = 1{\cdot}0$ lb.in.	$P_4 = 1{\cdot}0$ lb.in.
100·0	P3/2FD	− 1·00665	24·70518	0·5	1·00665
	P1/2FD	− 1·00326	24·64784	0·5	1·00326
	Difference	− 0·00339	0·05734	0·0	0·00339
	Percentage error	0·337	0·232	0·0	0·337
150·0	P3/2FD	− 377·26227	7210·69450	0·5	377·26227
	P1/2FD	− 67·17779	1290·79024	0·5	67·17779
	Difference	− 310·08448	5919·90426	0·0	310·08448
	Percentage error	82·4	82·1	0·0	82·4
200·0	P3/2FD	0·96274	− 12·81956	0·5	− 0·96274
	P1/2FD	0·98651	− 13·36510	0·5	− 0·98651
	Difference	− 0·02377	− 0·54554	0·0	0·02377
	Percentage error	− 2·47	− 4·25	0·0	− 2·47

Percentage error in bending moment q_4 when using Element Type P1/2FD.

Table 11 .

Mode number.	Eigenvalue ω radians per second.			
	Hybrid Method. (Levien[23,24])	Displacement Method. (Burch[40])	Rank Force Method.	
			Element Type P3/3FD	Element Type P1/3FD
1	194·8	188·5 – 194·8	194·3	194·4
2	766·7	754·1 – 785·4	766·6	860·2
3	1250	1226 – 1257	1250·5	1532·4
4	1357	1358	1354·0	–

Comparison of eigenvalues for a single storey plane frame.

Table 12 .

Mode number.	Eigenvalue ω radians per second.			
	Hybrid Method. (Levien[23,24])	Displacement Method. (Burch[40])	Rank Force Method.	
			Element Type P3/3FD	Element Type P1/3FD
1	103·7	100·6 – 113·1	100·2	100·2
2	314·2	307·9 – 314·2	307·3	300·0
3	754·1	741·5 – 754·1	740·2	739·2
4	1081	1100 – 1131	1080·9	1190·0

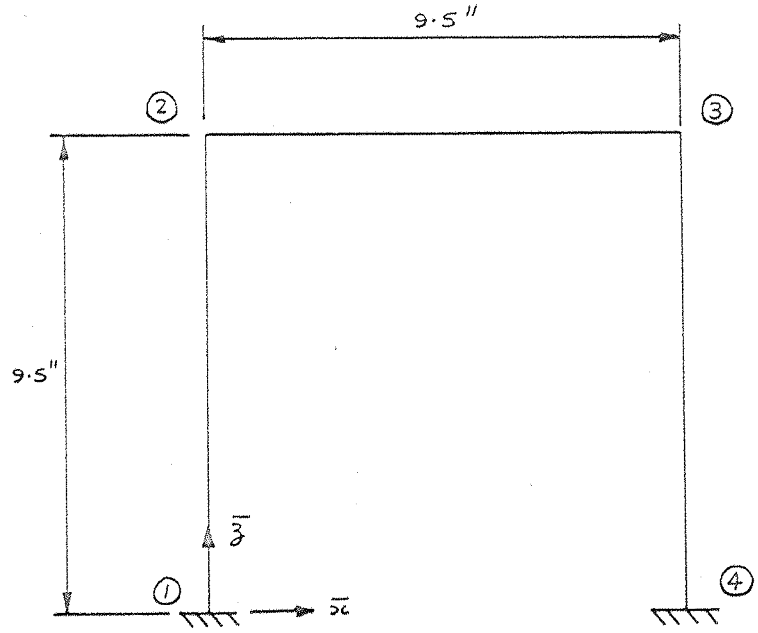
Comparison of eigenvalues for a two storey plane frame.

Table 13.

Cross sectional area.	0.09396 in^2.
Second moment of area.	0.00006866 in^4.
Material density.	0.283 lb per in^3.
Young's modulus.	30.0×10^6 lb per in^2.

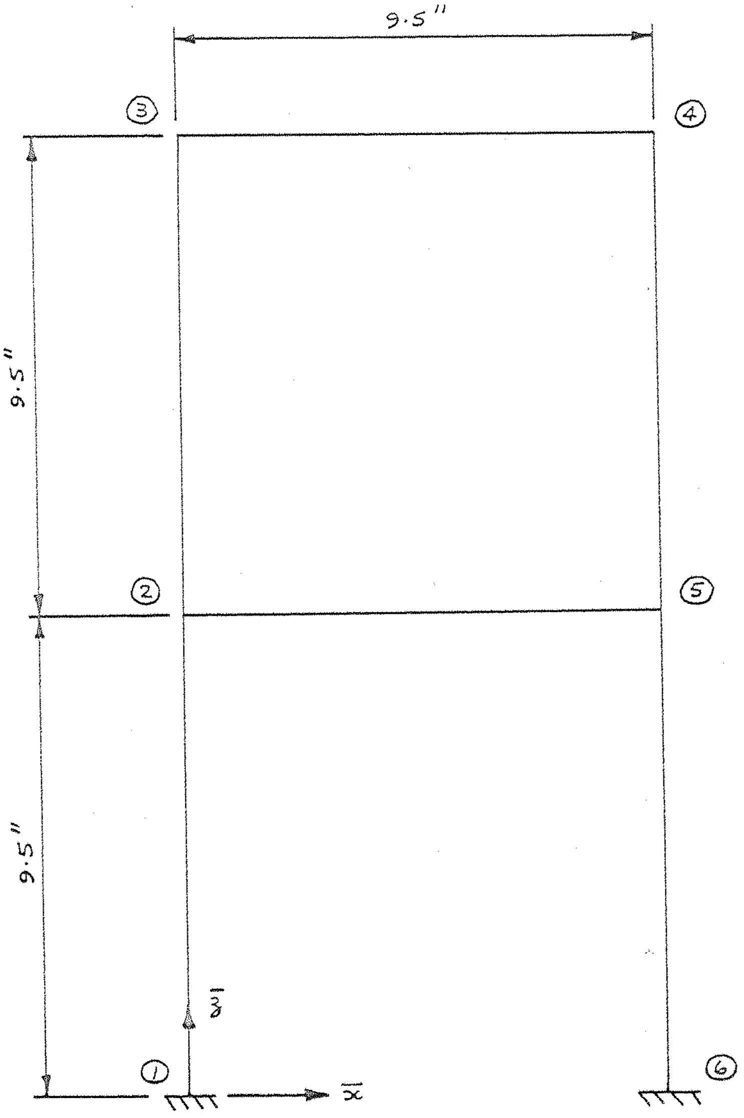
Frame element properties.

Table 14 .



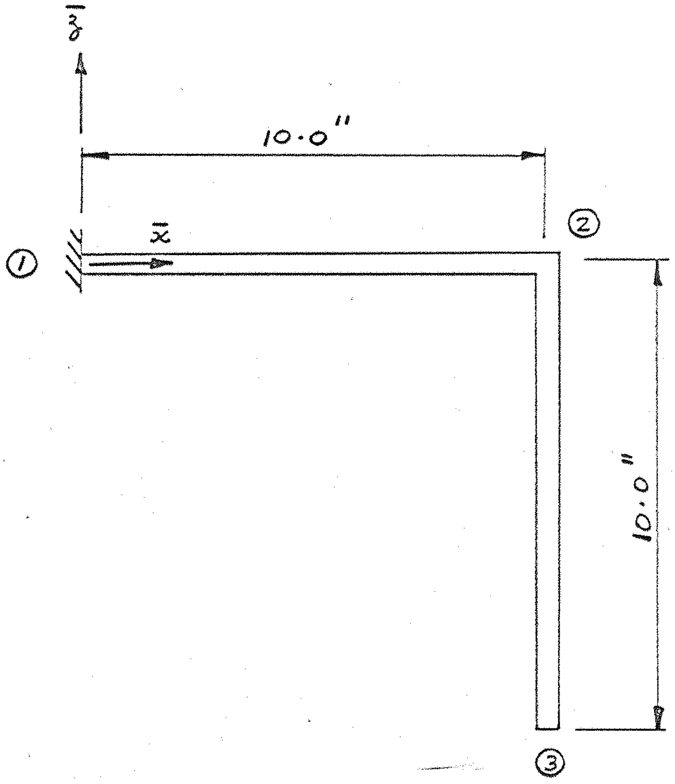
Single storey plane frame.

Fig. 32 .



Two storey plane frame.

Fig. 33.



Bent cantilever.

Fig. 34 .

Cross sectional area.	0.0313 in^2.
Second moment of area.	0.00004068 in^4.
Material density.	0.283 lb per in^3.
Young's modulus.	30.0×10^6 lb per in^2.

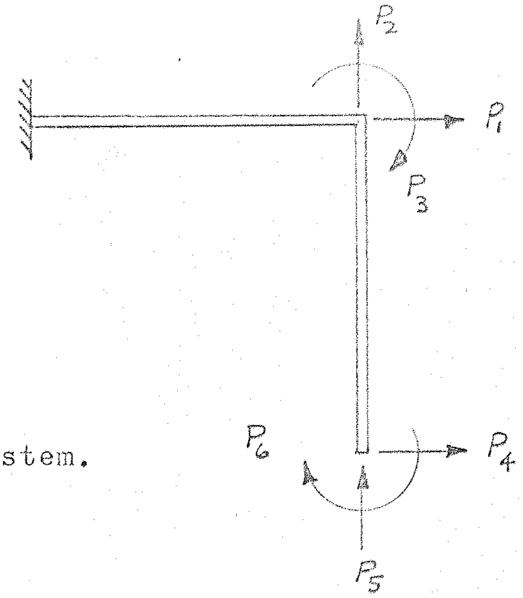
Element properties for bent cantilever.

Table 15 .

Mode number	Eigenvalue ω radians per second.	
	Bishop and Johnson[3]	Rank Force Method. Element Type P3/3FD.
1	83·2	85·6
2	226	233

Comparison of eigenvalues for a bent cantilever.

Table 16.

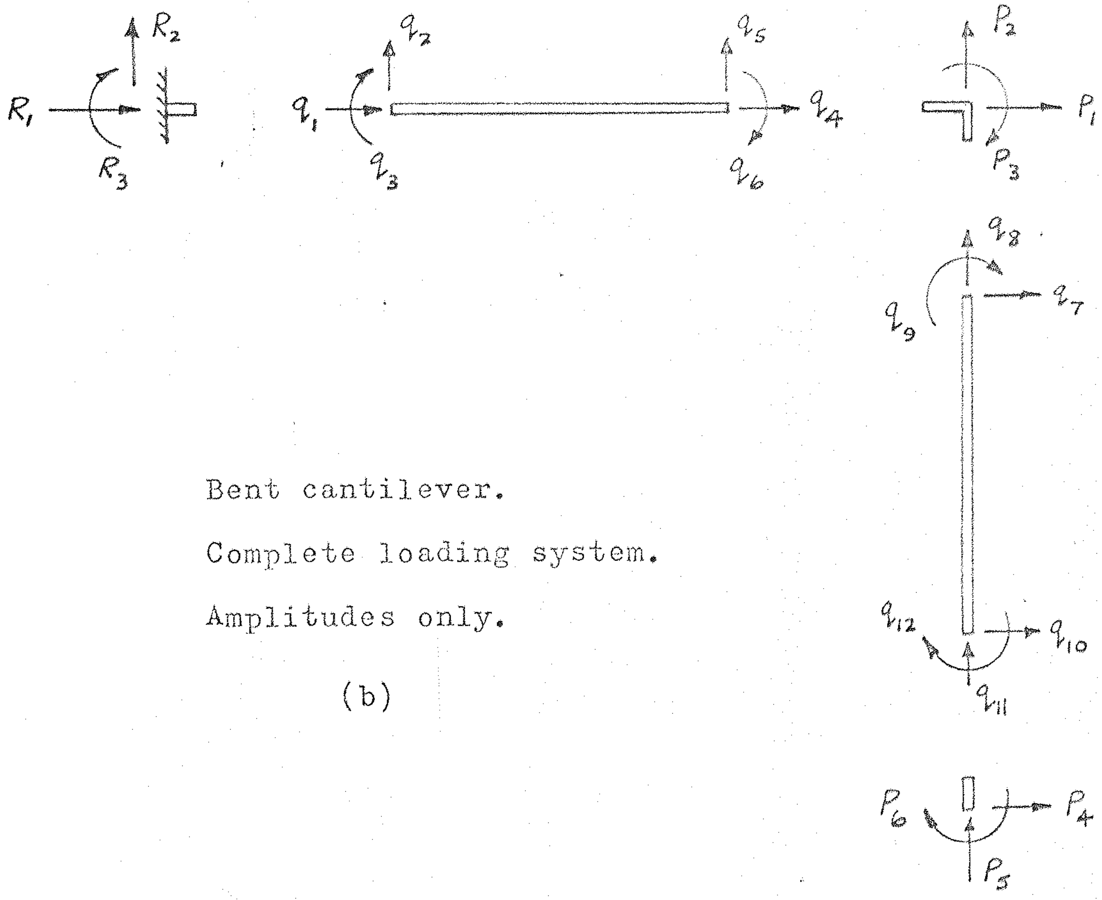


Bent cantilever.

Applied loading system.

Amplitudes only.

(a)



Bent cantilever.

Complete loading system.

Amplitudes only.

(b)

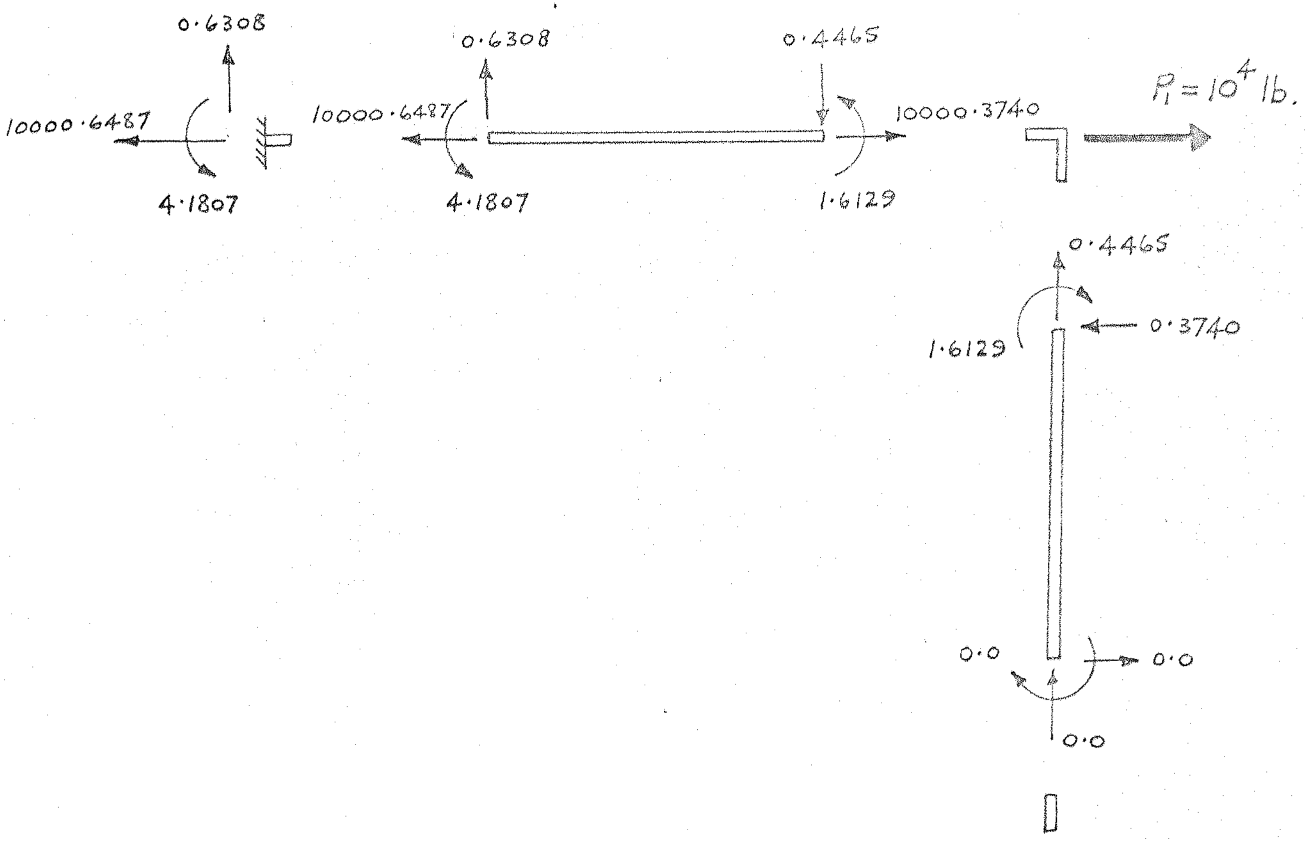
Loading system for a bent cantilever.

Fig. 35 .

q_α, R_e	$P_1=10^4$ lb.	$P_2=10^4$ lb.	$P_3=10^4$ lb.in.	$P_4=10^4$ lb.	$P_5=10^4$ lb.	$P_6=10^4$ lb.in.
q_1	−10000·6487	8129·19	−873·25	−4177·67	8129·41	−491·35
q_2	0·6308	−5597·80	−1570·59	24942·49	−5597·96	−2834·03
q_3	−4·1807	1904·56	10408·66	−165299·52	1904·61	18781·77
q_4	10000·3740	−8128·96	873·22	4177·56	−8129·19	491·34
q_5	−0·4465	6449·87	1111·72	−17655·28	6450·04	2006·04
q_6	−1·6129	55708·95	4015·66	−63772·47	55710·48	7245·99
q_7	−0·3740	8128·96	−873·22	−4177·56	8129·19	−491·34
q_8	0·4465	3550·13	−1111·72	17655·28	−6450·04	−2006·04
q_9	1·6129	−55708·95	5984·34	63772·47	−55710·48	−7245·99
q_{10}	0·0	0·0	0·0	10000·00	0·0	0·0
q_{11}	0·0	0·0	0·0	0·0	10000·00	0·0
q_{12}	0·0	0·0	0·0	0·0	0·0	10000·00
R_1	−10000·6487	8129·19	−873·25	−4177·67	8129·41	−491·35
R_2	0·6308	−5597·80	−1570·59	24942·49	−5597·96	−2834·03
R_3	−4·1807	1904·56	10408·66	−165299·52	1904·61	18781·77

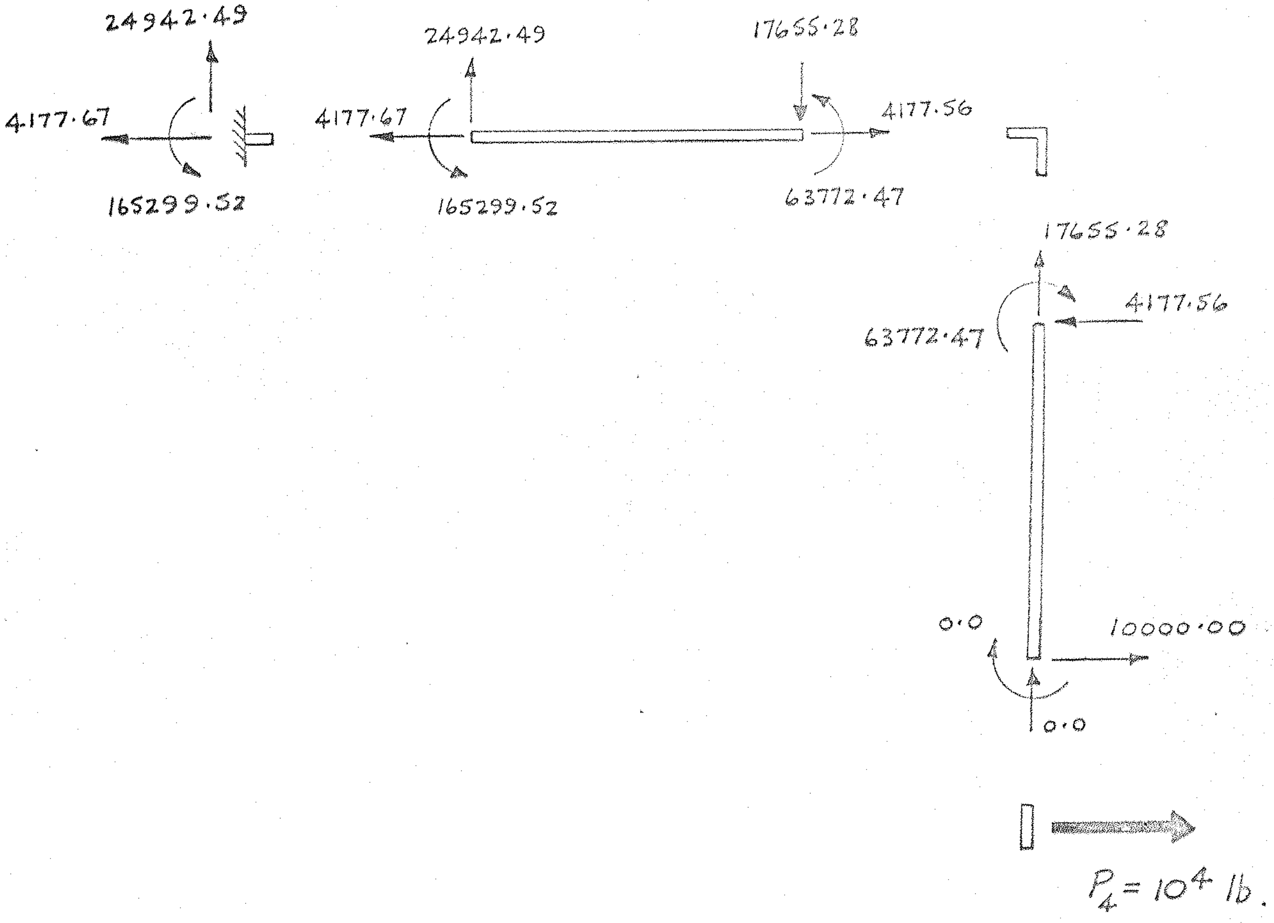
Unit matrix of unknowns for a bent cantilever at a frequency of 150 radians per second using Element Type P3/3FD. Amplitudes only.

Table 17 .



Bent cantilever. Element loads and structural reactions for an applied load of $P_1 = 10^4$ lb and a frequency of 150 radians per second using Element Type P3/3FD. Amplitudes only.

Fig. 36 .



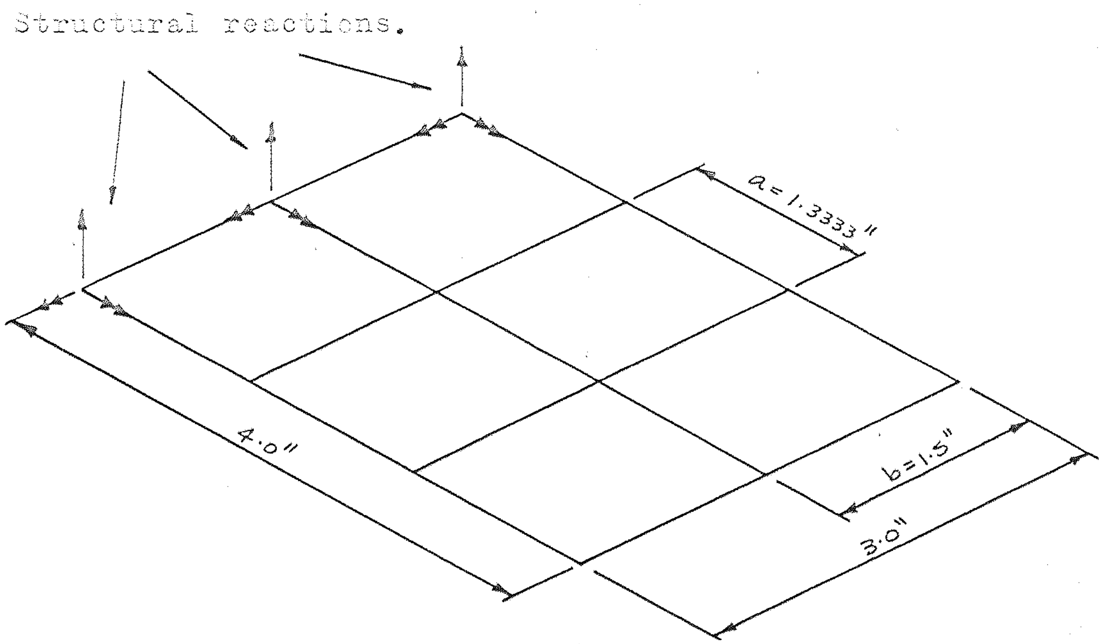
Bent cantilever. Element loads and structural reactions for an applied load of $P_4 = 10^4$ lb and a frequency of 150 radians per second using Element Type P3/3FD. Amplitudes only.

Fig. 37 .

Mode	Eigenvalues , radians per second .					
	Barton (Ritz Method) $\nu = 0{\cdot}3$	Zienkiewicz. Experimental. $\nu = 0{\cdot}34$	Rank Force Method. 6 finite elements. $\nu = 0{\cdot}34$		Zienkiewicz. 12 finite elements. $\nu = 0{\cdot}34$	
			q-system	s-system. (3 equations)	Displacement Method	Lumped mass.
1	2359	2356	2300	2400	2362	2300
2	7293	7176	7556	7537	7521	6763

Eigenvalues of a rectangular cantilever plate.

Table 18 .



t	0.19 in.
μ	0.098 lb per in^3.
ν	0.34
E	9×10^6 lb per in^2.

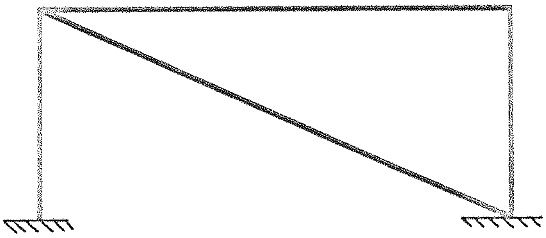
Rectangular cantilever plate idealized as six finite elements.

Fig. 38 .

Rank Force Method	
Joint equilibrium equations	14
Element boundary loads	24
Structural reactions	2
Total number of unknowns	26
Dynamic redundancy = 26-14	12
Static redundancy	0
Direct Stiffness Method	
Order of matrix to be inverted (unconstrained degrees of freedom. Same for static and dynamic analyses.)	12

Simply supported beam idealized as six finite elements.

Fig. 39 .



Rank Force Method	
Joint equilibrium equations	12
Element boundary loads	24
Structural reactions	6
Total number of unknowns	30
Dynamic redundancy 30-12	18
Static redundancy	6
Direct Stiffness Method	
Order of matrix to be inverted (unconstrained degrees of freedom. Same for static and dynamic analyses.)	6

Simple plane frame consisting of four elements.

Fig. 40 .

Collinear beam structures.	20 elements. 4 structural reactions.
General plane frames.	13 elements. 11 joints. 9 structural reactions.
Two dimensional rectangular plate structures.	6 elements. 12 structural reactions.

Structural limitations for analyses within the computer core storage.

Table 19 .

APPENDIX 1.

EIGENVALUE EVALUATION BY THE RANK FORCE METHOD USING A HIGHLY REDUCED STRUCTURAL DYNAMIC FLEXIBILITY MATRIX.

Synopsis.

In the rank force method for vibration analyses the eigenvalue evaluation requires the inversion of a large matrix, structural dynamic flexibility matrix. A method is proposed which only requires the inversion of a very small submatrix which is contained in the overall structural dynamic flexibility matrix and can be extracted immediately. Results are given which help to substantiate the method and which encourage further research.

Introduction.

In the present rank force method for vibration analyses the eigenvalues are evaluated by investigating the sign of the determinant value of the structural dynamic stiffness matrix. This is obtained by inverting the structural dynamic flexibility matrix for a given frequency. The order of these matrices is given by the number of generalized coordinates (applied loads or structural displacements) assumed at the unconstrained nodes for the constrained structure. In the rank force method no difficulty has been encountered in obtaining a symmetric structural dynamic flexibility matrix but on inverting this matrix nonsymmetry has resulted. However, the nonsymmetrical coefficients have been small compared with the large coefficients. The biggest impediment in this method of eigenvalue evaluation is the mere fact of having to invert large matrices. The following problem therefore presents itself ;

> How can the order of the overall structural dynamic flexibility matrix be reduced and still obtain the desired eigenvalues ?

A1.1 Proposed method.

The authors basic idea was stimulated by the "dummy load" method as used for static structural analysis. If a structure is being statically analysed to obtain the structural displacements at various locations on the structure, in order to obtain displacements at the unloaded points dummy loads must be applied. The dummy loads are then set to zero at the final stage of the analysis. Consider the structure shown in figure 41 . The actual applied load system consists of one load (P) but it is required to calculate the displacements $\Delta_1, \Delta_2, \ldots\ldots \Delta_8$ To do this the applied load system has to be considered as $P_1, P_2, \ldots\ldots\ldots\ldots, P_8$, where $P_2, P_3, \ldots\ldots\ldots, P_8$ are dummy loads. Therefore, the structural displacements are finally given by,

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \vdots \\ \Delta_8 \end{bmatrix} = \begin{bmatrix} \\ \\ \mathcal{F} \\ \\ \\ \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ \vdots \\ P_8 \end{bmatrix} \qquad \text{A1.1.1}$$

This equation can now be partitioned to give,

$$\left[\begin{array}{c} \Delta_1 \\ \hline \Delta_2 \\ \Delta_3 \\ \vdots \\ \vdots \\ \Delta_8 \end{array}\right] = \left[\begin{array}{c|c} \mathcal{F}_{11} & \mathcal{F}_{12} \\ \hline & \\ & \\ \mathcal{F}_{21} & \mathcal{F}_{22} \\ & \\ & \end{array}\right] \left[\begin{array}{c} P_1 = P \\ \hline P_2 \\ P_3 \\ \vdots \\ \vdots \\ P_8 \end{array}\right] \qquad \text{A1.1.2}$$

or

$$\left[\begin{array}{c} \Delta_a \\ \hline \Delta_u \end{array}\right] = \left[\begin{array}{c|c} \mathcal{F}_{11} & \mathcal{F}_{12} \\ \hline \mathcal{F}_{21} & \mathcal{F}_{22} \end{array}\right] \left[\begin{array}{c} P_a \\ \hline P_d \end{array}\right] \qquad \text{A1.1.3}$$

where,

$\{\Delta_a\}$ = displacements at loaded nodes in the direction of the corresponding actual applied loads.

$\{\Delta_u\}$ = displacements at unloaded nodes in the direction of the corresponding dummy loads.

$\{P_a\}$ = vector of actual generalized applied loads.

$\{P_d\}$ = vector of generalized dummy loads.

The dummy loads are now set to zero, $\{P_d\} = \{0\}$, therefore,

$$\{\Delta_a\} = [\mathcal{F}_{11}]\{P_a\} \qquad \text{A1.1.4}$$

and

$$\{\Delta_u\} = [\mathcal{F}_{21}]\{P_a\} \qquad \text{A1.1.5}$$

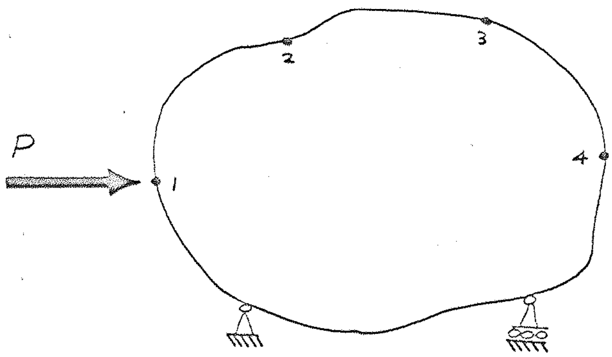
where $\{\Delta_u\}$ is now the vector of actual displacements at the unloaded nodes, in the direction of the corresponding dummy loads, due to the actual generalized applied loads. In the example the structural flexibility matrix is given by $[\mathcal{F}_{11}]$, order 1 × 1. Therefore, for a given structure and the same nodes a range of structural flexibility matrices can be obtained, each with its corresponding applied load system. Because of this the author intuitively felt that a reduced structural dynamic flexibility matrix could be used for the eigenvalue evaluation. This now presents the problem of how to reduce the structural dynamic flexibility matrix. In two dimensional problems two generalized applied loads can excite all the modes. Therefore, the simplest way is to select a reduced matrix which corresponds to the second leading submatrix of the overall structural dynamic flexibility matrix. See figure 42 . This procedure was applied to a simply supported beam idealized as six finite elements using Element Type P1/2FD. The beam properties are given in table 5 . Obviously, to compute the first eigenvalue for a simply supported beam it would not be idealized as six finite elements but this was chosen since, for example, in a two storey frame, figure 33 , the minimum idealization consists of six elements and therefore the total structure has to be considered even for the first eigenvalue. In the simply supported beam example the first five eigenvalues were evaluated using different order leading submatrices. These are compared with those obtained using the full structural dynamic flexibility matrix, order 12 × 12, and are given in table 20 . It can be seen that by investigating the first leading submatrix no eigenvalues were obtained but for all the remaining leading submatrices the same results were computed. This shows that two loads can excite all the modes and for the collinear beam problem these correspond to a vertical load and a moment as shown in figure 43(a). This method presents a simple inversion and notable time saving on the computer. The first three eigenvalues were obtained for a single storey plane frame using a 2 x 2 leading submatrix and Element Type P1/3FD. The frame dimensions and element properties are given in figure 32 and table 14 . The results are shown in table 21 and there is agreement with those obtained using the full matrix. The loads corresponding to the reduced matrix are shown in figure 43 (b). The elements of the structural dynamic flexibility matrix are functions of the frequency but it is virtually impossible to obtain these analytical expressions since the structural dynamic flexibility matrix, full or reduced, is obtained for a given frequency. If the eigenvalue evaluation was the only consideration this reduced matrix approach would also save considerable computer storage. This is because the coefficient matrix corresponding to the applied loads in the general case can be reduced to two columns in the two dimensional problems. The system of joint equilibrium equations is given by,

$$[\Omega_\alpha \vdots \Omega_e]\{{}_tq_\alpha \vdots {}_tR_e\} + \underset{\substack{\uparrow \\ \text{Two columns} \\ \text{for eigenvalue} \\ \text{evaluation only.}}}{[\Psi_\lambda]}\{{}_tP_\lambda\} = \{0\}$$

In the general case the number of columns in matrix $[\Psi_\lambda]$ is given by,

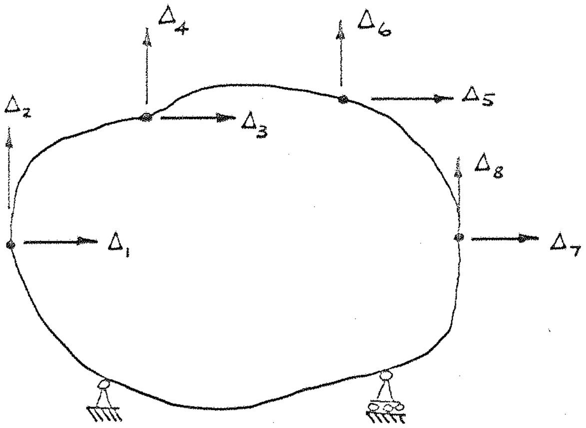
(number of nodes) × (number of assumed coordinates per node-
possible applied loads or displacements)
- (number of reactions)

Summarizing, to evaluate the element boundary loads, structural reactions and structural displacements due to a general applied load system loads are assumed to act at all nodes. This results in an overall structural dynamic flexibility matrix. However, the eigenvalues can be evaluated using a 2 × 2 submatrix for two dimensional structures and a 3 × 3 for three dimensional structures (to be investigated). This reduced matrix has been extracted immediately from the upper left hand corner of the overall matrix. The applied loads corresponding to the 2 × 2 leading submatrix are the first two that appear in the applied load vector for the constrained structure. Although this submatrix has given the desired results it is very likely that in the two dimensional frame structures these two loads, figure 43 (b), would not excite all the modes. Perhaps a moment and load would be required. Therefore, for different two dimensional configurations combinations of two loads may have to be considered in order to evaluate all the eigenvalues. In fact a criteria may exist for the choice of loads such that the numerical work is an optimum. This is an area for future investigations.



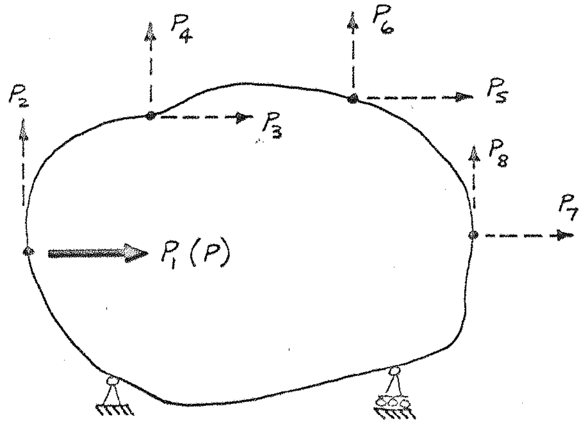
Actual applied loading system.

(a)



Structural displacements required.

(b)

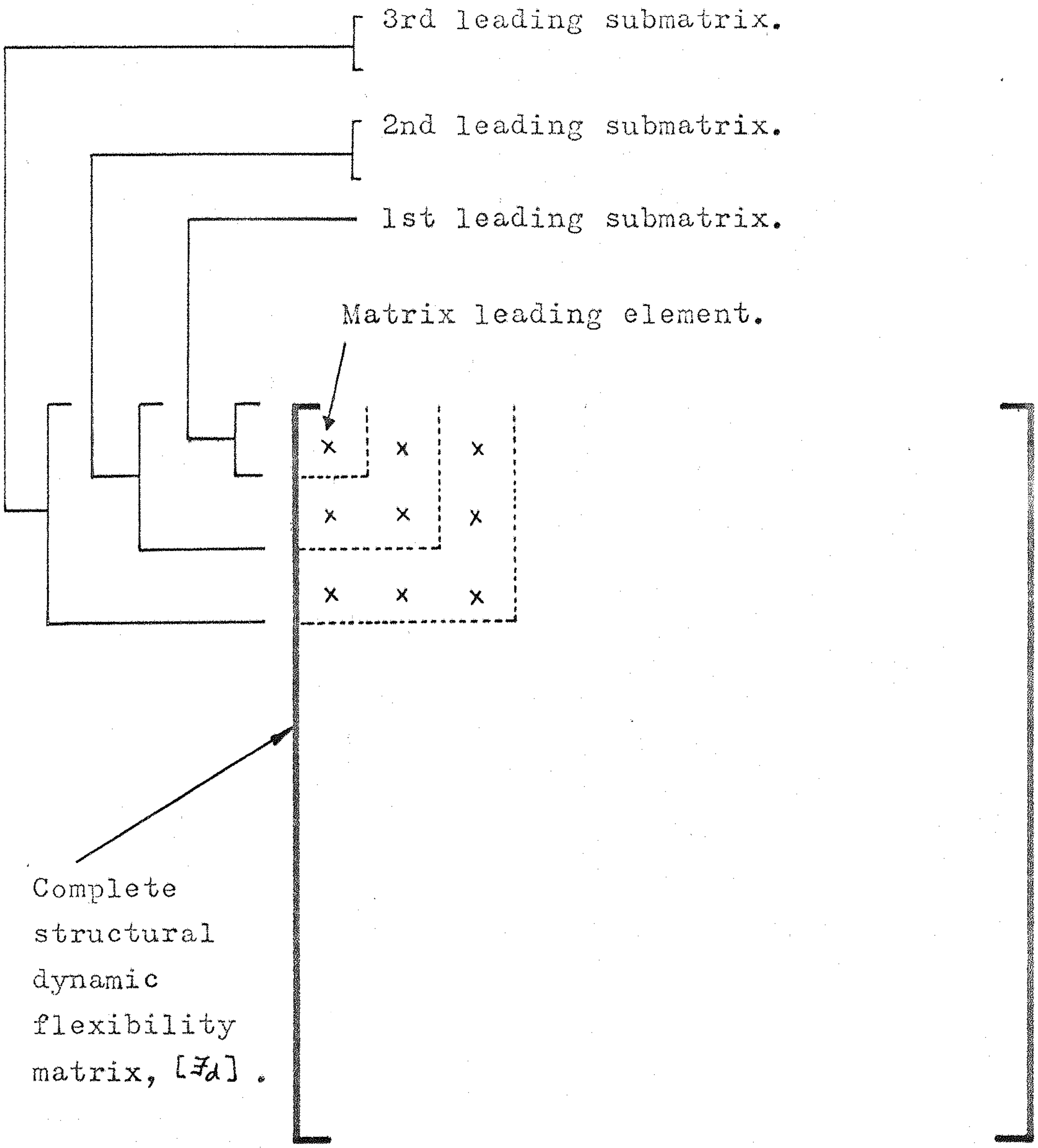


Dummy load approach.

Fig. 41 .

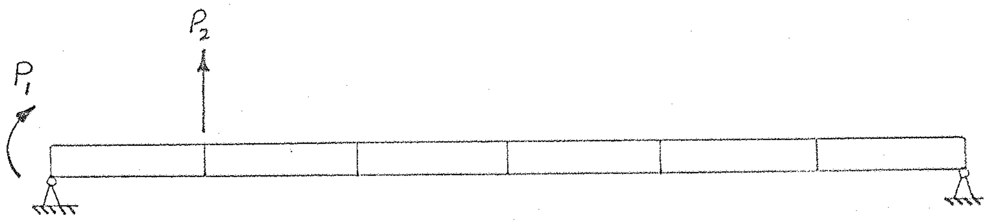
Applied loading system required to obtained desired displacements.

(c)



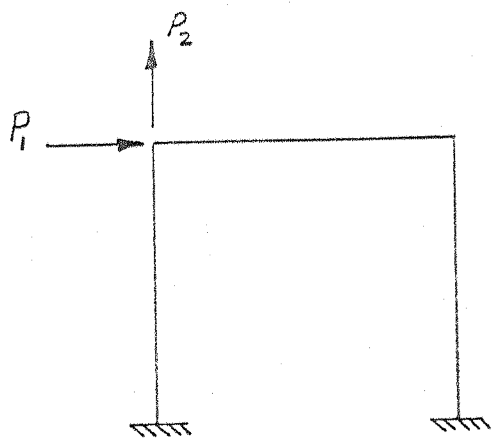
Leading submatrices.

Fig. 42 .



Collinear beam.

(a)



Plane frame.

(b)

Loads corresponding to the second leading submatrix.

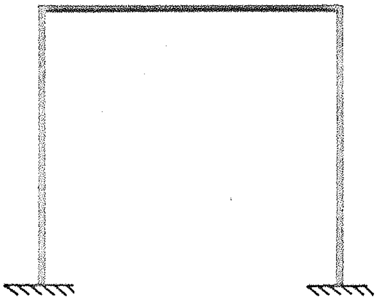
Fig. 43 .

Six finite elements.

Reference eigenvalues in rad/sec. Element Type P3/2FD	Eigenvalues using Element Type P1/2FD for different order leading submatrices.					
	Full matrix. 12×12	1×1	2×2	3×3	4×4	5×5
150·12	150·13	148 — 228 no zero crossing	150·13	150·13	150·13	150·13
600·50	600·99	590 - 670 no zero crossing	600·99	600·99	600·99	600·99
1351·13	1356·47	1350 - 1430 no zero crossing	1356·47	1356·47	1356·47	1356·47
2402·02	2430·41	2428 - 2508 no zero crossing	2430·41	2430·41	2430·41	2430·41
3753·16	3852·79	3840 - 3920 no zero crossing	3852·79	3852·79	3852·79	3852·79

Eigenvalues of a simply supported beam using the overall structural dynamic flexibility matrix and leading submatrices contained in this matrix.

Table 20 .



Reference eigenvalues in rad/sec. Element Type P3/3FD.	Eigenvalues using Element Type P1/3FD.	
	Full matrix. 6×6	Leading submatrix. 2×2
194·25	194·4	194·4
766·6	860·2	860·2
1250·5	1532·4	1532·4

Eigenvalues of a single storey plane frame using the overall structural dynamic flexibility matrix and a leading submatrix of order 2 × 2.

Table 21 .

APPENDIX 2.

DELAYED IMPOSITION OF ALL THE GENERALIZED STRUCTURAL REACTIONS IN THE RANK FORCE METHOD FOR VIBRATION ANALYSES.

Synopsis.

In this appendix a procedure is presented for the delayed imposition of the generalized structural reactions in the rank force method for structural vibration analysis. This procedure enables unconstrained structural dynamic flexibility matrices to be generated which are ideal for the vibration analysis of large practical structural configurations using "block elements" . A block element being itself an assembly of discrete elements, or in other words a substructure.

Introduction.

In the displacement approach the structural constraints need not be imposed until the latter stages of both the static and vibration formulations. This means, for a given structural configuration, that the unconstrained structural stiffness matrix (static or dynamic) can be generated and stored and then various constraint patterns considered without regeneration of the structural stiffness matrix for each case. This is also ideal for the analysis of large practical structures by the method of substructures since the unconstrained structural stiffness matrix for each substructure can be generated separately. Each substructure then becomes an element ("block element") with its own stiffness matrix. These block elements are then assembled in the normal manner (Direct Stiffness Method) to give the overall unconstrained structural stiffness matrix (static or dynamic). At this stage the actual constraints are imposed.

In the static rank force method at least the rigid body reactions have to be considered immediately in the system of equilibrium equations otherwise no solution can be obtained. In fact in reference 31 all reactions are considered immediately. If the system of equilibrium equations were assembled without at least the rigid body reactions and then investigated using the rank technique it would be found that there was no solution to this system of equations. In other words the rank of the coefficient matrix and the augmented matrix would be unequal. Because of this the analysis of large structures using block elements becomes more involved than in the direct stiffness method.

In the present dynamic rank force method all the structural reactions have been considered immediately in the generation of the joint equilibrium equations as influenced by the static formulation in reference 31. However, it is now realized that because of inertia loading a delayed imposition of all the generalized discrete structural reactions can be made for vibration analyses. This is now consistent with the dynamic stiffness method. The rank force method now becomes ideal for the vibration analysis of large structures using block elements since unconstrained dynamic flexibility matrices can now be adopted.

A2.1 Delayed imposition.

Initially when generating the system of joint equilibrium equations all applied loads are assumed to act at the nodal points, no reactions are considered. The next step is then to consider the structural reactions and to isolate the corresponding applied loads. The unknowns are now generalized element boundary loads and structural reactions. However, if this latter step is omitted and the rank force procedure continued an unconstrained solution will result.

The generalized element boundary loads (unknowns) will be given by,

$$\{ {}_t\bar{q}_\alpha \} = [\bar{\Delta}_{\alpha\lambda}] \{ {}_t\bar{P}_\lambda \} \qquad \text{A2.1.1}$$

and the generalized structural displacements by,

$$\{ {}_t\bar{\Delta}_\lambda \} = [\bar{\mathcal{F}}_d] \{ {}_t\bar{P}_\lambda \} \qquad \text{A2.1.2}$$

In equations A2.1.1 and A2.1.2 the bar is used to denote the unconstrained structure.

The unconstrained structural dynamic flexibility matrix, $[\bar{\mathcal{F}}_d]$, can now be partitioned such that the applied loads corresponding to reactions are separated from the possible applied loads. Therefore,

$$\left[\begin{array}{c} {}_t\Delta_\lambda \\ \hline {}_t\Delta_e \end{array} \right] = \left[\begin{array}{c|c} \bar{\mathcal{F}}_{11} & \bar{\mathcal{F}}_{12} \\ \hline \bar{\mathcal{F}}_{21} & \bar{\mathcal{F}}_{22} \end{array} \right] \left[\begin{array}{c} {}_tP_\lambda \\ \hline {}_tR_e \end{array} \right] \qquad \text{A2.1.3}$$

Assuming that the displacements corresponding to the structural reactions, $\{{}_t\Delta_e\}$, are zero equation A2.1.3 can be expanded to give the equations,

$$\{{}_t\Delta_\lambda\} = [\bar{\mathcal{F}}_{11}]\{{}_tP_\lambda\} + [\bar{\mathcal{F}}_{12}]\{{}_tR_e\} \qquad \text{A2.1.4}$$

and

$$\{0\} = [\bar{\mathcal{F}}_{21}]\{{}_tP_\lambda\} + [\bar{\mathcal{F}}_{22}]\{{}_tR_e\} \qquad \text{A2.1.5}$$

Therefore, from equation A2.1.5, the structural reactions are given by,

$$\{{}_tR_e\} = -[\mathcal{F}_{22}]^{-1}[\bar{\mathcal{F}}_{21}]\{{}_tP_\lambda\} \qquad \text{A2.1.6}$$

where, as for the previous formulation, Chapter 2,

$$[\Delta_{e\lambda}] = -[\bar{\mathcal{F}}_{22}]^{-1}[\bar{\mathcal{F}}_{21}] \qquad \text{A2.1.7}$$

Substituting for the structural reactions into equation A2.1.4 gives,

$$\{{}_t\Delta_\lambda\} = \left([\bar{\mathcal{F}}_{11}] - [\bar{\mathcal{F}}_{12}][\bar{\mathcal{F}}_{22}]^{-1}[\bar{\mathcal{F}}_{21}]\right)\{{}_tP_\lambda\} \qquad \text{A2.1.8}$$

where, as for the previous formulation, the constrained structural dynamic flexibility matrix is given by,

$$[\mathcal{F}_\alpha] = \left([\bar{\mathcal{F}}_{11}] - [\bar{\mathcal{F}}_{12}][\bar{\mathcal{F}}_{22}]^{-1}[\bar{\mathcal{F}}_{21}]\right) \qquad \text{A2.1.9}$$

Equation A2.1.2 can now be partitioned to give,

$$\{{}_tq_\alpha\} = [\bar{\Delta}_{11} \vdots \bar{\Delta}_{12}]\{{}_tP_\lambda \vdots {}_tR_e\}$$

or

$$\{{}_tq_\alpha\} = [\bar{\Delta}_{11}]\{{}_tP_\lambda\} + [\bar{\Delta}_{12}]\{{}_tR_e\} \qquad \text{A2.1.10}$$

Substituting for the structural reactions gives the generalized element boundary loads for the constrained structure, that is,

$$\{ {}_t q_\alpha \} = \left([\,\bar{\Delta}_{11}\,] - [\,\bar{\Delta}_{12}\,][\,\bar{\mathcal{F}}_{22}\,]^{-1}[\,\bar{\mathcal{F}}_{21}\,] \right) \{ {}_t P_\lambda \} \qquad \text{A2.1.11}$$

Therefore, as for the previous formulation,

$$[\,\Delta_{\alpha\lambda}\,] = \left([\,\bar{\Delta}_{11}\,] - [\,\bar{\Delta}_{12}\,][\,\bar{\mathcal{F}}_{22}\,]^{-1}[\,\bar{\mathcal{F}}_{21}\,] \right) \qquad \text{A2.1.12}$$

This procedure has been applied to a collinear beam structure consisting of two finite elements and using Element Type P1/2FD. These results were compared with those obtained by applying the structural reactions immediately and agreement was established.

APPENDIX 3.

RECTANGULAR PLATE ELEMENT (Element Type P1/6FD).

A procedure will now be suggested for future plate development. This procedure adopts the s-system as used for Element Type P1/5FD (figure 16) but the equilibrium equations will not be based on total resultants (3 equations) but on boundary resultants (8 equations). The boundary resultant equilibrium equations for the loading systems shown in figure 20 are given by,

$$a\, q_{\alpha+4} + \tfrac{1}{2}\, q_{\alpha+8} + \tfrac{1}{2}\, q_{\alpha+9} - s_{\beta} - s_{\beta+3} = 0$$

$$a\, q_{\alpha} - s_{\beta+1} - s_{\beta+4} = 0$$

$$b\, q_{\alpha+7} + \tfrac{1}{2}\, q_{\alpha+8} + \tfrac{1}{2}\, q_{\alpha+11} - s_{\beta} - s_{\beta+9} = 0$$

$$b\, q_{\alpha+3} - s_{\beta+2} - s_{\beta+11} = 0$$

$$b\, q_{\alpha+5} + \tfrac{1}{2}\, q_{\alpha+9} + \tfrac{1}{2}\, q_{\alpha+10} - s_{\beta+3} - s_{\beta+6} = 0$$

$$b\, q_{\alpha+1} - s_{\beta+5} - s_{\beta+8} = 0$$

$$a\, q_{\alpha+6} + \tfrac{1}{2}\, q_{\alpha+10} + \tfrac{1}{2}\, q_{\alpha+11} - s_{\beta+6} - s_{\beta+9} = 0$$

$$a\, q_{\alpha+2} - s_{\beta+7} - s_{\beta+10}$$

or in matrix form,

q-loads												s-loads											
α	$\alpha+1$	$\alpha+2$	$\alpha+3$	$\alpha+4$	$\alpha+5$	$\alpha+6$	$\alpha+7$	$\alpha+8$	$\alpha+9$	$\alpha+10$	$\alpha+11$	β	$\beta+1$	$\beta+2$	$\beta+3$	$\beta+4$	$\beta+5$	$\beta+6$	$\beta+7$	$\beta+8$	$\beta+9$	$\beta+10$	$\beta+11$
				a				$\frac{1}{2}$	$\frac{1}{2}$			-1			-1								
a													-1			-1							
							b	$\frac{1}{2}$			$\frac{1}{2}$	-1									-1		
			b											-1									-1
					b				$\frac{1}{2}$	$\frac{1}{2}$					-1			-1					
	b																-1			-1			
						a				$\frac{1}{2}$	$\frac{1}{2}$							-1			-1		
		a																	-1			-1	

$$= \{0\}$$

A3.1

The procedure is then the same as for Element Type P1/5FD. This approach has so far not been investigated with an example but a listing of the subroutine is given in table 22 . See Appendix 4, Subroutine FMD50 (Element Type P1/5FD) for argument description and other comments.

```
      SUBROUTINE FMD50(A,B,T,EM,XMUM,XNUM,OMEGA,FMD)
C     JOHN ROBINSON  I.S.V.R.
C     RECTANGULAR PLATE ELEMENT
C     ELEMENT TYPE P1/6FD
C     ELEMENT DYNAMIC FLEXIBILITY MATRIX
      DIMENSION D(12,24),ID(12),XM(12),PAR(4,12),PHI(4,12),IR(12)
      DIMENSION DELQS(12,12),DELQST(12,12),FMD(12,12),G(12,12)
      EQUIVALENCE (DELQS(1,1),D(1,1)),(DELQST(1,1),D(1,13))
      DO5 I=1,12
      DO5 J=1,24
    5 D(I,J)=0.0
      D(1,5),D(2,1),D(7,7),D(8,3)=A
      D(3,8),D(4,4),D(5,6),D(6,2)=B
      D(1,9),D(1,10),D(3,9),D(3,12),D(5,10),D(5,11),D(7,11),D(7,12)=0.5
      D(1,13),D(1,16),D(2,14),D(2,17),D(3,13),D(3,22),D(4,15),D(4,24),
     1D(5,16),D(5,19),D(6,18),D(6,21),D(7,19),D(7,22),D(8,20),D(8,23)=
     2-1.0
      CALL RANTEC(D,8,12,24,12,12,24,ID,XM,IR)
      CALL PARDER(D,IR,8,12,12,24,NN)
      DO10 M=1,4
      DO10 N=1,12
   10 PAR(M,N)=D(8+M,N)
      CALL FMD40(A,B,T,EM,XMUM,XNUM,OMEGA,FMD)
      CALL MATMULT(PAR,FMD,PHI,4,12,12,4,12,12)
      DO12 M=1,4
      DO12 N=1,12
   12 D(8+M,N)=PHI(M,N)
      CALL RANTEC(D,12,12,24,12,12,24,ID,XM,IR)
      CALL REAR(D,12,24,12,24,XCH)
      DO14 I=1,12
      DO14 J=13,24
   14 DELQS(I,J-12)=-D(I,J)
      DO16 I=1,12
      DO16 J=1,12
   16 DELQST(I,J)=DELQS(J,I)
      CALL MATMULT(FMD,DELQS,G,12,12,12,12,12,12)
      CALL MATMULT(DELQST,G,FMD,12,12,12,12,12,12)
      RETURN
      END
```

A.S.A. Fortran listing of subroutine FMD50, Element Type P1/6FD.

Table 22 .

APPENDIX 4.

A COMPUTERIZED STRUCTURAL ANALYSIS RESEARCH SYSTEM TO STUDY THE RANK FORCE METHOD.

Synopsis.

In this appendix a computerized structural research system to study the rank force method for vibration analysis is described in detail. The basic concept of any computerized system is to write the programme as a series of small programmes (subroutines) which are connected together by a master programme. Each subroutine carries out a specific step in the analysis. All subroutines used in this research system are described and were necessary examples of their usage and capabilities are given. Some subroutines are only applicable to a computerized system which adopts the rank force method, these are referred to as special subroutines. However, some are applicable for all systems, even nonstructural, these are referred to as standard subroutines.

Master programmes are then described for the vibration analysis of collinear beam structures, general plane frames and two dimensional plate structures. Many of the subroutines are commom to all of the master programmes. In order to analyse a structure using the master programmes certain input data must be prepared by the user and in a certain way. Input data preparation is described for each master programme and typical examples are given. The size of structural problem which can be analysed by the various programmes has been limited by the computer core storage (I.C.T. 1900 Computer). All structures are analysed completely in core but this is a self imposed restriction. Within the computer programme system efficient use is made of the EQUIVALENCE statement and temporary transfer of parts of matrices, these are described.

Introduction.

In any computerized structural analysis system the basic concept is to break down the computer programme into a series of smaller programmes called subroutines. These subroutines are connected together to form one system by a master programme. Each subroutine carries out a specific step in the analysis. Some subroutines are common to one system only, these will be referred to as special subroutines, others are common to all structural systems and even nonstructural work, these will be referred to as standard subroutines. Since subroutines are continually being improved this chain approach to writing a computer programme enables a subroutine to be replaced by an improved version with very little, if any, alterations to the overall system.

The computerized system written to study the rank force method is by no means optimized and the numerical methods and programme formulation are not necessarily the best possible. It is also possible that errors exist in the system which have not been brought to light by the examples used for checkout purposes. The author also realizes that numerical difficulties could arise when analysing a structure in which a wide range of element properties existed and when analysing larger configurations than those investigated in this work.

However, the aim of this research work is to assess the rank force method for vibration analysis to see if it offers anything over the popular displacement approach. It was felt that the assessment could be made on the basis of simple structural configurations for which results existed in the published literature which were obtained using the displacement approach. The master programmes and subroutines developed to study the rank force method have been written in A.S.A. Fortran for an I.C.T. 1900 Computer. These will now be listed and then described individually.

Subroutines.

1. RANTEC.
2. REAR.
3. MATINV.
4. MATMULT.
5. PARDER.
6. FMD10............Element Type P1/2FD.
7. FMD10............Element Type P2/2FD.
8. FMD10............Element Type P3/2FD.
9. FMD30............Element Type P1/3FD.
10. FMD30............Element Type P3/3FD.
11. FMD40............Element Type P1/4FD.
12. VARDET.
13. MODE.
14. FORCEB.
15. FORCEF.
16. FORCEP.
17. FMD50............Element Type P1/5FD.

Functions.

1. KINT.

Master Programmes.

1. FORCE-BEAM.
2. FORCE-PLANE FRAME.
3. FORCE-RECTANGULAR PLATE.

A4.1 Subroutines.

1. Subroutine RANTEC.

This subroutine has been described in Chapter 1.

2. Subroutine REAR.

2.1 Description of subroutine.

After applying subroutine RANTEC to a system of linear equations, containing the same number of independent equations as unknowns, the coefficient matrix will, in general, be a permuted unit matrix. Subroutine REAR rearranges the final augmented matrix to give a unit coefficient matrix. The required solution of the system of equations is then immediately available.

2.2 Subroutines called by REAR.

This subroutine calls no other subroutines.

2.3 Subroutine listing.

The listing of subroutine REAR is given in table 23 .

2.4 Description of subroutine arguments.

The first card of any subroutine contains the word SUBROUTINE, then the subroutine name followed by its arguments, given in parentheses. That is, for subroutine REAR,

```
SUBROUTINE REAR(XKD,N9,N7,N9MAX,N7MAX,XCH)
```

where,

XKD = rectangular(or square) array.

N9 = number of rows in XKD.

N7 = total number of columns in XKD.

$\left.\begin{array}{l}\text{N9MAX}\\ \text{N7MAX}\end{array}\right\}$ corresponding maximum values.

XCH = interchange constant, that is, $(-1)^{\text{NOINT}}$

NOINT = number of row interchanges to give a unit coefficient matrix. This can be used to evaluate a determinant.

See figure 44(a) for further clarification.

2.5 Example of usage.

After applying RANTEC to a system of linear equations the resulting system can be written as,

$$[A]\{x\} + [B]\{y\} = \{0\}$$

where,

$\{x\}$ = vector of unknowns (internal loads and structural reactions as an example).

$\{y\}$ = vector of knowns (applied loads).

$[A]$ = permuted unit matrix (square).

$[B]$ = matrix corresponding to knowns (rectangular or square).

Let;

1. actual number of unknowns = 100
2. actual number of knowns = 20
3. maximum possible number of unknowns likely to be considered = 1000
4. maximum number of knowns likely to be considered = 200

and

$[C] =$ augmented matrix $[A, B]$.

Therefore, in this case, the call statement to apply subroutine REAR would be,

```
CALL REAR(C,100,120,1000,1200,XCH)
```

See figure (b) for further clarification.

As a further example, if,

$$[A] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad [B] = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 4 \\ 2 & 1 & 2 \end{bmatrix}$$

then,

$$[C] = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 4 \\ 1 & 0 & 0 & 2 & 1 & 2 \end{array}\right]$$

After applying REAR,

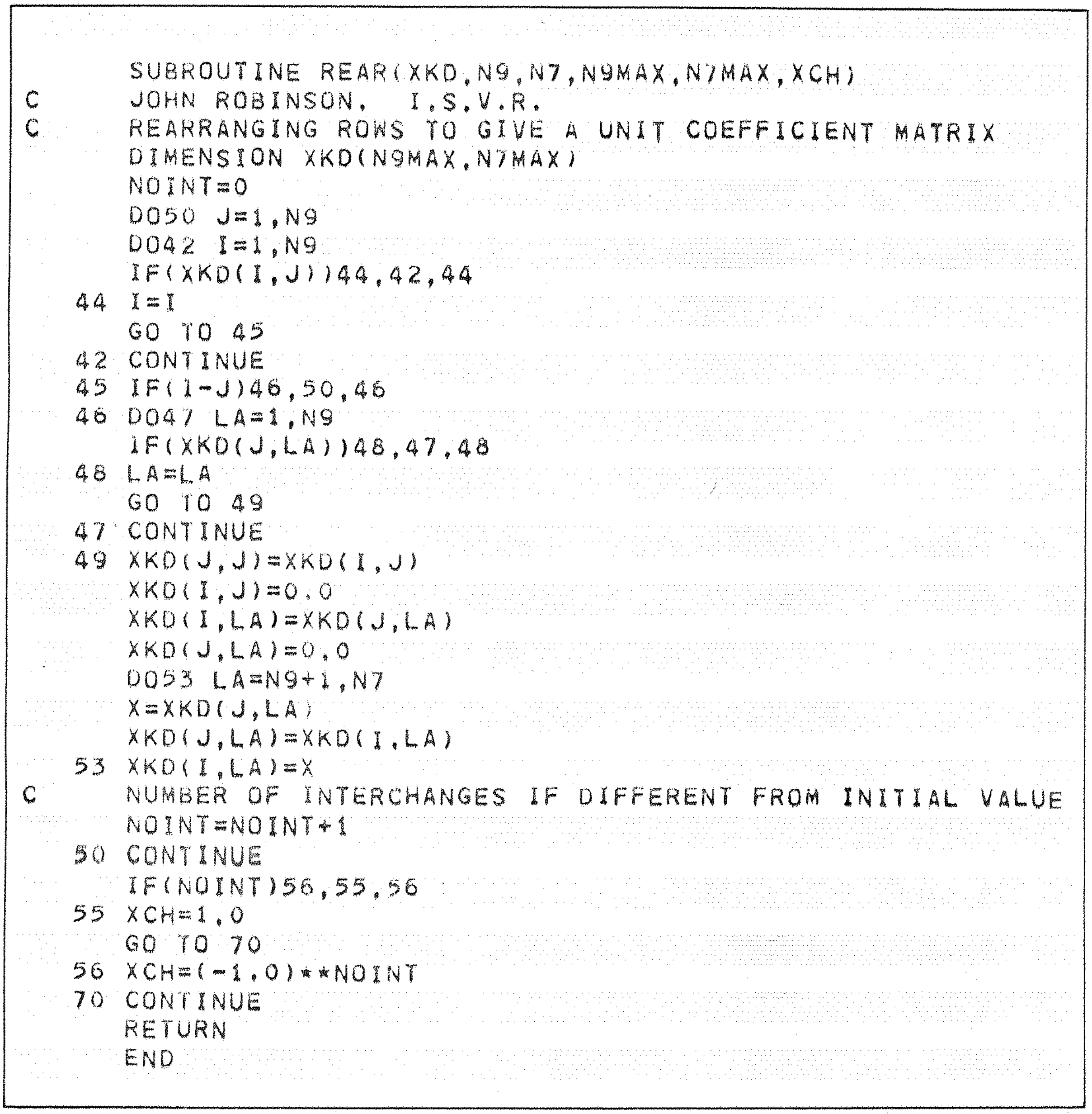
$$[C] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 4 \end{array}\right]$$

Therefore, expanding into the original form gives,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \{x\} + \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 0 & 4 \end{bmatrix} \{y\} = \{0\}$$

and the solution for $\{x\}$ is,

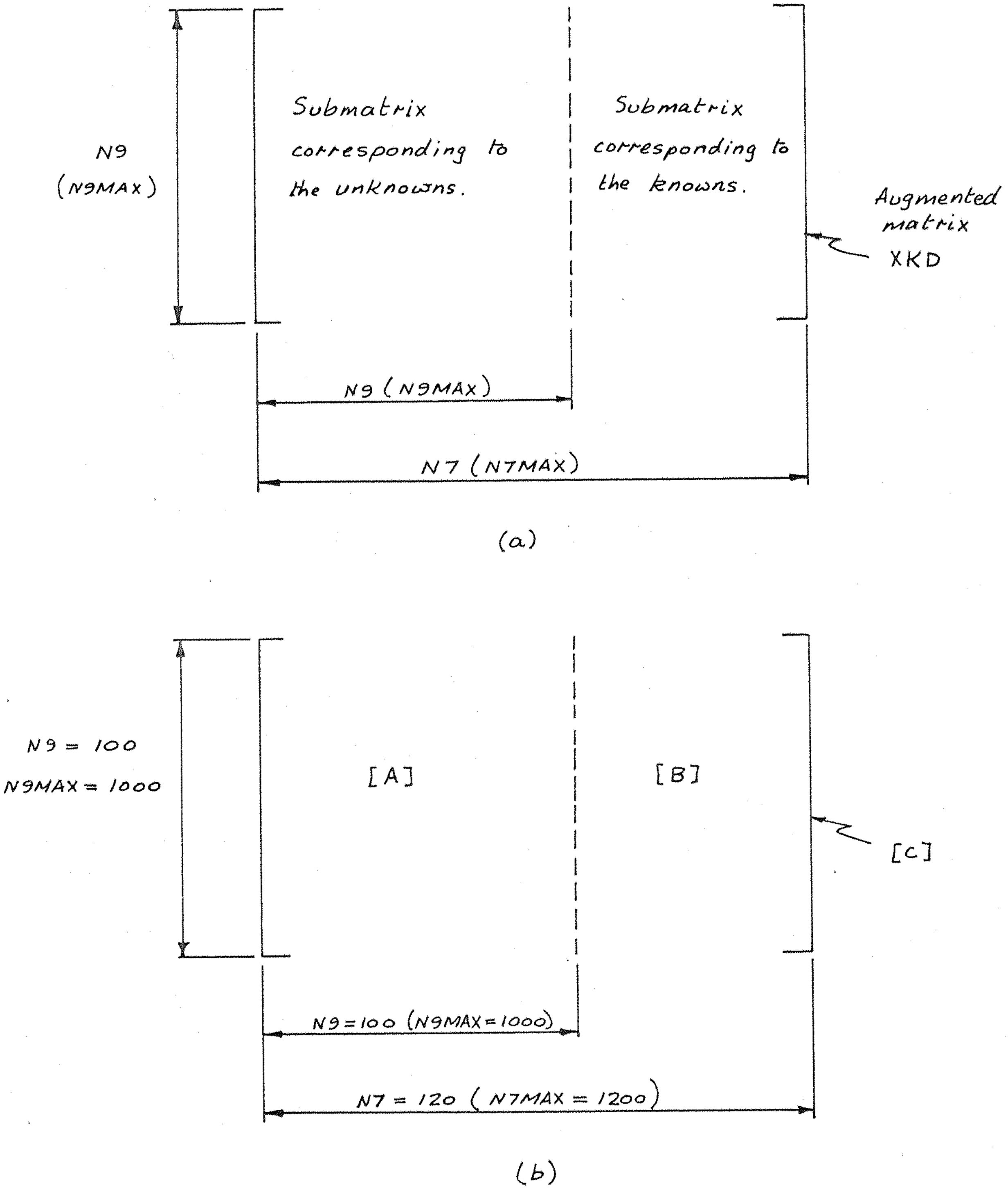
$$\{x\} = - \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 0 & 4 \end{bmatrix} \{y\}$$



```
      SUBROUTINE REAR(XKD,N9,N7,N9MAX,N7MAX,XCH)
C     JOHN ROBINSON.  I.S.V.R.
C     REARRANGING ROWS TO GIVE A UNIT COEFFICIENT MATRIX
      DIMENSION XKD(N9MAX,N7MAX)
      NOINT=0
      DO50 J=1,N9
      DO42 I=1,N9
      IF(XKD(I,J))44,42,44
   44 I=I
      GO TO 45
   42 CONTINUE
   45 IF(I-J)46,50,46
   46 DO47 LA=1,N9
      IF(XKD(J,LA))48,47,48
   48 LA=LA
      GO TO 49
   47 CONTINUE
   49 XKD(J,J)=XKD(I,J)
      XKD(I,J)=0.0
      XKD(I,LA)=XKD(J,LA)
      XKD(J,LA)=0.0
      DO53 LA=N9+1,N7
      X=XKD(J,LA)
      XKD(J,LA)=XKD(I,LA)
   53 XKD(I,LA)=X
C     NUMBER OF INTERCHANGES IF DIFFERENT FROM INITIAL VALUE
      NOINT=NOINT+1
   50 CONTINUE
      IF(NOINT)56,55,56
   55 XCH=1.0
      GO TO 70
   56 XCH=(-1.0)**NOINT
   70 CONTINUE
      RETURN
      END
```

A.S.A. Fortran listing of subroutine REAR.

Table 23 .



Argument definitions for subroutine REAR.

Fig. 44.

3. Subroutine MATINV.

3.1 Description of subroutine.

This subroutine forms an augmented matrix using the matrix to be inverted and a unit matrix of the same order. The Jordan elimination procedure is applied to this assembled matrix and after some rearranging the inverse of the original matrix is found in the submatrix corresponding to the original unit matrix. The matrix to be inverted is not required to have any special properties other than being real. The inversion procedure used in this subroutine was adopted because of the existence of RANTEC. A disadvantage of this procedure is that the computer storage required to invert a matrix is twice that required by the matrix to be inverted.

3.2 Subroutines called by MATINV.

This subroutine calls subroutines RANTEC and REAR. See figure 46.

3.3 Subroutine listing.

The listing of subroutine MATINV is given in table 24.

3.4 Description of subroutine arguments.

The first card of this subroutine is,

```
SUBROUTINE MATINV(A,B,C,N9,N9MAX,N8MAX,N7MAX,IDEP,XMAX,IQ)
```

where,

A = matrix to be inverted.

B = inverse of matrix A.

C = augmented matrix formed, within the subroutine, by matrix A and a unit matrix of the same order as A.

N9 = actual order of matrix A.

N9MAX = maximum possible order of matrix A likely to be considered.

N8MAX = N9MAX, this is used since it is passed over by RANTEC.

N7MAX = 2 × N9MAX, maximum number of columns likely to be considered in the augmented matrix.

IDEP = vector of dependent equations(row numbers).

XMAX = normvector, vector of normalized row elements. This can be used to evaluate a determinant.

IQ = vector of redundant load numbers.

If the matrix has no inverse a statement NO SOLUTION is printed out. Vectors XMAX,IDEP, and IQ are formed within the subroutine, see RANTEC. See also figure 45(a)for further clarification.

3.5 Example of usage.

To invert a matrix [F] of order 200 × 200 and maximum possible order of 2000 × 2000 would require the call statement,

```
CALL MATINV(F,K,D,200,2000,2000,4000,MDEP,WMAX,LQ)
```

where,

$[K]$ = inverse of $[F] = [F]^{-1}$

$[D]$ = augmented matrix $[F,U]$, formed within the subroutine. Note $[U]$ is a unit matrix.

See figure 45 (b) for further clarification.

To demonstrate the procedure adopted for matrix inversion consider the following example.

Let,

$$[F] = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Therefore, the original augmented matrix [D] would be,

$$[D] = \left[\begin{array}{ccc|ccc} 4 & 3 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Matrix to be inverted.

Unit matrix.

Applying RANTEC to [D] would give,

$$[D] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 1 & 0 & -2 \end{array}\right]$$

Subroutine REAR is now applied to give,

$$[D] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

Unit matrix.

Inverse of [F].

Therefore, the inverse of matrix [F] is given by,

$$[K] = [F]^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

In this work on the rank force method the evaluation of a determinant, in particular the structural dynamic stiffness matrix for a given frequency, is carried out using subroutine VARDET (described later). However, it will now be shown how subroutines RANTEC and REAR can be used for determinant evaluation.

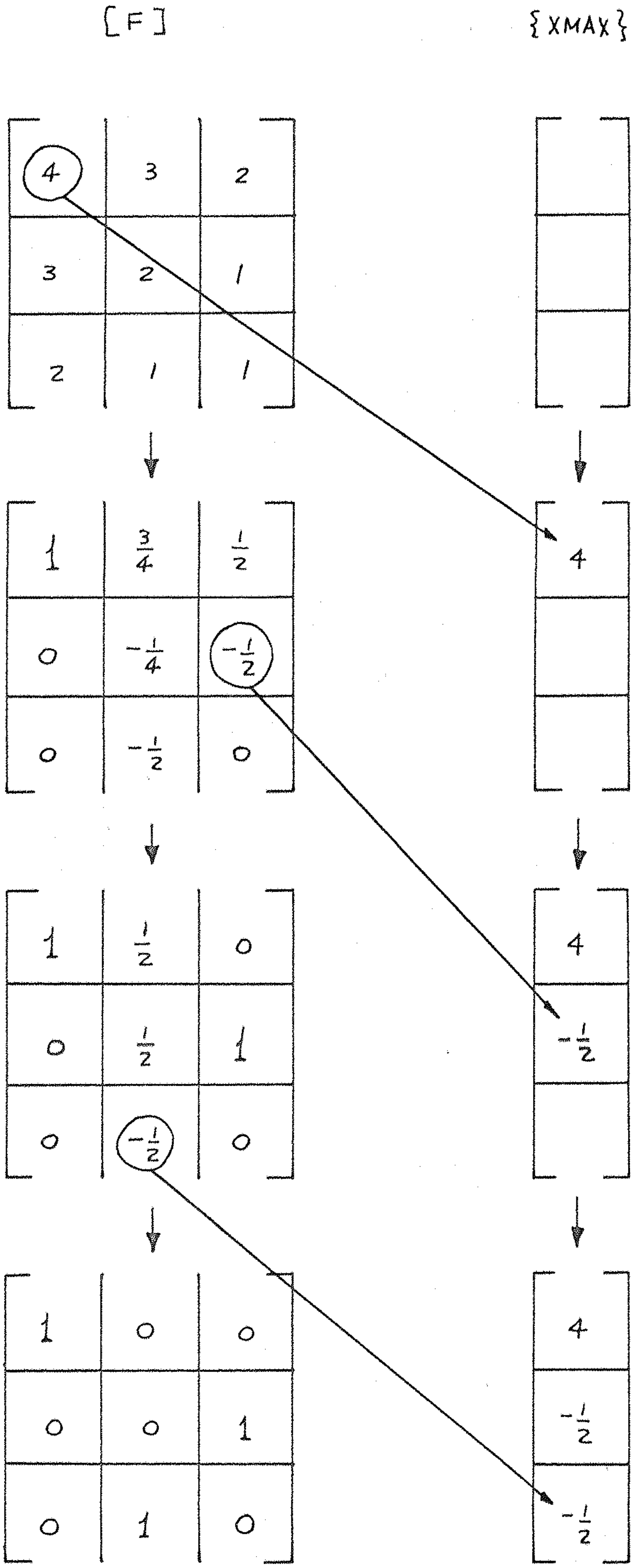
At each stage of the elimination procedure, when using RANTEC, a normvector is being generated, XMAX(see RANTEC), and when applying subroutine REAR an interchange constant, XCH(see REAR), is being computed. The determinant value is given by,

$$\det [A] = XCH \times (\text{continued product of the normvector terms})$$
$$= XCH \times (XMAX(1) \times XMAX(2) \times \ldots\ldots\ldots\ldots \times XMAX(N9))$$

To demonstrate the procedure consider the last example.

$$[F] = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

RANTEC is now applied to matrix [F] . The procedure of forming the normvector is best shown in a step by step presentation as follows.



[F] {XMAX}

$$\begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \qquad \begin{Bmatrix} \; \\ \; \\ \; \end{Bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} \\ 0 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{bmatrix} \qquad \begin{Bmatrix} 4 \\ \; \\ \; \end{Bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & 0 \end{bmatrix} \qquad \begin{Bmatrix} 4 \\ -\frac{1}{2} \\ \; \end{Bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{Bmatrix} 4 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{Bmatrix}$$

Applying REAR to give a unit matrix would require rows two and three to be interchanged. Therefore,

$$\text{NOINT} = 1$$

hence,

$$\text{XCH} = (-1)^1 = -1$$

Therefore,

$$\det[F] = \begin{vmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = (-1)\left(4 \times -\tfrac{1}{2} \times -\tfrac{1}{2}\right) = -1$$

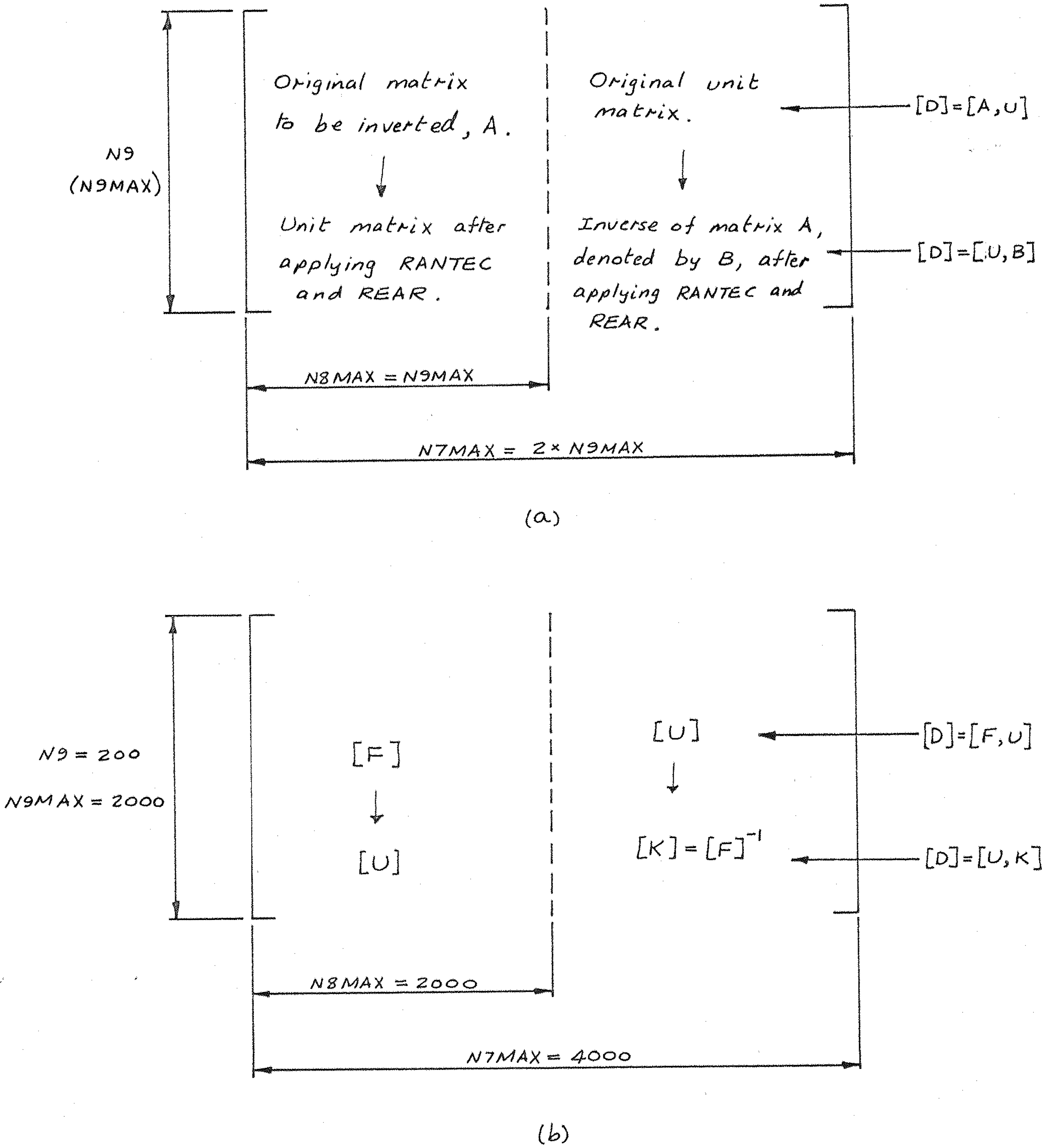
Note :

1. If a zero appeared in the normvector then the determinant would be zero and in fact the matrix would be singular. The degree of degeneracy would be given by the number of zeros. The location of the dependent rows (or equations) is given by vector IDEP.
2. $(-1)^0 = 1$

```
      SUBROUTINE MATINV(A,B,C,N9,N9MAX,N8MAX,N7MAX,IDEP,XMAX,IQ)
C     JOHN ROBINSON.  I.S.V.R.
C     MATRIX INVERSION.
      DIMENSION A(N9MAX,N9MAX),B(N9MAX,N9MAX),C(N9MAX,N7MAX)
      DIMENSION IDEP(N9MAX),XMAX(N9MAX),IQ(N8MAX)
      N7=2*N9
      DO8 I=1,N9
      DO5 J=1,N9
      C(I,J)=A(I,J)
      C(I,J+N9)=0.0
    5 CONTINUE
      C(I,I+N9)=1.0
    8 CONTINUE
      CALL RANTEC(C,N9,N9,N7,N9MAX,N8MAX,N7MAX,IDEP,XMAX,IQ)
      CALL REAR(C,N9,N7,N9MAX,N7MAX,XCH)
      DO9 I=1,N9
      DO9 J=N9+1,N7
    9 B(I,J-N9)=C(I,J)
      RETURN
      END
```

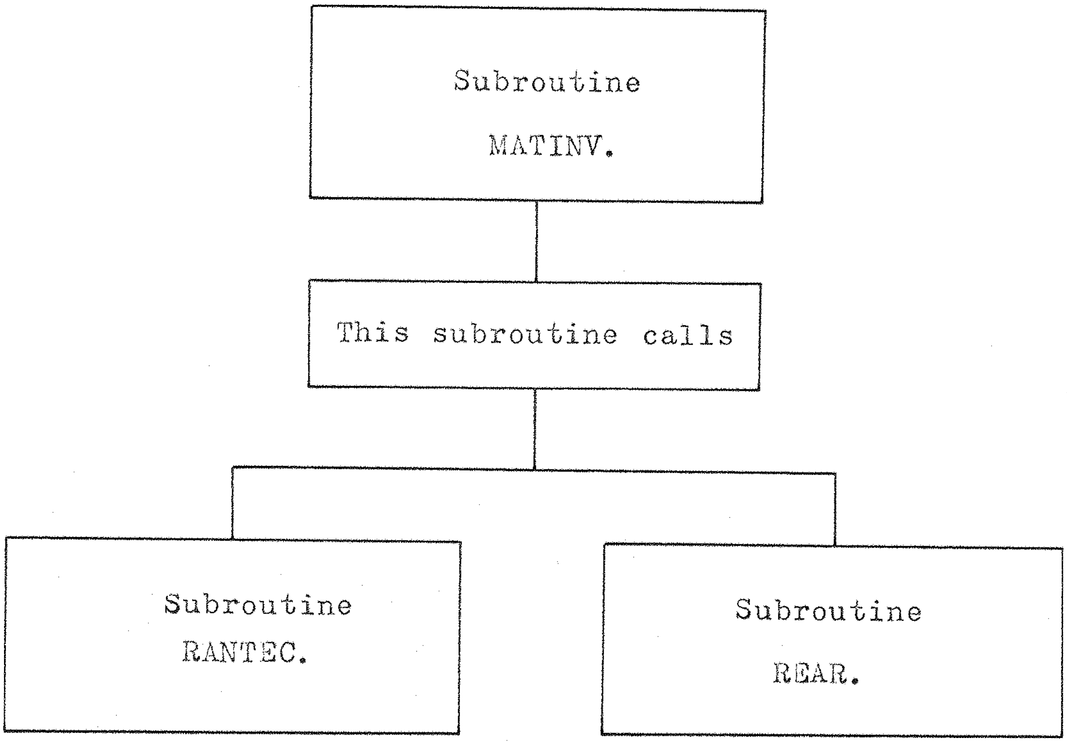
A.S.A. Fortran listing of subroutine MATINV.

Table 24 .



Argument definitions for subroutine MATINV.

Fig. 45 .



Subroutines called by subroutine MATINV.

Fig. 46 .

4. Subroutine MATMULT.

4.1 Description of subroutine.

This subroutines multiplies two matrices together.

4.2 Subroutines called by MATMULT.

This subroutine calls no other subroutines.

4.3 Subroutine listing.

The listing of subroutine MATMULT is given in table 25.

4.4 Description of subroutine arguments.

The first card of this subroutine is,

```
SUBROUTINE MATMULT(A,B,C,NI,NJ,NK,NIMAX,NJMAX,NKMAX)
```

where,

A = matrix to be postmultiplied by matrix B.

C = continued matrix product, A × B.

NI = number of rows in matrices C and A.

NJ = number of columns in matrices C and B.

NK = number of columns in matrix A and number of rows in matrix B.

$\left.\begin{array}{l}\text{NIMAX}\\ \text{NJMAX}\\ \text{NKMAX}\end{array}\right\}$ corresponding maximum values, maximum dimensions.

See figure 47(a) for further clarification.

4.5 Example of usage.

To evaluate the matrix product,

$$[FDA] = [FMD][DELA]$$

where,

[FMD] is of order 6 × 6 with maximum dimensions, that is, maximum possible order likely to be considered, 6 × 6.

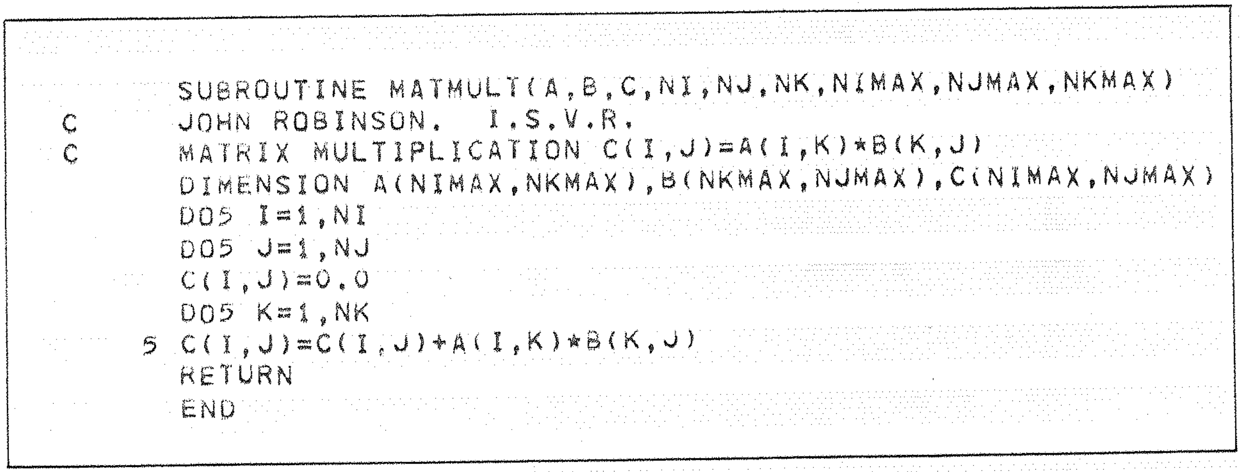
and

[DELA] is of order 6 × 9 with maximum dimensions 6 × 33.

the required call statement would be,

```
CALL MATMULT(FMD,DELA,FDA,6,9,6,6,33,6)
```

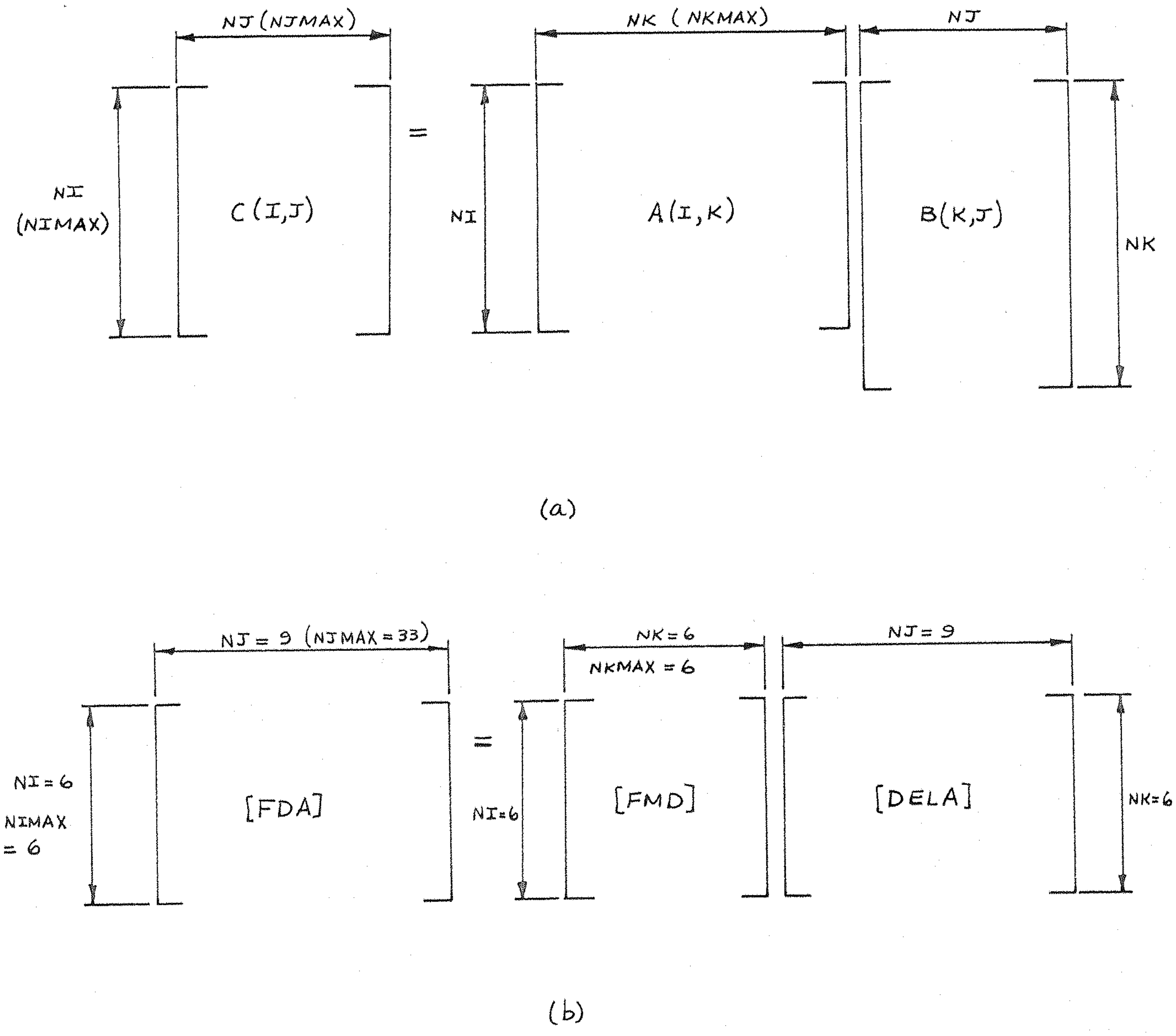
See figure 47(b) for further clarification.



```
      SUBROUTINE MATMULT(A,B,C,NI,NJ,NK,NIMAX,NJMAX,NKMAX)
C     JOHN ROBINSON.  I.S.V.R.
C     MATRIX MULTIPLICATION C(I,J)=A(I,K)*B(K,J)
      DIMENSION A(NIMAX,NKMAX),B(NKMAX,NJMAX),C(NIMAX,NJMAX)
      DO5 I=1,NI
      DO5 J=1,NJ
      C(I,J)=0.0
      DO5 K=1,NK
    5 C(I,J)=C(I,J)+A(I,K)*B(K,J)
      RETURN
      END
```

A.S.A. Fortran listing of subroutine MATMULT.

Table 25.



Argument definitions for subroutine MATMULT.

Fig. 47.

5. Subroutine PARDER.

5.1 Description of subroutine.

This subroutine generates the matrix of partial derivatives, $[{}_{\alpha}\partial_k \,\vdots\, {}_{e}\partial_k]$, which is required to generate the energy equations,

$$[{}_{\alpha}\partial_k][F_d]\{{}_{t}q_{\alpha}\} = \{0\} \qquad 2.1.8$$

This matrix is assembled from the system of equations given by,

$$[\gamma_{\alpha} \,\vdots\, \gamma_e]\{{}_{t}q_{\alpha} \,\vdots\, {}_{t}R_e\} + [\gamma_{\lambda}]\{{}_{t}P_{\lambda}\} = \{0\} \qquad 2.1.2$$

whose augmented matrix is given by,

$$[\gamma_{\alpha} \,\vdots\, \gamma_e \;,\; \gamma_{\lambda}]$$

The final augmented matrix for a unique solution of the unknowns, generalized element boundary loads and generalized structural reactions, is obtained from equation 2.1.10 and is given by,

$$[OM] = \left[\begin{array}{c|cc} \gamma_{\alpha} & \gamma_e & , \; \gamma_{\lambda} \\ \hline \varphi & 0 & 0 \end{array}\right]$$

where,

$$[\varphi] = [{}_{\alpha}\partial_k][F_d]$$

However, before generating submatrix $[\varphi]$ this augmented matrix appears in the computer storage as,

$$[OM] = \left[\begin{array}{c|cc} \gamma_{\alpha} & \gamma_e & , \; \gamma_{\lambda} \\ \hline 0 & 0 & 0 \end{array}\right]$$

which is input to subroutine PARDER for generation of the matrix of partial derivatives. The $[{}_\alpha\partial_k \,\vdots\, {}_e\partial_k]$ matrix is assembled immediately into the storage space allowed for submatrix $[\mathcal{Q}]$ and the null submatrix directly below matrix $[\gamma_e]$, that is,

$$[OM] = \left[\begin{array}{c:cc} \gamma_\alpha & \gamma_e & \gamma_\lambda \\ \hdashline \underbrace{{}_\alpha\partial_k}_{} & \underbrace{{}_e\partial_k}_{} & 0 \end{array}\right]$$

Temporary storage.

Matrix $[{}_\alpha\partial_k \,\vdots\, {}_e\partial_k]$ will finally be replaced by matrix $[\mathcal{Q} \,\vdots\, 0]$. Carrying out this method of temporary storage saves considerable storage space. Subroutine PARDER is only called when the system of joint equilibrium equations has an infinite number of solutions, that is, dynamic redundancy exists. In other words, additional independent linear equations are required for a unique solution.

5.2 Subroutines called by PARDER.

This subroutine calls no other subroutines.

5.3 Subroutine listing.

The listing of subroutine PARDER is given in table 26 .

5.4 Description of subroutine arguments.

The first card of this subroutine is,

```
SUBROUTINE PARDER(OM,IQ,N1,MC,N9MAX,N7MAX,N)
```

where,

$$OM = \left[\begin{array}{c|cc} \gamma_\alpha & \gamma_e , & \gamma_\lambda \\ \hline \mathbf{0} & \mathbf{0} \cdot & \mathbf{0} \end{array}\right]$$

This matrix is formed by first of all generating a null matrix and then superimposing the system of joint equilibrium equations. RANTEC is then applied to investigate this system which gives an independent system of equations, the degree of redundancy and a set of redundancies. This matrix is generated by the FORCE-subroutines which are described later.

IQ = vector of redundant load numbers, generated by RANTEC.

N1 = number of joint equilibrium equations, independent equations.

MC = number of unknowns, generalized element boundary loads and generalized structural reactions.

N9MAX = maximum row dimension for matrix OM.

N7MAX = maximum column dimension for matrix OM.

N = actual number of redundancies, this is determined within subroutine PARDER.

See figure 48(a) for further clarification.

5.5 Example of usage.

Consider the stage of analysis where it is required to form the matrix of partial derivatives. Having generated matrices OM and IQ for a given structure with 24 generalized element boundary loads, 2 generalized structural reactions, 14 joint equilibrium equations and maximum OM dimensions of 84 × 122 (rows × columns) the call statement would be,

```
CALL PARDER(OM,IQ,14,26,84,122,N)
```

See figure 48 (b) for further clarification.

The matrix of partial derivatives will now be discussed further using a simple example. The matrix of partial derivatives is given by,

$$[{}_{\alpha}\partial_k \,\vdots\, {}_{e}\partial_k] = \left[\begin{array}{cccccc:cccccc} \frac{\partial q_1}{\partial q^1} & \frac{\partial q_2}{\partial q^1} & \cdots & \frac{\partial q_\alpha}{\partial q^1} & \cdots & & \frac{\partial R_1}{\partial q^1} & \frac{\partial R_2}{\partial q^1} & \cdots & \frac{\partial R_e}{\partial q^1} & \cdots \\ \vdots & \vdots & & \vdots & & & \vdots & \vdots & & \vdots & \\ \frac{\partial q_1}{\partial q^k} & \frac{\partial q_2}{\partial q^k} & \cdots & \frac{\partial q_\alpha}{\partial q^k} & \cdots & & \frac{\partial R_1}{\partial q^k} & \frac{\partial R_2}{\partial q^k} & \cdots & \frac{\partial R_e}{\partial q^k} & \cdots \\ \vdots & \vdots & & \vdots & & & \vdots & \vdots & & \vdots & \end{array}\right]$$

The elements of this matrix are the partial derivatives of the generalized element boundary loads, $\{{}_t q_\alpha\}$, and the generalized structural reactions, $\{{}_t R_e\}$, with respect to each of the automatically selected redundancies, $\{{}_t q^k\}$. The redundancies are given by vector IQ in subroutine RANTEC. Let,

$$[OM] = \left[\begin{array}{c:cc} \gamma_\alpha & \gamma_e , & \gamma_\lambda \\ \hdashline O & O & O \end{array}\right]$$

$$
= \left.\begin{array}{cccccc|c|cc}
\multicolumn{6}{c|}{\{{}_tq_\alpha\}} & \{{}_tR_e\} & \multicolumn{2}{c}{\{{}_tP_\lambda\}} \\
q_1 & q_2 & q_3 & q_4 & q_5 & q_6 & R_1 & P_1 & P_2 \\
\hline
0 & a_{12} & 0 & 1 & a_{15} & 0 & a_{17} & b_{11} & b_{12} \\
1 & a_{22} & 0 & 0 & a_{25} & 0 & a_{27} & b_{21} & b_{22} \\
0 & a_{32} & 1 & 0 & a_{35} & 0 & a_{37} & b_{31} & b_{32} \\
0 & a_{42} & 0 & 0 & a_{45} & 1 & a_{47} & b_{41} & b_{42} \\
\hline
\multicolumn{6}{c|}{O} & O & \multicolumn{2}{c}{O}
\end{array}\right\} ,
$$

Joint
equilibrium
equations
after
applying
RANTEC.

The redundancies are given by,

$$
\{{}_tq^k\} = \begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix} = \begin{bmatrix} q_2 \\ q_5 \\ R_1 \end{bmatrix}
$$

Writing the investigated system of joint equilibrium equations in expanded form gives,

$$
[\,\gamma_\alpha \;\vdots\; \gamma_e\,]\{\,{}_tq_\alpha \;\vdots\; {}_tR_e\,\} + [\,\gamma_\lambda\,]\{\,{}_tP_\lambda\,\} = \{O\}
$$

which for the example are given by,

$$
\begin{array}{l}
(0)\,q_1 + a_{12}q_2 + (0)\,q_3 + q_4 + a_{15}q_5 + (0)\,q_6 + a_{17}R_1 + b_{11}P_1 + b_{12}P_2 = 0 \\
q_1 + a_{22}q_2 + (0)\,q_3 + (0)\,q_4 + a_{25}q_5 + (0)\,q_6 + a_{27}R_1 + b_{21}P_1 + b_{22}P_2 = 0 \\
(0)\,q_1 + a_{32}q_2 + q_3 + (0)\,q_4 + a_{35}q_5 + (0)\,q_6 + a_{37}R_1 + b_{31}P_1 + b_{32}P_2 = 0 \\
(0)\,q_1 + \underset{\uparrow\, q^1}{a_{42}q_2} + (0)\,q_3 + (0)\,q_4 + \underset{\uparrow\, q^2}{a_{45}q_5} + q_6 + \underset{\uparrow\, q^3}{a_{47}R_1} + b_{41}P_1 + b_{42}P_2 = 0
\end{array}
$$

Rearranging these equations and making the relevant substitutions results in,

$$q_1 = -a_{22}\,q^1 - a_{25}\,q^2 - a_{27}\,q^3 - b_{21}P_1 - b_{22}P_2$$
$$q_3 = -a_{32}\,q^1 - a_{35}\,q^2 - a_{37}\,q^3 - b_{31}P_1 - b_{32}P_2$$
$$q_4 = -a_{12}\,q^1 - a_{15}\,q^2 - a_{17}\,q^3 - b_{11}P_1 - b_{12}P_2$$
$$q_6 = -a_{42}\,q^1 - a_{45}\,q^2 - a_{47}\,q^3 - b_{41}P_1 - b_{42}P_2$$

Therefore,

$$\frac{\partial q_1}{\partial q^1} = -a_{22}\ ,\quad \frac{\partial q_3}{\partial q^1} = -a_{32}\ ,\quad \frac{\partial q_4}{\partial q^1} = -a_{12}\ ,\quad \frac{\partial q_6}{\partial q^1} = -a_{42}$$

$$\frac{\partial q_1}{\partial q^2} = -a_{25}\ ,\quad \frac{\partial q_3}{\partial q^2} = -a_{35}\ ,\quad \frac{\partial q_4}{\partial q^2} = -a_{15}\ ,\quad \frac{\partial q_6}{\partial q^2} = -a_{45}$$

$$\frac{\partial q_1}{\partial q^3} = -a_{27}\ ,\quad \frac{\partial q_3}{\partial q^3} = -a_{37}\ ,\quad \frac{\partial q_4}{\partial q^3} = -a_{17}\ ,\quad \frac{\partial q_6}{\partial q^3} = -a_{47}$$

Now,

$$\frac{\partial q_2}{\partial q^1} = 1 \quad \text{since} \quad q_2 = q^1$$

$$\left.\begin{aligned}\frac{\partial q_2}{\partial q^2} &= 0\\ \frac{\partial q_2}{\partial q^3} &= 0\end{aligned}\right\}$$

since $q_2 = q^1$, the partial derivative of a redundancy with respect to another redundancy is zero. Partial differentiation of a quantity with respect to one of a set of variables means the variation of that quantity with the selected variable whilst keeping all the remaining variables constant. Therefore, when partially differentiating a redundancy with respect to another redundancy this is the same as differentiating a constant with respect to a variable, which is of course zero. Note that the P's are constants.

Hence,

$$\frac{\partial q_5}{\partial q^1} = 0 \quad , \quad \frac{\partial R_1}{\partial q^1} = 0$$

$$\frac{\partial q_5}{\partial q^2} = 1 \quad , \quad \frac{\partial R_1}{\partial q^2} = 0$$

$$\frac{\partial q_5}{\partial q^3} = 0 \quad , \quad \frac{\partial R_1}{\partial q^3} = 1$$

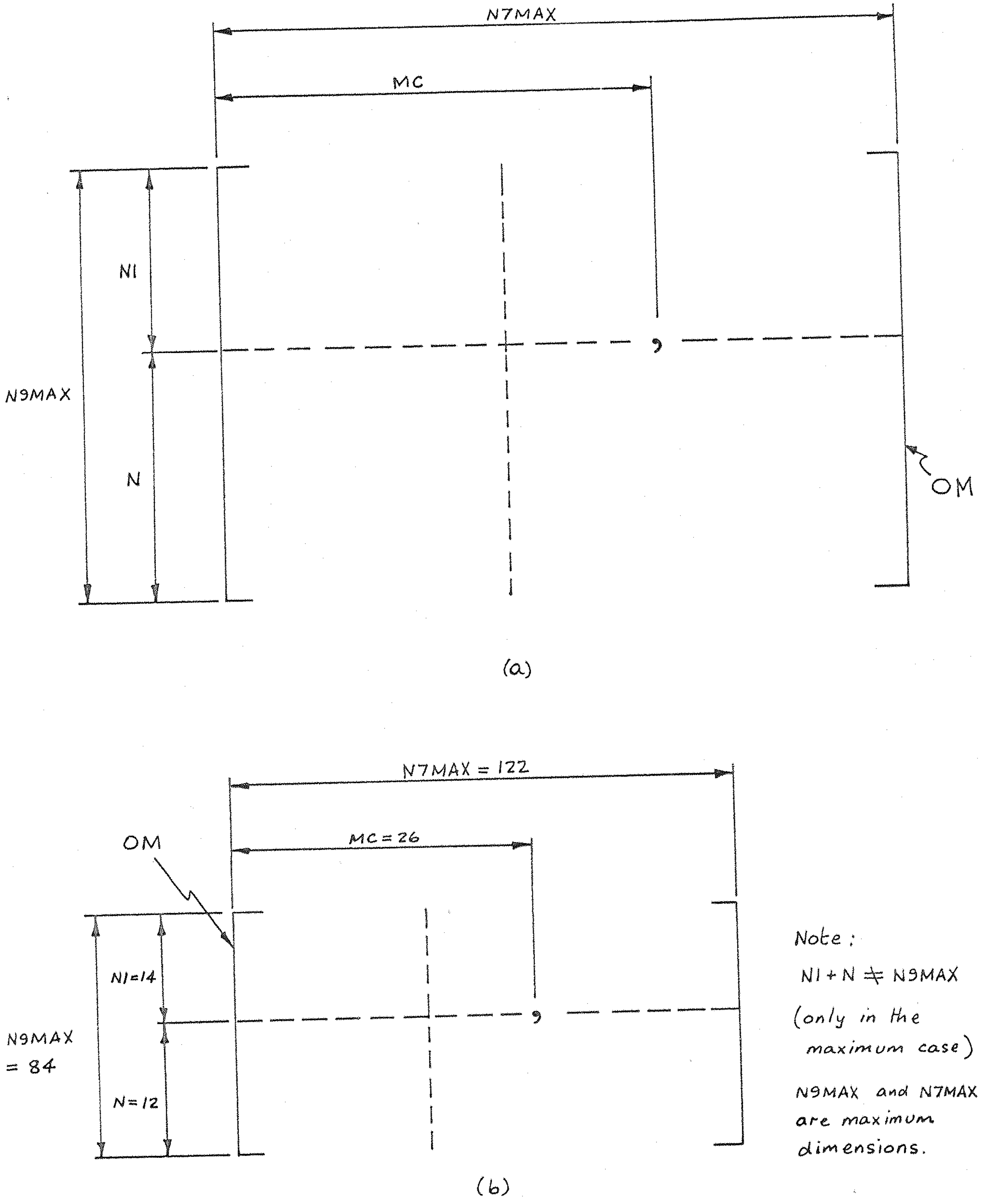
Therefore, the matrix of partial derivatives is given by,

$$[{}_\alpha\partial_k \,\vdots\, {}_e\partial_k] = \left[\begin{array}{cccccc|c} -a_{22} & 1 & -a_{32} & -a_{12} & 0 & -a_{42} & 0 \\ -a_{25} & 0 & -a_{35} & -a_{15} & 1 & -a_{45} & 0 \\ -a_{27} & 0 & -a_{37} & -a_{17} & 0 & -a_{47} & 1 \end{array}\right]$$

```
      SUBROUTINE PARDER(OM,IQ,N1,MC,N9MAX,N7MAX,N)
C     JOHN ROBINSON.  I.S.V.R.
C     MATRIX OF PARTIAL DERIVATIVES FOR ENERGY EQUATIONS
C     ALSO IMMEDIATE ASSEMBLY INTO MATRIX OM
      DIMENSION OM(N9MAX,N7MAX),IQ(N9MAX)
      N=0
      DO2 M=1,MC
      IF(IQ(M))3,2,3
    3 N=N+1
    2 CONTINUE
      DO12 I=1,N1
      DO9 J=1,MC
      IF(ABS(OM(I,J)).LE.1.0E-08)GO TO 9
      IF(J-IQ(J))9,9,8
    8 JJ=J
      GO TO 10
    9 CONTINUE
   10 NN=1
      DO12 M=1,MC
      IF(IQ(M))13,12,13
   13 OM(NN+N1,JJ)=-OM(I,M)
      NN=NN+1
   12 CONTINUE
      NN=1
      DO15 M=1,MC
      IF(IQ(M))16,15,16
   16 OM(NN+N1,M)=1.0
      NN=NN+1
   15 CONTINUE
      RETURN
      END
```

A.S.A. Fortran listing of subroutine PARDER.

Table 26.



Note:
$N1 + N \neq N9MAX$
(only in the maximum case).
N9MAX and N7MAX are maximum dimensions.

Argument definitions for subroutine PARDER.

Fig. 48.

6. Subroutine FMD10..........Element Type P1/2FD.

6.1 Description of subroutine.

This subroutine generates the dynamic flexibility matrix for a plane beam element, Element Type P1/2FD. See 3.1.2 and 4.1.2.

This element dynamic flexibility matrix was derived using a 3rd degree polynomial for the element internal bending moment distribution. The displacement function used for the inertia loading was obtained using the incremental loading equation, that is,

$$w_z(x) = \frac{d^2M(x)}{dx^2} = \rho\omega^2 u_z(x) \qquad 3.1.2.5$$

The generalized element boundary loads consist of a shear and a moment at each node. This type of element is used for the analysis of collinear beam structures.

6.2 Subroutines called by FMD10.

This subroutine calls no other subroutines.

6.3 Subroutine listing.

The listing of subroutine FMD10 is given in table 27 .

6.4 Description of subroutine arguments.

The first card of this subroutine is,

```
SUBROUTINE FMD10(XLM,XIM,EM,XMUM,CSAM,OMEGA,FMD)
```

where,

XLM = length of beam element (in).

XIM = second moment of area of beam cross section (in^4).

EM = Young's modulus of elasticity for the element material (lb per in^2).

XMUM = density of element material (lb per in^3).

CSAM = cross sectional area of beam element (in^2).

OMEGA = angular frequency (radians per second).

FMD = element dynamic flexibility matrix (order 4 × 4).

6.5 Example of usage.

The structural element properties for the analysis of a complete structure are read by the master programme in matrix form. A typical read statement would be,

```
READ(5,85)(XL(M),XI(M),E(M),XMU(M),CSA(M),M = 1,NE)
```

where,

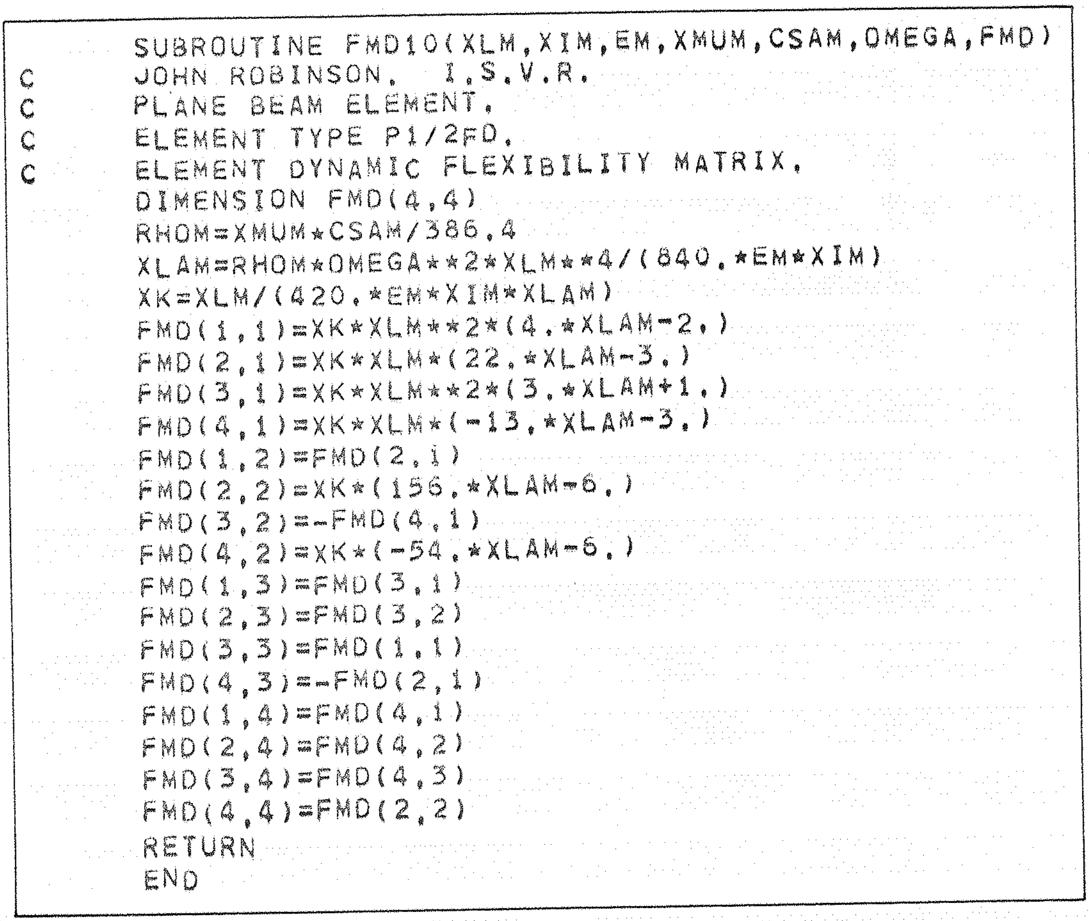
NE = total number of structural elements.

Therefore, the call statement required to generate the element dynamic flexibility matrix for element 1 (M = 1) would be,

```
CALL FMD10(XL(1),XI(1),E(1),XMU(1),CSA(1),OMEGA,FMD)
```

However, the generation of the element dynamic flexibility matrices is usually in the form of a DO-loop. In this case the call statement would be,

```
CALL FMD10(XL(M),XI(M),E(M),XMU(M),CSA(M),OMEGA,FMD)
```



```
      SUBROUTINE FMD10(XLM,XIM,EM,XMUM,CSAM,OMEGA,FMD)
C     JOHN ROBINSON.  I.S.V.R.
C     PLANE BEAM ELEMENT.
C     ELEMENT TYPE P1/2FD.
C     ELEMENT DYNAMIC FLEXIBILITY MATRIX.
      DIMENSION FMD(4,4)
      RHOM=XMUM*CSAM/386.4
      XLAM=RHOM*OMEGA**2*XLM**4/(840.*EM*XIM)
      XK=XLM/(420.*EM*XIM*XLAM)
      FMD(1,1)=XK*XLM**2*(4.*XLAM-2.)
      FMD(2,1)=XK*XLM*(22.*XLAM-3.)
      FMD(3,1)=XK*XLM**2*(3.*XLAM+1.)
      FMD(4,1)=XK*XLM*(-13.*XLAM-3.)
      FMD(1,2)=FMD(2,1)
      FMD(2,2)=XK*(156.*XLAM-6.)
      FMD(3,2)=-FMD(4,1)
      FMD(4,2)=XK*(-54.*XLAM-6.)
      FMD(1,3)=FMD(3,1)
      FMD(2,3)=FMD(3,2)
      FMD(3,3)=FMD(1,1)
      FMD(4,3)=-FMD(2,1)
      FMD(1,4)=FMD(4,1)
      FMD(2,4)=FMD(4,2)
      FMD(3,4)=FMD(4,3)
      FMD(4,4)=FMD(2,2)
      RETURN
      END
```

A. S. A. Fortran listing of subroutine FMD10,

Element Type P1/2FD.

Table 27.

7. Subroutine FMD10..........Element Type P2/2FD.

7.1 Description of subroutine.

This subroutine generates the dynamic flexibility matrix for a plane beam element, Element Type P2/2FD. See 3.2.2 and 4.2.2.

This element dynamic flexibility matrix was derived using a 3rd degree polynomial for the element internal bending moment distribution but the incremental loading equation is not satisfied. The displacement function used for the inertia loading is obtained by double integration of the bending moment expression and evaluating the integration constants by applying d'Alembert's principle for the overall element, that is,

$$u_z(x) = \frac{1}{EI}\left(\left(\iint M(x)\,dx\,dx\right) + C_1 x + C_2\right) \qquad 3.2.2.1$$

and

$$w_z(x) = \rho\omega^2 u_z(x)$$

The gneralized element boundary loads consist of a shear and a bending moment at each node. This type of element is used for the analysis of collinear beam structures.

7.2 Subroutines called by FMD10.

This subroutine calls no other subroutines.

7.3 Subroutine listing.

The listing of subroutine FMD10 is given in table 28.

7.4 Description of subroutine arguments.

This is the same as subroutine FMD10.....Element Type P1/2FD.

```
      SUBROUTINE FMD10(XLM,XIM,EM,XMUM,CSAM,OMEGA,FMD)
C     JOHN ROBINSON.  I.S.V.R.
C     PLANE BEAM ELEMENT.
C     ELEMENT TYPE P2/2FD.
C     ELEMENT DYNAMIC FLEXIBILITY MATRIX.
      DIMENSION FMD(4,4)
      RHOM=XMUM*CSAM/386.4
      XLAM=RHOM*OMEGA**2*XLM**4/(840.*EM*XIM)
      XK=XLM/(420.*EM*XIM*XLAM)
      YY=840.*XLAM**2
      FMD(1,1)=XK*XLM**2*(4.*XLAM-2.-(71.*YY/10395.))
      FMD(2,1)=XK*XLM*(22.*XLAM-3.-(223.*YY/6930.))
      FMD(3,1)=XK*XLM**2*(3.*XLAM+1.-(1097.*YY/166320.))
      FMD(4,1)=XK*XLM*(-13.*XLAM-3.+(1681.*YY/55440.))
      FMD(1,2)=FMD(2,1)
      FMD(2,2)=XK*(156.*XLAM-6.-(118.*YY/770.))
      FMD(3,2)=-FMD(4,1)
      FMD(4,2)=XK*(-54.*XLAM-6.+(2951.*YY/21560.))
      FMD(1,3)=FMD(3,1)
      FMD(2,3)=FMD(3,2)
      FMD(3,3)=FMD(1,1)
      FMD(4,3)=-FMD(2,1)
      FMD(1,4)=FMD(4,1)
      FMD(2,4)=FMD(4,2)
      FMD(3,4)=FMD(4,3)
      FMD(4,4)=FMD(2,2)
      RETURN
      END
```

A.S.A. Fortran listing of subroutine FMD10,
Element Type P2/2FD.

Table 28.

8. Subroutine FMD10..........Element Type P3/2FD.

8.1 Description of subroutine.

This subroutine generates the dynamic flexibility matrix for a plane beam element, Element Type P3/2FD. See 3.3.2 and 4.3.2.

This element dynamic flexibility matrix is derived using a differential equation approach (transcendental functions). This requires the solution of the equation,

$$EI \frac{\partial^4 u_z(x,t)}{\partial x^4} + \rho \frac{\partial^2 u_z(x,t)}{\partial t^2} = 0 \qquad 3.3.2.3$$

where,

$$u_z(x,t) = u_z(x) \sin \omega t$$

and

$$\rho = \frac{\mu A}{386.4}$$

and the relationships,

$$M(x,t) = EI \frac{\partial^2 u_z(x,t)}{\partial x^2} \qquad 3.1.2.1$$

and

$$Q(x,t) = - \frac{\partial M(x,t)}{\partial x} = - EI \frac{\partial^3 u_z(x,t)}{\partial x^3} \qquad 3.1.2.2$$

The generalized element boundary loads consist of a shear and a moment at each node. This type of element is used for the analysis of collinear beam structures.

8.2 Subroutines called by FMD10.

This subroutine calls no other subroutines.

8.3 Subroutine listing.

The listing of subroutine FMD10 is given in table 29 .

8.4 Description of subroutine arguments.

This is the same as subroutine FMD10..........Element Type P1/2FD.

```
      SUBROUTINE FMD10(XLM,XIM,EM,XMUM,CSAM,OMEGA,FMD)
C     JOHN ROBINSON.  I.S.V.R.
C     PLANE BEAM ELEMENT.
C     ELEMENT TYPE P3/2FD.
C     ELEMENT DYNAMIC FLEXIBILITY MATRIX.
      DIMENSION FMD(4,4)
      RHOM=XMUM*CSAM/386.4
      XLAM=(OMEGA**2*RHOM/(EM*XIM))**0.25
      XA=XLAM*XLM
      XB=-XA
      XCOSH=COSH(XA)
      XSINH=SINH(XA)
      XSIN=SIN(XA)
      XCOS=COS(XA)
      XD=EM*XIM*XLAM**3*(XCOS*XCOSH-1.)
      FMD(1,1)=(XSIN*XCOSH-XCOS*XSINH)/XD
      FMD(2,1)=XLAM*XSIN*XSINH/XD
      FMD(3,1)=(XSIN-XSINH)/XD
      FMD(4,1)=XLAM*(XCOSH-XCOS)/XD
      FMD(1,2)=FMD(2,1)
      FMD(2,2)=XLAM**2*(XCOS*XSINH+XSIN*XCOSH)/XD
      FMD(3,2)=-FMD(4,1)
      FMD(4,2)=XLAM**2*(XSIN+XSINH)/XD
      FMD(1,3)=FMD(3,1)
      FMD(2,3)=FMD(3,2)
      FMD(3,3)=FMD(1,1)
      FMD(4,3)=-FMD(2,1)
      FMD(1,4)=FMD(4,1)
      FMD(2,4)=FMD(4,2)
      FMD(3,4)=FMD(4,3)
      FMD(4,4)=FMD(2,2)
      RETURN
      END
```

A.S.A. Fortran listing of subroutine FMD10,
Element Type P3/2FD.

Table 29 .

9. Subroutine FMD30.............Element Type P1/3FD.

9.1 Description of subroutine.

This subroutine generates the dynamic flexibility matrix for an inclined plane beam element, Element Type P1/3FD. See 4.1.3. This element dynamic flexibility matrix is derived by assembling the matrices for element types P1/1FD and P1/2FD and then carrying out a transformation from local axes to global axes. Therefore, for this element the internal bending moment is assumed to vary as a 3rd degree polynomial and the endload as a 1st degree polynomial (linear variation). The generalized element boundary loads consist of a load in the $\bar{x}$-direction, a load in the $\bar{z}$-direction and a moment at each node, see figure 17(b). This type of element is used for the analysis of general plane frames.

9.2 Subroutines called by FMD30.

This subroutine calls no other subroutines.

9.3 Subroutine listing.

The listing of subroutine FMD30 is given in table 30.

9.4 Description of subroutine arguments.

The first card of this subroutine is,

```
SUBROUTINE FMD30(XBI,ZBI,XBJ,ZBJ,XIM,EM,XMUM,CSAM,OMEGA,FMD)
```

where,

XBI = $\bar{x}$-ordinate of node i (in).

ZBI = $\bar{z}$-ordinate of node i (in).

XBJ = $\bar{x}$-ordinate of node j (in).

ZBJ = $\bar{z}$-ordinate of node j (in).

XIM = second moment of area of beam cross section (in^4).

EM = Young's modulus of elasticity for the element material (lb per in^2).

XMUM = density of element material (lb per in^3).

CSAM = cross sectional area of beam element (in^2).

OMEGA = angular frequency (radians per second).

FMD = element dynamic flexibility matrix (order 6×6).

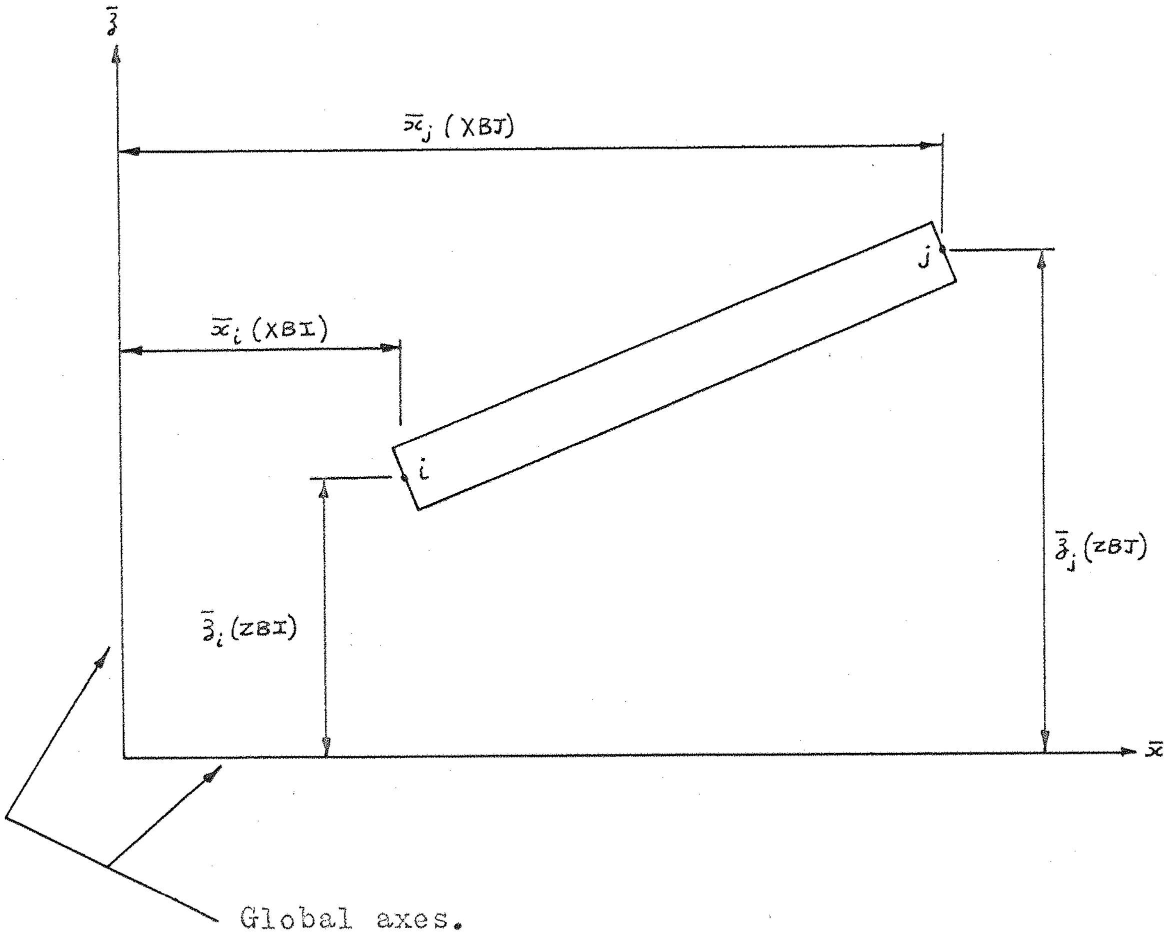
See figure 49 for further clarification.

9.5 Example of usage.

Assuming that the generation of the element dynamic flexibility matrices is in the form of a DO-loop a typical call statement would be,

```
 CALL FMD30(XBI(M),ZBI(M),XBJ(M),ZBJ(M),XI(M),E(M),XMU(M),
1CSA(M),OMEGA,FMD)
```

Continuation line.



Node coordinates for an inclined plane beam element.

Fig. 49 .

```
      SUBROUTINE FMD30(XBI,ZBI,XBJ,ZBJ,XIM,EM,XMUM,CSAM,OMEGA,FMD)
C     JOHN ROBINSON.  I.S.V.R.
C     INCLINED PLANE BEAM ELEMENT.
C     ELEMENT TYPE P1/3FD.
C     ELEMENT DYNAMIC FLEXIBILITY MATRIX.
      DIMENSION FMD(6,6)
      XDIF=XBJ-XBI
      YDIF=ZBJ-ZBI
      A1=XDIF**2
      A2=YDIF**2
      XLMSQ=A1+A2
      XLM=SQRT(XLMSQ)
      A=XDIF/XLM
      B=YDIF/XLM
      RHO=XMUM*CSAM/386.4
      XLAM=(RHO*OMEGA**2*XLM**4)/(840.0*EM*XIM)
      X=XLM/(420.0*EM*XIM*XLAM)
      Y=1.0/(XLM*RHO*OMEGA**2)
      Z=XLM/(CSAM*EM)
      FMD(1,1)=A**2*(Z/3.0-Y)+B**2*X*XLM**2*(4.0*XLAM-2.0)
      FMD(1,2)=A*B*(Z/3.0-Y-X*XLM**2*(4.0*XLAM-2.0))
      FMD(1,3)=-B*X*XLM*(22.0*XLAM-3.0)
      FMD(1,4)=A**2*(-Z/6.0-Y)+B**2*X*XLM**2*(3.0*XLAM+1.0)
      FMD(1,5)=A*B*(-Z/6.0-Y-X*XLM**2*(3.0*XLAM+1.0))
      FMD(1,6)=B*X*XLM*(13.0*XLAM+3.0)
      FMD(2,1)=FMD(1,2)
      FMD(2,2)=B**2*(Z/3.0-Y)+A**2*X*XLM**2*(4.0*XLAM-2.0)
      FMD(2,3)=A*X*XLM*(22.0*XLAM-3.0)
      FMD(2,4)=FMD(1,5)
      FMD(2,5)=B**2*(-Z/6.0-Y)+A**2*X*XLM**2*(3.0*XLAM+1.0)
      FMD(2,6)=-A*X*XLM*(13.0*XLAM+3.0)
      FMD(3,1)=FMD(1,3)
      FMD(3,2)=FMD(2,3)
      FMD(3,3)=X*(156.0*XLAM-6.0)
      FMD(3,4)=-FMD(1,6)
      FMD(3,5)=-FMD(2,6)
      FMD(3,6)=X*(-54.0*XLAM-6.0)
      FMD(4,1)=FMD(1,4)
      FMD(4,2)=FMD(2,4)
      FMD(4,3)=FMD(3,4)
      FMD(4,4)=FMD(1,1)
      FMD(4,5)=FMD(1,2)
      FMD(4,6)=-FMD(1,3)
      FMD(5,1)=FMD(1,5)
      FMD(5,2)=FMD(2,5)
      FMD(5,3)=FMD(3,5)
      FMD(5,4)=FMD(4,5)
      FMD(5,5)=FMD(2,2)
      FMD(5,6)=-FMD(2,3)
      FMD(6,1)=FMD(1,6)
      FMD(6,2)=FMD(2,6)
      FMD(6,3)=FMD(3,6)
      FMD(6,4)=FMD(4,6)
      FMD(6,5)=FMD(5,6)
      FMD(6,6)=FMD(3,3)
      RETURN
      END
```

A.S.A. Fortran listing of subroutine FMD30, Element Type P1/3FD.

Table 30.

10. Subroutine FMD30............Element Type P3/3FD.

10.1 Description of subroutine.

This subroutine generates the dynamic flexibility matrix for an inclined plane beam element, Element Type P3/FD3. See 4.3.3. This element dynamic flexibility matrix is derived by assembling the matrices of element types P3/1FD and P3/2FD and then carrying out a transformation from local axes to global axes. Therefore, for this element the internal bending moment and endload distributions are expressed in terms of transcendental functions, see 3.3.1,3.3.2,4.3.1 and 4.3.2. The generalized element boundary loads are the same as Element Type P1/3FD.

10.2 Subroutines called by FMD30.

This subroutine calls no other subroutines.

10.3 Subroutine listing.

The listing of subroutine FMD30 is given in table 31 .

10.4 Description of subroutine arguments.

This is the same as subroutine FMD30....... Element Type P1/3FD.

A.S.A. Fortran listing of subroutine FMD30, Element Type P3/3FD.

Table 31 .

```
      SUBROUTINE FMD30(XBI,ZBI,XBJ,ZBJ,XIM,EM,XMUM,CSAM,OMEGA,FMD)
C     JOHN ROBINSON.  I.S.V.R.
C     INCLINED PLANE BEAM ELEMENT.
C     ELEMENT TYPE P3/3FD.
C     ELEMENT DYNAMIC FLEXIBILITY MATRIX.
      DIMENSION FMD(6,6)
      XDIF=XBJ-XBI
      YDIF=ZBJ-ZBI
      A1=XDIF**2
      A2=YDIF**2
      XLMSQ=A1+A2
      XLM=SQRT(XLMSQ)
      XLAM1=OMEGA*SQRT(XMUM/(386.4*EM))
      XLAM2=(OMEGA**2*CSAM*XMUM/(386.4*EM*XIM))**0.25
      XA1=XLAM1*XLM
      XA2=XLAM2*XLM
      S2=SIN(XA2)
      C2=COS(XA2)
      SH=SINH(XA2)
      CH=COSH(XA2)
      S1=SIN(XA1)
      F1=S2*SH
      F3=C2*CH-1.0
      F5=C2*SH-S2*CH
      F6=C2*SH+S2*CH
      F7=S2+SH
      F8=S2-SH
      F10=C2-CH
      F11=-COS(XA1)/S1
      F12=-1.0/S1
      A3=A1/XLMSQ
      A4=A2/XLMSQ
      A5=XDIF*YDIF/XLMSQ
      A6=XDIF/XLM
      A7=YDIF/XLM
      B1=CSAM*EM*XLAM1
      C2=EM*XIM*F3
      D1=C2*XLAM2
      D2=D1*XLAM2
      D3=D2*XLAM2
      FMD(1,1)=A3*F11/B1-A4*F5/D3
      FMD(1,2)=A5*(F11/B1+F5/D3)
      FMD(1,3)=-A7*F1/D2
      FMD(1,4)=A3*F12/B1+A4*F8/D3
      FMD(1,5)=A5*(F12/B1-F8/D3)
      FMD(1,6)=A7*F10/D2
```

Subroutine FMD30 listing continued.

```
      FMD(2,1)=FMD(1,2)
      FMD(2,2)=A4*F11/B1-A3*F5/D3
      FMD(2,3)=A6*F1/D2
      FMD(2,4)=FMD(1,5)
      FMD(2,5)=A4*F12/B1+A3*F8/D3
      FMD(2,6)=-A6*F10/D2
      FMD(3,1)=FMD(1,3)
      FMD(3,2)=FMD(2,3)
      FMD(3,3)=F6/D1
      FMD(3,4)=-FMD(1,6)
      FMD(3,5)=-FMD(2,6)
      FMD(3,6)=F7/D1
      FMD(4,1)=FMD(1,4)
      FMD(4,2)=FMD(2,4)
      FMD(4,3)=FMD(3,4)
      FMD(4,4)=FMD(1,1)
      FMD(4,5)=FMD(1,2)
      FMD(4,6)=-FMD(1,3)
      FMD(5,1)=FMD(1,5)
      FMD(5,2)=FMD(2,5)
      FMD(5,3)=FMD(3,5)
      FMD(5,4)=FMD(4,5)
      FMD(5,5)=FMD(2,2)
      FMD(5,6)=-FMD(2,3)
      FMD(6,1)=FMD(1,6)
      FMD(6,2)=FMD(2,6)
      FMD(6,3)=FMD(3,6)
      FMD(6,4)=FMD(4,6)
      FMD(6,5)=FMD(5,6)
      FMD(6,6)=FMD(3,3)
      RETURN
      END
```

11. Subroutine FMD40............Element Type P1/4FD.

11.1 Description of subroutine.

This subroutine generates the dynamic flexibility matrix corresponding to a q-system of generalized element boundary loads for a rectangular plate element, Element Type P1/4FD, see Chapter 3 (3.1.3(a)) and Chapter 4(4.1.4). The distributed boundary loadings for this element are assumed uniform along the respective boundaries. The moment distributions have been taken in the form of polynomials. In deriving the element dynamic flexibility matrix an equivalent plate loading system has been adopted. The generalized element boundary loads consist of a uniform distributed shear and moment along each boundary and a concentrated load at each node, see figure 19 . This gives a total of 12 generalized element boundary loads. This type of element is used for the analysis of two dimensional plate structures.

11.2 Subroutines called by FMD40.

This subroutine calls no other subroutines.

11.3 Subroutine listing.

The listing of subroutine FMD40 is given in table 32.

11.4 Description of subroutine arguments.

The first card of this subroutine is,

```
SUBROUTINE FMD40(A,B,T,EM,XMUM,XNUM,OMEGA,FMD)
```

where,

A = length of the plate boundary in the x-direction (in).

B = length of the plate boundary in the z-direction (in).

T = plate thickness (in).

EM = Young's modulus of elasticity for the element material (lb per in^2).

XMUM = density of element material (lb per in^3).

XNUM = Poisson's ratio for the element material.

OMEGA = angular frequency (radians per second).

FMD = element dynamic flexibility matrix.

11.5 Example of usage.

Assuming that the generation of the element dynamic flexibility matrices is in the form of a DO-loop and that A,B and T are constant for all elements a typical call statement would be,

```
CALL FMD40(A,B,T,E(M),XMU(M),XNU(M),OMEGA,FMD)
```

A.S.A. Fortran listing of subroutine FMD40, Element Type P1/4FD.

Table 32.

```
      SUBROUTINE FMD40(A.B.T.EM.XMUM.XNUM.OMEGA.FMD)
C     JOHN ROBINSON.  I.S.V.R.
C     RECTANGULAR PLATE ELEMENT.
C     ELEMENT TYPE P1/4FD.
C     ELEMENT DYNAMIC FLEXIBILITY MATRIX.
      DIMENSION FMD(12.12)
      P=(A*B)/(EM*T**3*35.0)
      R=386.4/(XMUM*T*OMEGA**2)
      C=1.0/A**2
      D=1.0/B**2
      G=A/B
      FMD(1.1)=156.0*P-12.0*G*D*R
      FMD(1.2)=-105.0*XNUM*P
      FMD(1.3)=-54.0*P-12.0*G*D*R
      FMD(1.4)=-FMD(1.2)
      FMD(1.5)=22.0*B*P-6.0*A*D*R
      FMD(1.6)=-17.5*A*XNUM*P
      FMD(1.7)=13.0*B*P+6.0*A*D*R
      FMD(1.8)=FMD(1.6)
      FMD(1.9)=(22.0/G-17.5*G*XNUM)*P-6.0*D*R
      FMD(1.10)=FMD(1.9)
      FMD(1.11)=(13.0/G-17.5*G*XNUM)*P+6.0*D*R
      FMD(1.12)=FMD(1.11)
      FMD(2.2)=156.0*P-12.0*C*R/G
      FMD(2.3)=FMD(1.4)
      FMD(2.4)=-54.0*P-12.0*C*R/G
      FMD(2.5)=-17.5*B*XNUM*P
      FMD(2.6)=22.0*A*P-6.0*B*C*R
      FMD(2.7)=FMD(2.5)
      FMD(2.8)=13.0*A*P+6.0*B*C*R
      FMD(2.9)=(13.0*G-17.5/G*XNUM)*P+6.0*C*R
      FMD(2.10)=(22.0*G-17.5/G*XNUM)*P-6.0*C*R
      FMD(2.11)=FMD(2.10)
      FMD(2.12)=FMD(2.9)
      FMD(3.3)=FMD(1.1)
      FMD(3.4)=FMD(1.2)
      FMD(3.5)=-FMD(1.7)
      FMD(3.6)=-FMD(1.6)
      FMD(3.7)=-FMD(1.5)
      FMD(3.8)=FMD(3.6)
      FMD(3.9)=-FMD(1.11)
      FMD(3.10)=FMD(3.9)
      FMD(3.11)=-FMD(1.9)
      FMD(3.12)=FMD(3.11)
      FMD(4.4)=FMD(2.2)
      FMD(4.5)=-FMD(2.5)
      FMD(4.6)=-FMD(2.8)
      FMD(4.7)=FMD(4.5)
```

Subroutine FMD40 listing continued.

```
      FMD(4.8)=-FMD(2.6)
      FMD(4.9)=-FMD(2.10)
      FMD(4.10)=-FMD(2.9)
      FMD(4.11)=FMD(4.10)
      FMD(4.12)=FMD(4.9)
      FMD(5.5)=4.0*B**2*P-4.0*G*R
      FMD(5.6)=-35,0/12,0*A*B*XNUM*P-R
      FMD(5.7)=3.0*B**2*P+2.0*G*R
      FMD(5.8)=FMD(5.6)
      FMD(5.9)=(4.0*B/G-35.0/12.0*A*XNUM)*P-4.0/B*R
      FMD(5.10)=FMD(5.9)
      FMD(5.11)=(3.0*B/G-35.0/12.0*A*XNUM)*P+2.0/B*R
      FMD(5.12)=FMD(5.11)
      FMD(6.6)=4.0*A**2*P-4.0/G*R
      FMD(6.7)=FMD(5.6)
      FMD(6.8)=3.0*A**2*P+2.0/G*R
      FMD(6.9)=(3.0*A*G-35.0/12.0*B*XNUM)*P+2.0/A*R
      FMD(6.10)=(4.0*A*G-35.0/12.0*B*XNUM)*P-4.0/A*R
      FMD(6.11)=FMD(6.10)
      FMD(6.12)=FMD(6.9)
      FMD(7.7)=FMD(5.5)
      FMD(7.8)=FMD(5.6)
      FMD(7.9)=FMD(5.11)
      FMD(7.10)=FMD(7.9)
      FMD(7.11)=FMD(5.9)
      FMD(7.12)=FMD(7.11)
      FMD(8.8)=FMD(6.6)
      FMD(8.9)=FMD(6.10)
      FMD(8.10)=FMD(6.9)
      FMD(8.11)=FMD(8.10)
      FMD(8.12)=FMD(8.9)
      FMD(9.9)=(4.0*(D/C+C/D)+70.0/3.0*(1.0+0.75*XNUM))*P-7.0*R/(A*B)
      FMD(9.10)=(3.0*D/C+4.0*C/D-35.0/3.0*(1.0+1.5*XNUM))*P-R/(A*B)
      FMD(9.11)=(3.0*(D/C+C/D)+35.0/6.0)*P+5.0*R/(A*B)
      FMD(9.12)=(4.0*D/C+3.0*C/D-35.0/3.0*(1,0+1,5*XNUM))*P-R/(A*B)
      FMD(10.10)=FMD(9.9)
      FMD(10.11)=FMD(9.12)
      FMD(10.12)=FMD(9.11)
      FMD(11.11)=FMD(9.9)
      FMD(11.12)=FMD(9.10)
      FMD(12.12)=FMD(9.9)
      DO7 I=1.12
      DO6 J=1.12
      IF(I.EQ.J)GO TO 7
    6 FMD(I.J)=FMD(J.I)
    7 CONTINUE
      RETURN
      END
```

12. Subroutine VARDET.

12.1 Description of subroutine.

This subroutine was written by Dr.C.A.Mercer, I.S.V.R. Using the Gaussian elimination procedure and partial pivoting this subroutine evaluates the determinant of a matrix stored in conventional manner. In the case of a band matrix the evaluation can be speeded up by specifying the bandwidth and number of nonzero subdiagonals below the main diagonal.Note, this subroutine can evaluate the determinant of the first (N × N) elements of a square matrix [A] .

12.2 Subroutines called by VARDET.

This subroutine calls no other subroutines.

12.3 Subroutine listing.

The listing of subroutine VARDET is given in table 33.

12.4 Description of subroutine arguments.

N = actual order of matrix whose determinant is to be evaluated. This can be the order of the actual matrix [A] or any submatrix (square) contained in [A] whose determinant is to be evaluated. Note, only submatrices with leading element A(1,1) can be considered.

NR = number of nonzero subdiagonals.

NC = matrix bandwidth.

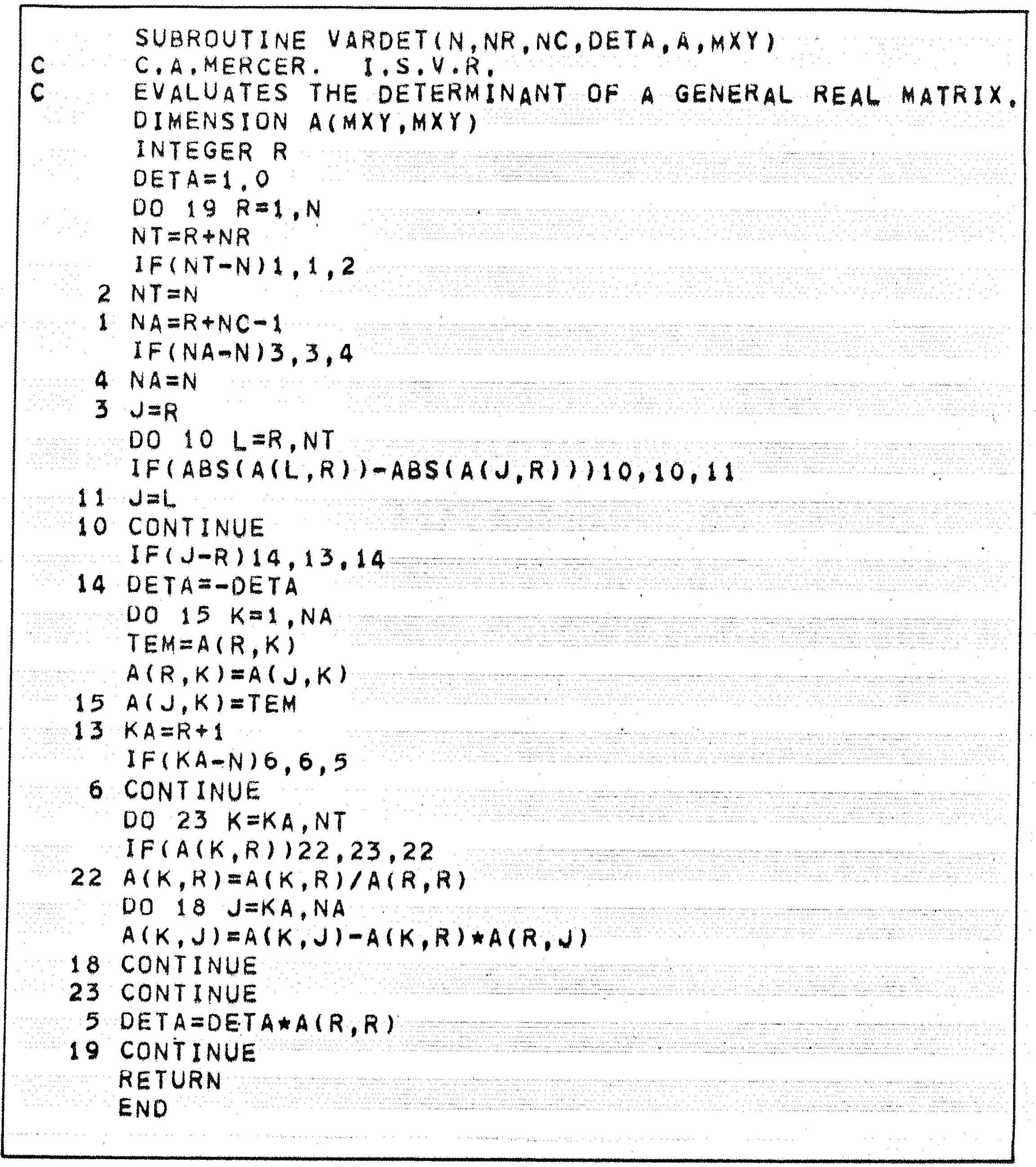
DETA = determinant of matrix (or submatrix).

A = actual matrix whose determinant (or subdeterminant) is to be evaluated.

MXY = maximum possible order of [A] , maximum dimension.

Note, for a general matrix put NR = NC = N.

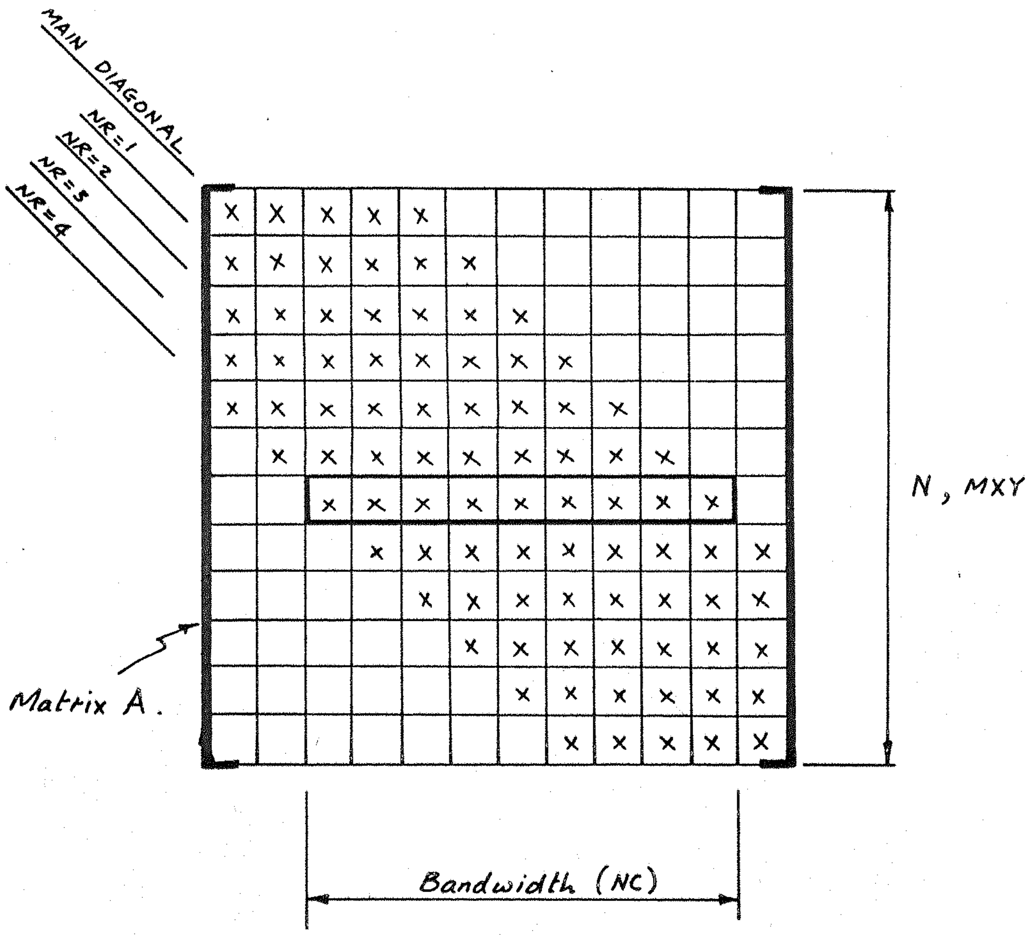
See figure 50 for further clarification.



```
      SUBROUTINE VARDET(N,NR,NC,DETA,A,MXY)
C     C.A.MERCER.  I.S.V.R.
C     EVALUATES THE DETERMINANT OF A GENERAL REAL MATRIX.
      DIMENSION A(MXY,MXY)
      INTEGER R
      DETA=1.0
      DO 19 R=1,N
      NT=R+NR
      IF(NT-N)1,1,2
    2 NT=N
    1 NA=R+NC-1
      IF(NA-N)3,3,4
    4 NA=N
    3 J=R
      DO 10 L=R,NT
      IF(ABS(A(L,R))-ABS(A(J,R)))10,10,11
   11 J=L
   10 CONTINUE
      IF(J-R)14,13,14
   14 DETA=-DETA
      DO 15 K=1,NA
      TEM=A(R,K)
      A(R,K)=A(J,K)
   15 A(J,K)=TEM
   13 KA=R+1
      IF(KA-N)6,6,5
    6 CONTINUE
      DO 23 K=KA,NT
      IF(A(K,R))22,23,22
   22 A(K,R)=A(K,R)/A(R,R)
      DO 18 J=KA,NA
      A(K,J)=A(K,J)-A(K,R)*A(R,J)
   18 CONTINUE
   23 CONTINUE
    5 DETA=DETA*A(R,R)
   19 CONTINUE
      RETURN
      END
```

A.S.A. Fortran listing of subroutine VARDET.

Table 33.



Argument definitions for subroutine VARDET.

Fig. 50 .

13. Subroutine MODE.

13.1 Description of subroutine.

This subroutine was written by Dr.C.A.Mercer and C.Seavey, I.S.V.R. Using the Gaussian elimination procedure and backsubstitution this subroutine computes the eigenvector of a system of homogeneous linear equations whose coefficient matrix has a zero determinant (must be one degree degenerate). For the normal mode analysis of structures the coefficient matrix corresponds to the structural dynamic stiffness matrix at the respective eigenvalue. The homogeneous equations are of the form,

$$[A]\{X\} = \{0\}$$

Both the determinant (det A) and ERROR SUM are crude measures of the error involved in the process. These two items are printed out. If the computed eigenvector is substituted back into the original system of homogeneous equations, in general, the following result is obtained,

$$[A]\{X\} = \{\varepsilon\} \neq \{0\}$$

The ERROR SUM is the summation of the epsilon vector terms, that is, $\varepsilon(1) + \varepsilon(2) + \ldots\ldots\ldots\ldots\ldots$

13.2 Subroutines called by MODE.

This subroutine calls subroutine VARDET.

13.3 Subroutine listing.

The listing of subroutine MODE is given in table 34 .

13.4 Description of subroutine arguments.

X = eigenvector corresponding to [A] . If the eigenvector of a submatrix formulation of [A] is being computed, say order (N × N), then the result is stored in the first N locations of vector X. See VARDET description.

XLAM = eigenvalue of system $[A]\{X\}=\{0\}$

The remaining arguments are the same as subroutine VARDET.

```
      SUBROUTINE MODE(N,NR,NC,A,MXY,X,XLAM)
C     C.A.MERCER AND C.SEAVEY.  I.S.V.R.
C     EIGENVECTOR EVALUATION.
      DIMENSION A(MXY,MXY),X(MXY)
      CALL VARDET(N,NR,NC,DETA,A,MXY)
C     BACKSUBSTITUTION
      X(N)=1.0E0
      DO1 I=2,N
      M=N-I+1
      MA=M+1
      X(M)=0.0
      DO32 K=MA,N
   32 X(M)=X(M)+A(M,K)*X(K)
    1 X(M)=-X(M)/A(M,M)
C     NORMALIZATION
      BIG=X(1)
      DO 4 I=2,N
      IF(ABS(X(I))-ABS(BIG))4,4,5
    5 BIG=X(I)
    4 CONTINUE
      DO6 K=1,N
      X(K)=X(K)/BIG
    6 CONTINUE
C     ERROR ROUTINE
      ER=0.0E0
      DO18 I=1,N
      DO18 K=1,I
      DO18 J=K,N
      IF(K-I)21,22,2
   21 ER=ER+A(I,K)*A(K,J)*X(J)
      GO TO 2
   22 ER=ER+A(K,J)*X(J)
    2 CONTINUE
   18 CONTINUE
      WRITE(6,104) XLAM,ER,DETA
  104 FORMAT(1H ///12H EIGENVALUE=,E14.7///11H ERROR SUM=,E15.7,13H DETE
     1RMINANT=,E15.7///12H EIGENVECTOR,1H //)
      WRITE(6,101) (X(I),I=1,N)
  101 FORMAT(E15.7)
      RETURN
      END
```

A.S.A. Fortran listing of subroutine MODE.

Table 34 .

FORCE-Series of Subroutines.

This series of subroutines have much in common and therefore the general points will be discussed first before considering the individual subroutines.

The FORCE-subroutines were written to investigate particular vibration characteristics of specific types of structures using the rank force method. Since this work was part of a research project all computer programming was core limited, that is, all the analyses were carried out within the computer core storage. The types of structures considered are,

1. Collinear beam structures (FORCEB).
2. General plane frames (FORCEF).
3. Two dimensional rectangular plate structures (FORCEP).

A programme flow chart which covers all the FORCE-subroutines is shown in figure 51.

Matrix OM is formed initially as a null matrix of order MC×LM, it can be seen from the flow chart that this matrix is continually changing throughout the analysis. The flow chart also defines the various submatrices which are contained in matrix OM at the various stages. In the latter part of an analysis other matrices are stored in matrix OM by taking advantage of EQUIVALENCE (storage assignment statement), the use of this will be presented when discussing the individual FORCE-subroutines. Figure 52 shows the integer parameters used in the FORCE-subroutines to define the order of matrix OM and its submatrices. These parameters will now be defined;

N2 = number of equilibrium equations.

M1 = number of generalized element boundary loads, that is, (number of generalized element boundary loads per element) × (number of elements).

N1 = (number of generalized applied loads allowed at each node) × (number of nodes), no structural reactions are considered in this parameter.

NC = number of structural reactions.

NL = number of possible applied loads, N1-NC.

MC = total number of unknowns, element loads and structural reactions, M1 + NC.

N = degree of redundancy, MC-N2.

LM = (total number of unknowns) + (number of possible applied loads) = MC + NL = M1 + N1.

The subroutines called by the series of FORCE-subroutines is shown in figure 53.

Flow chart for FORCE-series of subroutines.

Fig. 51 .

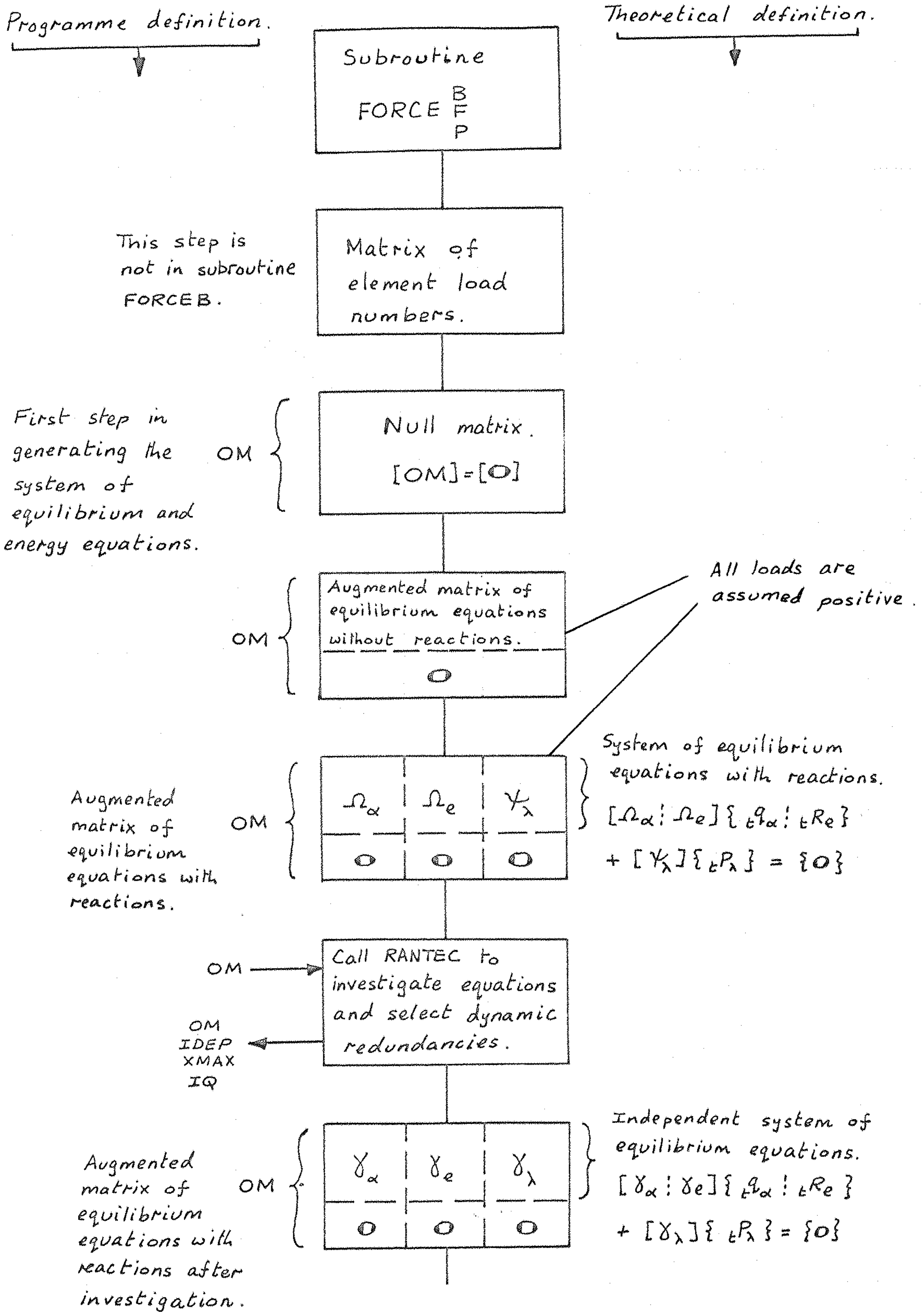


Fig. 51 continued.

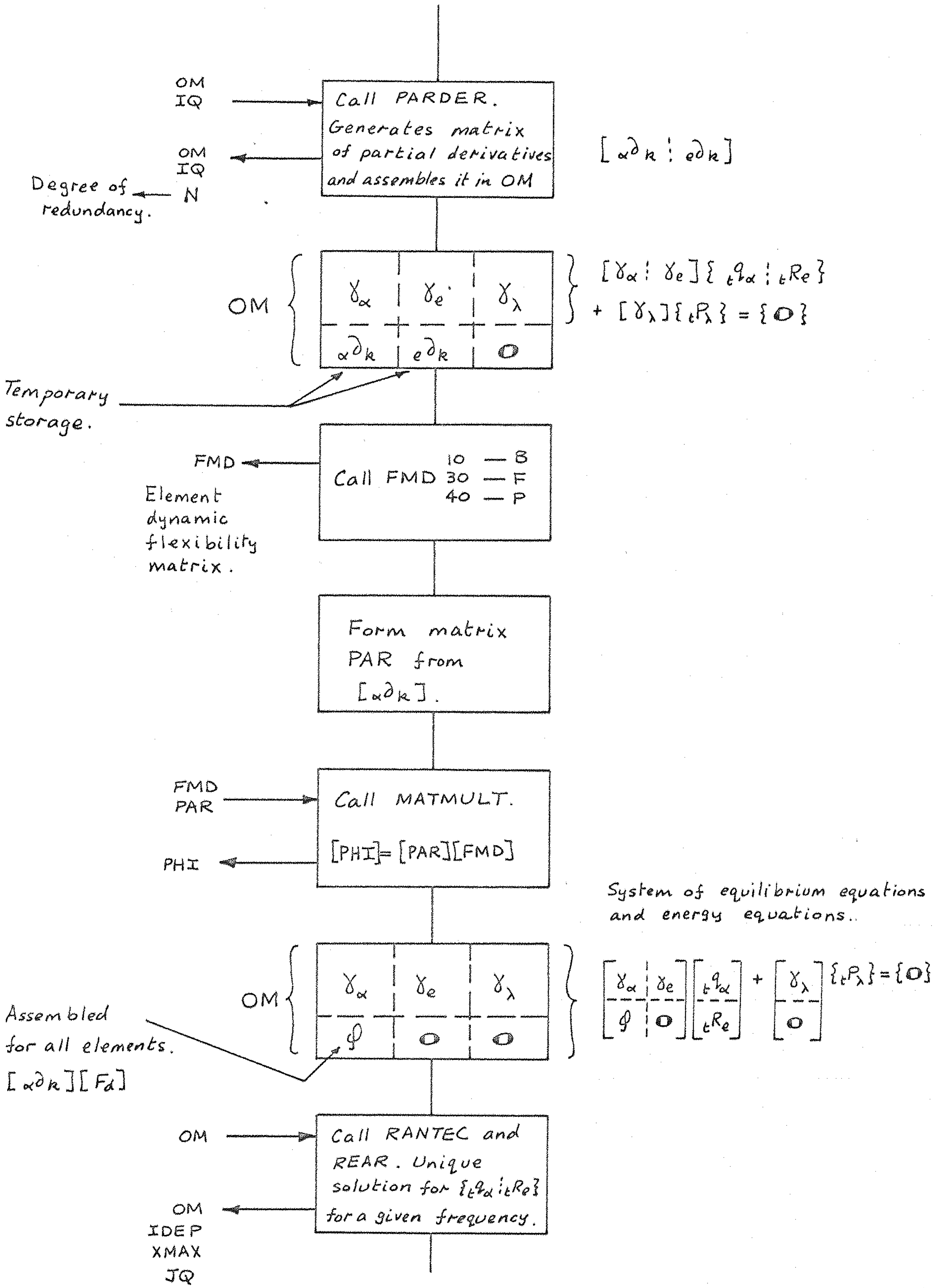


Fig. 51 continued.

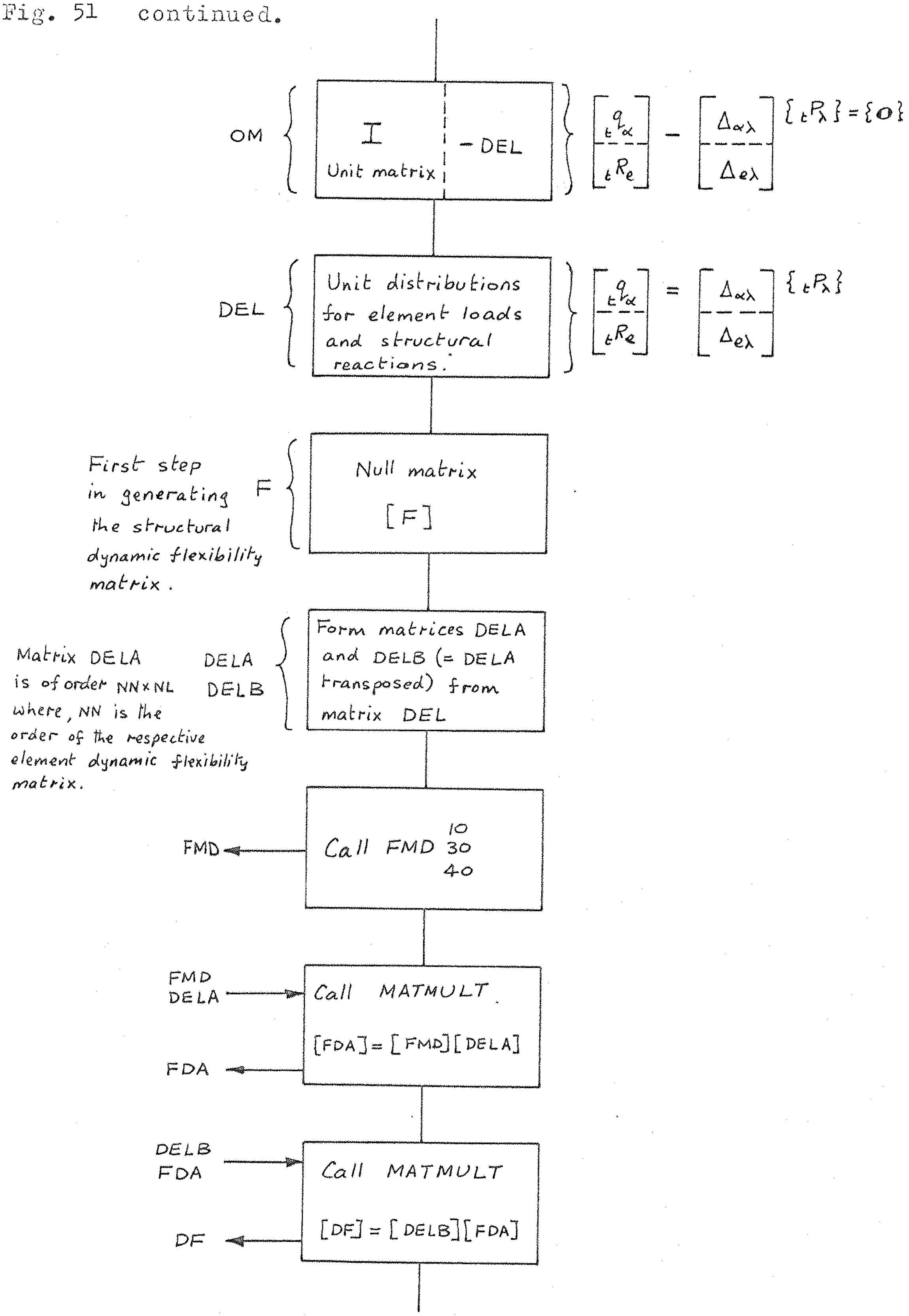
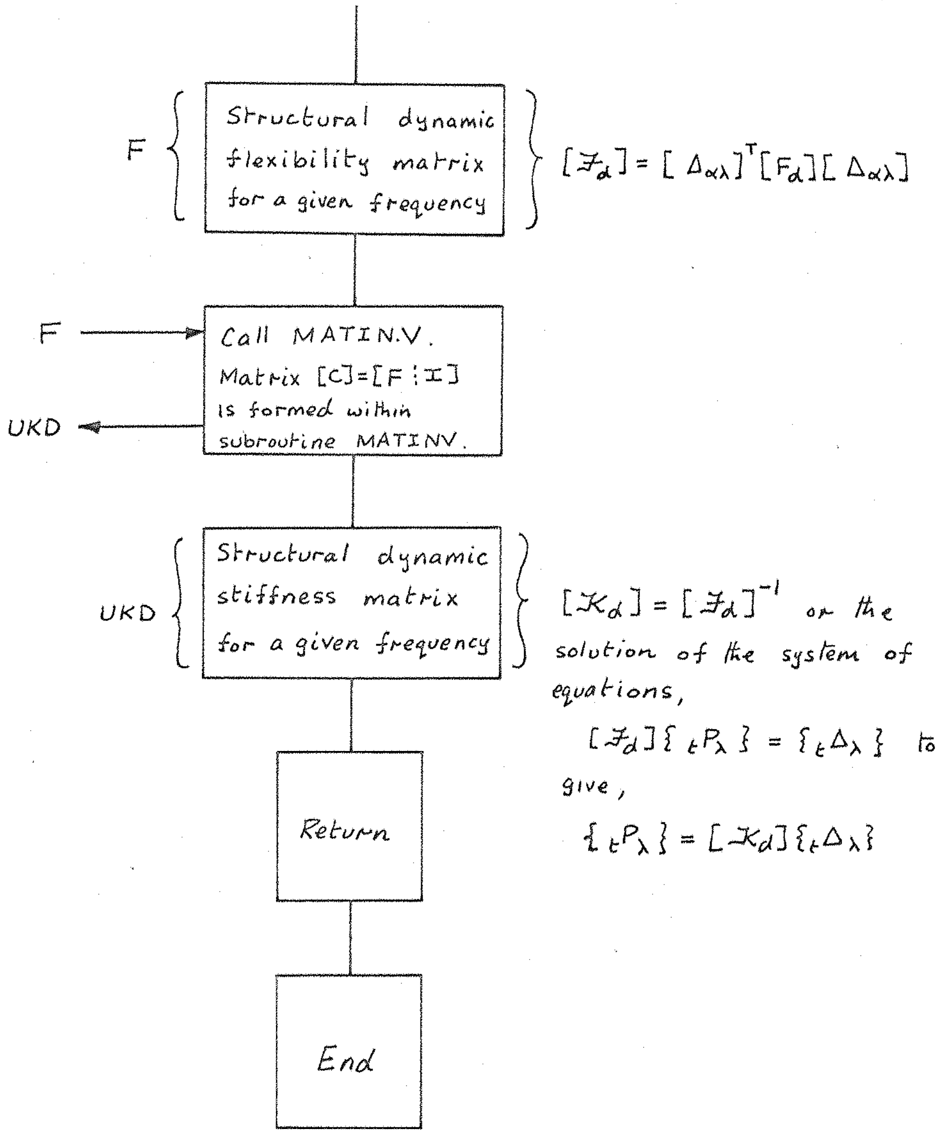


Fig.51 continued.

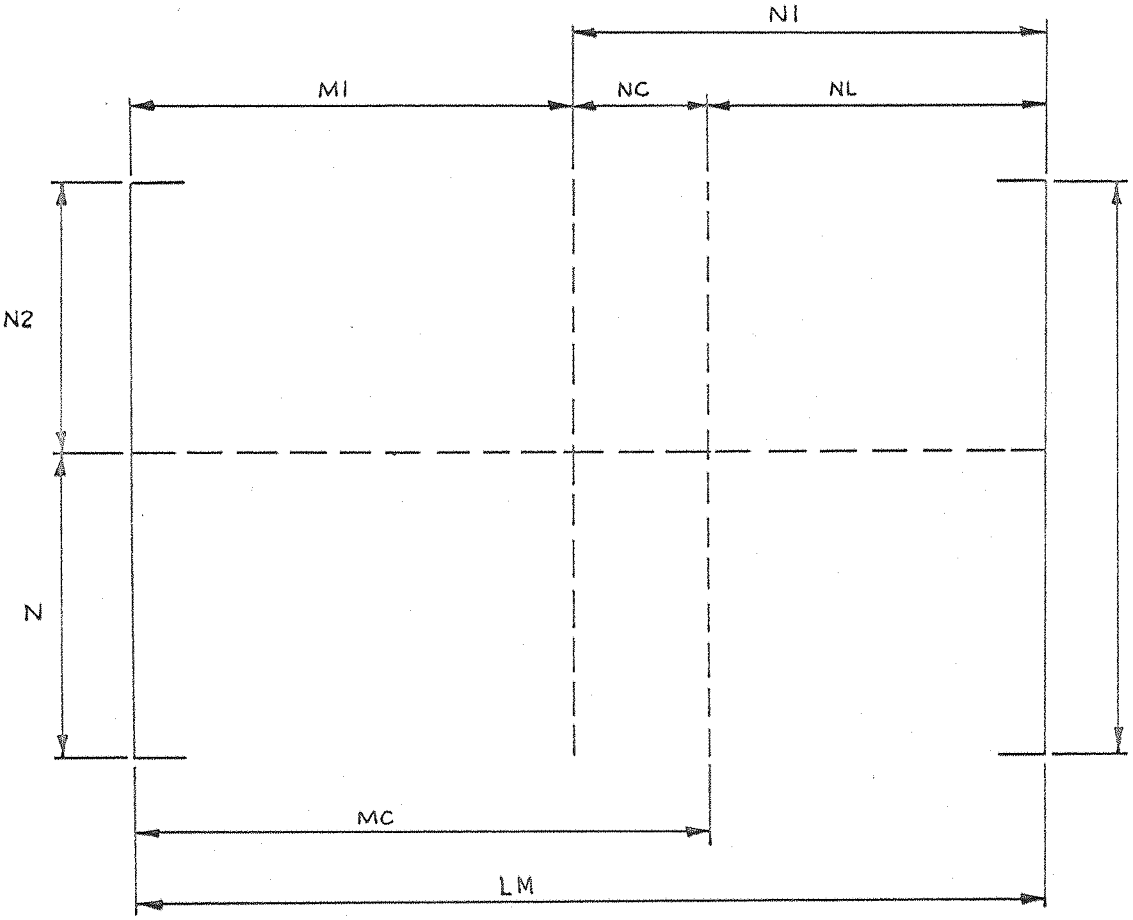


$[\mathcal{F}_d] = [\Delta_{\alpha\lambda}]^T[F_d][\Delta_{\alpha\lambda}]$

$[\mathcal{K}_d] = [\mathcal{F}_d]^{-1}$ or the solution of the system of equations,

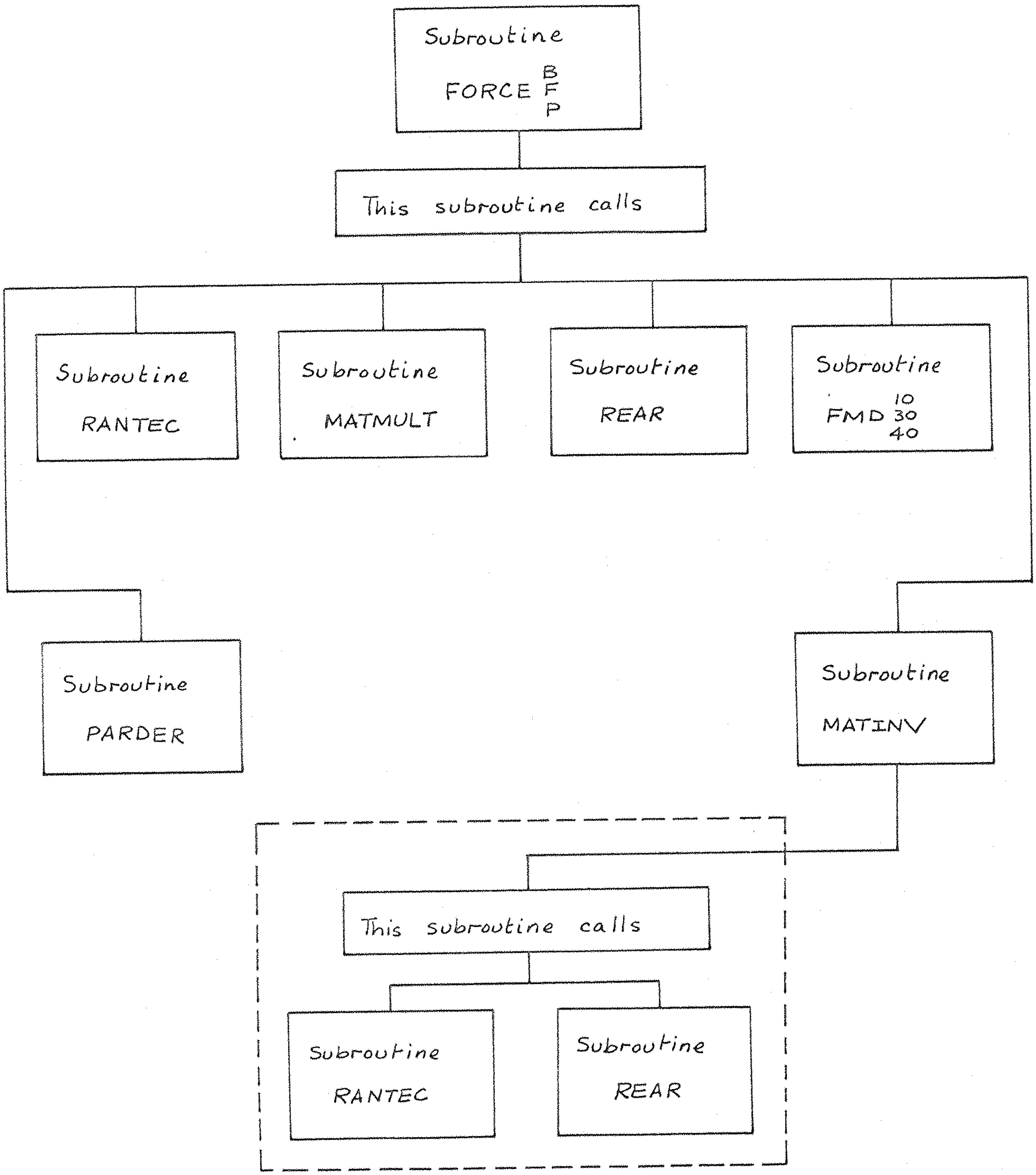
$[\mathcal{F}_d]\{{}_tP_\lambda\} = \{{}_t\Delta_\lambda\}$ to give,

$\{{}_tP_\lambda\} = [\mathcal{K}_d]\{{}_t\Delta_\lambda\}$



Integer parameters used for matrix OM and its submatrices.

Fig. 52 .



Subroutines called by FORCE-series of subroutines.

Fig.53 .

14. Subroutine FORCEB.

14.1 Description of subroutine.

This subroutine is used for the vibration analysis of collinear beam structures.

The subroutine generates the transformation matrix which relates the generalized applied loads to the generalized element boundary loads and structural reactions for a given frequency. The subroutine then generates the structural dynamic flexibility and stiffness matrices, again for a given frequency. The latter matrix is required for the eventual eigenvalue problem which is solved in the master programme FORCE-BEAM which is described later. A theoretical description for this subroutine is given in chapter 2. A flow chart for the programming formulation is given in the general discussion for the FORCE-series of subroutines, see figure 51 .

In this subroutine,

N2 = N1

M1 = 4×NE, four generalized element boundary loads per element.

N1 = 2×(NE+1)

LM = (4×NE)+(2×(NE+1)) = (6×NE) + 2

The following limitations have also been imposed,

NE = 20

NC = 4

Figure 54 (a) shows the applied load system which is assumed acting on a collinear beam structure. The initial formulation of the joint equilibrium equations considers no reactions, this formulation is then general for all collinear beam structures. The specific reaction system is then considered and the applied load system amended accordingly, see figure 54(b). Figure 55 shows the equivalencing carried out in this subroutine and in the master programme which calls it (FORCE-BEAM, described later), also in this figure are shown the maximum order of the respective matrices. It can be seen that certain matrices use the same storage but one should consider hierarchy of matrices as decided by the order of appearance but keeping in mind the period over which a matrix is detained in the programme. To accommodate for the equivalencing of matrices PAR and PHI a temporary transfer of part of matrix OM is carried out. See figure 56 (a) and (b). Equivalencing is carried out on maximum matrix dimensions and therefore it can be seen that this temporary transfer is not necessary for all collinear beam structures that can be analysed by this subroutine. The complete transfer is accomplished using the DO-loops with statement numbers 29 and 46, see subroutine listing.

14.2 Subroutines called by FORCEB.

This subroutine calls subroutines RANTEC, PARDER, MATMULT, REAR, FMD10 and MATINV. See figure 53. When calling subroutine FMD10 the appropiate subroutine should be used, that is;

1. Element Type P1/2FD.
2. Element Type P2/2FD.
3. Element Type P3/2FD.

Subroutine FORCEB has been written assuming that all elements in the structure are of the same type.

14.3 Subroutine listing.

The listing of subroutine FORCEB is given in table 35.

14.4 Description of subroutine arguments.

The first card of this subroutine is,

```
SUBROUTINE FORCEB(OMEGA,NE,IS,JS,XL,XI,E,XMU,CSA,NC,NL,
                  UKD,DEL,F,MC,IQ)
```

where,

OMEGA = angular frequency(radians per second).

NE = number of structural elements(finite elements).

IS = vector....first specifying node for each element.

JS = vector....second specifying node for each element.

XL = vector....length for each element(in).

XI = vector....second moment of area for each element(in^4).

E = vector....Young's modulus for each element (lb per in^2).

XMU = vector....material density for each element (lb per in^3).

CSA = vector....cross sectional area for each element(in^2).

NC = number of structural reactions.

IC = vector of applied load numbers which are to be considered as structural reactions. The applied load numbers are those taken from figure 54(a) using figure 54 (b) to give the corresponding reactions. The applied load numbering shown in figure 54(b) is the amended system. This is carried out within the subroutine.

NL = order of the structural dynamic flexibility and stiffness matrices(constrained structure).

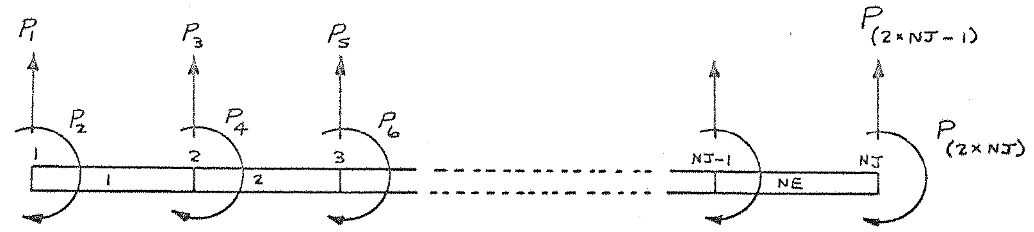
UKD = structural dynamic stiffness matrix(constrained) for a given frequency.

DEL = transformation matrix which relates the generalized applied loads to the generalized element boundary loads and structural reactions for a given frequency.

F = structural dynamic flexibility matrix for a given frequency.

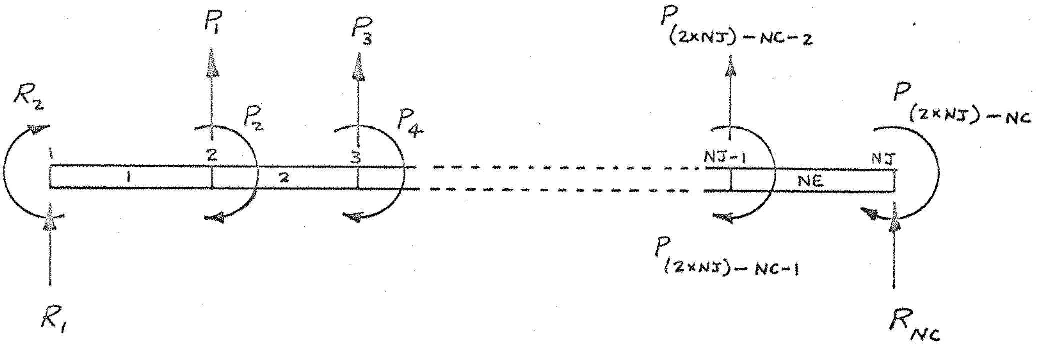
MC = total number of unknowns, that is, generalized element boundary loads and structural reactions.

IQ = vector of automatically selected redundancies. See chapter 1,part 3, that is 1.3.



Initial applied loading system without reactions, note NJ=NE+1.

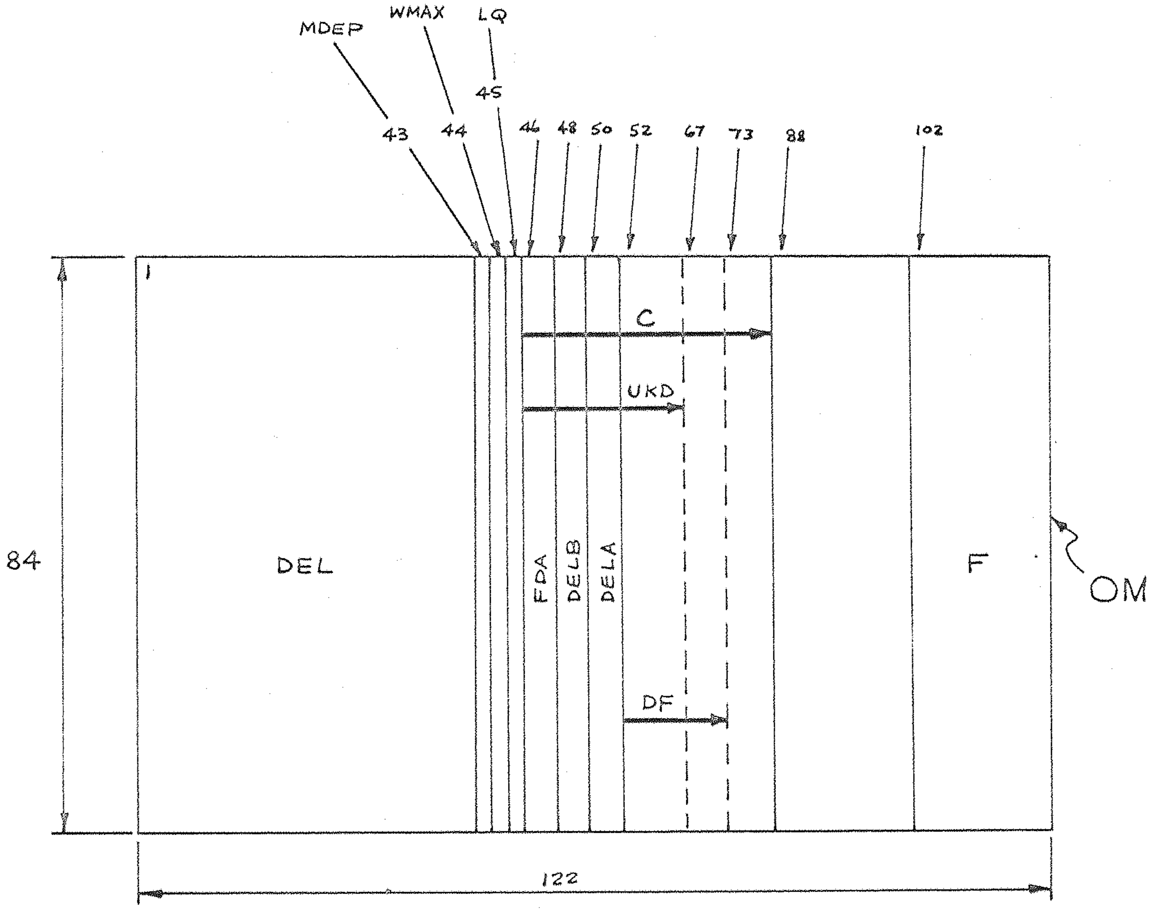
(a)



Possible applied load system when considering reactions.

(b)

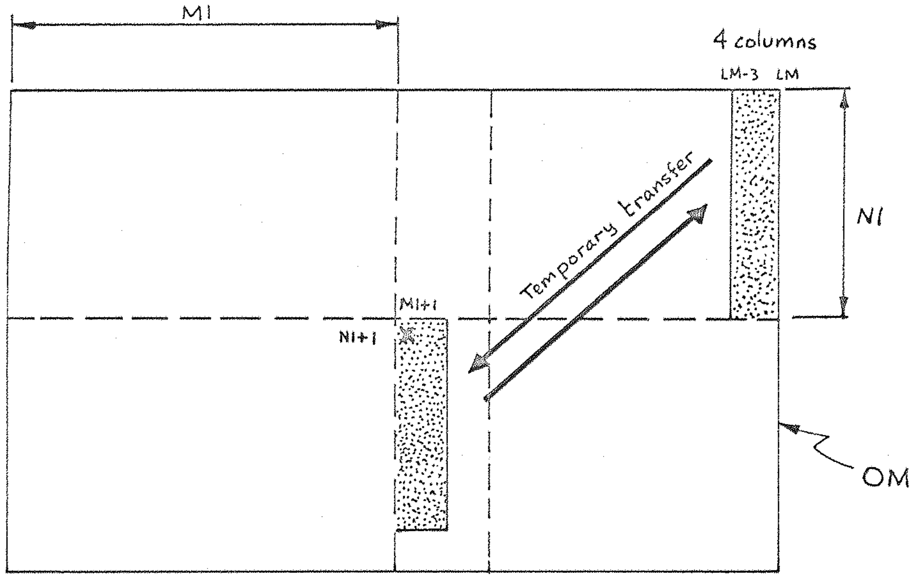
Fig.54 .



MATRIX	OM	DEL	F	DELA	DELB	FDA	DF	UKD	C	MDEP	WMAX	LQ
MAXIMUM ORDER	84×122	84×42	42×42	4×42	42×4	4×42	42×42	42×42	42×84	42×1	42×1	42×1

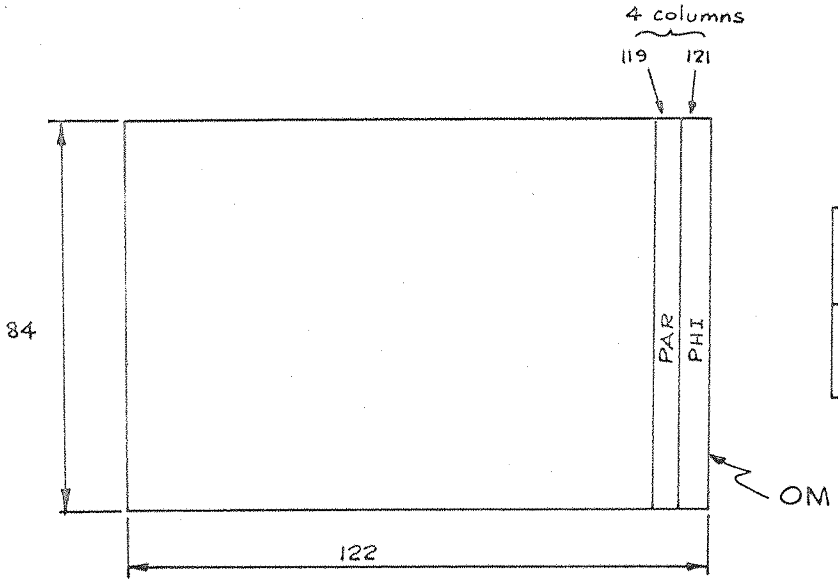
Matrices equivalenced in matrix OM and the maximum order of the respective matrices.

Fig. 55 .



Temporary transfer before equivalencing matrices PAR and PHI.

(a)



MATRIX	PAR	PHI
MAXIMUM ORDER	42×4	42×4

Matrices equivalenced in matrix OM and the maximum order of the respective matrices.

(b)

Fig. 56.

A.S.A. Fortran listing of subroutine FORCEB.

Table 35 .

```
      SUBROUTINE FORCEB(OMEGA,NE,IS,JS,XL,XI,E,XMU,CSA,NC,IC,NL,UKD,DEL,
     1F,MC,IQ)
C     JOHN ROBINSON.  I.S.V.R.
C     THE RANK FORCE METHOD FOR COLLINEAR BEAM STRUCUTRES,
C     VIBRATION ANALYSIS.
      DIMENSION IS(20),JS(20),XL(20),XI(20),E(20),XMU(20),CSA(20)
      DIMENSION OM(84,122),IDEP(84),XMAX(84),IQ(84),FMD(4,4)
      DIMENSION PAR(42,4),PHI(42,4),DEL(84,42),F(42,42),DELA(4,42)
      DIMENSION DELB(42,4),FDA(4,42),DF(42,42),C(42,84)
      DIMENSION MDEP(42),WMAX(42),LQ(42),UKD(42,42),IC(4),JQ(84)
      COMMON OM
      EQUIVALENCE(MDEP(1),OM(1,43)),(WMAX(1),OM(1,44)),(LQ(1),OM(1,45)),
     1(FDA(1,1),OM(1,46)),(DELB(1,1),OM(1,48)),(DELA(1,1),OM(1,50)),
     2(DF(1,1),OM(1,52)),(C(1,1),OM(1,46)),(PAR(1,1),OM(1,119)),
     3(PHI(1,1),OM(1,121))
C     JOINT EQUILIBRIUM EQUATIONS,
      LM=6*NE+2
      M1=4*NE
      MC=M1+NC
C     INITIAL NULL OM(I,J) MATRIX
      DO10 J=1,LM
      DO10 I=1,MC
   10 OM(I,J)=0.0
C     OMEGA ALPHA JOINT
      OM(1,1)=1.0
      OM(2,2)=1.0
      OM(2*NE+1,4*NE-1)=1.0
      OM(2*NE+2,4*NE)=1.0
C     NO FURTHER OMEGA ALPHA JOINT EQUATIONS FOR ONE ELEMENT
      IF(NE-1)20,20,11
C     ADDITIONAL OMEGA ALPHA JOINT EQUATIONS FOR MORE THAN ONE ELEMENT
   11 N2=4*(NE-1)
      N3=3
      N4=4
      DO16 K=3,N2,4
      M2=K
      DO14 I=N3,N4
      M3=M2+2
      DO12 J=M2,M3,2
   12 OM(I,J)=1.0
   14 M2=M2+1
      N3=N3+2
   16 N4=N4+2
   20 N1=2*(NE+1)
      NL=N1-NC
C     REACTIONS AND APPLIED LOADS
      IF(NC.EQ.0)GO TO 4
      NNJ=1
      DO6 N=1,NC
    6 OM(IC(N),M1+N)=-1.0
```

Subroutine FORCEB listing continued.

```
      DO7 I=1,N1
      DO9 N=1,NC
      IF(I.EQ.IC(N))GO TO 7
    9 CONTINUE
      OM(I,MC+NNJ)=-1.0
      NNJ=NNJ+1
    7 CONTINUE
      GO TO 5
    4 DO21 I=1,N1
      M5=MC+I
      DO21 J=M5,M5
   21 OM(I,J)=-1.0
    5 CONTINUE
C     SOLUTION OF EQUATIONS
      CALL RANTEC(OM,N1,MC,LM,84,84,122,IDEP,XMAX,IQ)
C     CHECK FOR DYNAMIC REDUNDANCY
      DO22 M=1,MC
      IF(IQ(M).EQ.0)GO TO 22
      GO TO 24
   22 CONTINUE
      GO TO 600
   24 CALL PARDER(OM,IQ,N1,MC,84,122,N)
      KP=4-NC-NL
      DO29 I=1,N1
      DO29 J=LM-3,LM
   29 OM(N1+I,J+KP)=OM(I,J)
C     ENERGY EQUATIONS
      JJ=0
      DO33 M=1,NE
      CALL FMD10(XL(M),XI(M),E(M),XMU(M),CSA(M),OMEGA,FMD)
   27 DO28 I=1,4
      DO28 J=1,4
   28 FMD(I,J)=10000.0*FMD(I,J)
      DO30 I=1,N
      DO30 J=1,4
      JK=J+JJ
   30 PAR(I,J)=OM(N1+I,JK)
      CALL MATMULT(PAR,FMD,PHI,N,4,4,42,4,4)
      DO32 I=1,N
      DO32 J=1,4
      JK=J+JJ
   32 OM(I+N1,JK)=PHI(I,J)
      JJ=JJ+4
   33 CONTINUE
```

Subroutine FORCEB listing continued.

```
      DO46 I=1,N1
      DO46 J=LM-3,LM
   46 OM(I,J)=OM(N1+I,J+KP)
      DO48 I=N1+1,MC
      DO48 J=M1+1,LM
   48 OM(I,J)=0.0
      CALL RANTEC(OM,N1+N,MC,LM,84,84,122,IDEP,XMAX,JQ)
  600 CONTINUE
      CALL REAR(OM,MC,LM,84,122,XCH)
C     DEL MATRIX
      DO34 I=1,MC
      DO34 J=MC+1,LM
      L=J-MC
   34 DEL(I,L)=-OM(I,J)
      DO23 I=1,NL
      DO23 J=1,NL
   23 F(I,J)=0.0
      II=0
      DO44 M=1,NE
      CALL FMD10(XL(M),XI(M),E(M),XMU(M),CSA(M),OMEGA,FMD)
   37 DO38 I=1,4
      DO38 J=1,NL
      IK=I+II
   38 DELA(I,J)=DEL(IK,J)
      DO40 I=1,4
      DO40 J=1,NL
   40 DELB(J,I)=DELA(I,J)
      CALL MATMULT(FMD,DELA,FDA,4,NL,4,4,42,4)
      CALL MATMULT(DELB,FDA,DF,NL,NL,4,42,42,4)
      DO42 I=1,NL
      DO42 J=1,NL
   42 F(I,J)=F(I,J)+DF(I,J)
      II=II+4
   44 CONTINUE
C     STRUCTURAL DYNAMIC STIFFNESS MATRIX
      CALL MATINV(F,UKD,C,NL,42,42,84,MDEP,WMAX,LQ)
      GO TO 400
    2 STOP
  400 RETURN
      END
```

15. Subroutine FORCEF.

15.1 Description of subroutine.

This subroutine is used for the vibration analysis of general plane frames. The formulation assumes rigid joints but the programme can readily be amended to allow various joint conditions. Example, in the case of a pinned joint connection the column in the $[\Omega_\alpha]$ matrix corresponding to the respective element load (moment) is nulled. See figure 57 . It should be noted however that there is no restriction on the structural reaction system. Typical systems are shown in figure 58 .

The first step in this subroutine is to generate a matrix of element load numbers. The first row of this matrix gives the element boundary load numbers assigned to the first element, the second row are for the second element, and so on. The elements are not numbered in an actual idealization but are accepted in the order in which the element input data are assembled (to be described later in the master programme FORCE-PLANE FRAME). The element load number matrix is of order (NE × 6). The element specifying nodes are denoted by i and j. The first three coefficients in any row of the element load number matrix give the generalized element boundary load numbers at node i for the respective element. The remaining three coefficients in a row are for node j. The load numbers in a row correspond to,

1. load in the $\bar{x}$-direction.
2. load in the $\bar{z}$-direction.
3. moment.

node i

4. load in the $\bar{x}$-direction.
5. load in the $\bar{z}$-direction.
6. moment.

node j

The loads are taken in this order.

This subroutine also generates,

1. equilibrium and energy equations.
2. matrix $[\mathcal{F}_d]$, structural dynamic flexibility matrix.
3. matrix $[\mathcal{K}_d]$, structural dynamic stiffness matrix.
4. matrix $\left[\begin{array}{c}\Delta_{\alpha\lambda}\\ \hline \Delta_{e\lambda}\end{array}\right]$, transformation matrix which relates the unknowns and knowns.

For a given frequency.

See description of subroutine FORCEB.

In subroutine FORCEF,

N2 = N1

M1 = 6 × NE, six generalized element boundary loads per element.

N1 = 3 × NJ (NJ = number of nodes).

LM = (6 × NE) + (3 × NJ).

The following limitations have been imposed,

NE = 13

NJ = 11

NC = 9

Figure 59 shows a plane frame consisting of three elements. Figure 60(a) shows the applied load system which is assumed acting on the frame, the initial formulation of the joint equilibrium equations considers no reactions. The specific reaction system is then imposed, see figure 60(b).

Figure 61 shows the equivalencing carried out in this subroutine and in the master programme which calls it (FORCE-PLANE FRAME, described later), also in this figure are shown the maximum order of the respective matrices. To accommodate for the equivalencing of matrices PAR and PHI a temporary transfer of part of matrix OM is carried out. See figure 62(a) and (b). The complete transfer is accomplished using the DO-loops with statement numbers 29 and 46, see subroutine listing.

15.2 Subroutines called by FORCEF.

This subroutine calls subroutines RANTEC, PARDER, MATMULT, REAR, FMD30 and MATINV. See figure 53 . When calling subroutine FMD30 the appropiate subroutine should be used, that is;

1. Element Type P1/3FD.
2. Element Type P3/3FD.

Subroutine FORCEF has been written assuming that all elements in the structure are of the same type.

15.3 Subroutine listing.

The listing of subroutine FORCEF is given in table 36 .

15.4 Description of subroutine arguments.

The first card of this subroutine is,

```
SUBROUTINE FORCEF(OMEGA,NE,NJ,XB,ZB,IS,JS,XI,E,XMU,CSA,
                NC,IC,NL,UKD,DEL,F,MC,IQ)
```

where,

OMEGA = angular frequency (radians per second).

NE = number of structural elements(finite elements).

NJ = number of nodes.

XB = vector......$\bar{x}$-ordinate for each node(in).

ZB = vector......$\bar{z}$-ordinate for each node(in).

IS = vector......first specifying node for each element.

JS = vector......second specifying node for each element.

XI = vector......second moment of area for each element(in^4).

E = vector......Young's modulus for each element(lb per in^2).

XMU = vector......material density for each element(lb per in^3).

CSA = vector......cross sectional area for each element(in^2).

NC = number of structural reactions.

IC = vector of applied load numbers which are to be considered as structural reactions. The applied load numbers are those taken from figure 60 (a) using figure 60 (b) to give the corresponding reactions. The applied load numbering shown in figure 60 (b) is the amended system. This is carried out within the subroutine.

NL = order of the structural dynamic flexibility and stiffness matrices(constrained structure).

UKD = structural dynamic stiffness matrix (constrained) for a given frequency.

DEL = transformation matrix which relates the generalized applied loads to the generalized element boundary loads and structural reactions for a given frequency.

F = structural dynamic flexibility matrix for a given frequency.

MC = total number of unknowns, that is, generalized element boundary loads and structural reactions.

IQ = vector of automatically selected redundancies.

A.S.A. Fortran listing of subroutine FORCEF.

Table 36 .

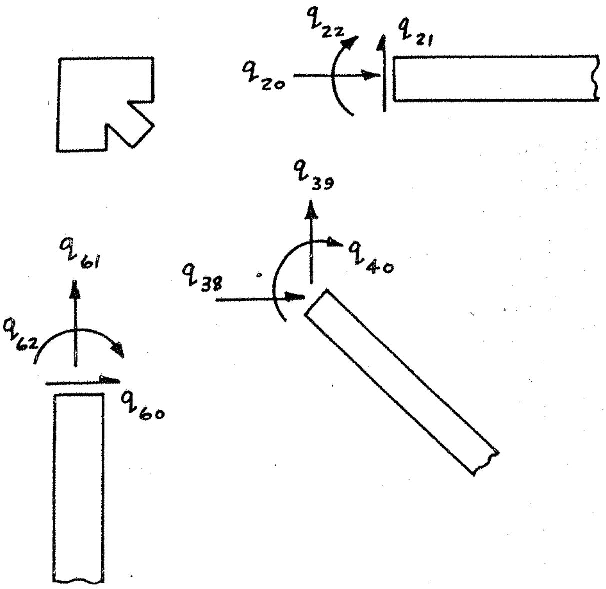
```
      SUBROUTINE FORCEF(OMEGA,NE,NJ,XB,ZB,IS,JS,XI,E,XMU,CSA,NC,IC,NL,UK
     1D,DEL,F,MC,IQ)
C     JOHN ROBINSON.  I.S.V.R.
C     THE RANK FORCE METHOD FOR GENERAL PLANE FRAMES. VIBRATION
C     ANALYSIS. ALL JOINTS HAVE BEEN ASSUMED RIGID.
      DIMENSION NOQ(13,6),OM(87,111),IDEP(87),XMAX(87),IQ(87),JQ(87)
     1 ,IS(13),JS(13),XI(13),E(13),XMU(13),CSA(13),FMD(6,6),IC(9),MDEP(3
     23),WMAX(33),LQ(33),DF(33,33),PAR(54,6),PHI(54,6),DEL(87,33),F(33,3
     33),C(33,66),UKD(33,33),DELA(6,33),DELB(33,6),FDA(6,33),XB(11),
     4ZB(11)
      COMMON OM
      EQUIVALENCE (MDEP(1),OM(1,34)),(WMAX(1),OM(1,35)),(LQ(1),OM(1,36))
     1,(FDA(1,1),OM(1,37)),(DELB(1,1),OM(1,40)),(DELA(1,1),OM(1,43)),
     2(DF(1,1),OM(1,46)),(C(1,1),OM(1,46)),(PAR(1,1),OM(1,104)),(PHI(1,1
     3),OM(1,108))
C     JOINT EQUILIBRIUM EQUATIONS FOR GENERAL PLANE FRAMES (VIB. ANALYS)
C     MATRIX OF ELEMENT LOAD NUMBERS
      JJJ=0
      DO12 I=1,NE
      DO11 J=1,6
   11 NOQ(I,J)=JJJ+J
   12 JJJ=JJJ+6
C     JOINT EQUILIBRIUM EQUATIONS
      N1=3*NJ
      LM=6*NE+3*NJ
      M1=6*NE
      MC=M1+NC
      NL=N1-NC
C     INITIAL NULL MATRIX
      DO16 J=1,LM
      DO16 I=1,MC
   16 OM(I,J)=0.0
      LLL=1
      DO28 III=1,NJ
      DO27 M=1,NE
      DO27 J=1,3
      IF(IS(M)-III)26,22,26
   22 OM (LLL+J-1,NOQ(M,J))=1.0
   26 IF(JS(M)-III)27,24,27
   24 OM (LLL+J-1,NOQ(M,J+3))=1.0
   27 CONTINUE
      LLL=LLL+3
   28 CONTINUE
C     REACTIONS AND APPLIED LOADS
      IF(NC.EQ.0)GO TO 4
      NNJ=1
      DO6 N=1,NC
    6 OM(IC(N),M1+N)=-1.0
```

Subroutine FORCEF listing continued.

```
      DO7 I=1,N1
      DO9 N=1,NC
      IF(I.EQ.IC(N))GO TO 7
    9 CONTINUE
      OM(I,MC+NNJ)=-1.0
      NNJ=NNJ+1
    7 CONTINUE
      GO TO 5
    4 DO18 I=1,N1
      M5=MC+I
      DO18 J=M5,M5
   18 OM(I,J)=-1.0
    5 CONTINUE
C     SOLUTION OF EQUATIONS
      CALL RANTEC(OM,N1,MC,LM,87,87,111,IDEP,XMAX,IQ)
C     CHECK FOR DYNAMIC REDUNDANCY
      DO63 M=1,MC
      IF(IQ(M).EQ.0)GO TO 63
      GO TO 65
   63 CONTINUE
      GO TO 600
   65 CALL PARDER(OM,IQ,N1,MC,87,111,N)
      KP=8-NC-NL
      DO29 I=1,N1
      DO29 J=LM-7,LM
   29 OM(N1+I,J+KP)=OM(I,J)
C     OM MATRIX INCLUDING ENERGY EQUATIONS
      JJ=0
      DO60 M=1,NE
      IN=IS(M)
      JN=JS(M)
      XBI=XB(IN)
      ZBI=ZB(IN)
      XBJ=XB(JN)
      ZBJ=ZB(JN)
      CALL FMD30(XBI,ZBI,XBJ,ZBJ,XI(M),E(M),XMU(M),CSA(M),OMEGA,FMD)
      DO53 I=1,6
      DO53 J=1,6
   53 FMD(I,J)=10000.0*FMD(I,J)
      DO58 I=1,N
      DO58 J=1,6
      JK=J+JJ
   58 PAR(I,J)=OM(N1+I,JK)
      CALL MATMULT(PAR,FMD,PHI,N,6,6,54,6,6)
      DO59 I=1,N
      DO59 J=1,6
      JK=J+JJ
   59 OM(I+N1,JK)=PHI(I,J)
      JJ=JJ+6
   60 CONTINUE
```

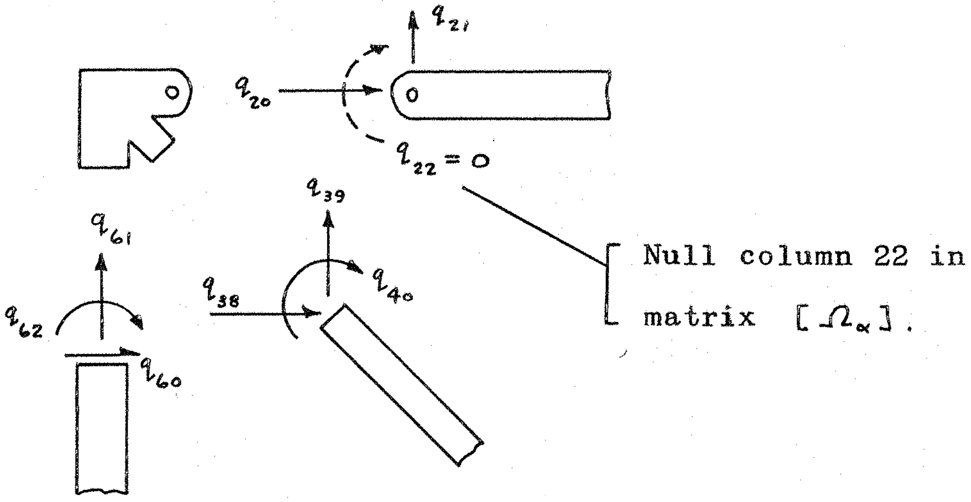
Subroutine FORCEF listing continued.

```
      DO46 I=1.N1
      DO46 J=LM-7.LM
   46 OM(I.J)=OM(N1+I.J+KP)
      DO48 I=N1+1.MC
      DO48 J=M1+1.LM
   48 OM(I.J)=0.0
      CALL RANTEC(OM.N1+N.MC.LM.87.87.111.IDEP.XMAX.JQ)
  600 CONTINUE
      CALL REAR(OM.MC.LM.87.111.XCH)
C     DEL MATRIX
      DO68 I=1.MC
      DO68 J=MC+1.LM
      L=J-MC
   68 DEL(I.L)=-OM(I.J)
      DO31 I=1.NL
      DO31 J=1.NL
   31 F(I.J)=0.0
      II=0
      DO44 M=1.NE
      IN=IS(M)
      JN=JS(M)
      XBI=XB(IN)
      ZBI=ZB(IN)
      XBJ=XB(JN)
      ZBJ=ZB(JN)
      CALL FMD30(XBI.ZBI.XBJ.ZBJ.XI(M).E(M).XMU(M).CSA(M).OMEGA.FMD)
      DO38 I=1.6
      DO38 J=1.NL
      IK=I+II
   38 DELA(I.J)=DEL(IK.J)
      DO40 I=1.6
      DO40 J=1.NL
   40 DELB(J.I)=DELA(I.J)
      CALL MATMULT(FMD.DELA.FDA.6.NL.6.6.33.6)
      CALL MATMULT(DELB.FDA.DF.NL.NL.6.33.33.6)
      DO42 I=1.NL
      DO42 J=1.NL
   42 F(I.J)=F(I.J)+DF(I.J)
      II=II+6
   44 CONTINUE
C     STRUCTURAL DYNAMIC STIFFNESS MATRIX
      CALL MATINV(F.UKD.C.NL.33.33.66.MDEP.WMAX.LQ)
      RETURN
      END
```



Rigid joint connections.

(a)

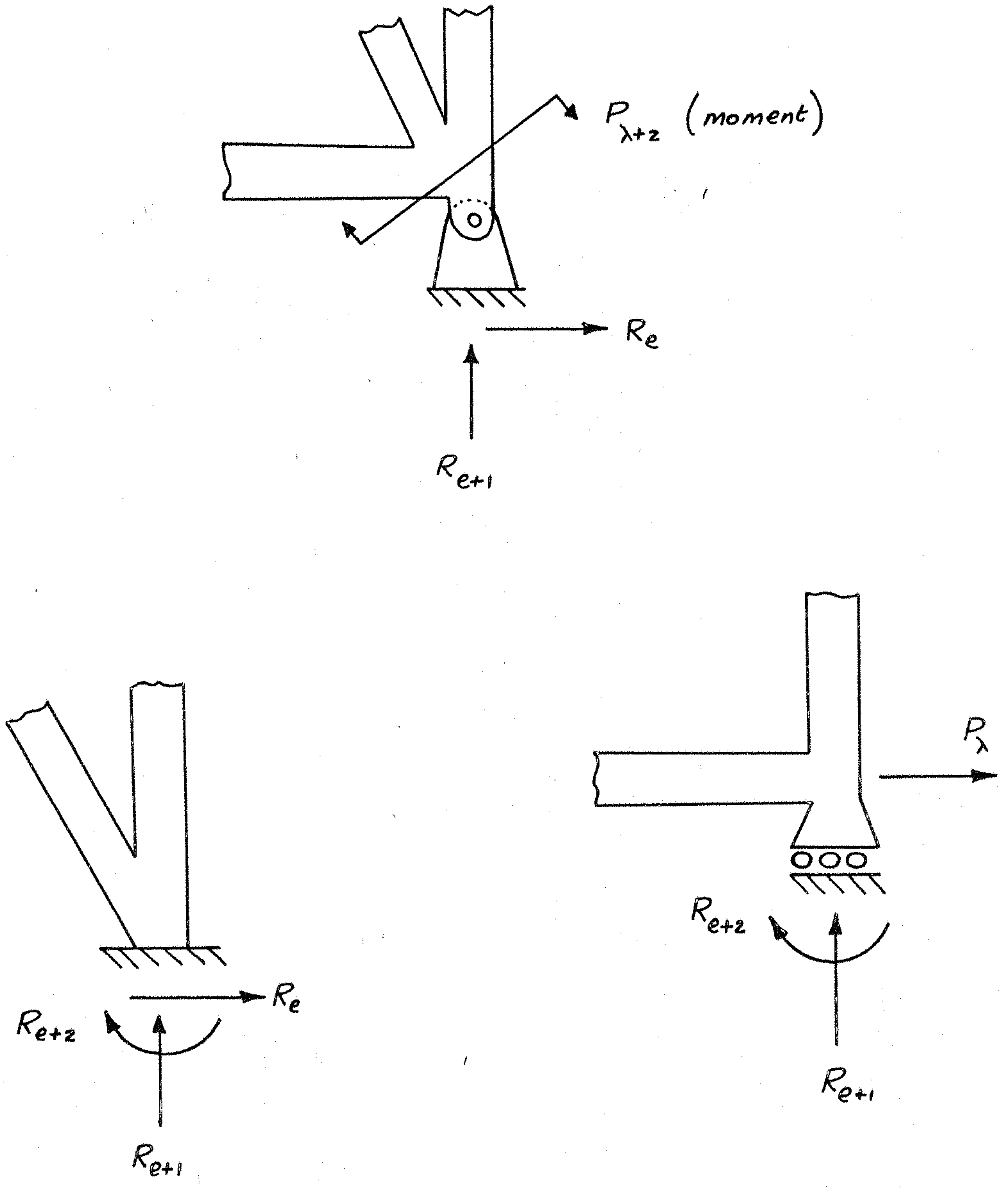


Null column 22 in matrix $[\Omega_\alpha]$.

Joint with pinned connection.

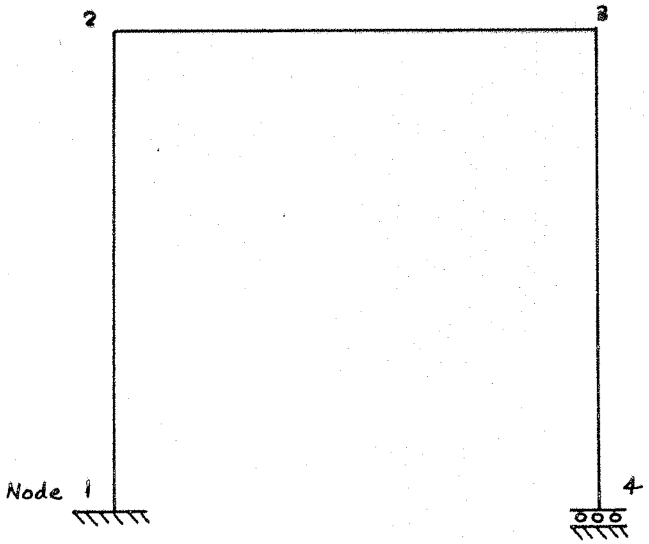
(b)

Fig. 57 .



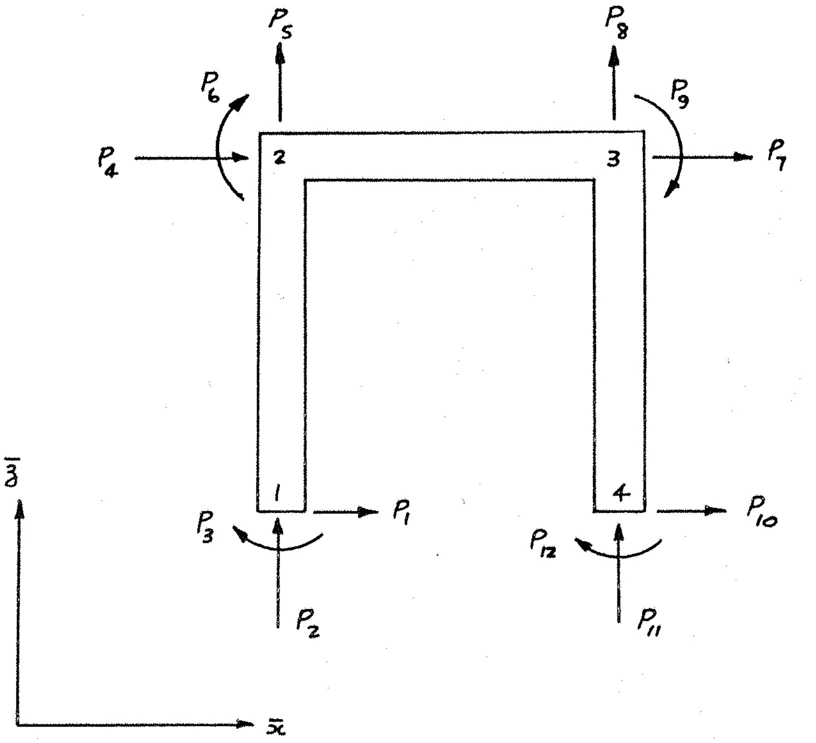
Typical structural reaction systems.

Fig. 58 .



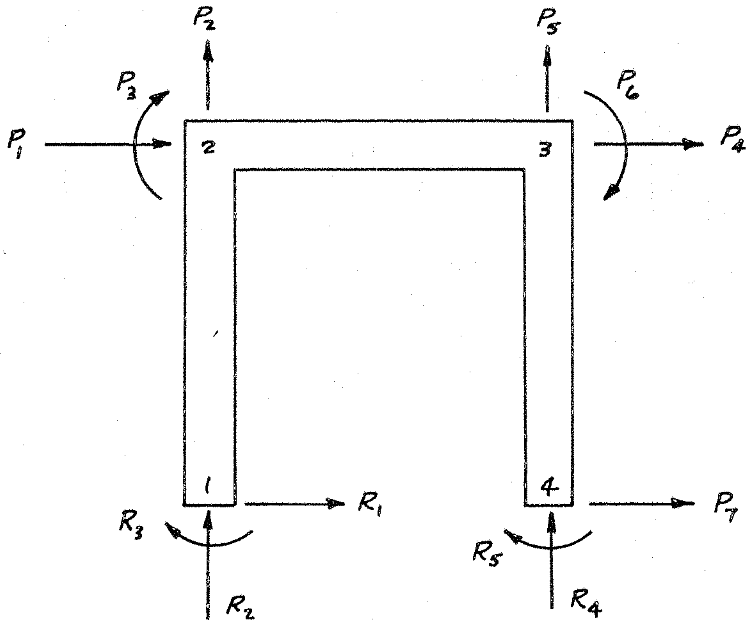
Simple plane frame, three elements.

Fig. 59 .



Initial applied loading system without reactions.

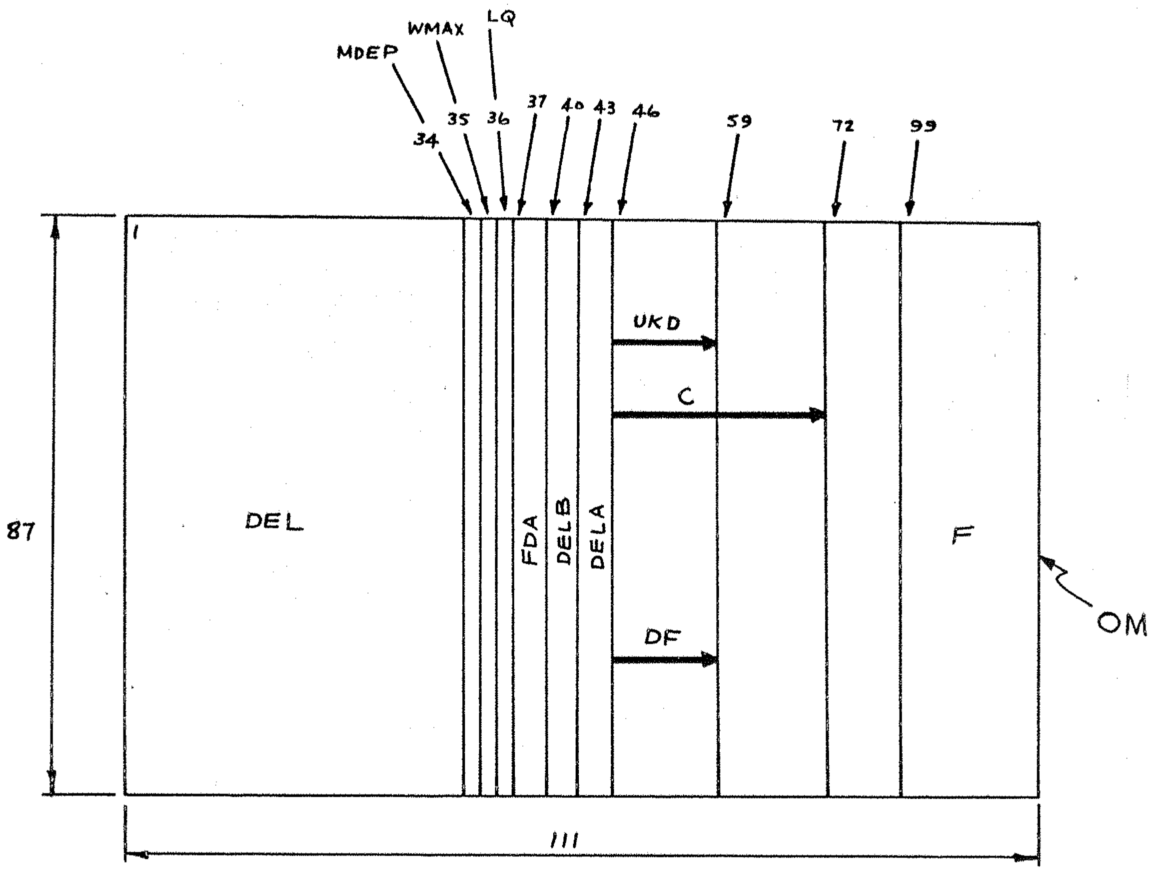
(a)



Possible applied load system when considering reactions.

(b)

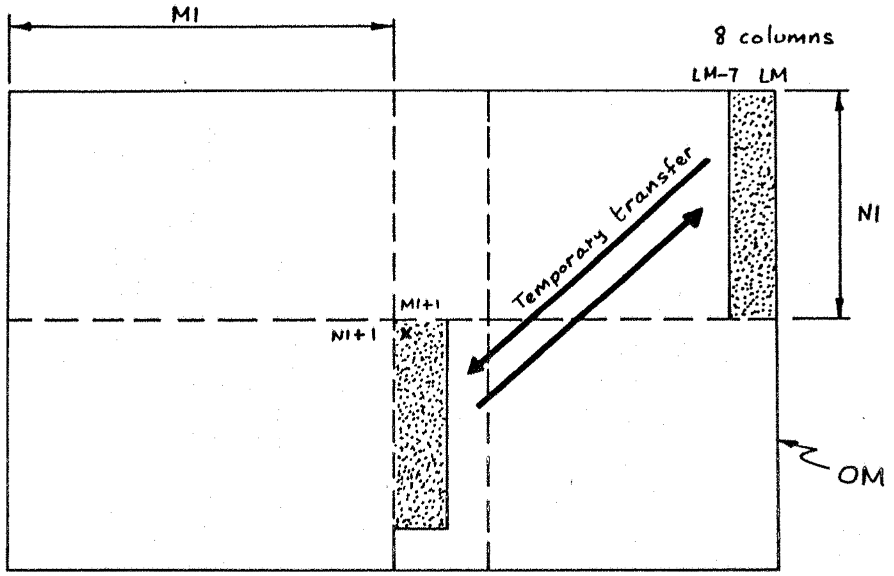
Fig. 60 .



MATRIX	OM	DEL	F	DELA	DELB	FDA	DF	UKD	C	MDEP	WMAX	LQ
MAXIMUM ORDER	87x111	87x33	33x33	6x33	33x6	6x33	33x33	33x33	33x66	33x1	33x1	33x1

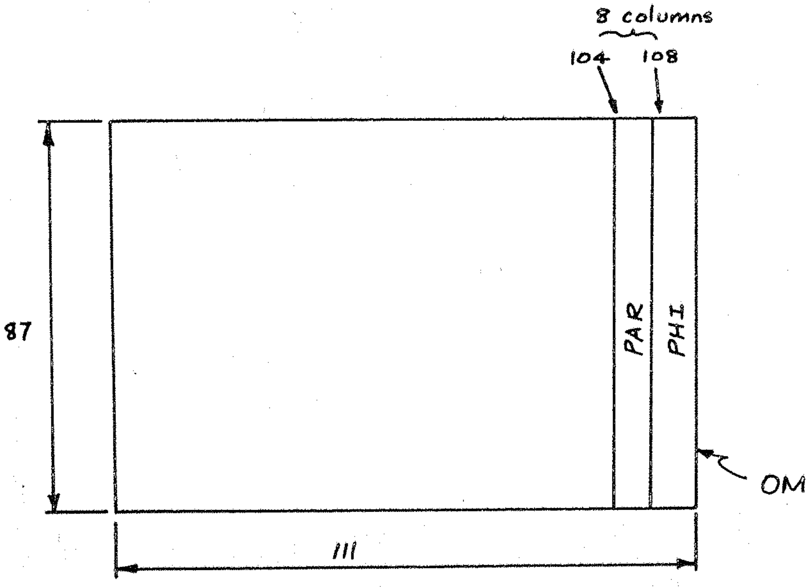
Matrices equivalenced in matrix OM and the maximum order of the respective matrices.

Fig. 61.



Temporary transfer before equivalencing matrices PAR and PHI.

(a)



MATRIX	PAR	PHI
MAXIMUM ORDER	54x6	54x6

Matrices equivalenced in matrix OM and the maximum order of the respective matrices.

(b)

Fig. 62.

16. Subroutine FORCEP.

See master programme

FORCE-RECTANGULAR PLATE.

17. Subroutine FMD50...............Element Type P1/5FD.

17.1 Description od subroutine.

This subroutine generates the dynamic flexibility matrix corresponding to an s-system of generalized element boundary loads for a rectangular plate element, Element Type P1/5FD, see Chapter 3(3.1.3(b)) and Chapter 4(4.1.5).

This type of element is used for the analysis of two dimensional plate structures.

17.2 Subroutines called by FMD50.

This subroutine calls subroutines RANTEC, PARDER, FMD40, MATMULT and REAR, see figure 63.

17.3 Subroutine listing.

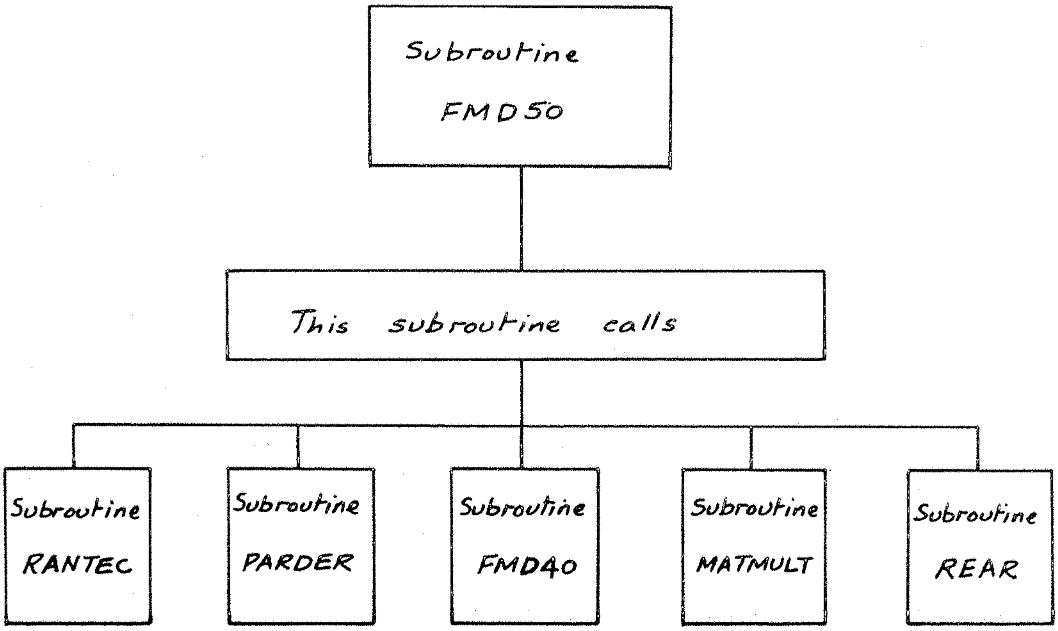
The listing of subroutine FMD50 is given in table 37 .

17.4 Description of subroutine arguments.

The first card of this subroutine is,

```
SUBROUTINE FMD50(A,B,T,EM,XMUM,XNUM,OMEGA,FMD)
```

where the argument definitions are the same as for subroutine FMD40 (A1.4.1(11.1)).



Subroutines called by subroutine FMD50.

Fig. 63 .

A.S.A. Fortran listing of subroutine FMD50, Element Type P1/5FD.

Table 37 .

```
      SUBROUTINE FMD50(A,B,T,EM,XMUM,XNUM,OMEGA,FMD)
C     JOHN ROBINSON  I.S.V.R.
C     RECTANGULAR PLATE ELEMENT
C     ELEMENT TYPE P1/5FD
C     ELEMENT DYNAMIC FLEXIBILITY MATRIX
      DIMENSION D(12,24),ID(12),XM(12),PAR(9,12),PHI(9,12),IR(12)
      DIMENSION DELQS(12,12),DELQST(12,12),FMD(12,12),G(12,12)
      EQUIVALENCE (DELQS(1,1),D(1,1)),(DELQST(1,1),D(1,13))
      DO5 I=1,12
      DO5 J=1,24
    5 D(I,J)=0.0
      D(1,5),D(1,7),D(2,1),D(2,3),D(3,10),D(3,11)=-A
      D(3,16),D(3,19)=A
      D(1,6),D(1,8),D(2,19),D(2,22),D(3,2),D(3,4)=-B
      D(2,11),D(2,12)=B
      D(2,7)=A*B
      D(3,6)=-A*B
      D(3,5),D(3,7)=-A**2/2.0
      D(2,6),D(2,8)=B**2/2.0
      D(1,9),D(1,10),D(1,11),D(1,12)=-1.0
      LL=0
      DO6 K=1,3
      DO8 L=13,24,3
    8 D(K,L+LL)=1.0
      LL=LL+1
    6 CONTINUE
      CALL RANTEC(D,3,12,24,12,12,24,ID,XM,IR)
      CALL PARDER(D,IR,3,12,12,24,NN)
      DO10 M=1,9
      DO10 N=1,12
   10 PAR(M,N)=D(3+M,N)
      CALL FMD40(A,B,T,EM,XMUM,XNUM,OMEGA,FMD)
      CALL MATMULT(PAR,FMD,PHI,9,12,12,9,12,12)
      DO12 M=1,9
      DO12 N=1,12
   12 D(3+M,N)=PHI(M,N)
      CALL RANTEC(D,12,12,24,12,12,24,ID,XM,IR)
      CALL REAR(D,12,24,12,24,XCH)
      DO14 I=1,12
      DO14 J=13,24
   14 DELQS(I,J-12)=-D(I,J)
      DO16 I=1,12
      DO16 J=1,12
   16 DELQST(I,J)=DELQS(J,I)
      CALL MATMULT(FMD,DELQS,G,12,12,12,12,12,12)
      CALL MATMULT(DELQST,G,FMD,12,12,12,12,12,12)
      RETURN
      END
```

A4.2 Functions.

1. Function KINT.

1.1 Description of function.

This function is used in the master programmes to be described later. It appears in the eigenvalue evaluation and was written by C. Seavey. This function uses function INT which takes the sign of a real number, say A, and multiplys it by the largest integer ≤ A. Function INT is an intrinsic function, I.C.T. 1900 Computer.

Examples,

A	INT	KINT would give
2.235	2	2
1.235	1	1
0.235	0	0
-1.235	-1	-2

1.2 Functions called by KINT.

this function uses function INT.

1.3 Function listing.

The listing of function KINT is given in table 38 .

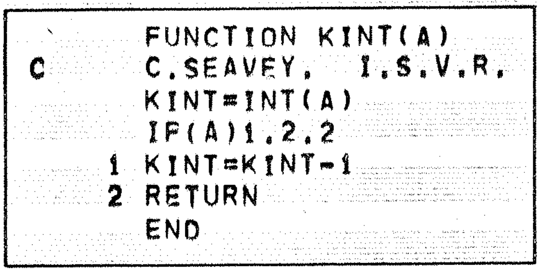
1.4 Description of function argument.

The first card of this function is,

```
FUNCTION KINT(A)
```

where,

A = real number.



```
      FUNCTION KINT(A)
C     C.SEAVEY,  I.S.V.R.
      KINT=INT(A)
      IF(A)1,2,2
    1 KINT=KINT-1
    2 RETURN
      END
```

A.S.A. Fortran listing of function KINT.

Table 38 .

A4.3 Master Programmes.

Master programmes have been written for the vibration analysis of collinear beam structures, general plane frames and two dimensional plate structures. The master programmes contain the procedure for eigenvalue evaluation which is the same for all programmes except that the respective FORCE-subroutine is called to obtain the structural dynamic stiffness matrix for a given frequency. It should be remembered that in the force formulation the structural dynamic stiffness matrix cannot be separated into the familiar iteration form

$$\left([\mathcal{K}] - \lambda\,[\mathcal{M}]\right)\{{}_t\Delta_\lambda\} = \{0\}$$

The eigenvalue formulation in the force approach is

$$[\mathcal{K}_d]\{{}_t\Delta_\lambda\} = \{0\}$$

where the frequency parameter λ is contained in the structural dynamic stiffness matrix $[\mathcal{K}_d]$.

The procedure adopted to find the eigenvalues of the system of homogeneous equations is based on a stepping method to find a change in sign of a determinant and is as follows ;

1. Calculate the structural dynamic stiffness matrix for an assumed value of the frequency parameter, λ .
2. Evaluate the determinant of matrix $[\mathcal{K}_d]$. If the determinant value is zero an eigenvalue

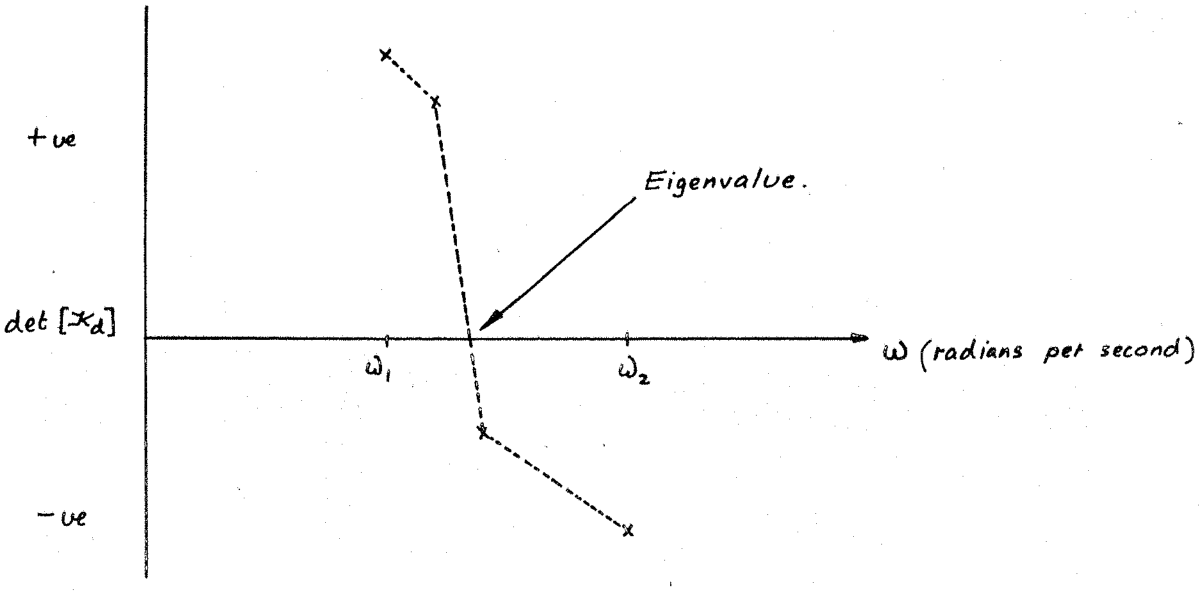
has been found, otherwise continue through a range of assumed values of λ until a sign change of the determinant value occurs. If no sign change is found a message NO ZERO CROSSING FOUND IN RANGE is printed out.

3. On finding a sign change the two adjacent λ-values are isolated and termed λ_1 and λ_2. These values are then fed into an iteration routine for calculating the actual eigenvalue within specified limits. If after ten iterations the eigenvalue has not been found to the desired accuracy the message ITERATION TERMINATED AFTER 10 STEPS is printed out. The previous estimate of the eigenvalue and the determinant are printed along with the current estimates. The current estimates are then used to find the eigenvector.
4. This cycle is repeated to obtain a range of eigenvalues.

The stepping procedure adopted is the same as that contained in subroutine PRVS1 which was written by Dr.C.A.Mercer, M.Petyt and C.Seavey.

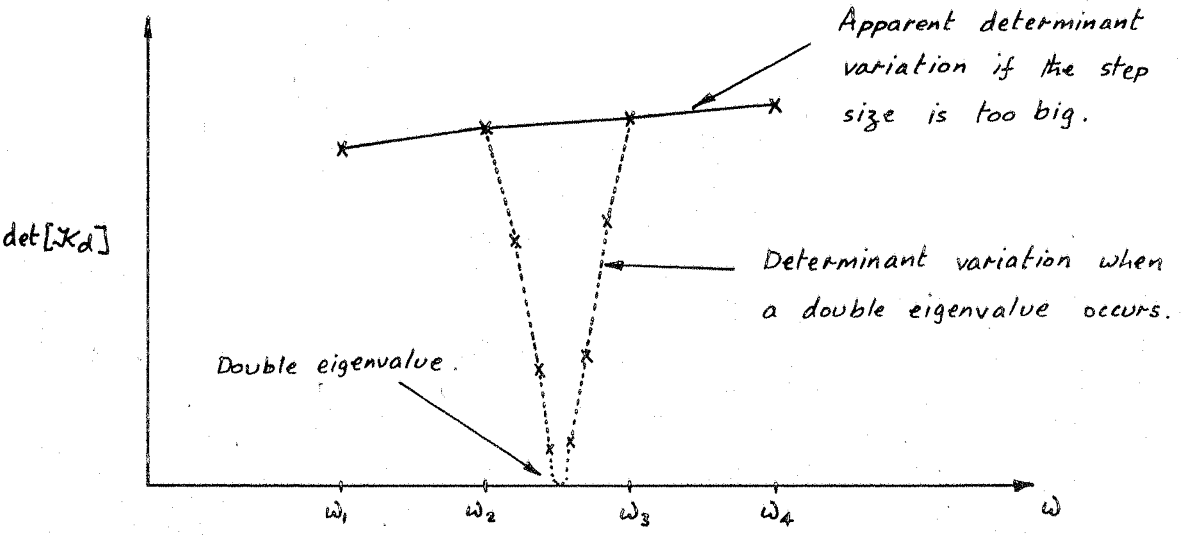
As structural configurations become more complex the more difficult the task of finding eigenvalues. Even when an eigenvalue is located it may not be known which one it corresponds to until the corresponding eigenvector has been computed. Although a sign change has occured between λ_1 and λ_2 it is quite possible that more than one eigenvalue is contained within these boundaries. It should also be realized that eigenvalues can exist without a sign change occuring . In the procedure adopted in the master programmes they are located by examining the variation of the determinant value and can easily be missed if the step size is too big. This situation of no sign change constitutes a double eigenvalue. The symptoms to observe are that the determinant value is increasing, it then drops and then increases again. At each step the sign of the determinant value is the same. Figure 64 shows eigenvalue conditions. Having found an eigenvalue it is fed into subroutine MODE to evaluate the corresponding eigenvector.

The master programmes also control the allocation of computer storage, reading of input data and the output of desired results.



Iteration to find single eigenvalue.

(a)



Iteration to find double eigenvalue.

(b)

Fig. 64 .

1. Master programme FORCE-BEAM.

This programme is for the vibration analysis of collinear beam structures using the rank force method. The programme has been written so that a number of problems can be solved together, that is, the programme has multiple case capability. The vibration analysis of a structure has been divided into two forms of analyses,

(a) Determination of element boundary loads, structural reactions, structural dynamic flexibility matrix (this gives the unit structural responses) and the structural dynamic stiffness matrix, all for a given frequency and a general system of harmonic forcing functions.

(b) Eigenvalue and eigenvector evaluation.

Both analyses can be carried out for a range of frequencies. The collinear beam structures can be idealized into any number of discrete elements and uses the plane beam type of element (FMD10 series of subroutines). Each discrete element can have a different length, Young's modulus, material density and section properties and the structure can be constrained in any way consistent with the basic assumptions.

The listing of master programme FORCE-BEAM is given in table 39 .

1.1 Input data.

KASES = number of cases being run (maximum = 9).

MR = form of analysis being undertaken, that is,

MR = 1 for element boundary loads, etc.

MR = 2 for eigenvalue and eigenvector evaluation.

KE = number of frequencies being considered (maximum = 42). When MR = 2 this maximum is based on the number of lowest frequency estimates, this has nothing to do with the incremental frequencies.

XXAM(K),DDLAM(K),NNST(K),IIEPS(K) = frequency data, where K = 1,KE.

This data is punched as one frequency data per card, that is,

XXAM(1),DDLAM(1),NNST(1),IIEPS(1)..........first card.
XXAM(2),DDLAM(2),NNST(2),IIEPS(2)..........second card.
⋮ ⋮ ⋮ ⋮ ⋮
XXAM(KE),DDLAM(KE),NNST(KE),IIEPS(KE)......KEth card.

When MR = 1 ;

XXAM = frequency at which the element boundary loads, structural reactions, etc., are required (radians per second).

DDLAM = frequency increment or step size (radians per second) = 1.0

NNST = maximum number of increments to be considered = 1

IIEPS = number of significant figures for answers (maximum = 8) = 1

DDLAM, NNST, IIEPS are not actually used in this analysis so they are quite arbitrary. These values (1.0,1,1) can be adopted for all considered frequencies but they must be punched on each card.

When MR = 2 ;

XXAM = lowest estimate of the eigenvalue being evaluated (radians per second).

DDLAM = frequency increment or step size (radians per second).

NNST = maximum number of increments to be considered. It is quite possible that an eigenvalue may be located after only a few increments and therefore this maximum will not be reached. On the other hand if no eigenvalue is located in this range NNST will have reached its maximum value.

IIEPS = number of significant figures for the answers (maximum = 8). In general this maximum is adopted.

Note ; the frequency data will take KE cards.

NE = number of discrete (finite) structural elements (maximum = 20).

IS(M),JS(M),XL(M),XI(M),E(M),XMU(M),CSA(M) = element data, where M = 1,NE.

This data is punched as one element data per card, that is,

IS(1),JS(1),XL(1),XI(1),E(1),XMU(1),CSA(1)......first card.

IS(2),JS(2),XL(2),XI(2),E(2),XMU(2),CSA(2)......second card.

⋮

IS(NE),JS(NE),XL(NE),XI(NE),E(NE),XMU(NE),CSA(NE)...NEth card.

IS = first element specifying node.

JS = second element specifying node.

XL = discrete element length (in).

XI = element second moment of area (in^4).

E = Young's modulus for the element (lb per in^2).

XMU = density of element material (lb per in^3).

CSA = element cross sectional area (in^2).

Note ; the element data will take NE cards.

NC, IC(N) = constraint data, where N = 1,NC.

NC = number of structural reactions (= 0 (zero) for no reactions). Maximum = 4.

IC = vector of applied load numbers which are considered as structural reactions. These numbers are established assuming initially that no reactions exist.

The input data formats and data card columns used for the respective parameters are given in table 40 . The method of data deck assembly for a multiple case analysis is shown in figure 65 and the complete programme/data assembly is shown in figure 66 .

A.S.A. Fortran listing of master programme FORCE-BEAM.

Table 39 .

```
      MASTER FORCE-BEAM
C     JOHN ROBINSON.  I.S.V.R.
C     VIBRATION ANALYSIS OF COLLINEAR BEAM STRUCTURES USING THE
C     RANK FORCE METHOD.
      DIMENSION IS(20),JS(20),XL(20),XI(20),E(20),XMU(20),CSA(20),IC(4),
     1UKD(42,42),X(42),XXAM(42),DDLAM(42),NNST(42),IIEPS(42),DEL(84,42),
     2F(42,42),OM(84,122),IO(84)
      COMMON OM
      EQUIVALENCE (DEL(1,1),OM(1,1)),(F(1,1),OM(1,102))
      EQUIVALENCE (UKD(1,1),OM(1,46))
      READ(5,85)KASES
   85 FORMAT(I1)
      DO500 KA=1,KASES
      WRITE(6,87)KA
   87 FORMAT(14H CASE NUMBER   ,I1)
      WRITE(6,3000)
 3000 FORMAT(23H JOHN ROBINSON I.S.V.R.)
      WRITE(6,900)
  900 FORMAT(22H PROGRAM   FORCE-BEAM.)
      READ(5,86)MR
   86 FORMAT(I1)
      READ(5,88)KE
   88 FORMAT(I2)
      WRITE(6,16)KE
   16 FORMAT(42H NUMBER OF FREQUENCIES BEING INVESTIGATED=,I4)
      READ(5,90)(XXAM(K),DDLAM(K),NNST(K),IIEPS(K),K=1,KE)
   90 FORMAT(2F12.6,I4,I3)
      WRITE(6,18)(XXAM(K),DDLAM(K),NNST(K),IIEPS(K),K=1,KE)
   18 FORMAT(23H ASSUMED FREQUENCY DATA///33H LOWER ESTIMATE STEP SIZE N
     1ST SIG///(2F20.6,I4,I3))
      READ(5,92)NE
   92 FORMAT(I2)
      WRITE(6,20)NE
   20 FORMAT(20H NUMBER OF ELEMENTS=,I3)
      READ(5,94)(IS(M),JS(M),XL(M),XI(M),E(M),XMU(M),CSA(M),M=1,NE)
   94 FORMAT(I2,I5,F12.4,F12.4,E9.1,F12.4,F12.4)
      WRITE(6,22)(IS(M),JS(M),XL(M),XI(M),E(M),XMU(M),CSA(M),M=1,NE)
   22 FORMAT(13H ELEMENT DATA///62H  NODES    LENGTH    SEC MOM AREA YUNG
     1MOD  DENSITY      C.S.AREA///(I3,I5,2F12.4,E9.1,2F12.4))
      READ(5,96)NC,(IC(N),N=1,NC)
   96 FORMAT(I1,4I4)
      WRITE(6,24)NC,(IC(N),N=1,NC)
   24 FORMAT(23H NUMBER OF CONSTRAINTS=,I4///20H IMPOSED CONSTRAINTS///(
     14I4))
```

Master programme FORCE-BEAM listing continued.

```
      DO310 K=1,KE
      XAM=XXAM(K)
      IF(MR-1)15,354,15
   15 DLAM=DDLAM(K)
      NST=NNST(K)
      IEPS=IIEPS(K)
      WRITE(6,14)K,XAM,DLAM,NST,IEPS
   14 FORMAT(15H FREQUENCY DATA,I4///5H XAM=,F12.6,3X,6H DLAM=,F12.6,3X,
     15H NST=,I4,6H IEPS=,I3)
      CALL FORCEB(XAM,NE,IS,JS,XL,XI,E,XMU,CSA,NC,IC,NL,UKD,DEL,F,MC,IQ)
      CALL VARDET(NL,NL,NL,DET1,UKD,42)
      WRITE(6,400)XAM,DET1
  400 FORMAT(17H LOWER FREQUENCY=,F12.6///13H DETERMINANT=,E15.7)
      DO101 J=1,NST
      XAM=XAM+DLAM
      CALL FORCEB(XAM,NE,IS,JS,XL,XI,E,XMU,CSA,NC,IC,NL,UKD,DEL,F,MC,IQ)
      CALL VARDET(NL,NL,NL,DET2,UKD,42)
      WRITE(6,402)J,XAM,DET2
  402 FORMAT(13H STEP NUMBER=,I3///19H CURRENT FREQUENCY=,F12.6///13H DE
     1TERMINANT=,E15.7)
      IF(DET1*DET2)200,209,102
  102 DET1=DET2
  101 CONTINUE
      NST=-1
      WRITE(6,103)
  103 FORMAT(33H NO ZERO CROSSINGS FOUND IN RANGE)
      GO TO 310
C     ITERATION (PHASE 2)
  200 F1=XAM-DLAM
      F2=XAM
      IF(IEPS-8)203,203,202
  202 IEPS=8
  203 CONTINUE
      DO207 I=1,10
      IF(I-1)205,205,204
  204 F1=XAM
      DET1=DET
  205 CONTINUE
      XAM=(F1*DET2-F2*DET1)/(DET2-DET1)
      A=ALOG10(XAM)
      IEX=KINT(A)
      EPS=10.0**(IEPS-IEX-1)
      HSUB=(XAM-F1)
      IF(XAM-HSUB*EPS)206,209,209
  206 CONTINUE
      IF(I-1)250,250,255
  250 FINT=F2-XAM
      CF=0.2
  251 FR=XAM+CF*FINT
```

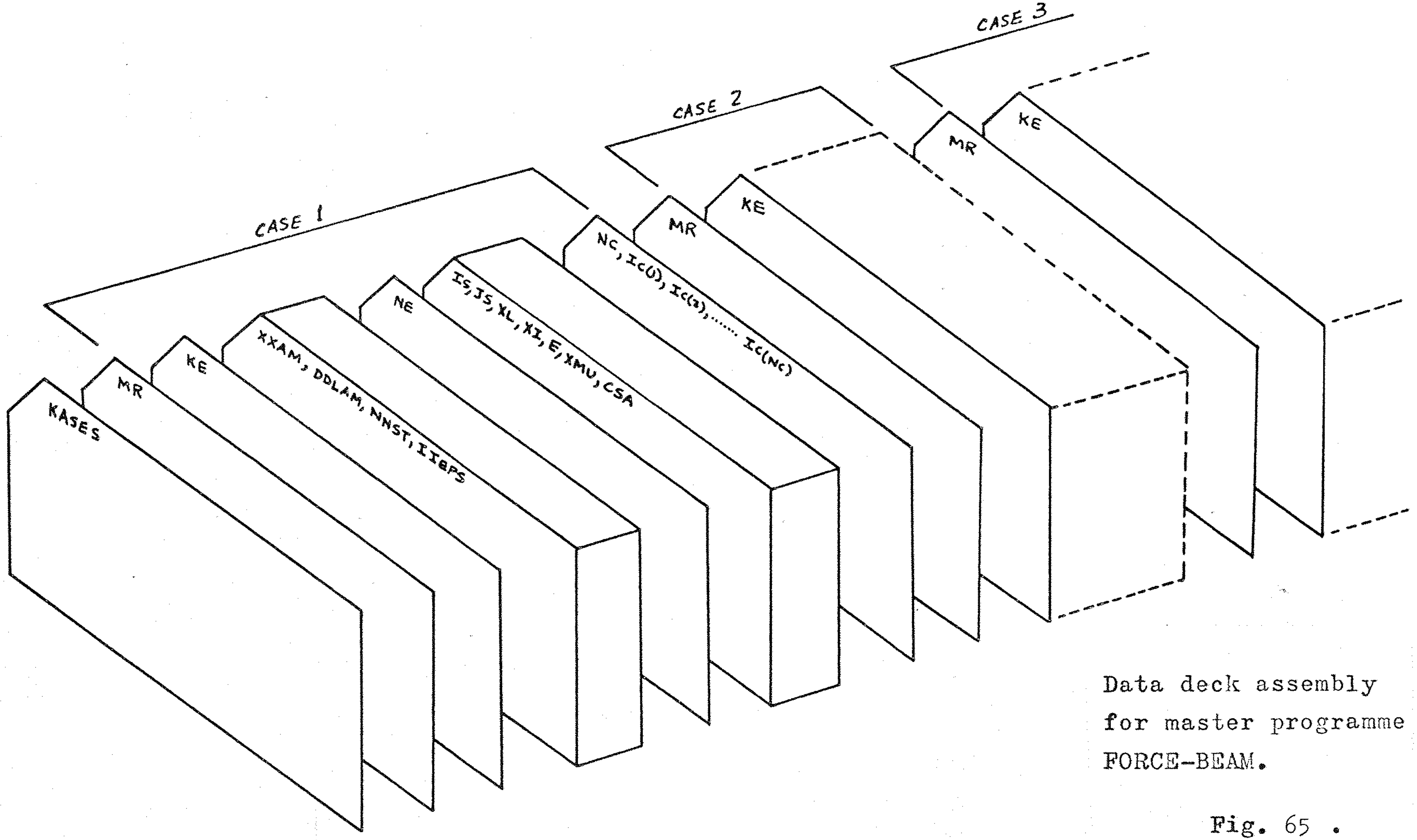
Master programme FORCE-BEAM listing continued.

```
      CALL FORCEB(FR.NE,IS.JS,XL,XI,E,XMU,CSA,NC.IC,NL,UKD,DEL.F.MC,IQ)
      CALL VARDET(NL,NL,NL.DET,UKD,42)
      IF(DET*DET2)252.253.254
  252 CF=CF+0.1
      GO TO 251
  253 XAM=FR
      GO TO 209
  254 F2=FR
      DET2=DET
  255 CALL FORCEB(XAM,NE.IS,JS,XL,XI,E,XMU.CSA.NC,IC,NL,UKD.DEL.F,MC,IQ)
      CALL VARDET(NL.NL,NL.DET,UKD,42)
  207 CONTINUE
      WRITE(6,208)F1.DET1.XAM,DET
  208 FORMAT(///36H ITERATION TERMINATED AFTER 10 STEPS//22H PREVIOUS
     1ESTIMATE =,E16.8,13H DETERMINANT=,E16.8///18H CURRENT ESTIMATE=,
     2E16.8,13H DETERMINANT=,E16.8//22H CURRENT ESTIMATE USED)
  209 CONTINUE
C     VECTOR EVALUATION (PHASE 3)
  354 CALL FORCEB(XAM.NE.IS,JS,XL,XI,E,XMU,CSA.NC,IC.NL,UKD.DEL,F,MC,IQ)
      IF(MR.EQ.1)GO TO 356
      CALL MODE(NL.NL.NL,UKD,42,X,XAM)
      GO TO 310
  356 IF(K.NE.1)GO TO 406
      DO410 I=1,MC
      IF(I,NE,1)GO TO 412
      WRITE(6,414)
  414 FORMAT(13H REDUNDANCIES)
  412 CONTINUE
      IF(IQ(I).EQ.0)GO TO 410
      WRITE(6,401)IQ(I)
  401 FORMAT(I6)
  410 CONTINUE
  406 WRITE(6,360)XAM
  360 FORMAT(11H FREQUENCY=,F12.6)
      WRITE(6,358)((I,J,DEL(I,J),J=1,NL),I=1,MC)
  358 FORMAT(24H UNIT MATRIX OF UNKNOWNS//4(I4,2H ,,I2,3X,E19.11))
      WRITE(6,359)((I,J,F(I,J),J=1,NL),I=1,NL)
  359 FORMAT(38H STRUCTURAL DYNAMIC FLEXIBILITY MATRIX///3(I4.2H ,,I2,3X
     1,E19,11,2X))
      WRITE(6,350)((KK,J,UKD(KK,J),J=1,NL),KK=1,NL)
  350 FORMAT(36H STRUCTURAL DYNAMIC STIFFNESS MATRIX///4(I4.2H ..I2,3X,E
     119,11))
  310 CONTINUE
  500 CONTINUE
      STOP
      END
```

Input parameter	Type of number	FORMAT statement	Data card columns (inclusive).	Remarks.	
KASES	Integer	I1	1		one card
MR	Integer	I1	1		One card
KE	Integer	I2	1 to 2	Right adjusted. KE < 10 punch in column 2.	One card
XXAM	Floating point	F12·6	1 to 12	Punch anywhere in this column range.	One card per frequency data.
DDLAM	Floating point	F12·6	13 to 24	Punch anywhere in this column range.	
NNST	Integer	I4	25 to 28	Right adjusted. NNST < 10 punch in column 28, 10 ≤ NNST < 100 Columns 27,28	
IIEPS	Integer	I3	29 to 31	Right adjusted. Punch in column 31 (IIEPS = 8 maximum).	
NE	Integer	I2	1 to 2	Right adjusted. NE < 10 punch in colum 2.	One card.
IS	Integer	I2	1 to 2	Right adjusted. IS < 10 punch in column 2	one card per element data
JS	Integer	I5	3 to 7	Right adjusted. JS < 10 punch in column 7	
XL	Floating point	F12·4	8 to 19	Punch anywhere in this column range.	
XI	Floating point	F12·4	20 to 31	Punch anywhere in this column range.	
E	Floating point	E9·1	32 to 40	$E = 30.0 \times 10^6$ punch 30·0E+06 $E = 4{\cdot}1 \times 10^6$ punch 41·0E+05 starting in column 33	
XMU	Floating point	F12·4	41 to 52	Punch anywhere in this column range.	
CSA	Floating point	F12·4	53 to 64	Punch anywhere in this column range.	
NC	Integer	I1	1		One card
IC(1)	Integer	I4	2 to 5	Right adjusted.	
⋮	⋮	⋮			
⋮	⋮	⋮			
IC(NC)	Integer	I4		Right adjusted.	

Input data for master programme FORCE-BEAM.

Table 40 .



Data deck assembly for master programme FORCE-BEAM.

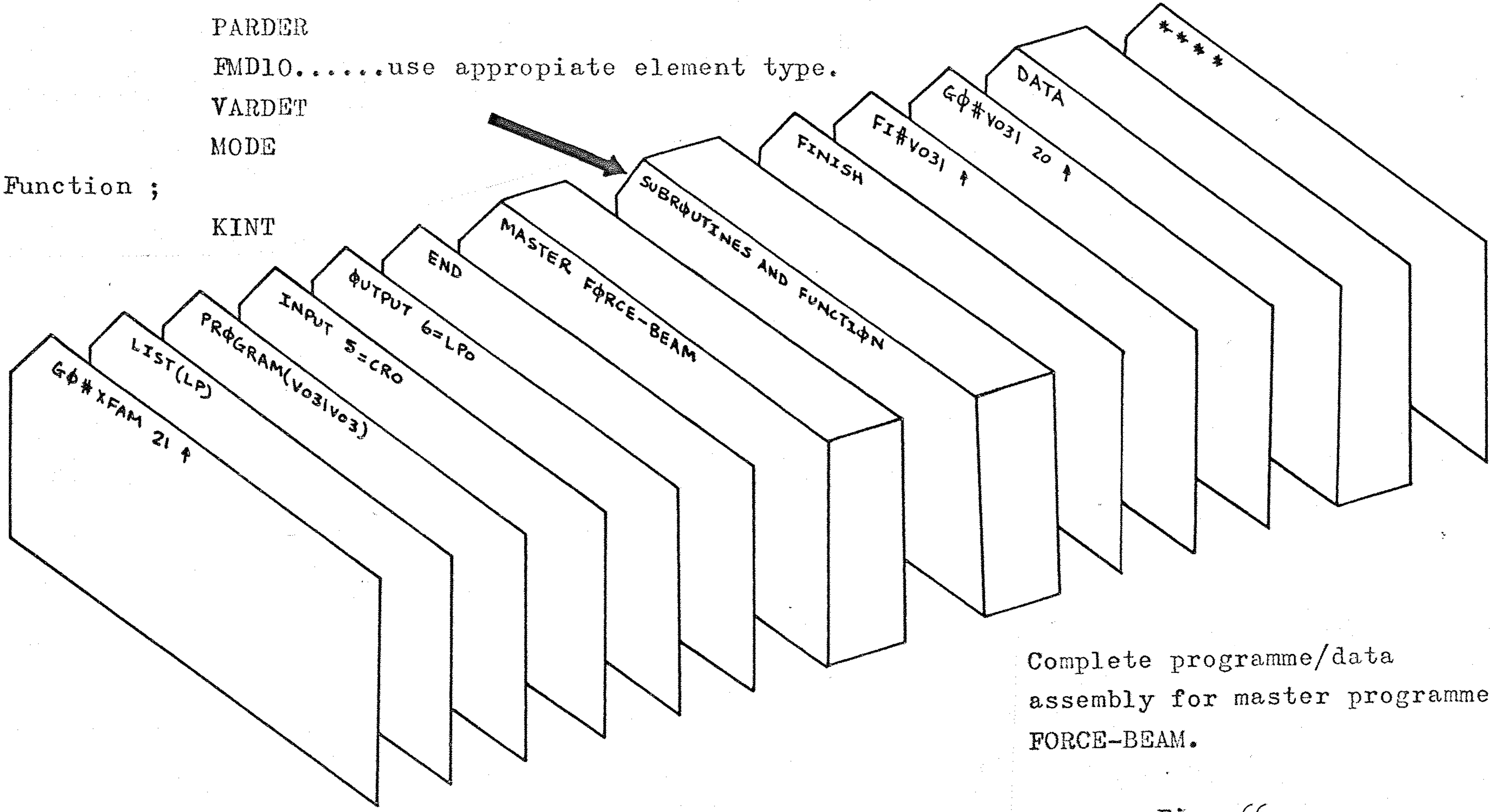
Fig. 65 .

Subroutines ;

FORCEB
RANTEC
REAR
MATINV
MATMULT
PARDER
FMD10......use appropiate element type.
VARDET
MODE

Function ;

KINT



Complete programme/data assembly for master programme FORCE-BEAM.

Fig. 66 .

1.2 Output data.

The first part of the output data consists of the title data and the input data for each case, that is,

```
CASE NUMBER    __
JOHN ROBINSON   I.S.V.R.
PROGRAM FORCE-BEAM.
NUMBER OF FREQUENCIES BEING INVESTIGATED = __ __
ASSUMED FREQUENCY DATA
LOWER ESTIMATE     STEP SIZE     NST     SIG
      |                |          |       |
      |                |          |       |
      |                |          |       |
      |                |          |       |
      |                |          |       |
NUMBER OF ELEMENTS = __ __
ELEMENT DATA
NODES   LENGTH   SEC MOM AREA   YUNG MOD   DENSITY   C.S.AREA
|   |     |          |              |          |          |
|   |     |          |              |          |          |
|   |     |          |              |          |          |
|   |     |          |              |          |          |
|   |     |          |              |          |          |
NUMBER OF CONSTRAINTS = __
IMPOSED CONSTRAINTS

    __ __ __ __
```

The second part of the output consists of the required results depending on the value of MR.

MR = 1

1. Automatically selected redundancies.

The dynamic redundancies which are isolated by the rank technique are printed out for each case.

REDUNDANCIES

2. Frequency being considered.

When MR = 1 the analysis for one case is carried out for one frequency or a range of frequencies. The frequency at each stage is printed out.

FREQUENCY =

3. Unit element loads and structural reactions.

The element boundary loads and structural reactions are given in the form of a DEL-matrix which is generated for a given frequency and a general system of harmonic forcing functions. In the programme formulation all possible applied loads are assumed to exist. Therefore,

$$\left[\begin{array}{c} {}_t q_\alpha \\ \hdashline {}_t R_e \end{array}\right] = \left[\begin{array}{c} \Delta_{\alpha\lambda} \\ \hdashline \Delta_{e\lambda} \end{array}\right] \{ {}_t P_\lambda \}$$

Vector of unknowns.

DEL-matrix (this gives the unit unknown distributions).

Vector of all possible applied loads.

The DEL-matrix is printed out and the actual unknowns corresponding to an actual applied load system are obtained by multiplying this matrix by the actual vector of applied harmonic forcing functions. This is for a given frequency. This latter operation is not presently in the programme but requires very little effort to incorporate it.

The DEL-matrix is printed out in the following manner,

UNIT MATRIX OF UNKNOWNS

I , J DEL(I,J) —— four per line.

Row number. Column number. Matrix coefficient.

E-FORMAT. Examples ;

1. -0.1667E 3
   = -166.7
2. 0.3857E-1
   = 0.03857

4. Structural dynamic flexibility matrix.

The unit structural responses for a given frequency and a general system of harmonic forcing functions are given by the structural dynamic flexibility matrix, that is,

$$\{{}_t\Delta_\lambda\} = [\mathcal{F}_d]\{{}_tP_\lambda\}$$

Structural response vector.

F-matrix. Structural dynamic flexibility matrix.

Vector of all possible applied loads.

The structural dynamic flexibility matrix is printed out, the actual structural response vector corresponding to an actual applied load system is obtained by multiplying the F-matrix by the actual vector of applied harmonic forcing functions. This is for a given frequency. This latter operation is not presently in the programme but again this is a simple amendment.

The F-matrix is printed out in the following manner,

STRUCTURAL DYNAMIC FLEXIBILITY MATRIX

I , J F(I,J) four per line.

Row number.

Column number.

Matrix coefficient. E-FORMAT.

5. Structural dynamic stiffness matrix.

The unit harmonic forcing functions for a given frequency and a general system of structural responses are given by the structural dynamic stiffness matrix, that is,

$$\{{}_tP_\lambda\} = [\mathcal{K}_d]\{{}_t\Delta_\lambda\}$$

Vector of harmonic forcing functions.

UKD-matrix. Structural dynamic stiffness matrix.

Vector of all possible structural responses.

The structural dynamic stiffness matrix only is printed out and in the following manner,

KK , J UKD(KK,J) four per line.

Row number.

Column number.

Matrix coefficient. E-FORMAT.

MR = 2.

FREQUENCY DATA —

XAM =	DLAM =	NST =	IEPS =
↑	↑	↑	↑
Lowest estimate.	Step size.	Number of steps.	Number of significant figures for answers.

```
LOWER FREQUENCY = --------
DETERMINANT = ------------
STEP NUMBER = 1
CURRENT FREQUENCY = --------
DETERMINANT = ------------
STEP NUMBER = 2
CURRENT FREQUENCY = --------
DETERMINANT = -------------
```

and so on until a change in sign of the determinant value has occured. Two possible conditions exist ;

(a) No change of sign.

When the sign of the determinant doesn't change through the range of frequencies considered (decided by the value of NNST in the input) the following statement is printed out,

```
NO ZERO CROSSING FOUND IN RANGE
```

When this statement is printed out investigate the variation in the determinant value to see if a double eigenvalue is indicated.

(b) Change of sign.

After locating a change of sign an iteration routine is entered and the eigenvalue is computed to the desired accuracy. This is then used to find the corresponding eigenvector. The output is as follows,

```
EIGENVALUE = --------
ERROR SUM = --------   DETERMINANT = ----------
EIGENVECTOR
```

See subroutine MODE (A1.4.1......13).

If during the iteration phase the eigenvalue is not found to the desired accuracy the statement

```
ITERATION TERMINATED AFTER 10 STEPS
```

is printed out along with,

```
PREVIOUS ESTIMATE =--------
DETERMINANT = --------------
CURRENT ESTIMATE =---------- ← This is fed into
                                subroutine MODE.
DETERMINANT = --------------
CURRENT ESTIMATE USED
```

After computing the eigenvector the following is printed out,

```
EIGENVALUE = ------
ERROR SUM = ------    DETERMINANT = -------
EIGENVECTOR
```

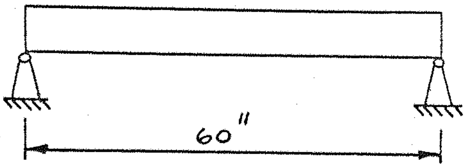
1.3 Example of usage.

The simply supported beam shown in figure 67 is idealized as two discrete elements, using Element Type P3/2FD evaluate ;

(a) The first and second eigenvalues.

(b) The unit distributions for the element boundary loads and structural reactions, structural dynamic flexibility and stiffness matrices at a frequency of 200 radians per second.

The input data preparation for this problem is given in table 41 .



$E = 30.0 \times 10^6$ lb per $in.^2$

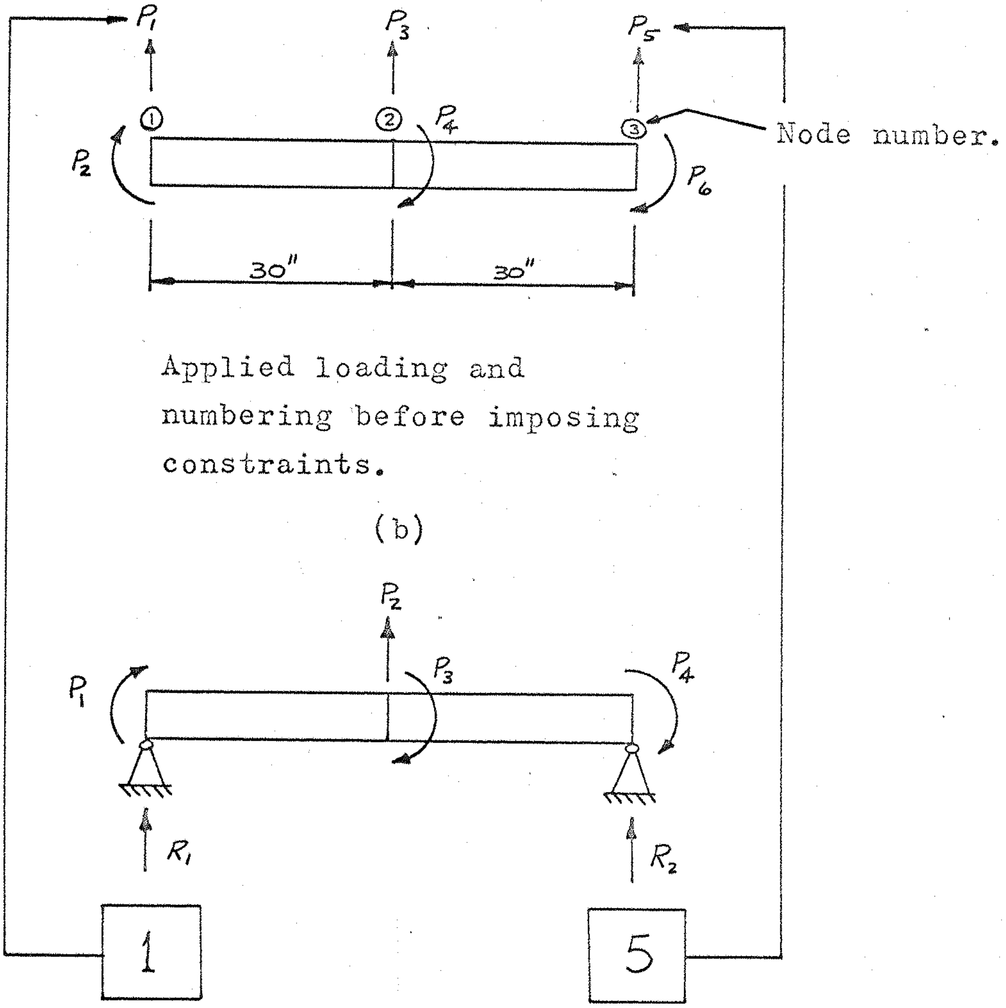
$A = 1.366 \; in^2$.

$I = 0.1 \; in^4$.

$\mu = 0.283$ lb per in^3.

Simply supported beam.

(a)



Applied loading and numbering before imposing constraints.

(b)

Applied load number corresponding to reaction.

Applied load number corresponding to reaction.

Possible applied loading and numbering with reactions.

(c)

Fig. 67 .

AME. John Robinson

DEPARTMENT. I.S.V.R.

PROBLEM. Simply supported beam idealized as two finite elements. Element Type P3/2FD.

ATE. May 1967

PAGE 1 OF 1

```
2
49.0       2.0           10  8
98.0       2.0           10  8
2
1    2        30.0         0.1 30.0E+06       0.283       1.366
2    3        30.0         0.1 30.0E+06       0.283       1.366
   1   5


1
00.0       1.0            1  1
2
1    2        30.0         0.1 30.0E+06       0.283       1.366
2    3        30.0         0.1 30.0E+06       0.283       1.366
   1   5
```

Table

2. Master programme FORCE-PLANE FRAME.

This programme is for the vibration analysis of general plane frames, this is presently restricted to rigid element joints but can readily be amended for arbitrary joint connections as indicated in the description for subroutine FORCEF, page 298 . The programme has been written for multiple case capability and the vibration analysis has been divided in the same way as for master programme FORCE-BEAM, page 322 . The general plane frame structure can be idealized into any number of discrete elements within the specified limits and uses the inclined plane beam type of element (FMD30 series of subroutines). Each discrete element can have a different length, the length of each element being calculated within the computer programme using the coordinates of the element specifying nodes. Each element can also have a different Young's modulus, material density and section properties and the structure can be constrained in any way consistent with the basic assumptions.

The listing the master programme FORCE-PLANE FRAME is given in table 42 .

2.1 Input data.

KASES = number of cases being run (maximum = 9).

MR = form of analysis being undertaken.

KE = number of frequencies being considered (maximum = 33).

XXAM(K),DDLAM(K),NNST(K),IIEPS(K) = frequency data, where K = 1,KE.

This data is define in the same way as master programme FORCE-BEAM.

NE = number of discrete structural elements (maximum = 13).

NJ = number of nodes (maximum = 11).

NODE, XB(NODE), ZB(NODE) = nodal data, where NODE = 1,NJ.

This data is punched as one nodal data per card, that is,

1, XB(1), ZB(1)................first card.

2, XB(2), ZB(2)................second card.

⋮

NJ,XB(NJ),ZB(NJ)...............NJth card.

NODE = node number.

XB = $\bar{x}$-ordinate of respective node (in).

ZB = $\bar{z}$-ordinate of respective node (in).

Note ; the nodal data will take NJ cards.

IS(M),JS(M),XI(M),E(M),XMU(M),CSA(M) = element data,
where M = 1,NE.

This data is defined in the same way as master programme FORCE-BEAM. It should be noted that in the frame programme no element length is given in the element data.

NC, IC(N) = constraint data, where NC = 1,NC.

NC = number of structural reactions (= 0(zero) for no reactions). Maximum = 9.

IC = vector of applied numbers which are considered as structural reactions. These numbers are established assuming initially that no reactions exist.

The input data formats and data card columns used for the respective parameters are given in table 43. The method of data deck assembly for a multiple case analysis is shown in figure 68 and the complete programme/data assembly is shown in figure 69 .

2.2 Output data.

The first part of the output data consists of the title data and the input data for each case, that is,

```
CASE NUMBER __
JOHN ROBINSON  I.S.V.R.
PROGRAM FORCE-PLANE FRAME.
NUMBER OF FREQUENCIES BEING INVESTIGATED = __
ASSUMED FREQUENCY DATA
LOWER ESTIMATE          STEP SIZE      NST       SIG
     ¦                      ¦            ¦         ¦
     ¦                      ¦            ¦         ¦
```

```
NUMBER OF ELEMENTS = _ _

NUMBER OF JOINTS   = _ _

NODAL COORDINATES

NODE     XB-ORDINATE      ZB-ORDINATE
 |            |                |
 |            |                |
 |            |                |
 |            |                |

ELEMENT DATA

NODE   NODE   SEC MOM  AREA    YOUNGS MOD    DENSITY    C. S. AREA
 |      |           |               |            |            |
 |      |           |               |            |            |
 |      |           |               |            |            |
 |      |           |               |            |            |
NUMBER OF CONSTRAINTS = __

IMPOSED CONSTRAINTS


_ _ _ _ _ _ _ _ _
```

The second part of the output consists of the required results depending on the value of MR. This is the same as for master programme FORCE-BEAM, page 332 .

A.S.A. Fortran listing of master programme FORCE-PLANE FRAME.

Table 42 .

```
       MASTER FORCE-PLANE FRAME
C      JOHN ROBINSON.   I.S.V.R.
C      VIBRATION ANALYSIS OF GENERAL PLANE FRAMES USING THE
C      RANK FORCE METHOD.
       DIMENSION XX(33),XXAM(33),DDLAM(33),NNST(33),IIEPS(33)
      1,IS(13),JS(13),XI(13),E(13),XMU(13),CSA(13),IC(9),UKD(33,33),OM(87
      2,111),DEL(87,33),F(33,33),IQ(87),XB(11),ZB(11)
       COMMON OM
       EQUIVALENCE (DEL(1,1),OM(1,1)),(F(1,1),OM(1,99)),(UKD(1,1),OM(1,86
      1))
       READ(5,85)KASES
    85 FORMAT(I1)
       DO500 KA=1,KASES
       WRITE(6,87)KA
    87 FORMAT(14H CASE NUMBER   ,I1)
       WRITE(6,3000)
  3000 FORMAT(23H JOHN ROBINSON I.S.V.R.)
       WRITE(6,3001)
  3001 FORMAT(27H PROGRAM FORCE-PLANE FRAME.)
       READ(5,86)MR
    86 FORMAT(I1)
       READ(5,88)KE
    88 FORMAT(I2)
       WRITE(6,16)KE
    16 FORMAT(42H NUMBER OF FREQUENCIES BEING INVESTIGATED=,I4)
       READ(5,90)(XXAM(K),DDLAM(K),NNST(K),IIEPS(K),K=1,KE)
    90 FORMAT(2F12.6,I4,I3)
       WRITE(6,18)(XXAM(K),DDLAM(K),NNST(K),IIEPS(K),K=1,KE)
    18 FORMAT(23H ASSUMED FREQUENCY DATA///33H LOWER ESTIMATE STEP SIZE N
      1ST SIG///(2F20.6,I4,I3))
       READ(5,6)NE,NJ
     6 FORMAT(I2,I4)
       WRITE(6,3)NE,NJ
     3 FORMAT(21H NUMBER OF ELEMENTS =,I4//19H NUMBER OF JOINTS =,I4)
       READ(5,8)(NODE,XB(NODE),ZB(NODE),NODE=1,NJ)
     8 FORMAT(I2,2F12.4)
       WRITE(6,4)(NODE,XB(NODE),ZB(NODE),NODE=1,NJ)
     4 FORMAT(18H NODAL COORDINATES//1X,5H NODE,2X,13H XB-ORDINATE ,2X,13
      1H ZB-ORDINATE //(I4,3X,F12.4,2X,F12.4))
       READ(5,7)(IS(M),JS(M),XI(M),E(M),XMU(M),CSA(M),M=1,NE)
     7 FORMAT(I2,I5,F12.4,E9.1,2F12.4)
       WRITE(6,5)(IS(M),JS(M),XI(M),E(M),XMU(M),CSA(M),M=1,NE)
     5 FORMAT(13H ELEMENT DATA//1X,56H NODE  NODE  SEC MOM AREA  YOUNGS M
      1OD  DENSITY  C.S.AREA//(I4,2X,I4,3X,F12.9,3X,E12.3,2X,F7.5,2X,F8.4
      2))
       READ(5,10)NC,(IC(N),N=1,NC)
    10 FORMAT(I1,9I4)
       WRITE(6,1)NC,(IC(N),N=1,NC)
     1 FORMAT(23H NUMBER OF CONSTRAINTS=,I4///20H IMPOSED CONSTRAINTS///(
      19I4))
```

Master programme FORCE-PLANE FRAME listing continued.

```
      DO310 K=1,KE
      XAM=XXAM(K)
      IF(MR-1)15,354,15
   15 DLAM=DDLAM(K)
      NST=NNST(K)
      IEPS=IIEPS(K)
      WRITE(6,14)K,XAM,DLAM,NST,IEPS
   14 FORMAT(15H FREQUENCY DATA,I4///5H XAM=,F12.6,3X,6H DLAM=,F12.6,3X,
     15H NST=,I4,6H IEPS=,I3)
      CALL FORCEF(XAM,NE,NJ,XB,ZB,IS,JS,XI,E,XMU,CSA,NC,IC,NL.UKD,DEL,F,
     1MC,IQ)
      CALL VARDET(NL,NL,NL,DET1,UKD,33)
      WRITE(6,400)XAM,DET1
  400 FORMAT(17H LOWER FREQUENCY=,F12.6///13H DETERMINANT=,E15.7)
      DO101 J=1,NST
      XAM=XAM+DLAM
      CALL FORCEF(XAM,NE,NJ,XB,ZB,IS,JS,XI,E,XMU,CSA,NC,IC,NL.UKD,DEL,F,
     1MC,IQ)
      CALL VARDET(NL,NL,NL,DET2,UKD,33)
      WRITE(6,402)J,XAM,DET2
  402 FORMAT(13H STEP NUMBER=,I3///19H CURRENT FREQUENCY=,F12.6///13H DE
     1TERMINANT=,E15.7)
      IF(DET1*DET2)200,209,102
  102 DET1=DET2
  101 CONTINUE
      NST=-1
      WRITE(6,103)
  103 FORMAT(33H NO ZERO CROSSINGS FOUND IN RANGE)
      GO TO 310
C     ITERATION (PHASE 2)
  200 F1=XAM-DLAM
      F2=XAM
      IF(IEPS-8)203,203,202
  202 IEPS=8
  203 CONTINUE
      DO207 I=1,10
      IF(I-1)205,205,204
  204 F1=XAM
      DET1=DET
  205 CONTINUE
      XAM=(F1*DET2-F2*DET1)/(DET2-DET1)
      A=ALOG10(XAM)
      IEX=KINT(A)
      EPS=10.0**(IEPS-IEX-1)
      HSUB=(XAM-F1)
      IF(XAM-HSUB*EPS)206,209,209
  206 CONTINUE
      IF(I-1)250,250,255
  250 FINT=F2-XAM
      CF=0.2
  251 FR=XAM+CF*FINT
```

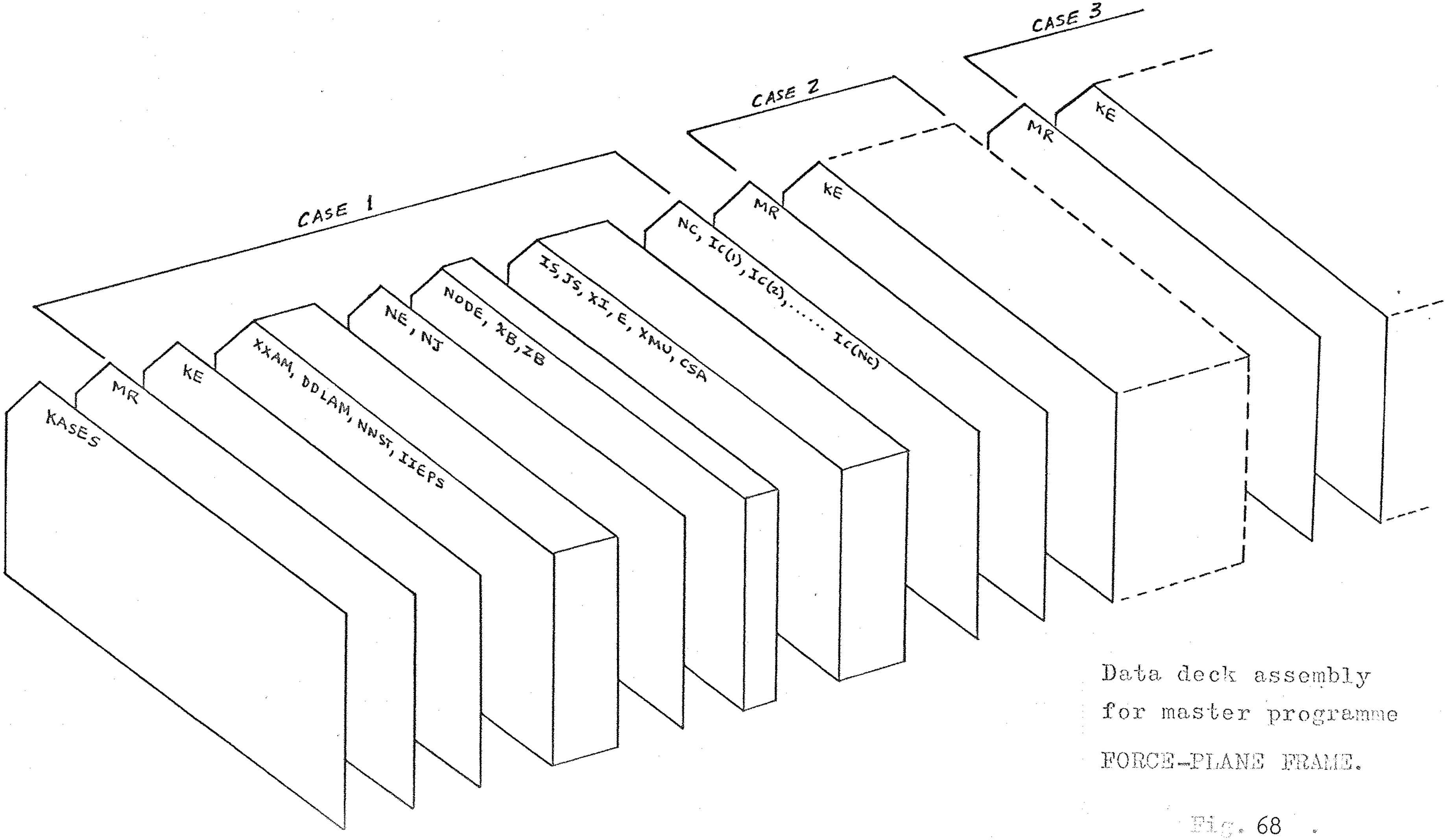
Master programme FORCE-PLANE FRAME listing continued.

```
      CALL FORCEF(FR ,NE,NJ,XB,ZB,IS,JS,XI,E,XMU,CSA,NC,IC,NL,UKD,DEL,F,
     1MC,IQ)
      CALL VARDET(NL,NL,NL,DET ,UKD,33)
      IF(DET*DET2)252,253,254
  252 CF=CF+0.1
      GO TO 251
  253 XAM=FR
      GO TO 209
  254 F2=FR
      DET2=DET
  255 CALL FORCEF(XAM,NE,NJ,XB,ZB,IS,JS,XI,E,XMU,CSA,NC,IC,NL,UKD,DEL,F,
     1MC,IQ)
      CALL VARDET(NL,NL,NL,DET ,UKD,33)
  207 CONTINUE
      WRITE(6,208)F1,DET1,XAM,DET
  208 FORMAT(///36H ITERATION TERMINATED AFTER 10 STEPS//22H PREVIOUS
     1ESTIMATE =,E16.8,13H DETERMINANT=,E16.8///18H CURRENT ESTIMATE=,
     2E16.8,13H DETERMINANT=,E16.8//22H CURRENT ESTIMATE USED)
  209 CONTINUE
C     VECTOR EVALUATION (PHASE 3)
  354 CALL FORCEF(XAM,NE,NJ,XB,ZB,IS,JS,XI,E,XMU,CSA,NC,IC,NL,UKD,DEL,F,
     1MC,IQ)
      IF(MR.EQ.1)GO TO 356
      CALL MODE(NL,NL,NL,UKD,33,XX,XAM)
      GO TO 310
  356 IF(K.NE.1)GO TO 406
      DO410 I=1,MC
      IF(I.NE.1)GO TO 412
      WRITE(6,414)
  414 FORMAT(13H REDUNDANCIES)
  412 CONTINUE
      IF(IQ(I).EQ.0)GO TO 410
      WRITE(6,401)IQ(I)
  401 FORMAT(I6)
  410 CONTINUE
  406 WRITE(6,360)XAM
  360 FORMAT(11H FREQUENCY=,F12.6)
      WRITE(6,358)((I,J,DEL(I,J),J=1,NL),I=1,MC)
  358 FORMAT(24H UNIT MATRIX OF UNKNOWNS//3(I4,2H .,I2,3X,E19.11,2X))
      WRITE(6,359)((I,J,F(I,J),J=1,NL),I=1,NL)
  359 FORMAT(38H STRUCTURAL DYNAMIC FLEXIBILITY MATRIX///3(I4.2H .,I2,3X
     1,E19.11,2X))
      WRITE(6,350)((KK,J,UKD(KK,J),J=1,NL),KK=1,NL)
  350 FORMAT(36H STRUCTURAL DYNAMIC STIFFNESS MATRIX///3(I4.2H .,I2,3X,E
     119.11,2X))
  310 CONTINUE
  500 CONTINUE
      STOP
      END
```

Input parameter	Type of number	FORMAT statement	Data card columns (inclusive).	Remarks.	
KASES	Integer	I1	1		One card
MR	Integer	I1	1		One card
KE	Integer	I2	1 to 2		One card
XXAM	Floating point	F12·6	1 to 12		One card per frequency data.
DDLAM	Floating point	F12·6	13 to 24	NOTE:	
NNST	Integer	I4	25 to 28	When punching the	
IIEPS	Integer	I3	29 to 31	input data consider	
NE	Integer	I2	1 to 2	the type of number.	One card
NJ	Integer	I4	3 to 6	See input data for	
NODE	Integer	I2	1 to 2	master programme	One card per nodal data
XB	Floating point	F12·4	3 to 14	FORCE-BEAM,	
ZB	Floating point	F12·4	15 to 26	table , for	
IS	Integer	I2	1 to 2	general remarks.	One card per element data
JS	Integer	I5	3 to 7		
XI	Floating point	F12·4	8 to 19		
E	Floating point	E9·1	20 to 28		
XMU	Floating point	F12·4	29 to 40		
CSA	Floating point	F12·4	41 to 52		
NC	Integer	I1	1		One card
IC(1)	Integer	I4	2 to 5		
⋮	⋮	⋮			
IC(NC)	Integer	I4			

Input data for master programme FORCE-PLANE FRAME.

Table 43 .



Data deck assembly
for master programme
FORCE-PLANE FRAME.

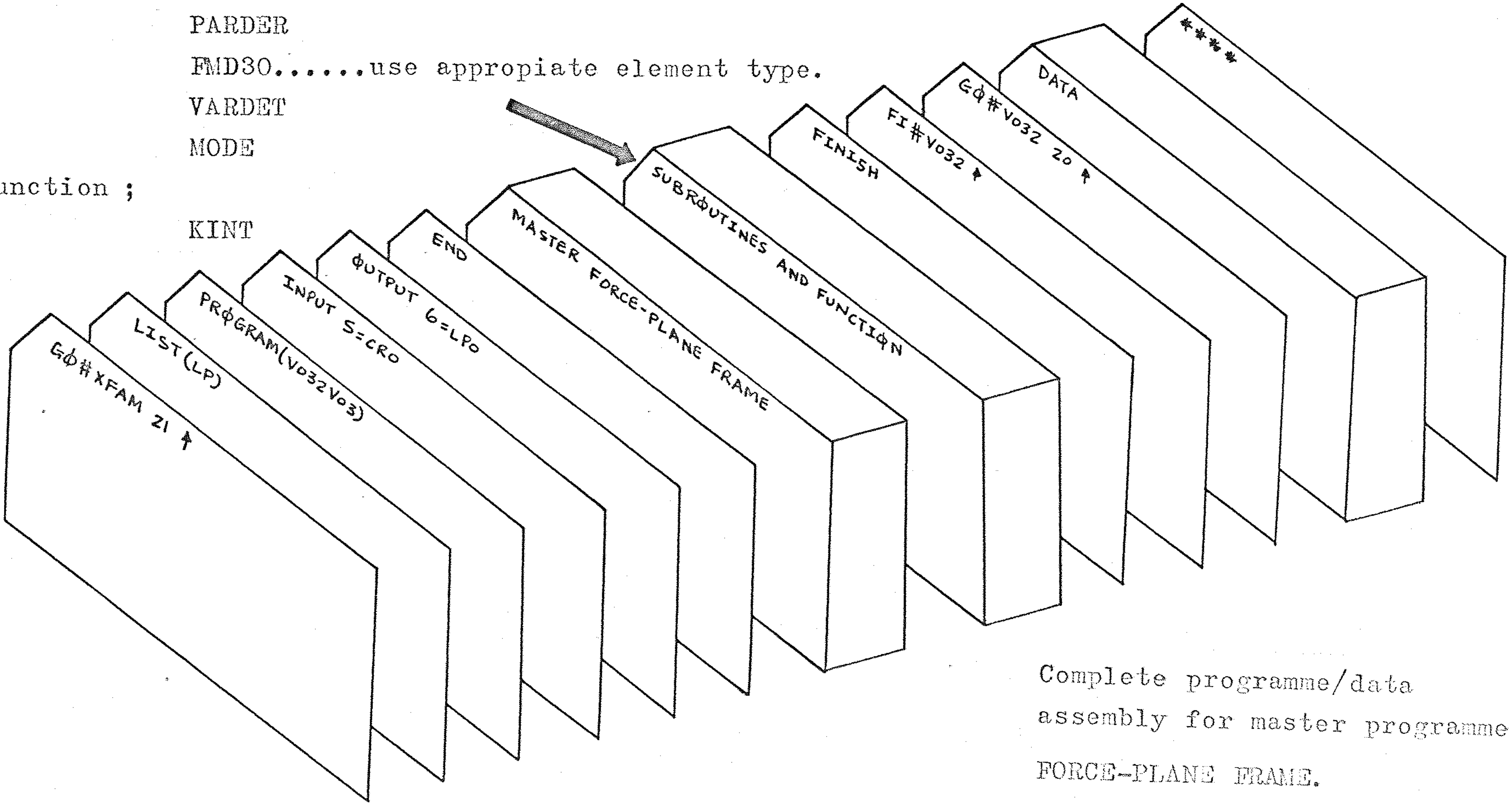
Fig. 68

Subroutines ;

FORCEF
RANTEC
REAR
MATINV
MATMULT
PARDER
FMD30......use appropiate element type.
VARDET
MODE

Function ;

KINT



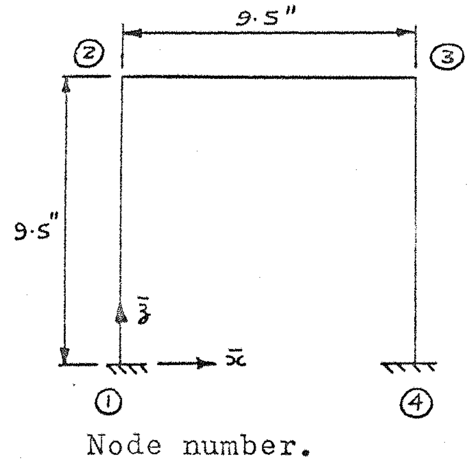
Complete programme/data assembly for master programme FORCE-PLANE FRAME.

Fig. 69 .

2.3 Example of usage.

The simple plane frame shown in figure 70 is considered as being idealized in its present form, that is, as three discrete elements. Using Element Type P3/3FD evaluate the second eigenvalue of the frame assuming a lower estimate of 760.0 radians per second.

The input data preparation for this problem is given in table 44 .



Constant frame properties.

$E = 30.0 \times 10^{6}$ lb per in^{2}.

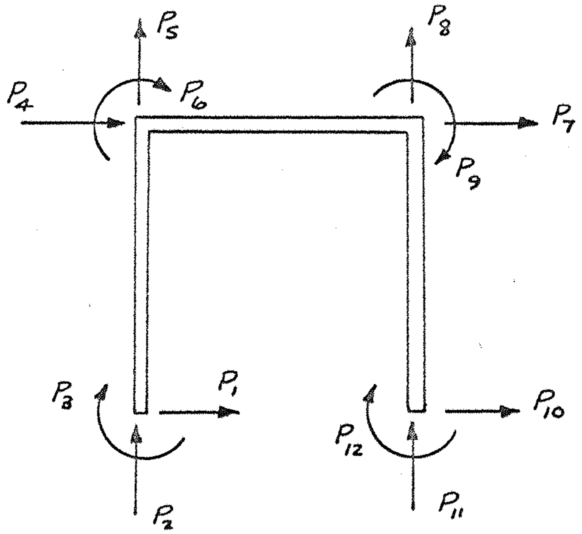
$A = 0.09396$ in^{2}.

$I = 0.00006866$ in^{4}.

$\mu = 0.283$ lb per in^{3}.

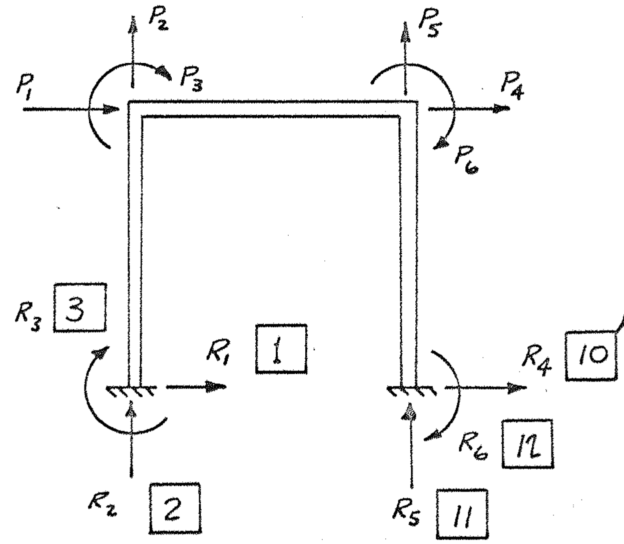
Simple plane frame.

(a)



Applied loading and numbering before imposing constraints.

(b)



Applied load number corresponding to reaction, obtained from figure (b).

Possible applied loading and numbering with reactions.

(c)

Fig. 70 .

NAME. John Robinson
DEPARTMENT. I.S.V.R.
PROBLEM. Simple plane frame. Element Type P3/3FD.
DATE. May 1967
PAGE 1 OF 1

```
1
60·0       1·0           10  8

3   4

1         0·0         0·0
2         0·0         9·5
3         9·5         9·5
4         9·5         0·0

1    2  0·00006866 30·0E+06       0·283     0·09396
2    3  0·00006866 30·0E+06       0·283     0·09396
3    4  0·00006866 30·0E+06       0·283     0·09396

   1   2   3  10  11  12
```

Table 44.

3. Master programme FORCE-RECTANGULAR PLATE.

Introduction.

The research work carried out for plate structures has been limited to in core problems. The plate programmes are very restricted and only of use for this initial research project, therefore, they will not be described in the same detail as for collinear beam and plane frame structures. These programmes were written purely to obtain some results to show that the suggested derivation procedures were valid. Further research will be continued in this area and more rigorous investigations carried out.

3.1 Plate programmes.

3.1.1 q-system.

The first attempt to derive a dynamic flexibility matrix for a rectangular plate element under shear, bending and twist was to adopt distributed boundary loads and four concentrated nodal loads as the generalized element unknowns. In order to incorporate such an element (Element Type P1/4FD) in a plate structure a method of formulating a system of joint equilibrium equations had to be established. The master programme using this type of element was written to analyse a cantilever plate structure only, this consisted of six finite elements and fixed properties. The assembled structure and loading is shown in figure 71 .

The first method investigated to formulate the joint equilibrium equations was to distribute equally the resultants of the uniformly distributed element boundary loads to the nodes. Example, for node 1,

$$\frac{a}{2} q_5 + \frac{b}{2} q_8 + q_9 - P_1 = 0$$

$$\frac{a}{2} q_1 - P_2 = 0$$

$$\frac{b}{2} q_4 - P_3 = 0$$

After completion of all node equations the assembled system is investigated using the rank technique. It was found that this system had no solution, the reason for this was that a dependency existed between certain applied loads. This can be seen clearly by investigating the moment equilibrium equations about the z-axis for nodes 2,6 and 10. Now for a = 1.5

$$0.75\, q_{38} + 0.75\, q_{52} - P_{30} = 0$$

$$0.75\, q_2 + 0.75\, q_{16} - P_6 = 0$$

and

$$0.75\, q_2 + 0.75\, q_{16} + 0.75\, q_{38} + 0.75\, q_{52} - P_{18} = 0$$

Therefore,

$$P_6 + P_{30} - P_{18} = 0$$

The next method to be investigated was that of equilibrium of orthogonal grid lines. Examples;

<u>1. Grid line 1,2,3,4.</u>

See figure 72(a).

Vertical equilibrium.

$$a(q_5 + q_{17} + q_{29}) + \tfrac{1}{2}(q_9 + q_{10} + q_{21} + q_{22} + q_{33} + q_{34}) - P_1 - P_4 - P_7 - P_{10} = 0$$

Moment equilibrium about the x-axis.

$$a(q_1 + q_{13} + q_{25}) - P_2 - P_5 - P_8 - P_{11} = 0$$

Moment equilibrium about node 1 (z-axis).

$$\frac{a^2}{2} q_5 + \frac{a}{2}(q_{10} + q_{21}) + \frac{3}{2} a^2 q_{17} + a(q_{22} + q_{33}) + \frac{5}{2} a^2 q_{29} + \frac{3}{2} a\, q_{34}$$

$$- P_3 - a P_4 - P_6 - 2a P_7 - P_9 - 3a P_{10} - P_{12} = 0$$

2. Grid line 2,6,10.

See figure 72 (b).

Vertical equilibrium.

$$\frac{1}{2}(q_{47} + q_{60}) + b(q_{42} + q_{56}) + \frac{1}{2}(q_{11} + q_{24} + q_{46} + q_{57}) + b(q_6 + q_{20})$$

$$+ \frac{1}{2}(q_{10} + q_{21}) - P_4 - P_{16} - P_{28} = 0$$

Moment equilibrium about the z-axis.

$$b(q_2 + q_{16} + q_{38} + q_{52}) - P_6 - P_{18} - P_{30} = 0$$

Moment equilibrium about node 10 (x-axis).

$$\frac{b^2}{2}(q_{42} + q_{56}) + \frac{b}{2}(q_{11} + q_{24} + q_{46} + q_{57}) + \frac{3}{2} b^2 (q_6 + q_{20}) + b(q_{10} + q_{21})$$

$$- 2b P_4 - P_5 - b P_{16} - P_{17} - P_{29} = 0$$

After assembling the equations for all grid lines the system was investigated and a solution was indicated. The rank force method was then formulated using these equations and the element dynamic flexibility matrix of Element Type P1/4FD . The listing of the master programme is given in table 45 and the corresponding FORCE-subroutine in table 46 .

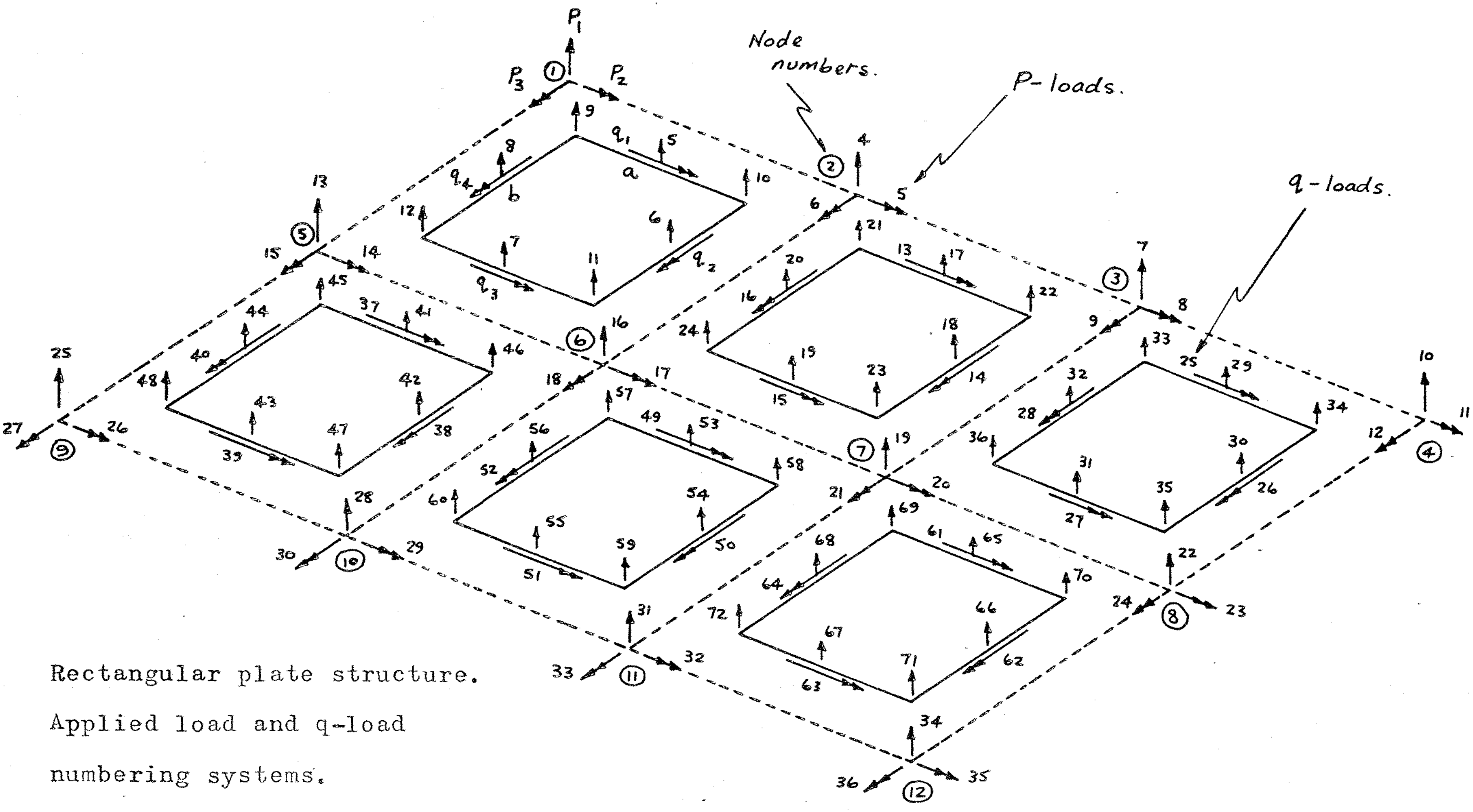
3.1.2 s-system.

Clearly, the q-system is not practical for the analysis of plate structures and an attempt was made to transform this system into a more convenient equivalent one using discrete nodal loads as the generalized unknowns.

A suggested procedure has been presented in Chapter 3 (3.1.3(b)), see also Appendix 3. The joint equilibrium equations using an s-system consist of Boolean matrices and are established simply from equilibrium of the individual joints as for collinear beam and plane frame structures. The assembled structure and loading is shown in figure 73 . The rank force method is formulated using these equations and Element Type P1/5FD. The listing of the master programme is given in table 47 and the corresponding FORCE-subroutine in table 48 .

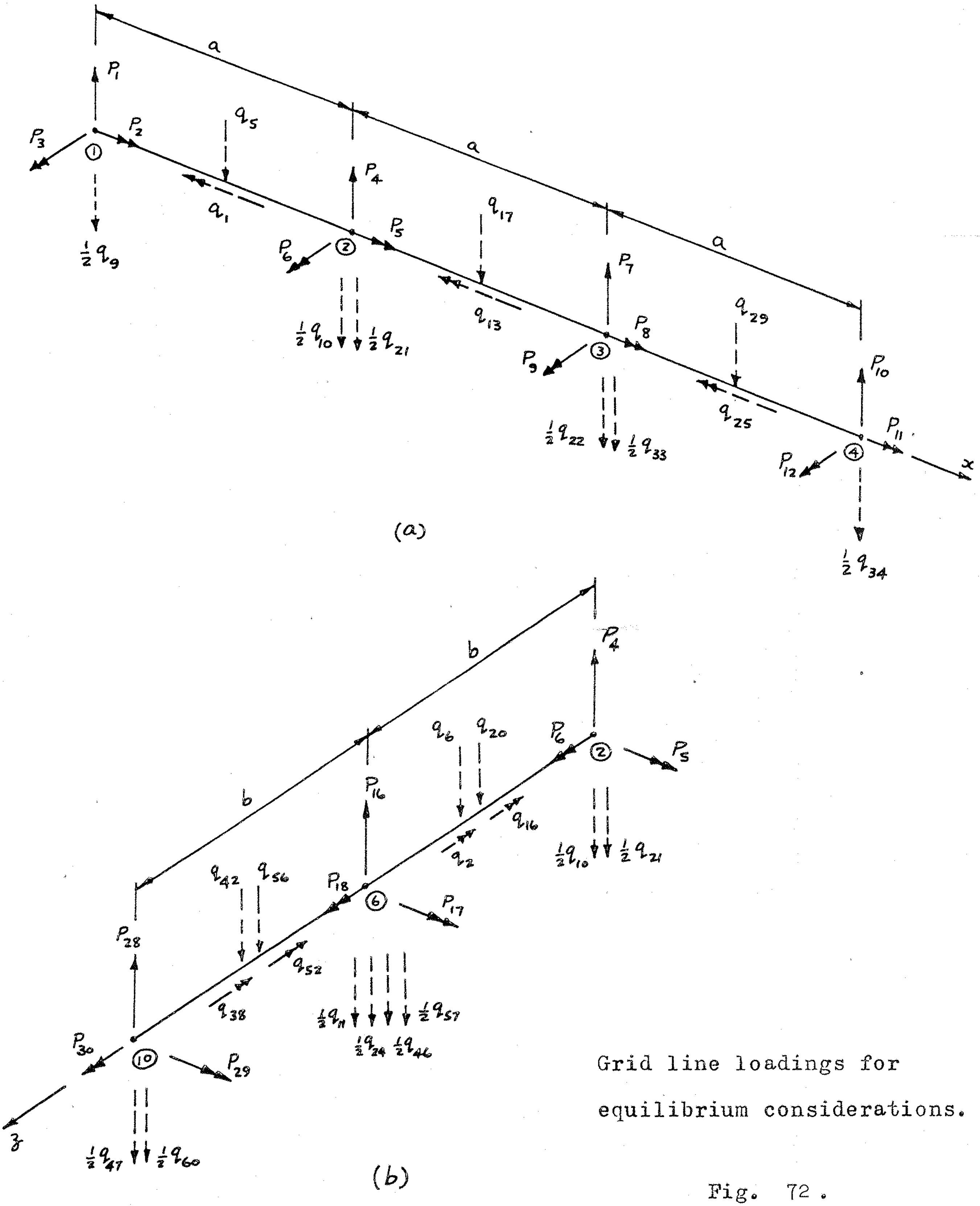
3.1.3 General comment.

It should be noted that when boundary equilibrium equations are being formulated each concentrated nodal load of the q-system is in actual fact a resultant of two equal loads due to twisting moments. See figure 74 .



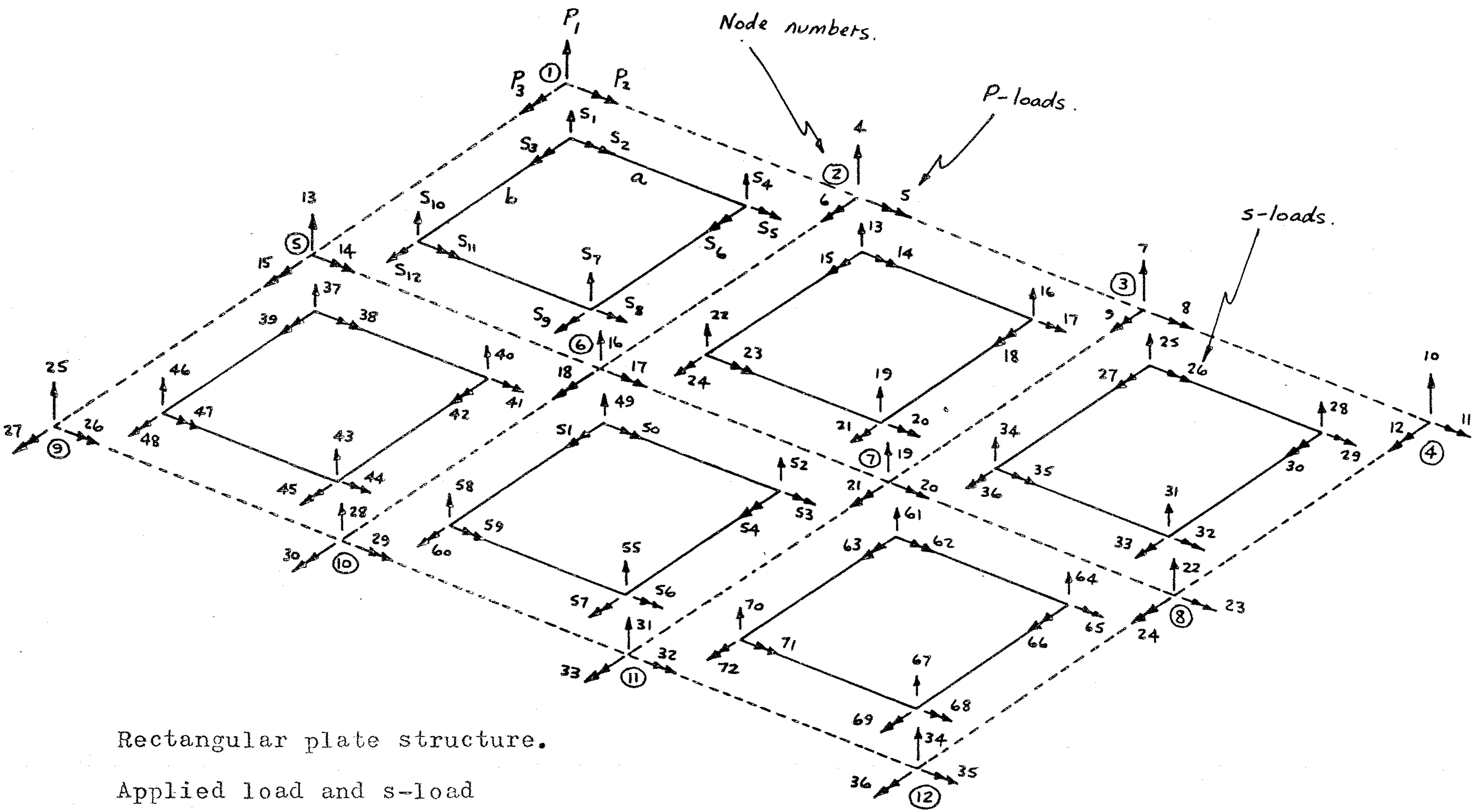
Rectangular plate structure.
Applied load and q-load
numbering systems.

Fig. 71.



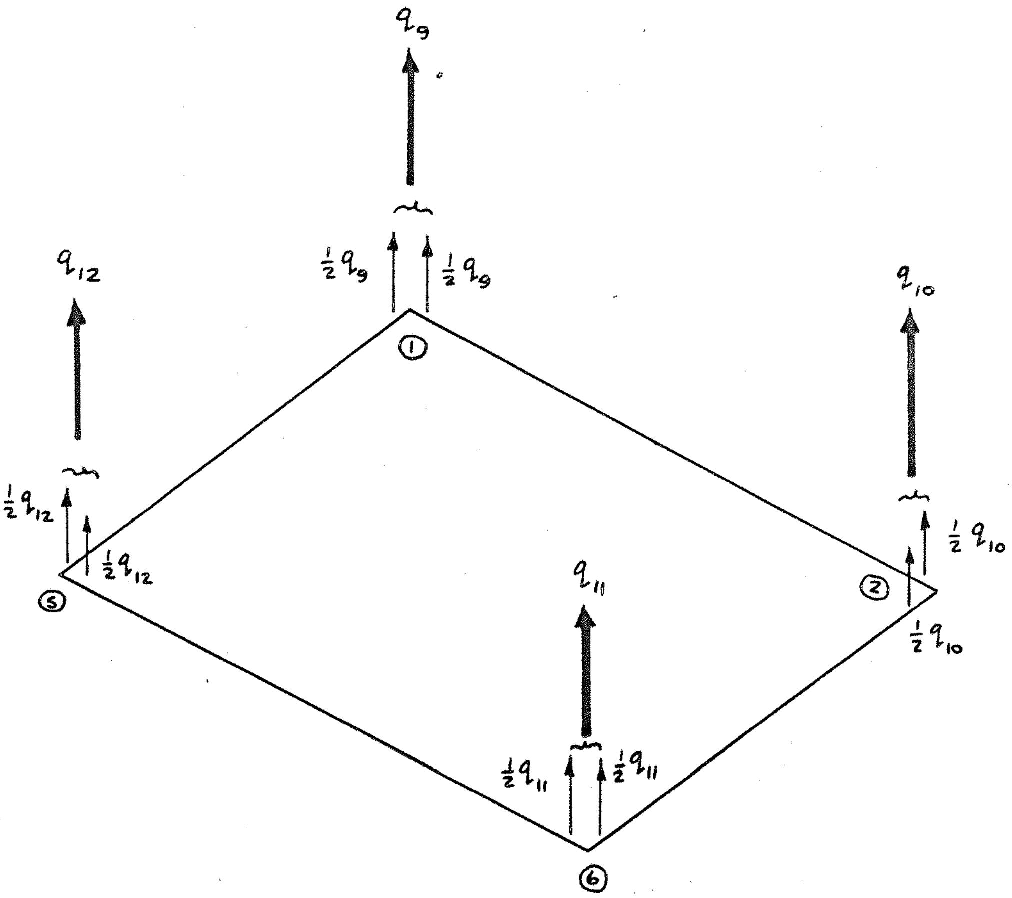
Grid line loadings for equilibrium considerations.

Fig. 72.



Rectangular plate structure.
Applied load and s-load
numbering system.

Fig. 73 .



Actual form of the concentrated nodal loads in the q-system.

Fig. 74.

A.S.A. Fortran listing of master programme FORCE-RECTANGULAR PLATE. (q-system).

Table 45 .

```
      MASTER FORCE-RECTANGULAR PLATE
C     JOHN ROBINSON.  I.S.V.R.
C     VIBRATION ANALYSIS OF TWO DIMENSIONAL RECTANGULAR PLATE
C     STRUCTURES USING THE RANK FORCE METHOD.
      DIMENSION X(36),XXAM(12),DDLAM(12),NNST(12),IIEPS(12)
      DIMENSION       UKD(36,36),DEL(84,36),F(36,36),OM(84.108),IQ(84)
      COMMON OM
      EQUIVALENCE (DEL(1,1),OM(1,1)),(UKD(1,1),OM(1,37)),(F(1.1),OM(1,58
     1))
      READ(5,85)KASES
   85 FORMAT(I1)
      DO500 KA=1,KASES
      WRITE(6,87)KA
   87 FORMAT(14H CASE NUMBER  ,I1)
      WRITE(6,406)
  406 FORMAT(23H JOHN ROBINSON I.S.V.R.)
      WRITE(6,900)
  900 FORMAT(35H PROGRAM   FORCE-RECTANGULAR PLATE.)
      READ(5,86)MR
   86 FORMAT(I1)
      READ(5,88)KE
   88 FORMAT(I2)
      WRITE(6,16)KE
   16 FORMAT(42H NUMBER OF FREQUENCIES BEING INVESTIGATED=,I4)
      READ(5,90)(XXAM(K),DDLAM(K),NNST(K),IIEPS(K),K=1,KE)
   90 FORMAT(2F12.6,I4,I3)
      WRITE(6,18)(XXAM(K),DDLAM(K),NNST(K),IIEPS(K),K=1,KE)
   18 FORMAT(23H ASSUMED FREQUENCY DATA///33H LOWER ESTIMATE STEP SIZE N
     1ST SIG///(2F20.6,I4,I3))
      NE=6
      NJ=12
      A=1,3333
      B=1,5
      T=0,19
      XMU=0,098
      XNU=0,34
      E=0,9*10,0**7
      NC=9
      DO310 K=1,KE
      XAM=XXAM(K)
      IF(MR-1)15,354,15
   15 DLAM=DDLAM(K)
      NST=NNST(K)
      IEPS=IIEPS(K)
      WRITE(6,14)K,XAM,DLAM,NST,IEPS
   14 FORMAT(15H FREQUENCY DATA,I4///5H XAM=,F12.6,3X,6H DLAM=,F12.6,3X,
     15H NST=,I4,6H IEPS=,I3)
```

Master programme FORCE-RECTANGULAR PLATE listing continued.

(q-system).

```
      CALL FORCEP(XAM,NE,NJ,T,XMU,XNU,E,A,B,NC,NL,UKD,DEL,F,MC,IQ)
      CALL VARDET(NL,NL,NL,DET1,UKD,36)
      WRITE(6,400)XAM,DET1
  400 FORMAT(17H LOWER FREQUENCY=,F12.6///13H DETERMINANT=,E15.7)
      DO101 J=1,NST
      XAM=XAM+DLAM
      CALL FORCEP(XAM,NE,NJ,T,XMU,XNU,E,A,B,NC,NL,UKD,DEL,F,MC,IQ)
      CALL VARDET(NL,NL,NL,DET2,UKD,36)
      WRITE(6,402)J,XAM,DET2
  402 FORMAT(13H STEP NUMBER=,I3///19H CURRENT FREQUENCY=,F12.6///13H DE
     1TERMINANT=,E15.7)
      IF(DET1*DET2)200,209,102
  102 DET1=DET2
  101 CONTINUE
      NST=-1

      WRITE(6,103)
  103 FORMAT(33H NO ZERO CROSSINGS FOUND IN RANGE)
      GO TO 310
C     ITERATION (PHASE 2)
  200 F1=XAM-DLAM
      F2=XAM
      IF(IEPS-8)203,203,202
  202 IEPS=8
  203 CONTINUE
      DO207 I=1,10
      IF(I-1)205,205,204
  204 F1=XAM
      DET1=DET
  205 CONTINUE
      XAM=(F1*DET2-F2*DET1)/(DET2-DET1)
      A=ALOG10(XAM)
      IEX=KINT(A)
      EPS=10.0**(IEPS-IEX-1)
      HSUB=(XAM-F1)
      IF(XAM-HSUB*EPS)206,209,209
  206 CONTINUE
      IF(I-1)250,250,255
  250 FINT=F2-XAM
      CF=0.2
  251 FR=XAM+CF*FINT
      CALL FORCEP(FR ,NE,NJ,T,XMU,XNU,E,A,B,NC,NL,UKD,DEL,F,MC,IQ)
      CALL VARDET(NL,NL,NL,DET,UKD,36)
      IF(DET*DET2)252,253,254
  252 CF=CF+0.1
      GO TO 251
  253 XAM=FR
      GO TO 209
  254 F2=FR
      DET2=DET
```

Master programme FORCE-RECTANGULAR PLATE listing continued.

(q-system).

```
  255 CALL FORCEP(XAM,NE,NJ,T,XMU,XNU,E,A,B,NC,NL,UKD,DEL,F,MC,IQ)
      CALL VARDET(NL,NL,NL,DET,UKD,36)
  207 CONTINUE
      WRITE(6,208)F1,DET1,XAM,DET
  208 FORMAT(///36H ITERATION TERMINATED AFTER 10 STEPS//22H PREVIOUS
     1ESTIMATE =,E16.8,13H DETERMINANT=,E16.8///18H CURRENT ESTIMATE=,
     2E16.8,13H DETERMINANT=,E16.8//22H CURRENT ESTIMATE USED)
  209 CONTINUE
C     VECTOR EVALUATION (PHASE 3)
  354 CALL FORCEP(XAM,NE,NJ,T,XMU,XNU,E,A,B,NC,NL,UKD,DEL,F,MC,IQ)
      IF(MR.EQ.1)GO TO 356
      CALL MODE(NL,NL,NL,UKD,36,X,XAM)
      GO TO 310
  356 IF(K.NE.1)GO TO 408
      DO410 I=1,MC
      IF(I.NE.1)GO TO 412
      WRITE(6,414)
  414 FORMAT(13H REDUNDANCIES)
  412 CONTINUE
      IF(IQ(I).EQ.0)GO TO 410
      WRITE(6,401)IQ(I)
  401 FORMAT(I6)
  410 CONTINUE
  408 WRITE(6,360)XAM
  360 FORMAT(11H FREQUENCY=,F12.6)
      WRITE(6,358)((I,J,DEL(I,J),J=1,NL),I=1,MC)
  358 FORMAT(24H UNIT MATRIX OF UNKNOWNS//3(I4,2H .,I2,3X,E19.11,2X))
      WRITE(6,359)((I,J,F(I,J),J=1,NL),I=1,NL)
  359 FORMAT(38H STRUCTURAL DYNAMIC FLEXIBILITY MATRIX///3(I4,2H .,I2,3X
     1,E19.11,2X))
      WRITE(6,350)((KK,J,UKD(KK,J),J=1,NL),KK=1,NL)
  350 FORMAT(36H STRUCTURAL DYNAMIC STIFFNESS MATRIX///3(I4,2H .,I2,3X,E
     119.11,2X))
  310 CONTINUE
  500 CONTINUE
      STOP
      END
```

A.S.A. Fortran listing of subroutine FORCEP. (q-system).

Table 46 .

```
      SUBROUTINE FORCEP(OMEGA,NE,NJ,T,XMU,XNU,E,A,B,NC,NL,UKD,DEL,F,MC,
     1IQ)
C     JOHN ROBINSON.  I.S.V.R.
C     THE RANK FORCE METHOD FOR TWO DIMENSIONAL RECTANGULAR
C     PLATE STRUCTURES. VIBRATION ANALYSIS.
      DIMENSION OM(84,108),FMD(12,12)
      DIMENSION UKD(36,36),DEL(84,36),F(36,36),IDEP(84),XMAX(84),IQ(84)
      DIMENSION DELA(12,36),DELB(36,12),FDA(12,36),DF(36,36),C(36,72)
      DIMENSION MDEP(36),WMAX(36),LQ(36),PAR(48,12),PHI(48,12),JQ(84)
      COMMON OM
      EQUIVALENCE(MDEP(1),OM(1,74)),(WMAX(1),OM(1,75)),(LQ(1),OM(1,76)),
     1(FDA(1,1),OM(1,80)),(DELB(1,1),OM(1,86)),(DELA(1,1),OM(1,92)),(DF(
     21,1),OM(1,92)),(C(1,1),OM(1,77)),(PAR(1,1),OM(1,95)),(PHI(1,1),OM(
     31,102))
      M1=12*NE
      N1=3*NJ
      LM=M1+N1
      MC=M1+NC
      NL=N1-NC
      N2=21
C     1, INITIAL NULL MATRIX
      DO12 I=1,MC
      DO12 J=1,LM
   12 OM(I,J)=0.0
C     2, EQUILIBRIUM EQUATIONS BY METHOD OF SECTIONS.
         OM(1,5),OM(1,17),OM(1,29),OM(2,1),OM(2,13),OM(2,25),OM(3,22),
     1OM(3,33),OM(4,7),OM(4,19),OM(4,31),OM(4,41),OM(4,53),OM(4,65),OM(5
     2,3),OM(5,15),OM(5,27),OM(5,37),OM(5,49),OM(5,61),OM(6,23),OM(6,36)
     3,OM(6,58),OM(6,69),OM(7,43),OM(7,55),OM(7,67),OM(8,39),OM(8,51),OM
     4(8,63),OM(9,59),OM(9,72)=1.3333
         OM(3,82),OM(6,91),OM(9,100)=-1.3333
         OM(3,10),OM(3,21),OM(6,11),OM(6,24),OM(6,46),OM(6,57),OM(9,47
     1),OM(9,60)=0,6667
         OM(3,5),OM(6,7),OM(6,41),OM(9,43)=0,8889
         OM(3,34),OM(6,35),OM(6,70),OM(9,71)=2.0
         OM(3,17),OM(6,19),OM(6,53),OM(9,55)=2,6667
         OM(3,29),OM(6,31),OM(6,65),OM(9,67)=4,4445
         OM(3,85),OM(6,94),OM(9,103)=-2.6667
         OM(3,88),OM(6,97),OM(9,106)=-4.0
         OM(10,8),OM(10,44),OM(11,4),OM(11,40),OM(12,9),OM(13,6),OM(13
     1,20),OM(13,42),OM(13,56),OM(14,2),OM(14,16),OM(14,38),OM(14,52),OM
     2(15,10),OM(15,21),OM(16,18),OM(16,32),OM(16,54),OM(16,68),OM(17,14
     3),OM(17,28),OM(17,50),OM(17,64),OM(18,22),OM(18,33),OM(19,30),OM(1
     49,66),OM(20,26),OM(20,62),OM(21,34)=1.5
         OM(12,76),OM(15,91),OM(18,94),OM(21,97)=-1.5
         OM(12,12),OM(12,45),OM(15,11),OM(15,24),OM(15,46),OM(15,57),O
     1M(18,23),OM(18,36),OM(18,58),OM(18,69),OM(21,35),OM(21,70)=0,75
```

Subroutine FORCEP listing continued. (q-system).

```
         OM(12,44),OM(15,42),OM(15,56),OM(18,54),OM(18,68),OM(21,66)=
    11,125
         OM(12,8),OM(15,6),OM(15,20),OM(18,18),OM(18,32),OM(21,30)=
    13,375
         OM(12,73),OM(15,82),OM(18,85),OM(21,88)=-3.0
         OM(1,9),OM(1,10),OM(1,21),OM(1,22),OM(1,33),OM(1.34),OM(4,11)
    1,OM(4,12),OM(4,23),OM(4,24),OM(4,35),OM(4,36),OM(4,45),OM(4,46),OM
    2(4,57),OM(4,58),OM(4,69),OM(4,70),OM(7,47),OM(7,48),OM(7,59),OM(7,
    360),OM(7,71),OM(7,72),OM(10,9),OM(10,12),OM(10,45),OM(10,48),OM(13
    4,10),OM(13,11),OM(13,21),OM(13,24),OM(13,46),OM(13,47),OM(13,57),O
    5M(13,60),OM(16,22),OM(16,23),OM(16,33),OM(16,36),OM(16,58),OM(16,5
    69),OM(16,69),OM(16,72),OM(19,34),OM(19,35),OM(19,70),OM(19,71)=0.5
         OM(1,73),OM(1,82),OM(1,85),OM(1,88),OM(2,74),OM(2,83),OM(2,86
    1),OM(2,89),OM(3,75),OM(3,84),OM(3,87),OM(3,90),OM(4,76),OM(4,91),O
    2M(4,94),OM(4,97),OM(5,77),OM(5,92),OM(5,95),OM(5,98),OM(6,78),OM(6
    3,93),OM(6,96),OM(6,99),OM(7,79),OM(7,100),OM(7.103),OM(7,106),OM(8
    4,80),OM(8,101),OM(8,104),OM(8,107),OM(9,81),OM(9,102),OM(9,105),OM
    5(9,108),OM(10,73),OM(10,76),OM(10,79),OM(11,75),OM(11.78),OM(11,81
    6),OM(12,74),OM(12,77),OM(12,80),OM(13,82),OM(13,91),OM(13,100),OM(
    714,84),OM(14,93),OM(14,102),OM(15,83),OM(15,92),OM(15,101),OM(16,8
    85),OM(16,94),OM(16,103),OM(17,87),OM(17,96),OM(17,105),OM(18,86),O
    9M(18,95),OM(18,104),OM(19,88),OM(19,97),OM(19,106),OM(20,90),OM(20
    1,99),OM(20,108),OM(21,89),OM(21,98),OM(21,107)=-1.0
C     SOLUTION OF EQUATIONS
      CALL RANTEC(OM,N2,MC,LM,84,84,108,IDEP,XMAX,IQ)
      WRITE(6,800)
  800 FORMAT(6H MARK1)
   26 CALL PARDER(OM,IQ,N2,MC,84,108,N)
      KP=14-NC-NL
      DO29 I=1,N2
      DO29 J=LM-13,LM
   29 OM(N2+I,J+KP)=OM(I,J)
C     ENERGY EQUATIONS
      CALL FMD40(A,B,T,E,XMU,XNU,OMEGA,FMD)
      DO27 I=1,12
      DO27 J=1,12
   27 FMD(I,J)=10.0**4*FMD(I,J)
      JJ=0
      DO33 M=1,NE
      DO30 I=1,N
      DO30 J=1,12
      JK=J+JJ
   30 PAR(I,J)=OM(N2+I,JK)
      CALL MATMULT(PAR,FMD,PHI,N,12,12,48,12,12)
      DO32 I=1,N
      DO32 J=1,12
      JK=J+JJ
   32 OM(I+N2,JK)=PHI(I,J)
      JJ=JJ+12
   33 CONTINUE
```

Subroutine FORCEP listing continued. (q-system).

```
      DO46 I=1,N2
      DO46 J=LM-13,LM
   46 OM(I,J)=OM(N2+I,J+KP)
      DO48 I=N2+1,MC
      DO48 J=M1+1,LM
   48 OM(I,J)=0.0
      CALL RANTEC(OM,N2+N,MC,LM,84,64,108,IDEP,XMAX,JQ)
      WRITE(6,802)
  802 FORMAT(6H MARK2)
  600 CONTINUE
      CALL REAR(OM,MC,LM,84,108,XCH)
C     DEL MATRIX
      DO34 I=1,MC
      DO34 J=MC+1,LM
      L=J-MC
   34 DEL(I,L)=-OM(I,J)
      DO25 I=1,NL
      DO25 J=1,NL
   25 F(I,J)=0.0
      II=0
      DO44 M=1,NE
      DO38 I=1,12
      DO38 J=1,NL
      IK=I+II
   38 DELA(I,J)=DEL(IK,J)
      DO40 I=1,12
      DO40 J=1,NL
   40 DELB(J,I)=DELA(I,J)
      CALL MATMULT(FMD,DELA,FDA,12,NL,12,12,36,12)
      CALL MATMULT(DELB,FDA,DF,NL,NL,12,36,36,12)
      DO42 I=1,NL
      DO42 J=1,NL
   42 F(I,J)=F(I,J)+DF(I,J)
      II=II+12
   44 CONTINUE
C     STRUCTURAL DYNAMIC STIFFNESS MATRIX
      CALL MATINV(F,UKD,C,NL,36,36,72,MDEP,WMAX,LQ)
      WRITE(6,804)
  804 FORMAT(6H MARK3)
      RETURN
      END
```

A.S.A. Fortran listing of master programme FORCE-RECTANGULAR PLATE. (s-system).

Table 47.

```
      MASTER FORCE-RECTANGULAR PLATE
C     JOHN ROBINSON.  I.S.V.R.
C     VIBRATION ANALYSIS OF TWO DIMENSIONAL RECTANGULAR PLATE
C     STRUCTURES USING THE RANK FORCE METHOD.
      DIMENSION X(36),XXAM(2),DDLAM(2),NNST(2),IIEPS(2),LSN(6,4)
      DIMENSION IC(12),UKD(36,36),DEL(84,36),F(36,36),OM(84,108),IQ(84)
      COMMON OM
      EQUIVALENCE (DEL(1,1),OM(1,1)),(UKD(1,1),OM(1,37)),(F(1,1),OM(1,58
     1))
      READ(5,85)KASES
   85 FORMAT(I1)
      DO500 KA=1,KASES
      WRITE(6,87)KA
   87 FORMAT(14H CASE NUMBER  ,I1)
      WRITE(6,406)
  406 FORMAT(23H JOHN ROBINSON I.S.V.R.)
      WRITE(6,900)
  900 FORMAT(35H PROGRAM   FORCE-RECTANGULAR PLATE.)
      READ(5,88)KE
   88 FORMAT(I2)
      WRITE(6,16)KE
   16 FORMAT(42H NUMBER OF FREQUENCIES BEING INVESTIGATED=,I4)
      READ(5,90)(XXAM(K),DDLAM(K),NNST(K),IIEPS(K),K=1,KE)
   90 FORMAT(2F12.6,I4,I3)
      WRITE(6,18)(XXAM(K),DDLAM(K),NNST(K),IIEPS(K),K=1,KE)
   18 FORMAT(23H ASSUMED FREQUENCY DATA///33H LOWER ESTIMATE STEP SIZE N
     1ST SIG///(2F20.6,I4,I3))
      READ(5,700)NE,NJ
  700 FORMAT(I2,I4)
      WRITE(6,800)NE,NJ
  800 FORMAT(20H NUMBER OF ELEMENTS=,I3//18H NUMBER OF JOINTS=,I3)
      READ(5,702)((LSN(I,J),J=1,4),I=1,NE)
  702 FORMAT(I2,3I4)
      WRITE(6,802)((LSN(I,J),J=1,4),I=1,NE)
  802 FORMAT(25H ELEMENT SPECIFYING NODES//(4I4))
      READ(5,704)T,XMU,XNU,E
  704 FORMAT(3F10.3,E9.1)
      WRITE(6,804)T,XMU,XNU,E
  804 FORMAT(17H PANEL THICKNESS=,F7.4//18H MATERIAL DENSITY=,F7.4//17H
     1POISSON'S RATIO=,F7.4//17H YOUNG'S MODULUS=,E9.1)
      READ(5,706)A,B
  706 FORMAT(2F10.3)
      WRITE(6,806)A,B
  806 FORMAT(21H FINITE ELEMENT SIZES//3H A=,F7.3//3H B=,F7.3)
      READ(5,708)NC,(IC(N),N=1,NC)
  708 FORMAT(I2,12I4)
```

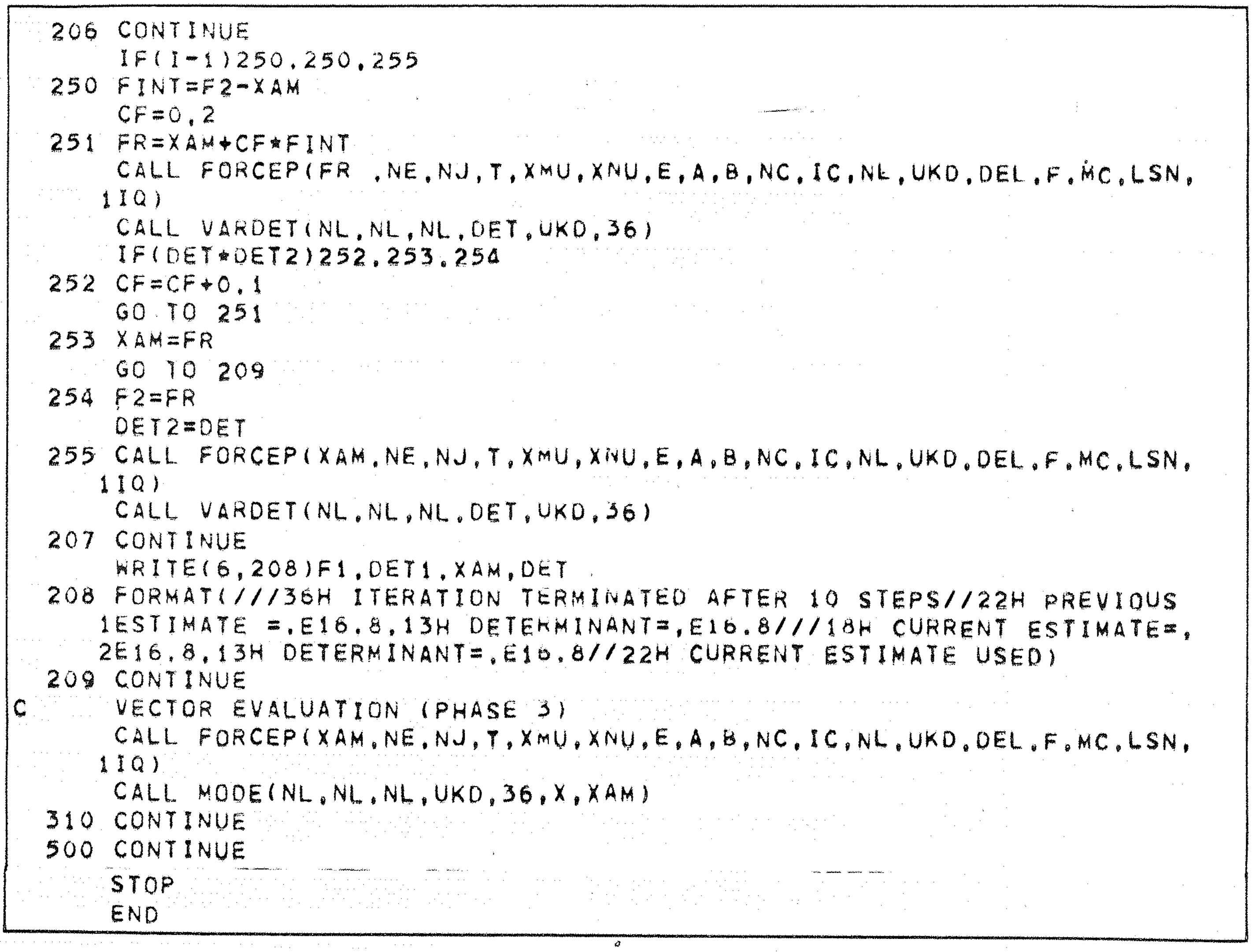
Master programme FORCE-RECTANGULAR PLATE listing continued.

(s-system).

```
      WRITE(6,808)NC,(IC(N),N=1,NC)
  808 FORMAT(23H NUMBER OF CONSTRAINTS=,I4///20H IMPOSED CONSTRAINTS///(
     112I4))
      DO310 K=1,KE
      XAM=XXAM(K)
   15 DLAM=DDLAM(K)
      NST=NNST(K)
      IEPS=IIEPS(K)
      WRITE(6,14)K,XAM,DLAM,NST,IEPS
   14 FORMAT(15H FREQUENCY DATA,I4///5H XAM=,F12.6,3X,6H DLAM=,F12.6,3X,
     15H NST=,I4,6H IEPS=,I3)
      CALL FORCEP(XAM,NE,NJ,T,XMU,XNU,E,A,B,NC,IC,NL,UKD,DEL,F,MC,LSN,
     1IQ)
      CALL VARDET(NL,NL,NL,DET1,UKD,36)
      WRITE(6,400)XAM,DET1
  400 FORMAT(17H LOWER FREQUENCY=,F12.6///13H DETERMINANT=,E15.7)
      DO101 J=1,NST
      XAM=XAM+DLAM
      CALL FORCEP(XAM,NE,NJ,T,XMU,XNU,E,A,B,NC,IC,NL,UKD,DEL,F,MC,LSN,
     1IQ)
      CALL VARDET(NL,NL,NL,DET2,UKD,36)
      WRITE(6,402)J,XAM,DET2
  402 FORMAT(13H STEP NUMBER=,I3///19H CURRENT FREQUENCY=,F12.6///13H DE
     1TERMINANT=,E15.7)
      IF(DET1*DET2)200,209,102
  102 DET1=DET2
  101 CONTINUE
      NST=-1
      WRITE(6,103)
  103 FORMAT(33H NO ZERO CROSSINGS FOUND IN RANGE)
      GO TO 310
C     ITERATION (PHASE 2)
  200 F1=XAM-DLAM
      F2=XAM
      IF(IEPS-8)203,203,202
  202 IEPS=8
  203 CONTINUE
      DO207 I=1,10
      IF(I-1)205,205,204
  204 F1=XAM
      DET1=DET
  205 CONTINUE
      XAM=(F1*DET2-F2*DET1)/(DET2-DET1)
      A=ALOG10(XAM)
      IEX=KINT(A)
      EPS=10.0**(IEPS-IEX-1)
      HSUB=(XAM-F1)
      IF(XAM-HSUB*EPS)206,209,209
```

Master programme FORCE-RECTANGULAR PLATE listing continued.

(s-system).



```
  206 CONTINUE
      IF(I-1)250,250,255
  250 FINT=F2-XAM
      CF=0,2
  251 FR=XAM+CF*FINT
      CALL FORCEP(FR  ,NE,NJ,T,XMU,XNU,E,A,B,NC,IC,NL,UKD,DEL,F,MC,LSN,
     1IQ)
      CALL VARDET(NL,NL,NL,DET,UKD,36)
      IF(DET*DET2)252,253,254
  252 CF=CF+0,1
      GO TO 251
  253 XAM=FR
      GO TO 209
  254 F2=FR
      DET2=DET
  255 CALL FORCEP(XAM,NE,NJ,T,XMU,XNU,E,A,B,NC,IC,NL,UKD,DEL,F,MC,LSN,
     1IQ)
      CALL VARDET(NL,NL,NL,DET,UKD,36)
  207 CONTINUE
      WRITE(6,208)F1,DET1,XAM,DET
  208 FORMAT(///36H ITERATION TERMINATED AFTER 10 STEPS//22H PREVIOUS
     1ESTIMATE =,E16.8,13H DETERMINANT=,E16.8///18H CURRENT ESTIMATE=,
     2E16.8,13H DETERMINANT=,E16.8//22H CURRENT ESTIMATE USED)
  209 CONTINUE
C     VECTOR EVALUATION (PHASE 3)
      CALL FORCEP(XAM,NE,NJ,T,XMU,XNU,E,A,B,NC,IC,NL,UKD,DEL,F,MC,LSN,
     1IQ)
      CALL MODE(NL,NL,NL,UKD,36,X,XAM)
  310 CONTINUE
  500 CONTINUE
      STOP
      END
```

A.S.A. Fortran listing of subroutine FORCEP. (s-system).

Table 48 .

```
      SUBROUTINE FORCEP(OMEGA,NE,NJ,T,XMU,XNU,E,A,B,NC,IC,NL,UKD,DEL,F,
     1MC,LSN,IQ)
C     JOHN ROBINSON.  I.S.V.R.
C     THE RANK FORCE METHOD FOR TWO DIMENSIONAL RECTANGULAR
C     PLATE STRUCTURES, VIBRATIUN ANALYSIS,
      DIMENSION LN(6,12),OM(84,108),LSN(6,4),FMD(12,12),IC(12)
      DIMENSION UKD(36,36),DEL(84,36),F(36,36),IDEP(84),XMAX(84),IQ(84)
      DIMENSION DELA(12,36),DELB(36,12),FDA(12,36),DF(36,36),C(36,72)
      DIMENSION MDEP(36),WMAX(36),LQ(36),PAR(48,12),PHI(48,12)
      COMMON OM
      EQUIVALENCE(MDEP(1),OM(1,74)),(WMAX(1),OM(1,75)),(LQ(1),OM(1,76)),
     1(FDA(1,1),OM(1,80)),(DELB(1,1),OM(1,86)),(DELA(1,1),OM(1,92)),(DF(
     21,1),OM(1,92)),(C(1,1),OM(1,77)),(PAR(1,1),OM(1,95)),(PHI(1,1),OM(
     31,102))
C     MATRIX OF ELEMENT LOAD NUMBERS
      N=0
      DO10 M=1,NE
      DO10 NN=1,12
      N=N+1
      LN(M,NN)=N
   10 CONTINUE
C     1. INITIAL NULL MATRIX
      M1=12*NE
      N1=3*NJ
      LM=M1+N1
      MC=M1+NC
      NL=N1-NC
      DO12 I=1,MC
      DO12 J=1,LM
   12 OM(I,J)=0.0
C     2. JOINT EQUILIBRIUM EQUATIONS
      LL=1
      DO18 JN=1,NJ
      DO15 M=1,NE
      DO17 L=1,4
      IF(LSN(M,L)-JN)17,16,17
   16 GO TO(19,20,21,22),L
   19 LLL=1
      GO TO 3
   20 LLL=4
      GO TO 3
   21 LLL=7
      GO TO 3
   22 LLL=10
    3 OM(LL,LN(M,LLL))=1.0
      OM(LL+1,LN(M,LLL+1))=1.0
      OM(LL+2,LN(M,LLL+2))=1.0
      GO TO 15
```

Subroutine FORCEP listing continued. (s-system).

```
   17 CONTINUE
   15 CONTINUE
      LL=LL+3
   18 CONTINUE
C     REACTIONS AND APPLIED LOADS
      IF(NC.EQ.0)GO TO 4
      NNJ=1
      DO6 N=1,NC
    6 OM(IC(N),M1+N)=-1.0
      DO7 I=1,N1
      DO9 N=1,NC
      IF(I.EQ.IC(N))GO TO 7
    9 CONTINUE
      OM(I,MC+NNJ)=-1.0
      NNJ=NNJ+1
    7 CONTINUE
      GO TO 5
    4 DO23 I=1,N1
      M5=MC+I
      DO23 J=M5,M5
   23 OM(I,J)=-1.0
    5 CONTINUE
C     SOLUTION OF EQUATIONS
      CALL RANTEC(OM,N1,MC,LM,84,84,108,IDEP,XMAX,IQ)
      WRITE(6,800)
  800 FORMAT(6H MARK1)
   26 CALL PARDER(OM,IQ,N1,MC,84,108,N)
      KP=14-NC-NL
      DO29 I=1,N1
      DO29 J=LM-13,LM
   29 OM(N1+I,J+KP)=OM(I,J)
C     ENERGY EQUATIONS
      CALL FMD50(A,B,T,E,XMU,XNU,OMEGA,FMD)
      DO27 I=1,12
      DO27 J=1,12
   27 FMD(I,J)=10.0**4*FMD(I,J)
      JJ=0
      DO33 M=1,NE
      DO30 I=1,N
      DO30 J=1,12
      JK=J+JJ
   30 PAR(I,J)=OM(N1+I,JK)
      CALL MATMULT(PAR,FMD,PHI,N,12,12,48,12,12)
      DO32 I=1,N
      DO32 J=1,12
      JK=J+JJ
   32 OM(I+N1,JK)=PHI(I,J)
      JJ=JJ+12
   33 CONTINUE
      DO46 I=1,N1
      DO46 J=LM-13,LM
```

Subroutine FORCEP listing continued. (s-system).

```
   46 OM(I,J)=OM(N1+I,J+KP)
      DO48 I=N1+1,MC
      DO48 J=M1+1,LM
   48 OM(I,J)=0.0
      CALL RANTEC(OM,N1+N,MC,LM,84,84,108,IDEP,XMAX,IQ)
      WRITE(6,802)
  802 FORMAT(6H MARK2)
  600 CONTINUE
      CALL REAR(OM,MC,LM,84,108,XCH)
C     DEL MATRIX
      DO34 I=1,MC
      DO34 J=MC+1,LM
      L=J-MC
   34 DEL(I,L)=-OM(I,J)
      DO25 I=1,NL
      DO25 J=1,NL
   25 F(I,J)=0.0
      II=0
      DO44 M=1,NE
      DO38 I=1,12
      DO38 J=1,NL
      IK=I+II
   38 DELA(I,J)=DEL(IK,J)
      DO40 I=1,12
      DO40 J=1,NL
   40 DELB(J,I)=DELA(I,J)
      CALL MATMULT(FMD,DELA,FDA,12,NL,12,12,36,12)
      CALL MATMULT(DELB,FDA,DF,NL,NL,12,36,36,12)
      DO42 I=1,NL
      DO42 J=1,NL
   42 F(I,J)=F(I,J)+DF(I,J)
      II=II+12
   44 CONTINUE
C     STRUCTURAL DYNAMIC STIFFNESS MATRIX
      CALL MATINV(F,UKD,C,NL,36,36,72,MDEP,WMAX,LQ)
      WRITE(6,804)
  804 FORMAT(6H MARK3)
      RETURN
      END
```

APPENDIX 5.

PRACTICAL APPLICATION OF AN AIRCRAFT STATIC STRUCTURAL ANALYSIS SYSTEM.

Synopsis.

The first part of this appendix discusses practical structural idealization, that is, the transformation of an actual structure into a practical model. Examples of typical aircraft structural idealizations are given. The practical model is established so that it meets the requirements of the computer programme which will be used to analyse the given structure. In order to use a structural computer programme certain data are prepared and input to the programme. Typical data required and comments on its preparation are given, examples are used for further clarification. The general discussion is based on static structural analysis but it is equally applicable to structural vibration analysis. The discussions are typical of a practical computerized system used in the aircraft industry, however, even at the research stage of analysis one should consider the possible future application. Initial structural research using finite element techniques is carried out using simple structural models, even then the work has to be computerized, particularly for vibration investigations. Therefore, a great deal of the discussion for practical analysis is relevant at the research stage since this helps in developing the work from the standpoint of theoretical formulation, numerical techniques and computer programme formulation.

Introduction.

To analyse a structure using finite element techniques requires a computerized system. In aircraft engineering, and to a lesser extent in civil engineering, computers are actively employed for development and design purposes. It is essential that an engineer thinks in terms of computerized design since a great deal of laborious work can be removed from his normal activities, thus allowing more time for new thinking. A computerized system enables many more alternative designs to be considered in a much shorter time than would normally be possible. A large number of designs are carried out using experience and by comparison with similar previously designed structures. When the structural configuration is a new concept experience in selecting the critical loading case for a particular piece of structure becomes questionable. This is when a computerized system is most effective. On the other hand one must appreciate that all development and design activities are not easy to computerize, if at all, and in any case the results obtained from any computerized system are influenced very strongly by engineering judgement. The biggest disadvantage of a computerized system is in acquiring one in the first place, and having acquired one establishing confidence in its capabilities. It must then be maintained to meet changing requirements. To develop a comprehensive system capable of analysing large structures efficiently costs a great deal of money and takes many years to write and check out. The development of such a system also requires experienced and qualified theorists, engineers and computer programmers. An adequate computer and supporting devices are also essential including a reliable staff. In order to apply a computer programme it is absolutely necessary to have good documentation for the users manual. This point cannot be emphasized enough. Certain checks can be incorporated into a programme but it is impossible to include a number of important ones. They are, ensuring that the structural element properties (cross sectional area, second moment of area, plate thickness, material constants), nodal coordinates and applied loads are correct or that the structural constraints have been imposed in the right manner. This responsibility lies with the engineer who is preparing the programme input data. Much money is wasted on bad computer runs because of these types of errors and wrong interpretation of the users manual. If an analysis was run on an I.B.M. 7094 Computer which took one hour to obtain a solution it would cost $450(£161 as a straight conversion). If the input was wrong this would be a write-off, and this doesn't include the delay in obtaining the results.

To try and give at least some indication of computer utilization for structural analysis in the aircraft industry a world survey was undertaken by the author[30,31].

Some of the results are shown in figure 75 . This figure shows the computer running time for structural analysis as a percentage of the total computer running time, denoted by C%. The numbers in parentheses give the years over which the percentage is based. In the survey it was requested that ;

1. The computer running time for structural analysis should include the phases ;
    - 1.1 Stress and deflection distributions.
    - 1.2 Generation of the structural flexibility or stiffness matrix for dynamic analysis, based on lumped masses.
    - 1.3 Phases 1.1 and 1.2 should include research, development and production work.
2. The total computer running time should be based on engineering work only, that is, stress dynamics, aerodynamics, weights and loading.

Figure 75 can be misleading and references 30 and 31 should be consulted for further details, limitations and remarks for the survey.

A5.1 Practical structural idealization.

Probably the most important step in an analysis is the transformation of an actual structure into a practical model, this is known as "structural idealization". The degree and nature of the idealization obviously depends on the computer available, programme capabilities, type of structure being analysed, time and money allocated and the desired accuracy of the results. However, the most valuable asset in structural idealization is engineering judgement. The first step in an analysis is to obtain drawings of the structure to be analysed and with members of the various groups involved discuss the structural problem. Points for discussion are ;

1. Stage of design.

   If the structure is at the project stage the concept will be continually changing. Under these circumstances the idealization would not be as rigorous as in a production analysis.

2. Hierarchy of importance.

   If for a particular analysis critical structural areas exist then these will influence the idealization. In areas of importance a more rigorous idealization is adopted, this would include using a larger number of discrete (finite) elements and a more complex structural element representation as compared with the unimportant areas.

3. Loading cases.

   Much effort can be saved if the critical loading cases for a particular structure can be isolated. This is not always possible, particularly with new concepts.

4. Results required.

   Time can be saved, both manually and computerwise, by only asking for results which are essential. Some items which can also be taken advantage of, if applicable, are ;

   4.1 Material of structural elements is constant.

   4.2 Only certain structural displacements are required.

   4.3 No vibration characteristics are requested.

   4.4 Constant temperature enviroment.

   4.5 Using as simple an element representation as possible which is compatible with the immediate requirements.

   4.6 No element stresses are required.

Numerous saving devices exist but are obviously dependent on the programme capabilities. A large amount of work is required to idealize a structure but the effort can be reduced by good work planning. If many loading cases are to be considered each case will generally design a particular part of the structure. By careful consideration a few idealizations can be used to satisfy many loading cases. In the case of cellular semi-monocoque structures the section properties of the idealized structure should be the same as the actual structure. However, for this type of structure, this requirement cannot be satisfied, a numerical example showing this is given in reference 31. Therefore, when idealizing such a structure the loading case becomes a strong criteria. For example, in aircraft design if a "rolling case" is being considered in the design of a fin then it is desirable that the section properties about the rolling axis are correct; in this case the section properties about the pitching axis would be in error, but this is perhaps not too drastic.

To discuss structural idealization further consider an aircraft wing, see figures 76 to 80 . Typical endload carrying elements are the stringers, rib caps, spar caps and, depending on the element representation, the panels. To idealize a wing structure stringers are lumped together to form an equivalent endload carrying member. If the plate type elements are assumed to carry shear only then an equivalent amount of skin area is added to form the equivalent area, see figure 81. Reference 31 should be consulted for further details. In the case of a wing surface design a grid system is formed by the equivalent endload carrying elements and chordwise ribs, plate elements being bounded by the grid lines, see figures 76 and 77 . The intersection of such lines designate nodal points, however, if for example a displacement was required at a point were no intersection of elements existed a nodal point can be established by using ficticious grid lines. These ficticious lines don't represent endload carrying elements but are used to advantage in acquiring information. Therefore, nodal points designate junction points, load application points, and any other points were information is desired.

Having established an idealization the nodal points are numbered. The method of numbering is something which improves with experience, in more sophisticated structural computer programmes the numbering can be quite arbitrary but avoid using the same number twice. However, even in the general programme a well prepared numbering system can save considerable time and increase the size of problem which can be analysed. To prepare an efficient numbering system requires more concentrated effort from the engineer but once a routine is established, by experience, there will be very little difference in effort compared with numbering in an arbitrary manner. One convenient guide line is to number the nodes in as cyclic a manner as possible. Figures 76, 77, 78, 79 and 80 show the idealization and numbering system for various parts of a project wing design. These were established by the author at the Boeing Company, Seattle, U.S.A. for static and lumped mass vibration analyses. Other idealizations are shown in figures 82 and 83. When idealizing a structure the placing of a node on an element boundary which is not one of the element specifying nodes must be avoided. This error often occurs, particularly when first meeting the subject of idealization. This error can be seen more clearly by referring to figure 84 . One method of avoiding this error is shown in figure 85 , the plate element 1,10,2,12 is now subdivided into two plate elements, 1,10,11 and 1,11,2,12.

In selecting the various types of elements for an idealization the engineer has to be very careful that he selects the best type possible for a particular analysis. The type of element adopted can affect the results considerably and one can see why engineering judgement and experience are an asset. Choosing the best elements for an idealization presents a problem which only lessens with experience based on continual usage and in many cases on "suck it and see" approaches. Mr. J. Rotter, Dynamics Group, Airplane Division, The Boeing Company, Seattle, U.S.A., carried out a very useful study using a cantilever spar structure, figure 86, and the displacement approach of analysis. A computer programme called "COSMOS"[35] was used to obtain the results. Three types of element representations were used ;

1. Element stiffness matrix derived by application of beam theory but extending the derivation to include shear web flexibility. Element type A, see figure 87(a).
2. Element stiffness matrix derived by assembling

the stiffnesses of constant endload elements and a web. Continuity between web and chords does not exist between nodes. Element type B, see figure 87(b).

3. Element stiffnesses derived by assembling the stiffnesses of constant endload elements and an isotropic plate element. The stiffness of the isotropic plate element is itself assembled using four isotropic triangular plate elements.[10] Continuity between web and chords does not exist between nodes. Element type C, see figure 87(c).

The vertical deflection of the free end of the spar shown in figure 86 was evaluated using the various representations and two applied loading systems. This deflection is compared with that obtained using engineers bending theory for a range of element aspect ratios, that is, span of element/ depth of element. The results are shown in figures 88 and 89. Therefore, when selecting the various structural element representations it is essential to consider the applied loading system. Further, if for example the generalized boundary vectors for elements meeting at a particular node did not contain moments or rotations this would mean that no moments can be applied to the structure at this particular node.

A5.2 Programme input data.

Having established an idealization, numbering system and applied loading system(from the loading group) the next step is to prepare the computer programme input data. The best method of presenting data to a keypunch operator is to write the data down on input sheets which are arranged similar to an enlarged computer data card, figure 90. The data is written according to a predetermined format, decided by the programmer and coordinating engineer. One method is to write the various input parameters in certain specified columns on the input sheet, the number of columns being chosen to accomodate the largest expected value. This approach works but a more convenient method which is less prone to error is to write down the parameters allocated to a line (card) as they come but separating the individual parameters by a comma. A blank could be used. The completion of data on a card is indicated by, say, two commas. Figure 91 shows data prepared to a predetermined format and figure 92 shows data separated by commas. By comparing the two forms it can be seen that the use of commas reduces the chance of error. With the former presentation one is inclined to use a different type of input sheet for each kind of data such that the columns to be used for the various parameters can be clearly marked and perhaps titled. The latter presentation gives a standard input sheet for all data and also reduces the computer time for reading data. It should be noted from figures 91 and 92 that some parts of the input data are written with a decimal point and others without. This is very important and the programme users manual should be carefully read regarding this point.

Input data for a static/vibration structural analysis would include ;

1. Nodal data (nodal point numbers and coordinates relative to a fixed set of axes). One node per card.
2. Element data (type designation, specifying nodes, section properties and material properties). One element per card.
3. Structural constraints (number of constraints, degrees of freedom to be constrained or reactions).
4. Generalized applied load system (node, generalized loads). One node loading per card.
5. Frequency data (number of frequencies, frequency parameters).

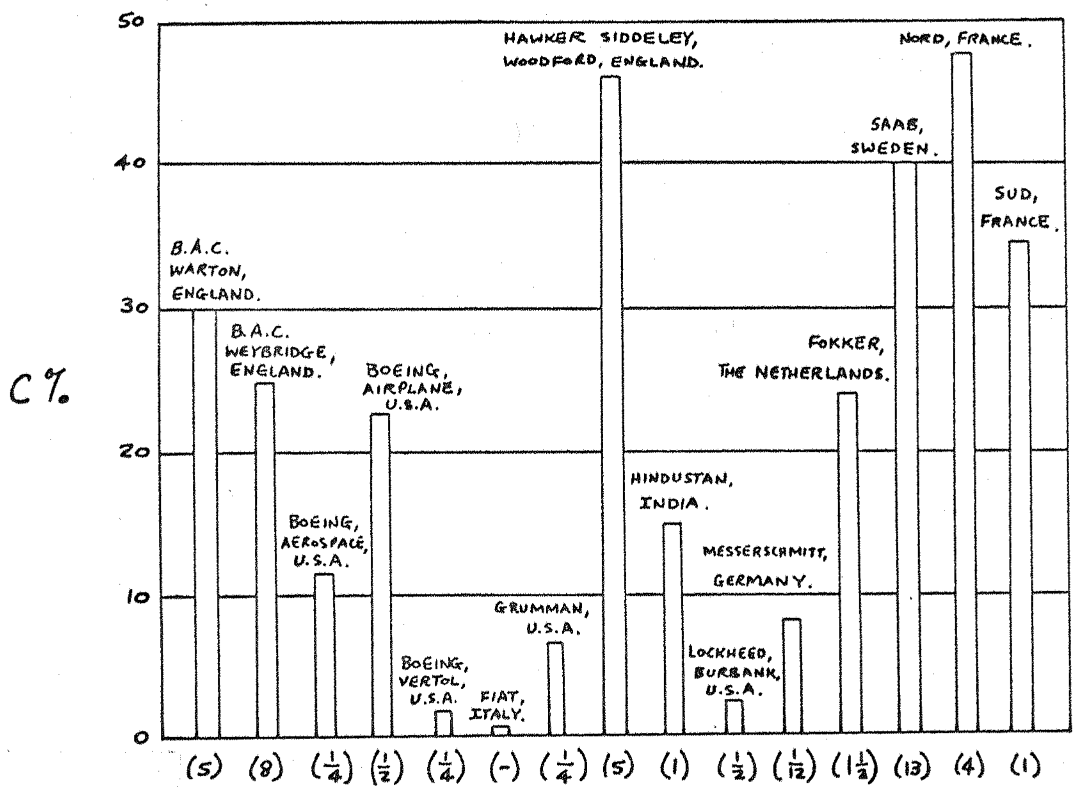
The nodal data is punched as one node per card, the element data as one element per card, the applied loading system as one node loading per card. This form of presentation enables amendments to be made readily and conveniently to the data. This can be extended to the other forms of data.

When writing down the element specifying nodes it is preferable to be systematic. One successful procedure will now be described. Start at the first node, say 1, the node numbering need not commence at 1, and write down the specifying nodes for all elements meeting at this node, moving in as cyclic a manner as possible. After an element is accounted for mark it on an idealization drawing, this helps in the book keeping. Next move to node 2 and write down the specifying nodes for elements meeting at this node, some elements may have been accounted for by consideration of the previous node. This is were a marking system helps. Continue this procedure until all nodes have been exhausted. The final system of element specifying nodes will now be orderly since the first specifying nodes will be in sequence. All elements with node 1 in its specifying nodes will come first, then those with node 2 (if not previously accounted for by node 1), then node 3, and so on. This helps in checking the work and is most convenient when making amendments. An example is given in table 49 which is compiled for the cantilever box structure shown in figure 93 . Having written down the element specifying nodes the element data can now be completed systematically. In writing down the element data many errors can be avoided and time saved by having two people doing this systematic procedure, that is, one reading and one writing. A simple check should now be made, sum the number of structural elements from the data, do the same using the idealization, and the two summations should be the same. This check has saved or produced many red faces. As a final check the whole procedure should be repeated without actually writing down the data but simply changing partners, that is, the person who was originally writing now does the reading, and going through the motions, checking with the previously compiled data. This is a boring task but for the time it takes compared with hte computer running time and cost, and schedule delays it is essential. All other data should have similar checks.

When all data are completed they are transferred from input sheets to computer data cards by a keypunch operator. This now presents a further and very common source of error. Therefore, a listing of the punched data should be obtained before running the programme and checked against the initial input sheet data. Further discussion on structural idealization and practical computer programme application can be found in references 31 and 10.

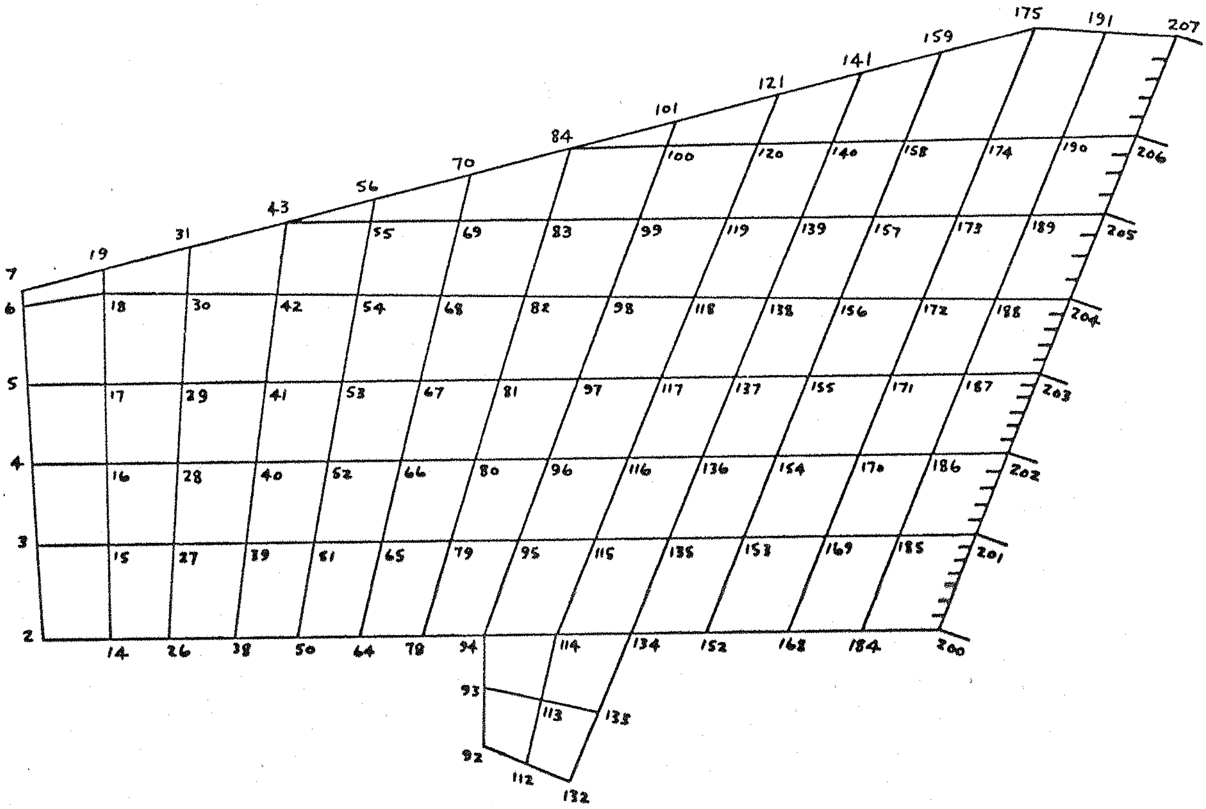
A5.3 Research stage of structural analysis.

The foregoing discussion has been concerned with practical computerized structural analysis. However, even at the research stage of structural analysis one should consider the possible future application, which is of course in a practical analysis system. Initial structural research using finite element techniques is carried out using simple models. Even then the work has to be computerized. Therefore, a great deal of the discussion for practical analysis applys at the research stage since this helps in developing the work from the standpoint of theoretical formulation, numerical techniques and computer programme formulation. Once a system has been shown to work on simple models it can then be expanded to large configurations. When carrying out research in new areas of analysis many unforseen problems present themselves. These may be theoretical, numerical or programming. Adopting simple structural models for initial research appears to be ideal since one can follow the various steps of an analysis more readily. Also, the more common simple structural models have either 'exact' solutions or have been analysed using alternative approximate procedures. Simple models can also be investigated experimentally at low cost. There is no point in going to large configurations until a new concept or new development has been tried out on simple known solution problems. To validate the theoretical work presented in this thesis a computerized structural analysis system was written and developed to analyse simple structural configurations.



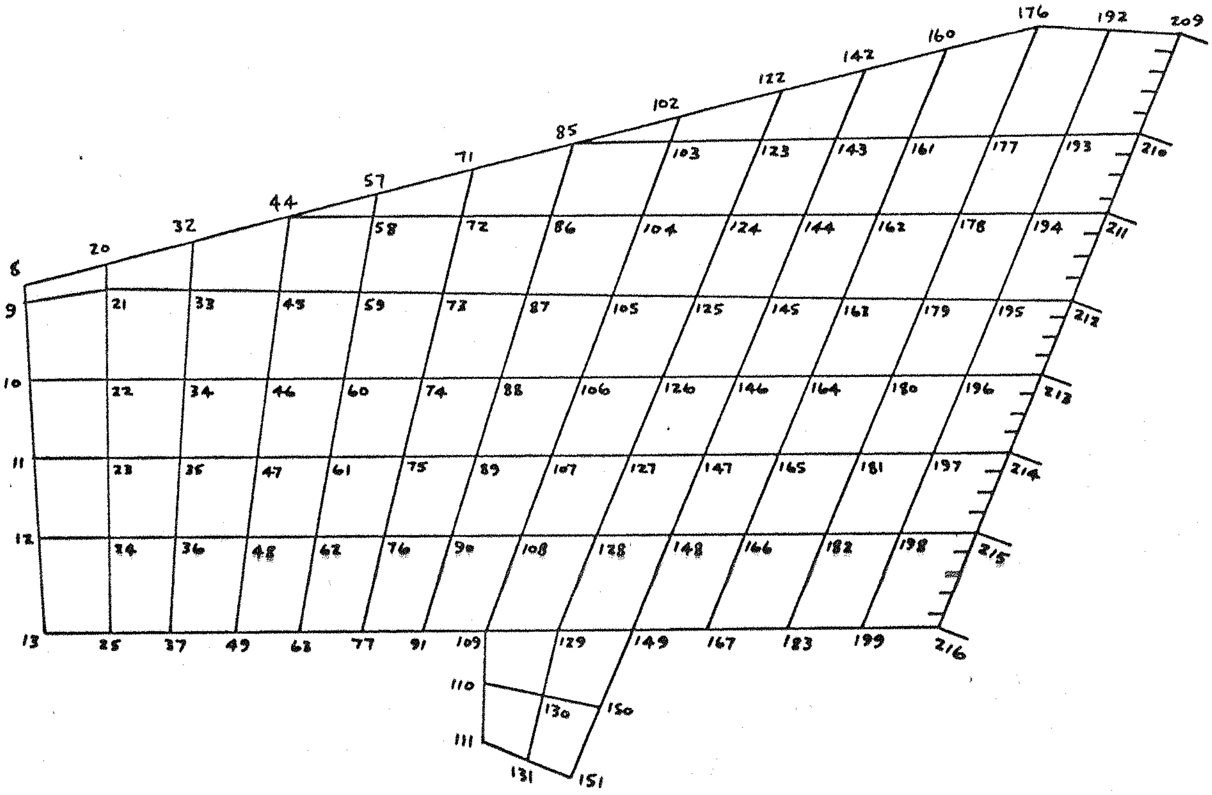
Computer utilization for structural analysis.

Fig. 75.



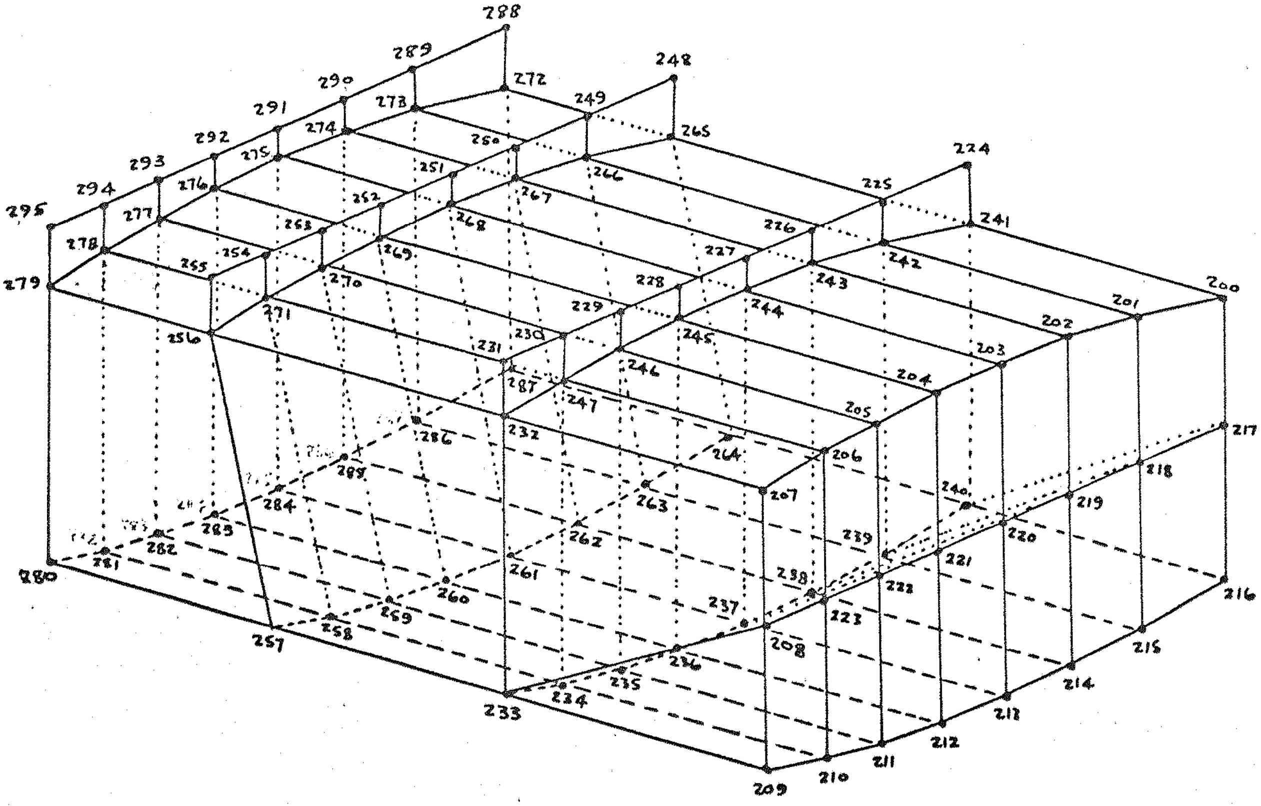
Structural idealization and nodal numbering system of the upper surface of a project wing design.

Fig. 76.



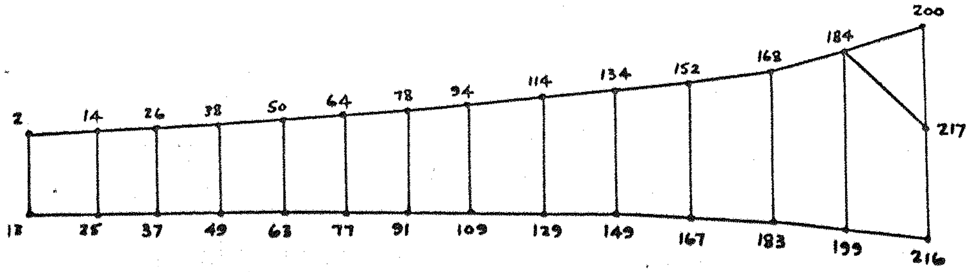
Structural idealization and nodal numbering system of the lower surface of a project wing design.

Fig. 77.

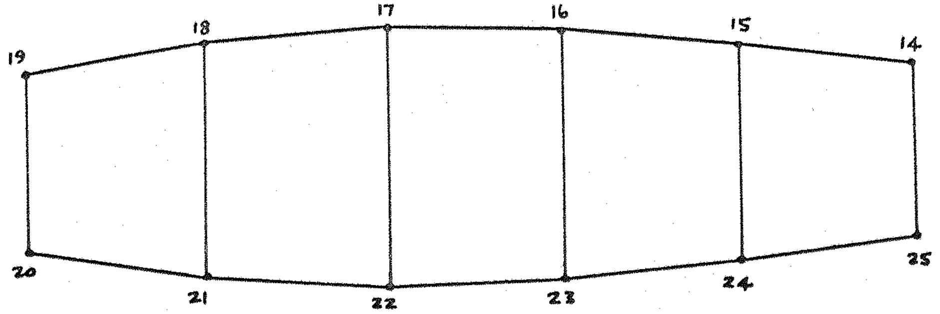


Structural idealization and nodal numbering system for the centre section of a project wing design.

Fig. 78.



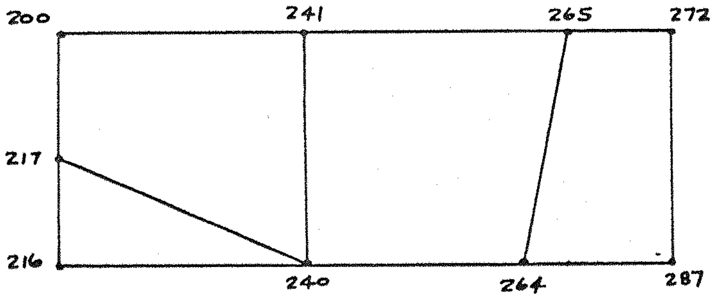
(a) Part rear spar.



(b) Rib.

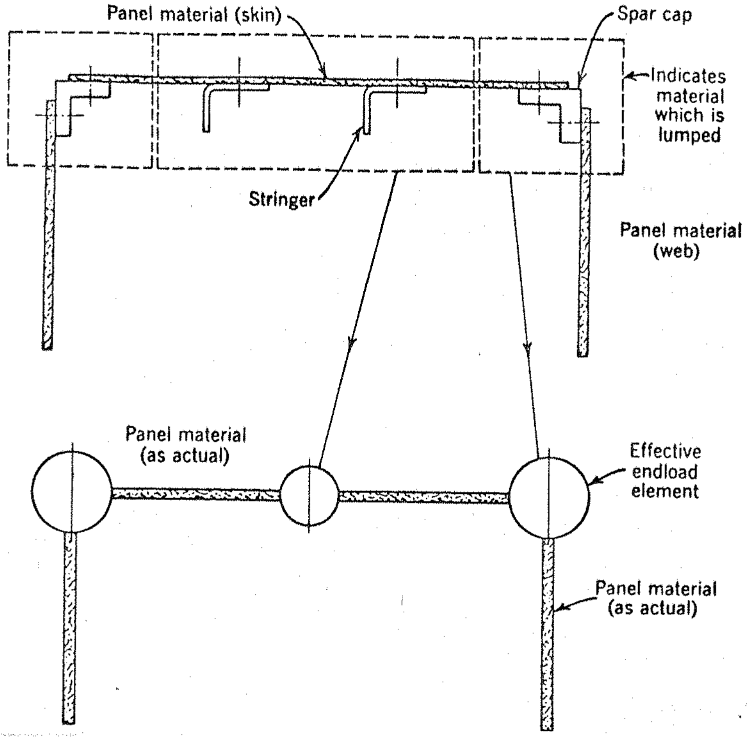
Structural idealization and nodal numbering system of a project wing design.

Fig. 79.



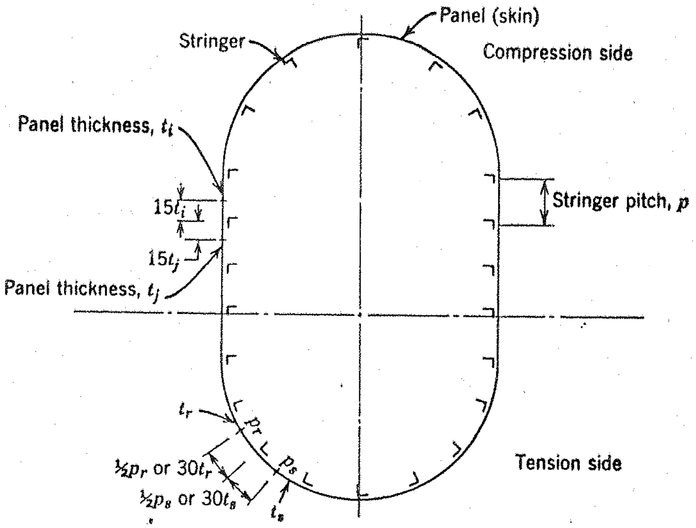
Spanwise beam in wing centre section.

Fig. 80.



Typical aircraft structural idealization.

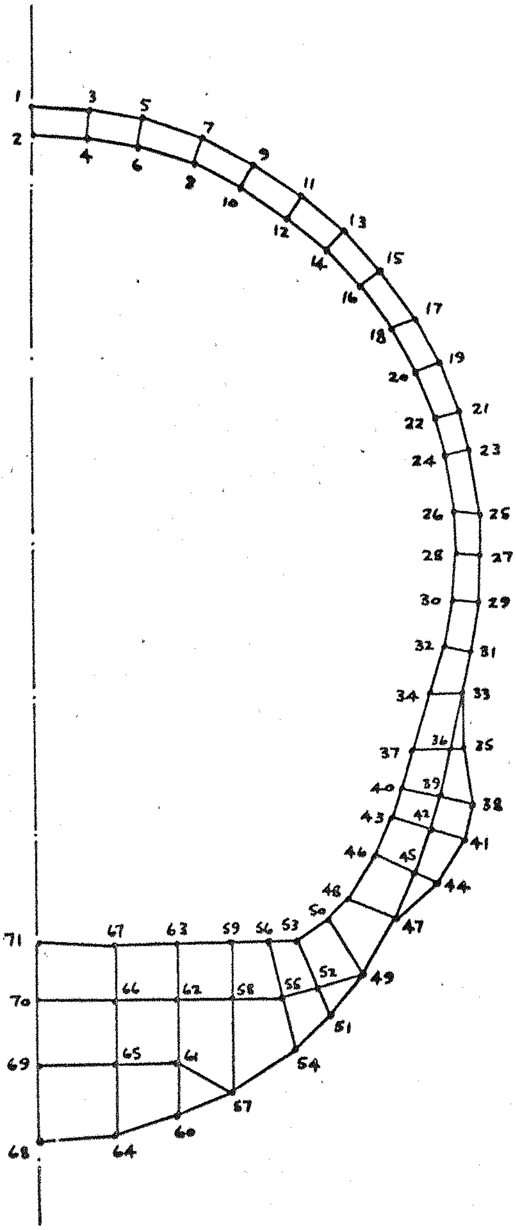
(a)



Fuselage cross section showing panel material which can be considered as effective in carrying endload.

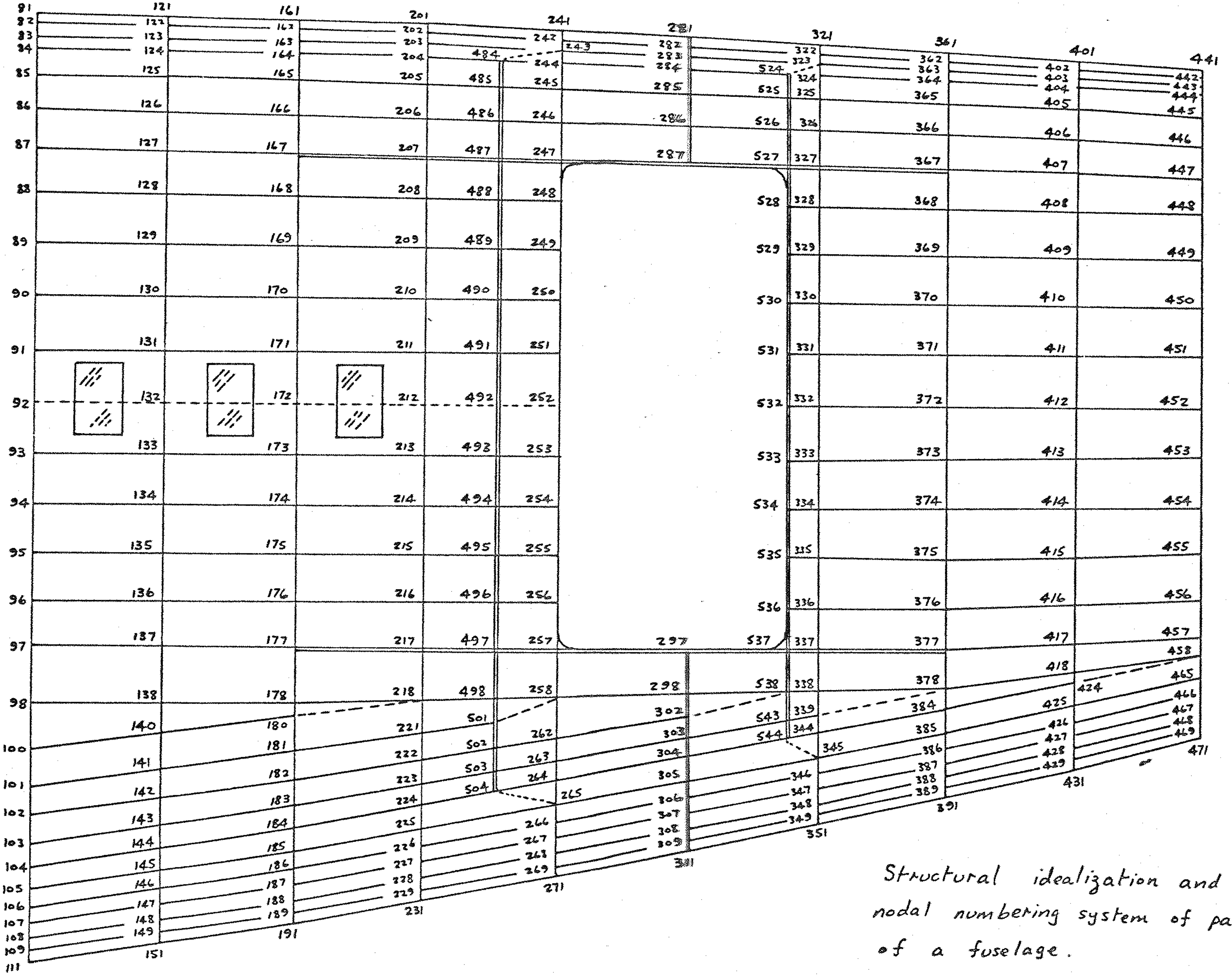
(b)

Fig. 81.



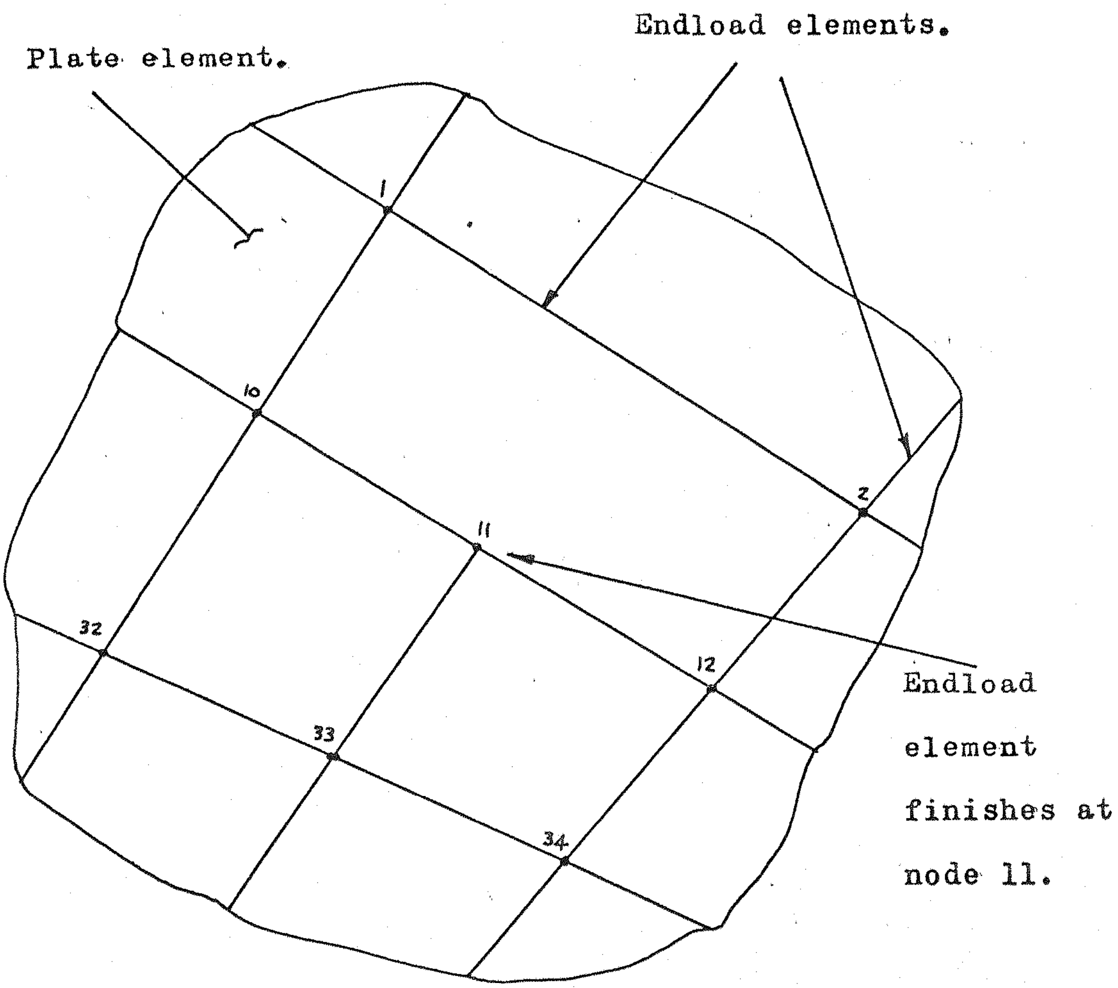
Structural idealization and nodal numbering system of a fuselage frame.

Fig. 82.



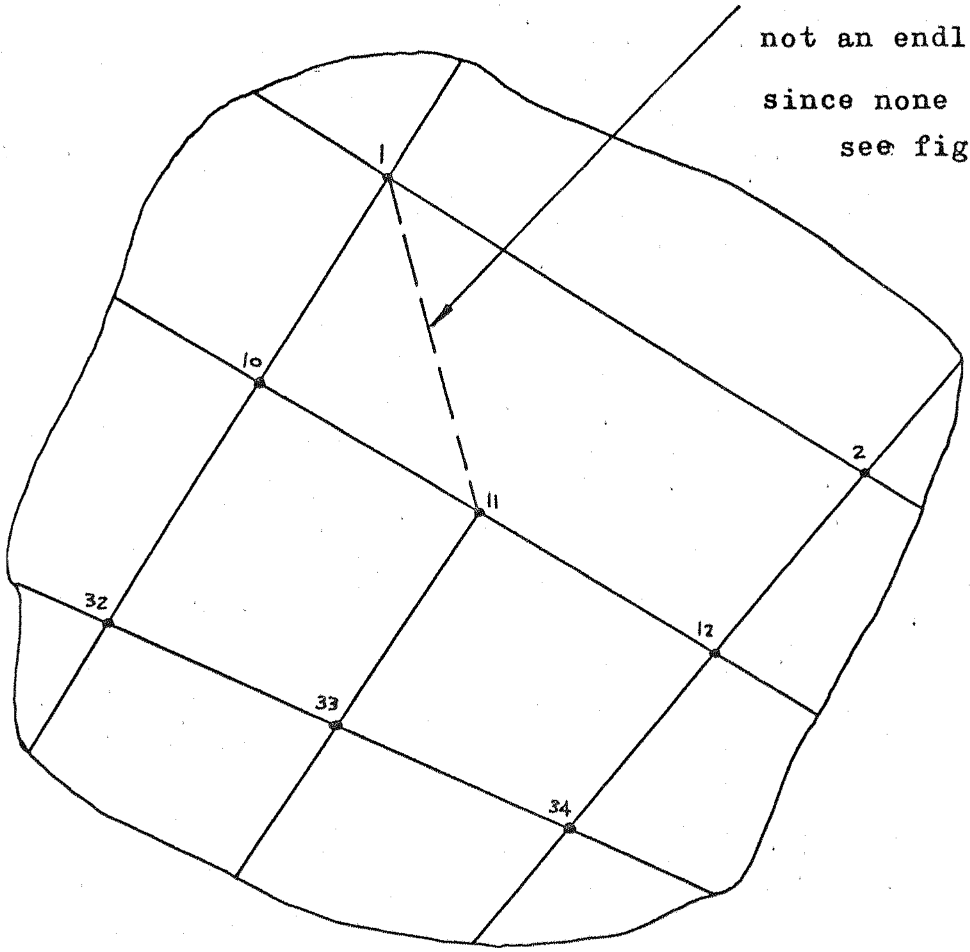
Structural idealization and nodal numbering system of part of a fuselage.

Fig. 83.



Idealization error. This would result in a discontinuity at node 11.

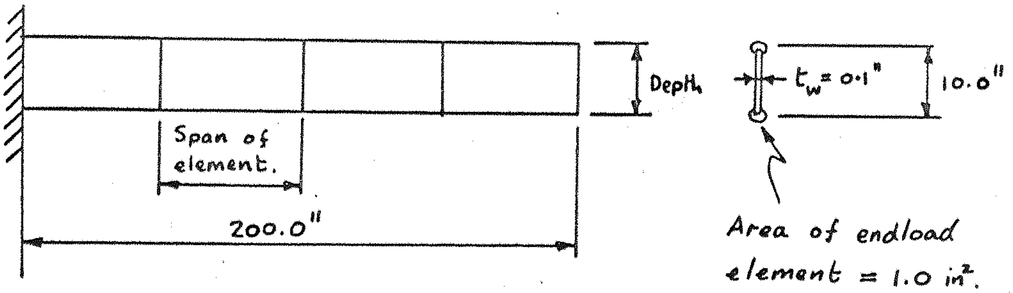
Fig. 84.



This is only a fictitious line used to give continuity. This is not an endload element since none exists, see figure 84.

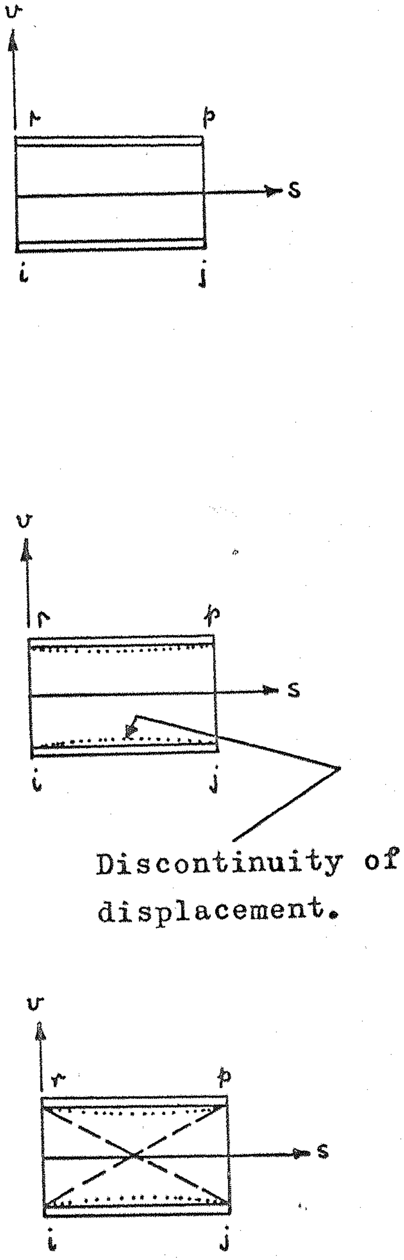
Correction of continuity error.

Fig. 85.



Cantilever spar structure.

Fig. 86.



Element type A.

(a)

Element type B.

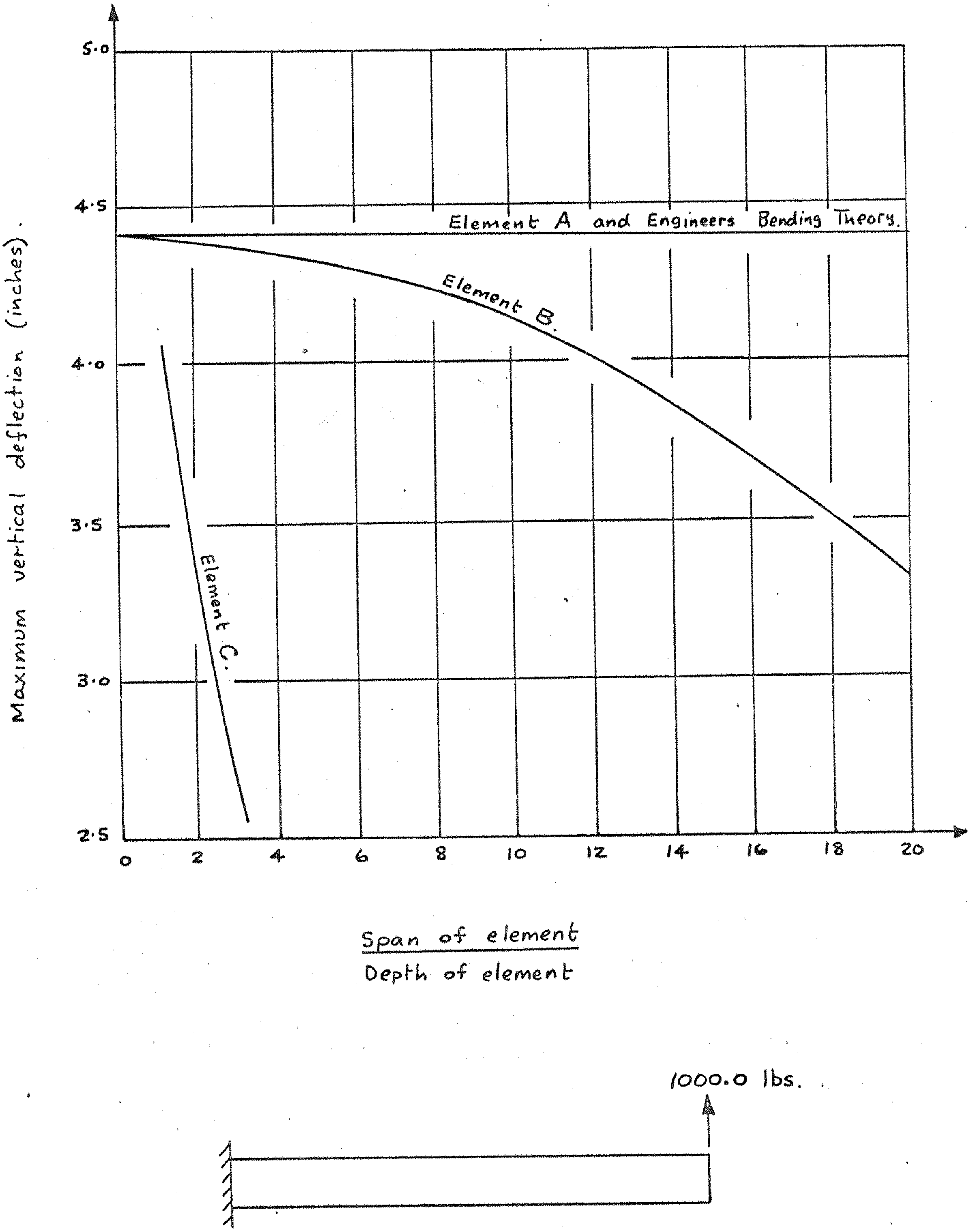
(b)

Element type C.

(c)

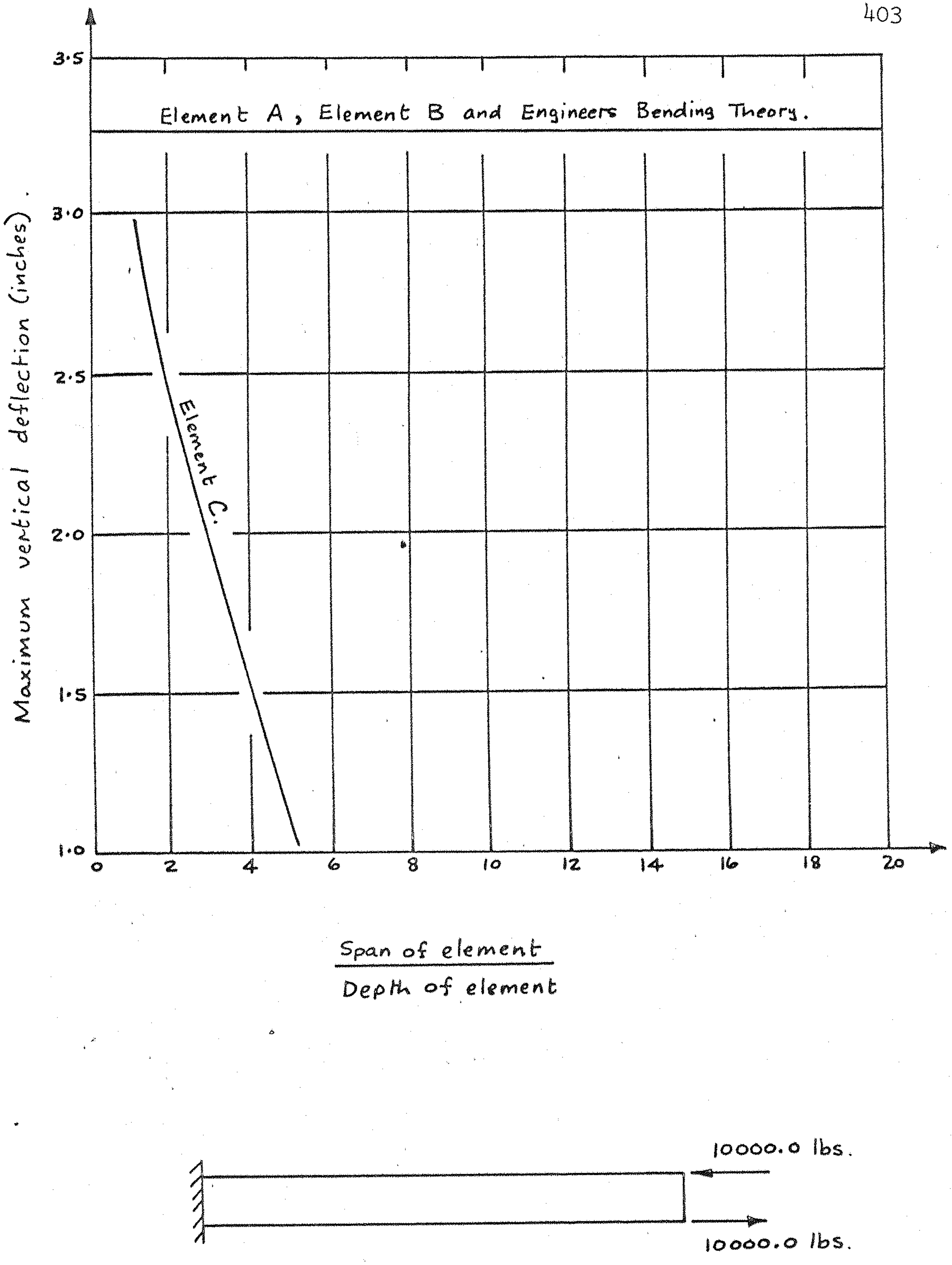
Various beam element representations.

Fig. 87.



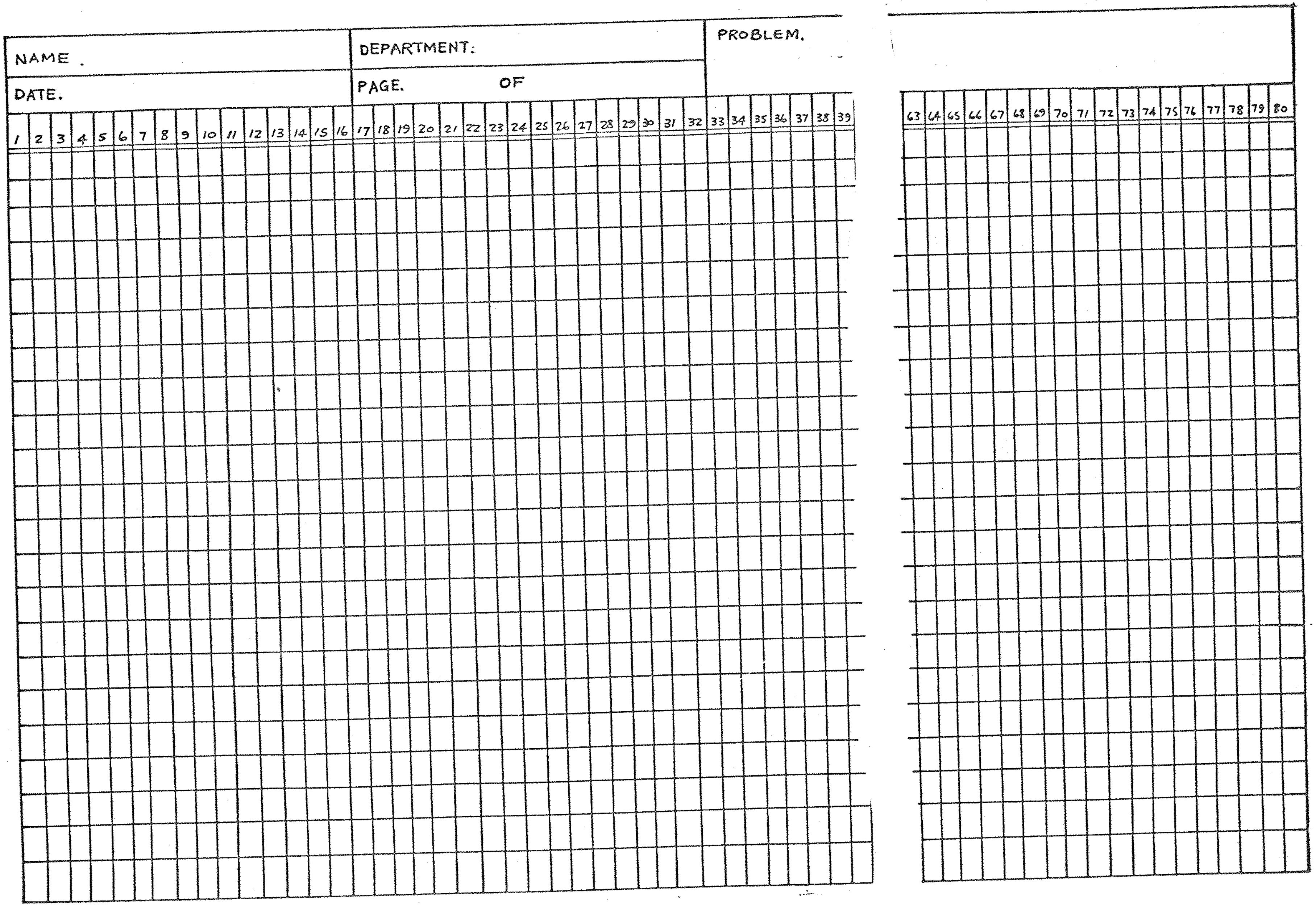
Beam deflection due to a lateral load applied at the free end.

Fig. 88.



Beam deflection due to a horizontal differential loading system applied at the free end.

Fig. 89.



NAME .

DEPARTMENT:

PROBLEM.

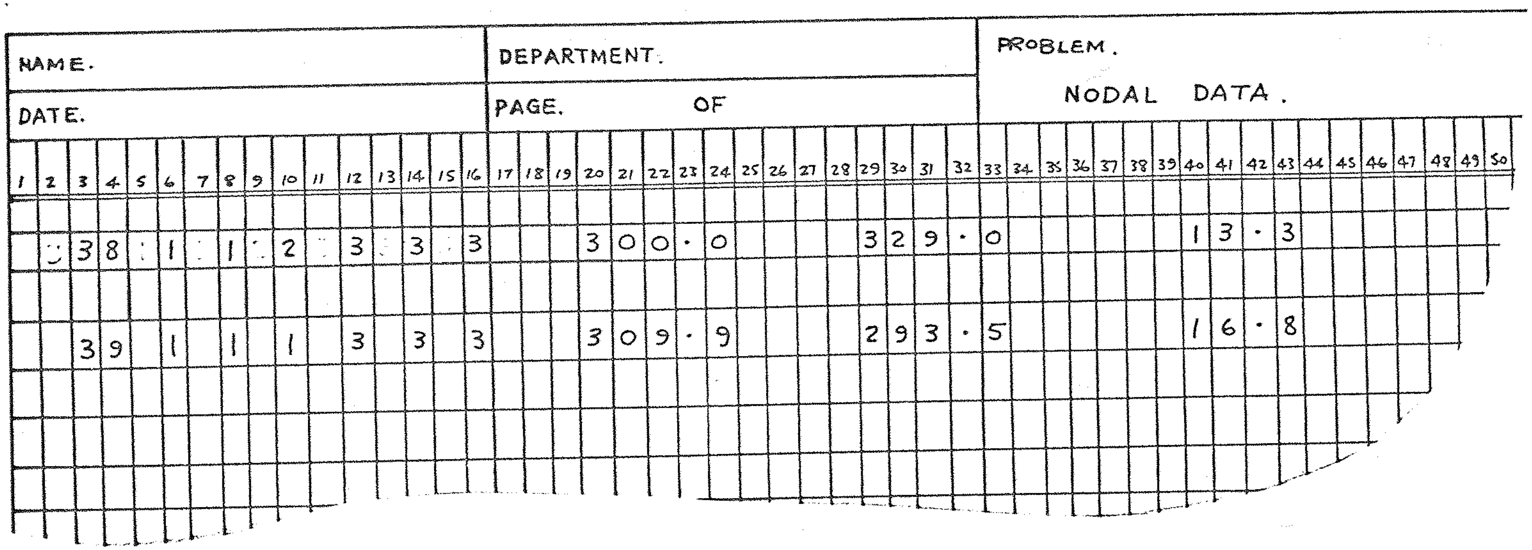
DATE.

PAGE. OF

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80

Fig. 90. Suggested standard input sheet.



NAME.

DEPARTMENT.

PROBLEM.
NODAL DATA.

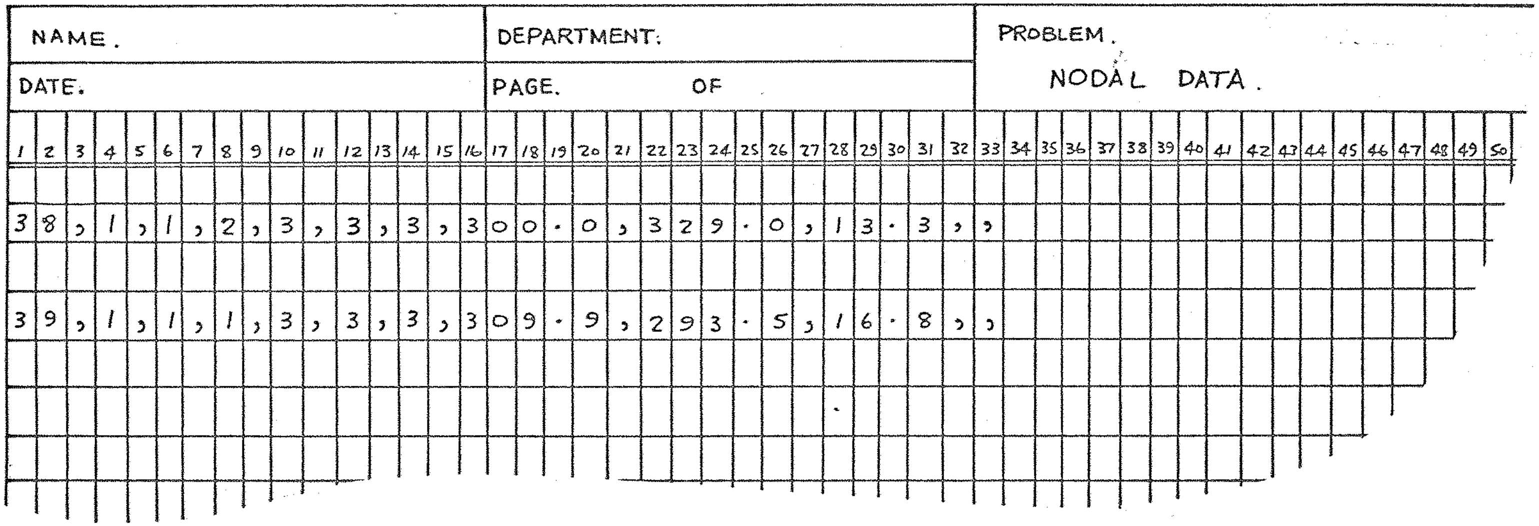
DATE.

PAGE. OF

```
  38 1 1 2 3 3 3   300.0    329.0      13.3
  39 1 1 1 3 3 3   309.9    293.5      16.8
```

Nodal data using a predetermined format.

Fig. 91.



NAME.
DATE.
DEPARTMENT:
PAGE. OF
PROBLEM.
NODAL DATA.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
3	8	,	1	,	1	,	2	,	3	,	3	,	3	,	3	0	0	.	0	,	3	2	9	.	0	,	1	3	.	3	,	,																	
3	9	,	1	,	1	,	1	,	3	,	3	,	3	,	3	0	9	.	9	,	2	9	3	.	5	,	1	6	.	8	,	,																	

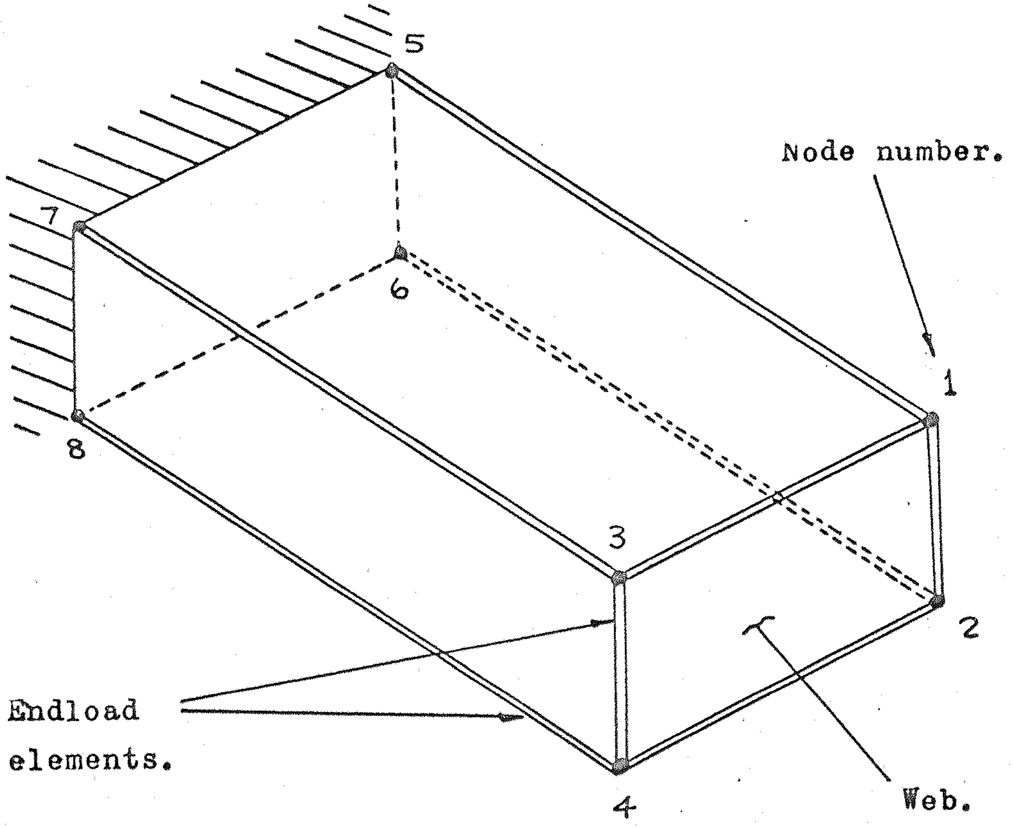
Nodal data using separating commas.

Fig. 92.

Element type.	Element specifying nodes				Material code	Area or thickness.
	i	j	r	p		
	1	5				
	1	5	3	7		
	1	3				
	1	3	2	4		
	1	2				
	1	2	5	6		
	2	6				
	2	6	4	8		
	2	4				
	3	7				
	3	7	4	8		
	3	4				
	4	8				

Element specifying nodes showing sequence. This table is only to clarify a point and is not the actual data format.

Table 49.



Cantilever box structure.

Fig. 93.

References

1. Rogers, Grover L.,
"An Introduction to the Dynamics of Framed Structures."
John Wiley and Sons, Inc., Publishers, New York, 1959.

2. Timoshenko, S. and D.H. Young,
"Vibration Problems in Engineering."
Third Edition.
D. Van Nostrand Company, 1955.

3. Bishop, R.E.D. and D.C. Johnson,
"The Mechanics of Vibration."
Cambridge University Press, 1960.

4. Scanlan, Robert H. and Robert Rosenbaum,
"Introduction to the Study of Aircraft Vibration and Flutter."
The Macmillan Company, 1962.

5. Wylie, C.R., Jr.,
"Advanced Engineering Mathematics."
McGraw-Hill Book Company, New York, 1960.

6. "Numerical Methods for Vibration Problems."
Volume 3, I.S.V.R. Symposium, July 1966.

7. Petyt, M.,
"The Application of Finite Element Techniques to Plate and Shell Problems."
I.S.V.R. Report, Number 120.

8. Archer, John S.,
"Consistent Mass Matrix for Distributed Mass Systems."
Journal of the Structural Division,
Proc.Am.Soc.Civ.Eng., August 1963.

9. Archer, John S.,
"A Stiffness Matrix Method of Normal Mode Analysis."
Proceedings, Natl. Specialists Meeting on Dynamics and Aeroelasticity,
I.A.S., November 1958.

10. Turner, M.J., R.W. Clough, H.C. Martin, and L.J. Topp,
"Stiffness and Deflection Analysis of Complex Structures."
J.Aeron.Sci., 23(9), 805, 1956.

11. Turner, M.J., H.C. Martin, and R.C. Weikel,
"Further Development and Applications of the Stiffness Method."
AGARD Meeting, Paris, France, July 1962.

12. Leckie, F.A. and G.M. Lindberg,
"The Effect of Lumped Parameters on Beam Frequencies."
The Aero.Quart., Volume 14, August 1963.

13. Bishop, R.E.D., G.M.L. Gladwell and S. Michaelson,
"The Matrix Analysis of Vibration."
Cambridge University Press, 1965.

14. Hessel, Arne,
"Analysis of Plates and Shells by Matrix Methods."
Saab Technical Note, No. 48, 1961.

15. Jaeger, L.G.
"Elementary Theory of Elastic Plates."
Pergamon Press, London, 1964.

16. Timoshenko, S. and S. Woinowsky-Krieger,
"Theory of Plates and Shells."
McGraw-Hill Book Company, New York, 1952.

17. Sechler, Ernest, E.,
"Elasticity in Engineering."
John Wiley and Sons, New York, 1952.

18. Pestel, Eduard, C.,
"Dynamic Stiffness Matrix Formulation by means of Hermitian Polynomials."
Proc.Conf.on "Matrix Methods in Structural Mechanics."

Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, October 1965.

19. Gallagher, Richard H.,
"A Correlation Study of Methods of Matrix Structural Analysis."
Report to the 14th Meeting, Structures and Materials Panel,
AGARD, France, July 6, 1962.

20. Zienkiewicz, O.C., B. Irons, and B. Nath,
"Natural Frequencies of Complex, Free or Submerged Structures by the Finite Element Method."
Symp.Vibr. in Civ. Eng., London, April, 1965.

21. Zienkiewicz, O.C. and Y.K. Cheung,
"Plate and Shell Problems, Finite Element-Displacement Approach."
Int.Symp. Computers in Structural Engineering,
Newcastle-upon-Tyne, July 1966.

22. Lillicrapp, R.,
"Frame Vibration."
University of Southampton,
Department of Aeronautics and Astronautics,
Hons. Project, 1964.

23. Levien, K.W.,
"Dynamic Analysis of Rigid Frames using the Force Method."
M.S. thesis in Civ. Eng., University of Washington, U.S.A., 1961.

24. Levien, K.W. and B.J. Hartz,
"Dynamic Flexibility Matrix Analysis of Frames."
Journal of the Structures Division,
Proc.Am.Soc.Civ. Eng., August 1963.

25. Pestel, Eduard C. and F.A. Leckie,
"Matrix Methods in Elastomechanics."
McGraw-Hill Book Company, New York, 1963.

26. Frazer, R.A., W.J. Duncan, and A.R. Collar,
"Elementary Matrices."
Cambridge University Press, 1963.

27. Robinson, John and Robert R. Regal,
"An Automated Matrix Analysis for General Plane Frames
('The Rank Technique')."
J.Am.Helicopter Soc., 8(4), October 1963.

28. Robinson, John,
"Automatic Selection of Redundancies in the Matrix Force Method
('The Rank Technique')."
Can.Aero.Space J., 11(1), January 1965.

29. Robinson, John,
"Dissertation on 'The Rank Technique' and its Application."
J.Roy.Aero.Soc., Technical Note, April 1965.

30. Robinson, John,
"World Survey, April 1965, Computer Utilization for Structural Analysis."
J.Roy.Aero.Soc., Technical Note, July 1966.

31. Robinson, John,
"Structural Matrix Analysis for the Engineer."
John Wiley and Sons, Inc., Publishers, New York, 1966.

32. Robinson, John,
"The Question of Redundancy in Structural Vibration."
Euromech Colloquium No.5, Vibration of Buildings.
I.S.V.R., University of Southampton, April 1967.

33. Robinson, John,
"Eigenvalues of Collinear Beam Structures using Finite Element Techniques and Various Dynamic Representations for the Structural Elements."
4th International Congress on the Application of Mathematics in Engineering, Weimar, East Germany, June 1967.

34. Robinson, John,
"Computerized Structural Analysis System Adopting the Rank Force Method for Vibration Analysis."
(I.C.T.1900, A.S.A. Fortran) I.S.V.R., University of Southampton, 1966-67.

35. "COSMOS, A Computer Program for Structural Analysis."
Document D2-4513, The Boeing Company, January 1962.

36. Hall, Arthur S., and Ronald W. Woodhead,
"Frame Analysis."
John Wiley and Sons, Inc., Publishers, New York, 1961.

37. Jennings, A., and K.I. Majid,
"The Computer Analysis of Space Frames Using Sparse Matrix Techniques."
International Conference on Space Structures, University of Surrey,
London, September 1966.

38. Barton, M.V.,
"Vibration of Rectangular and Skew Cantilever Plates."
J.App.Mech., June 1951.

39. Burch, G.T.,
"Digital Computer Solution for the Dynamic Analysis of Rigid Frames."
M.S. Thesis in Civ. Eng., University of Washington, U.S.A., 1959.

40. Robinson, John,
"Practical Application of a Computerized Structural Analysis System
which adopts a Finite Element Technique (Direct Stiffness Method)."
Conference on "Recent Advances in Stress Analysis", Royal
Aeronautical Society, March 1968, sponsored by the Joint British
Committee for Stress Analysis.

41. Zienkiewicz, O.C. and Y.K. Cheung,
"The Finite Element Method in the Analysis of Elastic Isotropic
and Orthotropic Slabs."
Proc. Inst. Civ.Eng. Vol 28, pp 471-488, 1964.

FIGURE AND PAGE NUMBERS.

TABLE AND PAGE NUMBERS.