ON GROUPS ACTING ON CONTRACTIBLE SPACES
WITH STABILIZERS OF PRIME POWER ORDER

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Abstract. Let $\mathfrak{F}$ denote the class of finite groups, and let $\mathfrak{P}$ denote the subclass consisting of groups of prime power order. We study group actions on topological spaces in which either (1) all stabilizers lie in $\mathfrak{P}$ or (2) all stabilizers lie in $\mathfrak{F}$. We compare the classifying spaces for actions with stabilizers in $\mathfrak{F}$ and $\mathfrak{P}$, the Kropholler hierarchies built on $\mathfrak{F}$ and $\mathfrak{P}$, and group cohomology relative to $\mathfrak{F}$ and to $\mathfrak{P}$. In terms of standard notations, we show that $\mathfrak{F} \subseteq \mathfrak{H}_1 \mathfrak{P} \subseteq \mathfrak{H}_1 \mathfrak{F}$, with all inclusions proper; that $\mathfrak{H}_0 \mathfrak{F} = \mathfrak{H}_0 \mathfrak{P}$; that $\mathfrak{F}H^*(G; -) = \mathfrak{P}H^*(G; -)$; and that $E_2G$ is finite-dimensional if and only if $E_3G$ is finite-dimensional and every finite subgroup of $G$ is in $\mathfrak{P}$.

1. Introduction

Let $\mathcal{F}$ denote a family of subgroups of a group $G$, by which we mean a collection of subgroups which is closed under conjugation and inclusion. A $G$-CW-complex $X$ is said to be a model for $E_\mathcal{F}G$, the classifying space for actions of $G$ with stabilizers in $\mathcal{F}$, if the fixed point set $X^H$ is contractible for $H \in \mathcal{F}$ and is empty for $H \notin \mathcal{F}$. The most common families considered are the family consisting of just the trivial group and the family $\mathfrak{F}$ consisting of all finite subgroups of $G$. In these cases $E_\mathcal{F}G$ is often denoted $EG$ and $\overline{EG}$ respectively. Note that $EG$ is the total space of the universal principal $G$-bundle, or equivalently the universal covering space of an Eilenberg-Mac Lane space for $G$. The space $\overline{EG}$ is called the classifying space for proper actions of $G$. Recently there has been much interest in finiteness conditions for classifying spaces for families, especially for $\overline{EG}$. Milnor and Segal’s constructions of $EG$ both generalize easily to construct models for any $E_\mathcal{F}G$, and one can show that any two models for $E_\mathcal{F}G$ are naturally equivariantly homotopy equivalent.

For some purposes the structure of the fixed point sets for subgroups in $\mathcal{F}$ is irrelevant. For example, a group is in Kropholler’s class $\mathfrak{H}_1 \mathcal{F}$

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if there is any finite-dimensional contractible $G$-CW-complex $X$ with all stabilizers in $\mathcal{F}$. The class $\mathbf{h}_1\mathcal{F}$ is the first stage of a hierarchy whose union is Kropholler’s class $\mathbf{h}\mathcal{F}$ of hierarchically decomposable groups [8]. (These definitions were first considered for the class $\mathbf{F}$ of all finite groups, but work for any family $\mathcal{F}$.)

A priori, the class $\mathbf{h}_1\mathcal{F}$ may contain groups $G$ that do not admit a finite-dimensional model for $E_\mathcal{F}G$, and we shall give such examples in the case when $\mathcal{F} = \mathbb{P}$, the class of groups of prime power order. By contrast, in the case when $\mathcal{F} = \mathbf{F}$, no group $G$ is known to lie in $\mathbf{h}_1\mathbf{F}$ without also admitting a finite-dimensional model for $E_\mathbf{G}$.

A construction due to Serre shows that every group $G$ in $\mathbf{h}_1\mathbf{F}$ that is virtually torsion-free has a finite-dimensional $E_\mathbf{G}$ [4], and the authors have given examples of $G$ for which the minimal dimension of a contractible $G$-CW-complex is lower than the minimal dimension of a model for $E_\mathbf{G}$ [12]. These examples $G$ also have the property that they admit a contractible $G$-CW-complex with finitely many orbits of cells, but that they do not admit any model for $E_\mathbf{G}$ with finitely many orbits of cells.

Throughout this paper, $\mathbf{F}$ will denote the family of finite groups, and $\mathbb{P}$ will denote the family of finite groups of prime power order. We compare the classifying space for $G$-actions with stabilizers in $\mathbb{P}$ with the more well-known $E_\mathbb{P}G$, and we compare the Kropholler hierarchies built on $\mathbf{F}$ and $\mathbb{P}$. We show that a finite group $G$ that is not of prime power order cannot admit a finite-dimensional $E_\mathbb{P}G$, but that every finite group is in $\mathbf{h}_1\mathbb{P}$. We also construct a group that is in $\mathbf{h}_1\mathbf{F}$ but not in $\mathbf{h}_1\mathbb{P}$, and we show that $\mathbf{h}_1\mathbb{P} = \mathbf{h}_1\mathbf{F}$.

In the final section we shall contrast this with cohomology relative to the family of all finite subgroups. The relative cohomological dimension can be viewed as a generalisation of the virtual cohomological dimension, since for virtually torsion free groups these are equal, see [15]. By a result of Bouc [2, 10] it follows that groups belonging to $\mathbf{h}_1\mathbf{F}$ have finite relative cohomological dimension, but the converse is not known. In contrast to our results concerning classifying spaces for families, we show that cohomology relative to subgroups in $\mathbf{F}$ is naturally isomorphic to cohomology relative to subgroups in $\mathbb{P}$.

2. Classifying spaces for the family of $\mathbb{P}$-subgroups

**Proposition 2.1.** Let $G$ be a finite group. Then $G$ has a finite dimensional model for $E_\mathbb{P}G$ if and only if $G$ has prime power order.

**Proof.** If $G$ has prime power order, then a single point may be taken as a model for $E_\mathbb{P}G$. Now let $G$ be an arbitrary finite group, let $p$ be a
prime dividing the order of $G$, and assume that there is a $p$-subgroup $P < G$, such that $N_G(P)$ is not a $p$-group. Then the Weyl-group $WP = N_G(P)/P$ contains a subgroup $H$ of order prime to $p$. Assume $G$ has a finite dimensional model for $E_8 G$, $X$ say. Then the augmented cellular chain complex of the $P$-fixed point set, $X^P$, is a finite length resolution of $\mathbb{Z}$ by free $H$-modules. This gives a contradiction, since $\mathbb{Z}$ has infinite projective dimension as an $H$-module for any non-trivial finite group $H$.

Therefore we may suppose that $G$ is not in $\mathfrak{P}$ and for all non-trivial subgroups $P \in \mathfrak{P}$, the normalizer $N_G(P)$ is also in $\mathfrak{P}$. Now let $N$ be a minimal normal subgroup of $G$. This cannot lie in $\mathfrak{P}$ and hence has the same properties as $G$. Thus, by minimality we may assume $N = G$ and $G$ is simple.

Choose two distinct Sylow $p$-subgroups $P$ and $Q$ of $G$, such that their intersection, $I$ say, is of maximal order. Now, the normalisers $N_P(I)$ and $N_Q(I)$ contain $I$ as a proper subgroup. Also, the group $\langle N_P(I), N_Q(I) \rangle$ does not contain $P$ and $Q$ and neither does $N_G(I) \geq \langle N_P(I), N_Q(I) \rangle$, which is a $p$-group by assumption. Hence there exists a Sylow $p$-subgroup $R$ containing $N_G(I)$ and $R \cap P \geq N_P(I)$. Thus $|R \cap P| > |I| = |P \cap Q|$, which contradicts the maximality of $I$. Therefore we may assume that in $G$ all Sylow $p$-subgroups intersect trivially. In such a group we have, for $P$ a Sylow $p$-subgroup:

$$H^*(G, \mathbb{F}_p) \cong H^*(P, \mathbb{F}_p),$$

see for example [4, Theorem III.10.3].

Any non-trivial $p$-group has non-trivial abelianization, and hence $H^1(P, \mathbb{F}_p)$, which is naturally isomorphic to $\text{Hom}(P, \mathbb{F}_p)$, is non-trivial. But this implies that $H^1(G, \mathbb{F}_p) \cong \text{Hom}(G, \mathbb{F}_p)$ is non-trivial, and so $G$ admits a surjective homomorphism to a group of order $p$. Since $G$ is not in $\mathfrak{P}$, it follows that $G$ cannot be simple, which gives the final contradiction. \hfill \Box

**Corollary 2.2.** For a group $G$, the following are equivalent.

(i) $G$ admits a finite-dimensional $E_8 G$;

(ii) Every finite subgroup of $G$ is in $\mathfrak{P}$ and $G$ admits a finite-dimensional $E_8 G$. \hfill \Box

**Remark 2.3.** We conclude the section with a remark on the type of $E_8 G$. It can proved analogously to Lück’s proof for $E_8 G$ [13] that a group $G$ admits a finite type model for $E_8 G$ if and only if $G$ has finitely many conjugacy classes of groups of prime power order and the Weyl-groups $N_G(P)/P$ for all subgroups $P$ of prime power order are finitely presented and of type $\text{FP}_\infty$. Hence any group admitting a finite
type \( \mathbb{E}G \) also admits a finite type \( \mathbb{E}P G \). Recall that a finite extension of a group admitting a finite model for \( \mathbb{E}G \) always has finitely many conjugacy classes of subgroups of prime power order \([4, IX.13.2]\). Hence the groups exhibited in \([12, Example 7.4]\) are groups admitting a finite type \( \mathbb{E}P G \) which do not admit a finite type \( \mathbb{E}G \).

This behaviour is in stark contrast to that of \( \mathbb{E}V C G \), the classifying space with virtually cyclic isotropy. Any group admitting a finite dimensional model for \( \mathbb{E}V C G \) admits a finite dimensional model for \( \mathbb{E}G \), see \([14]\) and the converse also holds for a large class of groups including all polycyclic-by-finite and all hyperbolic groups \([6, 14]\). Furthermore, any group admitting a finite type model for \( \mathbb{E}V C G \) also admits a finite type model for \( \mathbb{E}G \) \([7]\), but it is conjectured \([6]\) that any group admitting a finite model for \( \mathbb{E}V C G \) has to be virtually cyclic. This has been shown for a class of groups containing all hyperbolic groups \([6]\) and for elementary amenable groups \([7]\).

3. The Hierarchies \( h^G \) and \( h^G \)

**Proposition 3.1.** Let \( X \) be a finite dimensional contractible \( G \)-CW-complex such that all stabilizers are finite. If there is a bound on the orders of the stabilizers then there exists a finite dimensional contractible \( G \)-CW-complex \( Y \) and an equivariant map \( f : Y \to X \) such that \( Y^H = \emptyset \) if \( H \) is not a \( p \)-group.

**Proof.** Using the equivariant form of the simplicial approximation theorem, we may assume that \( X \) is a simplicial \( G \)-CW-complex. To simplify notation the phrase ‘\( G \)-space’ shall mean ‘simplicial \( G \)-CW-complex’ and ‘\( G \)-map’ will mean ‘\( G \)-equivariant simplicial map’ throughout the rest of the proof. The space \( Y \) will be a \( G \)-space in this sense and the map \( f : Y \to X \) will be a \( G \)-map in this sense. The \( G \)-space \( Y \) is constructed in two stages. Firstly, for each finite \( H \leq G \) we build a finite-dimensional contractible \( H \)-space \( Y_H \) with the property that all simplex stabilizers in \( Y_H \) lie in \( \mathfrak{P} \).

Suppose for now that each such \( H \)-space \( Y_H \) has been constructed. Using the \( G \)-equivariant form of the construction used in \([9, Section 8]\) the space \( Y \) is constructed as follows. Let \( I \) be an indexing set for the \( G \)-orbits of vertices in \( X \). For each \( i \in I \), let \( v_i \) be a representative of the corresponding orbit, and let \( H_i \) be the stabilizer of \( v_i \). Let \( X^0 \) denote the 0-skeleton of \( X \). Define a \( G \)-space \( Y^0 \) by

\[
Y^0 = \prod_{i \in I} G \times_{H_i} Y_{H_i},
\]
and define a $G$-map $f : Y^0 \to X^0$ by $f(g, y) = g. v_i$ for all $i \in I$, for all $g \in G$ and for all $y \in Y_H$. For each vertex $w$ of $X$, let $Y(w) = f^{-1}(w) \subset Y^0$. Each $Y(w)$ is a contractible subspace of $Y^0$, and the stabilizer of $w$ acts on $Y(w)$.

Now for $\sigma = (w_0, \ldots, w_n)$ an $n$-simplex of $X$, define a space $Y(\sigma)$ as the join

$$Y(\sigma) = Y(w_0) * Y(w_1) * \cdots * Y(w_n).$$

Each vertex of $Y(\sigma)$ is already a vertex of one of the $Y(w_i)$, and so the map $f : Y^0 \to X^0$ defines a unique simplicial map $f : Y(\sigma) \to \sigma$. By construction, whenever $\tau$ is a face of $\sigma$, the space $Y(\tau)$ is identified with a subspace of $Y(\sigma)$. This allows us to define $Y$ and $f : Y \to X$ as the colimit over the simplices $\sigma$ of $X$ of the subspaces $Y(\sigma)$, and to define $f : Y \to X$, which is a $G$-map of $G$-spaces. Since each $Y(\sigma)$ is contractible, it follows that $f$ is a homotopy equivalence, and hence $Y$ is also contractible (see [9, Corollary 8.6]).

It remains to build the $H$-space $Y_H$ for each finite group $H < G$. In the case when $H \in \mathfrak{P}$ we may take a single point to be $Y_H$, and so we may suppose that $H \notin \mathfrak{P}$. Fix such a subgroup $H$, and suppose that we are able to construct a finite-dimensional contractible $H$-space $Z_H$ in which each stabilizer is a proper subgroup of $H$. We may assume by induction that for each $K < H$ we have already constructed the $K$-space $Y_K$. The $H$-space $Y_H$ can now be constructed from $Z_H$ and the spaces $Y_K$ using a process similar to the construction of $Y$ from $X$ and the spaces $Y_H$. It remains to construct the $H$-space $Z_H$.

An explicit construction of an $H$-space $Z_H$ with the required properties is given in [11]. We therefore provide only a sketch of the argument. We may assume that $H$ is not in $\mathfrak{P}$. Let $S$ be the unit sphere in the reduced regular complex representation of $H$, so that $S$ is a topological space with $H$-action such that the stabilizer of every point of $S$ is a proper subgroup of $H$. Since $H$ is not in $\mathfrak{P}$, there are $H$-orbits in $S$ of coprime lengths. Using this property, it can be shown that the sphere $S$ admits an $H$-equivariant self-map $g : S \to S$ of degree zero. The $H$-space $Z_H$ is defined to be the infinite mapping telescope (suitably triangulated) of the map $g$. \hfill \Box

**Corollary 3.2.** If $G$ is in $\mathfrak{H}_1 \mathfrak{S}$ and there is a bound on the orders of the finite subgroups of $G$, then $G$ is in $\mathfrak{H}_1 \mathfrak{P}$. \hfill \Box

**Remark 3.3.** In Proposition 3.1, the bound on the orders of the stabilizers of $X$ is used only to give a bound on the dimensions of the spaces $Y_H$. In Corollary 3.8 we shall show that $\mathfrak{H}_1 \mathfrak{S} \neq \mathfrak{H}_1 \mathfrak{P}$. 


Remark 3.4. The construction in Proposition 3.1 does not preserve cocompactness, because for most finite groups $H$, the space $Y_H$ used in the construction cannot be chosen to be finite. A result similar to Proposition 3.1 but preserving cocompactness can be obtained by replacing $\mathfrak{P}$ by a larger class $\mathfrak{O}$ of groups. Here $\mathfrak{O}$ is defined to be the class of $\mathfrak{P}$-by-cyclic-by-$\mathfrak{P}$-groups. A theorem of Oliver [18] implies that any finite group $H$ that is not in $\mathfrak{O}$ admits a finite contractible $H$-CW-complex $Z'_H$ in which all stabilizers are proper subgroups of $H$. Applying the same argument as in the proof of Proposition 3.1, one can show that given any contractible $G$-CW-complex $X$ with all stabilizers in $\mathfrak{O}$, there is a contractible $G$-CW-complex $Y'$ with all stabilizers in $\mathfrak{O}$ and a proper equivariant map $f' : Y' \to X$. (By proper, we mean that the inverse image of any compact subset of $X$ is compact.)

For $X$ a $G$-CW-complex with stabilizers in $\mathfrak{O}$, and $p$ a prime, let $X_{\text{sing}(p)}$ denote the subcomplex consisting of points whose stabilizer has order divisible by $p$. For $G$ a group and $p$ a prime, let $S_p(G)$ denote the poset of non-trivial finite $p$-subgroups of $G$.

**Proposition 3.5.** Suppose that $X$ is a finite-dimensional contractible $G$-CW-complex with all stabilizers in $\mathfrak{O}$. For each prime $p$, the mod-$p$ homology of $X_{\text{sing}(p)}$ is isomorphic to the mod-$p$ homology of the (realization of the) poset $S_p(G)$.

**Proof.** Fix a prime $p$, and to simplify notation let $S$ denote the realization of the poset $S_p(G)$. For $P$ a non-trivial $p$-subgroup of $G$, let $X^P$ denote the points fixed by $P$, and let $S_{\geq P}$ denote the realization of the subposet of $S_p(G)$ consisting of all $p$-subgroups that contain $P$. By the P. A. Smith theorem [3], each $X^P$ is mod-$p$ acyclic. Each $S_{\geq P}$ is contractible since it is equal to a cone with apex $P$. Let $P$ and $Q$ be $p$-subgroups of $G$, and let $R = \langle P, Q \rangle$, the subgroup of $G$ generated by $P$ and $Q$. If $R$ is a $p$-group then $X^P \cap X^Q = X^R$, and otherwise $X^P \cap X^Q$ is empty. Similarly, $S_{\geq P} \cap S_{\geq Q} = S_{\geq R}$ if $R$ is a $p$-group and $S_{\geq P} \cap S_{\geq Q}$ is empty if $R$ is not a $p$-group.

Since each $X^P$ is mod-$p$ acyclic, the mod-$p$ homology $H_*(X_{\text{sing}(p)})$ is isomorphic to the mod-$p$ homology of the nerve of the covering $X_{\text{sing}(p)} = \bigcup P X^P$. Similarly, the mod-$p$ homology $H_*(S_p(G))$ is isomorphic to the mod-$p$ homology of the nerve of the covering $S_p(G) = \bigcup P S_{\geq P}$. By the remarks in the first paragraph, these two nerves are isomorphic. □

**Proposition 3.6.** Let $k$ be a finite field, and let $G$ be the group of $k$ points of a reductive algebraic group over $k$ of $k$-rank $n$. (For example,
$G = SL_{n+1}(k)$. Any finite-dimensional contractible $G$-CW-complex with stabilizers in $\mathcal{P}$ has dimension at least $n$.

Proof. The hypotheses on $G$ imply that $G$ acts on a spherical building $\Delta$ of dimension $n - 1$ [1, 5, Appendix on algebraic groups]. Any such building is homotopy equivalent to a wedge of $(n - 1)$-spheres. Quillen has shown that $\Delta$ is homotopy equivalent to the realization of $S_p(G)$, where $p$ is the characteristic of the field $k$ [19, Proposition 2.1 and Theorem 3.1]. It follows that $S_p(G)$ is homotopy equivalent to a wedge of $(n - 1)$-spheres, and in particular the mod-$p$ homology group $H_{n-1}(S_p(G))$ is non-zero.

Now suppose that $X$ is a finite-dimensional contractible $G$-CW-complex with stabilizers in $\mathcal{P}$. Using Proposition 3.5, one sees that the mod-$p$ homology group $H_{n-1}(X_{\text{sing}(p)})$ is non-zero. It follows that $X$ must have dimension at least $n$. \hfill \Box

Remark 3.7. In [11], it is shown that in the case when $G = SL_{n+1}(\mathbb{F}_p)$, every contractible $G$-CW-complex without a global fixed point has dimension at least $n$.

Corollary 3.8. There are the following strict containments and equalities between classes of groups:

(i) $\mathfrak{F} \subseteq H_1\mathcal{P}$;

(ii) $H_1\mathcal{P} \subseteq H_1\mathfrak{F}$;

(iii) $H_1\mathfrak{F} = H_1\mathcal{P}$.

Proof. Corollary 3.2 shows that $\mathfrak{F} \subseteq H_1\mathcal{P}$. The free product of two cyclic groups of prime order is in $H_1\mathcal{P}$ and is not finite. The claim that $H_1\mathfrak{F} = H_1\mathcal{P}$ follows from the inequalities $\mathcal{P} \subseteq \mathfrak{F} \subseteq H_1\mathcal{P}$, and the claim $H_1\mathfrak{F} \subseteq H_1\mathcal{P}$ follows from $\mathcal{P} \subseteq \mathfrak{F}$.

It remains to exhibit a group $G$ that is in $H_1\mathfrak{F}$ but not in $H_1\mathcal{P}$. Let $G = SL_\infty(\mathbb{F}_p)$, the direct limit of the groups $G_n = SL_n(\mathbb{F}_p)$, where $G_n$ is included in $G_{n+1}$ as the ‘top corner’. As a countable locally-finite group, $G$ acts with finite stabilizers on a tree. (Explicitly, the vertex set $V$ and edge set $E$ are both equal as $G$-sets to the disjoint union of the sets of cosets $G/G_1 \cup G/G_2 \cup \cdots$, with the edge $gG_i$ joining the vertex $gG_i$ to the vertex $gG_{i+1}$.) It follows that $G \in H_1\mathfrak{F}$. By Proposition 3.6, $G$ cannot be in $H_1\mathcal{P}$. \hfill \Box

Remark 3.9. Let $G$ be a group in $H_1\mathfrak{F}$ that is also of type $FP_\infty$. By a result of Kropholler [8], there is a bound on the orders of finite subgroups of $G$, and Kropholler-Mislin show that $G$ is in $H_1\mathfrak{F}$ [9]. Corollary 3.2 shows that $G$ is in $H_1\mathcal{P}$. 

4. Cohomology relative to a family of subgroups

Let $\Delta$ denote a $G$-set, and let $\mathbb{Z}\Delta$ denote the corresponding $G$-module. For $\delta \in \Delta$, we write $G_\delta$ for the stabilizer of $\delta$. A short exact sequence $A \rightarrow B \rightarrow C$ of $G$-modules is said to be $\Delta$-split if and only if it splits as a sequence of $G_\delta$-modules for each $\delta \in \Delta$. Equivalently, the sequence is $\Delta$-split if and only if the following sequence of $\mathbb{Z}G$-modules splits: $A \otimes \mathbb{Z}\Delta \rightarrow B \otimes \mathbb{Z}\Delta \rightarrow C \otimes \mathbb{Z}\Delta$ [16].

We say a $G$ module is $\Delta$-projective if it is a direct summand of a $G$-module of the form $N \otimes \mathbb{Z}\Delta$, where $N$ is an arbitrary $G$-module. $\Delta$-projectives satisfy analogue properties to ordinary projectives. Furthermore, for each $\delta$, and each $G_\delta$-module $M$, the induced module $\text{Ind}_{G_\delta}^G M$ is $\Delta$-projective. Given two $G$-sets $\Delta_1$ and $\Delta_2$ and a $G$-map $\Delta_1 \rightarrow \Delta_2$ then $\Delta_1$-projectives are $\Delta_2$-projective and $\Delta_2$-split sequences are $\Delta_1$-split. For more detail the reader is referred to [16].

Now suppose that $\mathcal{F}$ is a family of subgroups of $G$ closed under conjugation and taking subgroups. We consider $G$-sets $\Delta$ satisfying the following condition:

\[(*) \quad \Delta^H \neq \emptyset \iff H \in \mathcal{F}.\]

There are $G$-maps between any two $G$-sets satisfying condition $(* \cdot)$, and so we may define an $\mathcal{F}$-projective module to be a $\Delta$-split module for any such $\Delta$. Similarly, an $\mathcal{F}$-split exact sequence of $G$-modules is defined to be a $\Delta$-split sequence. If $\Delta$ satisfies $(* \cdot)$ and $M$ is any $G$-module, the module $M \otimes \mathbb{Z}\Delta$ is $\mathcal{F}$-projective and admits an $\mathcal{F}$-split surjection to $M$. This leads to a construction of homology relative to $\mathcal{F}$. An $\mathcal{F}$-projective resolution of a module $M$ is an $\mathcal{F}$-split exact sequence

$$
\cdots \rightarrow P_{n+1} \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0,
$$

where all $P_i$ are $\mathcal{F}$-projective. Group cohomology relative to $\mathcal{F}$, denoted $\mathcal{H}^*(G; N)$ can now be defined as the cohomology of the cochain complex $\text{Hom}_G(P_*, N)$, where $P_*$ is an $\mathcal{F}$-projective resolution of $\mathbb{Z}$.

We say that a module $M$ is of type $\mathcal{F}\text{FP}_n$ if $M$ admits an $\mathcal{F}$-projective resolution in which $P_i$ is finitely generated for $0 \leq i \leq n$. It has been shown that modules of type $\mathcal{F}\text{FP}_n$ are of type FP$_n$ [16]. We will say that a group $G$ is of type $\mathcal{F}\text{FP}_n$ if the trivial $G$-module $\mathbb{Z}$ is of type $\mathcal{F}\text{FP}_n$.

We now specialize to the cases when $\mathcal{F} = \mathfrak{F}$ and $\mathcal{F} = \mathfrak{P}$.

**Proposition 4.1.** The following properties hold.

(i) A short exact sequence of $G$-modules is $\mathfrak{F}$-split if and only if it is $\mathfrak{P}$-split.

(ii) A $G$ module is $\mathfrak{F}$-projective if and only if it is $\mathfrak{P}$-projective.
(iii) $\mathfrak{F}H^*(G, -) \cong \mathfrak{P}H^*(G, -)$

Proof: (i) It is obvious that any $\mathfrak{F}$-split sequence is $\mathfrak{P}$-split, and the converse follows from a standard averaging argument. Let $H$ be an arbitrary finite subgroup of $G$. Then $|H| = \prod_{i=1}^{n} p_i^{\alpha_i}$ where $p_i$ are distinct primes and $0 < a_i \in \mathbb{Z}$. For each $i$, let $n_i$ be the index $n_i = [H : P_i]$. Now consider a $\mathfrak{P}$-split surjection $A \rightarrow B$. Let $\sigma_i$ be a $P_i$-splitting of $\pi$, and define a map $s_i$ by summing $\sigma_i$ over the cosets of $P_i$:

$$s_i(b) = \sum_{t \in H/P_i} t\sigma_i(t^{-1}b).$$

For each $P_i$ we obtain a map $s_i : B \rightarrow A$, such that $\pi \circ s_i = n_i \times id_B$. There exist $m_i \in \mathbb{Z}$ so that $\sum_i m_i n_i = 1$, and the map $s = \sum_i m_i s_i$ is the required $H$-splitting.

(ii) It is obvious that a $\mathfrak{P}$-projective module is $\mathfrak{F}$-projective. Now let $P$ be $\mathfrak{F}$-projective. We may take a $\mathfrak{P}$-split surjection $M \rightarrow P$ with $M$ a $\mathfrak{P}$-projective. By (i) this surjection is $\mathfrak{F}$-split, and hence split. Thus $P$ is a direct summand of a $\mathfrak{P}$-projective and so is $\mathfrak{P}$-projective.

(iii) now follows directly from (i) and (ii).

$\square$

**Proposition 4.2.** A group $G$ is of type $\mathfrak{F}P_0$ if and only if $G$ has only finitely many conjugacy classes of subgroups of prime power order.

Proof: Suppose that $G$ has only finitely many conjugacy classes of subgroups in $\mathfrak{P}$. Let $I$ be a set of representatives for the conjugacy classes of $\mathfrak{P}$-subgroups and set

$$\Delta_0 = \bigsqcup_{P \in I} G/P.$$

This $G$-set satisfies condition $(\ast)$ for $\mathfrak{P}$ and therefore the surjection $\mathbb{Z}\Delta_0 \rightarrow \mathbb{Z}$ is $\mathfrak{F}$-split and also $\mathbb{Z}\Delta_0$ is finitely generated.

To prove the converse we consider an arbitrary $\mathfrak{F}$-split surjection $P_0 \rightarrow \mathbb{Z}$ with $P_0$ a finitely generated $\mathfrak{F}$-projective. As in [16, 6.1] we can show that $P_0$ is a direct summand of a module $\bigoplus_{\delta \in \Delta_f} \text{Ind}_{G_\delta}^G P_\delta$, where $\Delta_f$ is a finite $G$-set, the $G_\delta$ are finite groups and $P_\delta$ are finitely generated $G_\delta$-modules. Therefore we might assume from now on that $P_0$ is of the above form. Since there is a $G$-map $\Delta_f \rightarrow \Delta$, where $\Delta$ satisfies condition $(\ast)$ the $\mathfrak{F}$-split surjection $P_0 \rightarrow \mathbb{Z}$ is also $\Delta_f$-split [16]. Consider now the following commutative diagram:
That we can find such an \( \alpha \) follows from the fact that \( \varepsilon \) is \( \Delta_f \)-split, and \( \beta \) exists since \( P_0 \) is \( \Delta_f \)-projective being a direct sum of induced modules, induced from \( G_\delta, (\delta \in \Delta_f) \) to \( G \).

As a next step we’ll show that \( \varepsilon_f \) is \( \mathfrak{F} \)-split. Take an arbitrary finite subgroup \( H \) of \( G \) and show that \( \varepsilon_f \) splits when restricted to \( H \). Since \( \varepsilon \) is split by \( s \), say, when restricted to \( H \) we can define the required splitting by \( \beta \circ s \).

Now let \( P \) be an arbitrary \( p \)-subgroup of \( G \). Since the module \( \mathbb{Z}[G/P] \) is \( \mathfrak{F} \)-projective, there exists a \( G \)-map \( \varphi \), such that the following diagram commutes:

\[
\begin{array}{c}
\mathbb{Z}\Delta_f \xrightarrow{\varepsilon_f} \mathbb{Z} \\
\downarrow \varphi \\
\mathbb{Z}[G/P] \xrightarrow{\varepsilon_f} \mathbb{Z}
\end{array}
\]

The image \( \varphi(P) \) of the identity coset \( P \) is a point of \( \mathbb{Z}\Delta \) fixed by the action of \( P \). If \( H \) is any group and \( \mathbb{Z}\Omega \) is any permutation module, then the \( H \)-fixed points are generated by the orbit sums \( H.\omega \). Hence \( P \) must stabilize some point of \( \Delta_f \), since otherwise we would have that \( p \) divides \( \varepsilon_f \varphi(P) = \varepsilon \alpha(P) = 1 \), a contradiction. It follows that \( P \) is a subgroup of \( G_\delta \) for some \( \delta \in \Delta_f \).

\[ \square \]

Note that being of type \( \mathfrak{F}\text{FP}_0 \) does not imply that there are finitely many conjugacy classes of finite subgroups. In fact, the authors have examples with infinitely many conjugacy classes of finite subgroups, see [12]. Nevertheless this gives rise to the following conjecture:

**Conjecture 4.3.** A group \( G \) is of type \( \mathfrak{F}\text{FP}_\infty \) if and only if \( G \) is of type \( \text{FP}_\infty \) and has finitely many conjugacy classes of \( p \)-subgroups.

It is shown in [16] that any \( G \) of type \( \mathfrak{F}\text{FP}_\infty \) is of type \( \text{FP}_\infty \), which together with Proposition 4.2 proves one implication in the above conjecture.

**References**


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