ON GROUPS ACTING ON CONTRACTIBLE SPACES WITH STABILIZERS OF PRIME POWER ORDER

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Abstract. Let $\mathfrak{F}$ denote the class of finite groups, and let $\mathfrak{P}$ denote the subclass consisting of groups of prime power order. We study group actions on topological spaces in which either (1) all stabilizers lie in $\mathfrak{P}$ or (2) all stabilizers lie in $\mathfrak{F}$. We compare the classifying spaces for actions with stabilizers in $\mathfrak{F}$ and $\mathfrak{P}$, the Kropholler hierarchies built on $\mathfrak{F}$ and $\mathfrak{P}$, and group cohomology relative to $\mathfrak{F}$ and to $\mathfrak{P}$. In terms of standard notations, we show that $\mathfrak{F} \subset \mathbf{H}_1\mathfrak{P} \subset \mathbf{H}_1\mathfrak{F}$, with all inclusions proper; that $\mathbf{H}\mathfrak{F} = \mathbf{H}\mathfrak{P}$; that $\mathfrak{F}H^*(G;-) = \mathfrak{P}H^*(G;-)$; and that $\mathrm{E}_{\mathfrak{P}}G$ is finite-dimensional if and only if $\mathrm{E}_{\mathfrak{F}}G$ is finite-dimensional and every finite subgroup of G is in $\mathfrak{P}$.

1. Introduction

Let $\mathcal{F}$ denote a family of subgroups of a group G, by which we mean a collection of subgroups which is closed under conjugation and inclusion. A G-CW-complex X is said to be a model for $\mathrm{E}_{\mathcal{F}}G$, the classifying space for actions of G with stabilizers in $\mathcal{F}$, if the fixed point set X^H is contractible for $H \in \mathcal{F}$ and is empty for $H \notin \mathcal{F}$. The most common families considered are the family consisting of just the trivial group and the family $\mathfrak{F}$ consisting of all finite subgroups of G. In these cases $\mathrm{E}_{\mathcal{F}}G$ is often denoted $\mathrm{E}G$ and $\underline{\mathrm{E}}G$ respectively. Note that $\mathrm{E}G$ is the total space of the universal principal G-bundle, or equivalently the universal covering space of an Eilenberg-Mac Lane space for G. The space $\underline{\mathrm{E}}G$ is called the classifying space for proper actions of G. Recently there has been much interest in finiteness conditions for classifying spaces for families, especially for $\underline{\mathrm{E}}G$. Milnor and Segal's constructions of $\mathrm{E}G$ both generalize easily to construct models for any $\mathrm{E}_{\mathcal{F}}G$, and one can show that any two models for $\mathrm{E}_{\mathcal{F}}G$ are naturally equivariantly homotopy equivalent.

For some purposes the structure of the fixed point sets for subgroups in $\mathcal{F}$ is irrelevant. For example, a group is in Kropholler's class $\mathbf{H}_1\mathcal{F}$

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if there is any finite-dimensional contractible G-CW-complex X with all stabilizers in $\mathcal{F}$. The class $\mathbf{H}_1\mathcal{F}$ is the first stage of a hierarchy whose union is Kropholler's class $\mathbf{H}\mathcal{F}$ of hierarchically decomposable groups [8]. (These definitions were first considered for the class $\mathfrak{F}$ of all finite groups, but work for any family $\mathcal{F}$.)

A priori, the class $\mathbf{H}_1\mathcal{F}$ may contain groups G that do not admit a finite-dimensional model for $\mathrm{E}_{\mathcal{F}}G$, and we shall give such examples in the case when $\mathcal{F} = \mathfrak{P}$, the class of groups of prime power order. By contrast, in the case when $\mathcal{F} = \mathfrak{F}$, no group G is known to lie in $\mathbf{H}_1\mathfrak{F}$ without also admitting a finite-dimensional model for $\underline{\mathrm{E}}G$. A construction due to Serre shows that every group G in $\mathbf{H}_1\mathfrak{F}$ that is virtually torsion-free has a finite-dimensional $\underline{\mathrm{E}}G$ [4], and the authors have given examples of G for which the minimal dimension of a contractible G-CW-complex is lower than the minimal dimension of a model for $\underline{\mathrm{E}}G$ [12]. These examples G also have the property that they admit a contractible G-CW-complex with finitely many orbits of cells, but that they do not admit any model for $\underline{\mathrm{E}}G$ with finitely many orbits of cells.

Throughout this paper, $\mathfrak{F}$ will denote the family of finite groups, and $\mathfrak{P}$ will denote the family of finite groups of prime power order. We compare the classifying space for G-actions with stabilizers in $\mathfrak{P}$ with the more well-known $\underline{\mathrm{E}}G$, and we compare the Kropholler hierarchies built on $\mathfrak{F}$ and $\mathfrak{P}$. We show that a finite group G that is not of prime power order cannot admit a finite-dimensional $\mathrm{E}_{\mathfrak{P}}G$, but that every finite group is in $\mathbf{H}_1\mathfrak{P}$. We also construct a group that is in $\mathbf{H}_1\mathfrak{F}$ but not in $\mathbf{H}_1\mathfrak{P}$, and we show that $\mathbf{H}\mathfrak{P} = \mathbf{H}\mathfrak{F}$.

In the final section we shall contrast this with cohomology relative to the family of all finite subgroups. The relative cohomological dimension can be viewed as a generalisation of the virtual cohomological dimension, since for virtually torsion free groups these are equal, see [15]. By a result of Bouc [2, 10] it follows that groups belonging to $\mathbf{H}_1\mathfrak{F}$ have finite relative cohomological dimension, but the converse it not known. In contrast to our results concerning classifying spaces for families, we show that cohomology relative to subgroups in $\mathfrak{F}$ is naturally isomorphic to cohomology relative to subgroups in $\mathfrak{P}$.

2. Classifying spaces for the family of $\mathfrak{P}$-subgroups

Proposition 2.1. *Let G be a finite group. Then G has a finite dimensional model for $\mathrm{E}_{\mathfrak{P}}G$ if and only if G has prime power order.*

Proof. If G has prime power order, then a single point may be taken as a model for $\mathrm{E}_{\mathfrak{P}}G$. Now let G be an arbitrary finite group, let p be a prime dividing the order of G, and assume that there is a p-subgroup $P < G$, such that $N_G(P)$ is not a p-group. Then the Weyl-group $WP = N_G(P)/P$ contains a subgroup H of order prime to p. Assume G has a finite dimensional model for $\mathrm{E}_{\mathfrak{P}}G$, X say. Then the augmented cellular chain complex of the P-fixed point set, X^P, is a finite length resolution of $\mathbb{Z}$ by free H-modules. This gives a contradiction, since $\mathbb{Z}$ has infinite projective dimension as an H-module for any non-trivial finite group H.

Therefore we may suppose that G is not in $\mathfrak{P}$ and for all non-trivial subgroups $P \in \mathfrak{P}$, the normalizer $N_G(P)$ is also in $\mathfrak{P}$. Now let N be a minimal normal subgroup of G. This cannot lie in $\mathfrak{P}$ and hence has the same properties as G. Thus, by minimality we may assume $N = G$ and G is simple.

Choose two distinct Sylow p-subgroups P and Q of G, such that their intersection, I say, is of maximal order. Now, the normalisers $N_P(I)$ and $N_Q(I)$ contain I as a proper subgroup. Also, the group $\langle N_P(I), N_Q(I)\rangle$ does not contain P and Q and neither does $N_G(I) \geq \langle N_P(I), N_Q(I)\rangle$, which is a p-group by assumption. Hence there exists a Sylow p-subgroup R containing $N_G(I)$ and $R \cap P \geq N_P(I)$. Thus $|R \cap P| > |I| = |P \cap Q|$, which contradicts the maximality of I. Therefore we may assume that in G all Sylow p-subgroups intersect trivially. In such a group we have, for P a Sylow p-subgroup:

$$H^*(G, \mathbb{F}_p) \cong H^*(P, \mathbb{F}_p),$$

see for example [4, Theorem III.10.3].

Any non-trivial p-group has non-trivial abelianization, and hence $H^1(P, \mathbb{F}_p)$, which is naturally isomorphic to $\mathrm{Hom}(P, \mathbb{F}_p)$, is non-trivial. But this implies that $H^1(G, \mathbb{F}_p) \cong \mathrm{Hom}(G, \mathbb{F}_p)$ is non-trivial, and so G admits a surjective homomorphism to a group of order p. Since G is not in $\mathfrak{P}$, it follows that G cannot be simple, which gives the final contradiction. □

Corollary 2.2. *For a group G, the following are equivalent.*

(i) *G admits a finite-dimensional $\mathrm{E}_{\mathfrak{P}}G$;*
(ii) *Every finite subgroup of G is in $\mathfrak{P}$ and G admits a finite-dimensional $\underline{\mathrm{E}}G$.* □

Remark 2.3. We conclude the section with a remark on the type of $\mathrm{E}_{\mathfrak{P}}G$. It can proved analogously to Lück's proof for $\underline{\mathrm{E}}G$ [13] that a group G admits a finite type model for $\mathrm{E}_{\mathfrak{P}}G$ if and only if G has finitely many conjugacy classes of groups of prime power order and the Weyl-groups $N_G(P)/P$ for all subgroups P of prime power order are finitely presented and of type FP_∞. Hence any group admitting a finite type $\underline{\mathrm{E}}G$ also admits a finite type $\mathrm{E}_{\mathfrak{P}}G$. Recall that a finite extension of a group admitting a finite model for $\mathrm{E}G$ always has finitely many conjugacy classes of subgroups of prime power order [4, IX.13.2]. Hence the groups exhibited in [12, Example 7.4] are groups admitting a finite type $\mathrm{E}_{\mathfrak{P}}G$ which do not admit a finite type $\underline{\mathrm{E}}G$.

This behaviour is in stark contrast to that of $\mathrm{E}_{\mathcal{VC}}G$, the classifying space with virtually cyclic isotropy. Any group admitting a finite dimensional model for $\mathrm{E}_{\mathcal{VC}}G$ admits a finite dimensional model for $\underline{\mathrm{E}}G$, see [14] and the converse also holds for a large class of groups including all polycyclic-by-finite and all hyperbolic groups [6, 14]. Furthermore, any group admitting a finite type model for $\mathrm{E}_{\mathcal{VC}}G$ also admits a finite type model for $\underline{\mathrm{E}}G$ [7], but it is conjectured [6] that any group admitting a finite model for $\mathrm{E}_{\mathcal{VC}}G$ has to be virtually cyclic. This has been shown for a class of groups containing all hyperbolic groups [6] and for elementary amenable groups [7].

3. The hierarchies $\mathbf{H}\mathfrak{F}$ and $\mathbf{H}\mathfrak{P}$

Proposition 3.1. *Let X be a finite dimensional contractible G-CW-complex such that all stabilizers are finite. If there is a bound on the orders of the stabilizers then there exists a finite dimensional contractible G-CW-complex Y and an equivariant map $f : Y \to X$ such that $Y^H = \varnothing$ if H is not a p-group.*

Proof. Using the equivariant form of the simplicial approximation theorem, we may assume that X is a simplicial G-CW-complex. To simplify notation the phrase 'G-space' shall mean 'simplicial G-CW-complex' and 'G-map' will mean 'G-equivariant simplicial map' throughout the rest of the proof. The space Y will be a G-space in this sense and the map $f : Y \to X$ will be a G-map in this sense. The G-space Y is constructed in two stages. Firstly, for each finite $H \leq G$ we build a finite-dimensional contractible H-space Y_H with the property that all simplex stabilizers in Y_H lie in $\mathfrak{P}$.

Suppose for now that each such H-space Y_H has been constructed. Using the G-equivariant form of the construction used in [9, Section 8] the space Y is constructed as follows. Let I be an indexing set for the G-orbits of vertices in X. For each $i \in I$, let v_i be a representative of the corresponding orbit, and let H_i be the stabilizer of v_i. Let X^0 denote the 0-skeleton of X. Define a G-space Y^0 by

$$Y^0 = \coprod_{i\in I} G \times_{H_i} Y_{H_i},$$

and define a G-map $f : Y^0 \to X^0$ by $f(g, y) = g.v_i$ for all $i \in I$, for all $g \in G$ and for all $y \in Y_{H_i}$. For each vertex w of X, let $Y(w) = f^{-1}(w) \subset Y^0$. Each $Y(w)$ is a contractible subspace of Y^0, and the stabilizer of w acts on $Y(w)$.

Now for $\sigma = (w_0, \ldots, w_n)$ an n-simplex of X, define a space $Y(\sigma)$ as the join

$$Y(\sigma) = Y(w_0) * Y(w_1) * \cdots * Y(w_n).$$

Each vertex of $Y(\sigma)$ is already a vertex of one of the $Y(w_i)$, and so the map $f : Y^0 \to X^0$ defines a unique simplicial map $f : Y(\sigma) \to \sigma$. By construction, whenever τ is a face of σ, the space $Y(\tau)$ is identified with a subspace of $Y(\sigma)$. This allows us to define Y and $f : Y \to X$ as the colimit over the simplices σ of X of the subspaces $Y(\sigma)$, and to define $f : Y \to X$, which is a G-map of G-spaces. Since each $Y(\sigma)$ is contractible, it follows that f is a homotopy equivalence, and hence Y is also contractible (see [9, Corollary 8.6]).

It remains to build the H-space Y_H for each finite group $H < G$. In the case when $H \in \mathfrak{P}$ we may take a single point to be Y_H, and so we may suppose that $H \notin \mathfrak{P}$. Fix such a subgroup H, and suppose that we are able to construct a finite-dimensional contractible H-space Z_H in which each stabilizer is a proper subgroup of H. We may assume by induction that for each $K < H$ we have already constructed the K-space Y_K. The H-space Y_H can now be constructed from Z_H and the spaces Y_K using a process similar to the construction of Y from X and the spaces Y_H. It remains to construct the H-space Z_H.

An explicit construction of an H-space Z_H with the required properties is given in [11]. We therefore provide only a sketch of the argument. We may assume that H is not in $\mathfrak{P}$. Let S be the unit sphere in the reduced regular complex representation of H, so that S is a topological space with H-action such that the stabilizer of every point of S is a proper subgroup of H. Since H is not in $\mathfrak{P}$, there are H-orbits in S of coprime lengths. Using this property, it can be shown that the sphere S admits an H-equivariant self-map $g : S \to S$ of degree zero. The H-space Z_H is defined to be the infinite mapping telescope (suitably triangulated) of the map g. □

Corollary 3.2. *If G is in $\mathbf{H}_1\mathfrak{F}$ and there is a bound on the orders of the finite subgroups of G, then G is in $\mathbf{H}_1\mathfrak{P}$.* □

Remark 3.3. In Proposition 3.1, the bound on the orders of the stabilizers of X is used only to give a bound on the dimensions of the spaces Y_H. In Corollary 3.8 we shall show that $\mathbf{H}_1\mathfrak{F} \neq \mathbf{H}_1\mathfrak{P}$.

Remark 3.4. The construction in Proposition 3.1 does not preserve cocompactness, because for most finite groups H, the space Y_H used in the construction cannot be chosen to be finite. A result similar to Proposition 3.1 but preserving cocompactness can be obtained by replacing $\mathfrak{P}$ by a larger class $\mathfrak{O}$ of groups. Here $\mathfrak{O}$ is defined to be the class of $\mathfrak{P}$-by-cyclic-by-$\mathfrak{P}$-groups. A theorem of Oliver [18] implies that any finite group H that is not in $\mathfrak{O}$ admits a *finite* contractible H-CW-complex Z'_H in which all stabilizers are proper subgroups of H. Applying the same argument as in the proof of Proposition 3.1, one can show that given any contractible G-CW-complex X with all stabilizers in $\mathfrak{F}$, there is a contractible G-CW-complex Y' with all stabilizers in $\mathfrak{O}$ and a proper equivariant map $f' : Y' \to X$. (By proper, we mean that the inverse image of any compact subset of X is compact.)

For X a G-CW-complex with stabilizers in $\mathfrak{F}$, and p a prime, let $X_{\mathrm{sing}(p)}$ denote the subcomplex consisting of points whose stabilizer has order divisible by p. For G a group and p a prime, let $S_p(G)$ denote the poset of non-trivial finite p-subgroups of G.

Proposition 3.5. *Suppose that X is a finite-dimensional contractible G-CW-complex with all stabilizers in $\mathfrak{P}$. For each prime p, the mod-p homology of $X_{\mathrm{sing}(p)}$ is isomorphic to the mod-p homology of the (realization of the) poset $S_p(G)$.*

Proof. Fix a prime p, and to simplify notation let S denote the realization of the poset $S_p(G)$. For P a non-trivial p-subgroup of G, let X^P denote the points fixed by P, and let $S_{\geq P}$ denote the realization of the subposet of $S_p(G)$ consisting of all p-subgroups that contain P. By the P. A. Smith theorem [3], each X^P is mod-p acyclic. Each $S_{\geq P}$ is contractible since it is equal to a cone with apex P. Let P and Q be p-subgroups of G, and let $R = \langle P, Q\rangle$, the subgroup of G generated by P and Q. If R is a p-group then $X^P \cap X^Q = X^R$, and otherwise $X^P \cap X^Q$ is empty. Similarly, $S_{\geq P} \cap S_{\geq Q} = S_{\geq R}$ if R is a p-group and $S_{\geq P} \cap S_{\geq Q}$ is empty if R is not a p-group.

Since each X^P is mod-p acyclic, the mod-p homology $H_*(X_{\mathrm{sing}(p)})$ is isomorphic to the mod-p homology of the nerve of the covering $X_{\mathrm{sing}(p)} = \bigcup_P X^P$. Similarly, the mod-$p$ homology $H_*(S_p(G))$ is isomorphic to the mod-p homology of the nerve of the covering $S_p(G) = \bigcup_P S_{\geq P}$. By the remarks in the first paragraph, these two nerves are isomorphic. □

Proposition 3.6. *Let k be a finite field, and let G be the group of k points of a reductive algebraic group over k of k-rank n. (For example,*

$G = SL_{n+1}(k)$.) Any finite-dimensional contractible G-CW-complex with stabilizers in $\mathfrak{P}$ has dimension at least n.

Proof. The hypotheses on G imply that G acts on a spherical building Δ of dimension $n-1$ [1, 5, Appendix on algebraic groups]. Any such building is homotopy equivalent to a wedge of $(n-1)$-spheres. Quillen has shown that Δ is homotopy equivalent to the realization of $S_p(G)$, where p is the characteristic of the field k [19, Proposition 2.1 and Theorem 3.1]. It follows that $S_p(G)$ is homotopy equivalent to a wedge of $(n-1)$-spheres, and in particular the mod-p homology group $H_{n-1}(S_p(G))$ is non-zero.

Now suppose that X is a finite-dimensional contractible G-CW-complex with stabilizers in $\mathfrak{P}$. Using Proposition 3.5, one sees that the mod-p homology group $H_{n-1}(X_{\mathrm{sing}(p)})$ is non-zero. It follows that X must have dimension at least n. □

Remark 3.7. In [11], it is shown that in the case when $G = SL_{n+1}(\mathbb{F}_p)$, every contractible G-CW-complex without a global fixed point has dimension at least n.

Corollary 3.8. *There are the following strict containments and equalities between classes of groups:*

(i) $\mathfrak{F} \subsetneq \mathbf{H}_1\mathfrak{P}$;
(ii) $\mathbf{H}_1\mathfrak{P} \subsetneq \mathbf{H}_1\mathfrak{F}$;
(iii) $\mathbf{H}\mathfrak{F} = \mathbf{H}\mathfrak{P}$.

Proof. Corollary 3.2 shows that $\mathfrak{F} \subseteq \mathbf{H}_1\mathfrak{P}$. The free product of two cyclic groups of prime order is in $\mathbf{H}_1\mathfrak{P}$ and is not finite. The claim that $\mathbf{H}\mathfrak{F} = \mathbf{H}\mathfrak{P}$ follows from the inequalities $\mathfrak{P} \subseteq \mathfrak{F} \subseteq \mathbf{H}_1\mathfrak{P}$, and the claim $\mathbf{H}_1\mathfrak{P} \subseteq \mathbf{H}_1\mathfrak{F}$ follows from $\mathfrak{P} \subseteq \mathfrak{F}$.

It remains to exhibit a group G that is in $\mathbf{H}_1\mathfrak{F}$ but not in $\mathbf{H}_1\mathfrak{P}$. Let $G = SL_\infty(\mathbb{F}_p)$, the direct limit of the groups $G_n = SL_n(\mathbb{F}_p)$, where G_n is included in G_{n+1} as the 'top corner'. As a countable locally-finite group, G acts with finite stabilizers on a tree. (Explicitly, the vertex set V and edge set E are both equal as G-sets to the disjoint union of the sets of cosets $G/G_1 \cup G/G_2 \cup \cdots$, with the edge gG_i joining the vertex gG_i to the vertex gG_{i+1}.) It follows that $G \in \mathbf{H}_1\mathfrak{F}$. By Proposition 3.6, G cannot be in $\mathbf{H}_1\mathfrak{P}$. □

Remark 3.9. Let G be a group in $\mathbf{H}\mathfrak{F}$ that is also of type FP_∞. By a result of Kropholler [8], there is a bound on the orders of finite subgroups of G, and Kropholler-Mislin show that G is in $\mathbf{H}_1\mathfrak{F}$ [9]. Corollary 3.2 shows that G is in $\mathbf{H}_1\mathfrak{P}$.

4. Cohomology relative to a family of subgroups

Let Δ denote a G-set, and let $\mathbb{Z}\Delta$ denote the corresponding G-module. For $\delta \in \Delta$, we write G_δ for the stabilizer of δ. A short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ of G-modules is said to be Δ-*split* if and only if it splits as a sequence of G_δ-modules for each $\delta \in \Delta$. Equivalently, the sequence is Δ-split if and only if the following sequence of $\mathbb{Z}G$-modules splits: $A \otimes \mathbb{Z}\Delta \rightarrowtail B \otimes \mathbb{Z}\Delta \twoheadrightarrow C \otimes \mathbb{Z}\Delta$ [16].

We say a G module is Δ-projective if it is a direct summand of a G-module of the form $N \otimes \mathbb{Z}\Delta$, where N is an arbitrary G-module. Δ-projectives satisfy analogue properties to ordinary projectives. Furthermore, for each δ, and each G_δ-module M, the induced module $\mathrm{Ind}_{G_\delta}^G M$ is Δ-projective. Given two G-sets Δ_1 and Δ_2 and a G-map $\Delta_1 \to \Delta_2$ then Δ_1-projectives are Δ_2-projective and Δ_2-split sequences are Δ_1-split. For more detail the reader is referred to [16].

Now suppose that $\mathcal{F}$ is a family of subgroups of G closed under conjugation and taking subgroups. We consider G-sets Δ satisfying the following condition:

$$(*) \qquad \Delta^H \neq \varnothing \iff H \in \mathcal{F}.$$

There are G-maps between any two G-sets satisfying condition $(*)$, and so we may define an $\mathcal{F}$-projective module to be a Δ-split module for any such Δ. Similarly, an $\mathcal{F}$-split exact sequence of G-modules is defined to be a Δ-split sequence. If Δ satisfies $(*)$ and M is any G-module, the module $M \otimes \mathbb{Z}\Delta$ is $\mathcal{F}$-projective and admits an $\mathcal{F}$-split surjection to M. This leads to a construction of homology relative to $\mathcal{F}$. An $\mathcal{F}$-projective resolution of a module M is an $\mathcal{F}$-split exact sequence

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_0 \to M \to 0,$$

where all P_i are $\mathcal{F}$-projective. Group cohomology relative to $\mathcal{F}$, denoted $\mathcal{F}H^*(G;N)$ can now be defined as the cohomology of the cochain complex $\mathrm{Hom}_G(P_*, N)$, where P_* is an $\mathcal{F}$-projective resolution of $\mathbb{Z}$.

We say that a module M is of type $\mathcal{F}\mathrm{FP}_n$ if M admits an $\mathcal{F}$-projective resolution in which P_i is finitely generated for $0 \leq i \leq n$. It has been shown that modules of type $\mathcal{F}\mathrm{FP}_n$ are of type FP_n [16]. We will say that a group G is of type $\mathcal{F}\mathrm{FP}_n$ if the trivial G-module $\mathbb{Z}$ is of type $\mathcal{F}\mathrm{FP}_n$.

We now specialize to the cases when $\mathcal{F} = \mathfrak{F}$ and $\mathcal{F} = \mathfrak{P}$.

Proposition 4.1. *The following properties hold.*

(i) *A short exact sequence of G-modules is $\mathfrak{F}$-split if and only if it is $\mathfrak{P}$-split.*
(ii) *A G module is $\mathfrak{F}$-projective if and only if it is $\mathfrak{P}$-projective.*

(iii) $\mathfrak{F}H^*(G,-) \cong \mathfrak{P}H^*(G,-)$

Proof: (i) It is obvious that any $\mathfrak{F}$-split sequence is $\mathfrak{P}$-split, and the converse follows from a standard averaging argument. Let H be an arbitrary finite subgroup of G. Then $|H| = \prod_{i=1,\dots,n} p_i^{a_i}$ where p_i are distinct primes and $0 < a_i \in \mathbb{Z}$. For each i, let n_i be the index $n_i = [H : P_i]$. Now consider a $\mathfrak{P}$-split surjection $A \overset{\pi}{\twoheadrightarrow} B$. Let σ_i be a P_i-splitting of π, and define a map s_i by summing σ_i over the cosets of P_i:

$$s_i(b) = \sum_{t \in H/P_i} t\sigma_i(t^{-1}b).$$

For each P_i we obtain a map $s_i : B \to A$, such that $\pi \circ s_i = n_i \times id_B$. There exist $m_i \in \mathbb{Z}$ so that $\sum_i m_i n_i = 1$, and the map $s = \sum_i m_i s_i$ is the required H-splitting.

(ii) It is obvious that a $\mathfrak{P}$-projective module is $\mathfrak{F}$-projective. Now let P be $\mathfrak{F}$-projective. We may take a $\mathfrak{P}$-split surjection $M \twoheadrightarrow P$ with M a $\mathfrak{P}$-projective. By (i) this surjection is $\mathfrak{F}$-split, and hence split. Thus P is a direct summand of a $\mathfrak{P}$-projective and so is $\mathfrak{P}$-projective.

(iii) now follows directly from (i) and (ii). □

Proposition 4.2. *A group G is of type $\mathfrak{F}\mathrm{FP}_0$ if and only if G has only finitely many conjugacy classes of subgroups of prime power order.*

Proof: Suppose that G has only finitely many conjugacy classes of subgroups in $\mathfrak{P}$. Let I be a set of representatives for the conjugacy classes of $\mathfrak{P}$-subgroups and set

$$\Delta_0 = \bigsqcup_{P \in I} G/P.$$

This G-set satisfies condition $(*)$ for $\mathfrak{P}$ and therefore the surjection $\mathbb{Z}\Delta_0 \twoheadrightarrow \mathbb{Z}$ is $\mathfrak{F}$-split and also $\mathbb{Z}\Delta_0$ is finitely generated.

To prove the converse we consider an arbitrary $\mathfrak{F}$-split surjection $P_0 \twoheadrightarrow \mathbb{Z}$ with P_0 a finitely generated $\mathfrak{F}$-projective. As in [16, 6.1] we can show that P_0 is a direct summand of a module $\bigoplus_{\delta \in \Delta_f} \mathrm{Ind}_{G_\delta}^G P_\delta$, where Δ_f is a finite G-set, the G_δ are finite groups and P_δ are finitely generated G_δ-modules. Therefore we might assume from now on that P_0 is of the above form. Since there is a G-map $\Delta_f \to \Delta$, where Δ satisfies condition $(*)$ the $\mathfrak{F}$-split surjection $P_0 \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}$ is also Δ_f-split [16]. Consider now the following commutative diagram:

$$\begin{array}{ccc} P_0 & \xrightarrow{\varepsilon} & \mathbb{Z} \\ {\scriptstyle\alpha}\uparrow\;\downarrow{\scriptstyle\beta} & & \| \\ \mathbb{Z}\Delta_f & \xrightarrow{\varepsilon_f} & \mathbb{Z} \end{array}$$

That we can find such an α follows from the fact that ε is Δ_f-split, and β exists since P_0 is Δ_f-projective being a direct sum of induced modules, induced from G_δ, $(\delta \in \Delta_f)$ to G.

As a next step we'll show that ε_f is $\mathfrak{F}$-split. Take an arbitrary finite subgroup H of G and show that ε_f splits when restricted to H. Since ε is split by s, say, when restricted to H we can define the required splitting by $\beta \circ s$.

Now let P be an arbitrary p-subgroup of G. Since the module $\mathbb{Z}[G/P]$ is $\mathfrak{F}$-projective, there exists a G-map φ, such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}\Delta_f & \xrightarrow{\varepsilon_f} & \mathbb{Z} \\ {\scriptstyle\varphi}\uparrow & & \| \\ \mathbb{Z}[G/P] & \longrightarrow & \mathbb{Z} \end{array}$$

The image $\varphi(P)$ of the identity coset P is a point of $\mathbb{Z}\Delta$ fixed by the action of P. If H is any group and $\mathbb{Z}\Omega$ is any permutation module, then the H-fixed points are generated by the orbit sums $H.\omega$. Hence P must stabilize some point of Δ_f, since otherwise we would have that p divides $\varepsilon_f\varphi(P) = \varepsilon\alpha(P) = 1$, a contradiction. It follows that P is a subgroup of G_δ for some $\delta \in \Delta_f$. □

Note that being of type $\mathfrak{F}\mathrm{FP}_0$ does not imply that there are finitely many conjugacy classes of finite subgroups. In fact, the authors have examples with infinitely many conjugacy classes of finite subgroups, see [12]. Nevertheless this gives rise to the following conjecture:

Conjecture 4.3. *A group G is of type $\mathfrak{F}\mathrm{FP}_\infty$ if and only if G is of type FP_∞ and has finitely many conjugacy classes of p-subgroups.*

It is shown in [16] that any G of type $\mathfrak{F}\mathrm{FP}_\infty$ is of type FP_∞, which together with Proposition 4.2 proves one implication in the above conjecture.

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