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On state-space elastostatics within a plane stress sectorial domain – the wedge and the curved beam

N.G. Stephen *

School of Engineering Sciences, Mechanical Engineering, University of Southampton, Highfield, Southampton SO17 1BJ, UK

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ABSTRACT

The plane stress sectorial domain is analysed according to a state-space formulation of the linear theory of elasticity. When loading is applied to the straight radial edges (flanks), with the circular arcs free of traction, one has the curved beam; when loading is applied to the circular arcs, with the flanks free of traction, one has the elastic wedge. A complete treatment of just one problem (the elastic wedge, say) requires two state-space formulations; the first describes radial evolution for the transmission of the stress resultants (force and moment), while the second describes circumferential evolution for determination of the rates of decay of self-equilibrated loading on the circular arcs, as anticipated by Saint-Venant's principle. These two formulations can be employed subsequently for the curved beam, where now radial evolution is employed for the Saint-Venant decay problem, and circumferential evolution for the transmission modes. Power-law radial dependence is employed for the wedge, and is quite adequate except for treatment of the so-called wedge paradox; for this, and the curved beam, the formulations are modified so that $\ln r$ takes the place of the radial coordinate r. The analysis is characterised by a preponderance of repeating eigenvalues for the transmission modes, and the state-space formulation allows a systematic approach for determination of the eigen- and principal vectors. The so-called wedge paradox is related to accidental eigenvalue degeneracy for a particular angle, and its resolution involves a principal vector describing the bending moment coupled to a decay eigenvector. Restrictions on repeating eigenvalues and possible Jordan canonical forms are developed. Finally, symplectic orthogonality relationships are derived from the reciprocal theorem.

1. Introduction

The present work is companion to a recent exposition (Stephen, 2004) of a state-space approach to the linear elastostatic problem for a prismatic beam-like structure. In that formulation, the equilibrium equations and Hooke's law were rearranged to describe the evolution of a state vector of cross-sectional stress and displacement components as one moves along the axis of the beam. Eigenanalysis provided a systematic approach to the determination of the transmission modes for the general cross-section – tension, torsion, bending moment and shearing force, coupled to rigid body displacements and rotations, these being the well-known Saint-Venant solutions. For the specific case of the plane stress strip (beam of thin rectangular cross-section), a second formulation describing evolution in the transverse direction was required for determination of the Saint-Venant decay rates of self-equilibrated end loading (the well-known Papkovitch–Fadle solution, see for example Timoshenko and Goodier, 1970, article 26).

* Tel.: +44 23 80592359; fax: +44 23 80593230.
E-mail address: ngs@soton.ac.uk

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Nomenclature

a, b, c	inner and outer arc radii, curved beam semi-depth
A, B, C, D	constants
$\mathbf{A}$, $\mathbf{B}$, $\mathbf{D}$	partitions of system matrix
$\mathbf{d}$	vector of displacement components
E	Young's modulus
G	shear modulus
$\mathbf{H}$	system matrix
i, $\mathbf{I}$	$\sqrt{-1}$, identity matrix
$\mathbf{J}$, $\mathbf{J}_{\mathrm{m}}$	Jordan canonical form, metric
k	eigenvalue associated with circumferential evolution
M, $\mathbf{M}$	moment, element of, matrix exponential
n, N, $\mathbf{N}$	direction cosine, element of, matrix exponential
$\mathbf{p}$	vector of stress components
Q	shearing force
r, θ	polar coordinates
R, $\mathbf{R}$	curved beam centre-line radius, resolvent matrix
s, S, $\mathbf{s}$, $\mathbf{S}$	Laplace variable, radius $\times$ stress, state vector, transformation matrix of eigen- and principal vectors for radial evolution
T	tensile force
u, U	displacement, strain energy
$\mathbf{v}$, $\mathbf{V}$	state vector, transformation matrix of eigen- and principal vectors for circumferential evolution
x, y, z	Cartesian coordinates
α	semi wedge angle
β	arbitrary multiple (of eigenvector)
ε, γ	direct, shearing strain
λ	eigenvalue associated with radial evolution
ν	Poisson's ratio
σ,τ	direct, shearing stress
ξ	$\ln r$, modified radial coordinate
Φ	Airy stress function
Δ	defined by Eqs. (78) and (93)

Here the plane stress elastic sector is considered using polar coordinates. According to where surface loading is applied, this geometry encompasses both the wedge and the curved beam, and this state-space approach allows analysis of the two distinct problems in an efficient manner. A complete treatment of just one problem, say the elastic wedge where the flanks are free of traction with loads applied to the inner and outer arcs only, requires two state-space formulations, one describing radial evolution for the transmission of the stress resultants (force and moment), and a second describing circumferential evolution for determination of the rates of decay of self-equilibrated loading, as anticipated by Saint-Venant's principle. These two formulations, modified such that $\ln r$ takes the place of the radial coordinate r, are then employed for the curved beam where loads are applied to the straight flanks (ends) only, and the curved arcs are free of traction; now, radial evolution is employed for the Saint-Venant decay problem, and circumferential evolution for the transmission modes.

Both Stephen (2004) and the present study were motivated by a series of papers in the Chinese literature, largely by Zhong and his co-workers (Zhong (1991, 1994, 1995), Zhong and Ouyang (1992), Zhong and Xu (1996), Wang and Tang (1995), Xu et al. (1997)), concerned with the methods of Hamiltonian mechanics applied to problems of the linear mathematical theory of elasticity, rather than the more usual stress function (for example, Airy and Papkovitch–Neuber) or semi-inverse methods described in most, if not all, of the well-known texts on elasticity. The Hamiltonian approach is more familiar within the study of both rigid body and quantum mechanics, and results in a first-order matrix differential equation. This, in itself, is not the distinctive feature as one can always trade the order of a single differential equation with the size of the system matrix; rather, it treats position and momentum as independent variables on an equal footing, and these extra degrees of freedom aid the search for canonical transformations (which preserve Hamiltonian structure) for which the set of differential equations are either fully or partially solved. For application to the elasticity of a beam-like structure, a state vector consisting of cross-sectional displacement and stress components naturally takes the place of position and momentum, and the governing equations then describe how these evolve spatially (rather than temporally) as one moves along the beam.

In general, an elastostatic solution must satisfy the Hooke's law, the boundary conditions and the force equilibrium equations; if the latter are expressed in terms of stress then one must also employ strain compatibility in one form or another. However, if the equilibrium equations are expressed in terms of displacement (the Navier equations), then strain compatibility equations are not required; one immediate advantage of the state-space approach is that it treats displacement and stress components simultaneously, so strain compatibility is satisfied naturally. A second advantage is that it unifies the two (usually) separate problems which bear the name of Saint-Venant – the transmission of load applied to the ends only, according to a prescribed distribution (Saint-Venant's problem), and the decay of self-equilibrated end loading (Saint-Venant's principle) – through the machinery of an eigenvalue problem.

Zhong (1994) and Yao and Xu (2001) have previously shown how the governing state-space equations for the wedge can be derived from a Lagrangian function and use of the calculus of variations; but since these governing equations are already well-known, all that is really required is their manipulation into an appropriate state-variable form. Moreover, these authors only considered a formulation for radial evolution; here, the formulation is extended to evolution in the circumferential direction, as is required for the curved beam, and for determination of the Saint-Venant decay rates for the wedge.

While recent papers by Zhong and his co-workers have drawn attention to the advantage of these modern (control) system state-variable methods, it appears that Bahar (1975) was the first to note that classical elasticity is well suited to the state-space approach, and applied it to the problem of the plane stress elastic strip; he also noted the relationship to the method of *initial function*, or *parameter*, as developed earlier by Vlasov (1957). Sosa and Bahar (1992) represented the fourth-order biharmonic (Airy) stress function approach as a first-order 4×4 matrix mixed state-variable problem, and indicated advantages in the treatment of the Flamant problem. Earlier, Johnson and Little (1965) also considered the strip problem, but introduced an auxiliary function in addition to the three stress components, rather than the mixture of cross-sectional stress and displacement components employed by later authors.

The paper is laid out as follows: the governing equations are presented in Section 2, and are cast into a state-space form suitable for radial evolution in Section 3. Initially, displacement and stress components are assumed to have a power-law radial dependence, leading to a Hamiltonian system matrix $\mathbf{H}$. Further assuming exponential θ-dependence leads to a characteristic equation, whose numerous roots are consistent with the compendium of solution forms for the biharmonic stress functions in polar coordinates, presented by Timoshenko and Goodier (1970), article 43, and Southwell (1941), articles 419, 420. This power-law formulation for radial evolution is adequate for solution of the force and moment transmission problem for the wedge, which is considered in Section 4. Associated with the double eigenvalue $\lambda = 0$ are the eigenvector describing a rigid body displacement in an arbitrary radial direction, coupled to a principal vector describing a force, again in some arbitrary direction. Each of these vectors can be split into symmetric and asymmetric parts, which are the known solutions for the wedge subjected to a tensile force, and a shearing force. The eigenvalue $\lambda = 1$ pertains to a rigid body rotation about the origin, while the eigenvalue $\lambda = -1$ pertains to the wedge subjected to pure bending, which is the well-known Carothers (1912) solution. This collection of transmitting eigen- and principal vectors is employed to transform the system matrix into a Jordan canonical form.

As with the plane stress strip considered by Stephen (2004), the "end problem" for the wedge – determination of the rates of decay of self-equilibrated loading as anticipated by Saint-Venant's principle – requires a second state-space formulation, now for evolution in the circumferential direction. This is presented in Section 5, again employing a power-law radial dependence. Modified formulations, involving the natural logarithm of r, for both radial and circumferential evolution are required for the transmission and decay problems for the curved beam, and treatment of the wedge paradox; these are developed in Section 6. In Section 7, this modified formulation for circumferential evolution is employed for determination of the transmission modes of the curved beam, while in Section 8, radial evolution is employed for the end-problem (Saint-Venant decay) for the curved beam. The wedge paradox, the pathological behaviour of the Carothers (1912) solution for the wedge angle $2\alpha^*$ defined by $\sin 2\alpha^* = 2\alpha^* \cos 2\alpha^*$, occurs when the eigenvalue $\lambda = -1$ describing the rate of decay of self-equilibrated loading repeats the eigenvalue pertaining to diffusion of a pure bending moment into a divergent domain, an accidental degeneracy; this is treated in Section 9. Strain energy arguments lead to restrictions on possible Jordan canonical forms – equivalently possible repeating eigenvalues, and these are considered in Section 10 for both the wedge and the curved beam. In Section 11, symplectic orthogonality relationships for the wedge and the curved beam are developed from the reciprocal theorem. Conclusions are drawn in Section 12.

Finally note that, strictly, solution terms such as r^λ and $\ln r$ should be non-dimensionalised with respect to some arbitrary radius r_0 to become $(r/r_0)^\lambda$ and $\ln(r/r_0)$, respectively. However in the interests of simplicity, and to provide maximum commonality with the majority of research papers and textbooks in this field, such modification has not been made in the present analysis; nevertheless, resulting stress fields are in agreement with the solutions in standard monographs, such as Timoshenko and Goodier (1970), and other sources.

2. Governing equations in polar coordinates

The stress equilibrium equations are

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \tag{1a}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0, \tag{1b}$$

with strains

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \tag{2a}$$

$$\varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \tag{2b}$$

$$\gamma_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \tag{2c}$$

and Hooke's law

$$\varepsilon_r = \frac{\sigma_r}{E} - \frac{\nu}{E}\sigma_\theta, \tag{3a}$$

$$\varepsilon_\theta = \frac{\sigma_\theta}{E} - \frac{\nu}{E}\sigma_r, \tag{3b}$$

$$\gamma_{r\theta} = \frac{\tau_{r\theta}}{G}. \tag{3c}$$

Eqs. (3a),(3b) may also be written as

$$\sigma_r = \frac{E}{(1-\nu^2)}(\varepsilon_r + \nu\varepsilon_\theta), \tag{4a}$$

$$\sigma_\theta = \frac{E}{(1-\nu^2)}(\varepsilon_\theta + \nu\varepsilon_r). \tag{4b}$$

3. State-space formulation for radial evolution

Define a state vector describing radial evolution as

$$\mathbf{s} = [u_r\, u_\theta\, \sigma_r\, \tau_{r\theta}]^{\mathrm{T}}, \tag{5}$$

consisting of the displacement and stress components on an arc of any given radius. First, one needs to eliminate σ_θ from the equilibrium Eqs. (1a),(1b), as it is not a state variable for an arc of the domain; from Eq. (2b) and the Hooke's law (3b) one has

$$\sigma_\theta = \nu\sigma_r + E\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right). \tag{6}$$

Now rearrange Eq. (4a) as

$$\frac{\partial u_r}{\partial r} = \frac{(1-\nu^2)}{E}\sigma_r - \nu\frac{u_r}{r} - \frac{\nu}{r}\frac{\partial u_\theta}{\partial \theta}, \tag{7}$$

and rearrange Eq. (2c) as

$$\frac{\partial u_\theta}{\partial r} = \frac{\tau_{r\theta}}{G} + \frac{u_\theta}{r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}. \tag{8}$$

Eliminate σ_θ from Eq. (1a), and rearrange to give

$$\frac{\partial \sigma_r}{\partial r} = -\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{E}{r^2}u_r + \frac{E}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{1-\nu}{r}\sigma_r. \tag{9}$$

Last, eliminate σ_θ from Eq. (1b), and rearrange to give

$$\frac{\partial \tau_{r\theta}}{\partial r} = -\frac{\nu}{r}\frac{\partial \sigma_r}{\partial \theta} - \frac{E}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{E}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} - 2\frac{\tau_{r\theta}}{r}. \tag{10}$$

Eqs. (7)–(10) may be written in the matrix form

$$\frac{\partial}{\partial r}\begin{bmatrix} u_r \\ u_\theta \\ \sigma_r \\ \tau_{r\theta} \end{bmatrix} = \begin{bmatrix} -\nu/r & -\nu\partial/r\partial\theta & (1-\nu^2)/E & 0 \\ -\partial/r\partial\theta & 1/r & 0 & 1/G \\ E/r^2 & E\partial/r^2\partial\theta & (\nu-1)/r & -\partial/r\partial\theta \\ -E\partial/r^2\partial\theta & -E\partial^2/r^2\partial\theta^2 - & -\nu\partial/r\partial\theta & -2/r \end{bmatrix}\begin{bmatrix} u_r \\ u_\theta \\ \sigma_r \\ \tau_{r\theta} \end{bmatrix}. \tag{11}$$

The presence of the coordinate r within the right-hand side of Eq. (11) means that separation of variables has not yet been achieved; however, introduction of barred quantities, which are functions of θ only, and radial power-law dependence according to the scheme

$$u_r = \bar{u}_r r^\lambda, \quad u_\theta = \bar{u}_\theta r^\lambda, \quad \sigma_r = \bar{\sigma}_r r^{\lambda-1}, \quad \tau_{r\theta} = \bar{\tau}_{r\theta} r^{\lambda-1}, \quad \sigma_\theta = \bar{\sigma}_\theta r^{\lambda-1}, \tag{12}$$

does achieve separation, and leads to the eigenproblem

$$\lambda\begin{bmatrix}\bar{u}_r\\ \bar{u}_\theta\\ \bar{\sigma}_r\\ \bar{\tau}_{r\theta}\end{bmatrix}=\begin{bmatrix}-v & -v\mathrm{d}/\mathrm{d}\theta & (1-v^2)/E & 0\\ -\mathrm{d}/\mathrm{d}\theta & 1 & 0 & 1/G\\ E & E\mathrm{d}/\mathrm{d}\theta & v & -\mathrm{d}/\mathrm{d}\theta\\ -E\mathrm{d}/\mathrm{d}\theta & -E\mathrm{d}^2/\mathrm{d}\theta^2 & -v\mathrm{d}/\mathrm{d}\theta & -1\end{bmatrix}\begin{bmatrix}\bar{u}_r\\ \bar{u}_\theta\\ \bar{\sigma}_r\\ \bar{\tau}_{r\theta}\end{bmatrix}, \tag{13}$$

or

$$\lambda\bar{\mathbf{s}}=\mathbf{H}\bar{\mathbf{s}}, \tag{14}$$

where the system matrix $\mathbf{H}$ and state vector $\bar{\mathbf{s}}$ are defined accordingly.

Matrix $\mathbf{H}$ in this form has been derived previously by both Zhong (1994) and Yao and Xu (2001), although these authors employ $\ln r$ as the radial coordinate, with the state vector modified accordingly. (Yao and Xu (2001) treated a paradox arising when tractions applied to the flanks of the wedge have a power-law distribution $r^{\lambda-1}$ ($\lambda\geqslant 1$), and λ coincides with an eigenvalue associated with the decay of self-equilibrated loading applied to an arc; although related, such secondary paradoxes are not treated in the present paper.) As will be seen, introduction of this modified radial coordinate is not necessary for solution of the transmission and Saint-Venant decay problems for the wedge, but is necessary for treatment of the wedge paradox, and both the transmission and decay problems for the curved beam. Matrix $\mathbf{H}$ is Hamiltonian, having the 2×2 partition block form

$$\mathbf{H}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{D} & -\mathbf{A}^{\mathrm{T}}\end{bmatrix}. \tag{15}$$

In the above, $\mathbf{A}^{\mathrm{T}}$ denotes the adjoint of $\mathbf{A}$, which for a matrix differential operator is its transposition with the sign changed for all odd order differentials, while for a constant matrix, the adjoint is the same as the transpose; one of the properties of a Hamiltonian matrix is that its eigenvalues occur as ± pairs. The characteristic equation associated with Eqs. (13),(14) is

$$((\lambda+1)^2+\mathrm{d}^2/\mathrm{d}\theta^2)((\lambda-1)^2+\mathrm{d}^2/\mathrm{d}\theta^2)=0. \tag{16}$$

Numerous possibilities follow from the assumption of circumferential dependence as $\exp(k\theta)$, when Eq. (16) becomes

$$((\lambda+1)^2+k^2)((\lambda-1)^2+k^2)=0. \tag{17}$$

Suppose initially that $k=0$, that is, apparent independence of θ; Eq. (17) reduces to $(\lambda+1)^2(\lambda-1)^2=0$, and hence $\lambda=\pm 1$ are repeated roots. According to the power law scheme in Eq. (12), $\lambda=1$ implies that stresses are independent of radius, so one would expect this to be associated with a rigid body displacement or rotation (it is the latter). The root $\lambda=-1$ implies stress varying as r^{-2} which is known to occur for pure bending of both the wedge (moment applied to the inner and outer arcs) and the curved beam (moment applied to the straight flanks). However, more can be gleaned: for $\lambda=\pm 1$, Eq. (17) reduces to $k^2(2^2+k^2)=0$, implying that $k=0$ also repeats, and additional terms involving θ, and $\sin 2\theta$ and $\cos 2\theta$, which are known to occur for pure bending of the curved beam and the wedge, respectively. This need to consider the implication of both single and repeating eigenvalues for evolution in both directions simultaneously is a feature of the present analysis, and arises because the displacement components u_r and u_θ and the stress component $\tau_{r\theta}$ appear in the state vectors for both circumferential and radial evolution.

One might also expect that $k=\pm\mathrm{i}$ are associated with the curved beam subject to shear and/or tension, as this leads to (co)sinusoidal dependence on θ; Eq. (17) then reduces to $((\lambda+1)^2-1)((\lambda-1)^2-1)=0$. The first term in this product leads to $\lambda=0,-2$, the second to $\lambda=0,2$; recall that stress varies as $r^{\lambda-1}$, so $\lambda=-2,0$, and 2 lead, respectively, to stress varying as r^{-3}, r^{-1} and r, which are known forms for shear. As will be seen, shear of the curved beam is a principal vector, while tension is a combination of shear and pure bending, which are principal vectors from different eigen-spaces. The double root $\lambda=0$ implies an $\ln r$ term which arises from the coupling of a principal vector to an eigenvector for a repeating eigenvalue, as described in Section 6, and occurs for pure bending of the curved beam. Moreover, this double root implies, from Eq. (17), that $k=\pm\mathrm{i}$ is also repeated, indicating additional terms involving $\theta\sin\theta$ and $\theta\cos\theta$; such terms occur for the force transmission problem within the wedge.

Complex k, or at least the real part, is associated with the Saint-Venant decay of self-equilibrated end loading for the curved beam; the first term in the product Eq. (17) leads to $\lambda=-1\pm ik$, and hence stress varying as $r^{-2}r^{\pm ik}$, the second to $\lambda=1\pm ik$ and hence $r^{\pm ik}$. Noting the relationship $r^{ik}=(\mathrm{e}^{\ln r})^{ik}=\mathrm{e}^{ik\ln r}$, this leads to terms such as $\sin(k\ln r)$, $r^{-2}\sin(k\ln r)$, together with their cosine counterparts. Such terms have been employed previously by Kitover (1952) and Stephen and Wang (1993). On the other hand, general real and complex λ are associated with the decay of self-equilibrated loading for the wedge, and give rise to terms of the form $\sin(\lambda+1)\theta$ and $\sin(\lambda-1)\theta$, together with their cosine counterparts. Such terms have been employed previously by Stephen and Wang (1999) in a study of the applicability of Saint-Venant's principle for the elastic wedge. As will be seen, the treatment of these decay problems is particularly simple with the present state-space formulation, use of the matrix exponential, and a symbolic computation package such as MAPLE.

Finally, note that repetition and concurrence of the roots $\lambda=\pm 1$ as occurs for the particular angle associated with the wedge paradox implies, from Eq. (17), that the root $k=\pm 2\mathrm{i}$ is also repeated, indicating additional terms involving $\theta\sin 2\theta$

and $\theta\cos 2\theta$. The root pertaining to decay from the inner arc, $\lambda = -1$ repeats that pertaining to a pure bending moment for the wedge, which leads to additional $\ln r$ terms; the root pertaining to decay from the outer arc, $\lambda = 1$ repeats that pertaining to a rigid body rotation. Both these roots herald the breakdown of Saint-Venant's principle for asymmetric loads, in different ways. These degeneracies may be regarded as accidental, as they occur for one specific wedge angle only.

The possible multiple eigenvalues, and their associated eigen- and principal vectors, (e) and (p) respectively, are shown in Table 1; apart from the accidental degeneracies, these have been listed previously by Joseph and Zhang (1998).

4. Transmission modes for the wedge: radial evolution

When loading is applied to the circular arcs, with the flanks free of traction, one has the elastic wedge, Fig. 1. The radial evolution of the state vector on some generic arc is described by Eqs. (13),(14) with characteristic Eq. (16).

4.1. Eigenvector associated with $\lambda = 0$: rigid body displacement

A zero eigenvalue implies that the eigenvector satisfies the equation

$$\mathbf{H}\bar{\mathbf{s}}_1^{(0)} = \mathbf{0}. \tag{18}$$

The characteristic Eq. (16) becomes

$$(1 + \mathrm{d}^2/\mathrm{d}\theta^2)^2 = 0, \tag{19}$$

and setting the circumferential dependence as $\exp(k\theta)$ leads to $k = \pm\mathrm{i}$ with a multiplicity of two, and hence terms involving $\sin\theta$, $\cos\theta$, $\theta\sin\theta$ and $\theta\cos\theta$. Within a Cartesian coordinate system, the obvious eigenvectors are the rigid body displacements in the x- and y-directions with stress components equal to zero, and these take a particularly simple form; within a polar coordinate system, a rigid body displacement in the x-direction of magnitude A, leads to the symmetric eigenvector

$$\bar{\mathbf{s}}_{1\mathrm{sym}}^{(0)} = [A\cos\theta \quad -A\sin\theta \quad 0 \quad 0]^{\mathrm{T}}, \tag{20}$$

while a rigid body displacement in the y-direction of magnitude B, leads to the asymmetric eigenvector

$$\bar{\mathbf{s}}_{1\mathrm{asym}}^{(0)} = [B\sin\theta \quad B\cos\theta \quad 0 \quad 0]^{\mathrm{T}}. \tag{21}$$

The above are special cases of what may be gleaned from Eq. (18), under the assumption that the stress components are zero; in particular, this leads to the three distinct equations

$$\bar{u}_r + \mathrm{d}\bar{u}_\theta/\mathrm{d}\theta = 0, \tag{22a}$$
$$-\mathrm{d}\bar{u}_r/\mathrm{d}\theta + \bar{u}_\theta = 0, \tag{22b}$$
$$\mathrm{d}\bar{u}_r/\mathrm{d}\theta + \mathrm{d}^2\bar{u}_\theta/\mathrm{d}\theta^2 = 0. \tag{22c}$$

Addition of the second two of these leads to the equation

$$\mathrm{d}^2\bar{u}_\theta/\mathrm{d}\theta^2 + \bar{u}_\theta = 0, \tag{23}$$

and the solutions may be written as

$$\bar{u}_\theta = -A\sin\theta + B\cos\theta, \quad \bar{u}_r = -\mathrm{d}\bar{u}_\theta/\mathrm{d}\theta = A\cos\theta + B\sin\theta, \tag{24}$$

with constants A and B as defined above. The most general eigenvector is therefore

$$\bar{\mathbf{s}}_1^{(0)} = \bar{\mathbf{s}}_{1\mathrm{sym}}^{(0)} + \bar{\mathbf{s}}_{1\mathrm{asym}}^{(0)} = [(A\cos\theta + B\sin\theta) \quad (-A\sin\theta + B\cos\theta) \quad 0 \quad 0]^{\mathrm{T}}, \tag{25}$$

and represents a rigid body displacement of the wedge of magnitude $\sqrt{A^2 + B^2}$ in the direction $\theta = \tan^{-1}(B/A)$.

Table 1
Multiple eigenvalue possibilities from the characteristic equation, Eq. (17), and accidental degeneracies associated with the wedge paradox

Radial eigenvalues, λ	Circumferential eigenvalues, k	Eigen- (e) and principal (p) vector
0, 0	±i, ± i	Rigid displacement of wedge (e), tension/shear of wedge (p)
1	0, 0, ± 2i	Rigid rotation of wedge (e)
−1	0, 0, ± 2i	Pure bending of wedge (e)
±1, ±1	±2i, ±2i	Associated with the critical wedge angle $2\alpha^* \approx 257°$
0, 0, ± 2	±i, ±i	Rigid displacement of curved beam (e), shear of curved beam (p)
±1, ± 1	0, 0	Rigid rotation of curved beam (e), pure bending of curved beam (p)

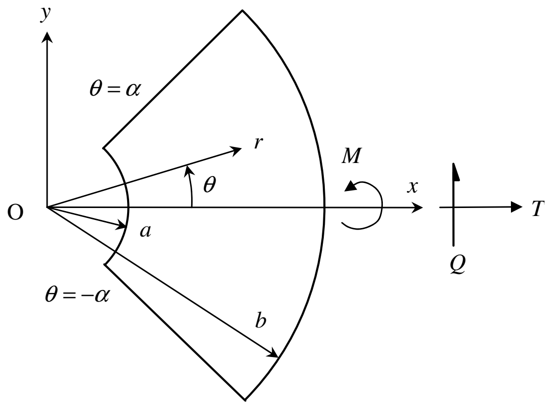

Fig. 1. Elastic wedge, showing positive stress resultants.

4.2. Principal vector associated with λ = 0: tensile and shearing forces

Coupled to this eigenvector is a principal vector, which again may be decomposed into symmetric and asymmetric components. The symmetric part

$$\bar{\mathbf{s}}^{(1)}_{1\mathrm{sym}} = \begin{bmatrix} A(1-\nu)\theta\sin\theta/2 - A(1+\nu)\cos\theta/2 \\ A(1-\nu)\theta\cos\theta/2 \\ EA\cos\theta \\ 0 \end{bmatrix}, \tag{26}$$

describes a tensile force in the x-direction, and is coupled to the rigid body displacement in the x-direction, $\bar{\mathbf{s}}^{(0)}_{1\mathrm{sym}}$, according to

$$\mathbf{H}\bar{\mathbf{s}}^{(1)}_{1\mathrm{sym}} = \bar{\mathbf{s}}^{(0)}_{1\mathrm{sym}}. \tag{27}$$

The requirement that the direct stress on the outer arc should constitute a tensile force T leads to $T = r\int_{-\alpha}^{\alpha}\sigma_r\cos\theta\,\mathrm{d}\theta = r\int_{-\alpha}^{\alpha}EA\cos^2\theta\,\mathrm{d}\theta$, from which one finds $A = 2T/(Er(2\alpha + \sin 2\alpha))$; note that this and all other stress resultants are per unit thickness. The constant A is now related to the deformation produced by the tensile force T. As is well-known, one can add any multiple of the generating eigenvector to a principal vector, and the result is still a principal vector. Physically, this represents the addition of an arbitrary rigid body displacement.

The asymmetric part

$$\bar{\mathbf{s}}^{(1)}_{1\mathrm{asym}} = \begin{bmatrix} -B(1-\nu)\theta\cos\theta/2 - B(1+\nu)\sin\theta/2 \\ B(1-\nu)\theta\sin\theta/2 \\ EB\sin\theta \\ 0 \end{bmatrix}, \tag{28}$$

describes a shearing force in the y-direction, and is coupled to the rigid body displacement in the y-direction, $\bar{\mathbf{s}}^{(0)}_{1\mathrm{asym}}$, according to

$$\mathbf{H}\bar{\mathbf{s}}^{(1)}_{1\mathrm{asym}} = \bar{\mathbf{s}}^{(0)}_{1\mathrm{asym}}. \tag{29}$$

The requirement that the direct stress on the outer arc should constitute a shearing force Q leads to $Q = r\int_{-\alpha}^{\alpha}\sigma_r\sin\theta\,\mathrm{d}\theta = r\int_{-\alpha}^{\alpha}EB\sin^2\theta\,\mathrm{d}\theta$, from which one finds $B = 2Q/(Er(2\alpha - \sin 2\alpha))$. The constant B is now related to the deformation produced by the shearing force Q. Again one can add multiples of the generating eigenvector to this principal vector, representing the addition of arbitrary rigid body displacement. The sum of the symmetric and asymmetric parts gives the most general principal vector as

$$\bar{\mathbf{s}}^{(1)}_{1} = \frac{T}{r(2\alpha+\sin 2\alpha)}\begin{bmatrix} ((1-\nu)\theta\sin\theta - (1+\nu)\cos\theta)/E \\ (1-\nu)\theta\cos\theta/E \\ 2\cos\theta \\ 0 \end{bmatrix} + \frac{Q}{r(2\alpha-\sin 2\alpha)}\begin{bmatrix} (-(1-\nu)\theta\cos\theta - (1+\nu)\sin\theta)/E \\ (1-\nu)\theta\sin\theta/E \\ 2\sin\theta \\ 0 \end{bmatrix}, \tag{30}$$

which describes a force in some arbitrary direction.

4.3. Eigenvector associated with $\lambda = 1$: rigid body rotation

For the strip problem using a Cartesian coordinate system (Stephen, 2004) a rigid body rotation is a principal vector coupled to a rigid body displacement in the y-direction, with $\lambda = 0$. For the present sector problem, rotation about the origin is simply $u_\theta = C_0 r$, where C_0 is a constant, and the eigenvector is

$$\bar{\mathbf{s}}_2 = [0 \quad C_0 \quad 0 \quad 0]^{\mathrm{T}}, \tag{31}$$

with $\lambda = 1$.

4.4. Eigenvector associated with $\lambda = -1$: pure bending

Pure bending is associated with the eigenvalue $\lambda = -1$; the characteristic Eq. (16) then takes the form

$$\mathrm{d}^2/\mathrm{d}\theta^2((-2)^2 + \mathrm{d}^2/\mathrm{d}\theta^2) = 0, \tag{32}$$

which suggests displacement components as

$$\begin{aligned} u_r &= \bar{u}_r r^{-1} = (C_1 + C_2\theta + C_3 \sin 2\theta + C_4 \cos 2\theta) r^{-1}, \\ u_\theta &= \bar{u}_\theta r^{-1} = (C_5 + C_6\theta + C_7 \sin 2\theta + C_8 \cos 2\theta) r^{-1}, \end{aligned} \tag{33}$$

where $C_{1,2,\ldots,8}$ are constants. If one calculates the strain and thence the stress components from the above, the first two rows of Eqs. (13) are satisfied identically (as they must – after all these are just the Hooke's law rearranged), while the third and fourth rows demand that the eigenvector should take the form

$$\begin{bmatrix} \bar{u}_r \\ \bar{u}_\theta \\ \bar{\sigma}_r \\ \bar{\tau}_{r\theta} \end{bmatrix} = \begin{bmatrix} C_1 + C_3 \sin 2\theta + C_4 \cos 2\theta \\ C_5 - C_4(1-\nu)\sin 2\theta/2 + C_3(1-\nu)\cos 2\theta/2 \\ -EC_1/(1+\nu) - EC_3 \sin 2\theta - EC_4 \cos 2\theta \\ EC_3 \cos 2\theta/2 - EC_4 \sin 2\theta/2 - EC_5/(1+\nu) \end{bmatrix}, \tag{34}$$

with circumferential stress $\bar{\sigma}_\theta = EC_1/(1+\nu)$; since the flanks of the wedge are free of traction, one must have $C_1 = 0$. Set $C_4 = 0$, which is consistent with asymmetric radial displacement and stress fields, and also the requirement that there should be no resultant in the x-direction, and imposing $\tau_{r\theta}$ on $\theta = \pm\alpha$, leads to $C_5 = C_3\,(1+\nu)\cos 2\alpha/2$; finally, the requirement that the shearing stress on the inner arc should constitute a moment M about the origin, that is $M = r^2 \int_{-\alpha}^{\alpha} \tau_{r\theta}\mathrm{d}\theta$, leads to $M = EC_3(\sin 2\alpha - 2\alpha\cos 2\alpha)/2$. The resulting eigenvector is then

$$\bar{\mathbf{s}}_3 = \begin{bmatrix} \bar{u}_r \\ \bar{u}_\theta \\ \bar{\sigma}_r \\ \bar{\tau}_{r\theta} \end{bmatrix} = \frac{M}{(\sin 2\alpha - 2\alpha \cos 2\alpha)} \begin{bmatrix} 2\sin 2\theta/E \\ ((1+\nu)\cos 2\alpha + (1-\nu)\cos 2\theta)/E \\ -2\sin 2\theta \\ \cos 2\theta - \cos 2\alpha \end{bmatrix}, \tag{35}$$

which corresponds to the well-known Carothers (1912) solution (see also Timoshenko and Goodier, 1970, article 39). Note that the stress components apparently become infinite for $\sin 2\alpha^* = 2\alpha^* \cos 2\alpha^*$, which occurs for $2\alpha^*$ equal to approximately 257°; this is the original wedge paradox, described first by Sternberg and Koiter (1958). In fact, for this particular wedge angle, it is clear from the above that the bending moment M becomes equal to zero, so the stress field is self-equilibrating. Moreover, $\lambda = -1$ is now repeating, an accidental degeneracy, as the decay problem has the same eigenvalue for this particular wedge angle. The wedge paradox is therefore resolved by the introduction of a coupled principal vector, and is treated in Section 9.

4.5. Jordan canonical form for the wedge transmission modes

Now construct a 4×4 transformation matrix $\mathbf{S}$ consisting of the eigen- and principal vectors as $\mathbf{S} = \left[\bar{\mathbf{s}}_1^{(0)} \quad \bar{\mathbf{s}}_1^{(1)} \quad \bar{\mathbf{s}}_2 \quad \bar{\mathbf{s}}_3\right]$, and one finds $\mathbf{HS} = \mathbf{SJ}$, where

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{36}$$

is the Jordan canonical form (JCF) for the wedge transmission modes, in which the single unity on the superdiagonal couples the eigen- and principal vector for the repeating zero eigenvalue. The eigenvalues occur as ± pairs, in accordance with the Hamiltonian property of $\mathbf{H}$. Now, despite the system matrix $\mathbf{H}$ having dimension 4×4, there are an infinite number of eigenvectors (beside the four transmission vectors considered above) associated with the decay of self-equilibrated loading. The characteristic equation for these decay eigenvalues is derived in Section 5, although the associated decay vectors are not determined explicitly. In principle however, once determined, any four vectors could be chosen at random to form a square transformation matrix **S**, and the system matrix **H** transformed into a new JCF. Two questions then spring to mind: first, would the eigenvalues on the leading diagonal of this new JCF still occur as ± pairs? Second, what repeating eigenvalues can give rise to coupling between eigen- and principal vectors, with an attendant unity element on the superdiagonal of the JCF? The answer to the first question is that while the decay eigenvalues, as will be seen, do indeed occur as ± pairs, the probability of choosing such a pair from the infinite set of decay vectors is extremely remote, so in general one should only expect the eigenvalues displayed on the leading diagonal of the JCF to occur as ± pairs for the transmission modes, as above. The answer to the second evolves from the treatment of the wedge paradox in Section 9, and strain energy arguments advanced in Section 10.1. Besides the repeating zero eigenvalue noted above, only repeating $\lambda = 1$ and $\lambda = -1$ can give rise to non-trivial Jordan blocks; these are accidental degeneracies that occur only for the critical wedge angle $2\alpha^*$.

5. Saint-Venant decay for the wedge: circumferential evolution

Define a state vector describing circumferential evolution as

$$\mathbf{v} = [u_r\, u_\theta\, \tau_{r\theta}\, \sigma_\theta]^{\mathrm{T}}. \tag{37}$$

First, one needs to eliminate σ_r from the equilibrium Eq. (1a), as it is not state-variable for this problem, and one employs the Hooke's law (3a), (2a) rearranged as

$$\sigma_r = \nu\sigma_\theta + E\partial u_r/\partial r. \tag{38}$$

Proceeding in a similar fashion to Section 3, initially one finds

$$\frac{\partial}{\partial\theta}\begin{bmatrix} u_r \\ u_\theta \\ \tau_{r\theta} \\ \sigma_\theta \end{bmatrix} = \begin{bmatrix} 0 & 1 - r\partial/\partial r & r/G & 0 \\ -1 - \nu r\partial/\partial r & 0 & 0 & r(1-\nu^2)/E \\ -E\partial/\partial r - Er\partial^2/\partial r^2 & 0 & 0 & (1-\nu) - \nu r\partial/\partial r \\ 0 & 0 & -2 - r\partial/\partial r & 0 \end{bmatrix}\begin{bmatrix} u_r \\ u_\theta \\ \tau_{r\theta} \\ \sigma_\theta \end{bmatrix}. \tag{39}$$

Again, introduce the barred quantities according to Eq. (12), to give

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{bmatrix} \bar{u}_r \\ \bar{u}_\theta \\ \bar{\tau}_{r\theta} \\ \bar{\sigma}_\theta \end{bmatrix} = \begin{bmatrix} 0 & 1-\lambda & 1/G & 0 \\ -(1+\nu\lambda) & 0 & 0 & (1-\nu^2)/E \\ -E\lambda^2 & 0 & 0 & (1-\nu\lambda) \\ 0 & 0 & -(1+\lambda) & 0 \end{bmatrix}\begin{bmatrix} \bar{u}_r \\ \bar{u}_\theta \\ \bar{\tau}_{r\theta} \\ \bar{\sigma}_\theta \end{bmatrix}, \tag{40}$$

or

$$\mathrm{d}\bar{\mathbf{v}}/\mathrm{d}\theta = \mathbf{H}'\bar{\mathbf{v}}, \tag{41}$$

which defines the primed system matrix; note that the characteristic equation associated with the above is identical to Eq. (17) for radial evolution. The formal solution may be written as

$$\bar{\mathbf{v}}(\theta) = \mathrm{e}^{\mathbf{H}'\theta}\bar{\mathbf{v}}(0), \tag{42}$$

where the matrix exponential $\mathbf{M} = \mathrm{e}^{\mathbf{H}'\theta}$ is calculated as the inverse Laplace transform of the resolvent matrix $\mathbf{R}' = (s\mathbf{I} - \mathbf{H}')^{-1}$, and s is the Laplace variable; this is readily accomplished using a symbolic computation package, such as MAPLE. The elements of matrix **M** are given in Appendix A. It is convenient, although not necessary, to redefine the coordinate system such that the wedge angle ranges from $\theta = 0$ to $\theta = 2\alpha$, rather than $\theta = \pm\alpha$. The traction-free boundary condition on the lower flank of the wedge, now $\alpha = 0$, implies that the "initial" state vector is $\bar{\mathbf{v}}(0) = [\,\bar{u}_r(0) \quad \bar{u}_\theta(0) \quad 0 \quad 0\,]^{\mathrm{T}}$, so Eq. (42) becomes, in more detail

$$\bar{\mathbf{v}}(\theta) = \begin{bmatrix} \bar{u}_r(\theta) \\ \bar{u}_\theta(\theta) \\ \bar{\tau}_{r\theta}(\theta) \\ \bar{\sigma}_\theta(\theta) \end{bmatrix} = \begin{bmatrix} M_{11}(\theta)\bar{u}_r(0) + M_{12}(\theta)\bar{u}_\theta(0) \\ M_{21}(\theta)\bar{u}_r(0) + M_{22}(\theta)\bar{u}_\theta(0) \\ M_{31}(\theta)\bar{u}_r(0) + M_{32}(\theta)\bar{u}_\theta(0) \\ M_{41}(\theta)\bar{u}_r(0) + M_{42}(\theta)\bar{u}_\theta(0) \end{bmatrix}. \tag{43}$$

But one also has traction-free conditions on the upper flank of the wedge, that is $\bar{\tau}_{r\theta}(2\alpha) = \bar{\sigma}_\theta(2\alpha) = 0$, giving

$$\begin{bmatrix} M_{31}(2\alpha) & M_{32}(2\alpha) \\ M_{41}(2\alpha) & M_{42}(2\alpha) \end{bmatrix}\begin{bmatrix} \bar{u}_r(0) \\ \bar{u}_\theta(0) \end{bmatrix} = \mathbf{0}. \tag{44}$$

Since the displacement components on the lower flank are not zero, the determinant must be equal to zero, leading to the eigenequation

$$\lambda^2(1 - \lambda^2 - \cos 4\lambda\alpha + \lambda^2\cos 4\alpha) = 0, \tag{45}$$

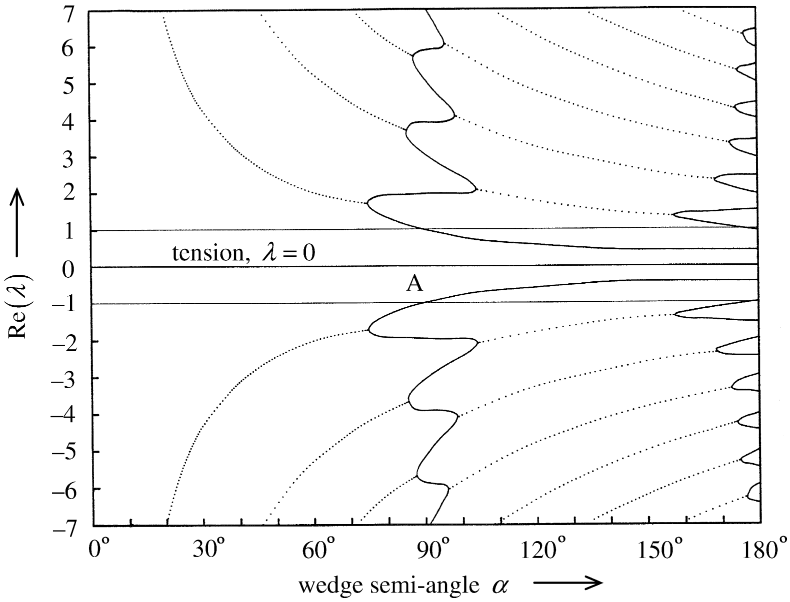

Fig. 2. Locus of symmetric decay eigenvalues; solid and dotted lines represent real and complex roots, respectively. The lines $\lambda = \pm 1$ are not roots, but are inserted as a reference.

which may be expressed in the more familiar form (see Timoshenko and Goodier, 1970, article 46) as

$$\lambda^2(\sin 2\lambda\alpha \pm \lambda \sin 2\alpha) = 0. \tag{46}$$

If one does not wish to redefine the coordinate system, then from Eq. (42) one has $\bar{\mathbf{v}}(-\alpha) = e^{\mathbf{H}'(-\alpha)}\bar{\mathbf{v}}(0)$, or $\bar{\mathbf{v}}(0) = e^{\mathbf{H}'\alpha}\bar{\mathbf{v}}(-\alpha)$. Eq. (42) can then be rewritten as $\bar{\mathbf{v}}(\theta) = e^{\mathbf{H}'\theta}e^{\mathbf{H}'\alpha}\bar{\mathbf{v}}(-\alpha) = e^{\mathbf{H}'(\theta+\alpha)}\bar{\mathbf{v}}(-\alpha)$. The traction-free condition on the upper flank (now $\theta = \alpha$) leads to Eq. (44), but with the displacement components in the column vector defined on the lower flank $\theta = -\alpha$; the governing eigenequation is obviously the same.

It is clear from Eq. (46) that if λ is an eigenvalue, then so too is $-\lambda$. The leading double root $\lambda = 0$ pertains to the rigid body displacement eigenvector and the coupled principal vector describing tensile and/or shearing force. The plus and minus signs pertain to the symmetric and asymmetric decay problems, respectively; the asymmetric case is also satisfied, for all α, by the roots $\lambda = -1$ which is associated with pure bending, and $\lambda = 1$ which is associated with a rigid body rotation.

The roots of Eq. (46) for varying wedge angle are plotted in Figs. 2 and 3, for the symmetric and asymmetric case, respectively. In Fig. 3, note the repeating eigenvalues $\lambda = \pm 1$ for the so-called critical wedge angle $2\alpha^*$, equal to approximately 257°; these are associated with the breakdown of Saint-Venant's principle (for asymmetric loading) and also the wedge paradox. This limit of applicability of Saint-Venant's principle has been attributed by Stephen and Wang (1999) to the effect of the diverging/converging geometry and, in fact, arises initially for the half-space, $2\alpha = \pi$. First recall that stress varies as $r^{\lambda-1}$; for asymmetric loads applied to the inner arc $r = a$, the divergent geometry leads to a reduction in stress associated with the moment, which diffuse as r^{-2} ($\lambda = -1$). Thus Saint-Venant's principle breaks down in the sense that self-equilibrating loads on the inner arc associated with $\lambda = -1$ decays at precisely the same rate as the moment diffuses; this occurs at $2\alpha = \pi$ (point A, Fig. 2) and $2\alpha^* = 1.43\pi$ (point B, Fig. 3) for symmetric and asymmetric self-equilibrated load, respectively.

Further, the eigenvalue $\lambda = 1$ implies that stress is independent of radius, and the associated vectors describing self-equilibrated loading now ceases to decay. This eigenvalue[1] also occurs at $2\alpha = \pi$ for symmetric loading, and $2\alpha^*$ for asymmetric loading. For larger angles, Saint-Venant's principle now breaks down in the sense that stress due to self-equilibrating loads applied to the outer arc, $r = b$, increases as one moves toward the origin due to the convergent geometry. For the maximum wedge angle, $2\alpha = 2\pi$, the unique modes I and II inverse square root stress singularities at a crack-tip, which lie at the heart of the discipline of *Linear elastic fracture mechanics* (*LEFM*), can be attributed to the breakdown of Saint-Venant's principle for just one symmetric and one asymmetric eigenvector, each associated with the eigenvalue $\lambda = 1/2$.

6. Radial and circumferential evolution employing ln *r*

The assumption of radial power-law dependence has been adequate for dealing with the transmission and Saint-Venant decay problems for the wedge. However, modified formulations for both radial and circumferential evolution are required

[1] Note that the eigenvalue $\lambda = 1$ in the present paper is equivalent to $\lambda = -2$ of Stephen and Wang (1999).

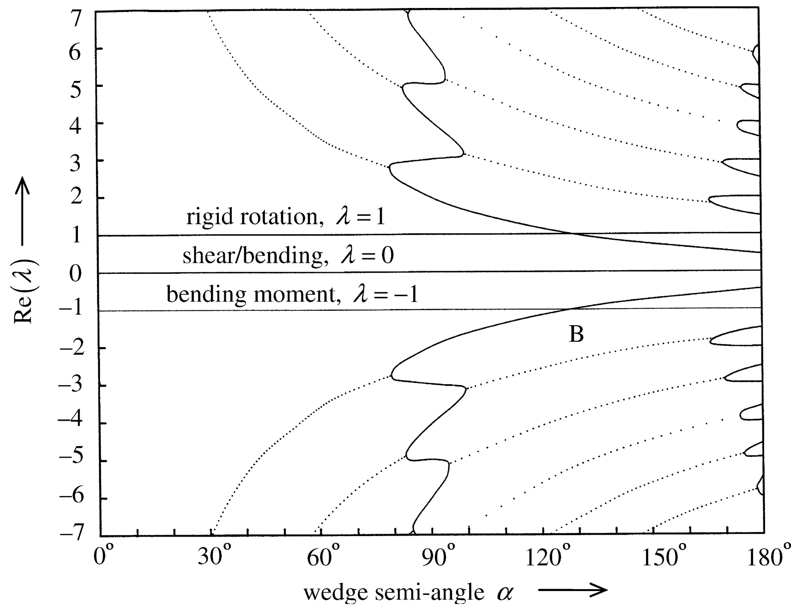

Fig. 3. Locus of asymmetric decay eigenvalues; solid and dotted lines represent real and complex roots, respectively.

for the transmission and decay problems for the curved beam, and treatment of the wedge paradox; these formulations are developed now.

6.1. Radial evolution

Introduce $\xi = \ln r$, then $r^\lambda = (e^{\ln r})^\lambda = e^{\lambda \ln r} = e^{\lambda\xi}$; further

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial \xi}\frac{d\xi}{dr} = \frac{1}{r}\frac{\partial}{\partial \xi}. \tag{47}$$

In addition

$$\frac{\partial^2}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \xi}\right) = -\frac{1}{r^2}\frac{\partial}{\partial \xi} + \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial \xi} = \frac{1}{r^2}\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial}{\partial \xi}\right). \tag{48}$$

Eq. (7) becomes

$$\frac{1}{r}\frac{\partial u_r}{\partial \xi} = \frac{(1-\nu^2)}{E}\sigma_r - \nu\frac{u_r}{r} - \frac{\nu}{r}\frac{\partial u_\theta}{\partial \theta}; \tag{49}$$

r-dependence can be removed by writing $S_r = r\sigma_r$, when this becomes

$$\frac{\partial u_r}{\partial \xi} = \frac{(1-\nu^2)}{E}S_r - \nu u_r - \nu\frac{\partial u_\theta}{\partial \theta}. \tag{50}$$

Similarly, by introducing $S_{r\theta} = r\tau_{r\theta}$, Eq. (8) becomes

$$\frac{\partial u_\theta}{\partial \xi} = \frac{S_{r\theta}}{G} + u_\theta - \frac{\partial u_r}{\partial \theta}. \tag{51}$$

Eqs. (9),(10) require greater attention: first note that $\partial S_r/\partial \xi = \partial(r\sigma_r)/\partial \xi = r\partial(r\sigma_r)/\partial r = r(\sigma_r + r\frac{\partial \sigma_r}{\partial r})$, so

$$r^2\frac{\partial \sigma_r}{\partial r} = \frac{\partial S_r}{\partial \xi} - S_r. \tag{52}$$

Similar relationships hold between $\tau_{r\theta}$ and $S_{r\theta}$, and σ_θ and S_θ. Eqs. (9),(10) are therefore multiplied by r^2, and with appropriate substitutions, Eq. (9) becomes

$$\frac{\partial S_r}{\partial \xi} = -\frac{\partial S_{r\theta}}{\partial \theta} + Eu_r + E\frac{\partial u_\theta}{\partial \theta} + \nu S_r, \tag{53}$$

while Eq. (10) becomes

$$\frac{\partial S_{r\theta}}{\partial \xi} = -\nu \frac{\partial S_r}{\partial \theta} - E\frac{\partial u_r}{\partial \theta} - E\frac{\partial^2 u_\theta}{\partial \theta^2} - S_{r\theta}. \tag{54}$$

In matrix form, Eqs. (50),(51),(53),(54) are

$$\frac{\partial}{\partial \xi}\begin{bmatrix} u_r \\ u_\theta \\ S_r \\ S_{r\theta} \end{bmatrix} = \begin{bmatrix} -\nu & -\nu\partial/\partial\theta & (1-\nu^2)/E & 0 \\ -\partial/\partial\theta & 1 & 0 & 1/G \\ E & E\partial/\partial\theta & \nu & -\partial/\partial\theta \\ -E\partial/\partial\theta & -E\partial^2/\partial\theta^2 & -\nu\partial/\partial\theta & -1 \end{bmatrix}\begin{bmatrix} u_r \\ u_\theta \\ S_r \\ S_{r\theta} \end{bmatrix}, \tag{55}$$

or more compactly $\partial \mathbf{s}/\partial\xi = \mathbf{Hs}$. This system matrix is identical to that in Eq. (13), except that the differentials with respect to θ are, at this stage, still partial; also, the state vector $\mathbf{s}$ is defined in a slightly different manner. Writing

$$\mathbf{s} = \bar{\mathbf{s}}(\theta)\mathrm{e}^{\lambda\xi} = [\bar{u}_r \quad \bar{u}_\theta \quad \bar{S}_r \quad \bar{S}_{r\theta}]^{\mathrm{T}} r^\lambda, \tag{56}$$

leads to the eigenequation

$$\lambda\bar{\mathbf{s}} = \mathbf{H}\bar{\mathbf{s}}, \tag{57}$$

and the system matrix $\mathbf{H}$ is now the same as in Eq. (13); again, note that an over bar indicates that the vector is independent of radius. When a principal vector is coupled to a generating eigenvector (designated by superscripts (1) and (0), respectively) with a repeating eigenvalue, one has the chain

$$\mathbf{H}\bar{\mathbf{s}}^{(1)} = \lambda\bar{\mathbf{s}}^{(1)} + \bar{\mathbf{s}}^{(0)}. \tag{58}$$

The state vector is then

$$\mathbf{s} = (\bar{\mathbf{s}}^{(1)} + \xi\bar{\mathbf{s}}^{(0)})\mathrm{e}^{\lambda\xi}. \tag{59}$$

Also note that one can add an arbitrary multiple (say β) of the generating eigenvector to a principal vector, and it is still a principal vector, that is

$$\mathbf{H}(\bar{\mathbf{s}}^{(1)} + \beta\bar{\mathbf{s}}^{(0)}) = \lambda(\bar{\mathbf{s}}^{(1)} + \beta\bar{\mathbf{s}}^{(0)}) + \bar{\mathbf{s}}^{(0)}, \tag{60}$$

and the state vector is then

$$\mathbf{s} = ((\bar{\mathbf{s}}^{(1)} + \beta\bar{\mathbf{s}}^{(0)}) + \xi\bar{\mathbf{s}}^{(0)})\mathrm{e}^{\lambda\xi}. \tag{61}$$

In more detail, this is

$$\begin{bmatrix} u_r \\ u_\theta \\ r\sigma_r \\ r\tau_{r\theta} \end{bmatrix} = \left(\begin{bmatrix} \bar{u}_r \\ \bar{u}_\theta \\ \bar{S}_r \\ \bar{S}_{r\theta} \end{bmatrix}^{(1)} + (\beta + \ln r)\begin{bmatrix} \bar{u}_r \\ \bar{u}_\theta \\ \bar{S}_r \\ \bar{S}_{r\theta} \end{bmatrix}^{(0)}\right) r^\lambda. \tag{62}$$

6.2. Circumferential evolution

Now one requires the use of Eqs. (47),(48) within Eq. (39), together with the modified stress components $S_{r\theta} = r\tau_{r\theta}$ and $S_\theta = r\sigma_\theta$, to give

$$\frac{\partial}{\partial \theta}\begin{bmatrix} u_r \\ u_\theta \\ S_{r\theta} \\ S_\theta \end{bmatrix} = \begin{bmatrix} 0 & 1-\partial/\partial\xi & 1/G & 0 \\ -1-\nu\partial/\partial\xi & 0 & 0 & (1-\nu^2)/E \\ -E\partial^2/\partial\xi^2 & 0 & 0 & 1-\nu\partial/\partial\xi \\ 0 & 0 & -1-\partial/\partial\xi & 0 \end{bmatrix}\begin{bmatrix} u_r \\ u_\theta \\ S_{r\theta} \\ S_\theta \end{bmatrix}, \tag{63}$$

or

$$\partial\mathbf{v}/\partial\theta = \mathbf{H}''\mathbf{v}. \tag{64}$$

Writing

$$\mathbf{v} = \tilde{\mathbf{v}}(\xi)\mathrm{e}^{k\theta} = \begin{bmatrix} \tilde{u}_r & \tilde{u}_\theta & \tilde{S}_{r\theta} & \tilde{S}_\theta \end{bmatrix}^{\mathrm{T}} \mathrm{e}^{k\theta}, \tag{65}$$

where tilde denotes independence of θ, leads to the eigenequation

$$k\tilde{\mathbf{v}} = \mathbf{H}''\tilde{\mathbf{v}}. \tag{66}$$

When a principal vector is coupled to a generating eigenvector with a repeating eigenvalue, one has the chain

$$\mathbf{H}''\tilde{\mathbf{v}}^{(1)} = k\tilde{\mathbf{v}}^{(1)} + \tilde{\mathbf{v}}^{(0)}, \tag{67}$$

where again superscripts (0) and (1) are employed for the eigen- and principal vectors, respectively. The state vector is then

$$\mathbf{v} = (\tilde{\mathbf{v}}^{(1)} + \theta\tilde{\mathbf{v}}^{(0)})\mathrm{e}^{k\theta}. \tag{68}$$

Again, an arbitrary multiple (β) of the generating eigenvector may be added, that is

$$\mathbf{H}''(\tilde{\mathbf{v}}^{(1)} + \beta\tilde{\mathbf{v}}^{(0)}) = k(\tilde{\mathbf{v}}^{(1)} + \beta\tilde{\mathbf{v}}^{(0)}) + \tilde{\mathbf{v}}^{(0)}, \tag{69}$$

and the state vector is

$$\mathbf{v} = ((\tilde{\mathbf{v}}^{(1)} + \beta\tilde{\mathbf{v}}^{(0)}) + \theta\tilde{\mathbf{v}}^{(0)})\mathrm{e}^{k\theta}. \tag{70}$$

In more detail, this is

$$\begin{bmatrix} u_r \\ u_\theta \\ r\tau_{r\theta} \\ r\sigma_\theta \end{bmatrix} = \left(\begin{bmatrix} \tilde{u}_r \\ \tilde{u}_\theta \\ \widetilde{S}_{r\theta} \\ \widetilde{S}_\theta \end{bmatrix}^{(1)} + (\beta+\theta) \begin{bmatrix} \tilde{u}_r \\ \tilde{u}_\theta \\ \widetilde{S}_{r\theta} \\ \widetilde{S}_\theta \end{bmatrix}^{(0)} \right)\mathrm{e}^{k\theta}. \tag{71}$$

7. Transmission modes for the curved beam: circumferential evolution

For the curved beam, Fig. 4, loading is confined to the straight radial edges (flanks) while the circular arcs are traction-free. Here we employ circumferential evolution according to Eq. (63).

7.1. Eigenvector associated with k = 0: rigid body rotation

The eigenvector describing a rigid body rotation about the origin may be stated simply as

$$\tilde{\mathbf{v}}_1^{(0)} = [0 \quad C_0 r \quad 0 \quad 0]^{\mathrm{T}} = [0 \quad C_0\mathrm{e}^{\xi} \quad 0 \quad 0]^{\mathrm{T}}, \tag{72}$$

where C_0 is a constant. Eq. (66) leads to the requirement $\mathbf{H}''\tilde{\mathbf{v}}_1^{(0)} = \mathbf{0}$, which reduces to $(1 - r\partial/\partial r)C_0 r = 0$, and is satisfied identically.

7.2. Principal vector associated with k = 0: pure bending

The repeated root $k = 0$ implies, from the characteristic Eq. (17), that $\lambda = \pm 1$ are also repeating roots, and suggests that the expressions for the displacement components should take the form

$$u_r = C_1 r + C_2 r\theta + C_3/r + C_4 r\ln r, \tag{73a}$$
$$u_\theta = C_5 r + C_6 r\theta + C_7/r + C_8 r\ln r. \tag{73b}$$

Two of the terms in each component arise from the radial dependence of r^λ, with $\lambda = \pm 1$; the $r\theta$ term originates from the coupling with the eigenvector as in Eq. (68), for the repeating root $k = 0$, while the $r\ln r$ ($=\xi\mathrm{e}^{\xi}$) term originates from the coupling

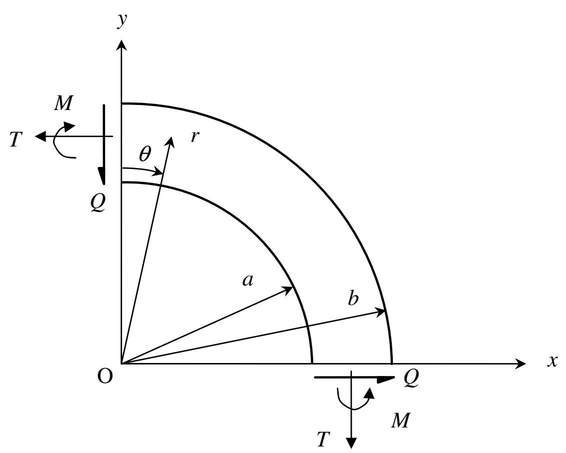

Fig. 4. Curved beam showing positive stress resultants.

with the eigenvector as in Eq. (59), for a repeating root $\lambda = 1$. Again, we note the necessity of considering the implication of repeating eigenvalues for evolution in both directions. From Eqs. (73) one may determine stress components as

$$\sigma_r = \frac{E}{(1-\nu)}(C_1 + C_4 \ln r) - \frac{E}{(1+\nu)}\frac{C_3}{r^2} + \frac{E}{(1-\nu^2)}(C_4 + \nu C_6), \tag{74a}$$

$$\sigma_\theta = \frac{E}{(1-\nu)}(C_1 + C_2\theta + C_4 \ln r) + \frac{E}{(1+\nu)}\frac{C_3}{r^2} + \frac{E}{(1-\nu^2)}(C_6 + \nu C_4), \tag{74b}$$

$$\tau_{r\theta} = \frac{E}{2(1+\nu)}(C_2 + C_8). \tag{74c}$$

First note that the shearing stress is apparently a constant, but as it is zero everywhere on the boundary, one must have $C_2 = -C_8$. Since $\tau_{r\theta} = 0, S_{r\theta} = 0$, so the first row of Eq. (64) reduces to $2C_7/r - C_8 r = 0$, and as this must be true for all values of the radius, one has $C_7 = C_8 = 0$, and hence $C_2 = 0$. The second row of Eq. (63) reduces to $C_6 = C_0$, the third row reduces to $C_4 = (1-\nu)C_0/2$, while the fourth row is satisfied identically as $S_{r\theta} = 0$. Finally, note that the term $C_5 r$ does not contribute to the stress; it represents the rigid body rotation about the origin, which is the eigenvector, multiples of which can always be added to a principal vector. The stress components are then

$$\sigma_r = -\frac{E}{(1+\nu)}\frac{C_3}{r^2} + \frac{E}{(1-\nu)}(C_1 + C_4 \ln r) + \frac{EC_0}{2(1-\nu)}, \tag{75a}$$

$$\sigma_\theta = \frac{E}{(1+\nu)}\frac{C_3}{r^2} + \frac{E}{(1-\nu)}(C_1 + C_4 \ln r) + \frac{EC_0(2-\nu)}{2(1-\nu)}, \tag{75b}$$

$$\tau_{r\theta} = 0. \tag{75c}$$

These are equivalent to the expressions given by Timoshenko and Goodier (1970), articles 28 and 29, derived from the axi-symmetric Airy stress function[2]

$$\Phi = A' \ln r + B' r^2 \ln r + C' r^2 + D', \tag{76}$$

which are

$$\sigma_r = A'/r^2 + B'(1 + 2\ln r) + 2C', \tag{77a}$$

$$\sigma_\theta = -A'/r^2 + B'(3 + 2\ln r) + 2C', \tag{77b}$$

$$\tau_{r\theta} = 0, \tag{77c}$$

if one sets $A' = -EC_3/(1+\nu)$, $B' = EC_0/4 = EC_4/(2(1-\nu))$, and $C' = (4EC_1 + EC_0\,(1+\nu))/(8(1-\nu))$. The various constants are evaluated from the requirements:

(a) radial stress $\sigma_r = 0$ on $r = a$, and $r = b$
(b) tensile force $T = \int_a^b \sigma_\theta \mathrm{d}r = 0$, and bending moment $M = -\int_a^b \sigma_\theta r \mathrm{d}r$.

The requirement that shearing stress $\tau_{r\theta}$ should be zero on the boundary, and not constitute a shearing force on the beam end, has already been satisfied. One finds

$$C_0 = -8M(b^2 - a^2)/E\Delta, \tag{76a}$$

$$C_1 = 4M(b^2 - a^2 + (1-\nu)(b^2 \ln b - a^2 \ln a))/E\Delta, \tag{76b}$$

$$C_3 = 4M(1+\nu)a^2 b^2 \ln(b/a)/E\Delta, \tag{76c}$$

$$C_4 = -4M(1-\nu)(b^2 - a^2)/E\Delta, \tag{76d}$$

where

$$\Delta = (b^2 - a^2)^2 - 4a^2 b^2 (\ln(b/a))^2. \tag{78}$$

The principal vector becomes

$$\tilde{\mathbf{v}}_1^{(1)} = \begin{bmatrix} \tilde{u}_r \\ \tilde{u}_\theta \\ \widetilde{S}_{r\theta} \\ \widetilde{S}_\theta \end{bmatrix}^{(1)} = \begin{bmatrix} C_1 r + C_3/r + C_4 r \ln r \\ 0 \\ 0 \\ EC_3/((1+\nu)r) + E(C_1 r + C_4 r \ln r)/(1-\nu) + EC_0(2-\nu)r/(2(1-\nu)) \end{bmatrix}, \tag{79}$$

or in terms of ξ

[2] A prime has been added to these constants to distinguish them from those already employed.

$$\tilde{\mathbf{v}}_1^{(1)} = \begin{bmatrix} \tilde{u}_r \\ \tilde{u}_\theta \\ \widetilde{S}_{r\theta} \\ \widetilde{S}_\theta \end{bmatrix}_1^{(1)} = \begin{bmatrix} C_1 e^{\xi} + C_3 e^{-\xi} + C_4 \xi e^{\xi} \\ 0 \\ 0 \\ EC_3 e^{-\xi}/(1+v) + E(C_1 e^{\xi} + C_4 \xi e^{\xi})/(1-v) + EC_0(2-v)e^{\xi}/(2(1-v)) \end{bmatrix}, \tag{80}$$

and satisfies the relationship $\mathbf{H}''\tilde{\mathbf{v}}_1^{(1)} = \tilde{\mathbf{v}}_1^{(0)}$.

7.3. Eigenvector associated with $k = \pm i$: rigid body displacement

First, assume a rigid body displacement A in the x-direction, then $u_r = A\sin\theta$ and $u_\theta = A\cos\theta$. Next, assume a rigid body displacement B in the y-direction, then $u_r = B\cos\theta$ and $u_\theta = -B\sin\theta$. An arbitrary rigid body displacement may therefore be written as

$$\begin{bmatrix} u_r \\ u_\theta \end{bmatrix} = \begin{bmatrix} A\sin\theta + B\cos\theta \\ A\cos\theta - B\sin\theta \end{bmatrix}. \tag{81}$$

Define a complex eigenvector as

$$\tilde{\mathbf{v}}_2^{(0)} = \begin{bmatrix} \tilde{u}_r & \tilde{u}_\theta & \widetilde{S}_{r\theta} & \widetilde{S}_\theta \end{bmatrix}^{\mathrm{T}} = [\,(-\mathrm{i}A + B) \quad (A + \mathrm{i}B) \quad 0 \quad 0\,]^{\mathrm{T}}, \tag{82}$$

then the rigid body displacements defined by Eq. (80) are the real part of

$$\mathbf{v} = \tilde{\mathbf{v}}_2^{(0)} e^{\mathrm{i}\theta} = \begin{bmatrix} -\mathrm{i}A + B \\ A + \mathrm{i}B \\ 0 \\ 0 \end{bmatrix} e^{\mathrm{i}\theta} = \begin{bmatrix} A\sin\theta + B\cos\theta + \mathrm{i}(-A\cos\theta + B\sin\theta) \\ A\cos\theta - B\sin\theta + \mathrm{i}(A\sin\theta + B\cos\theta) \\ 0 \\ 0 \end{bmatrix}. \tag{83}$$

This eigenvector satisfies the relationship

$$\mathbf{H}''\tilde{\mathbf{v}}_2^{(0)} = \mathrm{i}\tilde{\mathbf{v}}_2^{(0)}. \tag{84}$$

7.4. Principal vector associated with $k = \pm i$: shearing force

The repeated roots $k = \pm\mathrm{i}$ implies, from the characteristic Eq. (17), $\lambda = \pm 2$ together with repeating $\lambda = 0$, suggesting that the expressions for the displacement components should take the form:

$$u_r = e^{\mathrm{i}\theta}((-\mathrm{i}A + B)\theta + C_2 \ln r + C_3 r^2 + C_4 r^{-2} + C_5), \tag{85a}$$

$$u_\theta = e^{\mathrm{i}\theta}((A + \mathrm{i}B)\theta + C_7 \ln r + C_8 r^2 + C_9 r^{-2} + C_{10}), \tag{85b}$$

where the $\theta e^{\mathrm{i}\theta}$ terms arises through coupling of the eigen- and principal vectors according to Eq. (68), while the $\ln r$ terms arises from the repeating eigenvalue $\lambda = 0$ according to Eq. (59); the former terms contribute to the stress components, but not the displacement components within the principal vector which, in terms of ξ, are

$$\tilde{u}_r = C_2\xi + C_3 e^{2\xi} + C_4 e^{-2\xi} + C_5, \tag{86a}$$

$$\tilde{u}_\theta = C_7\xi + C_8 e^{2\xi} + C_9 e^{-2\xi} + C_{10}. \tag{86b}$$

From Eqs. (85), one may determine the modified stress components as

$$\widetilde{S}_{r\theta} = \frac{E}{2(1+v)}(C_7 - C_{10} + \mathrm{i}C_5 - \mathrm{i}A + B + \xi(\mathrm{i}C_2 - C_7) + e^{2\xi}(\mathrm{i}C_3 + C_8) + e^{-2\xi}(\mathrm{i}C_4 - 3C_9)), \tag{87a}$$

$$\widetilde{S}_\theta = \frac{E}{(1-v^2)}(C_5 + \mathrm{i}C_{10} + A + \mathrm{i}B + vC_2 + \xi(C_2 + \mathrm{i}C_7) + e^{2\xi}((1+2v)C_3 + \mathrm{i}C_8) + e^{-2\xi}((1-2v)C_4 + \mathrm{i}C_9)). \tag{87b}$$

The first two rows of the equation

$$\mathbf{H}''\tilde{\mathbf{v}}_2^{(1)} = \mathrm{i}\tilde{\mathbf{v}}_2^{(1)} + \tilde{\mathbf{v}}_2^{(0)}, \tag{88}$$

are satisfied identically; the third leads to the requirements $C_7 = \mathrm{i}C_2, C_8 = -\mathrm{i}(5+v)C_3/(1-3v), C_9 = -\mathrm{i}C_4$ and $(3-v)(C_{10} - \mathrm{i}C_5) = (1+v)(\mathrm{i}A - B + \mathrm{i}C_2)$, and the fourth is then satisfied identically. The stress components become

$$\sigma_r = \frac{Ee^{\mathrm{i}\theta}}{(1+v)}\left(\frac{2\mathrm{i}(1+v)C_3 r}{1-3v} + \frac{(3+v)C_2 + 2v(A+\mathrm{i}B)}{(3-v)r} - \frac{2C_4}{r^3}\right), \tag{89a}$$

$$\sigma_\theta = \frac{Ee^{\mathrm{i}\theta}}{(1+v)}\left(\frac{6(1+v)C_3 r}{1-3v} + \frac{2(A+\mathrm{i}B) - C_2(1-v)}{(3-v)r} + \frac{2C_4}{r^3}\right), \tag{89b}$$

$$\tau_{r\theta} = \frac{Ee^{i\theta}}{(1+\nu)}\left(\frac{-2i(1+\nu)C_3 r}{1-3\nu} + \frac{iC_2(1-\nu)+2(-iA+B)}{(3-\nu)r} + \frac{2iC_4}{r^3}\right). \tag{89c}$$

This is now applied to the curved beam, Fig. 4, subjected to a shearing force Q on the beam end $\theta = 0$. Take the imaginary parts of Eqs. (89) to give

$$\sigma_r = \frac{E}{(1+\nu)}\left(\frac{2(1+\nu)C_3 r}{1-3\nu} + \frac{(3+\nu)C_2 + 2\nu A}{(3-\nu)r} - \frac{2C_4}{r^3}\right)\sin\theta + \frac{2\nu EB}{(3-\nu)r}\cos\theta, \tag{90a}$$

$$\sigma_\theta = \frac{E}{(1+\nu)}\left(\frac{6(1+\nu)C_3 r}{1-3\nu} + \frac{2A - C_2(1-\nu)}{(3-\nu)r} + \frac{2C_4}{r^3}\right)\sin\theta + \frac{2EB}{(3-\nu)r}\cos\theta, \tag{90b}$$

$$\tau_{r\theta} = \frac{E}{(1+\nu)}\left(\frac{-2(1+\nu)C_3 r}{1-3\nu} + \frac{C_2(1-\nu) - 2A}{(3-\nu)r} + \frac{2C_4}{r^3}\right)\cos\theta + \frac{2EB}{(3-\nu)r}\sin\theta. \tag{90c}$$

The boundary condition $\sigma_\theta = 0$ on the end $\theta = 0$ leads to the requirement $B = 0$. The boundary condition $\sigma_r = \tau_{r\theta} = 0$ on either $r = a$ or $r = b$, leads to the requirement $C_2 = (1-\nu)A/2$. The above then reduce to

$$\sigma_r = E\left(\frac{2C_3 r}{1-3\nu} + \frac{A}{2r} - \frac{2C_4}{(1+\nu)r^3}\right)\sin\theta, \tag{91a}$$

$$\sigma_\theta = E\left(\frac{6C_3 r}{1-3\nu} + \frac{A}{2r} + \frac{2C_4}{(1+\nu)r^3}\right)\sin\theta, \tag{91b}$$

$$\tau_{r\theta} = -E\left(\frac{2C_3 r}{1-3\nu} + \frac{A}{2r} - \frac{2C_4}{(1+\nu)r^3}\right)\cos\theta. \tag{91c}$$

The constants are evaluated from the requirements $Q = \int_a^b \tau_{r\theta} dr$ on $\theta = 0$, and $\sigma_r = \tau_{r\theta} = 0$ on $r = a$ and $r = b$, leading to

$$A = -\frac{2(a^2+b^2)Q}{E\Delta}, \tag{92a}$$

$$C_3 = \frac{(1-3\nu)Q}{2E\Delta}, \tag{92b}$$

$$C_4 = -\frac{a^2 b^2(1+\nu)Q}{2E\Delta}, \tag{92c}$$

where Δ is now defined as

$$\Delta = a^2 - b^2 + (a^2+b^2)\ln(b/a). \tag{93}$$

The stress components become

$$\sigma_r = \frac{Q}{\Delta}(r - (a^2+b^2)/r + a^2b^2/r^3)\sin\theta, \tag{94a}$$

$$\sigma_\theta = \frac{Q}{\Delta}(3r - (a^2+b^2)/r - a^2b^2/r^3)\sin\theta, \tag{94b}$$

$$\tau_{r\theta} = -\frac{Q}{\Delta}(r - (a^2+b^2)/r + a^2b^2/r^3)\cos\theta, \tag{94c}$$

which is in agreement with Timoshenko and Goodier (1970) and Massonnet (1962). The constants C_5 and C_{10} have no effect on the stress components, and are rigid body displacements, multiples of which can always be added to the principal vector.

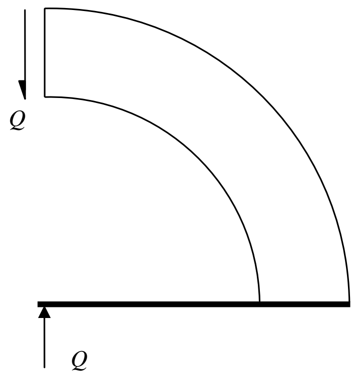

Fig. 5. Curved beam subject to shearing force on the end $\theta = 0$; the reaction on the end $\theta = \pi/2$, according to Eq. (94), is equivalent to a compressive force applied at the origin by means of a fictitious lever.

The principal vector can be written as

$$\tilde{\mathbf{v}}_2^{(1)} = \begin{bmatrix} \tilde{u}_r \\ \tilde{u}_\theta \\ \widetilde{S}_{r\theta} \\ \widetilde{S}_\theta \end{bmatrix}^{(1)} = \frac{Q}{\Delta} \begin{bmatrix} \frac{-(1-\nu)(a^2+b^2)\xi}{E} + \frac{(1-3\nu)}{2E}e^{2\xi} - \frac{(1+\nu)a^2b^2}{2E}e^{-2\xi} \\ \frac{-i(1-\nu)(a^2+b^2)\xi}{E} - \frac{i(5+\nu)}{2E}e^{2\xi} + \frac{i(1+\nu)a^2b^2}{2E}e^{-2\xi} - \frac{i(1+\nu)(a^2+b^2)}{E} \\ -i(e^{2\xi} - (a^2+b^2) + a^2b^2e^{-2\xi}) \\ 3e^{2\xi} - (a^2+b^2) - a^2b^2e^{-2\xi} \end{bmatrix}, \tag{95}$$

which satisfies Eq. (88).

For a curved beam forming one-quarter of a circle, the shearing stress $\tau_{r\theta}$ is zero on the end $\theta = \pi/2$, while the force resultant is $\int_a^b \sigma_\theta \mathrm{d}r$ which evaluates as $-Q$, as demanded by force equilibrium; moreover, the bending moment about the origin is $M = -\int_a^b r\sigma_\theta \mathrm{d}r$ which evaluates as zero. One may interpret the resultant on the end $\theta = \pi/2$ as consisting of a compressive force of magnitude Q applied at the origin by means of a fictitious rigid lever, as depicted in Fig. 5. The solution for a tensile force T applied on the beam centre-line at the end $\theta = \pi/2$ can then be found by replacing Q by $-T$, and superposing a pure bending moment as described in Section 7.2, of magnitude $M = T \times (a + b)/2$. Thus tension is described by a combination of principal vectors from different eigen-spaces.

7.5. Jordan canonical form for the curved beam transmission modes

Now construct a 4×4 transformation matrix $\mathbf{V}$ consisting of the eigen- and principal vectors as $\mathbf{V} = [\tilde{\mathbf{v}}_1^{(0)} \quad \tilde{\mathbf{v}}_1^{(1)} \quad \tilde{\mathbf{v}}_2^{(0)} \quad \tilde{\mathbf{v}}_2^{(1)}]$, and one finds $\mathbf{H}''\mathbf{V} = \mathbf{VJ}$, where

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & i \end{bmatrix}, \tag{96}$$

is the JCF for the curved beam transmission modes; now, the unity elements on the superdiagonal couple the eigen- and principal vector for the repeating zero, and the repeating imaginary unity eigenvalues, respectively. Much of the discussion regarding the JCF for the wedge, Section 4.5, is equally applicable here. Strain energy arguments presented in Section 10.2 lead to the conclusion that the above, Eq. (96), is the only possible JCF – one cannot have degenerate decay modes.

8. Saint-Venant decay for the curved beam: radial evolution

One now requires a formulation describing radial evolution. Write $\mathbf{s} = \tilde{\mathbf{s}}e^{-k\theta}$, then Eq. (55) becomes

$$\mathrm{d}\tilde{\mathbf{s}}/\mathrm{d}\xi = \overline{\mathbf{H}}\tilde{\mathbf{s}}, \tag{97}$$

where

$$\overline{\mathbf{H}} = \begin{bmatrix} -\nu & \nu k & (1-\nu^2)/E & 0 \\ k & 1 & 0 & 2(1+\nu)/E \\ E & -Ek & \nu & k \\ Ek & -Ek^2 & \nu k & -1 \end{bmatrix}. \tag{98}$$

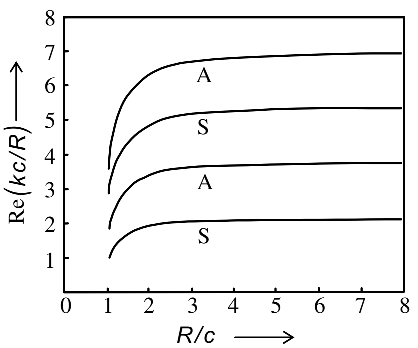

Fig. 6. Decay rates for the plane curved beam. The symbols A and S denote asymmetric and symmetric (with respect to the beam centre-line) self-equilibrated end-loads. The centre-line radius is $R = (a + b)/2$, beam semi-depth is $c = (b - a)/2$.

The formal solution may be written as

$$\tilde{\mathbf{s}}(\xi) = \mathrm{e}^{\overline{\mathbf{H}}\xi}\tilde{\mathbf{s}}(0), \tag{99}$$

where the matrix exponential $\mathbf{N} = \mathrm{e}^{\overline{\mathbf{H}}\xi}$ is calculated as the inverse Laplace transform of the resolvent matrix $\overline{\mathbf{R}} = (s\mathbf{I} - \overline{\mathbf{H}})^{-1}$; again this is readily accomplished using a symbolic computation package, such as MAPLE. The elements of matrix $\mathbf{N}$ are given in Appendix B. One has traction-free boundary conditions on the inner and outer arcs, that is $\sigma_r = \tau_{r\theta} = 0$ on $r = a$ and $r = b$, so first write $\tilde{\mathbf{s}}(\ln a) = \mathrm{e}^{\overline{\mathbf{H}}\ln a}\tilde{\mathbf{s}}(0)$, and rearrange as $\tilde{\mathbf{s}}(0) = \mathrm{e}^{-\overline{\mathbf{H}}\ln a}\tilde{\mathbf{s}}(\ln a)$. Now substitute into Eq. (99) to give

$$\tilde{\mathbf{s}}(\xi) = \mathrm{e}^{\overline{\mathbf{H}}\xi}\mathrm{e}^{-\overline{\mathbf{H}}\ln a}\tilde{\mathbf{s}}(\ln a) = \mathrm{e}^{\overline{\mathbf{H}}(\xi-\ln a)}\tilde{\mathbf{s}}(\ln a). \tag{100}$$

The state vector on the traction-free inner arc becomes

$$\tilde{\mathbf{s}}(\ln a) = [\,\tilde{u}_r(\ln a) \quad \tilde{u}_\theta(\ln a) \quad 0 \quad 0\,]^{\mathrm{T}}, \tag{101}$$

and Eq. (100) reduces to

$$\tilde{\mathbf{s}}(\xi) = \begin{bmatrix} \tilde{u}_r(\xi) \\ \tilde{u}_\theta(\xi) \\ \widetilde{S}_r(\xi) \\ \widetilde{S}_{r\theta}(\xi) \end{bmatrix} = \begin{bmatrix} N_{11}(\xi-\ln a)\tilde{u}_r(\ln a) + N_{12}(\xi-\ln a)\tilde{u}_\theta(\ln a) \\ N_{21}(\xi-\ln a)\tilde{u}_r(\ln a) + N_{22}(\xi-\ln a)\tilde{u}_\theta(\ln a) \\ N_{31}(\xi-\ln a)\tilde{u}_r(\ln a) + N_{32}(\xi-\ln a)\tilde{u}_\theta(\ln a) \\ N_{41}(\xi-\ln a)\tilde{u}_r(\ln a) + N_{42}(\xi-\ln a)\tilde{u}_\theta(\ln a) \end{bmatrix}. \tag{102}$$

Now require that the outer arc $r = b$ (or $\xi = \ln b$) be free of traction to give

$$\begin{bmatrix} N_{31}(\ln(b/a)) & N_{32}(\ln(b/a)) \\ N_{41}(\ln(b/a)) & N_{42}(\ln(b/a)) \end{bmatrix} \begin{bmatrix} \tilde{u}_r(\ln a) \\ \tilde{u}_\theta(\ln a) \end{bmatrix} = \mathbf{0}; \tag{103}$$

since the displacement components are not zero on the inner arc, the determinant must be zero, leading to the eigenequation

$$(1-(b/a)^2)^2k^2 - 2(b/a)^2(1-\cos(2k\ln(b/a))) = 0. \tag{104}$$

This eigenequation was derived first by Kitover (1952), and more recently by Stephen and Wang (1993) employing an Airy stress function approach, where it was shown that Eq. (104) reduces to the well-known Papkovitch–Fadle eigenequation for the plane-strain strip as $R \to \infty$. For example, self-equilibrated load on the end of the plane strain strip decays as e^{-kz}, where z is the axial coordinate; the two symmetric roots with smallest real part are $kc = 2.1061 \pm 1.1254\mathrm{i}$ and $kc = 5.3563 \pm 1.5516\mathrm{i}$, and the two asymmetric roots with smallest real part are $kc = 3.7488 \pm 1.3843\mathrm{i}$ and $kc = 6.9500 \pm 1.6761\mathrm{i}$. For the curved beam, it has been assumed that stress decay as $\mathrm{e}^{-k\theta}$ and to facilitate comparison write $z = R\theta$, where z is now the centre-line arc length from the loaded end, to give decay as $\mathrm{e}^{-kz/R}$. Decay rates are shown in Fig. 6, where it is seen that if the centre-line radius $R = (a + b)/2$ exceeds the curved beam depth $2c = (b - a)$, that is $R/c > 2$, the rates of decay differ little from those of the straight strip. However as internal radius approaches zero (when $R/c \to 1$), so too does the decay rate, implying that self-equilibrated loading does not decay at all. For example, consider a non-zero shearing stress $\tau_{r\theta}$ on an element located at the origin of an elastic quadrant: the complementary nature of shearing stress ($\tau_{r\theta} = \tau_{\theta r}$) demands that the Saint-Venant decay rate should be zero.

9. The wedge paradox

As noted in Section 4.4, for the particular wedge angle $2\alpha^*$ satisfying the equation $\sin 2\alpha^* = 2\alpha^*\cos 2\alpha^*$, equal to approximately 257°, one has the eigenvalue $\lambda = -1$ for the decay of self-equilibrated loading on the inner arc $r = a$; this repeats the eigenvalue for the diffusion of a bending moment for all wedge angles, and the Carothers solution for a bending moment breaks down for this particular angle when it describes a self-equilibrating field. This signals the breakdown of Saint-Venant's principle: self-equilibrated load on the inner arc decays at the same rate as moment diffuses into a divergent area. This repeating eigenvalue is an accidental degeneracy for this particular angle. From Eq. (17), $\lambda = -1$ implies circumferential roots $k = 0, 0, \pm 2\mathrm{i}$, leading to constant, θ, $\sin 2\theta$ and $\cos 2\theta$ terms, together with an $\ln r$ term coupling the Carothers eigenvector to the principal vector, according to Eq. (62). Moreover, for the wedge angle $2\alpha^*$ one has the eigenvalue $\lambda = 1$ for the decay of self-equilibrated loading on the outer arc $r = b$; since stress varies as $r^{\lambda-1}$, this implies a stress-field independent of radius, and signals a further breakdown of Saint-Venants's principle: self-equilibrated load on the outer arc does not decay as one moves towards the inner arc. The concurrence of these "decay" eigenvalues $\lambda = \pm 1$ for this angle implies, from Eq. (17), that $k = \pm 2\mathrm{i}$ is also repeating, leading to $\theta\sin 2\theta$ and $\theta\cos 2\theta$ term. For asymmetric radial displacement and stress fields, the displacement components then take the form

$$\begin{aligned} u_r &= \mathrm{e}^{-\xi}(C_3\xi\sin 2\theta + D_1\theta + C_6\sin 2\theta + C_9\theta\cos 2\theta), \\ u_\theta &= \mathrm{e}^{-\xi}\left(C_3\xi\left(\frac{1+v}{2}\cos 2\alpha^* + \frac{1-v}{2}\cos 2\theta\right) + F_2 + C_{11}\cos 2\theta + C_{12}\theta\sin 2\theta\right), \end{aligned} \tag{105}$$

and the modified stress components are calculated as

$$S_r = \frac{Ee^{-\xi}}{(1-\nu^2)}(C_3\xi(\nu^2-1)\sin 2\theta + (C_3+(\nu-1)C_6 - 2\nu C_{11} + \nu C_{12})\sin 2\theta + ((\nu-1)C_9 + 2\nu C_{12})\theta\cos 2\theta + (\nu-1)D_1\theta),$$
$$S_{r\theta} = \frac{Ee^{-\xi}}{2(1+\nu)}\begin{pmatrix}(1+\nu)C_3\xi(\cos 2\theta - \cos 2\alpha^*) + C_3(\frac{1-\nu}{2}\cos 2\theta + \frac{1+\nu}{2}\cos 2\alpha^*) + \\ (C_9+2C_6-2C_{11})\cos 2\theta - (C_9+C_{12})2\theta\sin 2\theta + D_1 - 2F_2\end{pmatrix}. \tag{106}$$

The principal vector then takes the form

$$\bar{\mathbf{s}}_3^{(1)} = \begin{bmatrix} D_1\theta + C_6\sin 2\theta + C_9\theta\cos 2\theta \\ F_2 + C_{11}\cos 2\theta + C_{12}\theta\sin 2\theta \\ \frac{E}{(1-\nu^2)}(D_1\theta(\nu-1) + (C_3+(\nu-1)C_6 - 2\nu C_{11} + \nu C_{12})\sin 2\theta + ((\nu-1)C_9 + 2\nu C_{12})\theta\cos 2\theta) \\ \frac{E}{2(1+\nu)}(D_1 - 2F_2 + C_3\frac{1+\nu}{2}\cos 2\alpha^* + (C_3\frac{1-\nu}{2} + 2C_6 + C_9 - 2C_{11})\cos 2\theta - 2(C_9+C_{12})\theta\sin 2\theta)\end{bmatrix}. \tag{107}$$

Now require that this vector should satisfy the relationship

$$\mathbf{H}\bar{s}_3^{(1)} = -\bar{\mathbf{s}}_3^{(1)} + \bar{\mathbf{s}}_3^{(0)}, \tag{108}$$

where $\bar{\mathbf{s}}_3^{(0)}$, in accordance with the Carothers solution, is the generating eigenvector

$$\bar{\mathbf{s}}_3^{(0)} = C_3\begin{bmatrix}\sin 2\theta \\ \frac{1+\nu}{2}\cos 2\alpha^* + \frac{1-\nu}{2}\cos 2\theta \\ -E\sin 2\theta \\ E(\cos 2\theta - \cos 2\alpha^*)/2\end{bmatrix}. \tag{109}$$

The first two rows are satisfied identically, while the third and fourth, respectively, yield

$$\frac{C_3(5-\nu^2)}{4} + C_6(1-\nu) + C_9(1-\nu) - 2C_{11} + C_{12} = 0, \quad 2C_{12} + (1-\nu)C_9 = 0, \tag{110}$$

and

$$\frac{C_3(1+4\nu-\nu^2)}{4} + C_6(1-\nu) + \frac{C_9(1-\nu)}{2} - 2C_{11} + 2C_{12} = 0, \quad D_1 = \frac{C_3(1+\nu)\cos 2\alpha^*}{2}, \tag{111}$$

from these, one finds $C_9 = -C_3$ and $C_{12} = C_3(1-\nu)/2$. The expression for S_θ becomes

$$S_\theta = \frac{EC_3e^{-\xi}}{4}(2\theta\cos 2\alpha^* - \sin 2\theta), \tag{112}$$

so the requirement that $\sigma_\theta = 0$ on $\theta = \pm\alpha$ is satisfied for the critical wedge angle $2\alpha^*$ only, while the requirement $\tau_{r\theta} = 0$ on $\theta = \pm\alpha^*$ leads to

$$F_2 = (1+\nu)C_3(2\alpha^*\sin 2\alpha^* + \cos 2\alpha^*)/4 + (C_6 - C_{11})\cos 2\alpha^*. \tag{113}$$

The tensile force resultant is $T = r\int_{-\alpha^*}^{\alpha^*}(\sigma_r\cos\theta - \tau_{r\theta}\sin\theta)d\theta$, and this is equal to zero since the integrand is an odd function of θ. On the other hand, the requirement that the shearing force $Q = r\int_{-\alpha^*}^{\alpha^*}(\sigma_r\sin\theta + \tau_{r\theta}\cos\theta)d\theta$ is zero for the wedge angle $2\alpha^*$ leads to

$$C_3(3-\nu)(1+\nu) - 12C_6(1-\nu) + 8C_{11}(1-2\nu) = 0; \tag{114}$$

this, together with results from above, yields $C_6 = C_{11} = C_3\ (3-\nu)/4$. Finally, the requirement $M = r^2\int_{-\alpha^*}^{\alpha^*}\tau_{r\theta}d\theta$ leads to $C_3 = -M/(E\alpha^{*2}\sin 2\alpha^*)$, and the stress components become

$$\sigma_r = -M((1-4\ln r)\sin 2\theta + 4\theta\cos 2\theta - 2\theta\cos 2\alpha^*)/(4r^2\alpha^{*2}\sin 2\alpha^*), \tag{115a}$$
$$\sigma_\theta = -M(2\theta\cos 2\alpha^* - \sin 2\theta)/(4r^2\alpha^{*2}\sin 2\alpha^*), \tag{115b}$$
$$\tau_{r\theta} = -M((1-2\ln r)(\cos 2\alpha^* - \cos 2\theta) + 2\theta\sin 2\theta - 2\alpha^*\sin 2\alpha^*)/(4r^2\alpha^{*2}\sin 2\alpha^*). \tag{115c}$$

This is in agreement with the special solution derived by Ting (1985). Note that only the term $2\alpha^*\sin 2\alpha^*$ within the above expression for $\tau_{r\theta}$ contributes to the moment; the other terms are, for this wedge angle, self-equilibrating. The displacement components within the principal vector are

$$\bar{u}_r = -M(2(1+\nu)\theta\cos 2\alpha^* + (3-\nu)\sin 2\theta - 4\theta\cos 2\theta)/(4E\alpha^{*2}\sin 2\alpha^*),$$
$$\bar{u}_\theta = -M((1+\nu)(2\alpha^*\sin 2\alpha^* + \cos 2\alpha^*) + (3-\nu)\cos 2\theta + 2(1-\nu)\theta\sin 2\theta)/(4E\alpha^{*2}\sin 2\alpha^*), \tag{116}$$

and the principal vector $\bar{\mathbf{s}}_3^{(1)}$ becomes

$$\bar{\mathbf{s}}_3^{(1)} = \begin{bmatrix} \bar{u}_r \\ \bar{u}_\theta \\ \bar{S}_r \\ \bar{S}_{r\theta} \end{bmatrix}^{(1)} = \frac{-M}{4\alpha^{*2}\sin 2\alpha^*}\begin{bmatrix} \{2(1+v)\theta\cos 2\alpha^* + (3-v)\sin 2\theta - 4\theta\cos 2\theta\}/E \\ \{(1+v)(2\alpha^*\sin 2\alpha^* + \cos 2\alpha^*) + (3-v)\cos 2\theta + 2(1-v)\theta\sin 2\theta\}/E \\ -2\theta\cos 2\alpha^* + \sin 2\theta + 4\theta\cos 2\theta \\ \cos 2\alpha^* - \cos 2\theta + 2\theta\sin 2\theta - 2\alpha^*\sin 2\alpha^* \end{bmatrix}. \tag{117}$$

Note that the stress components, Eqs. (115) may be determined also from the Airy stress function $\Phi = \Phi_1 + \Phi_2$ where

$$\Phi_1 = \frac{M}{12\alpha^{*2}\sin 2\alpha^*}(3\ln r(2\theta\cos 2\alpha^* - \sin 2\theta) + 3\theta\cos 2\theta - 4\sin 2\theta + 5\theta\cos 2\alpha^* + 6\alpha^*\theta\sin 2\alpha^*) \tag{118a}$$

$$\Phi_2 = \frac{M}{12\alpha^{*2}\sin 2\alpha^*}(\sin 2\theta - 2\theta\cos 2\alpha^*), \tag{118b}$$

and $\sigma_r = \partial\Phi/r\partial r + \partial^2\Phi/r^2\partial\theta^2$, $\sigma_\theta = \partial^2\Phi/\partial r^2$, $\tau_{r\theta} = -\partial/\partial r(\partial\Phi/r\partial\theta)$. The first of these stress functions, Φ_1, was presented by Sternberg and Koiter (1958) in their treatment of the wedge paradox, while the second, Φ_2, has as its primitive the original Carothers stress function for the wedge.

For this critical wedge angle, the eigenvalue $\lambda = 1$ also repeats, again an accidental degeneracy; the eigenvector is the rigid body rotation $\bar{\mathbf{s}}_2^{(0)} = [0 \quad C_0 \quad 0 \quad 0]^{\mathrm{T}}$ according to Eq. (31), in which the stress components are zero. The principal vector $\bar{\mathbf{s}}_2^{(1)}$ is self-equilibrating, and is coupled to the eigenvector according to

$$\mathbf{H}\bar{s}_2^{(1)} = \bar{\mathbf{s}}_2^{(1)} + \bar{\mathbf{s}}_2^{(0)}. \tag{119}$$

For an asymmetric radial field, the displacement components take the form

$$u_r = (C_1\theta + C_2\sin 2\theta)r = (C_1\theta + C_2\sin 2\theta)\mathrm{e}^{\xi}, \tag{120a}$$

$$u_\theta = (C_0\ln r + C_3\cos 2\theta)r = (C_0\xi + C_3\cos 2\theta)\mathrm{e}^{\xi}. \tag{120b}$$

The principal vector takes the form

$$\bar{\mathbf{s}}_2^{(1)} = \begin{bmatrix} \bar{u}_r \\ \bar{u}_\theta \\ \bar{S}_r \\ \bar{S}_{r\theta} \end{bmatrix}^{(1)} = \begin{bmatrix} C_1\theta + C_2\sin 2\theta \\ C_3\cos 2\theta \\ \frac{E}{1-v^2}[C_1(1+v)\theta + C_2(1+v)\sin 2\theta - 2vC_3\sin 2\theta] \\ \frac{E}{2(1+v)}[C_0 + C_1 + 2C_2\cos 2\theta] \end{bmatrix}. \tag{121}$$

The first and second rows of matrix Eq. (119) are satisfied identically; from the third row one finds $C_2 = C_3$, and from the fourth, $C_0 = -2C_1/(1-v)$. The shearing stress component becomes $\tau_{r\theta} = \frac{E}{2(1+v)}[C_0(1+v)/2 + 2C_2\cos 2\theta]$ and the requirement $\tau_{r\theta} = 0$ on $\theta = \pm\alpha^*$ leads to $C_0 = -4C_2\cos 2\alpha^*/(1+v)$. The stress components becomes

$$\sigma_r = \frac{EC_2}{(1+v)}[2\theta\cos 2\alpha^* + \sin 2\theta], \tag{122a}$$

$$\sigma_\theta = \frac{EC_2}{(1+v)}[2\theta\cos 2\alpha^* - \sin 2\theta], \tag{122b}$$

$$\tau_{r\theta} = \frac{EC_2}{(1+v)}[\cos 2\theta - \cos 2\alpha^*], \tag{122c}$$

from which one notes that the circumferential stress is zero on $\theta = \pm\alpha^*$, while the displacement components are

$$u_r = C_2\left(2\theta\cos 2\alpha^*\frac{(1-v)}{(1+v)} + \sin 2\theta\right)r, \tag{123a}$$

$$u_\theta = C_2\left(\cos 2\theta - \frac{4\cos 2\alpha^*}{(1+v)}\ln r\right)r. \tag{123b}$$

The principal vector becomes

$$\bar{\mathbf{s}}_2^{(1)} = \begin{bmatrix} \bar{u}_r \\ \bar{u}_\theta \\ \bar{S}_r \\ \bar{S}_{r\theta} \end{bmatrix}^{(1)} = C_2\begin{bmatrix} 2\theta\cos 2\alpha^*\frac{(1-v)}{(1+v)} + \sin 2\theta \\ \cos 2\theta \\ \frac{E}{(1+v)}[2\theta\cos 2\alpha^* + \sin 2\theta] \\ \frac{E}{(1+v)}[\cos 2\theta - \cos 2\alpha^*] \end{bmatrix}. \tag{124}$$

The tensile force $T = r\int_{-\alpha^*}^{\alpha^*}(\sigma_r\cos\theta - \tau_{r\theta}\sin\theta)\mathrm{d}\theta$ is zero, as the integrand is odd. The moment $M = r^2\int_{-\alpha^*}^{\alpha^*}\tau_{r\theta}\mathrm{d}\theta$ and the shearing force $Q = r\int_{-\alpha^*}^{\alpha^*}(\sigma_r\sin\theta + \tau_{r\theta}\cos\theta)\mathrm{d}\theta$ both reduce to zero for $\alpha = \alpha^*$, confirming that the stress field is self-equilibrating. Note that this stress field may also be derived from the Airy stress function

$$\Phi = \frac{-EC_2r^2}{2(1+v)}(\cos 2\theta - 2\theta\cos 2\alpha^*). \tag{125}$$

10. Strain energy: restrictions on Jordan canonical form

Strain energy arguments can be employed to exclude the possibility of certain eigenvalues, and possible JCF's. For a prismatic rod, with x, and y and z as axial, and cross-sectional coordinates, respectively, Synge (1945) considered exponential displacement solutions of the form $e^{kx}f(y,z)$. According to Synge, "*A purely imaginary k implies a periodic distribution of displacement and stress. Consider the energy in a length of cylinder equal to this period. It is equal to the work done by the terminal stress in passing from the natural state to the strained state. But from the periodicity, this is zero. Hence the energy of a strained state is zero, which is contrary to a basic postulate of elasticity. Hence there can be no purely imaginary eigenvalue k. It should be added that we cannot assert this if (Poisson's ratio) ν is arbitrary. It is necessarily only true if strain energy is positive definite, i.e. if* $-1< \nu < 1/2$", indicating that this "*simple and ingenious argument*" was originally put forward by Dougall (1913). More recently, and again for the prismatic rod, Stephen (2004) has shown that only the repeating eigenvalue $k = 0$ can give rise to a non-trivial Jordan block; in turn, degenerate Saint-Venant decay modes cannot exist. Similar arguments are now extended to the wedge and the curved beam.

10.1. The wedge

For the wedge with traction-free flanks, the strain energy stored within an arc element (circumferential strip) is equal to the work done on its surface, and may be written as

$$U = (1/2)\int_{-\alpha}^{\alpha} -(u_r\sigma_r + u_\theta\tau_{r\theta})_{\text{inner}} r\mathrm{d}\theta + (1/2)\int_{-\alpha}^{\alpha}(u_r\sigma_r + u_\theta\tau_{r\theta})_{\text{outer}}(r+\mathrm{d}r)\mathrm{d}\theta, \tag{126}$$

where the first integrand pertains to the inner arc (generic radius r), for which the direction cosine $n_r = -1$, and the second pertains to the outer arc where $n_r = +1$ and whose radius in now $r + \mathrm{d}r$. Let $\bar{\mathbf{s}}$ be an eigenvector, then the vector of displacement and stress components on the inner arc may be written as

$$\mathbf{s}_{\text{inner}} = \begin{bmatrix} u_r \\ u_\theta \\ \sigma_r \\ \tau_{r\theta} \end{bmatrix}_{\text{inner}} = \begin{bmatrix} \bar{u}_r r^\lambda \\ \bar{u}_\theta r^\lambda \\ \bar{\sigma}_r r^{\lambda-1} \\ \bar{\tau}_{r\theta} r^{\lambda-1} \end{bmatrix}. \tag{127}$$

On the outer arc, one has

$$\mathbf{s}_{\text{outer}} = \mathbf{s}_{\text{inner}} + (\partial\mathbf{s}_{\text{inner}}/\partial r)\mathrm{d}r, \tag{128}$$

and one finds

$$\mathbf{s}_{\text{outer}} = \begin{bmatrix} u_r \\ u_\theta \\ \sigma_r \\ \tau_{r\theta} \end{bmatrix}_{\text{outer}} = \begin{bmatrix} \bar{u}_r(r^\lambda + \lambda r^{\lambda-1}\mathrm{d}r) \\ \bar{u}_\theta(r^\lambda + \lambda r^{\lambda-1}\mathrm{d}r) \\ \bar{\sigma}_r(r^{\lambda-1} + (\lambda-1)r^{\lambda-2}\mathrm{d}r) \\ \bar{\tau}_{r\theta}(r^{\lambda-1} + (\lambda-1)r^{\lambda-2}\mathrm{d}r) \end{bmatrix}. \tag{129}$$

Substituting into Eq. (126), and ignoring terms involving $(\mathrm{d}\,r)^2$ and higher, one finds

$$U = \lambda r^{2\lambda-1}\mathrm{d}r\int_{-\alpha}^{\alpha}\bar{\mathbf{d}}^{\mathrm{T}}\bar{\mathbf{p}}\mathrm{d}\,\theta, \tag{130}$$

where the notation $\bar{\mathbf{d}} = [\bar{u}_r\,\bar{u}_\theta]^{\mathrm{T}}$ and $\bar{\mathbf{p}} = [\bar{\sigma}_r\,\bar{\tau}_{r\theta}]^{\mathrm{T}}$ has been employed. The following observations are made:

(a) The strain energy of an eigenvector associated with $\lambda = 0$ is zero – this is a rigid body translation in some arbitrary direction; on the other hand, a rigid body rotation has $\lambda = 1$, but the corresponding $\bar{\mathbf{p}} = \mathbf{0}$ and the strain energy is also zero.

(b) The strain energy stored within an arc element of thickness dr is independent of radius for $\lambda = 1/2$; since stress varies as $r^{\lambda-1}$, this pertains to the familiar inverse square root singularity at the tip of a crack within the discipline of *LEFM*. Note that the area of each arc element depends linearly on the radius, so the strain energy density is inversely proportional to the radius, leading to a theoretical infinite strain energy density at the crack tip.

Next, consider the implication of a principal vector coupled to an eigenvector for a repeating eigenvalue λ. From Section 6.1, assuming the inclusion of an arbitrary multiple (β) of the generating eigenvector, we may write the vector of displacement and stress components on the inner arc, from Eq. (61), as

$$\mathbf{s}^{\mathrm{p}}_{\text{inner}} = \begin{bmatrix} u_r \\ u_\theta \\ \sigma_r \\ \tau_{r\theta} \end{bmatrix}^{\mathrm{p}}_{\text{inner}} = \begin{bmatrix} \bar{u}_r r^\lambda \\ \bar{u}_\theta r^\lambda \\ \bar{S}_r r^{\lambda-1} \\ \bar{S}_{r\theta} r^{\lambda-1} \end{bmatrix}^{(1)} + (\beta + \ln r)\begin{bmatrix} \bar{u}_r r^\lambda \\ \bar{u}_\theta r^\lambda \\ \bar{S}_r r^{\lambda-1} \\ \bar{S}_{r\theta} r^{\lambda-1} \end{bmatrix}^{(0)}; \tag{131}$$

again determine the state-vector on the outer arc using Eq. (128) as

$$\mathbf{s}_{\text{outer}}^{\text{p}} = \begin{bmatrix} u_r \\ u_\theta \\ \sigma_r \\ \tau_{r\theta} \end{bmatrix}_{\text{outer}}^{\text{p}} = \begin{bmatrix} \bar{u}_r(r^\lambda + \lambda r^{\lambda-1}\text{d}r) \\ \bar{u}_\theta(r^\lambda + \lambda r^{\lambda-1}\text{d}r) \\ \bar{S}_r(r^{\lambda-1} + (\lambda-1)r^{\lambda-2}\text{d}r) \\ \bar{S}_{r\theta}(r^{\lambda-1} + (\lambda-1)r^{\lambda-2}\text{d}r) \end{bmatrix}^{(1)} + \begin{bmatrix} \bar{u}_r((\beta+\ln r)r^\lambda + (\lambda(\beta+\ln r)+1)r^{\lambda-1}\text{d}r) \\ \bar{u}_\theta((\beta+\ln r)r^\lambda + (\lambda(\beta+\ln r)+1)r^{\lambda-1}\text{d}r) \\ \bar{S}_r((\beta+\ln r)r^{\lambda-1} + ((\lambda-1)(\beta+\ln r)+1)r^{\lambda-2}\text{d}r) \\ \bar{S}_{r\theta}((\beta+\ln r)r^{\lambda-1} + ((\lambda-1)(\beta+\ln r)+1)r^{\lambda-2}\text{d}r) \end{bmatrix}^{(0)}. \tag{132}$$

The strain energy according to Eq. (126), again ignoring terms involving $(\text{d}\,r)^2$ and higher, is

$$U = r^{2\lambda-1}\text{d}r\int_{-\alpha}^{\alpha}[\lambda\bar{\mathbf{d}}^{(1)\text{T}}\bar{\mathbf{p}}^{(1)} + (\lambda(\beta+\ln r)+1/2)(\bar{\mathbf{d}}^{(1)\text{T}}\bar{\mathbf{p}}^{(0)} + \bar{\mathbf{d}}^{(0)\text{T}}\bar{\mathbf{p}}^{(1)}) + (\beta+\ln r)(\lambda(\beta+\ln r)+1)\bar{\mathbf{d}}^{(0)\text{T}}\bar{\mathbf{p}}^{(0)}]\text{d}\theta. \tag{133}$$

When β is equal to zero, this reduces to

$$U = r^{2\lambda-1}\text{d}r\int_{-\alpha}^{\alpha}[\lambda\bar{\mathbf{d}}^{(1)\text{T}}\bar{\mathbf{p}}^{(1)} + (\lambda\ln r+1/2)(\bar{\mathbf{d}}^{(1)\text{T}}\bar{\mathbf{p}}^{(0)} + \bar{\mathbf{d}}^{(0)\text{T}}\bar{\mathbf{p}}^{(1)}) + \ln r(\lambda\ln r+1)\bar{\mathbf{d}}^{(0)\text{T}}\bar{\mathbf{p}}^{(0)}]\text{d}\theta. \tag{134}$$

From Section 4.2, the principal vector for tension/shear has $\lambda = 0$, while the coupled eigenvector has $\bar{\mathbf{p}}^{(0)} = \mathbf{0}$, so Eq. (134) reduces to

$$U = r^{-1}\text{d}r(1/2)\int_{-\alpha}^{\alpha}\bar{\mathbf{d}}^{(0)\text{T}}\bar{\mathbf{p}}^{(1)}\text{d}\theta = r^{-1}\text{d}r(1/2)\int_{-\alpha}^{\alpha}\bar{u}_r^{(0)\text{T}}\bar{\sigma}_r^{(1)}\text{d}\theta, \tag{135}$$

since $\bar{\tau}_{r\theta}^{(1)} = 0$, and is in the expected form.

If one now requires that the strain energy should be independent of β, one finds

$$A\beta^2 + B\beta = 0, \tag{136}$$

where

$$A = \lambda\int_{-\alpha}^{\alpha}\bar{\mathbf{d}}^{(0)\text{T}}\bar{\mathbf{p}}^{(0)}\text{d}\theta, \tag{137a}$$

$$B = \lambda\int_{-\alpha}^{\alpha}(\bar{\mathbf{d}}^{(1)\text{T}}\bar{\mathbf{p}}^{(0)} + \bar{\mathbf{d}}^{(0)\text{T}}\bar{\mathbf{p}}^{(1)})\text{d}\theta + (2\lambda\ln r+1)\int_{-\alpha}^{\alpha}\bar{\mathbf{d}}^{(0)\text{T}}\bar{\mathbf{p}}^{(0)}\text{d}\theta. \tag{137b}$$

Since β is quite arbitrary, one must have both $A = B = 0$; for $A = 0$, one must have either $\lambda = 0$, or $\int_{-\alpha}^{\alpha}\bar{\mathbf{d}}^{(0)\text{T}}\bar{\mathbf{p}}^{(0)}\text{d}\theta = 0$, or perhaps both. If one chooses $\lambda = 0$, then the requirement that $B = 0$ leads to $\int_{-\alpha}^{\alpha}\bar{\mathbf{d}}^{(0)\text{T}}\bar{\mathbf{p}}^{(0)}\text{d}\theta = 0$. If one chooses $\lambda \neq 0$, then one must have $\int_{-\alpha}^{\alpha}\bar{\mathbf{d}}^{(0)\text{T}}\bar{\mathbf{p}}^{(0)}\text{d}\theta = 0$, and the requirement that $B = 0$, leads to $\int_{-\alpha}^{\alpha}(\bar{\mathbf{d}}^{(1)\text{T}}\bar{\mathbf{p}}^{(0)} + \bar{\mathbf{d}}^{(0)\text{T}}\bar{\mathbf{p}}^{(1)})\text{d}\theta = 0$. Thus one may conclude that

$$\int_{-\alpha}^{\alpha}\bar{\mathbf{d}}^{(0)\text{T}}\bar{\mathbf{p}}^{(0)}\text{d}\theta = 0, \tag{138}$$

always, and either

$$\text{(c)}\quad \lambda = 0, \tag{139}$$

or

$$\text{(d)}\quad \lambda \neq 0 \text{ and } \int_{-\alpha}^{\alpha}(\bar{\mathbf{d}}^{(1)\text{T}}\bar{\mathbf{p}}^{(0)} + \bar{\mathbf{d}}^{(0)\text{T}}\bar{\mathbf{p}}^{(1)})\text{d}\theta = 0. \tag{140}$$

Eq. (138) is satisfied for the rigid body displacement and rotation eigenvectors, as the stress components comprising $\bar{\mathbf{p}}^{(0)}$ within both are zero; it cannot be satisfied for a decay eigenmode as, from Eq. (130), the strain energy stored within the arc element would have to be zero.

Case (c) is in accord with what was found in Section 4.5, where a principal vector describing a tensile/shearing force is coupled to an eigenvector describing a rigid body displacement, both in arbitrary direction, with repeating eigenvalue $\lambda = 0$.

Case (d) is also consistent with the repeating eigenvalue $\lambda = 1$ leading to coupling of an eigen- and principal vector for the critical wedge angle $2\alpha^*$, Section 9. It has already been seen in Section 4.3 that the eigenvector describing a rigid body rotation, for which $\lambda = 1$, has zero stress components within the eigenvector, that is $\bar{\mathbf{p}}^{(0)} = 0$; hence the first integral in Eq. (140) would be zero, while the second reduces to

$$\int_{-\alpha}^{\alpha}\bar{\mathbf{d}}^{(0)\text{T}}\bar{\mathbf{p}}^{(1)}\text{d}\theta = 0. \tag{141}$$

Further, this possible principal vector, of which the stress component vector $\bar{\mathbf{p}}^{(1)}$ is a part, cannot be associated with a transmitting stress resultant: it has already been seen that tensile/shearing force is associated with the eigenvalue $\lambda = 0$ and this has been dealt with above, while bending moment has been seen to be an eigenvector associated with $\lambda = -1$. Hence the stress components $\bar{\mathbf{p}}^{(1)}$ can only be associated with a decay mode. Also note that the strain energy associated with a self-equilibrated load moving through some displacement will, in general, only be zero when that displacement is that of a rigid body – indeed, a crucial step in Toupin's (1965) proof of Saint-Venant's principle is "*the general proposition that any system of self-equilibrated forces does no work in a rigid motion of the points of action*." One concludes, therefore, that Eq. (141) will only be satisfied if the displacement vector $\bar{\mathbf{d}}^{(0)}$ is part of the rigid body rotation eigenvector, with $\lambda = 1$.

Case (d) is not consistent with the known repeating eigenvalue $\lambda = -1$, so one may conclude that arbitrary multiples of the generating eigenvector cannot be added to the principal vector in the special solution relating to the wedge paradox.

These arguments lead to the conclusion that $\lambda = -1$, $\lambda = 0$ and $\lambda = 1$ are the only eigenvalues that can lead to a non-trivial Jordan block for the radial evolution problem – one cannot have degenerate decay modes.

10.2. The curved beam

Now calculate the strain energy stored within an infinitesimal elastic sector, equal to the work done at its surface, as

$$U = (1/2)\int_a^b -(u_r\tau_{r\theta} + u_\theta\sigma_\theta)_\theta \mathrm{d}r + (1/2)\int_a^b (u_r\tau_{r\theta} + u_\theta\sigma_\theta)_{\theta+\mathrm{d}\theta}\mathrm{d}r. \tag{142}$$

The first integral pertains to some generic cross-section θ, where the direction cosine is $n_\theta = -1$, while the second, on $\theta + \mathrm{d}\theta$, has the direction cosine $n_\theta = 1$. Suppose that the state-vector on θ is $\mathbf{v} = \tilde{\mathbf{v}}\mathrm{e}^{k\theta}$, then on cross-section $\theta + \mathrm{d}\theta$ one has $\mathbf{v} + (\partial\mathbf{v}/\partial\theta)\mathrm{d}\theta = (1 + k\,\mathrm{d}\theta)\mathbf{v}$. Writing $\tilde{\mathbf{v}} = \begin{bmatrix}\tilde{\mathbf{d}}^{\mathrm{T}} & \tilde{\mathbf{p}}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$, with $\tilde{\mathbf{d}} = [\tilde{u}_r \quad \tilde{u}_\theta]^{\mathrm{T}}$ and $\tilde{\mathbf{p}} = \begin{bmatrix}\widetilde{S}_{r\theta} & \widetilde{S}_\theta\end{bmatrix}^{\mathrm{T}}$, and ignoring terms in $(\mathrm{d}\theta)^2$, one finds

$$U = k\mathrm{e}^{2k\theta}\mathrm{d}\theta\int_a^b \tilde{\mathbf{d}}^{\mathrm{T}}\tilde{\mathbf{p}}r^{-1}\mathrm{d}r. \tag{143}$$

The following observations are made:

(a) the strain energy of an eigenvector associated with $k = 0$ is zero – this is a rigid body rotation; on the other hand, a rigid body displacement has $k = \pm\mathrm{i}$, but the corresponding $\tilde{\mathbf{p}} = \mathbf{0}$, and the strain energy is also zero.

Next, consider the implication of a principal vector coupled to an eigenvector for a repeating eigenvalue k. From Section 6.2, assuming the inclusion of an arbitrary multiple (β) of the generating eigenvector, we may write the vector of displacement and stress components on the generic cross-section θ, from Eq. (70), as

$$\mathbf{v}_\theta^{\mathrm{p}} = \begin{bmatrix} u_r \\ u_\theta \\ \tau_{r\theta} \\ \sigma_\theta \end{bmatrix}^{\mathrm{p}} = \begin{bmatrix} \tilde{u}_r \\ \tilde{u}_\theta \\ \widetilde{S}_{r\theta}r^{-1} \\ \widetilde{S}_\theta r^{-1} \end{bmatrix}^{(1)} \mathrm{e}^{k\theta} + (\beta+\theta)\begin{bmatrix} \tilde{u}_r \\ \tilde{u}_\theta \\ \widetilde{S}_{r\theta}r^{-1} \\ \widetilde{S}_\theta r^{-1} \end{bmatrix}^{(0)} \mathrm{e}^{k\theta}. \tag{144}$$

On the section $\theta + \mathrm{d}\theta$ one has $\mathbf{v}_{\theta+\mathrm{d}\theta}^{\mathrm{p}} = \mathbf{v}_\theta^{\mathrm{p}} + (\partial\mathbf{v}_\theta^{\mathrm{p}}/\partial\theta)\mathrm{d}\theta$ which is

$$\mathbf{v}_{\theta+\mathrm{d}\theta}^{\mathrm{p}} = (1 + k\,\mathrm{d}\theta)\begin{bmatrix} u_r \\ u_\theta \\ \tau_{r\theta} \\ \sigma_\theta \end{bmatrix}^{\mathrm{p}} = \begin{bmatrix} \tilde{u}_r \\ \tilde{u}_\theta \\ \widetilde{S}_{r\theta}r^{-1} \\ \widetilde{S}_\theta r^{-1} \end{bmatrix}^{(1)} \mathrm{e}^{k\theta} + ((\beta+\theta)(1 + k\,\mathrm{d}\theta) + \mathrm{d}\theta)\begin{bmatrix} \tilde{u}_r \\ \tilde{u}_\theta \\ \widetilde{S}_{r\theta}r^{-1} \\ \widetilde{S}_\theta r^{-1} \end{bmatrix}^{(0)} \mathrm{e}^{k\theta}. \tag{145}$$

Ignoring terms in $(\mathrm{d}\theta)^2$ and higher, from Eq. (142) the strain energy becomes

$$U = \mathrm{e}^{2k\theta}\mathrm{d}\theta\int_a^b [k\tilde{\mathbf{d}}^{(1)\mathrm{T}}\tilde{\mathbf{p}}^{(1)} + (k(\beta+\theta) + 1/2)(\tilde{\mathbf{d}}^{(1)\mathrm{T}}\tilde{\mathbf{p}}^{(0)} + \tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(1)}) + (\beta+\theta)(k(\beta+\theta) + 1)\tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(0)}]r^{-1}\mathrm{d}r. \tag{146}$$

Note that the integral in Eq. (146) is identical in structure to the equivalent Eq. (133) for the wedge, except that now k replaces λ, θ replaces $\ln r$, and consistent with the definition of the state vector for the curved beam, one has the additional r^{-1} term.

When $\beta = 0$, Eq. (146) reduces to

$$U = \mathrm{e}^{2k\theta}\mathrm{d}\theta\int_a^b (k\tilde{\mathbf{d}}^{(1)\mathrm{T}}\tilde{\mathbf{p}}^{(1)} + (k\theta + 1/2)(\tilde{\mathbf{d}}^{(1)\mathrm{T}}\tilde{\mathbf{p}}^{(0)} + \tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(1)}) + (\theta + k\theta^2)\tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(0)})r^{-1}\mathrm{d}r. \tag{147}$$

This expression simplifies considerably for the case of the pure bending moment principal vector coupled to the eigenvector describing rigid body rotation, for which $k = 0$, leading to

$$U = (1/2)\mathrm{d}\theta\int_a^b \tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(1)}r^{-1}\mathrm{d}r = (1/2)\mathrm{d}\theta\int_a^b \tilde{u}_\theta^{(0)}\tilde{\sigma}_\theta^{(1)}\mathrm{d}r, \tag{148}$$

since $\tilde{\mathbf{p}}^{(0)} = \mathbf{0}$, and $\tilde{\tau}_{r\theta}^{(1)} = 0$, and is in the form one might expect. On the other hand, the repeating eigenvalue $k = \pm\mathrm{i}$ couples the shearing force principal vector to the rigid body displacement eigenvector, and setting $k = \mathrm{i}$, Eq. (147) reduces to

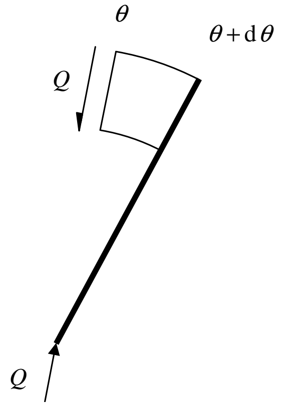

Fig. 7. Curved beam element subject to shearing force.

$$U = \mathrm{e}^{2\mathrm{i}\theta}\mathrm{d}\theta\int_a^b (\mathrm{i}\tilde{\mathbf{d}}^{(1)\mathrm{T}}\tilde{\mathbf{p}}^{(1)} + (\mathrm{i}\theta + 1/2)\tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(1)})r^{-1}\mathrm{d}r, \tag{149}$$

since $\tilde{\mathbf{p}}^{(0)} = \mathbf{0}$. Only the final term – that is $(1/2)\mathrm{e}^{2\mathrm{i}\theta}\mathrm{d}\theta\int_a^b \tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(1)}r^{-1}\mathrm{d}r$ – is in the form that one might expect; however, the stress vector $\tilde{\mathbf{p}}^{(1)}$acting on an element, Fig. 7, is effectively self-equilibrating on a radial line extending over the generic cross-section at θ to the origin. Thus the work done in moving through arbitrary rigid body displacement $\tilde{\mathbf{d}}^{(0)}$ is zero, and $\int_a^b \tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(1)}r^{-1}\mathrm{d}r = 0$. Rather, the strain energy reduces to $U = \mathrm{i}\mathrm{e}^{2\mathrm{i}\theta}\mathrm{d}\theta\int_a^b \tilde{\mathbf{d}}^{(1)\mathrm{T}}\tilde{\mathbf{p}}^{(1)}r^{-1}\mathrm{d}r$.

The condition that the strain energy should be independent of β is $A\beta^2 + B\beta = 0$, where

$$A = k\int_a^b \tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(0)}r^{-1}\mathrm{d}r, \tag{150a}$$

$$B = k\int_a^b (\tilde{\mathbf{d}}^{(1)\mathrm{T}}\tilde{\mathbf{p}}^{(0)} + \tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(1)})r^{-1}\mathrm{d}r + (2k\theta + 1)\int_a^b \tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(0)}r^{-1}\mathrm{d}r. \tag{150b}$$

Again, one must have $A = B = 0$; for $A = 0$, one must have either $k = 0$, or $\int_a^b \tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(0)}r^{-1}\mathrm{d}r = 0$, or perhaps both. If one chooses $k = 0$, then the requirement that $B = 0$ leads to $\int_a^b \tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(0)}r^{-1}\mathrm{d}r = 0$. If one chooses $k \neq 0$, then one must have $\int_a^b \tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(0)}r^{-1}\mathrm{d}r = 0$, and the requirement that $B = 0$, leads to $\int_a^b (\tilde{\mathbf{d}}^{(1)\mathrm{T}}\tilde{\mathbf{p}}^{(0)} + \tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(1)})r^{-1}\mathrm{d}r = 0$. Thus one may conclude that

$$\int_a^b \tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(0)}r^{-1}\mathrm{d}r = 0, \tag{151}$$

always, and either

$$\text{(b)}\quad k = 0, \tag{152}$$

or

$$\text{(c)}\quad k \neq 0, \text{and} \int_a^b (\tilde{\mathbf{d}}^{(1)\mathrm{T}}\tilde{\mathbf{p}}^{(0)} + \tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(1)})r^{-1}\mathrm{d}r = 0. \tag{153}$$

Eq. (151) is satisfied for the rigid body displacement and rotation eigenvectors, as the stress components comprising $\tilde{\mathbf{p}}^{(0)}$ within both are zero; it cannot be satisfied for a decay eigenmode as, from Eq. (143), the strain energy stored within the element would have to be zero.

Case (b) is in accord with what was found in Section 7, where a principal vector describing pure bending is coupled to an eigenvector describing rotation about the origin, with repeating eigenvalue $k = 0$.

Case (c) is consistent with repeating eigenvalue $k = \pm\mathrm{i}$; the rigid body displacement has $\tilde{\mathbf{p}}^{(0)} = \mathbf{0}$, and the integral in Eq. (153) reduces to $\int_a^b \tilde{\mathbf{d}}^{(0)\mathrm{T}}\tilde{\mathbf{p}}^{(1)}r^{-1}\mathrm{d}r = 0$, which is true as noted above.

These arguments lead to the conclusion that $k = 0$ and $k = \pm\mathrm{i}$ are the only eigenvalues that can lead to a non-trivial Jordan block for the radial evolution problem – again, one cannot have degenerate decay modes.

11. Reciprocal theorem and symplectic orthogonality

Zhong and Williams (1993) have shown for the prismatic structure that the statement of orthogonality is equivalent to the Betti–Maxwell reciprocal theorem, according to which the work done by stress components $\mathbf{p}_1$ acting through displacement components $\mathbf{d}_2$ is equal to the work done by stress components $\mathbf{p}_2$ acting through displacement components $\mathbf{d}_1$. Here, the orthogonality relationships are developed for both the wedge and the curved beam.

11.1. The wedge

Adapting Eq. (126) one has

$$\int_{-\alpha}^{\alpha} -(u_{r2}\sigma_{r1} + u_{\theta 2}\tau_{r\theta 1})_{\text{inner}} r\mathrm{d}\theta + \int_{-\alpha}^{\alpha} (u_{r2}\sigma_{r1} + u_{\theta 2}\tau_{r\theta 1})_{\text{outer}} (r+\mathrm{d}r)\mathrm{d}\theta$$
$$= \int_{-\alpha}^{\alpha} -(u_{r1}\sigma_{r2} + u_{\theta 1}\tau_{r\theta 2})_{\text{inner}} r\mathrm{d}\theta + \int_{-\alpha}^{\alpha} (u_{r1}\sigma_{r2} + u_{\theta 1}\tau_{r\theta 2})_{\text{outer}} (r+\mathrm{d}r)\mathrm{d}\theta. \tag{154}$$

Substituting from Eqs. (127),(129), and ignoring terms in $(\mathrm{d}r)^2$ and higher, one finds

$$(\lambda_1+\lambda_2)r^{\lambda_1+\lambda_2-1}\mathrm{d}r \int_{-\alpha}^{\alpha} (\bar{u}_{r2}\bar{\sigma}_{r1} + \bar{u}_{\theta 2}\bar{\tau}_{r\theta 1} - \bar{u}_{r1}\bar{\sigma}_{r2} - \bar{u}_{\theta 1}\bar{\tau}_{r\theta 2})\mathrm{d}\theta = 0, \tag{155}$$

or

$$(\lambda_1+\lambda_2)r^{\lambda_1+\lambda_2-1}\mathrm{d}r \int_{-\alpha}^{\alpha} (\bar{\mathbf{d}}_2^{\mathrm{T}}\bar{\mathbf{p}}_1 - \bar{\mathbf{d}}_1^{\mathrm{T}}\bar{\mathbf{p}}_2)\mathrm{d}\theta = 0, \tag{156}$$

where $\bar{\mathbf{d}}_n = [\,\bar{u}_{rn} \quad \bar{u}_{\theta n}\,]^{\mathrm{T}}$ and $\bar{\mathbf{p}}_n = [\,\bar{\sigma}_{rn} \quad \bar{\tau}_{r\theta n}\,]^{\mathrm{T}}, n = 1, 2$. For $\lambda_1 \neq -\lambda_2$, the integral must be zero

$$\int_{-\alpha}^{\alpha} (\bar{\mathbf{d}}_2^{\mathrm{T}}\bar{\mathbf{p}}_1 - \bar{\mathbf{d}}_1^{\mathrm{T}}\bar{\mathbf{p}}_2)\mathrm{d}\theta = 0, \tag{157}$$

which may be written as

$$\int_{-\alpha}^{\alpha} \bar{\mathbf{s}}_2^{\mathrm{T}}\mathbf{J}_{\mathrm{m}}\bar{\mathbf{s}}_1\mathrm{d}\theta = 0, \tag{158}$$

$\bar{\mathbf{s}}_n = [\,\bar{\mathbf{d}}_n^{\mathrm{T}} \quad \bar{\mathbf{p}}_n^{\mathrm{T}}\,]^{\mathrm{T}}, \quad \mathbf{J}_{\mathrm{m}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$ is known as the metric, and $\mathbf{I}$ is the 2×2 identity matrix. Eq. (158) is the statement of symplectic orthogonality for the wedge.

11.2. The curved beam

Adapting Eq. (142) one has

$$\int_a^b -(u_{r2}\tau_{r\theta 1} + u_{\theta 2}\sigma_{\theta 1})_{\theta}\mathrm{d}r + \int_a^b (u_{r2}\tau_{r\theta 1} + u_{\theta 2}\sigma_{\theta 1})_{\theta+\mathrm{d}\theta}\mathrm{d}r = \int_a^b -(u_{r1}\tau_{r\theta 2} + u_{\theta 1}\sigma_{\theta 2})_{\theta}\mathrm{d}r + \int_a^b (u_{r1}\tau_{r\theta 2} + u_{\theta 1}\sigma_{\theta 2})_{\theta+\mathrm{d}\theta}\mathrm{d}r. \tag{159}$$

Ignoring terms in $(\mathrm{d}\theta)^2$, one finds

$$(k_1+k_2)\mathrm{e}^{(k_1+k_2)\theta}\mathrm{d}\theta \int_a^b (\tilde{u}_{r2}\tilde{\tau}_{r\theta 1} + \tilde{u}_{\theta 2}\tilde{\sigma}_{\theta 1} - \tilde{u}_{r1}\tilde{\tau}_{r\theta 2} - \tilde{u}_{\theta 1}\tilde{\sigma}_{\theta 2})\mathrm{d}r = 0 \tag{160}$$

or

$$(k_1+k_2)\mathrm{e}^{(k_1+k_2)\theta}\mathrm{d}\theta \int_a^b (\tilde{\mathbf{d}}_2^{\mathrm{T}}\tilde{\mathbf{p}}_1 - \tilde{\mathbf{d}}_1^{\mathrm{T}}\tilde{\mathbf{p}}_2)r^{-1}\mathrm{d}r = 0 \tag{161}$$

where $\tilde{\mathbf{d}}_n = [\,\tilde{u}_{rn} \quad \tilde{u}_{\theta n}\,]^{\mathrm{T}}$ and $\bar{\mathbf{p}}_n = [\,\widetilde{S}_{r\theta n} \quad \widetilde{S}_{\theta n}\,]^{\mathrm{T}} = [\,r\tilde{\tau}_{r\theta n} \quad r\tilde{\sigma}_{\theta n}\,]^{\mathrm{T}}, n = 1, 2$. For $k_1 \neq -k_2$, one must have

$$\int_a^b \tilde{\mathbf{v}}_2^{\mathrm{T}}\mathbf{J}_{\mathrm{m}}\tilde{\mathbf{v}}_1 r^{-1}\mathrm{d}r = 0, \tag{162}$$

where $\tilde{\mathbf{v}}_n = [\,\tilde{\mathbf{d}}_n^{\mathrm{T}} \quad \tilde{\mathbf{p}}_n^{\mathrm{T}}\,]^{\mathrm{T}}$. Eq. (162) is the statement of symplectic orthogonality for the curved beam.

12. Concluding remarks

In this paper, the linear elasticity of a plane stress sector has been considered from a state-space point of view using polar coordinates; this geometry encompasses both the wedge and the curved beam. Since elasticity may be regarded as a classical field of study, one might imagine that there are no new insights to be gleaned; indeed there seems little point in deriving from a variational principle the governing equations, when these are already well-known, and need only be arranged into the required form to describe evolution in the radial and circumferential directions. On the other hand, the approach treats displacement and stress components on an equal footing, which obviates the need to employ strain compatibility equations. For both the wedge and the curved beam, the approach unifies the treatment of the transmission and decay problems, and benefits from the analytical machinery of an eigenproblem which provides a systematic means of dealing with the numerous multiple eigenvalues associated with the transmission modes – force, moment, often coupled to rigid body displacements and rotation. The Saint-Venant decay problem associated with self-equilibrated loading is elegantly solved through use of the matrix exponential of the system matrix, and is very straightforward with the use of a computer algebra package such as MAPLE.

One of the insights provided by this approach is the nature of the so-called wedge paradox. Much previous research has associated the paradox with the (fictional) notion of a moment applied at the apex of a wedge region – a moment clearly requires a dimension, which does not exist. Physically, this difficulty can be circumvented by requiring that the moment be applied over some small region close to the apex – either on an arc of small radius, or on the flanks. (A comparable difficulty applies to a force applied at the apex: since the cross-sectional area is effectively zero, the stress becomes infinite, and this can also be circumvented by limiting the minimum inner radius such that the stress on the inner arc does not exceed the elastic limit. However, since this symmetric solution does not display pathological behaviour for any particular angle, this has not attracted such attention.) Nevertheless, authors have employed rather esoteric (to an engineer) techniques including use of renormalisation group theory (Goldenfeld and Oono, 1991), and intermediate asymptotics (Barenblatt, 1996) to deal with this singularity at the apex. In fact, the paradox is associated with accidental eigenvalue degeneracy for the particular wedge angle $2\alpha^* \approx 257°$, which signals the breakdown of Saint-Venant's principle for self-equilibrated loading on the outer arc, and the decay of self-equilibrated loading on the inner arc at the same rate as the diffusion of bending moment into a divergent region; its resolution requires nothing other than a principal vector coupled to the decay eigenvector for a repeating eigenvalue.

One of the main advantages of this state-space approach is the ability to draw upon the extensive body of knowledge on system theory, and it is likely that this will prove beneficial in the analysis of the control of dynamical problems for continuum elastic structures.

Appendix A

The matrix exponential $\mathbf{M} = e^{\mathbf{H}'\theta}$ in Eq. (42) has elements

$$M_{11} = \cos\theta\cos\lambda\theta + (1-\lambda-\nu-\nu\lambda)\sin\theta\sin\lambda\theta/2$$
$$M_{12} = (1-\lambda/2-\nu\lambda/2)\sin\theta\cos\lambda\theta - (1-\nu)\cos\theta\sin\lambda\theta/2$$
$$M_{13} = (1+\nu)[(3-\nu+\lambda+\nu\lambda)\sin(\lambda+1)\theta + (3-\nu-\lambda-\nu\lambda)\sin(\lambda-1)\theta]/4E\lambda$$
$$M_{14} = (1+\nu)(3-\nu-\lambda-\nu\lambda)[\cos(\lambda-1)\theta - \cos(\lambda+1)\theta]/4E\lambda$$

$$M_{21} = (-1-\lambda/2-\nu\lambda/2)\sin\theta\cos\lambda\theta + (1-\nu)\cos\theta\sin\lambda\theta/2$$
$$M_{22} = \cos\theta\cos\lambda\theta + (1-\nu+\lambda+\nu\lambda)\sin\theta\sin\lambda\theta/2$$
$$M_{23} = (1+\nu)(3-\nu+\lambda+\nu\lambda)[\cos(\lambda+1)\theta - \cos(\lambda-1)\theta]/4E\lambda$$
$$M_{24} = (1+\nu)[(3-\nu-\lambda-\nu\lambda)\sin(\lambda+1)\theta + (3-\nu+\lambda+\nu\lambda)\sin(\lambda-1)\theta]/4E\lambda$$

$$M_{31} = E\lambda[(\lambda-1)\sin(\lambda-1)\theta - (\lambda+1)\sin(\lambda+1)\theta]/4$$
$$M_{32} = E\lambda(\lambda-1)[\cos(\lambda-1)\theta - \cos(\lambda+1)\theta]/4$$
$$M_{33} = \cos\theta\cos\lambda\theta + (-1+\nu-\lambda-\nu\lambda)\sin\theta\sin\lambda\theta/2$$
$$M_{34} = (1-\lambda/2-\nu\lambda/2)\sin\theta\cos\lambda\theta + (1-\nu)\cos\theta\sin\lambda\theta/2$$

$$M_{41} = E\lambda(\lambda+1)[\cos(\lambda-1)\theta - \cos(\lambda+1)\theta]/4$$
$$M_{42} = -E\lambda[(\lambda+1)\sin(\lambda-1)\theta - (\lambda-1)\sin(\lambda+1)\theta]/4$$
$$M_{43} = (-1-\lambda/2-\nu\lambda/2)\sin\theta\cos\lambda\theta - (1-\nu)\cos\theta\sin\lambda\theta/2$$
$$M_{44} = \cos\theta\cos\lambda\theta + (-1+\nu+\lambda+\nu\lambda)\sin\theta\sin\lambda\theta/2$$

Appendix B

The matrix exponential $\mathbf{N} = e^{\overline{\mathbf{H}}\xi}$ in Eq. (100) has elements

$$N_{11} = [k(1+\nu)(e^{\xi} - e^{-\xi})\sin k\xi + 2((1-\nu)e^{\xi} + (1+\nu)e^{-\xi})\cos k\xi]/4$$
$$N_{12} = [k(1+\nu)(e^{\xi} - e^{-\xi})\cos k\xi - 2(1-\nu)e^{\xi}\sin k\xi]/4$$
$$N_{13} = (1+\nu)[(2(1-\nu) - k^2(1+\nu))(e^{\xi} - e^{-\xi})\cos k\xi + k(3-\nu)(e^{\xi} + e^{-\xi})\sin k\xi]/(4E(1+k^2))$$
$$N_{14} = (1+\nu)[k(3-\nu)(e^{\xi} - e^{-\xi})\cos k\xi + (k^2(1+\nu)(e^{\xi} - e^{-\xi}) - 2(1-\nu)e^{\xi} - 4e^{-\xi})\sin k\xi]/(4E(1+k^2))$$

$$N_{21} = [k(1+\nu)(e^{\xi} - e^{-\xi})\cos k\xi + (4e^{\xi} - 2(1+\nu)e^{-\xi})\sin k\xi]/4$$
$$N_{22} = [4e^{\xi}\cos k\xi - k(1+\nu)(e^{\xi} - e^{-\xi})\sin k\xi]/4$$
$$N_{23} = (1+\nu)[-k(3-\nu)(e^{\xi} - e^{-\xi})\cos k\xi + (k^2(1+\nu)(e^{\xi} - e^{-\xi}) + 2(1-\nu)e^{-\xi} + 4e^{\xi})\sin k\xi]/(4E(1+k^2))$$
$$N_{24} = (1+\nu)[(k^2(1+\nu)+4)(e^{\xi} - e^{-\xi})\cos k\xi + k(3-\nu)(e^{\xi} + e^{-\xi})\sin k\xi]/(4E(1+k^2))$$

$$N_{31} = E[(2+k^2)(e^{\xi} - e^{-\xi})\cos k\xi - k(e^{\xi} + e^{-\xi})\sin k\xi]/4$$
$$N_{32} = -E[k(e^{\xi} - e^{-\xi})\cos k\xi + (2e^{\xi} + k^2(e^{\xi} - e^{-\xi}))\sin k\xi]/4$$
$$N_{33} = -E[2((1+\nu)e^{\xi} + (1-\nu)e^{-\xi})\cos k\xi - k((1-\nu)e^{\xi} + (1+\nu)e^{-\xi})\sin k\xi]/4$$
$$N_{34} = [k(1+\nu)(e^{\xi} - e^{-\xi})\cos k\xi - (2(1+\nu)e^{\xi} - 4e^{-\xi})\sin k\xi]/4$$

$$N_{41} = E[k(e^{\xi} - e^{-\xi})\cos k\xi - (k^2(e^{\xi} - e^{-\xi}) - 2e^{-\xi})\sin k\xi]/4$$
$$N_{42} = -Ek[(e^{\xi} + e^{-\xi})\sin k\xi + k(e^{\xi} - e^{-\xi})\cos k\xi]/4$$
$$N_{43} = [k(1+\nu)(e^{\xi} - e^{-\xi})\cos k\xi - 2(1-\nu)e^{-\xi}\sin k\xi]/4$$
$$N_{44} = [4e^{-\xi}\cos k\xi + k(1+\nu)(1 - e^{\xi})\sin k\xi]/4$$

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