On the Adams-Riemann-Roch theorem in positive characteristic

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with an appendix by B. Köck

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Abstract

We give a new proof of the Adams-Riemann-Roch theorem for a smooth projective morphism $X \to Y$, in the situation where Y is a scheme of characteristic $p > 0$, which is of finite type over a noetherian ring and carries an ample line bundle. This theorem implies the Hirzebruch-Riemann-Roch theorem in characteristic 0. We also answer a question of B. Köck.

1 Introduction

Let Y be a scheme, which is of finite type over an affine noetherian scheme. Suppose that there is an ample line bundle on Y. Let X be a scheme and let $f : X \to Y$ be a smooth projective morphism of schemes. Let $k \geqslant 2$ be a natural number and E an element of $K_0(X)$. A particular case of the Adams-Riemann-Roch theorem asserts that

$$\psi^k(R^\bullet f_*(E)) = R^\bullet f_*\big(\theta^k(\Omega_f)^{-1} \otimes \psi^k(E)\big) \tag{1}$$

in $K_0(Y)[\frac{1}{k}] := K_0(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{k}]$. The various symbols appearing in this formula are defined as follows.

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The Grothendieck group of locally free coherent sheaves (resp. coherent sheaves) on a scheme Z is denoted by $K_0(Z)$ (resp. $K_0'(Z)$).

For $f : X \to Y$ as above there is a unique group morphism $R^\bullet f_* : K_0(X) \to K_0(Y)$, which sends the class of an f-acyclic locally free coherent sheaf E on X to the sheaf $f_*(E)$ (see [13, Par. 4, Cor. 3, p. 103]). Recall that a locally free sheaf E on X is called f-acyclic if $R^l f_* E = 0$ for all $l > 0$. If E is any locally free sheaf on X, one can show that the image of the element $R^\bullet f_*(E) \in K_0(Y)$ under the natural map $K_0(Y) \to K_0'(Y)$ is $\sum_{l \geqslant 0} (-1)^l R^l f_*(E)$.

To define the symbol ψ^k, recall that for any scheme Z the tensor product $\otimes$ of $\mathcal{O}_Z$-modules makes the group $K_0(Z)$ into a commutative unitary ring and that the inverse image of coherent sheaves under any morphism of schemes $Z' \to Z$ induces a morphism of unitary rings $K_0(Z) \to K_0(Z')$ (see [10, Par. 1]).

In particular, $K_0(\cdot)$ may be viewed as a contravariant functor from the category of quasi-compact schemes to the category of commutative unitary rings. The symbol ψ^k refers to an endomorphism of this functor (sic!) that is uniquely determined by the further property that

$$\psi^k(L) = L^{\otimes k}$$

for any invertible sheaf L (see [10, Par. 16]).

The symbol θ^k refers to a different operation associating an element of $K_0(Z)$ to any locally free coherent sheaf on a quasi-compact scheme Z. It is uniquely determined by the properties:

(i) for any invertible sheaf L on Z we have

$$\theta^k(L) = 1 + L + \cdots + L^{k-1},$$

(ii) for any short exact sequence $0 \to E' \to E \to E'' \to 0$ of locally free coherent sheaves on Z we have

$$\theta^k(E) = \theta^k(E') \otimes \theta^k(E''),$$

(iii) for any morphism of quasi-compact schemes $g : Z' \to Z$ and any locally free coherent sheaf E on Z we have

$$g^*(\theta^k(E)) = \theta^k(g^*(E)).$$

If Z is of finite type over a noetherian ring and carries an ample line bundle, it is known that $\theta^k(E)$ is invertible in $K_0(Z)[\frac{1}{k}]$ for every locally free coherent sheaf E on Z (see [8, Lemma 4.3]). In that case θ^k extends to a unique map $K_0(Z) \to K_0(Z)[\frac{1}{k}]$ satisfying

$$\theta^k(E) = \theta^k(E') \otimes \theta^k(E'')$$

whenever $E = E' + E''$ in $K_0(Z)$.

As usual, Ω_f denotes the sheaf of differentials of X over Y.

This explains all the ingredients of the formula (1).

The formula (1) is classically proven using deformation to the normal cone and considering closed immersions and relative projective spaces separately (see [1]).

Our aim in this text is to provide a new and more direct proof of the formula (1) in the specific situation where k is a prime number p and Y is a scheme of characteristic p.

The search for this proof was motivated by the fact that for any quasi-compact scheme Z of characteristic p, the endomorphism $\psi^p : K_0(Z) \to K_0(Z)$ coincides with the endomorphism $F_Z^* : K_0(Z) \to K_0(Z)$ induced by pullback by the absolute Frobenius endomorphism $F_Z : Z \to Z$. This is a consequence of the splitting principle [10, Par. 5]. We asked ourselves whether in this case $\theta^p(\Omega_f)$ can also be represented by an explicit virtual bundle. If such a representative were available, one might try to give a direct proof of (1) that does not involve factorisation. The proof given in Section 3 shows that this is indeed possible.

In the article [7, sec. 5] by B. Köck, a different line of speculation led to a question (Question 5.2) in the context of a characteristic p interpretation of the Adams-Riemann-Roch formula. Our Proposition 2.6 and Proposition 3.2 show that the answer to this question is positive. See the Appendix for details.

Fix $k \geqslant 2$ and suppose that Y is the spectrum of a finite field. The formula (1) then formally implies the Hirzebruch-Riemann-Roch theorem for X over that field. This is explained for instance in [12, Intro.]. On the other hand, a specialization argument shows that the Hirzebruch-Riemann-Roch theorem for varieties over any field follows from the Hirzebruch-Riemann-Roch theorem for varieties over finite fields. Thus by reduction modulo primes our proof of (1) in positive characteristic leads to a proof of the Hirzebruch-Riemann-Roch formula in general.

The structure of the article is the following. In Section 2, we construct a canonical bundle representative for the element $\theta^p(E)$ for any locally free coherent sheaf E on a quasi-compact scheme of characteristic p. In Section 3, we give the computation proving (1) in the situation where $k = p$ and Y is a scheme of characteristic p.

After this article was completed, the second author discovered an unpublished text by M. Rost, where part of the material presented in this article is also described. Furthermore, M. Rost explains that some of these results were orally communicated to him by P. Deligne. See the proof of Proposition 3.2 below for references and details.

Acknowledgments. The second author thanks Reinhold Hübl for explanations on the relative Frobenius morphism. Both authors thank the referee for his work and for his suggestions.

2 A bundle representative for $\theta^p(E)$

Let p be a prime number and Z a scheme of characteristic p. Let E be a locally free coherent sheaf on Z. For any integer $k \geqslant 0$ let $\mathrm{Sym}^k(E)$ denote the k-th symmetric power of E. Then

$$\mathrm{Sym}(E) := \bigoplus_{k \geqslant 0} \mathrm{Sym}^k(E)$$

is a quasi-coherent graded $\mathcal{O}_Z$-algebra, called the symmetric algebra of E. Let $\mathcal{J}_E$ denote the graded sheaf of ideals of $\mathrm{Sym}(E)$ that is locally generated by the sections e^p of $\mathrm{Sym}^p(E)$ for all sections e of E, and set

$$\tau(E) := \mathrm{Sym}(E)/\mathcal{J}_E.$$

Locally this construction means the following. Consider an open subset $U \subset Z$ such that $E|U$ is free, and choose a basis $e_1, \ldots, e_r$. Then $\mathrm{Sym}(E)|U$ is the polynomial algebra over $\mathcal{O}_Z$ in the variables $e_1, \ldots, e_r$. Since Z has characteristic p, for any open subset $V \subset U$ and any sections $a_1, \ldots, a_r \in \mathcal{O}_Z(V)$ we have

$$\bigl(a_1 e_1 + \ldots + a_r e_r\bigr)^p = a_1^p e_1^p + \ldots + a_r^p e_r^p.$$

It follows that $\mathcal{J}_E|U$ is the sheaf of ideals of $\mathrm{Sym}(E)|U$ that is generated by $e_1^p, \ldots, e_r^p$. Clearly that description is independent of the choice of basis and compatible with localization; hence it can be used as an equivalent definition of $\mathcal{J}_E$ and $\tau(E)$.

The local description also implies that $\tau(E)|U$ is free over $\mathcal{O}_Z|U$ with basis the images of the monomials $e_1^{i_1}\cdots e_r^{i_r}$ for all choices of exponents $0 \leqslant i_j < p$. From this we deduce:

Lemma 2.1. *If E is a locally free coherent sheaf of rank r, then $\tau(E)$ is a locally free coherent sheaf of rank p^r.*

Now we go through the different properties that characterize the operation θ^p.

Lemma 2.2. *For any invertible sheaf L on Z we have*

$$\tau(L) \cong \mathcal{O}_Z \oplus L \oplus \cdots \oplus L^{\otimes(p-1)}.$$

Proof. In this case the local description shows that $\mathcal{J}_L$ is the sheaf of ideals of $\mathrm{Sym}(L)$ that is generated by $\mathrm{Sym}^p(L) = L^{\otimes p}$. The lemma follows at once. $\square$

Lemma 2.3. *For any morphism of schemes $g : Z' \to Z$ and any locally free coherent sheaf E on Z we have*

$$g^*(\tau(E)) \cong \tau(g^*(E)).$$

Proof. Direct consequence of the construction. $\square$

Lemma 2.4. *For any two locally free coherent sheaves E' and E'' on Z we have*

$$\tau(E' \oplus E'') \cong \tau(E') \otimes \tau(E'').$$

Proof. The homomorphism of sheaves

$$E' \oplus E'' \hookrightarrow \mathrm{Sym}(E') \otimes \mathrm{Sym}(E''),\ (e', e'') \mapsto e' \otimes 1 + 1 \otimes e''$$

induces an algebra isomorphism

$$\mathrm{Sym}(E' \oplus E'') \to \mathrm{Sym}(E') \otimes \mathrm{Sym}(E'').$$

The local description as polynomial rings in terms of bases of $E'|U$ and $E''|U$ shows that this is an isomorphism of sheaves of $\mathcal{O}_Z$-algebras. Since

$$(e' \otimes 1 + 1 \otimes e'')^p = e'^p \otimes 1 + 1 \otimes e''^p$$

for any local sections e' of E' and e'' of E'', this isomorphism induces an isomorphism of sheaves of ideals

$$\mathcal{J}_{E'\oplus E''} \to \mathcal{J}_{E'} \otimes \mathrm{Sym}(E'') \oplus \mathrm{Sym}(E') \otimes \mathcal{J}_{E''}.$$

The lemma follows from this by taking quotients. $\square$

Lemma 2.5. *For any short exact sequence* $0 \to E' \to E \to E'' \to 0$ *of locally free coherent sheaves on a quasi-compact scheme* Z *we have*

$$\tau(E) = \tau(E') \otimes \tau(E'')$$

in $K_0(Z)$.

Proof. Let $\widetilde{E}'$ and $\widetilde{E}''$ denote the inverse images of E' and E'' under the projection morphism $Z \times \mathbf{P}^1 \to Z$. Then there exists a short exact sequence

$$0 \to \widetilde{E}' \to \widetilde{E} \to \widetilde{E}'' \to 0$$

of locally free coherent sheaves on $Z \times \mathbf{P}^1$ whose restriction to the fiber above $0 \in \mathbf{P}^1$ is the given one and whose restriction to the fiber above $\infty \in \mathbf{P}^1$ is split (the construction is given in [2, I, Par. f)]). Thus the respective restrictions satisfy $\widetilde{E}_0 \cong E$ and $\widetilde{E}_\infty \cong E' \oplus E''$. Using Lemmata 2.3 and 2.4 this implies that

$$\tau(E) \cong \tau(\widetilde{E}_0) \cong \tau(\widetilde{E})_0$$

and

$$\tau(E') \otimes \tau(E'') \cong \tau(E' \oplus E'') \cong \tau(\widetilde{E}_\infty) \cong \tau(\widetilde{E})_\infty.$$

But the fact that $K_0(Z \times \mathbf{P}^1)$ is generated by the powers of $\mathcal{O}(1)$ over $K_0(Z)$ (see [13, chap. 8, Th. 2.1, p. 134]) implies that the restriction to 0 and ∞ induce the same map $K_0(Z \times \mathbf{P}^1) \to K_0(Z)$. Thus it follows that $\tau(\widetilde{E})_0 = \tau(\widetilde{E})_\infty$ in $K_0(Z)$, whence the lemma. $\square$

Remark. Lemma 2.5 can also be proved by an explicit calculation of sheaves. For a sketch consider the decreasing filtration of $\mathrm{Sym}(E)$ by the graded ideals $\mathrm{Sym}^i(E') \cdot \mathrm{Sym}(E)$ for all $i \geqslant 0$. One first shows that the associated bi-graded algebra is isomorphic to $\mathrm{Sym}(E') \otimes \mathrm{Sym}(E'')$. The filtration of $\mathrm{Sym}(E)$ also induces a filtration of $\tau(E)$ by graded ideals, whose associated bi-graded algebra is therefore a quotient to $\mathrm{Sym}(E') \otimes \mathrm{Sym}(E'')$. To prove that this quotient is isomorphic to $\tau(E')\otimes\tau(E'')$ one shows that the kernel of the quotient morphism $\mathrm{Sym}(E')\otimes\mathrm{Sym}(E'')\twoheadrightarrow\mathrm{Gr}(\tau(E))$ is precisely $\mathcal{J}_{E'}\otimes\mathrm{Sym}(E'')\oplus\mathrm{Sym}(E')\otimes\mathcal{J}_{E''}$. But this is a purely local assertion, for which one can assume that the exact sequence splits. The calculation then becomes straightforward, as in Lemma 2.4. Note that this argument does not require Z to be quasi-compact.

Proposition 2.6. *For any locally free coherent sheaf E on a quasi-compact scheme Z we have $\tau(E)=\theta^p(E)$ in $K_0(Z)$.*

Proof. Combination of Lemmata 2.2, 2.3, 2.5 and the defining properties (i), (ii), (iii) of $\theta^p(\cdot)$ in Section 1. $\square$

3 Proof of the Adams-Riemann-Roch formula

Let us now consider the morphism $f : X \to Y$ of the introduction. Let p be a prime number and make the supplementary hypothesis that Y is a scheme of characteristic $p > 0$. Let r be the the rank of Ω_f. This is a locally constant function on Y.

Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[rr, bend left, "F_X"] \arrow[dr, "f"'] & X' \arrow[r, "J"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& Y \arrow[r, "F_Y"] & Y
\end{tikzcd}$$

where F_X and F_Y are the respective absolute Frobenius morphisms and the square is cartesian. The morphism $F = F_{X/X'}$ is called the relative Frobenius morphism of X over Y. The following lemma summarizes the properties of F that we shall need.

Lemma 3.1. *The morphism F is finite and flat of constant degree p^r.*

For lack of a better reference, see [3, 1.1, p. 249].

Let I denote the kernel of the natural morphism of $\mathcal{O}_X$-algebras $F^*F_*\mathcal{O}_X\to\mathcal{O}_X$, which by construction is a sheaf of ideals of $F^*F_*\mathcal{O}_X$. Let

$$\mathrm{Gr}(F^*F_*\mathcal{O}_X) := \bigoplus_{k\geqslant 0} I^k/I^{k+1}$$

denote the associated graded sheaf of $\mathcal{O}_X$-algebras.

Proposition 3.2. [1] *There is a natural isomorphism of $\mathcal{O}_X$-modules*

$$I/I^2 \cong \Omega_f$$

and a natural isomorphism of graded $\mathcal{O}_X$-algebras

$$\tau(I/I^2) \cong \mathrm{Gr}(F^*F_*\mathcal{O}_X).$$

Proof. Since F is affine (see Lemma 3.1), there is a canonical isomorphism

$$\mathrm{Spec}\ F^*F_*\mathcal{O}_X \cong X \times_{X'} X,$$

for which the natural morphism of $\mathcal{O}_X$-algebras $F^*F_*\mathcal{O}_X \to \mathcal{O}_X$ corresponds to the diagonal embedding $X \hookrightarrow X \times_{X'} X$. We carry out these identifications throughout the remainder of this proof. Then I is the sheaf of ideals of the diagonal, and so I/I^2 is naturally isomorphic to the relative sheaf of differentials Ω_F. On the other hand we have $F^*\Omega_{f'} = F^*J^*\Omega_f = F_X^*\Omega_f$, which yields a natural exact sequence

$$F_X^*\Omega_f \to \Omega_f \to \Omega_F \to 0.$$

Here the leftmost arrow sends any differential dx to $d(x^p) = p \cdot x^{p-1} \cdot dx = 0$. Thus the exact sequence yields an isomorphism $\Omega_f \cong \Omega_F \cong I/I^2$, proving the first assertion.

For the second assertion observe that, by the universal property of the symmetric algebra $\mathrm{Sym}(\cdot)$, the embedding $I/I^2 \hookrightarrow \mathrm{Gr}(F^*F_*\mathcal{O}_X)$ extends to a unique morphism of $\mathcal{O}_X$-algebras

$$\rho : \mathrm{Sym}(I/I^2) \to \mathrm{Gr}(F^*F_*\mathcal{O}_X).$$

We want to compare the kernel of ρ with $\mathcal{J}_{I/I^2}$. For this recall that I, as the sheaf of ideals of the diagonal, is generated by the sections $s \otimes 1 - 1 \otimes s$ for all local sections s of $\mathcal{O}_X$. The p-th power of any such section is

$$(s \otimes 1 - 1 \otimes s)^p = s^p \otimes 1 - 1 \otimes s^p = 0$$

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[1] The special case of Proposition 3.2 where Y is assumed to be a field can be found in an unpublished text by M. Rost (see Lemma 2, p. 5 in the text *Frobenius, K-theory, and characteristic numbers*, available at the web address http://www.mathematik.uni-bielefeld.de/~rost/frobenius.html), who attributes it to P. Deligne. The authors discovered Proposition 3.2 independently.

in $F^*F_*\mathcal{O}_X$, because $s^p = F_X^*s$ is the pullback via F_X of a section of $\mathcal{O}_X$ and hence also the pullback via F of a section of $\mathcal{O}_{X'}$. Thus ρ sends the p-th powers of certain local generators of I/I^2 to zero. But in Section 2 we have seen that $\mathcal{J}_{I/I^2}$ is locally generated by the p-th powers of any local generators of I/I^2. Therefore $\rho(\mathcal{J}_{I/I^2}) = 0$, and so ρ factors through a morphism of $\mathcal{O}_X$-algebras

$$\bar{\rho} : \tau(I/I^2) \to \mathrm{Gr}(F^*F_*\mathcal{O}_X).$$

From the definition of $\mathrm{Gr}(F^*F_*\mathcal{O}_X)$ we see that ρ and hence $\bar{\rho}$ is surjective.

On the other hand the smoothness assumption on f implies that $I/I^2 \cong \Omega_f$ is locally free of rank r. Thus Lemma 2.1 shows that $\tau(I/I^2)$ is locally free of rank p^r.

We shall now prove[2] that $\bar{\rho}$ is also injective.

Let $x \in X$ and let $x' = F(x)$. A local computation shows that $\mathcal{O}_x \simeq X \times_{X'} \mathrm{Spec}\ \mathcal{O}_{x'}$. Thus, in the natural morphisms of rings

$$\mathcal{O}_{F_X(x)} \to \mathcal{O}_{x'} \to \mathcal{O}_x$$

the morphism on the right-hand side is injective and makes $\mathcal{O}_x$ a finite $\mathcal{O}_{x'}$-algebra. Furthermore, the image of $\mathcal{O}_{F_X(x)}$ in $\mathcal{O}_x$ is $\mathcal{O}_x^p$ by construction. This allows us to apply [9, Prop. 6.18, p.107], which implies that $\mathcal{O}_x$ has a p-basis of order r over $\mathcal{O}_{x'}$. By definition, this means that there exist $x_1, \dots x_r \in \mathcal{O}_x$ and $\xi_1, \dots \xi_r \in \mathcal{O}_{x'}$ such that

$$\mathcal{O}_x \simeq \mathcal{O}_{x'}[T_1, \dots T_r]/(T_1^p - \xi_1, \dots, T_r^p - \xi_r)$$

via the $\mathcal{O}_{x'}$-algebra morphism sending T_i on x_i. With this identification, the ideal I is given by the equations

$$(T_i - S_i)_{i\in\{1,\dots,r\}}$$

in the ring

$$\mathcal{O}_x \otimes_{\mathcal{O}_{x'}} \mathcal{O}_x \simeq \mathcal{O}_{x'}[T_1, \dots T_r, S_1, \dots S_r]/(T_1^p - \xi_1, \dots, T_r^p - \xi_r, S_1^p - \xi_1, \dots, S_r^p - \xi_r).$$

If we apply the $\mathcal{O}_{x'}$-algebra automorphism given by the formulae

$$T_i \mapsto T_i + S_i$$

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[2] This argument is a variant of an argument communicated to us by Reinhold Hübl.

and

$$S_i \mapsto S_i$$

to the ring $\mathcal{O}_{x'}[T_1, \dots T_r, S_1, \dots S_r]$, we obtain the following equivalent description: the ideal I is given by the equations $(T_i)_{i \in \{1,\dots,r\}}$ in the ring

$$\begin{aligned} & \mathcal{O}_{x'}[T_1, \dots T_r, S_1, \dots S_r]/(T_1^p + S_1^p - \xi_1, \dots, T_r^p + S_r^p - \xi_r, S_1^p - \xi_1, \dots, S_r^p - \xi_r) \\ = \ & \mathcal{O}_{x'}[T_1, \dots T_r, S_1, \dots S_r]/(T_1^p, \dots, T_r^p, S_1^p - \xi_1, \dots, S_r^p - \xi_r) \end{aligned}$$

Furthermore, the $\mathcal{O}_{x'}$-modules I^l/I^{l+1} ($l \in \mathbb{N}^*$) then have a $\mathcal{O}_{x'}$-basis given by the monomials

$$T_1^{l_1} \cdots T_r^{l_r} \cdot S_1^{s_1} \cdots S_r^{s_r}$$

with $l_1 + \cdots + l_r = l$ and $l_i, s_i < p$. This shows that $\mathrm{Gr}(F^* F_* \mathcal{O}_X)_x$ is locally free as an $\mathcal{O}_{x'}$-module. Its rank as an $\mathcal{O}_{x'}$-module must coincide with the rank of $(F^* F_* \mathcal{O}_X)_x$ as an $\mathcal{O}_{x'}$-module, which is p^{2r} by construction. Furthermore, $\tau(I/I^2)_x$ is also of rank p^{2r} over $\mathcal{O}_{x'}$. We deduce that $\bar{\rho}$ is injective at x and hence an isomorphism at x. Since x was arbitrary, we can conclude. $\square$

Remark. The assumption that f is projective was not used in the proof of Proposition 3.2. In particular, its conclusion is valid without this assumption.

Lemma 3.3. *Let Z be scheme, which has an ample sheaf and is of finite type over a noetherian ring. Let E be a locally free coherent sheaf of rank r on Z. Then the class of E is invertible in the ring $K_0(Z)[\frac{1}{r}]$.*

Proof. The infinite sum in $K_0(Z)[\frac{1}{r}]$

$$1/r + (r - E)/r^2 + (r - E)^{\otimes 2}/r^3 + \dots$$

only has a finite number of non-vanishing terms. This can be proved directly if Z is a Grassmann scheme and the general case is a consequence of this. A direct calculation with geometric series shows that this sum is an inverse of E in $K_0(Z)[\frac{1}{r}]$. $\square$

Remark. In [7, Question 5.2], B. Köck in particular asks the following question: is the equation

$$F_*(\theta^p(\Omega_f)^{-1}) = 1$$

valid in $K_0(Y)[\frac{1}{p}]$? Proposition 3.2 implies that the answer to this question is positive. Indeed, using the projection formula in K_0-theory, we compute

$$F_*(\theta^p(\Omega_g)^{-1}) = F_*((F^* F_* \mathcal{O}_Z)^{-1}) = F_*(F^*(F_* \mathcal{O}_Z)^{-1}) = (F_* \mathcal{O}_Z) \otimes (F_* \mathcal{O}_Z)^{-1} = 1.$$

This computation is partially repeated below.

We now come to the proof of the Adams-Riemann-Roch formula, which results from the following calculation in $K_0(X)[\frac{1}{p}]$. This calculation is in essence already in [7, Prop. 5.5]. It did not lead to a proof of the formula (1) there, because Proposition 3.2 was missing.

$$\begin{aligned}
\psi^p(R^\bullet f_*(E)) &= F_Y^* R^\bullet f_*(E) \\
&= R^\bullet f'_*(J^*(E)) \\
&= R^\bullet f'_*\big((F_*\mathcal{O}_X) \otimes (F_*\mathcal{O}_X)^{-1} \otimes J^*(E)\big) \\
&= R^\bullet f'_* F_*\big(F^*(F_*\mathcal{O}_X)^{-1} \otimes F^*J^*(E)\big) \\
&= R^\bullet f_*\big((F^*F_*\mathcal{O}_X)^{-1} \otimes F_X^*(E)\big) \\
&= R^\bullet f_*\big(\theta^p(\Omega_f)^{-1} \otimes \psi^p(E)\big).
\end{aligned}$$

Here the first equality uses the fact that $\psi^p = F_Y^*$ in $K_0(Y)$. The second equality follows from the fact that the formation of the relative Euler characteristic commutes with any base change (see [6, III, 7.7.5, 7.7.10, 7.8.4] or [11, chap. 0, par. 5, p. 19]). The third equality is the definition of $(F_*\mathcal{O}_X)^{-1}$ in $K_0(X')[\frac{1}{p}]$ using Lemmata 3.1 and 3.3. The fourth equality is justified by the projection formula in K_0-theory (see [10, Prop. 7.13]). The fifth equality is just a simplification. Finally, Proposition 3.2 and Proposition 2.6 imply that

$$F^*F_*\mathcal{O}_X = \mathrm{Gr}(F^*F_*\mathcal{O}_X) = \tau(I/I^2) = \theta^p(I/I^2) = \theta^p(\Omega_f) = \theta^p(L_f)$$

as elements of $K_0(X)$. This and the fact that $\psi^p = F_X^*$ in $K_0(X)$ prove the last equality, and we are done.

Appendix : Another formula for the Bott element

by Bernhard Köck[3]

The object of this appendix is to give another formula for the Bott element of a smooth morphism. This formula is analogous to the final displayed formula in the main part of this paper and extends a list of miraculous analogies explained in Section 5 of [7]. It is probably needless to say that this appendix is inspired by the

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elegant approach to the Adams-Riemann-Roch theorem in positive characteristic developed by Richard Pink and Damian Rössler in the main part of this paper.

We begin by setting up the context. Let l be a prime and let $f : X \to Y$ be a smooth quasi-projective morphism between Noetherian schemes of relative dimension d. We furthermore assume that there exists an ample invertible $\mathcal{O}_X$-module. Let Ω_f denote the locally free sheaf of relative differentials and let $\theta^l(\Omega_f) \in K_0(X)$ denote the l-th Bott element associated with Ω_f (see Introduction). Furthermore let $\Delta : X \to X^l$ denote the diagonal morphism from X into the l-fold cartesian product $X^l := X \times_Y \ldots \times_Y X$. We view Δ as a C_l-equivariant morphism where the cyclic group C_l of order l acts trivially on X and by permuting the factors on X^l. In particular we have a pull-back homomorphism $\Delta^* : K_0(C_l, X^l) \to K_0(C_l, X)$ between the corresponding Grothendieck groups of equivariant locally free sheaves on X^l and X, respectively. As the closed immersion Δ is also regular we furthermore have a push-forward homomorphism $\Delta_* : K_0(C_l, X) \to K_0(C_l, X^l)$ (see Section 3 in [8]). Let finally $([\mathcal{O}_X[C_l]])$ denote the principal ideal of $K_0(C_l, X)$ generated by the regular representation $[\mathcal{O}_X[C_l]]$. We have a natural map $K_0(X) \to K_0(C_l, X) \to K_0(C_l, X)/([\mathcal{O}_X[C_l]])$ which is in fact injective under certain rather general assumption (see Corollary 4.4 in [7]). The following theorem should be viewed as an analogue of the formula $\theta^p(\Omega_f) = F^*F_*(\mathcal{O}_X)$ proved at the very end of the main part of this paper.

Theorem. We have

$$\theta^l(\Omega_f) = \Delta^*(\Delta_*(\mathcal{O}_X))) \quad \text{in } K_0(C_l, X)/([\mathcal{O}_X[C_l]]).$$

Proof. Let $\mathcal{I}_\Delta$ denote the ideal sheaf corresponding to the regular closed immersion $\Delta : X \to X^l$. Then we have

$$\Delta^*(\Delta_*(\mathcal{O}_X)) = \lambda_{-1}(\mathcal{I}_\Delta/\mathcal{I}_\Delta^2) \quad \text{in } K_0(C_l, X)$$

by the equivariant self-intersection formula (see Corollary (3.9) in [8]); here $\lambda_{-1}(\mathcal{E})$ denotes the alternating sum $[\mathcal{O}_X] - [\mathcal{E}] + [\Lambda^2(\mathcal{E})] \pm \ldots$ for any locally free C_l-sheaf $\mathcal{E}$ on X. Furthermore we know that $\mathcal{I}_\Delta/\mathcal{I}_\Delta^2$ is C_l-isomorphic to $\Omega_f \otimes \mathcal{H}_{X,l}$ where $\mathcal{H}_{X,l} := \ker(\mathcal{O}_X[C_l] \xrightarrow{\text{sum}} \mathcal{O}_X)$ denotes the augmentation representation (see Lemma 3.5 in [7]). Finally we have $\lambda_{-1}(\mathcal{E} \otimes \mathcal{H}_{X,l}) = \theta^l(\mathcal{E})$ in $K_0(C_l, X)/([\mathcal{O}_X[C_l]])$ for any locally free C_l-module $\mathcal{E}$ on X (see Proposition 3.2 and Remark 3.9 in [7]). Putting these three facts together we obtain the desired equality of classes in $K_0(C_l, X)/([\mathcal{O}_X[C_l]])$.

Remark. The statements used in the above proof can also be found in Nori's paper [12].

The following table summarizes the astounding analogies mentioned at the beginning of this appendix. While the left hand column refers to the situation of the main part of this paper, the right hand column refers to the situation of this appendix and of Section 4 in [7]. The entries in the table are of a very symbolic nature; more detailed explanations can be found in Section 5 of [7]. For instance, $\tau^l : K_0(X) \to K_0(C_l, X)$ and $\tau^l_{\text{ext}} : K_0(X) \to K_0(C_l, X^l)$ denote the l-th tensor-power operation and l-th external-tensor-power operation, respectively.

$\psi^p = F_X^*$	$\psi^l = \tau^l$
relative Frobenius $F : X \to X'$	diagonal $\Delta : X \to X^l$
f is smooth $\Rightarrow F$ is flat $\Rightarrow$ We have $F_* : K_0(X) \to K_0(X')$	f is smooth $\Rightarrow \Delta$ is regular $\Rightarrow$ We have $\Delta_* : K_0(C_l, X) \to K_0(C_l, X^l)$
$f' : X' \to Y$	$f^l : X^l \to Y$
$J^* : K_0(X) \to K_0(X')$	$\tau^l_{\text{ext}} : K_0(X) \to K_0(C_l, X^l)$
Base change: $F_Y^* f_* = (f')_* J^*$	Künneth formula: $\tau^l f_* = f^l_* \tau^l_{\text{ext}}$
$F_X^* = F^* J^*$	$\tau^l = \Delta^* \tau^l_{\text{ext}}$
$\theta^p(\Omega_f) = F^*(F_*(\mathcal{O}_X))$	$\theta^l(\Omega_f) = \Delta^*(\Delta_*(\mathcal{O}_X))$
$F_*(\theta^p(\Omega_f)^{-1}) = 1$	$\Delta_* \left(\theta^p(\Omega_f)^{-1}\right) = 1$

The statements displayed in each of the two columns imply the Adams-Riemann-Roch theorem, see Section 3 of this paper and Section 4 of [7]. These two implications are entirely analogous to each other (see also [7, Proposition 5.5]) and they are purely formal, i.e. no further ingredients are needed.

All these analogies suggest that there should be a common reason or a general framework both of the two situations are special cases of. This hope is however tarnished by a certain discrepancy we are now going to explain.

While it is fairly easy to prove that $F_*(\mathcal{O}_X)$ is invertible in $K_0(X)[p^{-1}]$ (see Lemmas 3.1 and 3.3), the corresponding statement that $\Delta_*(\mathcal{O}_X)$ is invertible in $K_0(C_l, X^l)[l^{-1}]/(\mathcal{O}_{X^l}[C^l])$ follows in the absolute case (i.e. when $Y = \text{Spec}(k)$, k a perfect field) from rather involved K-theoretical results (see Section 2 of Nori's paper [12]) which unfortunately don't have a counterpart in the situation of the left hand column and which seem not to generalize to the general (relative) case. While the last statement in the left hand column of the above table is an immediate consequence of the penultimate formula and of the fact that $F_*(\mathcal{O}_X)$ is invertible in $K_0(X)[p^{-1}]$ (see Remark after Lemma 3.3), the analogous proof of the last formula in the right hand column (see [7, Theorem 3.1]) is in particular not (yet?) available in general.

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