LOWER SECONDARY SCHOOL STUDENTS' UNDERSTANDING OF ALGEBRAIC PROOF

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Secondary school students are known to face a range of difficulties in learning about proof and proving in mathematics. This paper reports on a study designed to address the issue of students' cognitive needs for conviction and verification in algebraic statements. Through an analysis of data from 418 students (206 from Grade 8, and 212 from Grade 9), we report on how students might be able to 'construct' a formal proof, yet they may not fully appreciate the significance of such formal proof. The students may believe that formal proof is a valid argument, while, at the same time, they also resort to experimental verification as an acceptable way of 'ensuring' universality and generality of algebraic statements.

INTRODUCTION

Evidence from a range of research studies indicates that across the world, secondary school students have difficulties in following and constructing formally presented deductive proofs, in understanding how such proofs differ from empirical evidence, and in using deductive proofs to derive further results (for recent reviews, see, for example, Mariotti, 2007). As part of a wider research initiative, we have researched such issues in the case of geometrical proofs (see, for example, Kunimune, 1987; Kunimune, Fujita and Jones, 2009). In this paper we address the issue of students' natural cognitive needs for conviction and verification in algebraic statements.

In what follows, we first provide some background from related existing research. We then outline our theoretical framework which seeks to capture secondary school students' understanding of algebraic proof. This leads to the presentation of our results in terms of how students in lower secondary schools perceive 'proof' in algebra through an analysis of data from 418 students (206 from Grade 8, and 212 from Grade 9) collected in Japan in 2005.

STUDENTS' UNDERSTANDING OF ALGEBRAIC PROOF

Of the range of research studies on students and algebraic proof, we highlight two studies that are particularly pertinent. Healy and Hoyles (2000) surveyed high-attaining 14- and 15-year-old students about proof in algebra and found that students simultaneously held two different conceptions of proof. On the one side, the students

Cite as: Kunimune, S., Kumakura, H., Jones, K. and Fujita, T. (2009) Lower secondary school students' understanding of algebraic proof. In, Tzekaki, M., Kaldrimidou, M. and Sakonidis, H. (eds.) *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (PME33), Volume 3, pp441-448.

viewed algebraic arguments as those they considered would receive the best mark from their teacher. On the other side, empirical argument predominated in students' own proof constructions, although most students were aware of the limitations of such arguments.

Similarly, Groves and Doig (2008) report that while over 35% of Year 8 students (aged 13) recognised the need for a logical mathematical explanation to prove Goldbach's conjecture, over 60% "believed that it was enough to show it true for at least 1000 randomly chosen numbers or as many as possible, or to find one number for which it was not true" (p 345).

Such studies illustrate the need to continue studying students' cognitive needs for conviction and verification in algebraic proofs.

THEORETICAL FRAMEWORK

Extensive research in algebra education (for recent reviews, see, for example, Kieran, 2006) suggests the following issues are relevant to students' understanding of algebraic proof:

- Cognitive gap between arithmetic and algebra
- Understanding of algebraic symbols

For example, consider the statement: 'Sums of three consecutive whole numbers such as 2, 3 & 4 or 7, 8&9 are always multiples of 3'. To 'prove' this statement (that is, to verifying its generality) a secondary school student might do one of the following:

- Use arithmetic examples, sometimes with large numbers, and check results. For example, the student might say 'I tried 4+5+6=15=3x5, 12+13+14=39=3x13, 23+24+25=72=3x24, and so one, and I found the answers are always multiples of 3.
- Use algebraic symbolisation to provide an argument which might say three consecutive numbers can be expressed as 'x', 'x+1' and 'x+2'; the sum is '3x+3'. Now '3x+3 = 3(x+1)'. This shows that the sum is always a multiple of three.

Given our focus on students' understanding of proof, and the knowledge that the transition from experimental/empirical verification to formal proof is not straightforward, in our research we capture students' understanding of proof in terms of the following two components: 'Generality of proof' and 'Construction of proof' (see, Kunimune, 1987; Kunimune, Fujita and Jones, 2009). In our work on students' understanding of algebraic proof, we refer to these two aspects of proof and proving as: 'Construction of algebraic proof' and 'Generality of algebraic proof'.

The first one of these, 'Generality of proof' recognizes that, on the one hand, students have to understand the generality of proof (including the generality of algebraic symbols, with, for example, 'x' as generalised number), the universality and generality of proved algebraic statements, the difference between formal proof and verification by examples, and so on. The second of these two components, ‘Construction of proof’, recognises that, on the other hand, students also have to learn how to ‘construct’ deductive arguments in algebra by knowing sufficient about definitions, assumptions, proofs, theorems, logical circularity, and so on.

In Table 1 we characterise the nature of the two aspects of student proof and proving in algebra: ‘Construction of algebraic proof’ and ‘Generality of algebraic proof’ using ideas related to the cognitive gap between arithmetic and algebra, and student understanding of algebraic symbols.

Construction of algebraic proof	**Generality of algebraic proof**
To follow, or construct, algebraic proof, students might have to: • Understand what is required to show/explain in given problems • Understand assumptions and conclusions in statements • Represent given word problems by using algebraic symbols, interpret algebraic results etc. • Undertake fundamental algebraic manipulations; for example: 3x+3 = 3(x+1), 2x+y+3x-6y = 5x-5y and so on	To appreciate or understand why formal proof is necessary, students might have to: • Understand the universality and generality of statements which are represented by algebraic symbols • Understand the universality and generality of algebraic symbols • Understand the universality and generality of proof • Understand difference between formal proof and experimental verification (inductive approach)

Table 1: The two aspects of students’ understanding of algebraic proof

Given these two aspects of student proof, our theoretical approach, informed by our work on proof in geometry (see, for example, Kunimune, 1987; Kunimune, Fujita and Jones, 2009) is to characterise four levels of students’ understanding of algebraic proof. This characterisation is presented in Table 2. We argue that this framework captures the increasing complexity in students’ attempts at construction of algebraic proof and generality of algebraic proof.

METHODOLOGY

The research design involved a survey of secondary school students’ understanding of algebraic proof. A sample of relevant questions, and the corresponding marking scheme, is provided in Appendix A.

	Construction of algebraic proof	Generality of algebraic proof
Level 0	At this level, students do not understand what they have to explain.	At this level, students do not understand what they have to explain.
Level I	At this level, students explain their argument without using any algebraic symbols	At this level, students do not understand neither why algebraic proof is necessary nor empirical verification is not enough to verify the universality and generality of algebraic statements
Level II	At this level, students start using algebraic symbols in their argument, but their use is incorrect	At this level, two things occur: a) students start recognising that empirical verification is not enough, but do not understand why they have to use algebraic symbols b) students start understanding why algebraic proof is necessary, but do not recognise that empirical verification is not enough
Level III	At this level, students use algebraic symbols properly to prove statements	At this level, students can understand why algebraic proof is necessary

Table 3: levels of students' understanding of algebraic proof

FINDINGS AND DISCUSSION

Our data is from 418 students (206 from Grade 8, and 212 from Grade 9) surveyed in Japan in 2005. The results for 'Construction of proof' are given in Table 4.

	Level 0	**Level I**	**Level II**	**Level III**	**N**
G8	64%	6%	10%	20%	G8=206
G9	29%	4%	14%	53%	G9 = 212

Table 4: results for 'Construction of proof'

As can be seen from Table 4, some 70% of Grade 8 students are at the Level I or below; that is, these students use empirical examples to verify the statements (Level I) or do not know what to do (Level 0). The results of Grade 9 students are superior. This is likely to be because students study more algebraic manipulation and proof in

Grade 9. Nevertheless, 33% of students remain at either Level 0 or I, which implies that the teaching of algebraic proof could be improved in Grades 8 and 9.

The results for 'Generality of proof' are given in Table 5.

	Level 0	**Level I**	**Level II(a) and (b)**	**Level III**	**N**
G8	15%	36%	4%&26%	19%	G8=206
G9	11%	23%	3%&24%	39%	G9 = 212

Table 5: results for 'Generality of proof

These results suggest that at Grades 8 and 9, students begin pondering the difference between empirical verifications and proof. This is, as indicated above, because students study more algebraic manipulations and proof in Grades 8 and 9. In general, more students are at Level II-b) than Level II-a). This implies that the students start understanding why algebraic proof is necessary in Grade 8, yet, at the same time, they do not recognise that empirical verification is insufficient for mathematical proof. Furthermore, 58% (11+23+24) of Grade 9 students remain at Level II-b), Level I or Level 0. These findings are very similar to the findings for a parallel study on geometrical proof (Kunimune, 1987, 2000; Kunimune, Fujita and Jones, 2009).

Thus, Grade 8 and 9 students are achieving in terms of 'Construction of proof', but not necessarily in terms of 'Generality of proof'. There is a gap between the two aspects. This means that students might be able to 'construct' a formal proof, yet they may not appreciate the significance of such a formal proof. They may believe that formal proof is a valid argument, while, at the same time, they also believe experimental verification is equally acceptable to 'ensure' universality and generality of algebraic statements.

We now compare students' Construction of proof (CoP) and their Generality of proof (GoP) at Grade 8 and Grade 9, see Table 6.

Grade 8 totals	**15%**	**36%**	**30%**	**19%**	**100%**	**N=206**
CoP III	0%	1%	6%	12%	19%	
CoP II	0%	3%	4%	3%	10%	
CoP I	0%	4%	2%	1%	7%	
CoP 0	15%	28%	18%	3%	64%	
Levels	GoP 0	GoP I	GoP II	GoP III	Total	

Grade 9 totals	11%	23%	27%	39%	100%	N=212
CoP III	0%	4%	14%	35%	53%	
CoP II	1%	4%	6%	3%	14%	
CoP I	1%	2%	1%	0%	4%	
CoP 0	9%	13%	5%	1%	29%	
Levels	GoP 0	GoP I	GoP II	GoP III	Total	

Table 6: compare students' Construction of proof (CoP) and their Generality of proof

The results in Table 6 show that, on the one hand, progressions from CoP I and CoG I to CoPII and CoG II are observed in Grade 9, when students study more algebra than in Grade 8. In addition, students are introduced to ideas of 'proof' in geometry, and this is likely to contribute to students' awareness of formal proof. On the other hand, in Grade 9 some 18% (=14+4) of students at CoP Level III are, at the same time, at GoP Level II or I (indicated in gray). This suggests that the teaching of algebra that the students might have experienced might have particularly emphasised the 'Construction of proof' aspects of algebra.

In general, more students are at Level II-b) than Level II-a). This implies that students start understanding why algebraic proof is necessary in Grade 8, but do not recognise that empirical verification is not enough. Furthermore, half of the Grade 9 students remain at Level II-b) or below. These findings are very similar to those that we have found with geometrical proof (Kunimune, 1987, 2000; Kunimune, Fujita and Jones, 2009).

CONCLUDING COMMENT

The Grade 8 and 9 students that we studied are achieving in terms of 'Construction of proof', but not necessarily so in terms of 'Generality of proof'. There is a gap between the two aspects. This means that students might be able to 'construct' a formal proof, yet they may not appreciate the significance of such formal proof. They may believe that formal proof is a valid argument, while, at the same time, they also resort to experimental verification as an acceptable way of 'ensuring' universality and generality of algebraic statements.

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Appendix A

Survey questions

Q8 Consider three consecutive whole numbers and their sums, e.g. 2+3+4=9=3x3, 7+8+9=24=3x8, and they are always the multiples of 3. In fact, if you consider any three consecutive numbers, then their sums are always the multiples of 3. Explain this.

Q9. See the calendar below carefully. You might notice that the sums of the three numbers in the boxes are as three times as the middle numbers (e.g. 2+9+16=27 = 3x9). Explain this is always true.

Sun	Mon	Tue	Wed	Thu	Fri	Sat
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

Q10. Read carefully the following three statements which explain a statement 'A sum of two odd numbers is an even number'.

Student A: 1+1=2, 3+3=6, 1+3=4, 3+7=10. So, I think a sum of two odd numbers is an even number.	Accept/Not accept
Student B: Let one odd number be 'm', and the other 'n'. The sum is 'm+n', and 'm+n' is an even number. I think a sum of two odd numbers is an even number.	Accept/Not accept

Student C: Let ‘m’ and ‘n’ be whole numbers. Two odd numbers are ‘2m+1’ and ‘2n+1’ The sum is (2m+1)+(2n+1) = 2m+2n+2=2(m+n+1). As ‘m+n+1 is a whole number, therefore 2(m+n+1) is an even number.	Accept/Not accept

Table 4: survey questions of students’ understanding of algebraic proof

	Construction of proof	Generality of proof
Level 0	Q8 & 9 • No answer • Does not make sense • Copy questions • Wrong explanation	Q10 • No answer
Level I	Q8 & 9 • Explanations with concrete examples, figures or words • Incomplete explanations with words • Explanations with concrete examples and arithmetic calculations	Q10 Answers such as: A: Accept; B: Accept; C: Not accept or A: Accept; B: Accept; C: Accept
Level II	Q8 & 9 • Incomplete or incorrect explanations with algebraic symbols	Q10 Level II(a) A: Not accept; B: Accept; C: Accept Level II(b) A: Accept; B: Not accept; C: Accept
Level III	Q8 & 9 • Explanations with algebraic symbols • Explanations with algebraic symbols with examples	Q10. A: Not accept; B: Not accept; C: Accept