ON GROUPS ACTING ON CONTRACTIBLE SPACES WITH STABILIZERS OF PRIME POWER ORDER

IAN J. LEARY AND BRITA E. A. NUCINKIS

Abstract. Let $\mathfrak{F}$ denote the class of finite groups, and let $\mathfrak{P}$ denote the subclass consisting of groups of prime power order. We study group actions on topological spaces in which either (1) all stabilizers lie in $\mathfrak{P}$ or (2) all stabilizers lie in $\mathfrak{F}$. We compare the classifying spaces for actions with stabilizers in $\mathfrak{F}$ and $\mathfrak{P}$, the Kropholler hierarchies built on $\mathfrak{F}$ and $\mathfrak{P}$, and group cohomology relative to $\mathfrak{F}$ and to $\mathfrak{P}$. In terms of standard notations, we show that $\mathfrak{F} \subset \mathbf{H}_1\mathfrak{P} \subset \mathbf{H}_1\mathfrak{F}$, with all inclusions proper; that $\mathbf{H}\mathfrak{F} = \mathbf{H}\mathfrak{P}$; that $\mathfrak{F}H^*(G; -) = \mathfrak{P}H^*(G; -)$; and that $\mathrm{E}_{\mathfrak{P}}G$ is finite-dimensional if and only if $\mathrm{E}_{\mathfrak{F}}G$ is finite-dimensional and every finite subgroup of G is in $\mathfrak{P}$.

1. Introduction

Let $\mathcal{F}$ denote a class of groups, by which we mean a collection of groups which is closed under isomorphism and taking subgroups. A G-CW-complex X is said to be a model for $\mathrm{E}_{\mathcal{F}}G$, the classifying space for actions of G with stabilizers in $\mathcal{F}$, if for each $H \leq G$, one has that the fixed point set X^H is contractible for $H \in \mathcal{F}$ and is empty for $H \notin \mathcal{F}$. The most common classes considered are the class of trivial groups and the class $\mathfrak{F}$ consisting of all finite groups. In these cases $\mathrm{E}_{\mathcal{F}}G$ is often denoted $\mathrm{E}G$ and $\underline{\mathrm{E}}G$ respectively. Note that $\mathrm{E}G$ is the total space of the universal principal G-bundle, or equivalently the universal covering space of an Eilenberg-Mac Lane space for G. The space $\underline{\mathrm{E}}G$ is called the classifying space for proper actions of G. Recently there has been much interest in finiteness conditions for the spaces $\mathrm{E}_{\mathcal{F}}G$, especially for $\underline{\mathrm{E}}G$. Milnor and Segal's constructions of $\mathrm{E}G$ both generalize easily to construct models for any $\mathrm{E}_{\mathcal{F}}G$, and one can show that any two models for $\mathrm{E}_{\mathcal{F}}G$ are naturally equivariantly homotopy equivalent.

For some purposes the structure of the fixed point sets for subgroups in $\mathcal{F}$ is irrelevant. For example, a group is in Kropholler's class $\mathbf{H}_1\mathcal{F}$ if there is any finite-dimensional contractible G-CW-complex X with

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all stabilizers in $\mathcal{F}$. The class $\mathbf{H}_1\mathcal{F}$ is the first stage of a hierarchy whose union is Kropholler's class $\mathbf{H}\mathcal{F}$ of hierarchically decomposable groups [10]. (These definitions were first considered for the class $\mathfrak{F}$ of all finite groups, but work for any class $\mathcal{F}$.)

A priori, the class $\mathbf{H}_1\mathcal{F}$ may contain groups G that do not admit a finite-dimensional model for $\mathrm{E}_{\mathcal{F}}G$, and we shall give such examples in the case when $\mathcal{F} = \mathfrak{P}$, the class of groups of prime power order. By contrast, in the case when $\mathcal{F} = \mathfrak{F}$, no group G is known to lie in $\mathbf{H}_1\mathfrak{F}$ without also admitting a finite-dimensional model for $\underline{\mathrm{E}}G$. A construction due to Serre shows that every group G in $\mathbf{H}_1\mathfrak{F}$ that is virtually torsion-free has a finite-dimensional $\underline{\mathrm{E}}G$ [4], and the authors have given examples of G for which the minimal dimension of a contractible G-CW-complex is lower than the minimal dimension of a model for $\underline{\mathrm{E}}G$ [14]. These examples G also have the property that they admit a contractible G-CW-complex with finitely many orbits of cells, but that they do not admit any model for $\underline{\mathrm{E}}G$ with finitely many orbits of cells.

Throughout this paper, $\mathfrak{F}$ will denote the class of finite groups, and $\mathfrak{P}$ will denote the class of finite groups of prime power order. We compare the classifying space for G-actions with stabilizers in $\mathfrak{P}$ with the more well-known $\underline{\mathrm{E}}G$, and we compare the Kropholler hierarchies built on $\mathfrak{F}$ and $\mathfrak{P}$. We show that a finite group G that is not of prime power order cannot admit a finite-dimensional $\mathrm{E}_{\mathfrak{P}}G$, but that every finite group is in $\mathbf{H}_1\mathfrak{P}$. We also construct a group that is in $\mathbf{H}_1\mathfrak{F}$ but not in $\mathbf{H}_1\mathfrak{P}$, and we show that $\mathbf{H}\mathfrak{P} = \mathbf{H}\mathfrak{F}$.

In the final section we shall contrast this with cohomology relative to all finite subgroups. The relative cohomological dimension can be viewed as a generalisation of the virtual cohomological dimension, since for virtually torsion free groups these are equal, see [17]. By a result of Bouc [2, 12] it follows that groups belonging to $\mathbf{H}_1\mathfrak{F}$ have finite relative cohomological dimension, but the converse it not known. In contrast to our results concerning classifying spaces, we show that cohomology relative to subgroups in $\mathfrak{F}$ is naturally isomorphic to cohomology relative to subgroups in $\mathfrak{P}$.

2. Classifying spaces for actions with stabilizers in $\mathfrak{P}$

Theorem 2.1. *Let G be a finite group. Then G has a finite dimensional model for $\mathrm{E}_{\mathfrak{P}}G$ if and only if G has prime power order.*

Proof. If G has prime power order, then a single point may be taken as a model for $\mathrm{E}_{\mathfrak{P}}G$. Now let G be an arbitrary finite group, let p be a prime dividing the order of G, and assume that there is a p-subgroup

$P < G$, such that $N_G(P)$ is not a p-group. Then the Weyl-group $WP = N_G(P)/P$ contains a subgroup H of order prime to p. Assume G has a finite dimensional model for $\mathrm{E}_{\mathfrak{P}}G$, X say. Then the augmented cellular chain complex of the P-fixed point set, X^P, is a finite length resolution of $\mathbb{Z}$ by free H-modules. This gives a contradiction, since $\mathbb{Z}$ has infinite projective dimension as an H-module for any non-trivial finite group H.

Therefore we may suppose that for each subgroup $P \leq G$ which lies in $\mathfrak{P}$, the normalizer $N_G(P)$ is also in $\mathfrak{P}$. It follows from the Frobenius normal p-complement theorem [7, 5.26] that G has a normal p-complement for each prime p. Hence G is nilpotent and equal to the direct product of its Sylow subgroups. If P is a non-trivial Sylow subgroup of G, it follows that $G = N_G(P)$ is in $\mathfrak{P}$. □

Remark 2.2. The above proof was suggested to the authors by Yoav Segev. It is considerably shorter than our original proof, which did not quote the Frobenius normal p-complement theorem.

Corollary 2.3. *For a group G, the following are equivalent.*

(i) *G admits a finite-dimensional $\mathrm{E}_{\mathfrak{P}}G$;*
(ii) *Every finite subgroup of G is in $\mathfrak{P}$ and G admits a finite-dimensional $\underline{\mathrm{E}}G$.* □

Remark 2.4. We conclude the section with a remark on the type of $\mathrm{E}_{\mathfrak{P}}G$. It can proved analogously to Lück's proof for $\underline{\mathrm{E}}G$ [15] that a group G admits a finite type model for $\mathrm{E}_{\mathfrak{P}}G$ if and only if G has finitely many conjugacy classes of groups of prime power order and the Weyl-groups $N_G(P)/P$ for all subgroups P of prime power order are finitely presented and of type FP_∞. Hence any group admitting a finite type $\underline{\mathrm{E}}G$ also admits a finite type $\mathrm{E}_{\mathfrak{P}}G$. Recall that a finite extension of a group admitting a finite model for $\mathrm{E}G$ always has finitely many conjugacy classes of subgroups of prime power order [4, IX.13.2]. Hence the groups exhibited in [14, Example 7.4] are groups admitting a finite type $\mathrm{E}_{\mathfrak{P}}G$ which do not admit a finite type $\underline{\mathrm{E}}G$.

This behaviour is in stark contrast to that of $\mathrm{E}_{\mathcal{VC}}G$, the classifying space with virtually cyclic isotropy. Any group admitting a finite dimensional model for $\mathrm{E}_{\mathcal{VC}}G$ admits a finite dimensional model for $\underline{\mathrm{E}}G$, see [16] and the converse also holds for a large class of groups including all polycyclic-by-finite and all hyperbolic groups [8, 16]. Furthermore, any group admitting a finite type model for $\mathrm{E}_{\mathcal{VC}}G$ also admits a finite type model for $\underline{\mathrm{E}}G$ [9], but it is conjectured [8] that any group admitting a finite model for $\mathrm{E}_{\mathcal{VC}}G$ has to be virtually cyclic. This has been shown for a class of groups containing all hyperbolic groups [8] and for elementary amenable groups [9].

3. The hierarchies $\mathbf{h}\mathfrak{F}$ and $\mathbf{h}\mathfrak{P}$

Proposition 3.1. *Let X be a finite dimensional contractible G-CW-complex such that all stabilizers are finite. If there is a bound on the orders of the stabilizers then there exists a finite dimensional contractible G-CW-complex Y and an equivariant map $f : Y \to X$ such that $Y^H = \varnothing$ if H is not a p-group.*

Proof. Using the equivariant form of the simplicial approximation theorem, we may assume that X is a simplicial G-CW-complex. To simplify notation the phrase 'G-space' shall mean 'simplicial G-CW-complex' and 'G-map' will mean 'G-equivariant simplicial map' throughout the rest of the proof. The space Y will be a G-space in this sense and the map $f : Y \to X$ will be a G-map in this sense. The G-space Y is constructed in two stages. Firstly, for each finite $K \leq G$ we build a finite-dimensional contractible K-space Y_K with the property that all simplex stabilizers in Y_K lie in $\mathfrak{P}$.

Suppose for now that each such K-space Y_K has been constructed. Using the G-equivariant form of the construction used in [11, Section 8] the space Y is constructed as follows. Let I be an indexing set for the G-orbits of vertices in X. For each $i \in I$, let v_i be a representative of the corresponding orbit, and let K_i be the stabilizer of v_i. Let X^0 denote the 0-skeleton of X. Define a G-space Y^0 by

$$Y^0 = \coprod_{i \in I} G \times_{K_i} Y_{K_i},$$

and define a G-map $f : Y^0 \to X^0$ by $f(g, y) = g.v_i$ for all $i \in I$, for all $g \in G$ and for all $y \in Y_{K_i}$. For each vertex w of X, let $Y(w) = f^{-1}(w) \subset Y^0$. Each $Y(w)$ is a contractible subspace of Y^0, and the stabilizer of w acts on $Y(w)$.

Now for $\sigma = (w_0, \ldots, w_n)$ an n-simplex of X, define a space $Y(\sigma)$ as the join

$$Y(\sigma) = Y(w_0) * Y(w_1) * \cdots * Y(w_n).$$

Each vertex of $Y(\sigma)$ is already a vertex of one of the $Y(w_i)$, and so the map $f : Y^0 \to X^0$ defines a unique simplicial map $f : Y(\sigma) \to \sigma$. By construction, whenever τ is a face of σ, the space $Y(\tau)$ is identified with a subspace of $Y(\sigma)$. This allows us to define Y and $f : Y \to X$ as the colimit over the simplices σ of X of the subspaces $Y(\sigma)$, and to define $f : Y \to X$, which is a G-map of G-spaces. Since each $Y(\sigma)$ is contractible, it follows that f is a homotopy equivalence, and hence Y is also contractible (see [11, Corollary 8.6]).

It remains to build the K-space Y_K for each finite group $K \leq G$. In the case when $K \in \mathfrak{P}$ we may take a single point to be Y_K, and so we may suppose that $K \notin \mathfrak{P}$. Fix such a subgroup K, and suppose that we are able to construct a finite-dimensional contractible K-space Z_K in which each stabilizer is a proper subgroup of K. We may assume by induction that for each $L < K$ we have already constructed the L-space Y_L. The K-space Y_K can now be constructed from Z_K and the spaces Y_L using a process similar to the construction of Y from X and the spaces Y_K. It remains to construct the K-space Z_K.

An explicit construction of an K-space Z_K with the required properties is given in [13]. We therefore provide only a sketch of the argument. We may assume that K is not in $\mathfrak{P}$. Let S be the unit sphere in the reduced regular complex representation of K, so that S is a topological space with K-action such that the stabilizer of every point of S is a proper subgroup of K. Since K is not in $\mathfrak{P}$, there are K-orbits in S of coprime lengths. Using this property, it can be shown that the sphere S admits an K-equivariant self-map $g : S \to S$ of degree zero. The K-space Z_K is defined to be the infinite mapping telescope (suitably triangulated) of the map g. □

Corollary 3.2. *If G is in $\mathbf{H}_1\mathfrak{F}$ and there is a bound on the orders of the finite subgroups of G, then G is in $\mathbf{H}_1\mathfrak{P}$.* □

Remark 3.3. In Proposition 3.1, the bound on the orders of the stabilizers of X is used only to give a bound on the dimensions of the spaces Y_K. In Theorem 3.8 we shall show that $\mathbf{H}_1\mathfrak{F} \neq \mathbf{H}_1\mathfrak{P}$.

Remark 3.4. The construction in Proposition 3.1 does not preserve cocompactness, because for most finite groups K, the space Y_K used in the construction cannot be chosen to be finite. A result similar to Proposition 3.1 but preserving cocompactness can be obtained by replacing $\mathfrak{P}$ by a larger class $\mathfrak{O}$ of groups. Here $\mathfrak{O}$ is defined to be the class of $\mathfrak{P}$-by-cyclic-by-$\mathfrak{P}$-groups. A theorem of Oliver [20] implies that any finite group K that is not in $\mathfrak{O}$ admits a *finite* contractible K-CW-complex Z'_K in which all stabilizers are proper subgroups of K. Applying the same argument as in the proof of Proposition 3.1, one can show that given any contractible G-CW-complex X with all stabilizers in $\mathfrak{F}$, there is a contractible G-CW-complex Y' with all stabilizers in $\mathfrak{O}$ and a proper equivariant map $f' : Y' \to X$. (By proper, we mean that the inverse image of any compact subset of X is compact.)

For X a G-CW-complex with stabilizers in $\mathfrak{F}$, and p a prime, let $X_{\mathrm{sing}(p)}$ denote the subcomplex consisting of points whose stabilizer has order divisible by p. For G a group and p a prime, let $S_p(G)$ denote the poset of non-trivial finite p-subgroups of G.

Proposition 3.5. *Suppose that X is a finite-dimensional contractible G-CW-complex with all stabilizers in $\mathfrak{P}$. For each prime p, the mod-p homology of $X_{\mathrm{sing}(p)}$ is isomorphic to the mod-p homology of the (realization of the) poset $S_p(G)$.*

Proof. Fix a prime p, and let S denote the realization of the poset $S_p(G)$. For P a non-trivial p-subgroup of G, let X^P denote the points fixed by P, and let $S_{\geq P}$ denote the realization of the subposet of $S_p(G)$ consisting of all p-subgroups that contain P. By the P. A. Smith theorem [3], each X^P is mod-p acyclic. Each $S_{\geq P}$ is contractible since it is equal to a cone with apex P. Let P and Q be p-subgroups of G, and let $R = \langle P, Q \rangle$, the subgroup of G generated by P and Q. If R is a p-group then $X^P \cap X^Q = X^R$, and otherwise $X^P \cap X^Q$ is empty. Similarly, $S_{\geq P} \cap S_{\geq Q} = S_{\geq R}$ if R is a p-group and $S_{\geq P} \cap S_{\geq Q}$ is empty if R is not a p-group.

Since each X^P is mod-p acyclic, the mod-p homology $H_*(X_{\mathrm{sing}(p)})$ is isomorphic to the mod-p homology of the nerve of the covering $X_{\mathrm{sing}(p)} = \bigcup_P X^P$. Similarly, the mod-$p$ homology $H_*(S)$ is isomorphic to the mod-p homology of the nerve of the covering $S = \bigcup_P S_{\geq P}$. By the remarks in the first paragraph, these two nerves are isomorphic. □

Proposition 3.6. *Let k be a finite field, and let G be the group of k-points of a reductive algebraic group over k, whose commutator subgroup has k-rank n. (For example, $G = SL_{n+1}(k)$, or $GL_{n+1}(k)$.) Any finite-dimensional contractible G-CW-complex with stabilizers in $\mathfrak{P}$ has dimension at least n.*

Proof. The hypotheses on G imply that G acts on a spherical building Δ of dimension $n-1$ [1, 5, Appendix on algebraic groups]. Any such building is homotopy equivalent to a wedge of $(n-1)$-spheres. Quillen has shown that Δ is homotopy equivalent to the realization of $S_p(G)$, where p is the characteristic of the field k [21, Proposition 2.1 and Theorem 3.1]. It follows that $S_p(G)$ is homotopy equivalent to a wedge of $(n-1)$-spheres, and in particular the mod-p homology group $H_{n-1}(S_p(G))$ is non-zero.

Now suppose that X is a finite-dimensional contractible G-CW-complex with stabilizers in $\mathfrak{P}$. Using Proposition 3.5, one sees that the mod-p homology group $H_{n-1}(X_{\mathrm{sing}(p)})$ is non-zero. It follows that X must have dimension at least n. □

Remark 3.7. In [22] it is shown that if G is a finite simple group of Lie type, of Lie rank n, then any contractible G-CW-complex of dimension strictly less than n contains a point fixed by G. (Theorem 1 of [22] contains the additional hypothesis that the G-CW-complex should be finite, but this is not used in the proof.) A similar argument to that used in [22, Theorem 2] was used in [13] to show that when $G = SL_{n+1}(\mathbb{F}_p)$, every contractible G-CW-complex without a global fixed point has dimension at least n. Note that Proposition 3.6 applies in greater generality than these results. For example, the Conner-Floyd construction [6] shows that whenever the multiplicative group of k does not have prime-power order, there is, for any $n \geq 1$, a 4-dimensional contractible $Gl_n(k)$-CW-complex without a global fixed point.

Theorem 3.8. *There are the following strict containments and equalities between classes of groups:*

(i) $\mathfrak{F} \subsetneq \mathbf{H}_1\mathfrak{P}$;
(ii) $\mathbf{H}_1\mathfrak{P} \subsetneq \mathbf{H}_1\mathfrak{F}$;
(iii) $\mathbf{H}\mathfrak{F} = \mathbf{H}\mathfrak{P}$.

Proof. Corollary 3.2 shows that $\mathfrak{F} \subseteq \mathbf{H}_1\mathfrak{P}$. The free product of two cyclic groups of prime order is in $\mathbf{H}_1\mathfrak{P}$ and is not finite. The claim that $\mathbf{H}\mathfrak{F} = \mathbf{H}\mathfrak{P}$ follows from the inequalities $\mathfrak{P} \subseteq \mathfrak{F} \subseteq \mathbf{H}_1\mathfrak{P}$, and the claim $\mathbf{H}_1\mathfrak{P} \subseteq \mathbf{H}_1\mathfrak{F}$ follows from $\mathfrak{P} \subseteq \mathfrak{F}$.

It remains to exhibit a group G that is in $\mathbf{H}_1\mathfrak{F}$ but not in $\mathbf{H}_1\mathfrak{P}$. Let $G = SL_\infty(\mathbb{F}_p)$, the direct limit of the groups $G_n = SL_n(\mathbb{F}_p)$, where G_n is included in G_{n+1} as the 'top corner'. As a countable locally-finite group, G acts with finite stabilizers on a tree. (Explicitly, the vertex set V and edge set E are both equal as G-sets to the disjoint union of the sets of cosets $G/G_1 \cup G/G_2 \cup \cdots$, with the edge gG_i joining the vertex gG_i to the vertex gG_{i+1}.) It follows that $G \in \mathbf{H}_1\mathfrak{F}$. By Proposition 3.6, G cannot be in $\mathbf{H}_1\mathfrak{P}$. □

Remark 3.9. Let G be a group in $\mathbf{H}\mathfrak{F}$ that is also of type FP_∞. By a result of Kropholler [10], there is a bound on the orders of finite subgroups of G, and Kropholler-Mislin show that G is in $\mathbf{H}_1\mathfrak{F}$ [11]. Corollary 3.2 shows that G is in $\mathbf{H}_1\mathfrak{P}$.

4. Cohomology relative to a class of groups

Let Δ denote a G-set, and let $\mathbb{Z}\Delta$ denote the corresponding G-module. For $\delta \in \Delta$, we write G_δ for the stabilizer of δ. A short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ of G-modules is said to be Δ-*split* if and only if it splits as a sequence of G_δ-modules for each $\delta \in \Delta$. Equivalently, the sequence is Δ-split if and only if the following sequence of $\mathbb{Z}G$-modules splits: $A \otimes \mathbb{Z}\Delta \rightarrowtail B \otimes \mathbb{Z}\Delta \twoheadrightarrow C \otimes \mathbb{Z}\Delta$ [18].

We say a G module is Δ-projective if it is a direct summand of a G-module of the form $N \otimes \mathbb{Z}\Delta$, where N is an arbitrary G-module. Δ-projectives satisfy analogue properties to ordinary projectives. Furthermore, for each δ, and each G_δ-module M, the induced module $\mathrm{Ind}_{G_\delta}^G M$ is Δ-projective. Given two G-sets Δ_1 and Δ_2 and a G-map $\Delta_1 \to \Delta_2$ then Δ_1-projectives are Δ_2-projective and Δ_2-split sequences are Δ_1-split. For more detail the reader is referred to [18].

Now suppose that $\mathcal{F}$ is a class of groups closed under taking subgroups. We consider G-sets Δ satisfying the following condition, for all $H \leq G$:

$$(*) \qquad \Delta^H \neq \varnothing \iff H \in \mathcal{F}.$$

There are G-maps between any two G-sets satisfying condition $(*)$, and so we may define an $\mathcal{F}$-projective module to be a Δ-split module for any such Δ. Similarly, an $\mathcal{F}$-split exact sequence of G-modules is defined to be a Δ-split sequence. If Δ satisfies $(*)$ and M is any G-module, the module $M \otimes \mathbb{Z}\Delta$ is $\mathcal{F}$-projective and admits an $\mathcal{F}$-split surjection to M. This leads to a construction of homology relative to $\mathcal{F}$. An $\mathcal{F}$-projective resolution of a module M is an $\mathcal{F}$-split exact sequence

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_0 \to M \to 0,$$

where all P_i are $\mathcal{F}$-projective. Group cohomology relative to $\mathcal{F}$, denoted $\mathcal{F}H^*(G;N)$ can now be defined as the cohomology of the cochain complex $\mathrm{Hom}_G(P_*, N)$, where P_* is an $\mathcal{F}$-projective resolution of $\mathbb{Z}$.

We say that a module M is of type $\mathcal{F}\mathrm{FP}_n$ if M admits an $\mathcal{F}$-projective resolution in which P_i is finitely generated for $0 \leq i \leq n$. It has been shown that modules of type $\mathcal{F}\mathrm{FP}_n$ are of type FP_n [18]. We will say that a group G is of type $\mathcal{F}\mathrm{FP}_n$ if the trivial G-module $\mathbb{Z}$ is of type $\mathcal{F}\mathrm{FP}_n$.

We now specialize to the cases when $\mathcal{F} = \mathfrak{F}$ and $\mathcal{F} = \mathfrak{P}$.

Theorem 4.1. *The following properties hold.*

(i) *A short exact sequence of G-modules is $\mathfrak{F}$-split if and only if it is $\mathfrak{P}$-split.*
(ii) *A G module is $\mathfrak{F}$-projective if and only if it is $\mathfrak{P}$-projective.*
(iii) $\mathfrak{F}H^*(G,-) \cong \mathfrak{P}H^*(G,-)$

Proof: (i) It is obvious that any $\mathfrak{F}$-split sequence is $\mathfrak{P}$-split, and the converse follows from a standard averaging argument. Let H be an arbitrary finite subgroup of G. Then $|H| = \prod_{i=1,\ldots,n} p_i^{a_i}$ where p_i are distinct primes and $0 < a_i \in \mathbb{Z}$. For each i, let n_i be the index $n_i = [H : P_i]$. Now consider a $\mathfrak{P}$-split surjection $A \overset{\pi}{\twoheadrightarrow} B$. Let σ_i be a P_i-splitting of π, and define a map s_i by summing σ_i over the cosets of P_i:

$$s_i(b) = \sum_{t \in H/P_i} t\sigma_i(t^{-1}b).$$

For each P_i we obtain a map $s_i : B \to A$, such that $\pi \circ s_i = n_i \times id_B$. There exist $m_i \in \mathbb{Z}$ so that $\sum_i m_i n_i = 1$, and the map $s = \sum_i m_i s_i$ is the required H-splitting.

(ii) It is obvious that a $\mathfrak{P}$-projective module is $\mathfrak{F}$-projective. Now let P be $\mathfrak{F}$-projective. We may take a $\mathfrak{P}$-split surjection $M \twoheadrightarrow P$ with M a $\mathfrak{P}$-projective. By (i) this surjection is $\mathfrak{F}$-split, and hence split. Thus P is a direct summand of a $\mathfrak{P}$-projective and so is $\mathfrak{P}$-projective.

(iii) now follows directly from (i) and (ii). □

Proposition 4.2. *A group G is of type $\mathfrak{F}\mathrm{FP}_0$ if and only if G has only finitely many conjugacy classes of subgroups of prime power order.*

Proof: Suppose that G has only finitely many conjugacy classes of subgroups in $\mathfrak{P}$. Let I be a set of representatives for the conjugacy classes of $\mathfrak{P}$-subgroups and set

$$\Delta_0 = \bigsqcup_{P \in I} G/P.$$

This G-set satisfies condition $(*)$ for $\mathfrak{P}$ and therefore the surjection $\mathbb{Z}\Delta_0 \twoheadrightarrow \mathbb{Z}$ is $\mathfrak{F}$-split and also $\mathbb{Z}\Delta_0$ is finitely generated.

To prove the converse we consider an arbitrary $\mathfrak{F}$-split surjection $P_0 \twoheadrightarrow \mathbb{Z}$ with P_0 a finitely generated $\mathfrak{F}$-projective. As in [18, 6.1] we can show that P_0 is a direct summand of a module $\bigoplus_{\delta \in \Delta_f} \mathrm{Ind}_{G_\delta}^G P_\delta$, where Δ_f is a finite G-set, the G_δ are finite groups and P_δ are finitely generated G_δ-modules. Therefore we might assume from now on that P_0 is of the above form. Since there is a G-map $\Delta_f \to \Delta$, where Δ satisfies condition $(*)$ the $\mathfrak{F}$-split surjection $P_0 \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}$ is also Δ_f-split [18]. Consider now the following commutative diagram:

$$\begin{array}{ccc} P_0 & \xrightarrow{\varepsilon} & \mathbb{Z} \\ {\scriptstyle\alpha}\uparrow\downarrow{\scriptstyle\beta} & & \| \\ \mathbb{Z}\Delta_f & \xrightarrow{\varepsilon_f} & \mathbb{Z} \end{array}$$

That we can find such an α follows from the fact that ε is Δ_f-split, and β exists since P_0 is Δ_f-projective being a direct sum of induced modules, induced from G_δ, $(\delta \in \Delta_f)$ to G.

As a next step we'll show that ε_f is $\mathfrak{F}$-split. Take an arbitrary finite subgroup H of G and show that ε_f splits when restricted to H. Since ε is split by s, say, when restricted to H we can define the required splitting by $\beta \circ s$.

Now let P be an arbitrary p-subgroup of G. Since the module $\mathbb{Z}[G/P]$ is $\mathfrak{F}$-projective, there exists a G-map φ, such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}\Delta_f & \xrightarrow{\varepsilon_f} & \mathbb{Z} \\ {\scriptstyle\varphi}\uparrow & & \| \\ \mathbb{Z}[G/P] & \longrightarrow & \mathbb{Z} \end{array}$$

The image $\varphi(P)$ of the identity coset P is a point of $\mathbb{Z}\Delta$ fixed by the action of P. If H is any group and $\mathbb{Z}\Omega$ is any permutation module, then the H-fixed points are generated by the orbit sums $H.\omega$. Hence P must stabilize some point of Δ_f, since otherwise we would have that p divides $\varepsilon_f\varphi(P) = \varepsilon\alpha(P) = 1$, a contradiction. It follows that P is a subgroup of G_δ for some $\delta \in \Delta_f$. $\square$

Note that being of type $\mathfrak{F}\mathrm{FP}_0$ does not imply that there are finitely many conjugacy classes of finite subgroups. In fact, the authors have examples with infinitely many conjugacy classes of finite subgroups, see [14]. Nevertheless this gives rise to the following conjecture:

Conjecture 4.3. *A group G is of type $\mathfrak{F}\mathrm{FP}_\infty$ if and only if G is of type FP_∞ and has finitely many conjugacy classes of p-subgroups.*

It is shown in [18] that any G of type $\mathfrak{F}\mathrm{FP}_\infty$ is of type FP_∞, which together with Proposition 4.2 proves one implication in the above conjecture.

References

[1] P. Abramenko and K. S. Brown, *Buildings: Theory and Applications*, Graduate Texts in Mathematics **248**, Springer Verlag (2008).
[2] S. Bouc, Le complexe de chaînes d'un G-complexe simplicial acyclique, *J. Algebra*, **220**, (1999), 415–436.
[3] G. E. Bredon, *Introduction to compact transformation groups*, Pure and Applied Math. **46**, Academic Press (1972).
[4] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics **87**, Springer Verlag (1982).
[5] K. S. Brown, *Buildings*, Springer Verlag (1989).

[6] P.E. Conner and S.E. Floyd, Differentiable periodic maps, *Bull. Am. Math. Soc.* **68**, (1962), 76–86.
[7] I.M. Isaacs, *Finite group theory*, Graduate Studies in Mathematics **92**, Providence, RI: American Mathematical Society (AMS). (2008).
[8] D. Juan-Pineda and I. J. Leary, On classifying spaces for the family of virtually cyclic subgroups, *Recent developments in algebraic topology* Contemporary Mathematics **407**, 135-145 (2006).
[9] D. H. Kochloukova, C. Martínez-Pérez and B.E.A. Nucinkis, Cohomological finiteness conditions in Bredon cohomology, preprint 2008.
[10] P. H. Kropholler, On groups of type FP_∞, *J. Pure Appl. Algebra* **90** (1993), 55–67.
[11] P. H. Kropholler and G. Mislin, Groups acting on finite dimensional spaces with finite stabilizers, *Comment. Math. Helv.* **73** (1998), 122–136.
[12] P. H. Kropholler and C. T. C. Wall A theorem of Bouc concerning finite group actions on finite dimensional acyclic spaces, preprint.
[13] I. J. Leary, On finite subgroups of groups of type VF, *Geom. Topol.* **9** (2005), 1953–1976.
[14] I. J. Leary and B. E. A. Nucinkis, Some groups of type VF, *Invent. Math.* **151** (2003), 135–165.
[15] W. Lück, The type of the classifying space for a family of subgroups, *J. Pure Appl. Algebra* **149** (2000), 177-203.
[16] W. Lück and M. Weiermann, On the classifying space of the family of virtually cyclic subgroups, arXiv:math.AT/0702646v1, 2007.
[17] C. Martinez-Pérez and B. E. A.Nucinkis, Cohomological Dimension of Mackey Functors for Infinite Groups, *J. London Math. Soc. (2)* **74** (2006), no. 2, 379–396.
[18] B. E. A. Nucinkis, Cohomology relative to a G-set and finiteness conditions, *Top. Appl.* **92**, (1999), 153–171.
[19] B. E. A. Nucinkis, Is there an easy algebraic characterisation of universal proper G-spaces?, *Manuscripta Math.* **102**, (2000), 335–345.
[20] R. Oliver, Fixed-point sets of group actions on finite acyclic complexes, *Comment. Math. Helvetici* **50** (1975), 155–177.
[21] D. Quillen, Homotopy properties of the poset of non-trivial p-subgroups of a group, *Advances in Math.* **28**, (1978), 101–128.
[22] Y. Segev, On the action of finite simple groups of Lie type and characteristic p on finite $\mathbb{Z}_p$-acyclic simplicial complexes, *Topology* **32** (1993) 665–675.
[23] J. Tits, On buildings and their applications, *Proc. Int. Cong. Mathematicians*, Vancouver, 1974.

Department of Mathematics, The Ohio State University, 231 W 18th Ave, Columbus Ohio 43210

Heilbronn Institute, University of Bristol, Royal Fort Annexe, Bristol BS8 1TW

E-mail address: `leary@math.ohio-state.edu`

School of Mathematics, University of Southampton, Southampton SO17 1BJ, UK

E-mail address: `B.E.A.Nucinkis@soton.ac.uk`