6th GRACM International Congress on Computational Mechanics
Thessaloniki, 19-21 June 2008
CD-ROM PROCEEDINGS ISBN: 978-960-6706-08-0

A FORMULATION FOR LARGE-SCALE TRUSS OPTIMIZATION

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Keywords: truss optimization, second-order cone programming, conic optimization

Abstract. *This paper is related to the optimization of trusses based on the minimum compliance method. There are two basic aims. First to show that the sizing problem of trusses can be cast as a second-order cone programming (SOCP) problem. This is feasible even if we consider multiple load cases and the design has to be based on the worst-case compliance. The benefit is that the problem can be solved easily using a standard software. The second aim is to show that the optimal connectivity can be obtained if we apply a heuristic which is based on SOCP duality. This means that we can simply consider a ground structure with some connections and after the solution of the sizing problem we can add only members which reduce the compliance. In this way we solve a sequence of numerical optimization problems. This is preferable than solving one sizing problem with all possible connections as this yields very large numerical optimization problems.*

1 INTRODUCTION

The optimal design of trusses is an important problem in the area of structural optimization. This is due to the simplicity of designing with these structures and the ease in constructing mathematical models of them. Truss optimization consists of finding the optimal nodal positions, connectivities and cross section areas. Of course the optimal design of a structure is defined by the target, the most popular target being either the minimum weight of a rigid plastic structure (under single loading) or the minimum compliance if the structure is made of elastic material. The reader can find some aspects on the similarities and the differences of these problems in references [2, 7]. Another objective function could be the minimization of the maximum stress.

This paper is concerned with minimum compliance methods. In general, the calculation of all the optimal parameters of a truss leads to non-linear problems which are very difficult to solve. A usual way to avoid including two of these parameters, the optimal positions of the nodes and connectivities, is to consider a dense grid of nodes with all possible connections (see Figure 1a) and solve the sizing optimization problem by mathematical programming techniques. Although the arising optimization problem is still non-linear it can be treated in various ways. It can be considered as a case of Mathematical Program with Complementary Constraints (MPCC) [5], or it can be solved by two level optimization problems as e.g. in [3]. However the sizing problem does not have a typical form for which optimizers are available. Moreover a dense grid where we consider all possible connections can result in a very large optimization problem. Another issue is that if we consider multiple load cases and particularly the "worst-case" compliance, the arising numerical optimization problem becomes even more complicated.

The first aim of this paper is to present an efficient formulation of the sizing problem as a case of second-order cone programming. This is advantageous because there are several efficient optimizers for this type of problem and therefore engineers do not have to construct their own optimizer. Also the formulation is fairly general as it can include the worst-case compliance. The second aim is to provide a heuristic so that we can avoid solving the problem with all possible connections. On the contrary we can consider a simple ground structure (see Figure 1b) and after the solution of the numerical problem add only the necessary members. The heuristic is based on SOCP duality and extends the

3 THE MINIMUM COMPLIANCE PROBLEM IN TRUSS OPTIMIZATION

Consider now a truss structure with NE members and NU degrees of freedom. For a given volume of material, V, our aim is to find the cross section areas of the bars such that the compliance will be the minimum. It would be more convenient to use the member volumes ξ_i as the unknowns instead of the cross sectional areas of the members. For the sake of simplicity we shall consider that they are not bounded.

The sizing optimization problem for the minimum compliance method reads as:

$$\begin{aligned} \min \quad & \mathbf{p}^T\mathbf{u} \\ \text{s.t.} \quad & (\sum_{i=1}^{NE} \xi_i \mathbf{K}_i)\mathbf{u} = \mathbf{p} \\ & \sum_{i=1}^{NE} \xi_i = V \end{aligned} \tag{9}$$

where $\mathbf{p} \in \Re^{NU}$ and $\mathbf{u} \in \Re^{NU}$ are the load and the displacement vector respectively. For each member

$$\mathbf{K}_i = \frac{E_i}{L_i^2}\mathbf{b}_i\mathbf{b}_i^T \tag{10}$$

where L_i, E_i are respectively the length and the Young's modulus of the ith member. The column vector $\mathbf{b}_i$ relates the elongation of the member i with the nodal displacement vector $\mathbf{u}$ so that

$$e_i = \mathbf{b}_i^T\mathbf{u}. \tag{11}$$

We notice that (9) is a non-convex problem, and therefore, the solution is rather difficult. For this reason alternative forms of this optimization problem have been presented (see e.g. [3, 2, 5]). Engineering structures are often subjected to multiple loading cases. In this case we are interested in calculating the "worst-case" compliance. Considering m load cases the problem reads:

$$\begin{aligned} \min_{\mathbf{u},\xi} \quad \max_{j=1,\dots,m} \quad & (\mathbf{p}^{(j)})^T\mathbf{u}^{(j)} \\ \text{s.t.} \quad & (\sum_{i=1}^{NE} \xi_i \mathbf{K}_i)\mathbf{u}^{(j)} = \mathbf{p}^{(j)} \quad . \\ & \sum_{i=1}^{NE} \xi_i = V \end{aligned} \tag{12}$$

This problem is even more complicated than (9). Here we will use the principle of the complementary energy - which gives the half of compliance - and also it will be shown that (12) - and consequantly (9) - can be formulated not only as a convex optimization problem but as a case of SOCP.

4 SIZING OPTIMIZATION OF TRUSSES FOR THE WORST-CASE COMPLIANCE

We consider that the structure is subjected to arbitrarily varying loads within a load domain $\mathscr{L}$. By applying the principle of complementary energy the optimization problem reads:

$$\begin{aligned} \min_{\xi,\mathbf{q}} \quad \max_{t\in T} \quad & \sum_{i=1}^{NE} \frac{L_i^2}{E_i}\frac{q_i^2(t)}{2\xi_i} \\ \text{s.t.} \quad & \mathbf{B}\mathbf{q}(t) = \mathbf{p}(t), \quad \forall t \in T \\ & \sum_{i=1}^{N} \xi_i = V \\ & \xi_i > 0 \end{aligned} \tag{13}$$

where t is the pseudo-time parameter such that $\mathbf{p}(t) \in \mathscr{L}$ and T is a set such that $T = \{t \ : \ \mathbf{p}(t) \in \mathscr{L}\}$. The vector $\mathbf{q}(t) \in \Re^{NE}$ is the internal axial forces vector, and $\mathbf{B} \in \Re^{NU\times NE}$, $\mathbf{B} = [\mathbf{b}_1 \dots \mathbf{b}_{NE}]$ is the equilibrium matrix. The worst compliance can be represented by a variable

$$r^* = \max_{t\in T} \sum_{i=1}^{NE} \frac{L_i^2}{E_i}\frac{q_i^2(t)}{2\xi_i} \tag{14}$$

or equivalently

$$\begin{aligned} r^* &\geq \sum_{i=1}^{NE} r_i(t), \qquad \forall t \in T \\ r_i(t) &\geq \frac{L_i^2}{E_i}\frac{q_i^2(t)}{2\xi_i}, \qquad \forall t \in T, \quad \forall i \in \{1,\dots,NE\}. \end{aligned} \tag{15}$$

- Only the worst-case compliance is minimized. The other compliances do not necessarily correspond to the minimum complementary energy for the calculated volumes and, therefore, the stresses would not correspond to their respective load cases.
- In the case of specific finite load cases i.e. $\mathbf{p}_1,\ldots,\mathbf{p}_N$ this problem corresponds to the worst compliance for a load domain whose load vector is given by

$$\mathbf{p}(t)=\sum_{j=1}^{N}\gamma_j(t)\mathbf{p}^{(j)}+\gamma_0(t)\mathbf{p}^{(0)} \tag{25}$$

with $\sum_{j=0}^{N}\gamma_j(t)=1,\quad \gamma_j(t)\geq 0$ and $\mathbf{p}^{(0)}=\mathbf{0}$ i.e. the load domain has the form of a hyper-triangle of $N+1$ load vertices. This case is also referred as alternating loading and this is the usual way that multiple load cases are considered.

- Consider minimizing the compliance for each load load vector $\mathbf{p}^{(j)}$. The lowest compliance arises, say, for the case $j=k$ which gives a compliance Π^* for $\xi=\xi^*$. If for this volume vector the compliance for the other load cases is still lower than Π^* the result for the single load case $\mathbf{p}^{(k)}$ will be the same as for the worst-case compliance.

5 A HEURISTIC FOR ADDING NEW MEMBERS

The dual problem of (24) - see duality betwen (7) and (8) - will have the form

$$\begin{array}{lll}
\max & \sum_{j=1}^{NV}(\mathbf{p}^{(j)})^T\mathbf{u}^{(j)}+Vz & \\
\text{s.t.} & (\psi_i^{(j)},\phi_i^{(j)},\bar{e}_i^{(j)})\in\mathcal{K}_r^{(3)}, & \forall i\in\{1,\ldots,NE\},\quad \forall j\in\{1,\ldots,NV\}\\
& \bar{\mathbf{B}}^T\mathbf{u}^{(j)}+\bar{\mathbf{e}}^{(j)}=\mathbf{0}, & \forall j\in\{1,\ldots,NV\}\\
& \bar{\psi}^{(j)}+\psi_i^{(j)}=0, & \forall i\in\{1,\ldots,NE\},\quad \forall j\in\{1,\ldots,NV\}\\
& \bar{\phi}_i^{(j)}+\phi_i^{(j)}=0, & \forall i\in\{1,\ldots,NE\},\quad \forall j\in\{1,\ldots,NV\}\\
& -\sum_{j=1}^{NV}\bar{\phi}_i^{(j)}+z=0, & \forall i\in\{1,\ldots,NE\}\\
& -\sum_{j=1}^{NV}\bar{\psi}^{(j)}=1 &
\end{array} \tag{26}$$

where $z\in\Re$ and $\bar{\mathbf{e}}^{(j)}\in\Re^{NE}$. The problem can be reduced to

$$\begin{array}{lll}
\max & \sum_{j=1}^{NV}(\mathbf{p}^{(j)})^T\mathbf{u}^{(j)}-z^* & \\
\text{s.t.} & \bar{\psi}^{*(j)}\phi_i^{(j)}\geq\frac{V}{2}(\bar{\mathbf{b}}_i^T\mathbf{u}^{(j)})^2, & \forall i\in\{1,\ldots,NE\},\quad \forall j\in\{1,\ldots,NV\}\\
& \sum_{j=1}^{NV}\phi_i^{(j)}=z^*, & \forall i\in\{1,\ldots,NE\}\\
& \sum_{j=1}^{NV}\bar{\psi}^{*(j)}=1 &
\end{array} \tag{27}$$

Now say that we connect two nodes that are existing but unconnected. By connecting them we have a new optimization problem and the question is whether this will lead to lower compliance. This will not occur if for the existing nodal solutions the set defined by the constraints remains feasible. To make it clearer by adding a member the dimension of the $\mathbf{u}$ and $\bar{\psi}^{*(j)}$ variables will remain the same. Now if we form the vector $\mathbf{b}$ and we have the additional constraints

$$\phi^{(j)}\geq V\frac{(\bar{\mathbf{b}}^T\mathbf{u}^{(j)})^2}{2\bar{\psi}^{*(j)}} \tag{28}$$

$$\sum_{j=1}^{NV}\phi_i^{(j)}=z^*. \tag{29}$$

Assuming that we give to $\mathbf{u}$ and $\bar{\psi}^{*(j)}$ the same values then the problem will remain feasible if for $\phi_i^{(j)}=V\frac{(\bar{\mathbf{b}}^T\mathbf{u}^{(j)})^2}{2\bar{\psi}^{*(j)}}$

$$\sum_{j=1}^{NV}\phi_i^{(j)}\leq z^*. \tag{30}$$

Grid	NE	single loading case			alternate loading case		
		It	CPU(s)	Π	It	CPU(s)	Π
13×13	14196	26	1.6	7.02887	23	8.6	1.79517
25×25	195000	34	42.7	6.77610	40	335.0	1.74816

Table 2: Results and statistics for all possible connections

Stage	13×13 grid				25×25 grid				37×37 grid			
	NE	It	CPU(s)	Π	NE	It	CPU(s)	Π	NE	It	CPU(s)	Π
1	600	17	0.1	8.82001	2352	17	0.2	8.82000	5256	19	0.7	8.82000
2	888	18	0.1	7.10646	6158	24	0.9	6.80826	23562	30	5.3	6.79149
3	894	17	0.1	7.10645	6232	23	0.9	6.77617	24122	28	5.1	6.76105
4	898	18	0.1	7.02887	6234	24	0.9	6.77617	24144	28	5.1	6.76104
5					6236	25	0.9	6.77617	24146	28	5.1	6.76106

Table 3: Results and statistics for the single loading

Stage	13×13 grid				25×25 grid				37×37 grid			
	NE	It	CPU(s)	Π	NE	It	CPU(s)	Π	NE	It	CPU(s)	Π
1	600	20	0.4	2.07324	2352	23	2.2	2.06590	5256	24	6.6	2.06864
2	1018	17	0.5	1,80378	7766	28	13.3	1,75434	31836	30	106.4	1,74383
3	1034	16	0.5	1.79517	8296	25	13.5	1.74821	35312	35	139.1	1.73819
4					8312	26	12.3	1.74820	35358	33	144.1	1.73813
5									35362	34	149.2	1.73813
6									35364	34	153.0	1.73813

Table 4: Results and statistics for the alternate loading

E.g. for the 13×13 grid we notice that just 1.50 sec is needed for the methodology that we propose and 8.60 sec if we solve the problem with all connections. The following general observations can be made:

- Most of the members are added after the first step.
- Probably three steps are sufficient to get a very good approximation of the optimal compliance.
- For the case of the 13×13 grid, we notice that although very few members were added in the last stage and the compliance was reduced by less than 0.5%, there is a clear difference between the quality of the solutions of these last two stages. This means that little difference between the values of the compliances does not mean that there will be strong similarity in the values of the design parameters (in our case the cross section areas).
- The solution of problems with multiple loading takes significantly more time. This feature can be improved if we customise the optimizer so that it can take advantage of the angular form of the matrix data. The optimiser behaved very well. One exception is the case of the 37×37 grid (single loading case), where there was a slight instability. However the error is limited to the last digit (the sixth) of the objective function.

7 CONCLUSIONS

In this paper it has been shown that the problem of truss optimization via worst-case compliance can be formulated as an SOCP problem. This is advantageous because several SOCP software exist and therefore all that the engineer has to do is to construct the elemental data and solve the arising optimization problem. Moreover the arising problems are sparse and this endorses the use of the interior point method on which most of the modern algorithms are based. The simplicity of the structure of SOCP allowed us to extend the work or Gilbert and Tyas [4] to the minimum compliance problem. In this way the sizing problems are significantly reduced and we can handle structures with even finer grids and obtain accurate topologies.

ACKNOWLEDGEMENT

This work is funded by an EPSRC grant (EP/E004547/1).

REFERENCES

[1] Andersen, E.D., Roos, C. and Terlaky, T. (2003), "On implementing a primal-dual interior-point method for conic quadratic optimization," *Mathematical Programming Series B,* Vol 95, pp. 249–277.

[2] Bendsoe, M.P. (1995), *Optimization of Structural Topology, Shape and Material,* Springer-Verlag, Berlin.

[3] Bendsoe, M.P., Ben-Tal, A. and Zowe, J. (1994), "Optimization methods for truss geometry and topology design," *Structural Optimization,* Vol. 7, pp. 141–159.

[4] Gilbert, M. and Tyas, A. (2003), "Layout optimization of large-scale pin jointed frames," *Engineering Computations,* Vol. 20, pp. 1044–1064.

[5] Kočvara, M. and Outrata, J.V. (2006), "Effective reformulations of the truss topology design problem," *Optimization and Engineering,* Vol. 7, pp. 201–219.

[6] MOSEK ApS, *The MOSEK optimization tools manual. Version 5.0, Revision 60.* Available from http://www.mosek.com

[7] Rozvany, G.I.N. (2001), "Stress ratio and compliance based methods in topology optimization – a critical review," *Structural and Multidisciplinary Optimization,* Vol. 21, pp. 109–119.