UNIVERSITY OF SOUTHAMPTON

FACULTY OF ENGINEERING, SCIENCE & MATHEMATICS

School of Mathematics

Classical and Non-Classical Schottky Groups

by

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Thesis for the degree of Doctor of Philosophy

February 2009

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

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This thesis looks at two disparate problems relating to Schottky groups, and in particular what it means for a Schottky group to be classical or non-classical.

The first problem focusses on the uniformization of Riemann surfaces using Schottky groups. We extend the retrosection theorem of Koebe by giving conditions on lengths of curves as to when a Riemann surface can be uniformized by a classical Schottky group.

The second section of this thesis examines a paper of Yamamoto ([40]), which gives the first example of a non-classical Schottky group. We firstly expand on the detail given in the paper, and then use this to give a second example of a non-classical Schottky group. We then take this second example and generalise to a two-variable family of non-classical Schottky groups.

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DECLARATION OF AUTHORSHIP

I, Jonathan Peter Williams, declare that the thesis entitled Classical and Non-Classical Schottky Groups, and the work presented in it are my own. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
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ACKNOWLEDGEMENTS

Many people have helped to make this thesis what it is, and they all deserve thanks for the support they provided in their own special ways.

Thanks must go to all at the School of Mathematics at Southampton, but particularly to my supervisor Jim Anderson for his encouragement, support and patience with me and my work. Special thanks also from throughout my time at Southampton to David Singerman and Vesna Perisic.

There have been many many office mates over the years, all of whom have been a pleasure to work with. There are so many great people that to highlight any would be unfair; thanks to them all. Having said that, special mention might just about be appropriate for Cormac Long. Just.

Not everyone that has helped with this thesis has been mathematically minded. Support has come from all directions, from Mum, Dad, other relatives, friends from work, uni and elsewhere. Thank you to you all - your support, deliberate and unintentional, has been much appreciated.

Finally there is of course someone else to whom I am hugely indebted. A thousand thank yous to my wife, Dr Kirsty Williams, for the support from the start of the PhD to the end, from before the thesis to long afterwards. This wouldn't have been possible without her.

Chapter 1

Introduction

Schottky groups were first constructed by Schottky in 1882, but were not studied in greater detail until work of Chuckrow [13], Maskit [26] and Marden [25], along with others, in the late 1960s/early 1970s, and then more recently by Maskit [30], Hidalgo [16] and Tan, Wong and Zhang [39] in the 21st Century.

A Schottky group is defined by its construction as follows. Let D be a region on the Riemann sphere bounded by $2n$ disjoint simple closed curves, $C_1, C_1', ..., C_n, C_n'$. The C_is are paired to the C_i's by loxodromic Möbius transformations, γ_i, such that $\gamma_i(C_i) = C_i'$ and that $\gamma_i(D) \cap D = \emptyset$. If Γ is the group generated by the γ_i then Γ is a Schottky group. Alternatively a purely loxodromic, free, finitely generated Kleinian group with non-empty domain of discontinuity is a Schottky group [26].

A Schottky group is classical if there exist some set of generators for the group such that there exist a set of curves $C_1, C_1', ..., C_n, C_n'$ as above such that the curves are Euclidean circles. Marden, [25], showed that not every

Schottky group is classical. Zarrow, [41] claimed to have discovered the first explicit example of a non-classical Schottky group, but this was later shown to be classical by Sato, [35]. The first explicit example of a non-classical Schottky group was given by Yamamoto [40].

In Chapter 2 we discuss some of the background of Schottky groups, looking initially at Kleinian groups as a natural precursor to studying Schottky groups. After looking at Schottky groups in general we look at some of the reasons why deciding if a Schottky group is classical or not is far from trivial. We also discuss Koebe's retrosection theorem, which shows that Schottky groups can be used to uniformize Riemann surfaces. With this in mind we give some background on Riemann surfaces and ring domains. We finally discuss non-classical Schottky groups in more detail.

Koebe's retrosection theorem states that all closed Riemann surfaces can be uniformized by Schottky groups, and it has been conjectured that every closed Riemann surface can be uniformized by a classical Schottky group. In Chapter 3 we work towards this by showing that there exists a value k such that a Riemann surface of genus g with g homologously independent simple closed curves of lengths less than k can be uniformized by a classical Schottky group.

It is particularly difficult to decide if a given Schottky group is classical or non-classical, and there are many questions for which more examples of non-classical Schottky groups would be useful. For example it is not known what properties Riemann surfaces uniformized only by non-classical Schottky groups have, or even if there are surfaces which only have non-classical uniformizations. As mentioned there is only one known example of a family of non-classical Schottky groups, given by Yamamoto. His paper [40] is not particularly easy to read, with many details omitted or left for the reader. We begin Chapter 4 by rewriting Yamamoto's paper with a different order, with details included and with any typographical errors corrected. We then go on to use this proof as a skeleton to find another family of non-classical Schottky groups, and then to generalise this approach to a two variable family of non-classical Schottky groups. We give the explicit bounds on the two variables in Appendix A.

Finally in Chapter 5 we discuss two areas for further study. We investigate finding inequalities involving fixed points and multipliers of Schottky generators which allow us to decide whether that given generator set has a set of classical SG-curves. We also discuss the effect of applying Nielsen transformations to the generators of Schottky groups, using the construction of Chuckrow [13], and build a graph analogous to the Andrews-Curtis graph for Schottky generators. We mention some interesting questions about this Schottky graph, and suggest that questions on Schottky groups might be answered by studying the Andrews-Curtis or Schottky graphs.

Chapter 2

Background

2.1 Kleinian Groups

We begin by defining Kleinian groups, and some important properties of these groups. We use [28] as a source for this section.

We denote the extended complex plane $\mathbb{C} \cup \{\infty\}$ as $\hat{\mathbb{C}}$. Möbius transformations are then defined as maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$f : z \mapsto \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. If we regard straight lines in $\mathbb{C}$ as circles in $\hat{\mathbb{C}}$ passing through ∞ then we can see that Möbius transformations send circles in $\hat{\mathbb{C}}$ to circles in $\hat{\mathbb{C}}$.

Möbius transformations can be classified into three different types, based on the number of fixed points of the transformation. A Möbius transformation has either one or two fixed points, which we obtain by solving $\frac{az+b}{cz+d} = z$.

A Möbius transformation, f, is said to be *parabolic* if f has exactly one fixed point. Every parabolic transformation is conjugate to $z \to z + 1$.

If a transformation has two fixed points then it is conjugate to one with fixed points at 0 and ∞, and hence there exists a Möbius transformation g such that $gfg^{-1}(z) = k^2 z$, with $k \in \mathbb{C}$ and $|k|^2 \geq 1$. We call k^2 the multiplier of f. If $|k|^2 = 1$ then the transformation is called *elliptic*, and is conjugate to a rotation $z \to e^{i\theta} z$ for θ real. Otherwise it is called a *loxodromic* Möbius transformation.

Within the loxodromic Möbius transformations if $k^2 \in \mathbb{R}^+$ then the transformation is called *hyperbolic*. The hyperbolic transformations can be thought of as dilations. Loxodromic Möbius transformations have two fixed points, one of which is referred to as the *attracting fixed point*, and the other the *repelling fixed point*. Given the distince fixed points x, y of a loxodromic transformation f we say that x is attracting if $\lim_{n\to\infty} f^n(z) \to x$ for all $z \neq y$. Then y is the repelling fixed point. The attracting fixed point of f is the repelling fixed point of f^{-1}, and vice versa. We will be concentrating on loxodromic transformations later.

We can write the transformation $f(z)$ in matrix form, for example as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where we again have that $ad - bc = 1$. This is very useful since we are able to compose two Möbius transformations simply by multiplying the corresponding matrices. These matrices are elements of

$$\mathrm{SL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{C}, \; ad - bc = 1 \right\}$$

We have though that the transformation f can be represented by two elements of $\mathrm{SL}_2(\mathbb{C})$, that is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

If we form the quotient group $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm I\}$ by factoring out the centre, where I is the identity matrix, then there exists an isomorphism from the group of Möbius transformations to $\mathrm{PSL}_2(\mathbb{C})$.

We can find a general form of a loxodromic Möbius transformation given its pair of fixed points, x, y, and its multiplier, k^2. We can choose a square root of the multiplier and then for $x \neq \infty \neq y$ the transformation can be written as:

$$f = \frac{1}{x-y}\begin{pmatrix} xk^{-1} - yk & xy(k-k^{-1}) \\ k^{-1} - k & xk - yk^{-1} \end{pmatrix}$$

and for $x = \infty$ as

$$f = \begin{pmatrix} k^{-1} & y(k-k^{-1}) \\ 0 & k \end{pmatrix}$$

and for $y = \infty$ as

$$f = \begin{pmatrix} k & x(k^{-1}-k) \\ 0 & k^{-1} \end{pmatrix}$$

A subgroup of $\mathrm{PSL}_2(\mathbb{C})$ is said to be *discrete* if it does not contain a sequence of distinct elements converging to I. We now define a Kleinian group, and some terminology related to Kleinian groups.

Definition 2.1.1. A *Kleinian group*, Γ, is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ [31].

Definition 2.1.2. A Kleinian group, Γ, acts *properly discontinously* at a point $x \in \hat{\mathbb{C}}$ if there exists a neighbourhood U_x about x such that $\gamma(U_x) \cap U_x = \emptyset$ for all but finitely many $\gamma \in \Gamma$.

Definition 2.1.3. The open set in $\hat{\mathbb{C}}$ which consists of all the points at which Γ acts discontinously is called the *domain of discontinuity* and denoted by $\Omega(\Gamma)$. Its complement in $\hat{\mathbb{C}}$ is called the *limit set*, and is denoted $\Lambda(\Gamma)$.

The limit set can also be defined in terms of accumulation points. A point $x \in \hat{\mathbb{C}}$ is called an *accumulation point* (or *limit point*) of a Kleinian group, Γ, if there exists a sequence of distinct elements of Γ, say $\{\gamma_i\}$, and a point $z \in \hat{\mathbb{C}}$ such that $\gamma_i(z) \to x$, [28]. The limit set $\Lambda(\Gamma)$ is then simply the set of all accumulation points of Γ.

A Kleinian group whose limit set consists of more than two points is called non-elementary. If Γ is non-elementary then Γ contains a loxodromic element, and the limit set of Γ is the closure of the set of loxodromic fixed points.

One of the uses of the domain of discontinuity is that in general $\hat{\mathbb{C}}/\Gamma$ is not Hausdorff, while $\Omega(\Gamma)/\Gamma$ is a Riemann surface, as discussed in §2.4. Often the easiest way to describe a Kleinian group is to describe $\Omega(\Gamma)/\Gamma$, usually in terms of the fundamental domain for Γ.

Definition 2.1.4. A *fundamental domain* for a Kleinian group Γ is an open subset D of the domain of discontinuity which (i) has the identity element in Γ as its stabilizer, (ii) satisfies $\gamma(D) \cap D = \emptyset$ for all γ in Γ not the identity, (iii) has sides paired by elements of Γ, (iv) for every $z \in \Omega(\Gamma)$ there is a $\gamma \in \Gamma$ with $\gamma(z) \in \overline{D}$, (v) the sides of D only accumulate at limit points, and (vi) only finitely many translates of D meet any compact subset of $\Omega(\Gamma)$.

2.2 Schottky Groups

Schottky groups are a class of Kleinian groups which are particularly interesting for a number of reasons. One such reason is their link to uniformizing Riemann surfaces, as will be seen in §2.4, and another is due to their simple construction which we detail below. We begin by defining SG-curves.

Definition 2.2.1. Take $2g$ disjoint Jordan curves in $\hat{\mathbb{C}}$, which are not nested and hence define an open region D with the $2g$ curves as boundary. We label these curves in pairs as $C_1, C_1', ..., C_g, C_g'$, and will refer to them as the defining curves for the Schottky group or for ease of reference as *SG-curves*.

We now detail the set-up to define a Schottky group. Suppose there exist loxodromic Möbius transformations $\gamma_1, ..., \gamma_g$ such that $\gamma_i(C_i) = C_i'$. An example with $g = 2$ is shown in Figure 2.1.

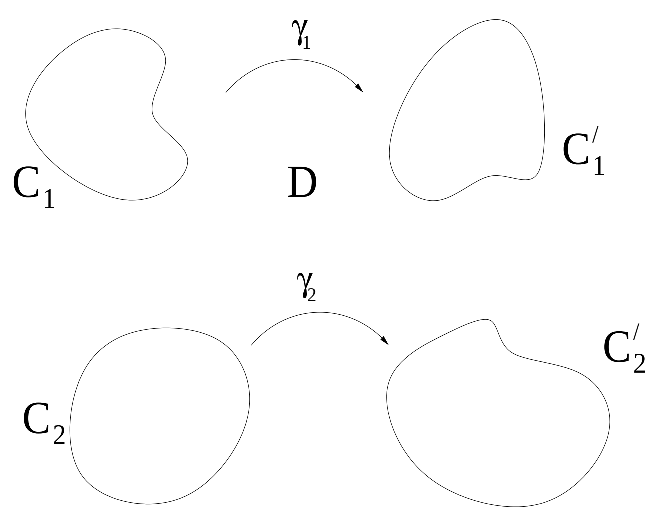

Figure 2.1: An example of four curves, two loxodromic Möbius transformations and the region D.

Each curve C_i (or C_i') separates $\hat{\mathbb{C}}$ into two regions, and we define the *outside* of C_i (or C_i') to be the part of $\hat{\mathbb{C}} - C_i$ (or $\hat{\mathbb{C}} - C_i'$) containing other SG-curves, and the *inside* to be the region containing no other SG-curves. We can also define the inside of an SG-curve to be the part of $\hat{\mathbb{C}} - C_i$ (or $\hat{\mathbb{C}} - C_i'$) containing only one of all of the fixed points for the generators of Γ. Explicitly, C_i has the repelling fixed point of γ_i inside it, and C_i' has the attracting fixed point of γ_i inside it.

We have that $D = \hat{\mathbb{C}} \setminus \bigcup_i (C_i \cup C_i')$. For any of the loxodromic Möbius transformations, we have that $\gamma_i(D)$ could either intersect D or have empty intersection with D. We have from the definition of the fundamental domain that $D \cap \gamma_i(D) = \emptyset$ for all γ_i not equal to the identity in Γ, and hence we have the property that each γ_i sends the inside of C_i to the outside of C_i', and the outside of C_i to the inside of C_i'. So, for example, in Figure 2.1, γ_1 sends C_1 to C_1' and sends C_2, C_1' and C_2' to curves inside C_1'. Figure 2.2 shows the image of D under γ_1, and we see that $D \cap \gamma_1(D) = \emptyset$.

Definition 2.2.2. A *Schottky group* is then simply defined as the group generated by the loxodromic transformations, $\Gamma = \langle \gamma_1, ...\gamma_g \rangle$.

We also define a Schottky system for use later:

Definition 2.2.3. A *Schottky system* is the name given to a Schottky group and a specified set of SG-curves. The set of SG-curves used is not unique to the group, and there are many Schottky systems for a given Schottky group.

We will see later in this section that given a Schottky group and specified SG-curves that changing the curves slightly will not alter the Schottky group, but will change Schottky system.

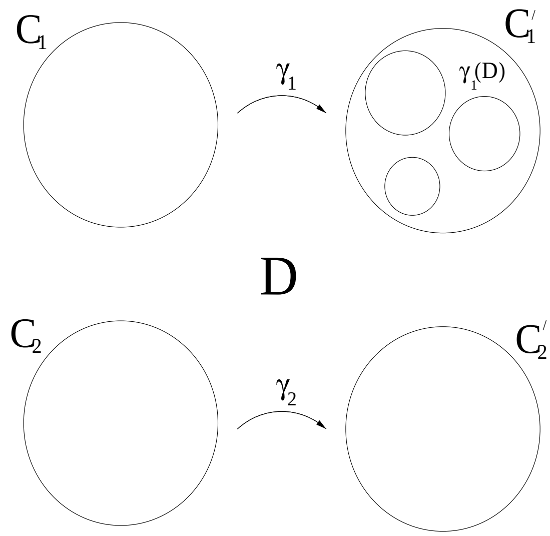

Figure 2.2: The image of D under γ_1.

The domain of discontinuity of a Schottky group can be written as

$$\Omega(\Gamma) = \bigcup_{\gamma \in \Gamma} \gamma(\overline{D})$$

where $\overline{D}$ denotes the closure of D. We have that as more and more Möbius transformations are applied, the diameters of the $\gamma(C)$, for C an SG-curve, tend to zero. These image curves will all be inside of a particular curve, and after a finite number, say k, of applications of γ to C we have that $\gamma^{n+1}(C)$ is inside $\gamma^n(C)$ for $n > k$. Thus we have that the limit set $\Lambda(\Gamma)$ can be seen as the set of accumulation points of this nesting of images of the SG-curves and is a Cantor set. This idea can be seen schematically in Figure 2.3, whereby each SG-curve has three curves nested inside it.

Schottky groups have some interesting properties as shown in a proposition of Maskit:

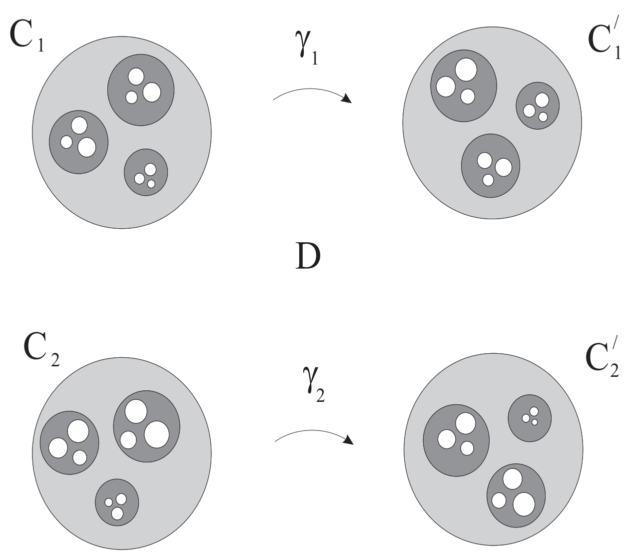

Figure 2.3: Images of D after two Möbius tranformations (inside C_1' are $\gamma_1(D)$ shaded lightly, and $\gamma_1\gamma_2(D)$, $\gamma_1\gamma_2^{-1}(D)$ and $\gamma_1\gamma_1(D)$ shaded more darkly; inside C_1 are $\gamma_1^{-1}(D)$ shaded lightly, and $\gamma_1^{-1}\gamma_2(D)$, $\gamma_1^{-1}\gamma_2^{-1}(D)$ and $\gamma_1^{-1}\gamma_1^{-1}(D)$ shaded more darkly; with similarly shaded regions inside C_2 and C_2').

Proposition 2.2.4. ([28] X.H.2) *Let Γ be a Schottky group on the generators $\gamma_1, ..., \gamma_n$. Then: Γ is free on the n generators; is purely loxodromic; has D as a fundamental domain; Γ is Kleinian with $\Omega(\Gamma)/\Gamma$ a finite Riemann surface.*

There is also a converse result to the above proposition. From the following theorem of Maskit we can see that we have necessary and sufficient conditions for a Kleinian group to be Schottky.

Theorem 2.2.5. ([26]) *Every Kleinian group which is purely loxodromic, finitely generated, free and having non-empty domain of discontinuity is a Schottky group.*

We can look at the effect that changing generators of our Schottky group has on a given Schottky system. Suppose our group Γ can be represented by two different sets of generators $\Gamma = \langle \gamma_1, ..., \gamma_g \rangle$ and $\Gamma = \langle \overline{\gamma}_1, ..., \overline{\gamma}_g \rangle$. The map $\gamma_i \to \overline{\gamma}_i$ extends to an automorphism $\Gamma \to \Gamma$.

There are three elementary automorphisms, the Nielsen transformations, which we look at.

(i) We can replace the first generator γ_1 by its inverse γ_1^{-1}:
$\langle \gamma_1, ..., \gamma_g \rangle \to \langle \gamma_1^{-1}, ..., \gamma_g \rangle$

(ii) We can swap the first generator γ_1 with any other generator γ_i:
$\langle \gamma_1, ..., \gamma_i, ..., \gamma_g \rangle \to \langle \gamma_i, ... \gamma_1, ..., \gamma_g \rangle$

(iii) We can replace the first generator γ_1 by the product of the first two generators:
$\langle \gamma_1, \gamma_2, ..., \gamma_g \rangle \to \langle \gamma_2 \gamma_1, \gamma_2, ..., \gamma_g \rangle$

We can then look at other generator sets for Γ as being multiple applications of Nielsen transformations (i) - (iii) on $\langle \gamma_1, \gamma_2 \rangle$ due to the following theorem of Nielsen:

Theorem 2.2.6. ([33]) *If Γ is free on $x_1, ..., x_n$ and also free on $y_1, ..., y_n$ then a finite sequence of Nielsen transformations will change $x_1, ..., x_n$ to $y_1, ..., y_n$.*

For each application of a Nielsen transformation applied to Γ we see that the Schottky system changes; either we find we have the same SG-curves as the original system, but with a new labelling or we have that some of the new SG-curves were not in the original Schottky system. We will look at the effect of the three automorphisms, (i)-(iii), in turn. Throughout this section we use C_i to refer to the SG-curves of the initial Schottky system for Γ, and K_i used for SG-curves for the Schottky system once a transformation has been applied. We begin with transformation (i).

If we look at transformation (i), we replace the first generator by its inverse:

$$\langle \gamma_1, ..., \gamma_g \rangle \to \langle \gamma_1^{-1}, ..., \gamma_g \rangle$$

In terms of the SG-curves then this can be seen as keeping the SG-curves exactly the same, but simply swapping the direction of the γ_1 arrow around, that is labelling C_1 as K_1' and labelling C_1' as K_1. This is the case because our new first generator sends C_i' to C_i now, rather than the other way around. This is shown in in Figure 2.4.

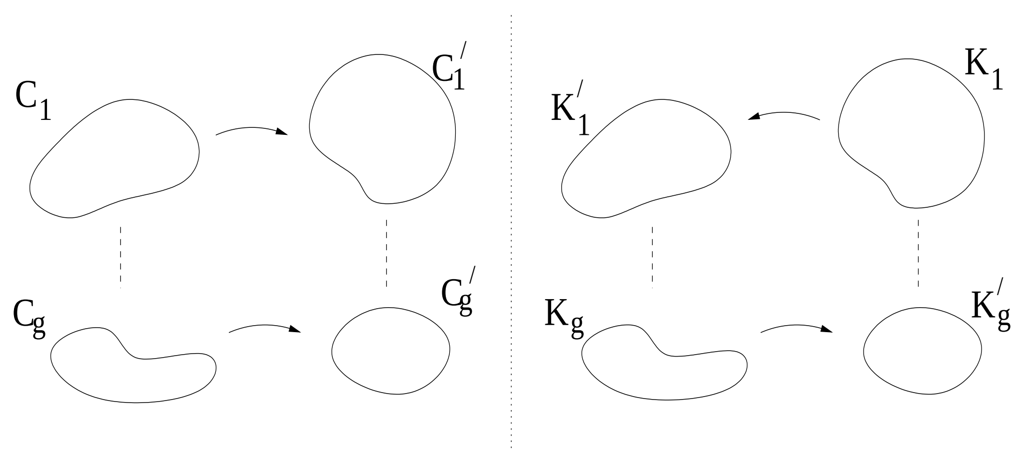

Figure 2.4: The original defining curves for a Schottky group and the new curves, after the automorphism (i).

Looking now at the second automorphism, (ii), we see this involves swapping the positions of two generators.

$$\langle \gamma_1, ..., \gamma_i, ..., \gamma_g \rangle \to \langle \gamma_i, ... \gamma_1, ..., \gamma_g \rangle$$

In terms of the SG-curves again we may keep the same SG-curves as in the initial Schottky system and just have a relabelling. We swap the subscripts on the labels for the curves, swapping i subscripts for 1 subscripts, and vice versa. Explicitly we relabel C_1 as K_i, C_1' as K_i', C_i as K_1 and C_i' as K_1' as can be seen in Figure 2.5.

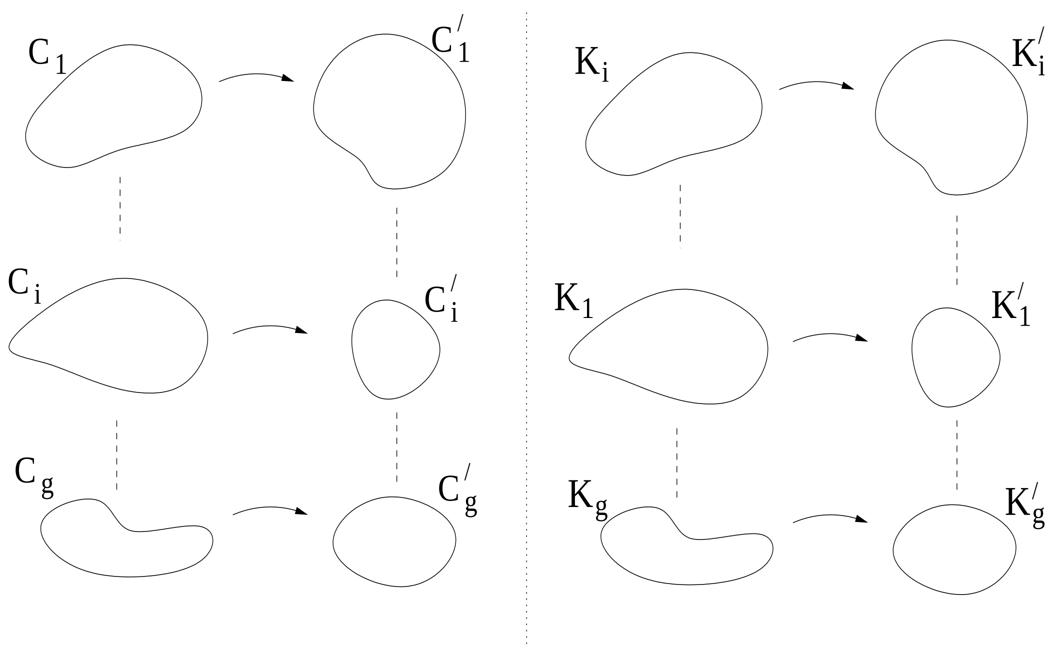

Figure 2.5: The original defining curves for a Schottky group and the new curves, after the automorphism (ii)

The third automorphism is more complicated - it actually produces a new set of curves, rather than just a relabelling. This construction of new curves comes from a paper of Chuckrow [13]. We have the transformation

$$\langle \gamma_1, \gamma_2, ..., \gamma_g \rangle \rightarrow \langle \gamma_2\gamma_1, \gamma_2, ..., \gamma_g \rangle$$

Given the SG-curves, from the initial Schottky system, for $\langle \gamma_1, \gamma_2, ..., \gamma_g \rangle$ we look for the new SG-curves for $\langle \gamma_2\gamma_1, \gamma_2, ..., \gamma_g \rangle$. The old curves and the new curves are shown on Figure 2.6, with explanations afterwards.

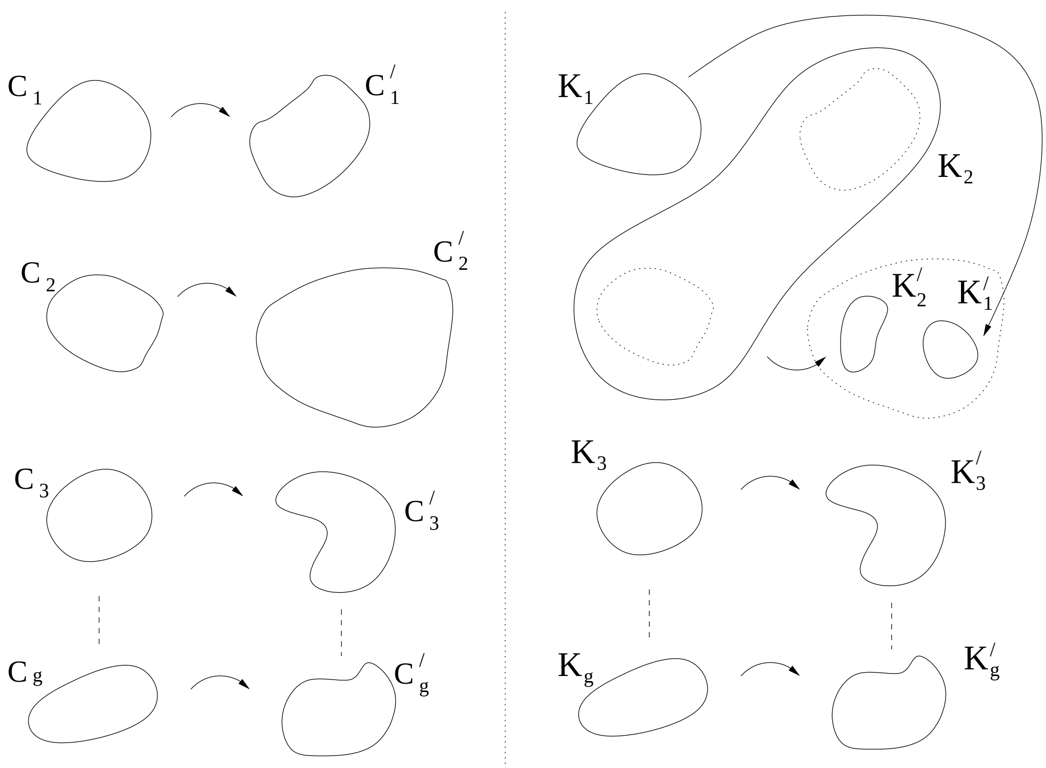

Figure 2.6: The original defining curves for a Schottky group and the new curves, after the automorphism (iii).

We take K_1 to be the curve C_1 from the initial Schottky system. Since the first generator for Γ is now $\gamma_2\gamma_1$ we can see that K_1' is by definition $\gamma_2\gamma_1(C_1)$. We can see that K_1' is therefore inside the original C_2'. We choose the new K_2 to be a curve which has C_1' and C_2 on its inside, and all other C_i curves on its outside. Then we have that by definition $K_2' = \gamma_2(K_2)$, and is thus inside C_2'. We also have that K_1' and K_2' are not nested. The remaining curves are unchanged, and so $K_i = C_i$ and $K_i' = C_i'$ for $3 \leq i \leq g$.

As can be seen from the diagram, the new curves may bear little resemblance to the old curves.

2.3 Classical Schottky Groups

We begin this section with the definition of a classical Schottky group.

Definition 2.3.1. A Schottky group, Γ is said to be *classical* if for at least one set of generators at least one set of SG-curves can be taken to be Euclidean circles in $\mathbb{C}$. That is that there exists a Schottky system which consists of Γ and Euclidean circles.

Definition 2.3.2. A particular generator of a Schottky group, γ_i, is referred to as a *classical generator* if the SG-curves C_i and C_i' for that generator are circles in $\mathbb{C}$.

There are two conditions in the definition of a classical Schottky group, the fact that we have 'at least one set of generators...' and the fact that we have 'at least one set of SG-curves...'. We shall briefly look at these conditions, and the reasons why they make deciding whether a given Schottky group is classical or not such a difficult question to answer.

2.3.1 'At least one set of SG-curves...'

For a fixed generator set for a Schottky group $\Gamma = \langle \gamma_1, ..., \gamma_g \rangle$ we can alter the SG-curves slightly and create a new set of SG-curves and hence new Schottky system. If we take the C_1 and C_2 curves and deform them, or keep the curves the same shape but move them, or a combination of both, then the image curves will also be slightly changed. As long as the fixed points of the generators of Γ and images of ∞ are still inside the new curves, and as long as the new curves do not intersect each other, or their images, then the new curves are new SG-curves by definition.

We can formalise this process as follows. Assume we have a specific Schottky system with a given set of SG-curves for our Schottky group, labelled in the usual way. We define a ring domain precisely in §2.5, but for this section we just define a ring domain to be the open region in $\mathbb{C}$ between two nested Jordan curves. By compactness arguments there exist disjoint ring domains d_i, one about each C_i, such that: the d_i do not intersect each other; the images of the ring domains under γ_i do not intersect each other; the images of the ring domains under γ do not intersect the d_i. For each d_i we can take any Jordan curve which separates the boundary curves, and these curves can be the new C_i for a different Schottky system. Their images under γ_i will be the new C_i', and we know that none of the curves will intersect since the ring domains do not intersect. We then have a set of $2g$ non-intersecting, non-nesting curves, paired by the generators of the group. These are therefore a new set of SG-curves for Γ.

Finally we give the following example which shows that a specified generator set can have more than one set of SG-curves, simply by moving one set of SG-curves.

Take our curves to be $C_1 = \{|z - 10i| = 1\}$ and $C_2 = \{|z| = 1\}$ with transformations

$$f(z) = \frac{(10 + 10i)z + 99 - 100i}{z - 10i}$$

and

$$g(z) = \frac{10z - 1}{z}$$

with C_1' and C_2' defined as $C_1' = f(C_1)$ and $C_2' = g(C_2)$. Then we have that $C_1' = \{|z-10-10i| = 1\}$ and $C_2' = \{|z-10| = 1\}$ as on the left of Figure 2.7. If we move C_1 and C_2 by $\frac{1}{2}$ to the right we see that the image curves move to

$C_1' = \{|z - \frac{32}{3} - 10i| = \frac{4}{3}\}$ and $C_2' = \{|z - \frac{32}{3}| = \frac{4}{3}\}$ as shown on the right of Figure 2.7. Also we can easily observe that the property of insides of curves going to outsides of curves etc (that is $\gamma(D) \cap (D) = \emptyset$ for $\gamma \in \Gamma = \langle f, g \rangle$ and $D = \hat{\mathbb{C}} \setminus \{\text{new curves}\}$) holds for the moved SG-curves. Thus we have two different Schottky systems for the same generator set.

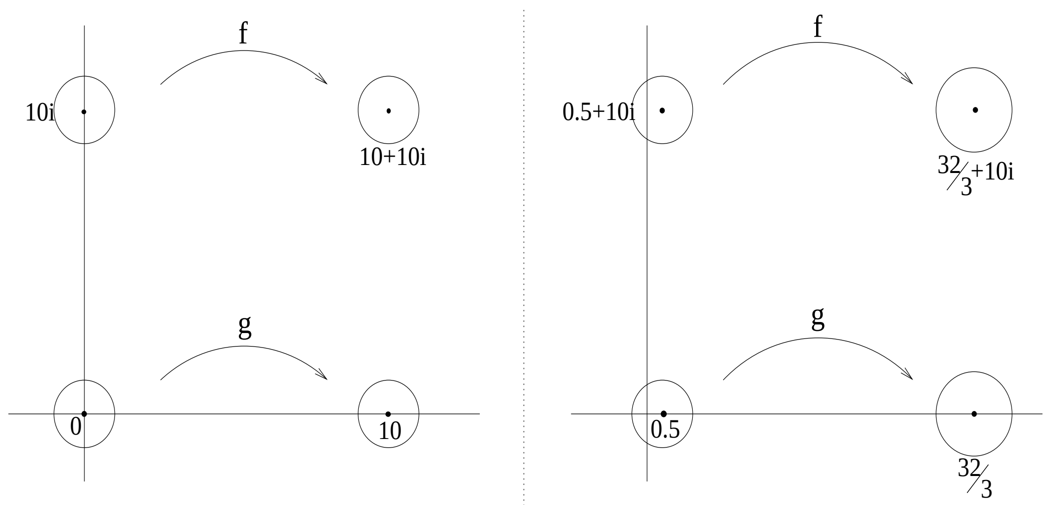

Figure 2.7: Two different sets of SG-curves for the Schottky group $\langle f, g \rangle$.

2.3.2 'At least one set of generators...'

From §2.2 we know that changing generators can alter the SG-curves quite significantly.

In a similar sense to Definition 2.3.1 we can define the notion of a Schottky group being *classical on a specific generator set* if at least one set of SG-curves for that particular generator set can be taken to be Euclidean circles. Obviously if we have that Γ is classical on a given generator set, then applications of the automorphisms (i) and (ii) ensure that Γ is classical on the new set, since these automorphisms do not change the SG-curves themselves, just the labelling. Automorphism (iii) can alter whether a Schottky group is classical on its generator set or not since it changes the curves dramatically. Clearly if Γ is classical on any specific generator set then Γ is classical. If Γ is not classical on a specific generator set then we cannot conclude that Γ is not classical. If we were to try to show that a Schottky group was not classical by analysing how changing generator sets alters SG-curves we would need to show that Γ is not classical on any of its generator sets. We discuss non-classical Schottky groups in more detail in §2.6.

2.4 Riemann Surfaces

Kleinian groups, and hence Schottky groups, are linked to Riemann surfaces as mentioned at the end of §2.1. Firstly we define a Riemann surface, via definitions of charts and an atlas.

Definition 2.4.1. A *chart* is a pair consisting of an open, simply connected region U_i on S and a homeomorphism $\phi_i : U_i \to D_i$ which maps U_i onto an open subset D_i of the complex plane $\mathbb{C}$. The homeomorphism ϕ gives a local co-ordinate system at each point on S.

Definition 2.4.2. An *atlas* is the name given to a family of charts.

When two open regions U_α and U_β on S intersect, we have that there are two images of the intersection in $\mathbb{C}$, namely $\phi_\alpha(U_\alpha \cap U_\beta)$ and $\phi_\beta(U_\alpha \cap U_\beta)$ as in Figure 2.8.

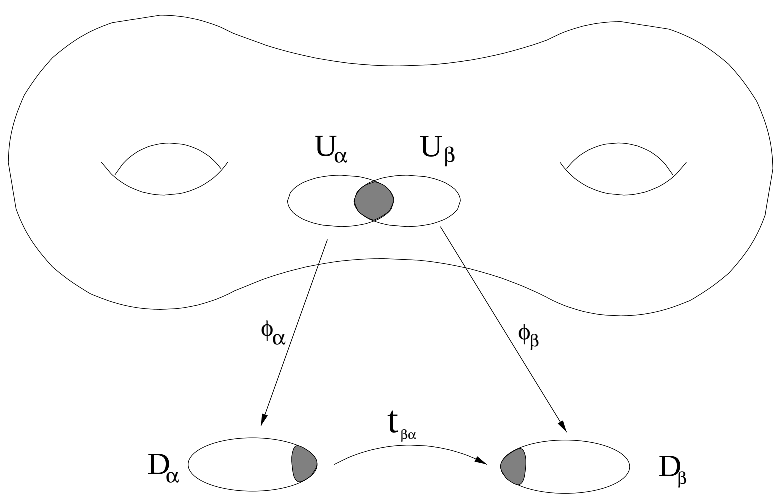

Figure 2.8: A Riemann surface, with intersecting charts and a transition function.

Definition 2.4.3. A *transition function* then takes one image of the intersection to the other by $t_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$. These functions represent the transition from one coordinate system on $(U_\alpha \cap U_\beta)$ to another.

Definition 2.4.4. A *Riemann surface* is a two-real-dimensional connected manifold, S, with a maximal atlas with analytic transition functions, that is that the transition functions are differentiable [5].

For Γ a Schottky group we have that $S = \Omega(\Gamma)/\Gamma$ is a connected Riemann surface. The proof of this can be found in [28] (II.F.6).

This link between Riemann surfaces and Schottky groups can be described as the process of using Schottky groups to uniformize closed Riemann surfaces.

Definition 2.4.5. A collection of g disjoint, homologically independent, sufficiently smooth, simple closed curves $s_1, ..., s_g$ on a closed Riemann surface S of genus g are defining curves for a Schottky uniformization, or *SU-curves*, if one can choose a Schottky group Γ, with generators $\gamma_1, ..., \gamma_g$, so that there is a fundamental region D bounded by SG-curves $C_1, C_1', ..., C_g, C_g' \in \Omega(\Gamma)$, with $\gamma_i(C_i) = C_i'$ such that s_i is the image of C_i (and C_i') under the map $\pi : \Omega(\Gamma) \to \Omega(\Gamma)/\Gamma$. We say then that S is *uniformized by a Schottky group.*

Theorem 2.4.6. Koebe Retrosection Theorem ([6],[18]) *Every closed Riemann surface can be uniformized by a Schottky group.*

We look to extend this theorem to classical Schottky groups in Chapter 3.

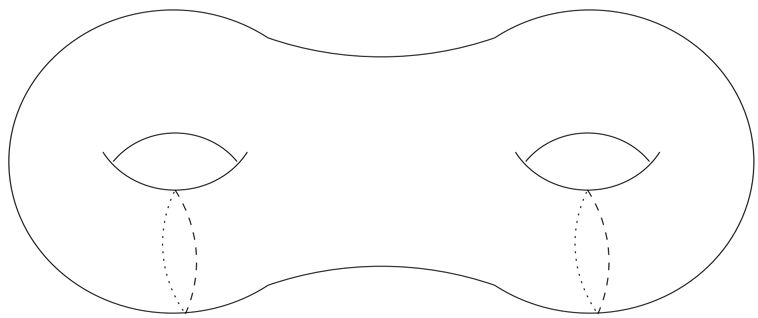

Figure 2.9: An example of SU-curves on a closed Riemann surface.

It could be that properties of the SG-curves relate to properties of the SU-curves, or of the surface, S. In [37] Sibner shows that a closed symmertic Riemann surface of genus g can be represented by a Schottky group which has a standard fundamental domain which exhibits the symmetry.

It will be necessary to look at collars about the SU-curves on S. The collar lemma states:

Theorem 2.4.7. **([11])** *Let S be a compact Riemann Surface of genus $g \geq 2$, and let $s_1, ..., s_g$ with lengths $l(s_1), ..., l(s_g)$ be pairwise disjoint simple closed geodesics on S. Then the collars*

$$\mathcal{C}(s_i) = \{p \in S \,|\, \mathrm{dist}(p, s_i) \leq w(s_i)\}$$

of widths

$$w(s_i) = \mathrm{arcsinh}\left[\frac{1}{\sinh\left(\frac{1}{2}l(s_i)\right)}\right]$$

are pairwise disjoint for $i = 1, ..., g$.

An example of a pair of collars is shown on Figure 2.10. We now look at the pre-image of such a collar, under π, in $\Omega(\Gamma)$.

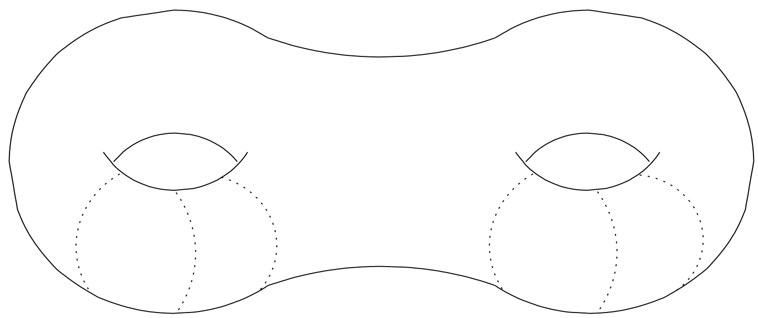

Figure 2.10: An example of a pair of collars about SU-curves on a closed Riemann surface.

2.5 Ring Domains

We first give two definitions.

Definition 2.5.1. A *ring domain* in $\mathbb{C}$ is a doubly connected domain, that is, the open region between a pair of nested disjoint Jordan curves k_1, k_2, with k_2 inside k_1 as in Figure 2.11.

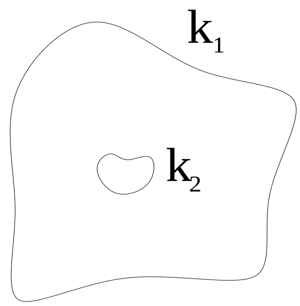

Figure 2.11: A Ring Domain.

Definition 2.5.2. An SG-curve c is said to *separate* the boundary components of a ring domain A if any line connecting one boundary component of A to the other crosses c an odd number of times.

Given a collar as described in Theorem 2.4.7, about a curve, s, on a closed Riemann surface, S, we can look at the pre-image of the collar on the domain of discontinuity, under $\pi : \Omega(\Gamma) \to \Omega(\Gamma)/\Gamma$. The collar has as pre-image an infinite collection of disjoint ring domains, paired by conjugate elements of Γ.

One such pair of ring domains will contain a pair of SG-curves, with the SG-curves separating the boundary components of the ring domain.

We briefly mention a pair of definitions of properties of ring domains.

Definition 2.5.3. A ring domain $B \subset \mathbb{C}$ is *round* if it is bounded by concentric Euclidean circles, and hence has the form $\{z \mid r < |z - c| < s\}$ for $r, s \in \mathbb{R}^+$, $c \in \mathbb{C}$. If a ring domain is round we will call it an *annulus*.

Definition 2.5.4. Given two ring domains, A and B with $B \subseteq A$, we say that B is *essential* with respect to A if B separates the boundary components of A.

We now look to define the module and modulus of a ring domain. The module and modulus are numbers assigned to a given ring domain, which measure the size of the ring domain, for some definition of size, and are conformal invariants. We shall briefly discuss conformal equivalence and invariance.

Definition 2.5.5. A map f is conformal if it preserves angles.

Examples of such functions include rotations, dilations, and in fact any Möbius tranformation.

Definition 2.5.6. Two objects A and B are *conformally equivalent* if there exists a conformal map f such that $B = f(A)$.

Definition 2.5.7. A property p, such as extremal length, module or modulus, is said to be a *conformal invariant* if, for any conformal map f, $p(A) = p(f(A))$. In other words, p is invariant under conformal maps, or that for two conformally equivalent objects A and B it is true that $p(A) = p(B)$.

Extremal length is an example of a conformal invariant, and we give some details about it here. Suppose we have a region Δ, with boundary consisting of a set of explicitly defined edges, and a set of rectifiable arcs Ξ in Δ, for example joining two edges of Δ or separating two components of the boundary of Δ. Extremal length is a property invariant under conformal mappings which we descibe shortly. We let m denote extremal length, with appropriate subscript denoting the specific region which is being considered. Let Δ' represent Δ after the conformal mapping, and let Ξ' represent Ξ after the mapping. To be a conformal invariant is the same as requiring that $m_{\Delta}(\Xi) = m_{\Delta'}(\Xi')$. We consider the family of Riemannian metrics which are conformally equivalent to the euclidean metric. We let P be the set of length elements, ρ, on the region Δ, which are then used to define metrics by $ds = \rho|dz|$. We may look at an arc $\xi \in \Xi$, which has a well defined ρ-length given by:

$$L(\xi, \rho) = \int_{\xi} \rho \, |dz|$$

The open set Δ has ρ-area given by:

$$A(\Delta, \rho) = \iint_{\Delta} \rho^2 \, dx \, dy$$

These functions are both invariant under change of metric by conformal mapping. Now we define the minimum length of any arc in Ξ for a given ρ by taking the infimum of the ρ-lengths over all possible arcs:

$$L(\Xi, \rho) = \inf_{\xi \in \Xi} L(\xi, \rho)$$

Scaling of the region Δ by a factor is a conformal map, and so extremal length must be unchanged when ρ is multiplied by a constant. We therefore take the function $L(\Xi, \rho)^2/A(\Delta, \rho)$ to form the definition of extremal length since replacing ρ by $c\rho$ for c a constant doesn't alter the value. We take the least upper bound of this function over all ρ, giving us that the extremal length of Ξ in Δ is defined [3] as:

$$m_\Delta(\Xi) = \sup_\rho \frac{L(\Xi,\rho)^2}{A(\Delta,\rho)}$$

It is useful to work through an example, to give an idea of how extremal length is calculated. If we take a rectangle R, with sides on $x = 0$, $x = a$, $y = 0$ and $y = b$ then we can calculate the extremal length of the arcs joining the vertical sides as follows. Let the set of arcs joining $x = 0$ with $x = a$ be denoted Ξ. We may get a lower bound on the extremal length of Ξ by initially choosing any metric that we like, say $\rho = 1$. In this case $L(\Xi, 1) = a$, and $A(R, 1) = ab$, and hence $\frac{L(\Xi,1)^2}{A(R,1)} = \frac{a}{b}$, and hence that

$$m_R(\Xi) \geq \frac{a}{b} \tag{2.1}$$

To find if this bound holds for all ρ we may pick an arbitrary ρ, although we may normalise by choosing for example that $L(\Xi, \rho) = a$. Therefore we have for a curve $\xi \in \Xi$:

$$\begin{aligned} L(\xi,\rho) &\geq a \\ \int_0^a \rho\,dx &\geq \int_0^a 1\,dx \\ \int_0^a (\rho-1)\,dx &\geq 0 \\ \iint_R (\rho-1)\,dx\,dy &\geq 0 \qquad (2.2) \end{aligned}$$

We may use this fact to calculate the area $A(R, \rho)$. It is clear that we have:

$$\iint_R (\rho-1)^2\,dx\,dy \geq 0 \tag{2.3}$$

Expanding the brackets in Equation (2.3), rearranging, and substituting in the result of Equation (2.2) we get:

$$\begin{aligned} A(R,\rho) = \iint_R \rho^2 \, dx\, dy &\geq \iint_R (2\rho - 1)\, dx\, dy \\ \iint_R \rho^2 \, dx\, dy &\geq 2\iint_R (\rho - 1)\, dx\, dy + \iint_R 1\, dx\, dy \\ A(R,\rho) &\geq \iint_R dx\, dy \\ &= ab \end{aligned}$$

So we have that for all ρ:

$$m_R(\Xi) = \sup_\rho \frac{L(\Xi,\rho)^2}{A(\Delta,\rho)} \leq \frac{a}{b} \tag{2.4}$$

We therefore can combine Equations (2.1) and (2.4) to see that $m_R(\Xi) = \frac{a}{b}$.

We return now to the concept of the modulus of a ring domain. In the literature the modulus can be defined in two equivalent ways, both of which are worth mentioning here. There is some difference of normalisation in the texts in this area, the main difference being a factor of 2π. The three sources I use here are Herron, Liu and Minda [15], Lehto and Virtanen [22] and McMullen [32]. These sources use the same notation for slightly different definitions, so I will use the subscripts HLM, LV and M respectively to denote which text the term relates to. Module is denoted by $M(A)$ for a ring domain A, and modulus will be denoted by $mod(A)$, with appropriate subscripts. The function μ to be defined later will also carry subscripts to denote source where appropriate. We will be using the definitions of Herron, Liu and Minda predominantly, but there is value in looking at all the different definitions and checking that they are consistant.

Definition 2.5.8. ([22]) A general ring domain A is conformally equivalent to an annulus of the form $\{z \mid r_1 < |z| < r_2\}$; let f be the map from A to such an annulus. The *module* of a ring domain is defined as $M_{LV}(A) = \ln\left(\frac{r_2}{r_1}\right)$.

Equivalently A is conformally equivalent to an annulus $A(R)$, that is, there exists an angle preserving map from A to $A(R)$, $A(R)$ defined as $A(R) = \{z \mid 1 < |z| < R\}$ for a unique R. The *module* of the ring domain is then defined to as $M_{LV}(A) = \ln(R)$

A second definition of module can be constructed as below. We firstly set up notation used throughout this section, and in Chapter 3.

Remark 2.5.9. ([22], I 6.2) Let $(-A)_1$ and $(-A)_2$ denote the components of the complement of A, and we can let $\mathscr{C}$ be the set of curves separating $(-A)_1$ and $(-A)_2$.

We now go on with the alternate definition. Then let $\mathscr{P}$ be the family of all possible metrics on A, defined by line elements ρ dz where ρ is a non-negative function on A such that the metrics are continuous enough for the following integrations, and that $\mathscr{P}$ contains the metric defined by the function $|f'/f|$, for f as defined in Definition 2.5.8. For $\rho \in \mathscr{P}$, let

$$a_\rho(A) = \iint_D \rho^2 \, \mathrm{d}\sigma$$

where $\mathrm{d}\sigma$ is the area element on A. Also for a curve $C \in \mathscr{C}$ we define

$$l_\rho(C) = \int_C \rho \, |\mathrm{d}z|$$

and $|\mathrm{d}z|$ is the length element. The alternate definition for module can then be expressed as

$$M_{LV}(A) = \inf_{\rho \in \mathscr{P}} \frac{2\pi a_\rho(A)}{\left(\inf\limits_{C \in \mathscr{C}} l_\rho(C)\right)^2}$$

This alternate definition will be useful later.

Definition 2.5.10. The *modulus* of a ring domain, A, which is conformally equivalent to an annulus $\{z \mid 1 < |z| < R\}$, is defined in [32] as $mod_M(A) = \frac{\ln R}{2\pi}$.

Thus we have that $mod_M(A) = M_{LV}(A)/2\pi$. Herron, Liu and Minda give the same definition of modulus, so $mod_M(A) = mod_{HLM}(A)$. It is worth noting then that from the above we have

$$mod_M(A) = mod_{HLM}(A) = \inf_{\rho \in \mathscr{P}} \frac{a_\rho(A)}{\left(\inf\limits_{C \in \mathscr{C}} l_\rho(C)\right)^2}$$

Calculating the modulus of a ring domain directly is not necessarily easy, but there exist bounds based on the modulus of a specific domain defined below.

Definition 2.5.11. *Grötzsch's Extremal Domain* is a domain, denoted $B(r)$, which is conformally equivalent to a ring domain (as detailed below), and has as its boundary the unit circle $|z| = 1$ and the segment of the real axis $0 \leq x \leq r$, $r < 1$. Its modulus is denoted $\mu(r)$.

Grötzsch's extremal domain is conformally equivalent to a ring domain, and in fact conformally equivalent to an annulus. The conformal map which maps $B(r)$ to an annulus is constructed as follows. We may map the upper half of Grötzsch's Extremal Domain conformally to the upper half of the particular annulus defined by $1 < |z| < e^{\mu_{LV}(r)}$. This map can then be extended to a map on the whole of $B(r)$ by following this with a reflection in the real axis [22]. We have Grötzsch's module theorem, given below, which

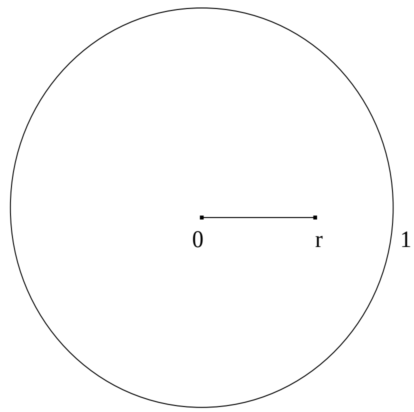

Figure 2.12: Grötzsch's Extremal Domain.

gives us a bound on the modulus (and hence module) of any ring domain which is essential with respect to Grötzsch's extremal domain.

Theorem 2.5.12. **([15])** *If a ring domain A separates the points 0 and r from the unit circle then $mod_{HLM}(A) \leq \mu_{HLM}(r)$.*

The μ function is discussed in all three texts and again there is some difference in normalisation. Grötzsch's extremal domain is denoted $B(r)$ in Lehto and Virtanen. They define the *module* of Grötzsch's extremal domain to be $M_{LV}(B(r)) = \mu_{LV}(r)$. In [15] the domain is defined in the same way, denoting it $R_G(r)$, so $B(r) = R_G(r)$. They denote its *modulus* using μ, so I will denote it as $\mu_{HLM}(r)$. Therefore we can write the following

$$\begin{aligned}
\mu_{LV}(r) &= M_{LV}(B(r)) \\
&= 2\pi mod_{HLM}(B(r)) \\
&= 2\pi mod_{HLM}(R_G(r)) \\
&= 2\pi \mu_{HLM}(r)
\end{aligned}$$

McMullen doesn't use Grötzsch's extremal domain to introduce μ but we can easily see that $\mu_{LV}(r) = \mu_M(r)$.

In order to get a better bound on the module or modulus of a general ring domain than the one in Theorem 2.5.12 we introduce Teichmüller's extremal domain.

Definition 2.5.13. *Teichmüller's Extremal Domain* is a domain in $\hat{\mathbb{C}}$ which has as its boundary the segment of the real axis $-r_1 \leq x \leq 0$ and the segment $r_2 \leq x \leq \infty$. Its modulus is given in terms of the modulus of Grötzsch's extremal domain as $2\mu_{HLM}\left(\sqrt{\frac{r_1}{r_1+r_2}}\right)$.

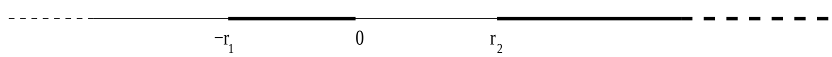

Figure 2.13: Teichmüller's Extremal Domain.

From this we can get Teichmüller's module theorem, which appears in Lehto and Virtanen ([22], II 1.3) and in McMullen ([32], pg 11). As before we take a ring domain A with complement $(-A)_1$ and $(-A)_2$, where $0 \in (-A)_1$ and $\infty \in (-A)_2$. Then to bound the modulus of A we simply need a point inside $(-A)_1$ and a point outside $(-A)_2$. We have:

Theorem 2.5.14. ([22],[32]) *If a ring domain B separates the points 0 and z_1 from z_2 and ∞, then we have*

$$M_{LV}(B) \leq 2\mu_{LV}\left(\sqrt{\frac{|z_1|}{|z_1|+|z_2|}}\right)$$

and in the notation of McMullen

$$mod_M(B) \leq \frac{1}{\pi}\mu_M\left(\sqrt{\frac{|z_1|}{|z_1|+|z_2|}}\right)$$

Using the conversions above, an equivalent formula can be written in the notation of Herron, Liu and Minda and so the version of the theorem we will use is:

Theorem 2.5.15. *If a ring domain B separates the points 0 and z_1 from z_2 and ∞, then we have*

$$mod_{HLM}(B) \leq 2\mu_{HLM}\left(\sqrt{\frac{|z_1|}{|z_1|+|z_2|}}\right)$$

We can see that the three different sets of notation are actually consistent with each other. We may convert from one to the other using rules such as $\mu_{LV} = 2\pi\mu_{HLM}$ as derived earlier, or $mod_M = mod_{HLM}$. Table 2.5 shows the different definitions based on the sources used, and after each definition gives a way to convert between the three texts.

We now need to define the μ function so that we can constructively use Teichmüller's module theorem. From here on, we will use the HLM definition of μ and mod, without explicit use of subscripts.

Remark 2.5.16. This function $\mu(r)$ is defined in terms of elliptic integrals, in that

$$\mu(r) = \frac{1}{4}\frac{K(\sqrt{1-r^2})}{K(r)}$$

where

$$K(r) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-r^2x^2)}}$$

There exist some useful bounds on the behaviour of $\mu(r)$ in [15], [22] and [34]. For instance $\mu(r) < \frac{1}{2\pi}\ln(\frac{4}{r})$ and $\mu(r) < \frac{1}{2\pi}\ln\frac{2(1+\sqrt{1-r^2})}{r}$ but we will be using the elliptic integral definition predominantly. This particular elliptic integral is known as the complete elliptic integral of the first kind.

	HLM	LV	M
Module		$M(A) = \ln(R)$	
Modulus	$mod(A) = \frac{1}{2\pi}\ln(R)$		$mod(A) = \frac{1}{2\pi}\ln(R)$
Conversion	$= mod_{HLM}(A)$	$M_{LV}(A) = 2\pi mod_{HLM}(A)$	$mod_M(A) = mod_{HLM}(A)$
$\mu(r)$	Modulus of GED	Module of GED	"positive decreasing function of r"
Conversion	$= \mu_{HLM}(r)$	$\mu_{LV}(r) = 2\pi\mu_{HLM}(r)$	$\mu_M(r) = 2\pi\mu_{HLM}(r)$
Teichmüller's Module Theorem	$mod(A) \leq 2\mu\left(\sqrt{\frac{\lvert z_1\rvert}{\lvert z_1\rvert+\lvert z_2\rvert}}\right)$	$M(A) \leq 2\mu\left(\sqrt{\frac{\lvert z_1\rvert}{\lvert z_1\rvert+\lvert z_2\rvert}}\right)$	$mod(A) \leq \frac{1}{\pi}\mu\left(\sqrt{\frac{\lvert z_1\rvert}{\lvert z_1\rvert+\lvert z_2\rvert}}\right)$

Table 2.1: Summarising the concepts of module, modulus and μ, with notes for comparing the texts.

Finally we will discuss the extremal length of the family of curves $\mathscr{C}$ mentioned in the second definition of the module of a ring domain.

For a ring domain extremal length can be rewritten in terms of the terms defined in Section 2.4. We can see quite easily that the terms in the definition of extremal length can be replaced by those in the definition of module, and hence modulus. We see that for a ring domain A with set of curves $\mathscr{C}$ as previously defined we have a definition for extremal length in terms of components of the definition of modulus:

$$m_A(\mathscr{C}) = \sup_{\rho \in \mathscr{P}} \frac{\left(\inf\limits_{C \in \mathscr{C}} l_\rho(C)\right)^2}{a_\rho(A)}$$

Thus we can see that

$$m_A(\mathscr{C}) = \frac{1}{mod(A)}$$

2.6 Non-Classical Schottky Groups

We return now to non-classical Schottky groups. It is obvious that classical Schottky groups exist, simply by construction, but it is not obvious that non-classical Schottky groups exist. In his paper [25] Marden proved the existence of non-classical Schottky groups using Schottky space. We give two equivalent definitions of Schottky space, and then briefly discuss Marden's proof, and survey what is known in this area.

Definition 2.6.1. The *Schottky space* of a given genus g is denoted S_g and is the set of all equivalence classes of Schottky groups with g generators, where two Schottky groups $\Gamma = \langle \gamma_1, ..., \gamma_g \rangle$ and $\Gamma' = \langle \gamma_1', ..., \gamma_g' \rangle$ are equivalent if there exists a Möbius transformation f with $f\gamma_i f^{-1} = \gamma_i'$ for all $i = 1, ..., g$. We put a topology on S_g by requiring that the equivalence class $[G_n]$ converges to $[G]$ iff there exists $\langle \gamma_1, ..., \gamma_g \rangle \in [G]$ and $\langle \gamma_{1,n}, ..., \gamma_{g,n} \rangle \in [G_n]$ such that $\gamma_{i,n}$ converges to γ_i, [29],[30].

In this sense we may think of a point in Schottky space as being a set of free generators for a Schottky group, modulo conjugation in $\mathrm{PSL}_2(\mathbb{C})$. Alternatively we may define Schottky space in relation to Riemann surfaces uniformized by Schottky groups.

Definition 2.6.2. [17] *Schottky space* can be defined as the set of equivalence classes of pairs (X, σ), where X is a Riemann surface of genus g and $\sigma : \Phi_g \longrightarrow \mathrm{PSL}_2(\mathbb{C})$ is an injective homomorphism, where Φ_g is the free group on $\gamma_1, ..., \gamma_g$, where $\Gamma := \sigma(\Phi_g)$ is a Schottky group, and where $\Omega(\Gamma)/\Gamma \cong X$. We have that (X, σ) and (X', σ') are equivalent if there exists some $A \in \mathrm{PSL}_2(\mathbb{C})$ with $\sigma(\gamma_i') = A\sigma(\gamma_i)A^{-1}$ for all $i = 1, ..., g$. We have then that X' is isomorphic to X.

We define classical Schottky space in a similar way to the first definition of Schottky space, that is the set of equivalence classes represented by classical Schottky groups with g generators. We say that a Schottky group Γ' is equivalent to a classical Schottky group Γ if there exists a Möbius transformation f with $f\gamma_i f^{-1} = \gamma_i'$ for all $i = 1, ..., g$. Note we have that if the γ_i have classical SG-curves then we do not necessarily have that γ_i' has classical SG-curves. We let S_g^c denote classical Schottky space.

In his paper, [25], Marden compares Schottky space with classical Schottky space, to show that non-classical Schottky groups exist. He shows that the intersection of the closure of classical Schottky space with Schottky space is not the whole of Schottky space, and hence there are Schottky groups which are not classical. We briefly summarise the idea of his main proof. Firstly Marden shows that if G is a group in the closure of classical Schottky space then G is discontinuous. Marden then takes a Schottky group H on the boundary of Schottky space which is not discontinuous; this group exists through a result of Chuckrow [13]. He then chooses a sequence G_n of Schottky groups, which are not on the boundary of classical Schottky space, and whose limit is H. He shows that at most a finite number of the G_n lie in classical Schottky space, and hence the remaining groups lie in $S_g - (\overline{S_g^c} \cap S_g)$, which is therefore non-empty, and therefore there exist non-classical Schottky groups.

It is worth mentioning here about Teichmüller and moduli space, and their links to Schottky space. In a similar vein to the second definition of

Schottky space we define these spaces in terms of surfaces. A simple way to define Teichmüller and moduli spaces involves first defining the mapping class group, [38].

Definition 2.6.3. Let $\mathrm{Diff}(X)$ be the set of orientation-preserving diffeomorphisms of a surface X, and let $\mathrm{Diff}_0(X)$ be the set of those diffeomorphisms isotopic to the identity. We define the *mapping class group* to be $\mathrm{MCG}(X) = \mathrm{Diff}(X)/\mathrm{Diff}_0(X)$.

We may also define the mapping class group in terms of the uniformizing group Γ:

Definition 2.6.4. *[10]* For a Schottky group Γ, we let the group of orientation preserving automorphisms be denoted by $\overline{Aut}(\Gamma)$ (where an orientation preserving automorphism is one which corresponds to an orientation preserving diffeomorphism). We then have that $\mathrm{MCG}(\Gamma) = \overline{Aut}(\Gamma)/Inn(\Gamma)$.

For a given oriented surface X we let $\mathcal{M}(X)$ denote the set of all complex structures on X which agree with the differentiable structure on X. We can then define the moduli space, M_g, and Teichmüller space, T_g, as follows:

Definition 2.6.5. *Moduli space*, M_g, of a surface X of genus g, is defined by $M_g = \mathcal{M}(X)/\mathrm{Diff}(X)$. Moduli space is therefore the space of all equivalence classes of compact Riemann surfaces of genus g, where two surfaces are equivalent if there is conformal diffeomorphism between them.

Definition 2.6.6. *Teichmüller space*, T_g, of a surface X of genus g, is defined by $T_g = \mathcal{M}(X)/\mathrm{Diff}_0(X)$. Teichmüller space is therefore the space of all equivalence classes of compact Riemann surfaces of genus g where two surfaces are equivalent if there is a conformal diffeomorphism between them which is isotopic to the identity.

A simple example to show the difference between the two is to consider a surface of genus $g > 0$ with a given complex structure. We consider the surface obtained by cutting one of the handles in two, twisting one section by 2π and then glueing them back together. This new surface would correspond to the same point as the original surface in moduli space, but would be a different point in Teichmüller space. This is because the operation described above (a "Dehn twist") is not isotopic to the identity but the surface obtained by the twist is still equivalent in moduli space.

It is simple to see that moduli space and Teichmüller space are related by $M_g = T_g/\mathrm{MCG}(X)$. The link with Schottky space is explained in [29]. As discussed in §2.4 we have a Riemann surface uniformized by any Schottky group. Let S be the surface uniformized by Γ. Looking at the generators of $\pi_1(S)$ we can see that a particular generator may also be a generator of Γ or not. Let N be the smallest normal subgroup of $\pi_1(S)$ containing those generators of $\pi_1(S)$ which are not generators of Γ. We can then define $\mathcal{N}_{alg}$ as the subgroup of the mapping class group which is the subgroup of outer automorphisms $\phi : \pi_1(S) \longrightarrow \pi_1(S)$ with the properties that $\phi(N) = N$ and that the induced isomorphism $\phi : \pi_1/N \longrightarrow \pi_1/N$ is the identity. Schottky space is then $S_g = T_g/\mathcal{N}_{alg}$.

We can define another space, known as unmarked Schottky space as follows.

Definition 2.6.7. Let $\mathcal{N}_{un}$ be the subgroup of the mapping class group consisting of all outer automorphisms $\phi : \pi_1(S) \longrightarrow \pi_1(S)$ with the property that

$\phi(N) = N$. *Unmarked Schottky space* is then defined as $S_g^{un} = T_g/\mathcal{N}_{un}$.

A point in unmarked Schottky space is a Schottky group modulo conjugation in $\mathrm{PSL}_2(\mathbb{C})$ [29].

Finally, in [29] Maskit defines S_g^{top}, which is topological Schottky space. This is done in a similar way, by defining another subgroup of the mapping class group.

Definition 2.6.8. Let $\mathcal{N}_{top}$ be the subgroup of the mapping class group consisting of all outer automorphisms $\phi : \pi_1(S) \longrightarrow \pi_1(S)$ with the following property. If $a_1, ..., a_g$ denote the generators of $\pi_1(S)$ which are not generators of Γ then $\phi(a_i)$ is conjugate to a_i for all $i = 1, ..., p$. We then define the *topological Schottky space* as $S_g^{top} = T_g/\mathcal{N}_{top}$.

A point in topological Schottky space can be regarded as consisting of a Schottky group with a fundamental domain bounded by $2g$ SG-curves. We can see that points in S_g^{top} carry the most information, then points in S_g, and then points in S_g^{un} which carry the least information about the group.

From the nature of defining these Schottky spaces using subgroups of the mapping class group we have a tower of coverings as shown below:

$$T_g \to S_g^{top} \to S_g \to S_g^{un} \to M_g$$

Work on the nature of these coverings can be found in [29], [30] and others.

The first concrete example of a non-classical Schottky group was given by Yamamoto [40]. An earlier example given by Zarrow [41] was then shown to be classical by Sato [35]. Yamamoto's group is a two generator Schottky group, with SG-curves C_1 the rectangle with corners $\sqrt{2}-1+i(1-\varepsilon/3)$, $\sqrt{2}-$

$1 - i(1-\varepsilon/3)$, $-\sqrt{2}+1+i(1-\varepsilon/3)$ and $-\sqrt{2}+1-i(1-\varepsilon/3)$, and C_1' is defined as $\gamma_1(C_1)$, $C_2 := \{|z+\sqrt{2}| = 1-\varepsilon\}$, $C_2' := \{|z-\sqrt{2}| = 1-\varepsilon\}$. A schematic picture of this arrangement is shown in Figure 2.14.

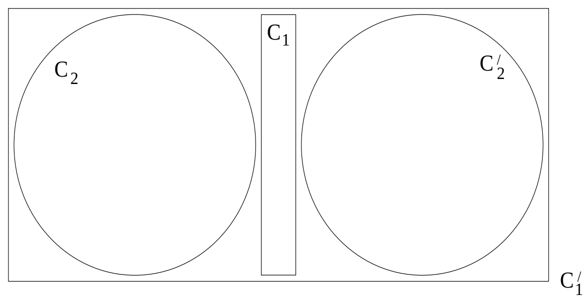

Figure 2.14: The defining curves for Yamamoto's non-classical Schottky group.

The two transformations are

$$\gamma_1(z) = i(\sqrt{2}+1)z$$

and

$$\gamma_2(z) = \frac{\sqrt{2}(1-\varepsilon)^{-1}z + (1-\varepsilon)(2(1-\varepsilon)^{-2}-1)}{(1-\varepsilon)^{-1}z + \sqrt{2}(1-\varepsilon)^{-1}}$$

and it is shown that for $\varepsilon < 10^{-20}$ Γ_ε is a non-classical Schottky group. This paper will be looked at in more detail in §4.1, and generalised in §4.2.

Finally in this section we discuss the idea of how classical a Schottky group can be. By definition a Schottky group is classical if it is classical on at least one set of generators. We can ask the question now as to, in some sense, how classical a Schottky group can be, or what it means for one Schottky group to be more classical than another. If a Schottky group is classical on a generator set then the group is classical, but equally if a Schottky group is classical on many different generator sets it is still classical. The natural question to ask is how many different generator sets can a Schottky group be classical on? Does there exist a Schottky group Γ_c which is classical on all generator sets? We refer to this group Γ_c as über-classical.

As mentioned previously, only one Nielsen transformation alters the SG-curves themselves, the transformation (iii) in Section 2.2. For a Schottky group Γ to be classical on all generator sets then first of all it would have to be classical on $\langle \gamma_2\gamma_1, \gamma_2 \rangle$, and also $\langle \gamma_2\gamma_2\gamma_1, \gamma_2 \rangle$. By the procedure of Chuckrow, as in Figure 2.6, this only allows for certain configurations. In the notation of §2.3 we would need to be able to find a circle K_2 around C_1' and C_2, which does not intersect K_1, K_1' or K_2'. This is possible, but we would also require that there was a circle around C_2 and C_1' so that $\langle \gamma_1\gamma_2, \gamma_1 \rangle$ is classical, and so on. These restrictions from the Chuckrow construction prohibit many arrangements of original SG-curves.

Aside from the difficulties of the Chuckrow construction there are also certain restrictions on how a generator set can have classical SG-curves, for instance locations of fixed points and $\gamma_i(\infty)$. For example in a two generator Schottky group $\Gamma = \langle \gamma_1, \gamma_2 \rangle$, with SG-curves C_1, C_1', C_2 and C_2', we have that both $\gamma_1^{-1}(\infty)$ and the repulsive fixed point of γ_1 must be on the inside of C_1. These points, along with similar restrictions for C_1', C_2 and C_2', can be positioned such that no classical SG-curves exist for those generators, but it is more difficult to show that for every group there is at least one generator set where such arrangements arise.

In [12] Button shows that a Fuchsian Schottky group is über-classical iff it has two generators which have intersecting axes.

Of course the equivalent question regarding non-classicality is much more simple. If we want to look for an über-non-classical Schottky group then we require that it is non-classical on all possible generator sets. But of course for a group to be non-classical it must be non-classical on all possible generator sets. Therefore any non-classical Schottky group is über-non-classical.

Chapter 3

Uniformization by Classical Schottky Groups

3.1 Theorem

As discussed in §2.4 Koebe's Retrosection Theorem states that every closed Riemann surface can be uniformized by a Schottky group. As we have also mentioned, Schottky groups exist in two distinct types, classical and non-classical. An interesting question to ask would be whether every closed Riemann surface can be uniformized by a classical Schottky group, or conversely what features do Riemann surfaces that are uniformized only by non-classical Schottky groups have. In this section we try to extend Koebe's theorem by looking at what can be said about surfaces uniformized by classical Schottky groups.

Work on Schottky uniformizations of surfaces has been done, particularly on those with certain symmetries, by people such as Hidalgo [16]. From

§2.6 we know Marden showed that non-classical Schottky groups exist, and an explicit example of a non-classical Schottky group is due to Yamamoto [40]. The natural question to ask is whether Koebe's theorem holds if we restrict to classical Schottky groups, that is Schottky groups where some set of defining curves can be taken to be circles.

In [29] Maskit states that a surface of genus p with p sufficiently small homologously independent simple disjoint geodesics can be uniformized by a classical Schottky group. We prove this in this chapter, with a numerical estimate to formalise 'sufficiently small'. We define the *Schottky uniformization constant*, k, to be the smallest positive solution of the following equation:

$$e^x \sin\left(\frac{x}{4}\right) + \sin\left(\frac{x}{4}\right) = 2e^{\frac{x}{2}} \tag{3.1}$$

We then have the following theorem:

Theorem 3.1.1. *Let S be a closed Riemann surface of genus $g \geq 2$, and let $s_1, ..., s_g$ be the defining curves for a Schottky uniformization. If these curves have length less than the Schottky uniformization constant, k, then, independent of the genus g, there exists a classical Schottky uniformization of S.*

Proof. First we give a brief outline of the proof. The aim is to show that given a condition on the length of curves on a closed Riemann surface, S, we are able to show the existence of a set of Euclidean circles which are SG-curves for a classical Schottky uniformization of S.

The surface will have a Schottky uniformization, $S = \Omega(\Gamma)/\Gamma$ by Koebe [18], and hence we have a set of SU-curves $s_1, ..., s_g$ on S and corresponding

SG-curves $C_1, C'_1, ..., C'_g$ on $\Omega(\Gamma)$, not necessarily circles. These SG-curves along with Γ form our initial Schottky system.

The aim of our proof is to show that for each given SG-curve C_i with a corresponding SU-curve s_i there exists a Euclidean circle on $\Omega(\Gamma)$ which is mapped to a curve s' on S such that s' is homotopic to s_i. If this is the case then we can use this circle as our new SG-curve in a new Schottky system for Γ. That is that s' will be an SU-curve for the Schottky uniformization with a classical generator.

We take each SU-curve, α, in the uniformization individually and find a classical SG-curve for it in the following way. We look at the collar about α on S and lift it to a pair of ring domains in $\Omega(\Gamma)$, one about C_i, call this A, and the the other about C'_i. Figure 3.1 shows an example.

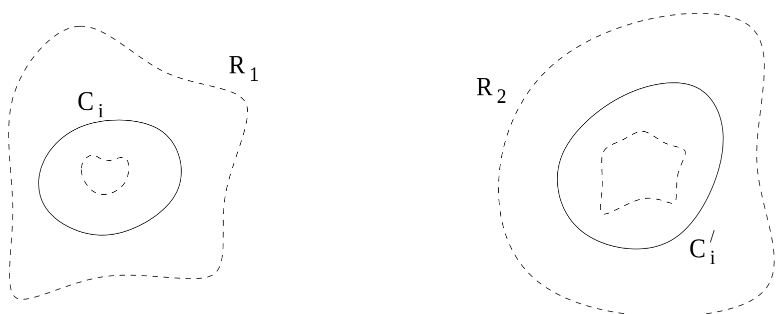

Figure 3.1: An example of the lift of a collar under $\Omega(\Gamma) \to \Omega(\Gamma)/\Gamma$ to a pair of ring domains R_1 and R_2 with C_i and C'_i respectively as separating curves

Given a condition on the length of α, this can then be adapted through work of Maskit to give a condition on the extremal length of the family of curves separating the components of the boundary of A. We will then relate extremal length to the modulus of A, and from this show that this bound on modulus gives us that there exists a circle in $\Omega(\Gamma)$ which can be mapped to a curve homotopic to α on S.

We are now able to look at the details of the proof. Let k be the smallest positive solution of $e^x \sin\left(\frac{x}{4}\right) + \sin\left(\frac{x}{4}\right) = 2e^{\frac{x}{2}}$. If we look at the rearranged equation $x = 4\arcsin\left(\frac{2e^{x/2}}{e^x+1}\right)$ we see that there is only one solution, namely the Schottky uniformization constant, k. We use this rearranged form later. Using Maple [24] we can use numerical methods to solve this rearranged form, and get that $k \approx 2.371776$. We assume that $l(s_i) < k$ for all s_i in the set of SU-curves on S and we want to show that this implies existence of a set of circular SG-curves.

If all the s_i on S lift to circles in $\Omega(\Gamma)$ then S is uniformized by a classical Schottky group. If this is not the case then some SU-curves do not lift to circles in $\Omega(\Gamma)$, so we can take one such curve and call it α. There exists a collar about α which then lifts to a pair of ring domains in $\Omega(\Gamma)$, one about some C_i and the other about C_i'. Let R be the ring domain about C_i. Using the notation of [22] and Remark 2.5.9 we can assume, through conjugation by an element of $PSL_2(\mathbb{C})$ if necessary, that $\overline{\mathbb{C}} - R$ consists of $(-R)_1$ and $(-R)_2$, with $0 \in (-R)_1$ and $\infty \in (-R)_2$. Let $z_1 \in (-R)_1$ be the (not necessarily unique) point which maximises $|z|$ over $(-R)_1$. Similarly let $z_2 \in (-R)_2$ be the (not necessarily unique) point which minimises $|z|$ over $(-R)_2$. We can see that for a circle to 'fit' inside a ring domain it suffices to show that $|z_1| < |z_2|$. If this is the case then clearly R contains the essential round annulus $B = \{z \mid |z_1| < |z| < |z_2|\}$. Examples of $|z_1| < |z_2|$ and $|z_1| > |z_2|$ are given in Figure 3.2. In the first we have an essential round annulus, and hence a circle fits inside the ring domain. In the second we see that $|z_1| > |z_2|$

and hence the round annulus B may not exist.

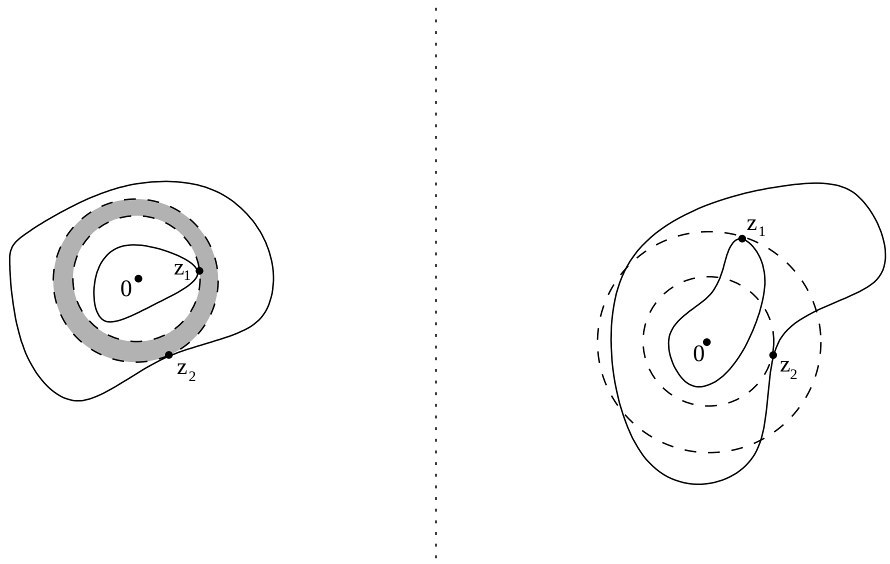

Figure 3.2: Two examples of ring domains, one which admits an essential round annulus centered at 0, and one which does not.

We want to find a link from a bound on $l(s_i)$ to the existence of annuli. To begin we use a paper of Maskit [27] to compare the hyperbolic length l with the extremal length, m, of the curve family $\mathcal{C}$ consisting of all curves separating $(-R)_1$ and $(-R)_2$. The paper of Maskit uses a different realization of S, rather than using $S = \Omega(\Gamma)/\Gamma$ to define S, it can also be written as a quotient of $\mathbb{H}^2$ as $S = \mathbb{H}^2/\Gamma$. A geodesic on S is lifted by the covering map to a set of hyperbolic lines $\gamma(\beta)$ for $\gamma \in \Gamma$ and β a hyperbolic line. Up to conjugation of the group, we can assume that β is a Euclidean straight line in $\mathbb{H}^2$. A collar of the type discussed in Theorem 2.4.7 about the geodesic β on S is symmetrical, and hence will be lifted to a symmetrical collar of the form $\left\{\frac{\pi}{2} - \phi < \arg z < \frac{\pi}{2} + \phi\right\}$ as in Figure 3.3.

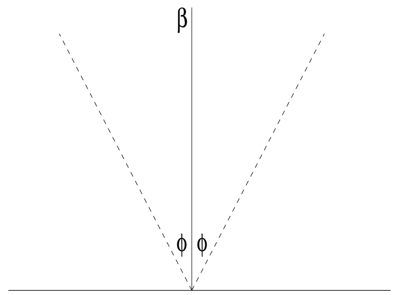

Figure 3.3: A topological collar about β

We have from [27] that $l \geq m\theta$ where the angle θ is the angle width for the collar, so, in Figure 3.3, we have that $\theta = 2\phi$.

The collar on S lifts to a collar in $\mathbb{H}^2$ as in Figure 3.4, where the sides of the collar are at a hyperbolic distance of $w = w(\alpha)$ from the vertical hyperbolic line.

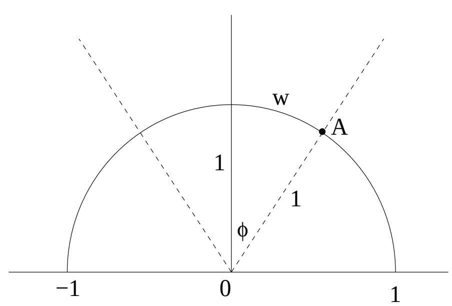

Figure 3.4: The topological collar

The point A, of distance w along the hyperbolic line through $-1, i, 1$ is

$$A = \frac{e^{2w} - 1}{e^{2w} + 1} + \frac{2e^w}{e^{2w} + 1} i$$

We can see that $\sin\phi = \mathrm{Re}(A)$, and hence $\sin\phi = \frac{e^{2w}-1}{e^{2w}+1}$. Now we want this angle in terms of l not w, so first we need to look at the definition of w from the collar lemma (Theorem 2.4.7), which states that

$$w = \operatorname{arcsinh}\left[1/\sinh(\tfrac{l}{2})\right]$$

First of all we let $x = 1/\sinh\frac{l}{2}$ so that we have $w = \operatorname{arcsinh}(x)$. Now we can rewrite w in terms of the logarithmic definition of arcsinh, so we have that $w = \ln(x + \sqrt{x^2+1})$. Also we have the standard exponential definition of sinh, which gives us that $x = 1/\sinh(\frac{l}{2}) = \dfrac{2}{e^{l/2} - e^{-l/2}}$. Combining all of this we get

$$\begin{aligned}
\sin\phi &= \frac{e^{2w}-1}{e^{2w}+1} \\
&= \frac{(x+\sqrt{x^2+1})^2 - 1}{(x+\sqrt{x^2+1})^2+1} \\
&= \frac{x^2 + x\sqrt{x^2+1}}{x^2+1+x\sqrt{x^2+1}} \\
&= \frac{2e^{l/2} + 4 + 2e^{-l/2}}{e^l + 2e^{l/2} + 2 + 2e^{-l/2} + e^{-l}} \\
&= \frac{2e^{l/2}}{e^l+1}
\end{aligned}$$

So we have $\theta = 2\phi = 2\arcsin\left(\frac{2e^{l/2}}{e^l+1}\right)$. Now we can rearrange the inequality of Maskit to get that

$$m \leq \frac{l}{\theta} = \frac{l}{2\arcsin\left(\frac{2e^{l/2}}{e^l+1}\right)}$$

Rearranging the expression in Theorem 3.1.1 we get that $\frac{k}{2} = 2\arcsin\left(\frac{2e^{k/2}}{e^k+1}\right)$, and so if we have that $l < k$ from our theorem then we have that $\frac{l}{\theta} < \frac{k}{k/2} = 2$, and hence $m < 2$.

The next detail we need to look at is that of the extremal length of the curve family $\mathcal{C}$ consisting of all curves separating $(-R)_1$ and $(-R)_2$. We have that this extremal length is $m < 2$. Now, from §2.5 we have that $mod(R) = \frac{1}{m}$. Hence if $m < 2$ then $mod(R) > \frac{1}{2}$.

Now, Theorem 2.5.15 tells us that $mod(R) \leq 2\mu\left(\sqrt{\frac{|z_1|}{|z_1|+|z_2|}}\right)$. So we have specifically that

$$\frac{1}{2} < mod(R) \leq 2\mu\left(\sqrt{\frac{|z_1|}{|z_1|+|z_2|}}\right)$$

From Remark 2.5.16 we get that:

$$\frac{1}{2} < mod(R) \leq 2\mu\left(\sqrt{\frac{|z_1|}{|z_1|+|z_2|}}\right) = \frac{1}{2}\frac{K(\sqrt{1-r^2})}{K(r)}$$

where

$$r = \sqrt{\frac{|z_1|}{|z_1|+|z_2|}}$$

Now recall that to have an essential round annulus we require that $|z_1| < |z_2|$. We can write $|z_2| = \delta|z_1|$ for some δ, and the condition $|z_1| < |z_2|$ is equivalent to $\delta > 1$. So now:

$$\frac{1}{2} < \frac{1}{2}\frac{K(\sqrt{1-r^2})}{K(r)}, \quad r = \sqrt{\frac{1}{1+\delta}} \tag{3.2}$$

We now look at the function $K(r)$ in more detail, to see how it behaves for $0 < r < 1$. Heuristically we can just plot $K(r)$ against r and see that it is an increasing function. Similarly we can see that $K(\sqrt{1-r^2})$ is a decreasing function, and so we have that $\frac{K(\sqrt{1-r^2})}{K(r)}$ is a decreasing function for $0 < r < 1$, which is precisely the range that r satisfies since r originates in the definition of Grötzsch's extremal domain.

More technically we can note that

$$K(r) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 r^{2n}$$

where the double factorial is defined using a recursive definition as

$$n!! = \begin{cases} 1 & \text{if } n=-1, n=0 \text{ or } n=1 \\ n[(n-2)!!] & \text{if } n \geq 2 \end{cases}$$

In [2] it is shown that $K(r)$ is strictly increasing and positive, and it is also shown that for $0 < r < 1$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{K(\sqrt{1-r^2})}{K(r)}\right) = \frac{-\pi}{2r(1-r^2)K(r)^2}$$

and so we have that $\frac{K(\sqrt{1-r^2})}{K(r)}$ is a strictly decreasing function on $0 < r < 1$.

Equation 3.2 now becomes $1 < \frac{K(\sqrt{1-r^2})}{K(r)}$, which we need to solve. When $r = \sqrt{2}/2$ we have that $1 - r^2 = r$, and so $K(\sqrt{1-r^2}) = K(r)$, and hence that $1 = \frac{K(\sqrt{1-r^2})}{K(r)}$. Now, since we know that this function is decreasing we have that:

$$1 < \frac{K(\sqrt{1-r^2})}{K(r)} \Rightarrow r < \sqrt{2}/2$$

Now we simply have that:

$$\begin{aligned} r &< \frac{\sqrt{2}}{2} \\ \sqrt{\frac{1}{1+\delta}} &< \frac{\sqrt{2}}{2} \\ 1 &< \delta \end{aligned}$$

Thus if the length of α on S is $l(\alpha) < k$ then $|z_1| < |z_2|$ and hence there exists an essential annulus. Hence we can find a curve homotopically equivalent to α which lifts to circle in $\Omega(\Gamma)$. We can then repeat this process for all other SU-curves. If the condition $l(s_i) < k$ holds for all SU-curves in the Schottky uniformization then S can be uniformized by a classical Schottky group.

$\square$

3.2 Implications

We now consider the implications of our result by looking at Bers' constant. It was hoped that the bound in Theorem 3.1.1 would be such that we could use work done with Bers' constant to show that certain types of Riemann surfaces were uniformizable by classical Schottky groups. The bound in our theorem is slightly too small for the particular result we were after, and in this section we briefly look at implications following our theorem and suggest further work.

In [7] Bers showed the existence of a constant $B(g)$, depending only upon the genus, g, of a closed Riemann surface, S, such that there exists a pants decomposition of S where the length of the $3g-3$ curves do not exceed $B(g)$. A pants decomposition is a way of splitting a closed Riemann surface into three-holed spheres, or 'pairs of pants' using $3g-3$ curves. Much is known about Bers' constant, for example, if we take a pair of pants decomposition for a genus two surface $\mathcal{S}$, that is three geodesics j_1, j_2 and j_3 which separate $\mathcal{S}$ into two three-holed spheres, we have several methods to get a bound on the lengths of these curves.

From [11] we have that for every compact Riemann surgace of genus g there exists a set of curves defining a pants decomposition which have lengths defined by $l(j_k) \leq 4k \ln\left(\frac{8\pi}{k}\right)$ for $k = 1, ..., 3g-3$. We can easily calculate that these lengths for a genus 2 surface to be approximately $l(j_1) \leq 12.90$, $l(j_2) \leq 20.25$ and $l(j_3) \leq 25.51$.

There are known bounds on Bers' constant given in terms of the genus of the surface, for example, [11], $B(g) \leq 21(g-1)$. A lower bound is also known for $B(g)$ in that $B(g) \geq \sqrt{6g} - 2$ for a genus g surface.

For our Schottky uniformization we only need g SU-curves, which could be a subset of the $3g-3$ curves such that the complement of the SU-curves is connected. We do know though there exist multiple sets of g such curves in any given pants decomposition (for $g \geq 2$), and hence many such sets of SU-curves with lengths less than $B(g)$.

If we had that Bers' constant for a given surface was less than the constant in Theorem 3.1.1 then we would have that that surface was uniformized by a classical Schottky group. For instance, taking the example given previously using the theorem from [11], we calculated that for any genus 2 surface we have three curves with lengths approximately $l(j_1) \leq 12.90$, $l(j_2) \leq 20.25$ and $l(j_3) \leq 25.51$ which define a pants decomposition. We would require just two of these for our Schottky uniformization, so we can take the shortest two, but we can see already that these bounds are a lot more than k which is the value we require for classical Schottky uniformization. We have that $k < 3$ and so these values are a lot greater.

We have the following conjecture related to Schottky uniformizations:

Conjecture 3.2.1. There exists a constant $S(g)$, analogous to $B(g)$, for which there exists a decomposition of S into a $2g$-holed spheres, where the lengths of the g curves do not exceed $S(g)$. Then $S(g) \leq B(g)$ for $g \geq 2$.

Looking back at Bers' constant, whilst Bers' proof does not give any information on $B(g)$ aside from its existence, work has been done as mentioned above on bounding $B(g)$ by others. A theorem of Grácio and Sousa Ramos [14] states that for genus 2 surfaces, $B(g) = 2\operatorname{arccosh}(2)$. If our bound k had been greater than $B(2)$ then we would have all genus 2 Riemann surfaces uniformized by classical Schottky groups. Unfortunately our bound, k, is less than this, but it is hoped that $S(2) < k$ then we would have had the desired result. The values that we have are very close to that stated in the Grácio and Sousa Ramos paper, in that $k \approx 2.371776$ and $B(2) = 2.633915794$ so the values are very close. Work on $S(g)$ is necessary to progress this further.

Papers such as [23] might be useful to find if the Riemann surfaces satisfying the conditions of Theorem 3.1.1 are known elsewhere. Other possible work in improving this could come from investigating whether it is the case that if just one Schottky uniformizing curve has length less than k satisfying $e^k \sin\left(\frac{k}{4}\right) + \sin\left(\frac{k}{4}\right) = 2e^{\frac{k}{2}}$ then there exist a full set of defining curves with shorter length, and hence if one curve has length less than k we have classical Schottky uniformization.

Chapter 4

Non-Classical Schottky Groups

In this chapter we find more examples of non-classical Schottky groups, using techniques from Yamamoto's paper [40]. Firstly we discuss this paper in detail, rewriting the paper with all details included. We then take this rewritten paper and use it as a skeleton for the proof that a different family of SG-curves produce a non-classical Schottky group. Finally we generalise the step from Yamamoto's example to our example and produce a two variable family of non-classical Schottky groups through the following theorem:

Theorem 4.0.1 *Let $J_{a,\varepsilon}$ be the free group generated by:*

$$\begin{aligned} l_a : z &\mapsto \frac{k+1}{a+1} iz \\ h_{a,\varepsilon} : z &\mapsto \frac{k(1-\varepsilon)^{-1}z + (1-\varepsilon)(k^2(1-\varepsilon)^{-2}-1)}{(1-\varepsilon)^{-1}z + k(1-\varepsilon)^{-1}} \end{aligned}$$

where $k = \sqrt{a^2+2a+2}$. Then $J_{a,\varepsilon}$ is non-classical for $0 \leq a < 1.4$ and for $\varepsilon < f(a)$ for some function f (given explicitly in Appendix A).

4.1 Yamamoto's Paper

As discussed in Chapter 2 the first example of a non-classical Schottky group was given in a paper of Yamamoto [40]. In this chapter we use notation from Yamamoto's paper, drawing comparisons to previously discussed notation where appropriate. The Schottky group G_ε is defined by Yamamoto, and is generated by the transformations l and h_ε below:

$$\begin{aligned} l : z &\mapsto \; = i(\sqrt{2}+1)z \\ h_\varepsilon : z &\mapsto \; \frac{\sqrt{2}(1-\varepsilon)^{-1}z + (1-\varepsilon)(2(1-\varepsilon)^{-2}-1)}{(1-\varepsilon)^{-1}z + \sqrt{2}(1-\varepsilon)^{-1}} \end{aligned}$$

Yamamoto shows that when $\varepsilon \leq 10^{-20}$ then G_ε is non-classical. The SG-curves are such that l sends $C_{1,\varepsilon}$ to $C_{3,\varepsilon}$ and h_ε sends $C_{2,\varepsilon}$ to $C_{4,\varepsilon}$, and are explicitly defined as:

$$\begin{aligned} C_{1,\varepsilon} &= \text{The rectangle with vertices:} && \sqrt{2}-1+i(1-\varepsilon/3) \\ &&& \sqrt{2}-1-i(1-\varepsilon/3) \\ &&& -\sqrt{2}+1+i(1-\varepsilon/3) \\ &&& -\sqrt{2}+1-i(1-\varepsilon/3) \\ C_{3,\varepsilon} &= l(C_1) \\ C_{2,\varepsilon} &= \{|z+\sqrt{2}| = 1-\varepsilon\} \\ C_{4,\varepsilon} &= \{|z-\sqrt{2}| = 1-\varepsilon\} \end{aligned}$$

The paper gives a proof that G_ε is non-classical for $\varepsilon \leq 10^{-20}$, using a proof by contradiction. The paper uses a number of technically dense lemmas, and proves these lemmas after proving the main theorem. A lot of the reasoning behind using the lemmas comes from the proofs, and so it would feel more natural to prove the lemmas en route to the proof of the main

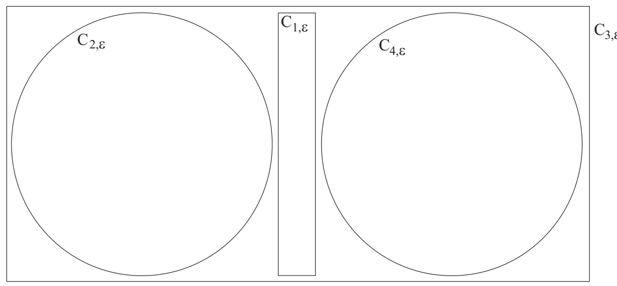

Figure 4.1: The defining curves for Yamamoto's non-classical Schottky group. Here ε is the distance between the inner rectangle and the circles

theorem. Some details in the proofs of the main theorem or lemmas are left to the reader of the paper, but these are generally not trivial calculations, especially due to the fact that in various places in [40] there are some incorrect details, through typographical error and occasional mathematical error. These details do not affect whether the theorem is true but are worth correcting. In this section we rewrite Yamamoto's paper, making some alterations to the order of results presented, and correcting the errors. We also include a number of figures which help with some of the explanation of the details of the proofs.

Since the proof of the theorem is technically dense in places and consists of several lemmas on the way we begin by giving an overview of the proof. We take the four SG-curves defined above, and assume we can find classical SG-curves for G_ε. From the definition of classical that means we assume that there exist four euclidean circles C_1, C_1', C_2 and C_2' which are also SG-curves for G_ε and which bound a fundamental domain for G_ε. Considering all possible images of these four circles under the group we find a particular set, $\widetilde{C}$, of image circles which are nested and intersect the real interval$(0, \sqrt{2}+1)$

once. We introduce Lemma 4.1.3 which shows that the distance between consecutive circles in $\widetilde{C}$ is less than 10^{-2}, where distance is measured along the real and imaginary axes. Lemma 4.1.3 is proven using Lemmas 4.1.4, 4.1.5 and 4.1.6. Lemmas 4.1.4 and 4.1.5 look at lengths of components of the domain of discontinuity which intersect the real and imaginary axes. Lemma 4.1.6 then relates the regions in Lemmas 4.1.4 and 4.1.5 with the distance between circles in $\widetilde{C}$. Finally we find a particular image of one of C_1, C_1', C_2 or C_2' which is not in $\widetilde{C}$, which intersects $(0, \sqrt{2}+1)$ twice and has diameter greater than 10^{-2}. This means that this circle intersects at least one of the $\widetilde{C}$, which means that the original circles cannot be SG-curves, and so our assumption that the group G_ε is classical is incorrect.

Theorem 4.1.1. **([40])** *The group G_ε generated by*

$$h_\varepsilon : z \mapsto \frac{\sqrt{2}(1-\varepsilon)^{-1}z + (1-\varepsilon)(2(1-\varepsilon)^{-2}-1)}{(1-\varepsilon)^{-1}z + \sqrt{2}(1-\varepsilon)^{-1}}$$

and

$$l : z \mapsto i(\sqrt{2}+1)z$$

is a non-classical Schottky group if $\varepsilon \leq 10^{-20}$.

The first step is to use the following lemma, which as stated by Yamamoto is a lemma of Marden, the proof of which is given in sufficient detail in Yamamoto's paper, and is therefore omitted here.

Lemma 4.1.2. **([25])** *Let Γ be a classical Schottky group generated by two Möbius transformations. Let γ be an element of Γ. Then there exists a fundamental domain for Γ surrounded by four circles, at least one of which separates the fixed points of γ.*

We are now able to begin the proof of Theorem 4.1.1, pausing to prove necessary lemmas on the way.

Proof. We prove that G_ε is non-classical by contradiction. Suppose that for $\varepsilon = 10^{-20}$ we have G_ε is classical. Then by Lemma 4.1.2 we have a fundamental domain D_ε for G_ε bounded by four circles, our proposed SG-curves, C_1, C_1', C_2, C_2', one of which separates 0 and ∞, the fixed points of l. Without loss of generality let C_1 be this curve. There will be another boundary curve of D_ε which also separates the fixed points of l. We let $\widetilde{C} = \{C_\varepsilon^1, C_\varepsilon^2, C_\varepsilon^3, ..., C_\varepsilon^N\}$ be the complete list of images of the SG-curves under G_ε which satisfy the following conditions:

(i) Each C_ε^j separates 0 from ∞.

(ii) Each C_ε^{j+1} separates C_ε^j from 0.

(iii) C_ε^1, C_ε^j (for $j = 2, 3 ... N-1$) and C_ε^N meet $[\sqrt{2}+1, \infty)$, $(\sqrt{2}-1, \sqrt{2}+1)$ and $(0, \sqrt{2}-1]$ respectively.

An example of a set of circles which satisfy the above is shown in Figure 4.2.

If C_a and C_b are curves in $\widetilde{C}$ which are not separated by other curves in $\widetilde{C}$ then C_a and C_b lie on the boundary of a fundamental domain, that is a translate of D_ε. If $C_a = C_\varepsilon^j$ and $C_b = C_\varepsilon^{j+1}$ then the translate of D_ε is called $D_{j\varepsilon}$.

Let $u_{jk\varepsilon}$ be the point $C_\varepsilon^j \cap i^{k-1}\mathbb{R}^+$, where $k = 1, ..., 4$ and $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. We then define $\nu_{j\varepsilon} = \max_k\{|u_{jk\varepsilon} - u_{(j+1)k\varepsilon}|\}$, the largest distance between two consecutive circles in $\widetilde{C}$, where distance is measured along the real or imaginary axes.

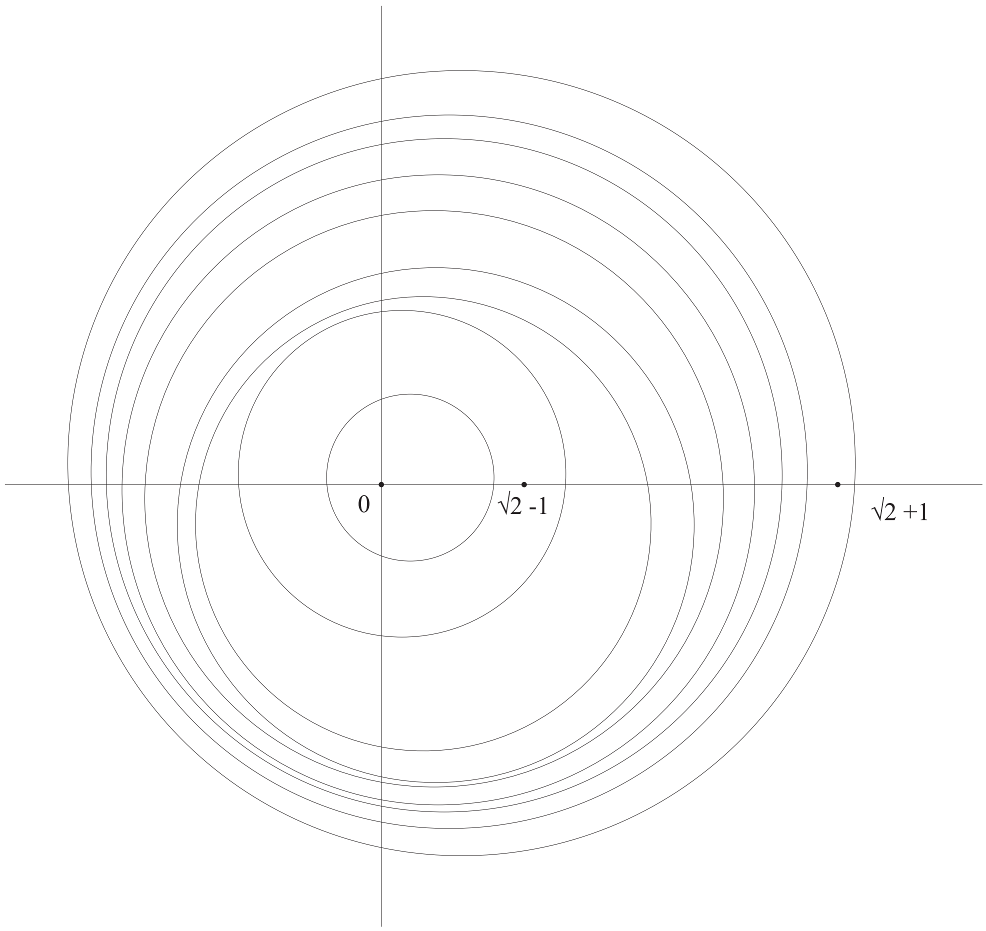

Figure 4.2: The set $\widetilde{C}$, with C_ε^1 on the outside and C_ε^N nearest to 0.

As mentioned in the overview we are looking to show that a particular circle defined later has diameter greater than the gaps between the circles in $\widetilde{C}$, so we now need to find a bound on the gaps between the circles, that is, a bound on $\nu_{j\varepsilon}$.

In particular we prove the following lemma which puts an upper bound on $\nu_{j\varepsilon}$.

Lemma 4.1.3. ([40], Lemma 2) *For every* $0 < \varepsilon \leq 10^{-20}$ *and every positive integer* $1 \leq j < N$,

$$\nu_{j\varepsilon} < 10^{-2}$$

Proof. To prove Lemma 4.1.3 we move to H_ε, the group generated by h_ε and $L = l^2$. This group is used as it preserves the real axis, and the intervals between the $u_{jk\varepsilon}$ points that we are looking at are simply segments of the real and imaginary axes. Following the standard notation we let $\Omega(H_\varepsilon)$ be the domain of discontinuity of this group. This allows us to look at the two axes separately. To prove this lemma we require a second lemma:

Lemma 4.1.4. ([40], Lemma 3) *The length of each component of* $(\Omega(H_\varepsilon)\cap\mathbb{R})\cup(\Omega(lH_\varepsilon l^{-1})\cap i\mathbb{R})$ *which meets the region made up of the union of segments given as* $[-(\sqrt{2}+1)^3, (\sqrt{2}+1)^3]\cup i[-(\sqrt{2}+1)^4, (\sqrt{2}+1)^4]$ *is less than* $2.01(\sqrt{2}+1)^3(2+\sqrt{2})\sqrt{\varepsilon}$.

Proof. To prove Lemma 4.1.4 it is sufficient to prove the following lemma. This simplifies the details by restricting to segments of the real line.

Lemma 4.1.5. ([40], Lemma 4) *The length of each component of* $\Omega(H_\varepsilon)\cap\mathbb{R}$ *which meets the region* $[-\sqrt{2}-1, -\sqrt{2}+1]\cup[\sqrt{2}-1, \sqrt{2}+1]$ *is less than* $2.01(2+\sqrt{2})\sqrt{\varepsilon}$.

Proof. Let $W = h_\varepsilon L h_\varepsilon L^{-1}$ with fixed points:

$$\begin{aligned} w_1 &= (1-2\varepsilon+\varepsilon^2)(1+\sqrt{2}) + (1+\sqrt{2})\sqrt{\varepsilon(1-2\varepsilon+\varepsilon^2)(2-\varepsilon)} \\ w_2 &= (1-2\varepsilon+\varepsilon^2)(1+\sqrt{2}) - (1+\sqrt{2})\sqrt{\varepsilon(1-2\varepsilon+\varepsilon^2)(2-\varepsilon)} \end{aligned}$$

We let $\mathscr{I}_1$ be the component of $\Omega(H_\varepsilon)\cap\mathbb{R}$ which is bounded by the fixed points of W. All components of $\Omega(H_\varepsilon)\cap\mathbb{R}$ are equivalent under H_ε, but we define three other regions $\mathscr{I}_2 = h_\varepsilon^{-1}(\mathscr{I}_1)$, $\mathscr{I}_3 = L^{-1}(\mathscr{I}_2)$ and $\mathscr{I}_4 = L^{-1}(\mathscr{I}_1)$ to make some of the explicit calculations in the proof below simpler.

Let $\mathscr{J}$ be a component of $\Omega(H_\varepsilon) \cap \mathbb{R}$ which is inside $C_{2,\varepsilon}$ or $C_{4,\varepsilon}$. We can write $\mathscr{J}$ as $\gamma_{2q}(\mathscr{I}) = h_\varepsilon^{p_{2q}} L^{p_{2q-1}} ... h_\varepsilon^{p_2} L^{p_1}(\mathscr{I})$, where $\mathscr{I}$ signifies one of $\mathscr{I}_1, \mathscr{I}_2, \mathscr{I}_3$ or $\mathscr{I}_4$ and $p_{2q}p_{2q-1}...p_2 \neq 0$.

Calculating the lengths of $\mathscr{I}_1, \mathscr{I}_2, \mathscr{I}_3$ or $\mathscr{I}_4$ explicitly from their definition we find that $\mathscr{I}_1$ and $\mathscr{I}_2$ have the same lengths as each other, and $\mathscr{I}_3$ and $\mathscr{I}_4$ have the same lengths as each other. The length of $\mathscr{I}_1$ (and hence $\mathscr{I}_2$) is $2(\sqrt{2}+1)\sqrt{(1-2\varepsilon+\varepsilon^2)(2-\varepsilon)\varepsilon}$, and the length of $\mathscr{I}_3$ (and hence $\mathscr{I}_4$) is $2(\sqrt{2}-1)\sqrt{(1-2\varepsilon+\varepsilon^2)(2-\varepsilon)\varepsilon}$. We can see that $\mathscr{I}_1$ and $\mathscr{I}_2$ are longer than $\mathscr{I}_3$ and $\mathscr{I}_4$. Looking again at the length of $\mathscr{I}_1$ we can see that

$$2(\sqrt{2}+1)\sqrt{(1-2\varepsilon+\varepsilon^2)(2-\varepsilon)\varepsilon} < (2+\sqrt{2})2.01\sqrt{\varepsilon}$$

and so the length of $\mathscr{I}$ is less than $(2+\sqrt{2})2.01\sqrt{\varepsilon}$. To complete the proof we need to show that the length of any image of $\mathscr{I}$ under γ_{2q} is no longer than the length of $\mathscr{I}$, which we do by showing that $|\gamma_{2q}'(x)| < 1$, $\forall x \in \mathscr{I}$. We prove this using induction.

Our method of induction involves first showing that $|\gamma_{2q}'(x)| < 1 \ \forall x \in \mathscr{I}$ is true for $q = 1$. We then show that assuming it is true for $q = n$ then this implies it is true for $q = n + 1$.

Equivalently we prove in some cases that the image of $\gamma_2(\mathscr{I})$ is no longer than $\mathscr{I}$, and that if we assume $\gamma_{2n}(\mathscr{I})$ is no longer than $\mathscr{I}$ then $\gamma_{2(n+1)}(\mathscr{I})$ is also no longer than $\mathscr{I}$. We will often be using induction to prove steps along the way, so we refer to the induction on q as the q-induction for ease of referencing.

Following the proof in the paper as a guide we now work through the details of the induction, labelling the sections of the induction for ease of reading.

The $\mathbf{q = 1}$ case. If we set $q = 1$ we have $\gamma_2 = h_\varepsilon^{p_2} L^{p_1}$. We look at three cases (i) $p_1 = 0$, (ii) $p_1 < 0$ and (iii) $p_1 > 0$.

For reference, since they are used frequently in this proof, we mention that

$$|h'_\varepsilon(x)| = \left|\frac{(1-\varepsilon)^2}{(x+\sqrt{2})^2}\right| \qquad |L'(x)| = \left|i(\sqrt{2}+1)\right| = \sqrt{2}+1$$

It is useful to briefly look at $|h'_\varepsilon(x)|$ to see where this function is greater than one, and where it is less that one. Inside $C_{2,\varepsilon}$ we have that $|h'_\varepsilon(x)| > 1$ since $\left|x+\sqrt{2}\right| < 1-\varepsilon$ and outside $C_{2,\varepsilon}$ we have that $|h'_\varepsilon(x)| < 1$ since $\left|x+\sqrt{2}\right| > 1-\varepsilon$.

Looking first at the case where **(i)** $\mathbf{p_1 = 0}$, we want that $|(h^{p_2})'(x)| < 1$ $\forall x \in \mathscr{I}$. For this we use induction on p_2. Looking at $p_2 > 0$ we see that for the initial step of the induction we have $|h'_\varepsilon(x)| < 1$ for all $x \in \mathscr{I}$ apart from some $x \in \mathscr{I}_2$, but we can see that the image of $\mathscr{I}_2$ under h_ε remains the same size (it is $\mathscr{I}_1$). Thus we need to look at h_ε^2 for our first step of the induction, essentially using a $p_2 = 2$ stage rather than an $p_2 = 1$ stage for the start of the induction.

If we look at $|(h_\varepsilon^2)'(x)|$ we see that

$$\left|(h_\varepsilon^2)'(x)\right| = \left|\frac{(1-\varepsilon)^4}{(-2\sqrt{2}x - 3 - 2\varepsilon + \varepsilon^2)^2}\right|$$

Analysing this function we see that if $x < -\sqrt{2}$ then $|(h_\varepsilon^2)'(x)| < 1$. Similarly if $x > \frac{1}{2}\sqrt{2}(-1-2\varepsilon+\varepsilon^2)$ then $|(h_\varepsilon^2)'(x)| < 1$. If we take the region where $|(h_\varepsilon^2)'(x)| > 1$, that is $-\sqrt{2} < x < \frac{1}{2}\sqrt{2}(-1-2\varepsilon+\varepsilon^2)$ we see that $\mathscr{I}_2$ is to the left of this region, $\mathscr{I}_1$ and $\mathscr{I}_3$ are obviously to the right of this region, and for small values of ε (specifically $\varepsilon < 1 \times 10^{-4}$) that $\mathscr{I}_4$ is also outside this region. Therefore we have that for all values of $x \in \mathscr{I}$ we have $|(h_\varepsilon^2)'(x)| < 1$. This completes the $p_2 = 2$ case of this induction.

Now we need to show that assuming $|(h_\varepsilon^{p_2})'(x)| < 1$ for $p_2 = n$ then it is true for $p_2 = n + 1$.

We can write $|(h_\varepsilon^{n+1})'(x)|$ as

$$\begin{aligned}
|(h_\varepsilon^{n+1})'(x)| &= \left|\frac{d}{dx}(h_\varepsilon^{n+1}(x))\right| \\
&= \left|\frac{d(h_\varepsilon^{n+1}(x))}{d(h_\varepsilon^{n}(x))}\right|\left|\frac{d(h_\varepsilon^{n}(x))}{dx}\right| \\
&< \left|\frac{d(h_\varepsilon^{n+1}(x))}{d(h_\varepsilon^{n}(x))}\right| \\
&= \frac{1 - 2\varepsilon + \varepsilon^2}{\left(h_\varepsilon^n(x) + \sqrt{2}\right)^2}
\end{aligned}$$

We only need to show this is true for $n \geq 3$, since we already know this to be true for $n = 2$. If we apply h_ε^n to any of the four regions which make up $\mathscr{I}$ we see that $h_\varepsilon^n(x)$ will be inside $C_{4,\varepsilon}$. If we take values of $h_\varepsilon^n(x)$ to be any value inside $C_{4,\varepsilon}$ and substitute these into the above we get that $|(h_\varepsilon^{n+1})'(x)| < 1$ as required.

The induction for $p_2 < 0$ follows very closely to the above. First we prove it for $p_2 = -2$ and then show that assuming $|(h_\varepsilon^{p_2})'(x)| < 1$ is true for $p_2 = n$, $n < 0$ then it is true for $p_2 = n - 1$. The details follow the exact method above. This proves case (i) above.

Looking now at **(ii)** $\mathbf{p_1 < 0}$ we use an induction method in a similar way to the previous case, where we can write

$$
\begin{aligned}
|(\gamma_2)'(x)| &= |(h_\varepsilon^{p_2} L^{p_1})'(x)| \\
&= \left| \frac{d}{dx}(h_\varepsilon^{p_2} L^{p_1}(x)) \right| \\
&= \left| \frac{dh_\varepsilon^{p_2} L^{p_1}(x)}{dL^{p_1}(x)} \right| \left| \frac{dL^{p_1}(x)}{dx} \right| \\
&< \frac{1}{\sqrt{2}+1} \left| \frac{dh_\varepsilon^{p_2} L^{p_1}(x)}{dL^{p_1}(x)} \right|
\end{aligned}
$$

Then if we let $u = L^{p_1}(x)$ and we have a similar situation as in the $p_1 = 0$ case in that we have $|(\gamma_2)'(x)| < \frac{1}{\sqrt{2}+1} \left| \frac{dh_\varepsilon^{p_2}(u)}{du} \right|$, requiring an induction on p_2. The induction follows that of the induction in case (i), except where before we tested for $x \in \mathscr{I}$ we now test for $u \in L^{p_1}(x)$. Since $p_1 < 0$ in this case we have that the regions we consider are all outside $C_{2,\varepsilon}$, and hence using these values we get that $|(h_\varepsilon^{p_2} L^{p_1})'(u)| < 1$, proving the case of $p_1 < 0$.

Finally for the first step of our q-induction we prove the case of **(iii)** $\mathbf{p_1 > 0}$ which we consider as having to show that the image of $\mathscr{I}$ becomes no longer under the transformation $h_\varepsilon^{p_2} L^{p_1}$ for p_1 positive. We first look at $p_2 > 0$, but the proof for $p_2 < 0$ is very similar. We can restrict our values of x to those only in $\mathscr{I}_1$ or $\mathscr{I}_2$ because if $p_1 = 1$ we know application of L sends $\mathscr{I}_3$ and $\mathscr{I}_4$ to $\mathscr{I}_2$ and $\mathscr{I}_1$ and we know that $|(h_\varepsilon^{p_2})'(x)| < 1$ for all $x \in \mathscr{I}$ from previous work. We therefore need to focus on $p_1 > 1$ and $x \in \mathscr{I}_1$ or $\mathscr{I}_2$. We can therefore see that any image of $\mathscr{I}_1$ or $\mathscr{I}_2$ under L^{p_1} will be outside $C_{2,\varepsilon}$, and we know that h_ε is contracting outside of $C_{2,\varepsilon}$ ($|h'_\varepsilon(x)| < 1$ for $\left|x + \sqrt{2}\right| > 1 - \varepsilon$, which is precisely those points outside $C_{2,\varepsilon}$). This means that the longest interval for $h_\varepsilon^{p_2} L^{p_1}(x)$ will be when $p_2 = 1$. All we need to show is that $h_\varepsilon L^{p_1}(\mathscr{I})$ is shorter than $\mathscr{I}$ for all p_1.

We want to look at

$$F_1 = \left| h_\varepsilon\left(\left(\sqrt{2}+1\right)^{2p_1}\mathscr{I}_{1a}\right) - h_\varepsilon\left(\left(\sqrt{2}+1\right)^{2p_1}\mathscr{I}_{1b}\right)\right|$$

and

$$F_2 = \left| h_\varepsilon\left(\left(\sqrt{2}+1\right)^{2p_1}\mathscr{I}_{2a}\right) - h_\varepsilon\left(\left(\sqrt{2}+1\right)^{2p_1}\mathscr{I}_{2b}\right)\right|$$

where $\mathscr{I}_{na}$ and $\mathscr{I}_{nb}$ are the end points of $\mathscr{I}_n$. We want to show that these are both less than $|\mathscr{I}_{1a} - \mathscr{I}_{1b}|$. For small values of ε we calculate $\max_{p_1} F_1$ and $\max_{p_1} F_2$, and we get that p_1 has absolute value less than 1, and that the lengths of F_1 and F_2 tend to zero as p_1 increases so all we need to specifically test is when p_1 is equal to 1.

The length of F_1 and F_2 is shorter than $\mathscr{I}$ for $p_1 = 1$ and therefore so will any image of $\mathscr{I}$ under $h_\varepsilon^{p_2}L^{p_1}$ for $p_1 > 0$. Thus we have proven the first step of the q-induction.

Now we have shown that for any combination of p_1 and p_2 we have that $|(\gamma_2)'(x)| < 1$ for any $x \in \mathscr{I}$. We now move on to the second step of the induction.

The induction step of the proof. We assume that $|\gamma_{2q}'(x)| < 1$ for $q = 1, 2, ..., n$, and try to prove it is true for $q = n+1$. We need to prove $|(h_\varepsilon^{p_{2n+2}}L^{p_{2n+1}}h_\varepsilon^{p_{2n}}L^{p_{2n-1}}...h_\varepsilon^{p_2}L^{p_1})'(x)| < 1$, $\forall x \in \mathscr{I}$ or equivalently that the image of $\mathscr{I}$ under $\gamma_{2(n+1)}$ is no longer than the image of $\mathscr{I}$ under γ_{2n}. To show that this is true we look at the case where (i) $p_{2n+1} < 0$ and the case where (ii) $p_{2n+1} > 0$ seperately.

Looking first at the case **(i)** $\mathbf{p_{2n+1} < 0}$ we see that from our assumption that the image of $\mathscr{I}$ under γ_{2n} is shorter than $\mathscr{I}$. Since $p_{2n+1} < 0$ we have that the image of $\mathscr{I}$ under $L^{p_{2n+1}}\gamma_{2n}$ will be shorter still. Moreover, since the image of $\mathscr{I}$ under γ_{2n} will be inside either $C_{2,\varepsilon}$ or $C_{4,\varepsilon}$ (depending on the sign of p_{2n}) we have that the image under $L^{p_{2n+1}}\gamma_{2n}$ will be inside $C_{1,\varepsilon}$ and hence outside $C_{2,\varepsilon}$ and $C_{4,\varepsilon}$. As mentioned previously h_ε (resp h_ε^{-1}) contracts regions outside of $C_{2,\varepsilon}$ (resp $C_{4,\varepsilon}$) so the image under $h_\varepsilon^{p_{2n+2}}L^{p_{2n+1}}\gamma_{2n}$ will in turn be shorter than the image under $L^{p_{2n+1}}\gamma_{2n}$. Thus the case of $p_{2n+1} < 0$ is proven.

Now finally we need to prove the case **(ii)** $\mathbf{p_{2n+1} > 0}$. In the paper Yamamoto describes two cases which we will look at here. The two cases are (a) $\xi \leq |\gamma_{2n}(x)| < \sqrt{2}+1$ and (b) $\sqrt{2}-1 \leq |\gamma_{2n}(x)| < \xi$, where $\xi = (2-(1-\varepsilon)^2)^{\frac{1}{2}}$ is the attractive fixed point of h_ε. If we look at case (a) we have that

$$\begin{aligned}
\left|\gamma_{2q+2}'(x)\right| &= \left|\frac{\mathrm{d}(h_\varepsilon^{p_{2q+2}}L^{p_{2q+1}}\gamma_{2q}(x))}{\mathrm{d}L^{p_{2q+1}}\gamma_{2q}(x)}\right| \left|\frac{\mathrm{d}(L^{p_{2q+1}}\gamma_{2q}(x))}{\mathrm{d}\gamma_{2q}(x)}\right| \left|\gamma_{2q}'(x)\right| \\
&< \left|\frac{\mathrm{d}(h_\varepsilon^{p_{2q+2}}L^{p_{2q+1}}\gamma_{2q}(x))}{\mathrm{d}L^{p_{2q+1}}\gamma_{2q}(x)}\right| (\sqrt{2}+1)^{p_{2q+1}} \\
&< \left|\frac{(1-e)^2(\sqrt{2}+1)^{p_{2q+1}}}{((\sqrt{2}+1)^{p_{2q+1}}+\sqrt{2})^2}\right| \\
&< 1
\end{aligned}$$

Now we need to look at case (b). The proof of this is similar to case (a) except that we need to note that because of the conditions of case (b) we have that $|L^{p_{2q-1}}\gamma_{2q-2}(x)| < \sqrt{2}-1$ and hence $p_{2q-1} < 0$. Taking the same method as above, but expanding back to $\left|\gamma_{2q-2}'(x)\right|$ rather than just $\left|\gamma_{2q}'(x)\right|$ we obtain that $\left|\gamma_{2q+2}'\right| < 1$ as required.

This completes the induction, and hence the proof of the lemma. $\square$

We now show that Lemma 4.1.5 proves Lemma 4.1.4. Using multiple applications of l we can see that we can extend the proof of Lemma 4.1.5

to the region stated in the statement of Lemma 4.1.4. Applying l twice and three times to the region in Lemma 4.1.5 we get that Lemma 4.1.4 is true for the parts of $(\Omega(H_\varepsilon) \cap \mathbb{R}) \cup (\Omega(lH_\varepsilon l^{-1}) \cap i\mathbb{R})$ which meet the region $[-(\sqrt{2}+1)^3, -\sqrt{2}+1] \cup [\sqrt{2}-1, (\sqrt{2}+1)^3] \cup i[-(\sqrt{2}+1)^4, -1] \cup i[1, (\sqrt{2}+1)^4]$. To show that we can extend this to the whole region stated in the lemma we simply need to observe that to fill in this region we would apply l^{-1} a number of times. Since l^{-1} is contracting this does not increase the size of intervals, and so we may extend the regions above to the regions of the real and imaginary axes in the statement of Lemma 4.1.4.

□

Now, to complete the proof of Lemma 4.1.3 we need one final lemma, given below with detailed proof.

Lemma 4.1.6. ([40], Lemma 5) *Let $\eta > 0$. If $|u_{jk\varepsilon} - u_{j+1k\varepsilon}| < \eta$ for at least two values of $k \in \{1, 2, 3, 4\}$ then $\nu_{j\varepsilon} < 10^6\eta$.*

Proof. We let $P_j = x_j + iy_j$ and R_j be the centre and radius of C_ε^j respectively. We let μ_j be the distance between the centres of C_ε^j and C_ε^{j+1}, that is $\mu_j = |P_j - P_{j+1}|$. We define a circle C' which is concentric to C_ε^{j+1} and tangent to C_j, then C' is given by $(x - x_{j+1})^2 + (y - y_{j+1})^2 = (R_j - \mu_j)^2$, and we let T be the point of tangency. We define S' to be the point on C' such that TS' is a diameter. Finally we set $u'_k = C' \cap i^{k-1}\mathbb{R}^+$ for $k = 1, ..., 4$. All this information is shown in Figure 4.3

Let k_1 and k_2 denote the two values of k for which $|u_{jk\varepsilon} - u_{(j+1)k\varepsilon}| < \eta$, which exist by the hypothesis of the lemma. Without loss of generality we may assume $\angle u'_{k_1} P_{j+1} S' \leq \angle u'_{k_2} P_{j+1} S'$. Let θ denote $\angle u'_{k_1} P_{j+1} S'$, and let us

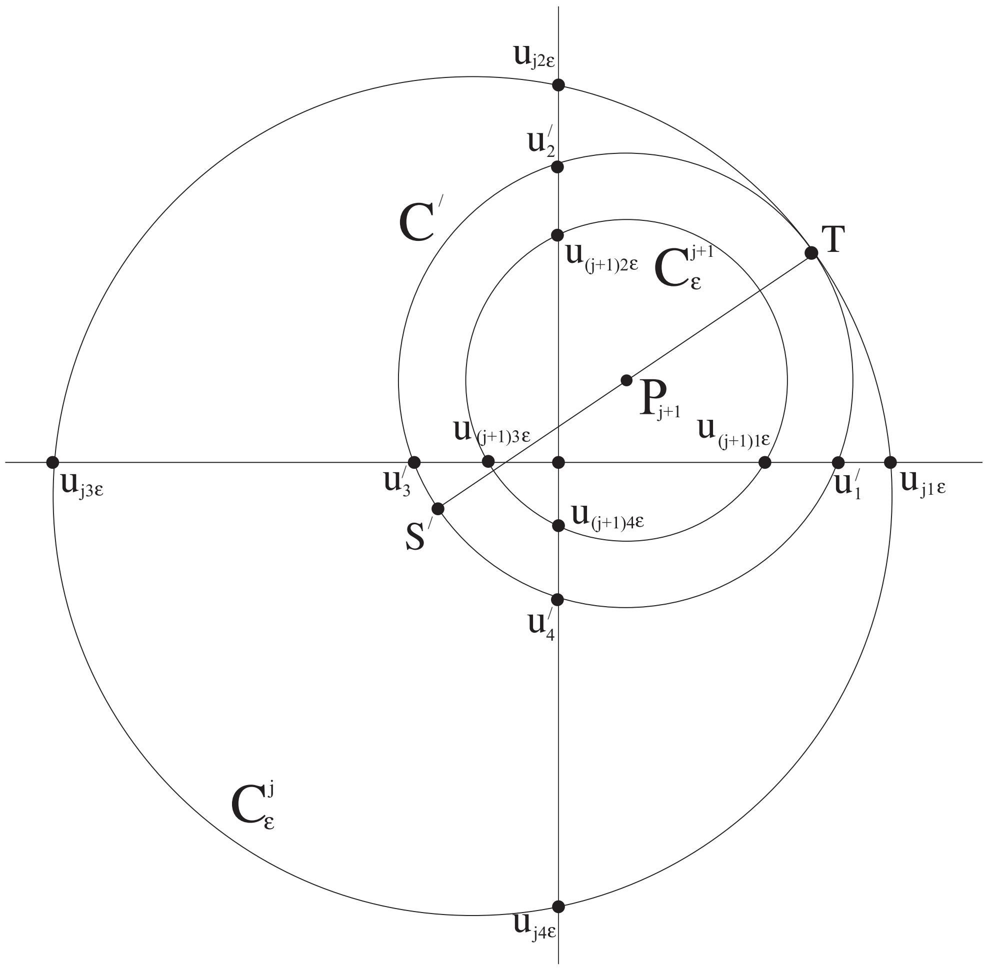

Figure 4.3: The circles C_ε^j, C_ε^{j+1} and C'.

first consider $\frac{\pi}{2} \leq \theta \leq \pi$.

Considering the triangle $\triangle S'Tu_{k_1}$ we can see, using the sine rule that $\sin\theta > \sin u_{k_1}S'T$, and similarly for $\triangle S'Tu_{k_2}$. Therefore we have

$$\sin\theta + \sin\angle u'_{k_2}P_{j+1}S' > \sin\angle u'_{k_1}S'T + \sin\angle u'_{k_2}S'T$$

Since $S'T$ is a diameter, and u'_{k_1} is on the circle C' we have that $\triangle S'Tu'_{k_1}$ is right-angled. Recalling that the radius of C' is $R'_j = R_j - \mu_j$ we have that

$\sin \angle u'_{k_1} S'T = \frac{|u'_{k_1} - T|}{2(R_j - \mu_j)}$ therefore we have

$$\sin\theta + \sin \angle u'_{k_2} P_{j+1} S' > \sum_{r=1}^{2} \frac{|u'_{k_r} - T|}{2(R_j - \mu_j)} \tag{4.1}$$

We now want to derive an inequality for $2(R_j - \mu_j)$ in terms of the lengths $|u'_k|$. From the definition of the u'_k we can see that the maximum the sum of the $|u'_k|$ can be is twice the diameter of C', and the minimum it can be is the diameter. Hence:

$$\begin{aligned} |u'_1| + |u'_2| + |u'_3| + |u'_4| &\geq \text{Diameter of } C' \\ \sum_{k=1}^{4} |u'_k| &\geq 2(R_j - \mu_j) \end{aligned} \tag{4.2}$$

By the triangle inequality we have

$$|u'_{k_1} - T| + |u'_{k_2} - T| > |u'_{k_1} - u'_{k_2}| > |u'_{k_1}| \tag{4.3}$$

the last part of the above line being due to the fact that the u'_{k_r} are on the axes.

Combining inequalities (4.2) and (4.3) into the right hand side of Equation (4.1) we get:

$$\sin\theta + \sin \angle u'_{k_2} P_{j+1} S' > \frac{|u'_{k_1}|}{\sum_{k=1}^{4} |u'_k|} \tag{4.4}$$

Since l has fixed points of 0 and ∞, and since C' separates them, any image of C' under l will also separate 0 and ∞ and thus be either one of the $\widetilde{C}$ or outside $C_{1,\varepsilon}$. The image cannot be inside C_j since l has 0 as a repulsive point. The image cannot be C_j since C_j is tangential to C', so therefore it must be outside C_j. Therefore we have that the images of the u_k under l must be outside the u_k themselves, so we have $|u'_{k+1}| < (\sqrt{2}+1)|u'_k|$.

We know that k_1 is just one of the k, and so u'_{k_1} is one of the u'_k. We can therefore write $\sum_{k=1}^{4} |u'_k|$ as $|u'_{k_1}| + |u'_{k_1+1}| + |u'_{k_1+2}| + |u'_{k_1+3}|$, where the subscript addition is cyclic through $\{1,2,3,4\}$. We therefore have that

$$\begin{aligned}
\sum_{k=1}^{4} |u'_k| &< |u'_{k_1}| + (\sqrt{2}+1)|u'_{k_1}| + (\sqrt{2}+1)^2|u'_{k_1}| + (\sqrt{2}+1)^3|u'_{k_1}| \\
&= |u'_{k_1}| \Big(1 + (\sqrt{2}+1) + (2\sqrt{2}+3) + (5\sqrt{2}+7)\Big) \\
&= |u'_{k_1}|(12 + 8\sqrt{2}) \\
&< 24|u'_{k_1}|
\end{aligned}$$

Therefore Equation (4.4) becomes

$$\sin\theta + \sin\angle u'_{k_2} P_{j+1} S' > \frac{1}{24} \tag{4.5}$$

Now, since we have assumed $\theta \leq \angle u'_{k_2} P_{j+1} S'$ and $\frac{\pi}{2} \leq \theta < \pi$ we have that $\sin\theta > \sin\angle u'_{k_2} P_{j+1} S'$ and hence we have from Equation (4.5) that $\sin\theta > \frac{1}{48}$. Later we shall want an inequality for $(1+\cos\theta)$ so from the above we have

$$(1+\cos\theta)^{-1} < 4608 \tag{4.6}$$

As mentioned earlier C_ε^j and C_ε^{j+1}, along with two other curves, bound a fundamental domain for G_ε, so in particular we have that $L(C_\varepsilon^j) \cap C_\varepsilon^{j+1} = \emptyset$. The transformation L preserves any line though the origin, so taking the line $\mathcal{L}$ through the origin and P_j we see that the line segment from the point A on C_ε^j which is $R_j - |P_j|$ away from the origin and the point B which is $R_j + |P_j|$ away from the origin is a diameter of C_ε^j. We can see that the image of A

under L will be on the line $\mathcal{L}$ and further from the origin than B. Hence we have:

$$\begin{aligned}(\sqrt{2}+1)^2(R_j-|P_j|) &> R_j+|P_j| \\ (2\sqrt{2}+2)R_j &> (2\sqrt{2}+4)|P_j| \\ R_j &> \sqrt{2}|P_j| \qquad (4.7)\end{aligned}$$

By the triangle inequality it is clear that

$$\mu_j \leq |P_j|+|P_{j+1}| \qquad (4.8)$$

Finally since the origin lies inside C' (since the origin lies inside C_ε^{j+1} which in turn lies inside C') we must have that

$$|P_{j+1}| < R_j-\mu_j \qquad (4.9)$$

Combining Equations (4.7), (4.8) and (4.9) we get that

$$\mu_j < \frac{\sqrt{2}+1}{2\sqrt{2}}R_j < \frac{9R_j}{10} \qquad (4.10)$$

We now let $\eta' = |u_{jk_1\varepsilon} - u'_{k_1}|$ and so we necessarily have that $\eta' < \eta$. We can see easily that $R_j < |P_j - u'_{k_1}| + \eta'$ from the triangle inequality, and so looking at $\triangle u'_{k_1}P_{j+1}P_j$ we use the cosine rule to get:

$$\begin{aligned}(R_j-\eta')^2 &< |P_j-u_{k_1}|^2 \\ &= \mu_j^2+(R_j-\mu_j)^2-2\mu_j(R_j-\mu_j)\cos\theta\end{aligned}$$

Therefore we have:

$$\begin{aligned}(\eta')^2-2R_j\eta' &< 2\mu_j^2-2R_j\mu_j-2\mu_j(R_j-\mu_j)\cos\theta \\ 2\mu_j(R_j-\mu_j)(1+\cos\theta) &< 2R_j\eta'-(\eta')^2 < 2R_j\eta' < 2R_j\eta \\ \mu_j &< R_j\eta(R_j-\mu_j)^{-1}(1+\cos\theta)^{-1} \qquad (4.11)\end{aligned}$$

From Equation (4.10) we get that $(R_j - \mu_j)^{-1} < \frac{10}{R_j}$ and combining this with (4.6) we see that Equation (4.11) becomes

$$\begin{aligned} \mu_j &< R_j \eta (R_j - \mu_j)^{-1} (1 + \cos\theta)^{-1} \\ \mu_j &< R_j \eta \frac{10}{R_j} 4608 \\ \mu_j &< 5 \times 10^4 \, \eta \end{aligned} \tag{4.12}$$

We now look at the case that $0 \leq \theta < \frac{\pi}{2}$. We have that $(1 + \cos\theta)^{-1} < 1$ simply because $\cos\theta$ is positive in this range, and so we have that the above all holds for this case, since $(1 + \cos\theta)^{-1} < 1 < 4608$.

We now turn our attention to $v_{j\varepsilon}$ and look to prove the lemma. Recalling the definition of $v_{j\varepsilon}$ from before Lemma 4.1.3 we can see that

$$\begin{aligned} v_{j\varepsilon} &= \max_k \{ |u_{jk\varepsilon} - u_{(j+1)k\varepsilon}| \} \\ &< \sum_{k=1}^{4} |u_{jk\varepsilon} - u_{(j+1)k\varepsilon}| \\ &= \sum_{k=1}^{2} \left(|u_{jk\varepsilon} - u_{j(k+2)\varepsilon}| - |u_{(j+1)k\varepsilon} - u_{(j+1)(k+2)\varepsilon}| \right) \end{aligned} \tag{4.13}$$

The last step of the above can be seen from Figure 4.4.

From the triangle shown in Figure 4.4 we can see using Pythagoras that $|u_{j1\varepsilon} - u_{j3\varepsilon}| = 2\sqrt{R_j^2 - y_j^2}$. We can do the same for other equivalent triangles and get that $|u_{(j+1)1\varepsilon} - u_{(j+1)3\varepsilon}| = 2\sqrt{R_{j+1}^2 - y_{j+1}^2}$, $|u_{j2\varepsilon} - u_{j4\varepsilon}| = 2\sqrt{R_j^2 - x_j^2}$ and $|u_{(j+1)2\varepsilon} - u_{(j+1)4\varepsilon}| = 2\sqrt{R_{j+1}^2 - y_{j+1}^2}$. Then some simple algebra gives

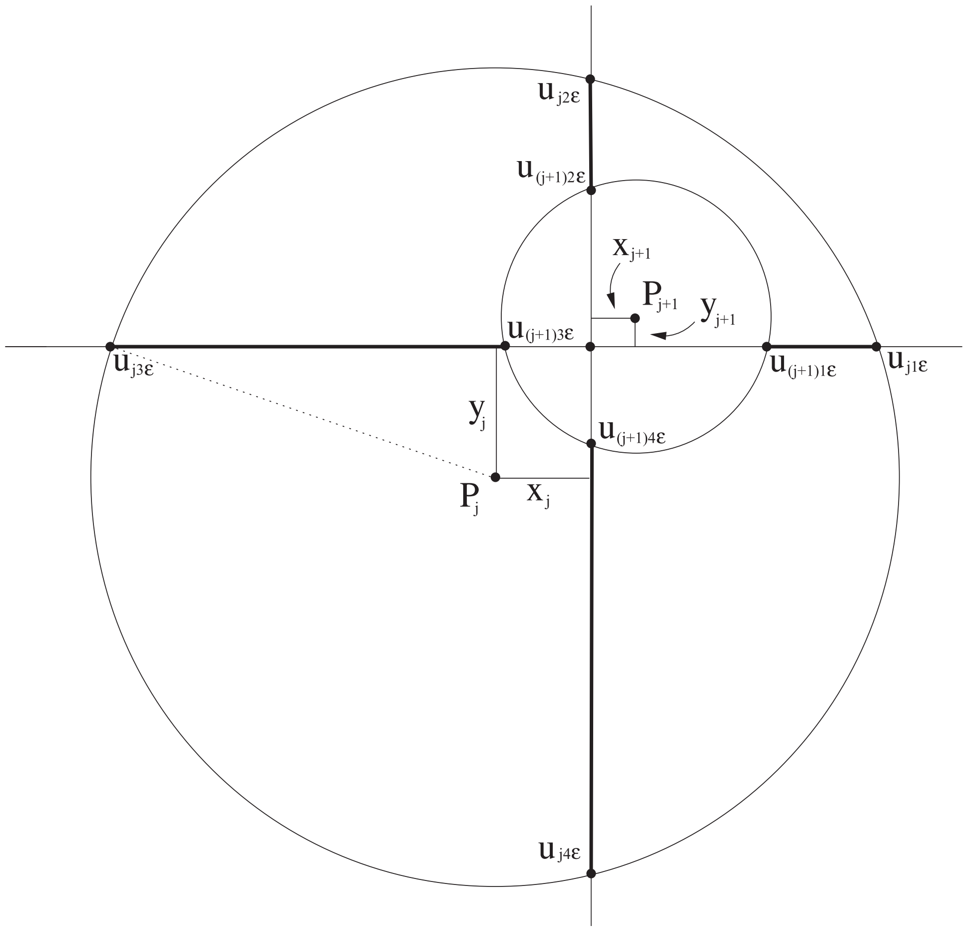

Figure 4.4: The circles C_ε^j and C_ε^{j+1} .

us that Equation (4.13) becomes

$$\begin{aligned}
v_{j\varepsilon} &< 2\left[\sqrt{R_j^2 - y_j^2} - \sqrt{R_{j+1}^2 - y_{j+1}^2} + \sqrt{R_j^2 - x_j^2} - \sqrt{R_{j+1}^2 - x_{j+1}^2}\,\right] \\
&= \frac{2\left(\sqrt{R_j^2 - x_j^2} - \sqrt{R_{j+1}^2 - x_{j+1}^2}\right)\left(\sqrt{R_j^2 - x_j^2} + \sqrt{R_{j+1}^2 - x_{j+1}^2}\right)}{\sqrt{R_j^2 - x_j^2} + \sqrt{R_{j+1}^2 - x_{j+1}^2}} \\
&\quad + \frac{2\left(\sqrt{R_j^2 - y_j^2} - \sqrt{R_{j+1}^2 - y_{j+1}^2}\right)\left(\sqrt{R_j^2 - y_j^2} + \sqrt{R_{j+1}^2 - y_{j+1}^2}\right)}{\sqrt{R_j^2 - y_j^2} + \sqrt{R_{j+1}^2 - y_{j+1}^2}} \\
&= \frac{2\left(R_j^2 - x_j^2 - R_{j+1}^2 + x_{j+1}^2\right)}{\sqrt{R_j^2 - x_j^2} + \sqrt{R_{j+1}^2 - x_{j+1}^2}} + \frac{2\left(R_j^2 - y_j^2 - R_{j+1}^2 + y_{j+1}^2\right)}{\sqrt{R_j^2 - y_j^2} + \sqrt{R_{j+1}^2 - y_{j+1}^2}} \qquad (4.14)
\end{aligned}$$

We now have that $v_{j\varepsilon}$ is less than the sum of two fractions, and by replacing the denominators of each fraction by the minimum of the two denominators we see that

$$\begin{aligned} v_{j\varepsilon} &< \frac{2\left(R_j^2 - x_j^2 - R_{j+1}^2 + x_{j+1}^2\right) + 2\left(R_j^2 - y_j^2 - R_{j+1}^2 + y_{j+1}^2\right)}{\min\left\{\sqrt{R_j^2 - x_j^2} + \sqrt{R_{j+1}^2 - x_{j+1}^2}\ ,\ \sqrt{R_j^2 - y_j^2} + \sqrt{R_{j+1}^2 - y_{j+1}^2}\right\}} \\ &= \frac{2\left(2(R_j^2 - R_{j+1}^2) + (-x_j^2 + x_{j+1}^2 - y_j^2 + y_{j+1}^2)\right)}{\min\left\{\sqrt{R_j^2 - x_j^2} + \sqrt{R_{j+1}^2 - x_{j+1}^2}\ ,\ \sqrt{R_j^2 - y_j^2} + \sqrt{R_{j+1}^2 - y_{j+1}^2}\right\}} \end{aligned} \tag{4.15}$$

Now we need a few more facts to finish the proof. Firstly we can write the following, using the triangle inequality and the geometry of Figure 4.4:

$$\begin{aligned} -x_j^2 + x_{j+1}^2 - y_j^2 + y_{j+1}^2 &\leq |x_j^2 - x_{j+1}^2 + y_j^2 - y_{j+1}^2| \\ &\leq |x_j - x_{j+1}||x_j + x_{j+1}| + |y_j - y_{j+1}||y_j + y_{j+1}| \\ &< (|x_j - x_{j+1}| + |y_j - y_{j+1}|)(R_j + R_{j+1}) \end{aligned} \tag{4.16}$$

Now, by taking a line through the centers of C_ε^j and C_ε^{j+1} and from Equation (4.12) we can see that $R_j - R_{j+1} < \mu_j + \eta < (5 \times 10^4 + 1)\eta$. From Equation (4.7) we get $|P_j| < R_j/\sqrt{2}$, and hence clearly $|x_j|, |y_j| < R_j/\sqrt{2}$. This gives us that, for example,

$$\begin{aligned} x_j &< \frac{R_j}{\sqrt{2}} \\ -x_j^2 &> -\frac{R_j^2}{2} \\ R_j^2 - x_j^2 &> \frac{R_j^2}{2} \\ \sqrt{R_j^2 - x_j^2} &> \frac{R_j}{\sqrt{2}} \end{aligned}$$

Finally we need to look at $|x_j - x_{j+1}| + |y_j - y_{j+1}|$. We consider a triangle with one vertex at P_j and one at P_{j+1}, with two of its edges parallel to the axes, and with edges of length $|x_j - x_{j+1}|$, $|y_j - y_{j+1}|$ and μ_j. We can see that the minimum that $|x_j - x_{j+1}| + |y_j - y_{j+1}|$ can be is μ_j, when one of the other two sides has length zero, and the maximum it can be is $\sqrt{2}\mu$ when the triangle is isosceles. Hence $|x_j - x_{j+1}| + |y_j - y_{j+1}| \le \sqrt{2}\mu < \sqrt{2} \cdot 5 \times 10^4\,\eta$.

Combining all of these comments together with Equation (4.16) we get from Equation (4.15) that

$$\begin{aligned}
v_{j\varepsilon} &< \frac{2\left[2(R_j^2 - R_{j+1}^2) + (-x_j^2 + x_{j+1}^2 - y_j^2 + y_{j+1}^2)\right]}{\min\left\{\sqrt{R_j^2 - x_j^2} + \sqrt{R_{j+1}^2 - x_{j+1}^2}\ ,\ \sqrt{R_j^2 - y_j^2} + \sqrt{R_{j+1}^2 - y_{j+1}^2}\right\}} \\
&< \frac{2(R_j + R_{j+1})\left[2(R_j - R_{j+1}) + (|x_j - x_{j+1}| + |y_j - y_{j+1}|)\right]}{\min\left\{\sqrt{R_j^2 - x_j^2} + \sqrt{R_{j+1}^2 - x_{j+1}^2}\ ,\ \sqrt{R_j^2 - y_j^2} + \sqrt{R_{j+1}^2 - y_{j+1}^2}\right\}} \\
&< \frac{2(R_j + R_{j+1})\left[2(R_j - R_{j+1}) + (|x_j - x_{j+1}| + |y_j - y_{j+1}|)\right]}{\frac{R_j}{\sqrt{2}} + \frac{R_{j+1}}{\sqrt{2}}} \\
&< 2\sqrt{2}\left[2(5 \times 10^4 + 1)\eta + (\sqrt{2} \cdot 5 \times 10^4)\eta\right] \\
&< 10^6 \eta \qquad (4.17)
\end{aligned}$$

as required. This ends the proof of Lemma 4.1.6.

□

We are now able to prove Lemma 4.1.3. We define $E_{j\varepsilon}$ to be the doubly-connected domain surrounded by C_ε^j and C_ε^{j+1}. Thus the boundary of $E_{j\varepsilon}$ is a subset of the boundary of a fundamental domain $D_{j\varepsilon}$, as in Figure 4.5.

At least two components of $E_{j\varepsilon} \cap (\mathbb{R} \cup i\mathbb{R})$ are included in $D_{j\varepsilon} \cap (\mathbb{R} \cup i\mathbb{R})$, in fact in Figure 4.5 only two such components coincide, due to the placing of the additional two circles in $D_{j\varepsilon}$. Let these components be A_1 and A_2 (there may

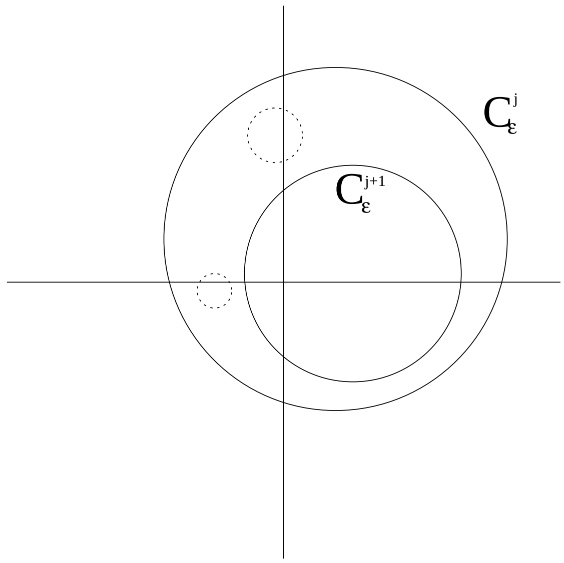

Figure 4.5: An example of the domain $E_{j\varepsilon}$ and two (dotted) circles which could be added to make a $D_{j\varepsilon}$.

be others). We can see that each of A_1 and A_2 lies in a component of $(\Omega(H_\varepsilon)\cap\mathbb{R})\cup(\Omega(lH_\varepsilon l^{-1})\cap i\mathbb{R})$ meeting $[-(\sqrt{2}+1)^3,(\sqrt{2}+1)^3]\cup i[-(\sqrt{2}+1)^4,(\sqrt{2}+1)^4]$ and so from Lemma 4.1.4 has length less than $2.01(\sqrt{2}+1)^3(2+\sqrt{2})\sqrt{\varepsilon}$. From Lemma 4.1.6 we have that $\nu_{j\varepsilon} < 10^6\, 2.01(\sqrt{2}+1)^3(2+\sqrt{2})\sqrt{\varepsilon} < 10^8\sqrt{\varepsilon}$. Hence if $0<\varepsilon\leq 10^{-20}$ then $\nu_{j\varepsilon}<10^{-2}$ as required. This proves Lemma 4.1.3.

$\square$

We are now able to prove the theorem in question, that is Theorem 4.1.1.

Let C be a circle meeting $[-(\sqrt{2}+1)^5,-(\sqrt{2}+1)]$ which is equivalent to C_1 under the group generated by l^4. We may then apply h_ε to C and look at the properties of this new circle. We see that since C_1 separates 0 and ∞ then so does C. Since C is outside $C_{2,\varepsilon}$ its image $h_\varepsilon(C)$ will meet $[\sqrt{2}-1,\sqrt{2}+1]$ twice, as illustrated by Figure 4.6.

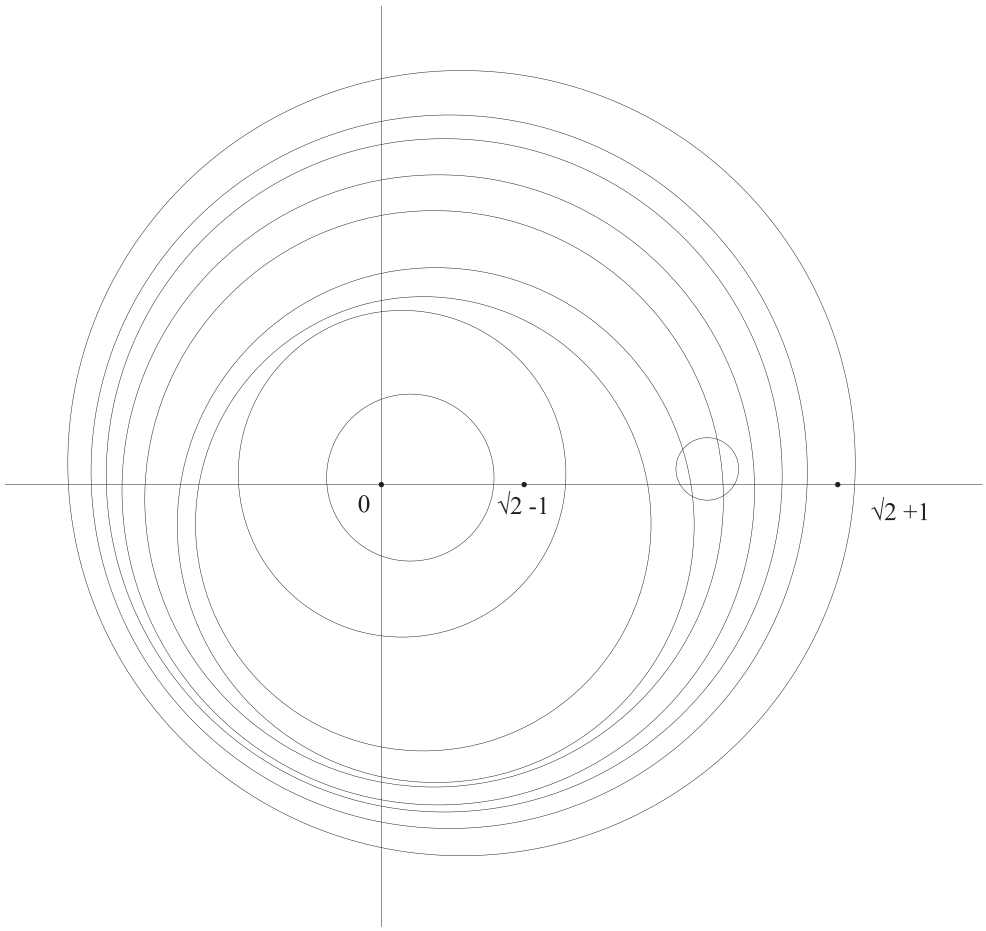

Figure 4.6: The set $\widetilde{C}$, from Figure 4.2, along with $h_\varepsilon(C)$.

We know that ∞ and $-(\sqrt{2}+1)^5$ are outside of C and so $h_\varepsilon(\infty)$ and $h_\varepsilon(-(\sqrt{2}+1)^5)$ will be inside $h_\varepsilon(C)$ and on the real axis. The diameter of $h_\varepsilon(C)$ will therefore be greater than $\left|h_\varepsilon(-(\sqrt{2}+1)^5) - h_\varepsilon(\infty)\right|$.

Calculating this explicitly we see that

$$\left|h_\varepsilon(-(\sqrt{2}+1)^5) - h_\varepsilon(\infty)\right| = \left|\frac{-(1-\varepsilon)^2}{-(\sqrt{2}+1)^5+\sqrt{2}}\right| > 10^{-2}$$

However from Lemma 4.1.3 we know that the gaps between the curves in

$\widetilde{C}$ is less than 10^{-2}, and therefore we see that $h_\varepsilon(C)$ meets some C_ε^j. This means that all images of C_1, C_1', C_2 and C_2' are not disjoint, and hence the assumption that the group is classical is incorrect.

$\square$

4.2 Generalising Yamamoto

We would like to have many more examples of non-classical Schotty groups, amongst other things to help progress in the work discussed in Chapter 5. The following is an example of a non-classical Schottky group, obtained using the methods of Yamamoto. We take his example, and experiment with ways of creating new examples. We would like to create a three generator non-classical Schottky group, but whilst a lot of the details follow through, the very last step of Yamamoto's proof does not hold, since we need to place four circles in Figure 4.5 rather than just two, and they may block all four sections of the axes. Instead we take Yamamoto's example and alter the diagram slightly by adding a gap of $\frac{1}{2}$ above and below each circle to get a new family of non-classical Schottky groups, described in Section 4.2.1. In section 4.2.2 we look to further this process by adding a distance of a above and below the circles, and then get a bound on a and note its effect on ε.

4.2.1 A new non-classical Schottky group

We look for a second example of a non-classical Schottky group, and prove the following theorem:

Theorem 4.2.1. *The Schottky group J_ε with generators:*

$$\begin{aligned} l : z &\mapsto \frac{2}{3}\left(\frac{1}{2}\sqrt{13}+1\right) iz \\ h_\varepsilon : z &\mapsto \frac{\frac{\sqrt{13}}{2}(1-\varepsilon)^{-1}z + (1-\varepsilon)(\frac{13}{4}(1-\varepsilon)^{-2}-1)}{(1-\varepsilon)^{-1}z + \frac{\sqrt{13}}{2}(1-\varepsilon)^{-1}} \end{aligned}$$

is non-classical for $\varepsilon < 5 \times 10^{-19}$.

The SG-curves are shown in Figure 4.7. We change the notation slightly from Yamamoto, and define the curves as:

$$\begin{aligned}
C_1 &= \text{The rectangle with vertices:} && \tfrac{1}{2}\sqrt{13} - 1 + i(\tfrac{3}{2} - \varepsilon/2)\\
& && \tfrac{1}{2}\sqrt{13} - 1 - i(\tfrac{3}{2} - \varepsilon/2)\\
& && -\tfrac{1}{2}\sqrt{13} + 1 + i(\tfrac{3}{2} - \varepsilon/2)\\
& && -\tfrac{1}{2}\sqrt{13} + 1 - i(\tfrac{3}{2} - \varepsilon/2)\\
C_1' &= l(C_1)\\
C_2 &= \{|z + \tfrac{\sqrt{13}}{2}| = 1 - \varepsilon\}\\
C_2' &= \{|z - \tfrac{\sqrt{13}}{2}| = 1 - \varepsilon\}
\end{aligned}$$

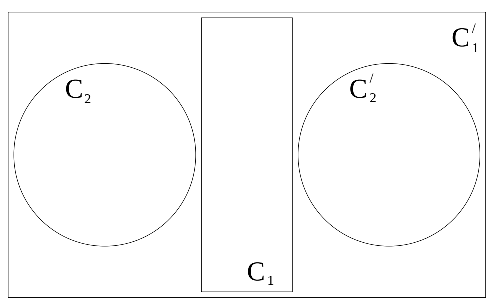

Figure 4.7: The defining curves for a non-classical Schottky group. The distance between the inner rectangle and the circles are ε and the distance above and below the circles are $\varepsilon + \frac{1}{2}$.

Proof. The proof follows that of Yamamoto's example, but with different details. We again assume that the group is classical and look for a contradiction. We assume the existance of $\hat{C}_1, \hat{C}_1', \hat{C}_2$ and $\hat{C}_2'$ which are classical SG-curves for J_ε.

We define $\widetilde{C}$ in a similar way as in the proof of Theorem 4.1.1 with different bounds:

$\widetilde{C} = \{C_\varepsilon^1, C_\varepsilon^2, C_\varepsilon^3, ..., C_\varepsilon^N\}$ which is a complete list of images of the SG-curves under G_ε which satisfy:

(i) Each C_ε^j separates 0 from ∞.

(ii) Each C_ε^{j+1} separates C_ε^j from 0.

(iii) C_ε^1, C_ε^j and C_ε^N meet the regions $\left[\frac{1}{2}\sqrt{13}+1, \infty\right)$, $\left(\frac{1}{2}\sqrt{13}-1, \frac{1}{2}\sqrt{13}+1\right)$ and $\left(0, \frac{1}{2}\sqrt{13}-1\right]$ respectively.

We again define $\nu_{j\varepsilon} = \max_k\{|u_{jk\varepsilon} - u_{(j+1)k\varepsilon}|\}$, and have equivalents to Lemma 4.1.3, Lemma 4.1.4, Lemma 4.1.5 and Lemma 4.1.6 for our group J_ε. We give these new lemmas with some details of the proofs.

Lemma 4.2.2. *For every* $0 < \varepsilon \leq 5 \times 10^{-19}$ *and every positive integer* $1 \leq j < N$*,*

$$\nu_{j\varepsilon} < 8 \times 10^{-3}$$

Proof. We define the real line preserving group K_ε as $K_\varepsilon = \langle h_\varepsilon, L = l^2 \rangle$. To prove Lemma 4.2.2 we need Lemma 4.2.3 and hence Lemma 4.2.4.

Lemma 4.2.3. *The length of each component of* $(\Omega(K_\varepsilon) \cap \mathbb{R}) \cup (\Omega(lK_\varepsilon l^{-1}) \cap i\mathbb{R})$ *which meets the region*

$$\left[-\frac{4}{9}\left(\frac{1}{2}\sqrt{13}+1\right)^3, \frac{4}{9}\left(\frac{1}{2}\sqrt{13}+1\right)^3\right] \cup \\ i\left[-\frac{8}{27}\left(\frac{1}{2}\sqrt{13}+1\right)^4, \frac{8}{27}\left(\frac{1}{2}\sqrt{13}+1\right)^4\right]$$

is less than $\frac{16}{81}(\frac{1}{2}\sqrt{13}+1)^4\, 2.01\sqrt{2\varepsilon}$*.*

Proof. To prove this we use an equivalent to Lemma 4.1.5:

Lemma 4.2.4. *The length of each component of* $\Omega(K_\varepsilon) \cap \mathbb{R}$ *which meets the region* $\left[-\frac{1}{2}\sqrt{13}-1, -\frac{1}{2}\sqrt{13}+1\right] \cup \left[\frac{1}{2}\sqrt{13}-1, \frac{1}{2}\sqrt{13}+1\right]$ *is less than* $\frac{1}{3}(2+\sqrt{13})\, 2.01\sqrt{2\varepsilon}$.

Proof. The proof of Lemma 4.2.4 follows that of Lemma 4.1.5, with differences in the numerical details, but not in the process used. We omit it here for simplicity. □

As in Yamamoto's paper, where Lemma 4.1.4 follows from Lemma 4.1.4, here we have Lemma 4.2.3 follows from Lemma 4.2.4 by the same reasoning. □

Next, to finish the proof of Lemma 4.2.2 we require the following.

Lemma 4.2.5. *If* $|u_{jk\varepsilon} - u_{j+1k\varepsilon}| < \eta$ *for at least two* $k \in \{1,2,3,4\}$ *then* $\nu_{j\varepsilon} < 2.5 \times 10^5 \eta$.

Proof. As in Yamamoto's proof of Lemma 4.1.6 we let $P_j = x_j + iy_j$ and R_j be the centre and radius of C^j_ε respectively and $\mu_j = |P_j - P_{j+1}|$. We define a circle C' concentric to C_{j+1} and tangent to C_j, then C' is given by $(x - x_{j+1})^2 + (y - y_{j+1})^2 = (R_j - \mu_j)^2$ and let T be the point of tangency. We define S' to be the point on C' such that TS' is a diameter and we set $u'_k = C' \cap i^{k-1}\mathbb{R}^+$.

We let k_1 and k_2 denote two values of k for which $|u_{jk\varepsilon} - u_{(j+1)k\varepsilon}| < \eta$. We assume $\angle u'_{k_1} P_{j+1} S' \leq \angle u'_{k_2} P_{j+1} S'$. Let θ denote $\angle u'_{k_1} P_{j+1} S'$, and let us first consider $\frac{\pi}{2} \leq \theta \leq \pi$.

Looking at the triangle $\triangle S' P_{j+1} u_{k_1}$ we see, using the sine rule, that $\sin\theta > \sin u_{k_1} S' T$, and similarly for $\triangle S' P_{j+1} u_{k_2}$. Therefore we have

$$\sin\theta + \sin\angle u'_{k_2} P_{j+1} S' > \sin\angle u'_{k_1} S' T + \sin\angle u'_{k_2} S' T$$

$S'T$ is a diameter and u'_{k_1} is on C', so we have that $\triangle S'Tu'_{k_1}$ is right-angled, so $\sin\angle u'_{k_1}S'T = \frac{|u'_{k_1}-T|}{2(R_j-\mu_j)}$ therefore we have the same equation as for Lemma 4.1.6:

$$\sin\theta + \sin\angle u'_{k_2}P_{j+1}S' > \sum_{r=1}^{2}\frac{|u'_{k_r}-T|}{2(R_j-\mu_j)} \tag{4.18}$$

Also, with the same reasoning as in Lemma 4.1.6 we have:

$$\begin{aligned} |u'_1|+|u'_2|+|u'_3|+|u'_4| &\geq \text{Diameter of } C' \\ \sum_{k=1}^{4}|u'_k| &\geq 2(R_j-\mu_j) \end{aligned} \tag{4.19}$$

And:

$$|u'_{k_1}-T|+|u'_{k_2}-T| > |u'_{k_1}-u'_{k_2}| > |u'_{k_1}| \tag{4.20}$$

Combining the inequalities of Equations (4.19) and (4.20) into the right hand side of Equation (4.18) we get:

$$\sin\theta + \sin\angle u'_{k_2}P_{j+1}S' > \frac{|u'_{k_1}|}{\sum_{k=1}^{4}|u'_k|} \tag{4.21}$$

In our proof we have the same reasoning to show that the images of the u_k under l must be outside the u_k themselves, but we have $|u'_{k+1}| < \frac{2}{3}(1+\frac{1}{2}\sqrt{13})|u'_k|$ because of the change in generators and therefore have that

$$\begin{aligned} \sin\theta + \sin\angle u'_{k_2}P_{j+1}S' &> \frac{1}{1+\frac{2}{3}(1+\frac{1}{2}\sqrt{13})+\frac{4}{9}(1+\frac{1}{2}\sqrt{13})^2+\frac{8}{27}(1+\frac{1}{2}\sqrt{13})^3} \\ &> \frac{1}{13} \end{aligned} \tag{4.22}$$

This implies that $\sin\theta > \frac{1}{26}$, and hence

$$(1+\cos\theta)^{-1} < 1359 \tag{4.23}$$

We know C_ε^j and C_ε^{j+1}, along with two other curves, bound a fundamental domain for G_ε, so in particular we have that $L(C_\varepsilon^j) \cap C_\varepsilon^{j+1} = \emptyset$. L preserves lines through the origin, so taking the line $\mathcal{L}$ through the origin and P_j we see that the line segment from the point A on C_ε^j which is $R_j - |P_j|$ from the origin and the point B which is $R_j + |P_j|$ from the origin is a diameter. We can see that the image of A under L will be on the line $\mathcal{L}$ and further from the origin than B. Hence we have:

$$\begin{aligned} \frac{4}{9}\left(1+\frac{1}{2}\sqrt{13}\right)^2 (R_j - |P_j|) &> R_j + |P_j| \\ R_j &> \frac{65+8\sqrt{13}}{47+8\sqrt{13}}|P_j| \end{aligned} \tag{4.24}$$

By the triangle inequality it is clear that

$$\mu_j \leq |P_j| + |P_{j+1}| \tag{4.25}$$

Finally since the origin lies inside C' we must have that

$$|P_{j+1}| < R_j - \mu_j \tag{4.26}$$

Combining the inequalities (4.24), (4.25) and (4.26) we get

$$\mu_j < \frac{8(7+\sqrt{13})}{65+8\sqrt{13}} R_j < \frac{91 R_j}{100} \tag{4.27}$$

Defining $\eta' = |u_{jk_1\varepsilon} - u'_{k_1}|$ (and hence $\eta' < \eta$), we can derive the same equation as in (4.11):

$$\mu_j < R_j \eta (R_j - \mu_j)^{-1} (1 + \cos\theta)^{-1} \tag{4.28}$$

From Equation (4.27) we get that $(R_j - \mu_j)^{-1} < \frac{100}{9R_j}$ and combining this with (4.23) we see that Equation (4.28) becomes

$$\begin{aligned}
\mu_j &< R_j\eta(R_j-\mu_j)^{-1}(1+\cos\theta)^{-1} \\
\mu_j &< R_j\eta\frac{100}{9R_j}1359 \\
\mu_j &< 1.51\times 10^4\,\eta \qquad (4.29)
\end{aligned}$$

We have trivially that $(1+\cos\theta)^{-1} < 1359$ for the case that $0 \le \theta < \frac{\pi}{2}$ since $\cos\theta$ is positive in this range. This means that the above holds for all θ.

We now turn our attention to $v_{j\varepsilon}$ and proving the lemma. The details of this section are the same as in Lemma 4.1.6, so we look straight at the following:

$$v_{j\varepsilon} < \frac{2(R_j+R_{j+1})\left[2(R_j-R_{j+1})+(|x_j-x_{j+1}|+|y_j-y_{j+1}|)\right]}{\min\left\{\sqrt{R_j^2-x_j^2}+\sqrt{R_{j+1}^2-x_{j+1}^2}\,,\ \sqrt{R_j^2-y_j^2}+\sqrt{R_{j+1}^2-y_{j+1}^2}\right\}} \qquad (4.30)$$

By taking a line through the centres of C_ε^j and C_ε^{j+1} we can see that $R_j - R_{j+1} < \mu_j + \eta < (1.51\times 10^4+1)\eta$. From Equation (4.24) we have a relation between $|P_j|$ and R_j, and so we have, in a similar way as in Yamamoto's proof, for example

$$\sqrt{R_j^2-x_j^2} > \frac{12}{13}\frac{\sqrt{13}(\sqrt{13}+1)(5\sqrt{13}+8)}{389+80\sqrt{13}}R_j$$

We have from work in Lemma 4.1.6 that in our case we have $|x_j - x_{j+1}| + |y_j - y_{j+1}| \le \sqrt{2}\mu < 1.51\times 10^4\sqrt{2}\eta$.

Combining all of these comments together we get from Equation (4.30) that

$$\begin{aligned} v_{j\varepsilon} &< \frac{26(389+80\sqrt{13})}{12\sqrt{13}(\sqrt{13}+1)(5\sqrt{13}+8)}\left[2(1.51\times 10^4+1)\eta + 1.51\times 10^4\sqrt{2}\eta\right] \\ &< 2.5\times 10^5\eta \qquad (4.31) \end{aligned}$$

as required. This ends the proof of Lemma 4.2.5

□

We conclude the proof to Lemma 4.2.2 by noting that by defining A_1 and A_2 in the same way as before, in that they are sections of the real or imaginary axes not covered by the boundary of the domains equivalent to $D_{j\varepsilon}$ in Figure 4.5. We have that each of A_1 and A_2 lies in a component of $(\Omega(H_\varepsilon)\cap\mathbb{R})\cup(\Omega(lH_\varepsilon l^{-1})\cap i\mathbb{R})$ meeting $\left[-\frac{4}{9}(\frac{1}{2}\sqrt{13}+1)^3, \frac{4}{9}(\frac{1}{2}\sqrt{13}+1)^3\right]\cup i\left[-\frac{8}{27}(\frac{1}{2}\sqrt{13}+1)^4, \frac{8}{27}(\frac{1}{2}\sqrt{13}+1)^4\right]$ and so from Lemma 4.2.3 has length less than $\frac{16}{81}(\frac{1}{2}\sqrt{13}+1)^4\, 2.01\sqrt{2\varepsilon}$. From Lemma 4.2.5 we have that $\nu_{j\varepsilon} < 2.5\times 10^5\frac{16}{81}(\frac{1}{2}\sqrt{13}+1)^4\, 2.01\sqrt{2\varepsilon} < 9\times 10^6\sqrt{\varepsilon}$. Hence if $0<\varepsilon\leq 5\times 10^{-19}$ then $\nu_{j\varepsilon} < 8\times 10^{-3}$ as required. □

Finally as in Yamamoto's proof we let C be a circle meeting the region $\left[-\left(\frac{2}{3}\right)^4(1+\frac{1}{2}\sqrt{13})^5, -(1+\frac{1}{2}\sqrt{13})\right]$ which is equivalent to $\hat{C}_1$ under the group generated by l^4. The diameter of the image of C under h_ε is greater than

$$\left|h_\varepsilon\left(-\left(\frac{2}{3}\right)^4(1+\frac{1}{2}\sqrt{13})^5\right) - h_\varepsilon(\infty)\right| = \frac{81(1-\varepsilon)^2}{1381+344\sqrt{13}} > 8\times 10^{-3}$$

Again, since C is outside C_2 its image under h_ε will meet $[\frac{1}{2}\sqrt{13}-1, \frac{1}{2}\sqrt{13}+1]$ twice. Since the distance between image circles $\widetilde{C}$ is less (from Lemma 4.2.2) than the diameter of this circle $h_\varepsilon(C)$ we see that $h_\varepsilon(C)$ meets some C_ε^j, contradicting that all images of $\hat{C}_1$, $\hat{C}_1'$, $\hat{C}_2$ and $\hat{C}_2'$ are disjoint, and hence the assumption that the group is classical is incorrect.

$\square$

4.2.2 Generalisation

We now look at adjusting the area above the circles, and see how far the proof holds. We look to get a two variable family of non-classical Schottky groups. The key distances in the diagrams of initial choice of curves in the previous two sections are those along the x-axis, and so we preserve the proximity of all the curves on the axes, and simply increase the space above and below the circles in §4.2. We study the following arrangement of SG-curves:

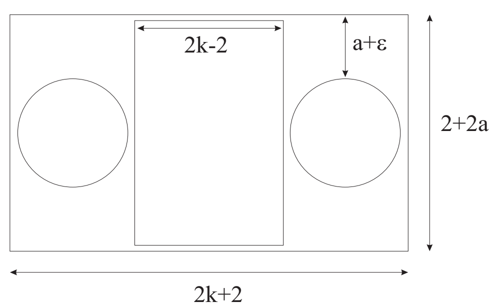

Figure 4.8: The defining curves for a 2-variable family of non-classical Schottky groups.

Where we have $k = \sqrt{2 + 2a + a^2}$. We keep the same notation as in §4.2, and define the curves more precisely as:

$$\begin{aligned}
C_1 &= \text{The rectangle with vertices:} \quad & k-1+i(1+a-\tfrac{\varepsilon}{3}) \\
& & k-1-i(1+a-\tfrac{\varepsilon}{3}) \\
& & -k+1+i(1+a-\tfrac{\varepsilon}{3}) \\
& & -k+1-i(1+a-\tfrac{\varepsilon}{3}) \\
C_1' &= l(C_1) \\
C_2 &= \{|z+k| = 1-\varepsilon\} \\
C_2' &= \{|z-k| = 1-\varepsilon\}
\end{aligned}$$

We have the following generators for these SG-curves:

$$\begin{aligned}
l_a : z &\mapsto \frac{k+1}{a+1} iz \\
h_{a,\varepsilon} : z &\mapsto \frac{k(1-\varepsilon)^{-1}z + (1-\varepsilon)(k^2(1-\varepsilon)^{-2}-1)}{(1-\varepsilon)^{-1}z + k(1-\varepsilon)^{-1}}
\end{aligned}$$

We now show that the group generated by these functions is non-classical, with particular values of a and ε. We prove the following:

Theorem 4.0.1 *The Schottky group* $J_{a,\varepsilon} = \langle l_a, h_{a,\varepsilon} \rangle$ *is non-classical for* $0 \leq a < 1.4$ *and for* $0 < \varepsilon < f(a)$ *for some function* f*.*

The function f is given explicitly in Appendix A, but we briefly describe it here. The function is the quotient of two expressions in integer powers of a, in integer powers of $\sqrt{k}$ for $k = \sqrt{2+2a+a^2}$ and in $J = (ak+3k+4+3a+a^2)^{\frac{1}{2}}(16k+7+10a-3a^2-16a^3-15a^4-6a^5-a^6+4k^3+4ak)^{\frac{1}{2}}$. the function f is positive increasing for $0 \leq a < 1.4$ and for inputs of this range of a it outputs numbers of the order of the bounds of ε in the two examples of non-classical Schottky groups in §4.1 and §4.2.

Proof. The proof follows the same skeleton as Yamamoto's proof, with different details. We assume that the group is classical and look for a contradiction. We assume the existance of $\hat{C}_1, \hat{C}'_1, \hat{C}_2$ and $\hat{C}'_2$ which are classical SG-curves for $J_{\alpha,\varepsilon}$. We state the lemmas which are equivalents to those in Yamamoto's paper, but in some cases we omit the proofs here. The methods of proof are the same, but the details are more unwieldy, with complicated regions and numbers. The notations of $\nu_{j\varepsilon}, u_{jk\varepsilon}$ and η are as in the previous sections, and $K_{a,\varepsilon}$ is the real line preserving group, $\langle h_{a,\varepsilon}, l_a^2\rangle$. We begin by listing the equivalents to Lemmas 4.1.4, 4.1.5 and 4.1.6 without proof, and then prove the equivalent to 4.1.3 afterwards for ease of notation and reading. We continue to use the notation of k defined above, and introduce other shorthands. The key feature is that all the shorthands are purely in terms of the variable a.

Lemma 4.2.6. *The length of each component of* $(\Omega(K_\varepsilon)\cap\mathbb{R})\cup(\Omega(lK_\varepsilon l^{-1})\cap i\mathbb{R})$ *which meets the intersection of intervals*

$$\left[-\frac{(k+1)^3}{(a+1)^2}, \frac{(k+1)^3}{(a+1)^2}\right] \cup i\left[-\frac{(k+1)^4}{(a+1)^3}, \frac{(k+1)^4}{(a+1)^3}\right]$$

is less than

$$\frac{2.01(k+1)^2\sqrt{\varepsilon}}{(1+a)^4}\sqrt{(1+a)^4+8k^2+4k(k^2+1)}$$

Lemma 4.2.7. *The length of each component of* $\Omega(K_\varepsilon)\cap\mathbb{R}$ *which meets the region* $[-k-1,-k+1]\cup[k-1,k+1]$ *is less than*

$$\frac{2.01\sqrt{\varepsilon}}{(1+a)(k+1)}\sqrt{(1+a)^4+8k^2+4k(k^2+1)}$$

Lemma 4.2.8. *If* $|u_{jk\varepsilon} - u_{j+1k\varepsilon}| < \eta$ *for at least two* $k \in \{1, 2, 3, 4\}$ *then*

$$\nu_{j\varepsilon} < \frac{2k(2(\mathcal{X}+1) + \sqrt{2}\mathcal{X})}{a+1}\eta$$

where $\mathcal{X}$ *is defined as:*

$$\frac{4k\sqrt{k}\sqrt{(a+3)k + k^2 + a + 2}(k-1)^{-1}}{2\sqrt{k}\sqrt{(a+3)k + k^2 + a + 2} - \sqrt{(1+a)(4k^2 - (1+a)^5) + 4k(4 + a + k^2)}}$$

Lemma 4.2.9. *For every* $1 \leq j < N$*,*

$$\nu_{j\varepsilon} < \frac{4.02k(2(\mathcal{X}+1) + \sqrt{2}\mathcal{X})(k+1)^2\sqrt{\varepsilon}}{(a+1)^5}\sqrt{(1+a)^4 + 8k^2 + 4k(k^2+1)}$$

We briefly explain how these tie together to prove the theorem. We let C be a circle meeting the region $\left[\frac{-(k+1)^5}{(1+a)^4}, -k-1\right]$ which is equivalent to $\hat{C}_1$ under the group generated by l^4. The diameter of the image of C under h_ε is greater than

$$\left| h_\varepsilon\left(\frac{-(k+1)^5}{(1+a)^4}\right) - h_\varepsilon(\infty)\right| = \left|\frac{-(a+1)^4(1-e)^2}{-(k+1)^5 + k(a+1)^4}\right| > 10^{-2}$$

Again, since C is outside C_2 its image under h_ε will meet $[k-1, k+1]$ twice.

The group is non-classical if the diameter of this circle is greater than the bound on $\nu_{j\varepsilon}$ above. That is that the group is non-classical if the following holds:

$$\frac{4.02k(2(\mathcal{X}+1) + \sqrt{2}\mathcal{X})(k+1)^2\sqrt{\varepsilon}}{(a+1)^5}\sqrt{(1+a)^4 + 8k^2 + 4k(k^2+1)} < 10^{-2}$$

This can be rearranged to give the function f in the statement of the theorem, thus proving that for a given ε (dependent on a) we have that $J_{a,\varepsilon}$ is non-classical.

Finally we need to show that we have $0 \leq a < 1.4$. Geometrically we need that $\nu_{j\varepsilon}$ is positive, which simplifies to

$$\mathcal{X} > \frac{-1}{2+\sqrt{2}}$$

This gives us the bound on a as in the theorem. $\square$

For completeness we give the definition of the function f in Appendix A, showing how ε relates to a.

Chapter 5

Further Work

5.1 Criterion for showing if a Schottky group is classical

Deciding whether a given Schottky group is classical or non-classical is a very difficult task, as discussed in §2.3, due to the freedom of choice of generator set and choice of SG-curves. We would like to be able to tell from any generator set whether the group is classical or not. One possible way would be to create an inequality into which we could enter information from the generator set, and if the inequality holds we have a classical Schottky group. Given information such as fixed points of the generators, and the multipliers we could insert this information into an inequality then from this decide if the group is classical on its given generators. We would expect that large multipliers with a long distance between fixed points would be classical, and small multipliers with close fixed points would indicate non-classical.

Taking this general theory and producing such inequalities is not trivial. We would not expect a complete answer from this method, simply a criterion for showing if a Schottky group was classical, another for if it was non-classical, and a grey area in between - that is to say that if a Schottky group didn't satisfy the classical inequality then it is not necessarily non-classical, and vice versa. Having looked at both an inequality to show classicality and an inequality to show non-classicality we have made some progress on the former, and so we mention this briefly now.

There is an obvious choice of curves to look at to give classical SG-curves for given generators, and that is to use isometric circles.

Definition 5.1.1. Given a loxodromic Möbius tranformation $g(z) = \frac{az+b}{cz+d}$, $ad - bc = 1$ we look at circles which are mapped to circles of the same radius by g. The point $\alpha = g^{-1}(\infty)$ is the centre of the isometric circle of g, and the point $\alpha' = g(\infty)$ is the centre of the isometric circle of g^{-1}. We have a unique circle, I, centred at α, which maps under g to a circle of the same radius centred at α'. This circle I is called the *isometric circle of g* and its image under g, $g(I) = I'$ is the *isometric circle of g^{-1}*.

Explicitly given in terms of a, b, c and d we can write the two isometric circles as

$$I = \left\{ \left| z + \frac{d}{c} \right| = \frac{1}{|c|} \right\}$$

$$I' = \left\{ \left| z - \frac{a}{c} \right| = \frac{1}{|c|} \right\}$$

If the isometric circles for a Schottky group do not intersect then the group is classical on its isometric circles.

If we have generator g_1 with fixed points a_1 and a_2 and multiplier γ^2 and generator g_2 with fixed points b_1 and b_2 multiplier σ^2 for a two generator Schottky group, $\Gamma = \langle g_1, g_2 \rangle$, we can look to get conditions from isometric circles. So that we have fewer variables in our inequality we apply a Möbius transformation to send the fixed points to -1 and 1 for g_1 and $-X$ and X for some X for g_2. From §2.1 we have that the transformations g_1 and g_2 can be written as:

$$g_1(z) = \frac{\frac{1}{2}(\gamma + \gamma^{-1})z + \frac{1}{2}(\gamma - \gamma^{-1})}{\frac{1}{2}(\gamma - \gamma^{-1})z + \frac{1}{2}(\gamma + \gamma^{-1})} \tag{5.1}$$

$$g_2(z) = \frac{\frac{1}{2X}(X\sigma + X\sigma^{-1})z + \frac{1}{2X}X^2(\sigma - \sigma^{-1})}{\frac{1}{2X}(\sigma - \sigma^{-1})z + \frac{1}{2X}(X\sigma + X\sigma^{-1})} \tag{5.2}$$

The simple conditions that will ensure that the isometric circles do not intersect are that the centres of the isometric circles must be more than the sums of the radii apart. We have six inequalities, one for each pair of circles. For simplicity, if we have $g_1(z) = \frac{az+b}{cz+d}$ and $g_2(z) = \frac{a'z+b'}{c'z+d'}$ then these six inequalities are:

$$\left|-\frac{d}{c} - \frac{a}{c}\right| > \frac{2}{|c|} \qquad \left|-\frac{d}{c} + \frac{d'}{c'}\right| > \frac{1}{|c|} + \frac{1}{|c'|} \qquad \left|-\frac{d}{c} - \frac{a'}{c'}\right| > \frac{1}{|c|} + \frac{1}{|c'|}$$
$$\left|-\frac{d'}{c'} - \frac{a'}{c'}\right| > \frac{2}{|c'|} \qquad \left|-\frac{d'}{c'} - \frac{a}{c}\right| > \frac{1}{|c|} + \frac{1}{|c'|} \qquad \left|\frac{a}{c} - \frac{a'}{c'}\right| > \frac{1}{|c|} + \frac{1}{|c'|}$$

When we substitute in the values of a, b, c, d, a', b', c' and d' from (5.1) and (5.2) into these inequalities, they simplify to give the following four inequalities.

$$\left|\gamma + \gamma^{-1}\right| > 2$$
$$\left|\sigma + \sigma^{-1}\right| > 2$$
$$\left|X(\sigma + \sigma^{-1})(\gamma - \gamma^{-1}) + (\gamma + \gamma^{-1})(\sigma - \sigma^{-1})\right| > |2X| \left|\gamma - \gamma^{-1}\right| + 2\left|\sigma - \sigma^{-1}\right|$$
$$\left|X(\sigma + \sigma^{-1})(\gamma - \gamma^{-1}) - (\gamma + \gamma^{-1})(\sigma - \sigma^{-1})\right| > |2X| \left|\gamma - \gamma^{-1}\right| + 2\left|\sigma - \sigma^{-1}\right|$$

If a Schottky group satisfies these conditions then it is classical with its isometric circles as its SG-curves.

The value of X is simple to find using the original four fixed points a_1, a_2, b_1 and b_2. If we let B denote the cross ratio of the four fixed points, that is $B = \frac{(a_1-b_2)(b_1-a_2)}{(a_1-a_2)(b_1-b_2)}$ then $X = 2B - 1 + 2\sqrt{B^2 - B}$.

As mentioned previously, if generators for a Schottky group do not satisfy the above equations then that does not mean that the group is non-classical, just that it is not classical on that generator set on isometric circles.

We now look at an example of a Schottky group satisfying the inequalities above, but not being classical on isometric circles for the given generators. We take the Schottky group $\Gamma_1 = \left\langle \frac{5z-3}{\frac{1}{3}z}, \frac{(4-30i)z+\frac{915}{2}i}{-2iz+(4+30i)} \right\rangle$, then we have multipliers $\frac{5+\sqrt{21}}{2}$ and $4+\sqrt{15}$ respectively. The fixed points are $\frac{15}{2} - \frac{3\sqrt{21}}{2}$ and $\frac{15}{2} + \frac{3\sqrt{21}}{2}$ for the first generator, and $15 - \frac{5i\sqrt{2}}{4}$ and $15 - \frac{5i\sqrt{2}}{4}$ for the second generator. Using the calculations above we see that $B = \frac{1}{2} + \frac{97i\sqrt{42}}{2520}$ and hence that $X = \frac{i\sqrt{42}}{1260}(97 + \sqrt{47209})$. We find that the inequalities are all satisfied, and so we know we have some set of isometric circles which are SG-curves for this group. The isometric circles however are not necessarily the isometric circles of the generators of Γ_1 since we have moved the fixed points to $\pm 1, \pm X$ and moving back to the fixed points of the generators above will preserve circles, but not whether they are isometric circles or not. We actually have isometric circles for different generators, but these in turn give us SG-curves which are circles. The isometric circles for the generators of Γ_1 given above intersect, but we do have classical SG-curves on these generators, as shown in Figure 5.1

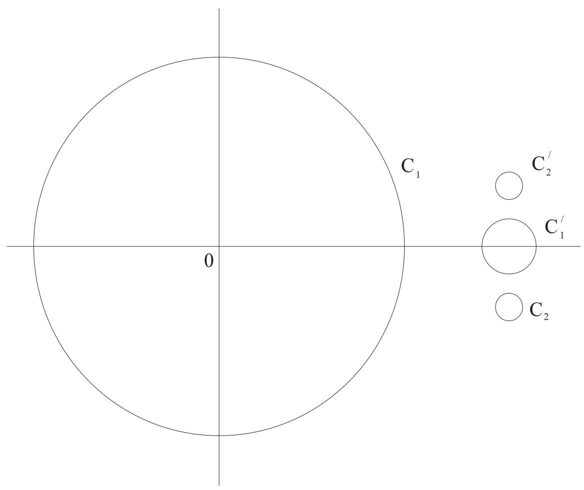

Figure 5.1: Classical SG-curves for Γ_1.

The circles in Figure 5.1 are $C_1 = \{|z| = 9\}$, $C_1' = \{|z-15| = 1\}$, $C_2 = \{|z-15+2i| = \frac{1}{2}$ and $C_2' = \{|z-15-2i| = \frac{1}{2}$.

Initial investigations at improving these inequalities or finding an inequality to show non-classicality have not yet been sucessful. This is partially due to the lack of non-classical examples on which to work, but particularly on the fact that showing a group is non-classical is a more difficult problem than showing that it is classical.

Finally it is worth noting that this question links to the comments on über-classical Schottky groups in §2.6. An über-classical Schottky group would be classical on all generators, so regardless of which generator set we used in improved inequalities we would find that they were satisfied. We might be able to use improved inequalities to define conditions which an über-classical Schottky group would satisfy, and hence prove or disprove their existence.

For an über-classical Schottky group to exist we would like to be able to use improved versions of these inequalities to investigate conditions for all possible generator sets to satisfy the inequalities.

5.2 Andrews-Curtis Graph

In §2.3.2 we described the three Nielsen transformations which are used to go from one generator set of a Schottky group to another. We look now at some questions which come about from thinking about these transformations in further detail. We turn our attention to a two generator Schottky group, Γ, and fix a base generator set as $\Gamma = \langle \gamma_1, \gamma_2 \rangle$. If we have a general generator set for Γ written as $\langle x_1, x_2 \rangle$, where each x_i is a word in $\gamma_1, \gamma_2, \gamma_1^{-1}$ and γ_2^{-1} then we can firstly label the three Nielsen transformations from §2.3.2 as:

A: $\langle x_1, x_2 \rangle \to \langle x_2, x_1 \rangle$

B: $\langle x_1, x_2 \rangle \to \langle {x_1}^{-1}, x_2 \rangle$

C: $\langle x_1, x_2 \rangle \to \langle x_2 x_1, x_2 \rangle$

We can then look at other generator sets for Γ as being multiple applications of Nielsen transformations A - C on $\langle \gamma_1, \gamma_2 \rangle$ due to the theorem of Nielsen [33] given previously (Theorem 2.2.6).

Any pair of generators that generate our group Γ can be thought of as being our base generator set with a finite number of Nielsen transformations applied. We are able to write any generator set for Γ in terms of applications of A, B and C to our base generator set $\langle \gamma_1, \gamma_2 \rangle$, for example, $\langle \gamma_2^{-1}, \gamma_2 \gamma_1 \rangle = BAC \langle \gamma_1, \gamma_2 \rangle$.

It may be of use to consider the graph $\mathcal{S}(\Gamma)$ constructed in the following way. The vertex set of $\mathcal{S}(\Gamma)$ corresponds to pairs of generators for Γ, where any incidences of xx^{-1} or $x^{-1}x$ have been simplified. Two vertices of $\mathcal{S}(\Gamma)$

are joined by an edge if you can get from one generator set to the other by A, B or C. The edges corresponding to A and B will not be directed edges since multiple application of either A or B simply moves back and forth between two pairs of generators. The edges corresponding to C will be a directed edge since multiple applications of C take us further from the original generator set. The graph will be 4-valent, since at any vertex v we can apply A, B and C, and there will also be a directed edge coming into the vertex corresponding to application of C to a different vertex v' such that $C(v') = v$. The following are some examples of rules that result in cycles on the graph, including:

$$ABAB = BABA \qquad CBCB = BCBC \qquad ACABA = CABCAB$$

It would firstly be interesting to know in greater detail properties of $\mathcal{S}(\Gamma)$.

If we take Γ to be a classical Schottky group we know from §2.3.2 that some generator sets for Γ may not necessarily have classical SG-curves. Chuckrow's construction [13] shows that Nielsen transformations can totally alter the shape of the SG-curves. It would be interesting to know which vertices of our graph $\mathcal{S}(\Gamma)$ correspond to generator sets with classical SG-curves. Let v_c be a vertex corresponding to a set of generators for which Γ is classical on those generators. Any vertex v_i joined to v_c by a sequence of A and B edges will also be classical on those generators, since A and B do not change the SG-curves. Some vertices joined to v_c by sequences including C may also be classical as in some cases the method of constructing new SG-curves can result in circles.

It would be interesting to know the answers to the following:

- Is the set of classical vertices connected?
- If not, is there a maximum/minimum number of connected classical sets of vertices?
- For a given base generator set for a classical Schottky group is there a maximum radius in terms of distinct edges travelled, after which the vertices correspond to generator sets without classical SG-curves?

A similar construction to this is used in studying the Andrews-Curtis conjecture, and it is interesting to see the links between the two graphs. We define the normal closure of a set first, and then state the Andrews-Curtis conjecture [4].

Definition 5.2.1. The *normal closure* of a set A in a group G is the smallest normal subgroup containing A.

Conjecture 5.2.2. Andrews-Curtis Conjecture [4] If F is free on the generators $x_1, ..., x_n$ and the normal closure of $\{r_1, ..., r_n\}$ is F, then $r_1, ..., r_n$ may be changed to $x_1, ..., x_n$ by a finite sequence of the operations below:

A'_i: $\langle a_1, ..., a_i, ..., a_n \rangle \to \langle a_i, ..., a_1, ..., a_n \rangle$

B': $\langle a_1, ..., a_n \rangle \to \langle {a_1}^{-1}, ..., a_n \rangle$

C': $\langle a_1, a_2, ..., a_n \rangle \to \langle a_1 a_2, a_2, ..., a_n \rangle$

D'_g: $\langle a_1, ..., a_n \rangle \to \langle g a_1 g^{-1}, ..., a_n \rangle \qquad$ for $\quad g \in G$

We can see that these transformations are related to our transformations A, B and C. A'_i and B' are just generalisations of our A and B. C' is obviously closely linked to our C, with right multiplication rather than left. Explicitly,

$$C'\langle x_1, x_2, ..., x_n\rangle \equiv A_2BA_2BCA_2BA_2B\langle x_1, x_2, ..., x_n\rangle$$

In [8] and [9] the authors introduce the Andrews-Curtis graph, which uses slightly different, but equivalent transformations:

(i) $\langle x_1, ..., x_i, ..., x_j, ..., x_k\rangle \to \langle x_1, ..., x_i x_j^{\pm 1}, ..., x_j, ..., x_k\rangle$

(ii) $\langle x_1, ..., x_i, ..., x_j, ..., x_k\rangle \to \langle x_1, ..., x_i x_j^{\pm 1}, ..., x_j, ..., x_k\rangle$

(iii) $\langle x_1, ..., x_i, ..., x_k\rangle \to \langle x_1, ..., x_i^{-1}, ..., x_k\rangle$

(iv) $\langle x_1, ..., x_i, ..., x_k\rangle \to \langle x_1, ..., x_i^w, ..., x_k\rangle \qquad \text{for} \quad w \in G$

These can easily be seen to be equivalent to A'_i - D'_g. Operation (iv) is equivalent to $A'_iD'_wA'_i$, (iii) is equivalent to $A'_iB'A'_i$. Depending on the sign in the index of x_j we have that

$$\begin{aligned} \text{(i)} \quad &\equiv \quad A'_jA'_2A'_iC'A'_iA'_2A'_j \\ \text{or} \quad & \quad\;\; A'_jB'A'_2A'_iC'A'_iA'_2B'A'_j \end{aligned}$$

and that

$$\begin{aligned} \text{(ii)} \quad &\equiv \quad A'_jA'_2A'_iB'C'B'A'_iA'_2A'_j \\ \text{or} \quad & \quad\;\; A'_jB'A'_2A'_iB'C'B'A'_iA'_2B'A'_j \end{aligned}$$

Similarly we can show that A'_i - D'_g can be written in terms of (i) - (iv). We can now look at the Andrews-Curtis graph $\triangle_n(G, N)$

We take a group G and $N \lhd G$, and look at the graph $\triangle_n(G, N)$ where the vertices are n-tuples of elements in N which generate N as a normal subgroup. We join two such vertices of $\triangle_n(G, N)$ by an edge if we can obtain one n-tuple from the other using one of the operations (i) - (iv). The link with the graph we were looking at for a Schottky group is that if we let the $N = G$ then the vertex set is those n-tuples which generate G. The edges are different, since they correspond to combinations of our operations A - C and extra edges due to the conjugation edges. The Andrews-Curtis conjecture can be rewritten as: *For $n \geq 2$, the Andrews-Curtis graph $\triangle_n(F_n, F_n)$ is connected.*

The Andrews-Curtis graph and the graph $\mathcal{S}(\Gamma)$ have many similarities, and there may be links between the two problems. It may be that one is a subgraph of the other. In [8] the authors mention that 'still virtually nothing is known about the properties of the Andrews-Curtis graph for free groups', and it would seem more likely that for any progress to be made with the classical Schottky group questions mentioned above, progress would first have to be made with properties of the graph $\mathcal{S}(\Gamma)$ or the Andrews-Curtis graph.

This problem also has links to über-classical Schottky groups. We could investigate the question of the existance of such groups via the graph $\mathcal{S}(\Gamma)$in §5.2. As discussed, it would be interesting to know if there is a maximum radius of classicality to the graph, and if the radius is infinite then we would have the existance of an über-classical Schottky group.

Appendix A

Explicit Formula for ε

As discussed in §4.2.2 we now give an explicit formula for the bound on ε in terms of a. We set a few preliminary shorthands for complicated expressions in a; firstly we use k to denote the following:

$$k = \sqrt{2 + 2a + a^2}$$

We then define J in terms of a and k as follows:

$$\begin{aligned} J &= \left(ak + 3k + 4 + 3a + a^2\right)^{\frac{1}{2}} \\ &\quad \left(16k + 7 + 10a - 3a^2 - 16a^3 - 15a^4 - 6a^5 - a^6 + 4k^3 + 4ak\right)^{\frac{1}{2}} \end{aligned}$$

Our bound on ε will be given in terms of three expressions, A, B and C which are in terms of a, J and k. We give the three definitions of A, B and C on the next few pages, and finally give the inequality giving us the bound on ε in terms of A, B and C.

$$
\begin{aligned}
A = \quad & -31 + 3640a^{11} + 552a^{13} + 16a^{15} + 120a^{14} - 352a - 3066a^{8} \\
& -1832a^{2} + 3440a^{9} - 18488a^{6} + 5096a^{10} - 12224a^{7} - 17976a^{5} \\
& -12180a^{4} - 5768a^{3} + a^{16} + 840a^{4}J\sqrt{k} + 4J\sqrt{k} \\
& -32k - 336ak + 480a^{3}J\sqrt{k} + 40a^{9}J\sqrt{k} + 1008a^{5}J\sqrt{k} \\
& -4k^{3} + 4a^{10}J\sqrt{k} - 8208a^{7}k - 4200a^{8}k - 1520a^{9}k - 840a^{6}k^{3} \\
& -372a^{10}k - 56a^{11}k - 480a^{7}k^{3} + 840a^{6}J\sqrt{k} - 8820a^{4}k \\
& -11904a^{5}k - 40ak^{3} - 180a^{2}k^{3} - 11592a^{6}k - 840a^{4}k^{3} \\
& -480a^{3}k^{3} - 180a^{8}k^{3} - 40a^{9}k^{3} - 4a^{10}k^{3} - 4a^{12}k - 4600a^{3}k \\
& +180a^{2}J\sqrt{k} + 480a^{7}J\sqrt{k} - 1604a^{2}k + 40aJ\sqrt{k} \\
& +180a^{8}J\sqrt{k} - 1008a^{5}k^{3} + 1708a^{12}
\end{aligned}
$$

$$
\begin{aligned}
B = \quad & -337918 + 8516a^{11} + 974a^{13} + 16a^{15} + 153a^{14} - 1628226a \\
& -674679a^{8} + 3983a^{12} - 3979981a^{2} - 153462a^{9} - 3918453a^{6} \\
& -9271a^{10} - 1905392a^{7} - 6085238a^{5} - 7176507a^{4} - 6318596a^{3} \\
& +a^{16} + 592\sqrt{2}aJ\sqrt{k^{5}} + 736\sqrt{2}aJ\sqrt{k^{7}} + 88\sqrt{2}a^{2}J\sqrt{k^{13}} \\
& +88\sqrt{2}a^{2}J\sqrt{k^{9}} + 16\sqrt{2}a^{3}J\sqrt{k^{13}} + 16\sqrt{2}a^{3}J\sqrt{k^{9}} \\
& +4\sqrt{2}a^{4}J\sqrt{k^{13}} + 4\sqrt{2}a^{4}J\sqrt{k^{9}} + 144\sqrt{2}aJ\sqrt{k^{13}} \\
& -4976\sqrt{2}a^{11} - 112216k + 680\sqrt{2}a^{2}J\sqrt{k^{5}} - 1976\sqrt{2}a^{10}k \\
& -477648ak - 136896\sqrt{2} - 49824\sqrt{2}k - 6292k^{7} \\
& -512\sqrt{2}a^{12} - 6316k^{3} - 2656\sqrt{2}k^{7} - 3258000\sqrt{2}a^{4} \\
& +768\sqrt{2}a^{2}J\sqrt{k^{7}} + 16\sqrt{2}a^{3}J\sqrt{k^{11}} + 464\sqrt{2}a^{3}J\sqrt{k^{7}} \\
& +4\sqrt{2}a^{4}J\sqrt{k^{11}} + 196\sqrt{2}a^{4}J\sqrt{k^{7}} + 144\sqrt{2}aJ\sqrt{k^{11}} \\
& +8\sqrt{2}a^{6}J\sqrt{k^{5}} - 1702144\sqrt{2}a^{2} + 144\sqrt{2}aJ\sqrt{k^{9}} \\
& +8\sqrt{2}a^{6}J\sqrt{k^{7}} + 48\sqrt{2}a^{5}J\sqrt{k^{7}} + 448\sqrt{2}a^{3}J\sqrt{k^{5}} \\
& +192\sqrt{2}a^{4}J\sqrt{k^{5}} + 48\sqrt{2}a^{5}J\sqrt{k^{5}} + 88\sqrt{2}a^{2}J\sqrt{k^{11}} \\
& -280\sqrt{2}a^{5}k^{5} - 152\sqrt{2}a^{11}k - 40\sqrt{2}a^{6}k^{5} - 34352a^{2}k^{3} \\
& -5336\sqrt{2}ak^{5} - 223968\sqrt{2}ak - 224\sqrt{2}a^{2}k^{9} - 672\sqrt{2}ak^{9} \\
& -789160\sqrt{2}a^{4}k - 24\sqrt{2}a^{13} + 12a^{4}J\sqrt{k^{13}} + 16a^{4}J\sqrt{k^{9}} \\
& +16a^{4}J\sqrt{k^{11}} + 432aJ\sqrt{k^{13}} + 576aJ\sqrt{k^{9}} + 576aJ\sqrt{k^{11}} \\
& -678304\sqrt{2}a - 2778816\sqrt{2}a^{3} - 266152a^{7}k - 85056a^{8}k \\
& +9136a^{6}k^{3} - 2860a^{10}k - 196a^{11}k + 7840a^{7}k^{3} + 352a^{2}J\sqrt{k^{9}} \\
& -19392a^{9}k - 1471244a^{4}k - 1098480a^{5}k - 20964ak^{3}
\end{aligned}
$$

$$
\begin{aligned}
C = \quad & -15800a^4k^3 + 1612a^5k^3 - 32360a^3k^3 + 3960a^8k^3 + 1344a^9k^3 \\
& +64a^3k^9 + 60a^4k^9 + 24a^5k^9 + 4a^6k^9 + 4a^{12}k^3 + 4a^{12}k \\
& +48a^{11}k^3 + 48a^3J\sqrt{k^{13}} + 64a^3J\sqrt{k^9} + 64a^3J\sqrt{k^{11}} \\
& -5352\sqrt{2}a^2k^5 - 1872\sqrt{2}a^3k^7 - 3912\sqrt{2}ak^7 - 362120\sqrt{2}a^6k \\
& -161544\sqrt{2}a^7k - 53720\sqrt{2}a^8k - 752448\sqrt{2}a^3k - 614504\sqrt{2}a^5k \\
& -12760\sqrt{2}a^9k - 510624\sqrt{2}a^2k - 3120\sqrt{2}a^3k^5 - 1200\sqrt{2}a^4k^5 \\
& -720\sqrt{2}a^4k^7 - 168\sqrt{2}a^5k^7 - 24\sqrt{2}a^6k^7 - 3448\sqrt{2}a^2k^7 \\
& +264a^2J\sqrt{k^{13}} - 252a^5k^7 - 436a^2k^9 - 32a^6k^7 + 404\sqrt{2}J\sqrt{k^7} \\
& +16\sqrt{2}J\sqrt{k^{17}} + 148\sqrt{2}J\sqrt{k^{11}} + 164\sqrt{2}J\sqrt{k^{13}} + 312a^{10}k^3 \\
& +148\sqrt{2}J\sqrt{k^9} + 16\sqrt{2}J\sqrt{k^{19}} + 32\sqrt{2}J\sqrt{k^{15}} - 1464276a^3k \\
& +272\sqrt{2}J\sqrt{k^5} + 1496J\sqrt{k^7} - 896\sqrt{2}k^9 + 772a^4J\sqrt{k^5} \\
& -13080ak^5 - 12860a^2k^5 - 84a^6k^5 - 600a^5k^5 - 2700a^4k^5 \\
& -1448ak^9 - 8876ak^7 - 1252a^4k^7 - 3656a^3k^7 - 32ak^{11} \\
& -131840\sqrt{2}a^9 - 1030992\sqrt{2}a^7 - 1952848\sqrt{2}a^6 - 2873800\sqrt{2}a^5 \\
& -2848\sqrt{2}k^5 + 352a^2J\sqrt{k^{11}} + 32a^6J\sqrt{k^5} + 48J\sqrt{k^{19}} \\
& +112J\sqrt{k^{15}} + 64J\sqrt{k^{17}} - 7124k^5 - 1039132a^2k - 96k^{11} \\
& -7384a^2k^7 + 524J\sqrt{k^{13}} + 2512aJ\sqrt{k^7} + 2504a^2J\sqrt{k^7} \\
& +592J\sqrt{k^9} + 608J\sqrt{k^{11}} + 2512aJ\sqrt{k^5} + 2808a^2J\sqrt{k^5} \\
& +1440a^3J\sqrt{k^7} + 1808a^3J\sqrt{k^5} + 600a^4J\sqrt{k^7} - 7296a^3k^5 \\
& +144a^5J\sqrt{k^7} + 192a^5J\sqrt{k^5} + 24a^6J\sqrt{k^7} + 1220J\sqrt{k^5} \\
& -30576\sqrt{2}a^{10} - 621272a^6k - 421872\sqrt{2}a^8 - 2012k^9
\end{aligned}
$$

We then combine these three terms to get our bound on ε which depends only on a:

$$\varepsilon < \frac{A}{646416(B+C)}$$

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