PROPERTY A AND CAT(0) CUBE COMPLEXES

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ABSTRACT. Property A is a non-equivariant analogue of amenability defined for metric spaces. Euclidean spaces and trees are examples of spaces with Property A. Simultaneously generalizing these facts, we show that finite dimensional CAT(0) cube complexes have Property A. We do not assume that the complex is locally finite. We also prove that given a discrete group acting properly on a finite dimensional CAT(0) cube complex the stabilisers of vertices at infinity are amenable.

INTRODUCTION

This paper is devoted to the study of Property A for finite dimensional CAT(0) cube complexes. These spaces, which are higher dimensional analogues of trees, appear naturally in many problems in geometric group theory and low dimensional topology [AR90, Ch07, HW07, Sag97, Wis04]. Property A was introduced by Yu as a non-equivariant generalisation of amenablity from the context of groups to the context of discrete metric spaces. It was used with great effect in his attack on the Baum Connes conjecture, in which he proved, among other things, that Gromov's δ-hyperbolic spaces, and hence hyperbolic groups, satisfy Property A, even though they may be very far from amenable [Yu00].

In this paper we prove:

Theorem. *Let X be a finite dimensional* CAT(0) *cube complex. Equipped with the geodesic metric, X has Property A. The vertex set of X, equipped with the edge-path metric has Property A.*

The proof of the theorem rests on the often used statement that intervals in a CAT(0) cube complex admit combinatorial embeddings into Euclidean spaces. While this fact appears several times in the literature no proof has been published and we take the opportunity to

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provide one here. Our proof of this generalises to intervals in measured wall spaces, though we omit the details here as this is not relevant to the current application.

While interval embeddings exist they are far from unique. Any given interval may admit a large number of such embeddings in spaces of varying dimensions and the embeddings may be very different from one another. For each embedding the target interval fibres over the image, and again these fiberings vary considerably. Nonetheless it is a remarkable fact that regardless of how we embed the interval into Euclidean space the norms of the functions we are computing on each fibre are independent of the embedding chosen.

Our technique may well have other applications and we present one here. A group acting properly on an Hadamard space, a building for example, fixing a point in a suitable refinement of the visual boundary is amenable [Cap07]. In the context of CAT(0) cube complexes the natural choice for the boundary is the combinatorial boundary.

Theorem. *A countable group acting properly on a finite dimensional* CAT(0) *cube complex and fixing a vertex at infinity is amenable.*

The advantage to working with the combinatorial boundary rather than the refined Hadamard boundary is that it is typically much smaller. One might expect the cost of this to be somewhat larger stabilisers at infinity, however our theorem shows that this is not the case. The stabilisers at infinity in both cases are virtually abelian of rank bounded by the dimension of the cube complex.

Our main theorem is known to be false for infinite dimensional cube complexes [Now07], thus our result is the best possible. While it is already known for finite dimensional CAT(0) cube complexes admitting a cocompact action by a countable discrete group [CN05], the approach taken there involved a deformation of the standard embedding of the cube complex in Hilbert space and rested on a functional analytic argument involving the uniform Roe algebra to conclude Property A (see [GK04] and [BNW07]). That approach is ultimately unsuitable for non-locally finite complexes. Here, we shall remove the assumption of local finiteness by offering a direct proof of Property A in which the asymptotically invariant functions called for in Yu's non-equivariant generalisation of the Følner criterion are explicitly constructed. Furthermore we do not require the existence of a group action to make this argument work. The problem of clarifying the relationship between Property A and coarse embeddability (in Hilbert space) has attracted some attention lately, and indeed was a motivation for our study. As a consequence of the above theorem, and the coarse invariance of Property A, we obtain the following corollaries.

Corollary. *A metric space that coarsely embeds in a finite dimensional* CAT(0) *cube complex has Property A.*

Corollary. *A countable discrete group acting metrically properly on a finite dimensional* CAT(0) *cube complex has Property A.*

Indeed to conclude property A for a group it would, according to our theorem, be sufficient for the group to embed uniformly in a finite dimensional CAT(0) cube complex, with no equivariance assumptions on the embedding.

Putting the corollaries in perspective, one can use an approximation argument to show that a metric space which coarsely embeds in Hilbert space coarsely embeds in an infinite dimensional CAT(0) cube complex. (This follows from the observations that the infinite dimensional Euclidean space $\mathbb{R}^\infty$ is an infinite dimensional cube complex and a dense subset of the Hilbert space ℓ^2.)

1. Preliminaries

1.1. **Property A.** In his work on the Novikov conjecture Yu introduced *Property A* [Yu00]. There are now several variants of the basic definition, all of which are equivalent for spaces of *bounded geometry*; see for example [HR00, Tu01, DG05]. We, however, intend to study spaces that do *not* have bounded geometry and shall restrict ourselves to the definition below. The definition we have chosen is the strongest, implying all others in full generality.

Before formally introducing Property A we recall some elementary notions from coarse geometry. Let X and Y be metric spaces. A function $\phi : X \to Y$ is a *coarse embedding* if:

(a) For every $A > 0$ there exists $B > 0$ such that

$$d(x, x') < A \Rightarrow d(\phi(x), \phi(x')) < B.$$

(b) For every $B > 0$ there exists $A > 0$ such that

$$d(\phi(x), \phi(x')) < B \Rightarrow d(x, x') < A.$$

A subset $Z \subset Y$ is *coarsely dense* if there exists $C > 0$ such that for every $y \in Y$ there exists $z \in Z$ such that $d(y, z) < C$. A coarse embedding $\phi : X \to Y$ is a *coarse equivalence* if its image is coarsely dense in Y. If there is a coarse equivalence $X \to Y$ the metric space X is *coarsely equivalent* to Y. Although not apparent, coarse equivalence is an equivalence relation.

Proposition 1.1. *Every metric space contains a discrete coarsely dense subset. In particular, every metric space is coarsely equivalent to a discrete metric space.*

Proof. A straightforward application of Zorn's lemma. $\square$

Definition 1.2. A discrete metric space X has *Property A* if for every $R > 0$ and every $\varepsilon > 0$ there exists an $S > 0$ and a family of finite non-empty subsets $A_x \subset X \times \mathbb{N}$, indexed by $x \in X$, such that:

(a) For every x, $x' \in X$ with $d(x,x') < R$ we have $\frac{|A_x \Delta A_{x'}|}{|A_x|} < \varepsilon$.
(b) For every $(x',n) \in A_x$ we have $d(x,x') \leq S$.

An arbitrary metric space X has Property A if it contains a discrete coarsely dense subset with Property A.

Remark. We shall see presently that if one discrete coarsely dense subset of a metric space has Property A then every such subset has Property A (see Proposition 1.4 below).

Proposition 1.3. *Let X and Y be discrete metric spaces. If X is coarsely embeddable in Y and Y has Property A then X has Property A.*

Proof. Let $\phi : X \to Y$ be a coarse embedding. Let $\psi : Y \to X$ be a function satisfying

$$d(\phi(\psi(y)), y) \leq d(\phi(X), y) + 1.$$

Let $R > 0$ and $\varepsilon > 0$. Since ϕ is a coarse embedding there exists $R' > 0$ such that

$$d(x,x') < R \Rightarrow d(\phi(x), \phi(x')) < R'.$$

Since Y has Property A there is a family $\{\, B_y \,\}_{y \in Y}$ and an S' satisfying the conditions of Definition 1.2 for R' and ε. Define

$$A_x = \{\, (x',n) \in X \times \mathbb{N} : n \leq |\{\, (y,m) \in B_{\phi(x)} : \psi(y) = x' \,\}| \,\}$$

and, using once more the fact that ϕ is a coarse embedding, we obtain S such that

$$d(\phi(x), \phi(x')) \leq 2S' + 1 \Rightarrow d(x,x') \leq S.$$

The family $\{\, A_x \,\}_{x \in X}$ and S satisfy the conditions of Definition 1.2 for R and ε. Indeed, if $(x',n) \in A_x$ then there exists $(y,m) \in B_{\phi(x)}$ such that $\psi(y) = x'$. It follows that $d(\phi(x), y) \leq S'$ and

$$d(\phi(x), \phi(x')) \leq d(\phi(x), y) + d(y, \phi(x')) = d(\phi(x), y) + d(y, \phi(\psi(y))) \leq 2S' + 1,$$

hence also $d(x,x') \leq S$. Finally, suppose $d(x,x') \leq R$. Then $d(\phi(x), \phi(x')) \leq R'$ so that

$$\frac{|A_x \Delta A_{x'}|}{|A_x|} \leq \frac{|B_{\phi(x)} \Delta B_{\phi(x')}|}{|B_{\phi(x)}|} < \varepsilon. \quad \square$$

Proposition 1.4. *Property A is a coarse invariant of discrete metric spaces. Precisely, if X and Y are coarsely equivalent discrete metric spaces then X has Property A if and only if Y has Property A.*

Proof. If X and Y are coarsely equivalent then each is coarsely embeddable in the other. □

We shall work exclusively with the following characterisation of Property A.

Proposition 1.5. *A discrete metric space X has Property A if and only if there exists a sequence of families of finitely supported functions $f_{n,x} : X \to \mathbb{N} \cup \{0\}$, indexed by $x \in X$, and a sequence of constants $S_n > 0$, such that:*

(a) *For every n and x the function $f_{n,x}$ is supported in $B_{S_n}(x)$.*
(b) *For every $R > 0$*

$$\frac{\|f_{n,x} - f_{n,x'}\|}{\|f_{n,x}\|} \to 0$$

uniformly on the set $\{(x, x') : d(x, x') \leq R\}$ as $n \to \infty$.

Furthermore, if X is the vertex set of a graph, equipped with the edge-path metric, it is sufficient to require (b) *only for $R = 1$.*

Remark. The norm $\|\cdot\|$ is the ℓ^1-norm on the space of (finitely supported) functions on X. This is the only norm we shall encounter.

Proof. Both Property A and the conditions in the proposition are equivalent to the following statement: for every $R > 0$ and $\varepsilon > 0$ there exists a family of finitely supported functions $f_x \colon X \to \mathbb{N} \cup \{0\}$, indexed by $x \in X$, and an $S > 0$ such that f_x is supported in $B_S(x)$, and

$$d(x, x') \leq R \Longrightarrow \frac{\|f_x - f_{x'}\|}{\|f_x\|} < \varepsilon.$$

The equivalence with the conditions of the proposition is elementary. The equivalence with Property A is given by mapping A_x to $f_x(y) = |A_x \cap (\{y\} \times \mathbb{N})|$, and conversely by mapping f_x to $A_x = \{(y, n) : 1 \leq n \leq f_x(y)\}$.

It remains to check that in the case of a metric graph (b) for $R = 1$ implies (b) for every $R > 0$. It follows from (b) for $R = 1$ that

$$\|f_{n,x}\| \, \|f_{n,x'}\|^{-1} \to 1 \tag{1}$$

as $n \to \infty$, uniformly on the set of pairs of adjacent vertices x and x'. Given two vertices x and x' with $d(x, x) \leq R$ we find an $r \leq R$ and a sequence of vertices $x = x_0, x_1, \ldots, x_r = x'$

comprising an edge-path from x to x'. Writing

$$\|f_{n,x}\| \, \|f_{n,x'}\|^{-1} = \|f_{n,x_0}\| \, \|f_{n,x_1}\|^{-1} \cdot \|f_{n,x_1}\| \, \|f_{n,x_2}\|^{-1} \cdots \left\|f_{n,x_{r-1}}\right\| \, \|f_{n,x_r}\|^{-1}$$

it follows that the convergence in (1) is in fact uniform on the set $\{ (x, x') : d(x, x') \leq R \}$. The condition (b) for R is now an application of the triangle inequality: writing

$$\frac{\|f_{n,x} - f_{n,x'}\|}{\|f_{n,x}\|} \leq \sum_{i=0}^{r-1} \frac{\left\|f_{n,x_i} - f_{n,x_{i+1}}\right\|}{\|f_{n,x}\|} = \sum_{i=0}^{r-1} \frac{\left\|f_{n,x_i} - f_{n,x_{i+1}}\right\|}{\|f_{n,x_i}\|} \cdot \frac{\|f_{n,x_i}\|}{\|f_{n,x}\|},$$

note that each summand converges to zero uniformly on the appropriate set. □

Definition 1.6. We shall refer to functions $f_{n,x}$ as in Proposition 1.5 as weight functions.

1.2. *CAT*(0) **cube complexes.** A *cube complex* is a polyhedral complex in which the cells are Euclidean cubes of side length one, the attaching maps are isometries identifying the faces of a given cube with cubes of lower dimension and the intersection of two cubes is a common face of each [Gro87, Sag95, BH99]. One dimensional cubes are called *edges*, two dimensional cubes are called *squares* and a cube complex is *finite dimensional* if there is a bound on the dimension of its cubes.

The Euclidean distance between points in a cube is well-defined, allowing us to define the length of a rectifiable path. If a cube complex is finite dimensional it is a complete geodesic metric space with respect to the *geodesic metric*, in which the distance between two points is defined to be the infimum of the lengths of rectifiable paths connecting them [BH99]. A finite dimensional cube complex is a CAT(0) *cube complex* if the geodesic metric satisfies the CAT(0) *inequality*, according to which a geodesic triangle in the complex is 'thinner' than a triangle in Euclidean space with the same side lengths. Equivalently, the underlying topological space of the complex is simply connected and the complex satisfies Gromov's *link condition*, [Gro87]; these requirements comprise the definition for infinite dimensional CAT(0) cube complexes.

The vertex set of a cube complex is also equipped with the *edge-path metric*, in which the distance between vertices is defined to be the minimum number of edges on an edge-path connecting them.

A CAT(0) cube complex possesses a rich combinatorial structure. A (geometric) *hyperplane* H divides the vertex set into two path connected subspaces which we shall refer to as *half-spaces*. Two hyperplanes provide four possible half-space intersections; the hyperplanes *intersect* if and only if each of these four half-space intersections is non-empty. Two vertices in a half-space are connected by an edge-path that does not cross H whereas an edge-path connecting a vertex in one half-space to one in the other must cross H. In the latter case we say that H *separates* the two vertices. The set of hyperplanes separating the vertices x and y is denoted $\mathfrak{H}(x, y)$. The *interval* from x to y, denoted $[x, y]$, is the intersection of all half-spaces containing both x and y. A set of vertices is *convex* if whenever it contains both x and y it contains the entire inverval $[x, y]$. Finally, the set of vertices of a CAT(0) cube complex is a *median space*; the *median* of the vertices w, x and y is the (unique) vertex in $[w, x] \cap [x, y] \cap [w, y]$ [Rol98].

Proposition 1.7. *Let X be a* CAT(0) *cube complex. The restriction of the geodesic metric to the vertex set is coarsely equivalent to the edge-path metric. Moreover, if X is finite dimensional the vertex set* (*with either metric*) *is coarsely equivalent to X.*

Proof. For the purposes of the proof denote the geodesic metric by d_2 and the edge-path metric by d_1. Let x and y be vertices in X. Let $x = x_0, x_1, \ldots, x_n = y$ be the ordered sequence of vertices on a shortest edge-path from x to y. By the triangle inequality,

$$d_2(x, y) \leq \sum_{i=1}^{n} d_2(x_{i-1}, x_i) = n = d_1(x, y).$$

Conversely, given two vertices x, y with $d_1(x, y) = k$ the interval between them is a CAT(0) cube complex with exactly k hyperplanes, and therefore embeds as a subcomplex of the k-dimensional unit cube. This embedding is an isometry for the edge-path metrics and a contraction at the level of the geodesic metrics. We denote the image of a point z under this embedding by $\overline{z}$, and abuse notation by letting d_1 and d_2 to refer to the edge-path and geodesic metrics in both cube complexes. We conclude $d_1(x, y) = d_1(\overline{x}, \overline{y}) = \sqrt{d_2(\overline{x}, \overline{y})} \leq \sqrt{d_2(x, y)}$. Thus, the metrics are coarsely equivalent as required.

If X is finite dimensional the vertex set is $\sqrt{\dim(X)}/2$-dense in X in the geodesic metric. Consequently, the vertex set with the (restriction of the) geodesic metric is coarsely equivalent to X. $\square$

A CAT(0) cube complex also possesses a *combinatorial boundary*, which we now describe. A function σ assigning to each hyperplane one of its two half-spaces is an *ultrafilter* if it satisfies the following condition: for two hyperplanes H and K the half-spaces $\sigma(H)$ and $\sigma(K)$ have non-trivial intersection. (The condition is vacuous when the hyperplanes H and K themselves intersect.)

A vertex $x \in X$ defines an assignment of half-spaces to hyperplanes as follows: assign to the hyperplane H the half-space H_x that contains x. The assignment is an ultrafilter since for two hyperplanes H and K we have $x \in H_x \cap K_x$. Further, distinct vertices define distinct ultrafilters; indeed, if $x \neq y$ then $H_x \neq H_y$ precisely when H separates x and y. We have thus described an injective function from vertices of X to ultrafilters. Ultrafilters that are *not* in the image of this map are *vertices at infinity*; these comprise the *ideal boundary* ∂X of X and we denote $\overline{X} = X \cup \partial X$.

The elementary combinatorics of hyperplanes and half spaces extends to $\overline{X}$. Let $z, w \in \overline{X}$. Being an ultrafilter, z associates to each hyperplane H one of its two half spaces; we denote this half space by H_z. A hyperplane H *separates* z and w if $H_z \neq H_w$; the set of these hyperplanes is denoted $\mathfrak{H}(z, w)$. We say that H_z *contains* z, and define the *interval* $[z, w]$ to be the intersection of of all half spaces containing both z and w. Observe that $[z, w] \subset X$.

Lemma 1.8. *Let x, $w \in X$ and $z \in \overline{X}$. If $w \in [x, z]$ then $[x, w] \subset [x, z]$.*

Proof. The intersection of convex sets is convex; in particular, intervals are convex. □

Lemma 1.9. *Let x, y, $w \in X$ and $z \in \overline{X}$. If $w \in [x, z]$ and $y \in [x, w]$ then $\mathfrak{H}(y, w) \subset \mathfrak{H}(y, z)$.*

Proof. If not there is a hyperplane H such that $H_z = H_y \neq H_w$. We must have either $H_x = H_z$ or $H_x = H_w$, but the first of these statements contradicts $w \in [x, z]$ and the second contradicts $y \in [x, w]$. □

The set $\overline{X}$ carries a natural topology. We shall require only the following, which we take as a definition: a sequence of vertices $z_j \in X$ converges to a vertex $z \in \overline{X}$ if and only if for every hyperplane H we have $H \notin \mathfrak{H}(z_j, z)$ for almost every $j \in \mathbb{N}$. (As usual, we say that a property holds for *almost every* $j \in \mathbb{N}$ if the set of those $j \in \mathbb{N}$ for which the property does *not* hold is finite.) We defer the question of whether or not there exist sequences converging to a given vertex at infinity until later. For now we note the following properties of such sequences.

Lemma 1.10. *Let $z_j \in X$, $z \in \overline{X}$ and let $z_j \to z$. A hyperplane H separates y from z percisely when it separates y from almost every z_j:*

$$\mathfrak{H}(y, z) = \bigcup_k \bigcap_{j \geq k} \mathfrak{H}(y, z_j).$$

Proof. A hyperplane H separates y from z means that $H_y \neq H_z$; $z_j \to z$ means that for every hyperplane H we have $H_z = H_{z_j}$ for almost every j. □

Lemma 1.11. *Let $z_j \in X$, $z \in \overline{X}$ and suppose $z_j \to z$. Let x and $y \in X$. Precisely one of the following two statements holds:*

(a) $y \in [x, z_j]$ *for almost every j,*

(b) *$y \notin [x, z_j]$ for almost every j.*

In the first case $y \in [x, z]$ whereas in the second $y \notin [x, z]$.

Proof. The first statement fails if and only if $y \notin [x, z_j]$ for infinitely many j; this is clearly implied by the second statement, and we must show it implies the second statement. Now, if $y \notin [x, z_j]$ there exists exists $H \in \mathfrak{H}(x, y)$ such that $H_x = H_{z_j}$. Assuming this is the case for infinitely many j then, since $\mathfrak{H}(x, y)$ is finite, there exists $H \in \mathfrak{H}(x, y)$ such that $H_x = H_{z_j}$ for infinitely many j. By the definition of convergence we have $H_z = H_{z_j}$ for almost every j. Thus, $H_x = H_z = H_{z_j}$ for almost every j. In particular, $y \notin [x, z_j]$ for almost every j, and $y \notin [x, z]$.

It remains only to see that the first statement implies $y \in [x, z]$. But, if $y \notin [x, z]$ there exists an $H \in \mathfrak{H}(x, y)$ such that $H_x = H_z$. By the definition of convergence, we have $H_z = H_{z_j}$ for almost every j, so that $y \notin [x, z_j]$ for almost every j. □

Lemma 1.12. *Let x, $y \in X$ and $z \in \overline{X}$. The intersection of the intervals $[x, y]$, $[x, z]$ and $[y, z]$ consists of a single vertex of X.*

Proof. To prove uniqueness suppose $m \neq m'$ are in $[x, y] \cap [x, z] \cap [y, z]$ and let H be a hyperplane separating m and m'. Two of the three half spaces H_x, H_y and H_z must be equal; suppose, for example, $H_x = H_z$. Since $H_m \neq H_{m'}$ only one of these can be H_x; if, for example, $H_m \neq H_x$ we have $m \notin [x, z]$, a contradiction.

To prove existence, let $z_j \in X$ be such that $z_j \to z$. The interval $[x, y]$ is finite and contains the medians $m_j = m(x, y, z_j)$. Hence there exists an $m \in [x, y]$ such that $m = m_j \in [x, z_j]$ for infinitely many j. By Lemma 1.11, $m \in [x, z_j]$ for almost every j and $m \in [x, z]$. Similarly, $m \in [y, z]$. □

Let $x \in X$ and $z \in \overline{X}$. Denote by $\mathfrak{N}_z(x)$ the set of hyperplanes separating x and z and adjacent to x. (The notation is inspired by [NR98]; when $z \in X$ the hyperplanes in $\mathfrak{N}_z(x)$ span the first cube on the *normal cube path* from x to z.)

Lemma 1.13. *Let X be a finite dimensional* CAT(0) *cube complex. Let $x \in X$ and $z \in \overline{X}$. The cardinality of $\mathfrak{N}_z(x)$ is bounded by the dimension of X.*

Proof. Since a family of pairwise intersecting hyperplanes have a common point of intersection the cardinality of such a family is bounded by the dimension of X [Sag95, Theorem 4.14]. Thus, it suffices to show that every pair of hyperplanes H and $K \in \mathfrak{N}_z(x)$ intersect. For such H and K we have $H_x \cap K_x \neq \emptyset$. Further the vertex immediately across H from x lies in $H_z \cap K_x$; similarly $H_x \cap K_z \neq \emptyset$. Finally, if $z_j \in X$ converge to z then for almost every j

we have $z_j \in H_z \cap K_z$. All four half-space intersections being nonempty, H and K intersect. Compare [NR98, Proposition 3.3]. $\square$

Finally we consider the geometry of intervals in CAT(0) cube complexes. We shall make extensive use of the following often used result; apparently no complete proof exists in the literature so we also provide a detailed discussion. Compare [CR05].

We view $\mathbb{R}^d$ as a cube complex in the obvious way; the vertex set is the integer grid $\mathbb{Z}^d$ and the (top dimensional) cubes are the translates of the unit cube with vertices $\{0,1\}^d$. An interval in $\mathbb{R}^d$ is a cuboid. Precisely, the interval $[\overline{x}, \overline{y}]$ for the vertices $\overline{x} = (x_1, \ldots, x_d)$ and $\overline{y} = (y_1, \ldots, y_d)$ is the product

$$\{x_1, \ldots, y_1\} \times \{x_2, \ldots, y_2\} \times \cdots \times \{x_d, \ldots, y_d\}, \tag{2}$$

where for simplicity we assume that $x_i \leq y_i$ for all i. To include vertices in the combinatorial bounday we allow the possibility that one or both of x and y are vertices at infinity, meaning that $x_i = -\infty$ or $y_i = \infty$ (or both) for some i.

Theorem 1.14. *Let X be a* CAT(0) *cube complex of dimension d and let x and y be vertices in $\overline{X}$. Then the interval $[x, y]$ admits an isometric embedding as an interval $[\overline{x}, \overline{y}]$ in the cube complex $\mathbb{R}^d$.*

For purposes of the proof we define a partial order on the set $\mathfrak{H}(x, y)$ of hyperplanes separating x and y as follows:

$$H \leq K \Leftrightarrow H_x \subset K_x.$$

Lemma 1.15. *Two hyperplanes H and $K \in \mathfrak{H}(x, y)$ are incomparable for the partial order precisely when they intersect.*

Proof. The intersections $H_x \cap K_x$ and $H_y \cap K_y$ are always non-empty since $H_x \cap K_x = \emptyset$ contradicts the fact that x defines an ultrafilter; further $H_x \cap K_y = \emptyset \Leftrightarrow H_x \subset K_x$ and $H_y \cap K_x = \emptyset \Leftrightarrow K_x \subset H_x$. Consequently, H and K are incomparable precisely when the four possible intersections of half-spaces determined by H and K are non-empty, in other words, when they intersect. $\square$

Lemma 1.16. *The partially ordered set $\mathfrak{H}(x, y)$ is a disjoint union of d (possibly empty) chains:*

$$\mathfrak{H}(x, y) = \mathfrak{P}_1 \cup \cdots \cup \mathfrak{P}_d \quad (\textit{disjoint}).$$

Proof. According to the previous lemma an anti-chain in $\mathfrak{H}(x,y)$ is a collection of pairwise intersecting hyperplanes. A collection of pairwise intersecting hyperplanes has a common intersection [Sag95, Theorem 4.14]. As a consequence, the cardinality of an anti-chain in $\mathfrak{H}(x,y)$ is bounded by the dimension of X. With this remark, the result is an immediate consequence of Dilworth's lemma [Dil50, Theorem 1.1]. □

Proof of Theorem 1.14. We shall require, and prove, the result only in the case x is a vertex of X. We use the decomposition of $\mathfrak{H}(x,y)$ given in the previous lemma to define a function $z \mapsto \overline{z}$ of the interval $[x,y] \subset X$ into $\overline{\mathbb{Z}^d}$ (the d-dimensional Euclidean cube complex together with its combinatorial boundary):

$$\overline{z} = (\overline{z}_1, \dots \overline{z}_d), \quad \overline{z}_i = |\{\, H \in \mathfrak{P}_i : z \in H_y \,\}|\,.$$

Note that $\overline{x} = 0$, whereas the coordinates of $\overline{y}$ are $\overline{y}_i = |\mathfrak{P}_i|$; we allow the possibility that some $\overline{y}_i = \infty$. For every $z \in [x,y]$ the coordinates of $\overline{z}$ are finite and further $\overline{z} \in [\overline{x}, \overline{y}]$. The function is an isometric embedding. Indeed, we calculate for v, $w \in [x,y]$,

$$d(\overline{v}, \overline{w}) = \sum_{i=1}^{d} |\{\, H \in \mathfrak{P}_i : H \in \mathfrak{H}(v,w) \,\}| = |\mathfrak{H}(v,w)| = d(v,w),$$

since $\mathfrak{H}(v,w) \subset \mathfrak{H}(x,y)$. □

Now we return to the question of the existence of sequences of vertices converging to a given vertex at infinity.

Lemma 1.17. *Let $x \in X$ and let $z \in \overline{X}$. There exists a sequence $(z_j)_{j\in\mathbb{N}}$ of vertices in $[x,z]$ such that $z_j \to z$.*

Proof. We follow the construction of normal cube paths as in [NR98]. Let $z_0 = x$. Assuming we have constructed the vertex z_i in the sequence we define the vertex z_{i+1} to be the vertex opposite to z_i on the unique cube adjacent to z_i crossed by all the hyperplanes adjacent to z_i separating z_i from z. Since no hyperplane separates z_{i+1} from both x and z all the vertices in the sequence lie in the interval $[x,z]$. It remains to show that given any hyperplane h there are only finitely many values i for which h separates z_i from z. To see this we note that when h separates z_i from z the set of hyperplanes separating z_i from h is properly contained in the set of hyperplanes separating z_{i-1} from h and that both sets are finite.

□

1.3. **Combinations.** The weights that we give to vertices in a CAT(0) cube complex will be defined in terms of the function $\binom{n}{r}$. *A priori* this function is defined on pairs of integers with $0 \le r \le n$. It is uniquely determined by the following properties:

(a) $\binom{n}{0} = \binom{n}{n} = 1$ for $n \ge 0$.
(b) $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for $1 \le r \le n$.

In fact the function $\binom{n}{r}$ can be defined for *all* pairs of integers. It is the unique function on $\mathbb{Z} \times \mathbb{Z}$ with the following properties

(a) $\binom{n}{0} = 1$ for $n \ge 0$, and $\binom{n}{n} = 1$ for all $n \in \mathbb{Z}$.
(b) $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for all $n, r \in \mathbb{Z}$.

It follows that $\binom{n}{r}$ vanishes when $r > n$ or $r < 0 \le n$. Moreover it satisfies the identity $\binom{n}{r} = (-1)^{n+r}\binom{-1-r}{-1-n}$, which allows one to compute $\binom{n}{r}$ for $r < 0$.

We will make use of $\binom{n}{r}$ for $r \ge -1$ and $n \in \mathbb{Z}$, where the function takes exclusively non-negative values. In particular note that $\binom{n}{-1} = (-1)^{n-1}\binom{0}{-1-n}$ which is 1 if $n = -1$ and vanishes otherwise.

2. The Euclidean case

The standard proof that $\mathbb{Z}^d$ has Property A proceeds as follows. The weight function $f_{n,x}$ is the characteristic function of the ball of radius n and center x. The variation property, condition (b) of Proposition 1.5, follows from the facts that balls are Følner sets for $\mathbb{Z}^d$ and that the weight functions $f_{n,x}$ are translates of the single function $f_{n,0}$.

In this section we shall offer a different proof of Property A for $\mathbb{Z}^d$. Our proof parallels the standard proof for $\mathbb{Z}^d$, but with several differences, each of which is important for generalising the argument to arbitrary finite dimensional CAT(0) cube complexes (which do not in general admit an action by an amenable group). First, our weight functions $f_{n,x}$ will be supported on a certain subset of the n-ball with center x, rather than the whole ball. Second, they will not be characteristic functions. Finally, for fixed n and varying x the $f_{n,x}$ will be defined separately, rather than being translates of a single function.

For the remainder of the section fix an ambient dimension $N \ge d-1$. In proving that $\mathbb{R}^d$ has Property A we will take $N \ge d$; it will nonetheless be useful to note that the definitions and some of the results remain valid in the case $N = d-1$ when the codimension is said to be -1.

2.1. **Construction of weight functions.** Our definition of weight functions for $\mathbb{Z}^d$, and indeed for general CAT(0) cube complexes, is motivated by the following example.

Example. Let X be a (simplicial) tree. To show that X has Property A one can use weight functions defined as follows. Fix a basepoint $O \in X$. For each vertex $x \in X$ place weights on the interval $[O, x]$ according to

$$f_{n,x}(y) = \begin{cases} 1 & \text{if } y \neq O \text{ and } d(x,y) \leq n \\ n - d(x,y) + 1 & \text{if } y = O \text{ and } d(x,y) \leq n \\ 0 & \text{if } d(x,y) > n. \end{cases}$$

Heuristically we imagine that a charge of $n+1$ units has been placed at the vertex x and has then flowed towards the origin, where, ultimately it 'piles up'.

In higher dimensions we take the same heuristic point of view, that we will 'flow' a charge from a vertex x towards the origin O, distributing it across the interval $[O, x]$. As with the tree case, excess charge will collect at the origin, but, unlike the tree case, there will be additional points at which the charge accumulates. This occurs wherever the charge reaches the boundary on its journey towards the origin, losing a degree of freedom in the routes it can travel as it continues to flow. This loss of freedom is quantified as a 'deficiency', defined below. Fix a basepoint $O = (0, 0, \dots, 0)$ of $\mathbb{R}^d$.

Definition 2.1. The deficiency $\delta(y)$ of a vertex $y = (y_1, \dots, y_d) \in \mathbb{Z}^d$ is N minus the number of non-zero coordinates of y.

Definition 2.2. For a vertex $x \in \mathbb{Z}^d$ define the weight function $f_{n,x} : \mathbb{Z}^d \to \mathbb{N} \cup \{0\}$ by

$$f_{n,x}(y) = \begin{cases} \binom{n-d(x,y)+\delta(y)}{\delta(y)}, & y \in [O, x] \\ 0, & \text{otherwise.} \end{cases}$$

We make several remarks on the definition. First, since $N \geq d - 1$ we have $\delta(y) \geq -1$ for all y, so that $f_{n,x}$ is non-negative integer valued. Second, $f_{n,x}$ is supported in the interval $[O, x]$ so that it lies in the space of finitely supported functions on the vertex set. Finally, although it is not reflected in the notation, the weight functions depend on the fixed ambient dimension N.

The definitions are motivated by the following geometric intuition. Imagine a vertex x in the ambient $\mathbb{R}^N$, all of whose coordinates exceed n. The intersection of the interval from x to the origin with the ball of radius n is an N-dimensional tetrahedron containing $\binom{n+N}{N}$ points of $\mathbb{Z}^N$. Projecting $\mathbb{R}^N$ onto a subspace $\mathbb{R}^d$ (supposing $d \leq N$) the image is a d-dimensional tetrahedron, and the fibre over a vertex y will be an $(N-d)$-dimensional tetrahedron, the sides of which have length $n - d(x,y)$. Hence each fibre contains $\binom{n-d(x,y)+N-d}{N-d}$ points of

$\mathbb{Z}^N$. We thus take a weighting of $\binom{n-d(x,y)+N-d}{N-d}$ on each point of the image tetrahedron in $\mathbb{Z}^d$. Now suppose that the coordinates of x do not all exceed n. Then the tetrahedron will cross outside the interval from x to the origin, and we must further project points of the tetrahedron onto the faces of the interval. This results in higher deficiencies than the standard $N-d$.

2.2. **Analysis of weight functions.** We conclude our proof of Property A for $\mathbb{Z}^d$. The first step it to show that the norm of the weight function $f_{n,x}$ depends only on n and N, and in particular does not depend on x or d. Indeed, as the intuition above indicates the norm is exactly the number of points of $\mathbb{Z}^N$ contained in a tetrahedron of side length n.

Proposition 2.3. *For every $N \geq d-1$ and $x \in \mathbb{Z}^d$, the ℓ^1-norm of $f_{n,x}$ is $\binom{n+N}{N}$.*

Proof. In the proof we write $f^d_{n,x}$ in place of $f_{n,x}$. We shall show that for every $0 \leq d \leq N+1$ and for every $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$

$$\sum_{y \in \mathbb{Z}^d} f^d_{n,x}(y) = \binom{n+N}{N}.$$

Recall that for d in the range considered $f^d_{n,x}$ is non-negative and integer valued.

The proof is by induction on d. In the case $d=0$ we also have $x=O$. The sum has the single term $y=O$ and, since the deficiency is N, we have $f^0_{n,O}(O) = \binom{n+N}{N}$.

Suppose $d>0$ and let $x=(x_1,\dots,x_d) \in \mathbb{Z}^d$. Denote the projection of $z=(z_1,\dots,z_d) \in \mathbb{Z}^d$ to $\mathbb{Z}^{d-1}$ by $\widehat{z} = (z_2,\dots,z_d)$. The decomposition of the interval $[O,x]$ as a product $[0,x_1] \times [\widehat{O},\widehat{x}]$ gives a natural fibring of $[O,x]$ over $[\widehat{O},\widehat{x}]$. The interval $[0,x_1]$ in $\mathbb{Z}$ is ordered from 0 to x_1, which is the usual order in $\mathbb{Z}$ when $x_1 \geq 0$ and is the reverse order when $x_1 < 0$. We enumerate the points in the fibre over $\widehat{y}$ in $[\widehat{O},\widehat{x}]$ in the order $y^0, y^1, \dots, y^{|x_1|}$ determined by the ordering of the interval $[0,x_1]$. This is illustrated in Figure 1.

We shall show that for every $\widehat{y} \in [\widehat{O},\widehat{x}]$

$$\sum_{j=0}^{|x_1|} f^d_{n,x}(y^j) = f^{d-1}_{n,\widehat{x}}(\widehat{y}) \underset{\text{def}}{=} \binom{n-d(\widehat{x},\widehat{y})+\delta(\widehat{y})}{\delta(\widehat{y})}. \tag{3}$$

Once we have established this equality, we can compute the ℓ^1-norm of $f^d_{n,x}$ as follows:

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d} f^d_{n,x}(z) = \sum_{z\in[O,x]} f^d_{n,x}(z) &= \sum_{\widehat{y}\in[\widehat{O},\widehat{x}]}\sum_{j=0}^{|x_1|} f^d_{n,x}(y^j)\\ &= \sum_{\widehat{y}\in[\widehat{O},\widehat{x}]} f^{d-1}_{n,\widehat{x}}(\widehat{y})\\ &= \sum_{\widehat{y}\in\mathbb{Z}^{d-1}} f^{d-1}_{n,\widehat{x}}(\widehat{y}) = \binom{n+N}{N},\end{aligned}$$

where the equality on the second line follows from equation (3) and the final equality follows from the induction hypothesis.

To establish (3) let $\widehat{y}\in[\widehat{O},\widehat{x}]$; we shall prove by induction on i that, for $0\le i\le |x_1|$,

$$\sum_{j=0}^{i} f^d_{n,x}(y^j) = \binom{n-d(x,y^i)+\delta(\widehat{y})}{\delta(\widehat{y})}. \tag{4}$$

In coordinates, $\widehat{y}=(y_2,\dots,y_d)$ so that $y^0=(0,y_2,\dots,y_d)$ and $y^j=(\pm j,y_2,\dots,y_d)$ for $j\ge 1$, where we choose $\pm$ according to whether x_1 is greater or less than zero. It follows that $\widehat{y}$ and y^0 have the same number of non-zero coordinates, and hence the same deficiency: $\delta(\widehat{y})=\delta(y^0)$. Similarly for $j\ge1$ we find that $\delta(y^j)=\delta(\widehat{y})-1$. In particular, we see that $f^d_{n,x}(y^0)=\binom{n-d(x,y^0)+\delta(\widehat{y})}{\delta(\widehat{y})}$ yielding equation (4) in the case $i=0$.

Assume that (4) holds for i. Split the sum for $i+1$ into the sum for i and the term for $i+1$, apply the induction hypothesis and the definition of $f^d_{n,x}$ to obtain

$$\begin{aligned}\sum_{j=0}^{i+1} f^d_{n,x}(y^j) &= \binom{n-d(x,y^i)+\delta(\widehat{y})}{\delta(\widehat{y})} + f^d_{n,x}(y^{i+1})\\ &= \binom{n-d(x,y^i)+\delta(\widehat{y})}{\delta(\widehat{y})} + \binom{n-d(x,y^{i+1})+\delta(y^{i+1})}{\delta(y^{i+1})}\\ &= \binom{n-d(x,y^{i+1})+\delta(\widehat{y})-1}{\delta(\widehat{y})} + \binom{n-d(x,y^{i+1})+\delta(\widehat{y})-1}{\delta(\widehat{y})-1}\\ &= \binom{n-d(x,y^{i+1})+\delta(\widehat{y})}{\delta(\widehat{y})},\end{aligned}$$

where we have used $\delta(y^{i+1})=\delta(\widehat{y})-1$ $(i\ge 0)$ and $d(x,y^i)=d(x,y^{i+1})+1$ in the third equality. The final equality is the binomial coefficient formula from Section 1.3.

The formula (3) follows from (4) taking $i=|x_1|$ and noting that $d(x,y^{|x_1|})=d(\widehat{x},\widehat{y})$. $\square$

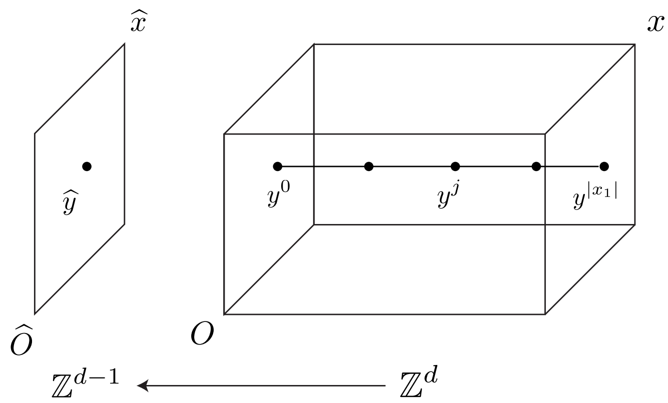

FIGURE 1. Projecting to $\mathbb{Z}^{d-1}$

The second step in our proof of Property A for $\mathbb{Z}^d$ is to estimate the norm of the difference $f_{n,x} - f_{n,x'}$ of weight functions when x and x' are adjacent vertices. We shall see that the norm of this difference depends only on n and N, and in particular does not depend on the points x and x' or on d.

Proposition 2.4. *For every $N \geq d$ and adjacent vertices x and $x' \in \mathbb{Z}^d$, the ℓ^1-norm of $f_{n,x} - f_{n,x'}$ is $2\binom{n+N-1}{N-1}$.*

Proof. In the proof we shall encounter weight functions for various values of ambient dimension N; we incorporate the ambient dimension into the notation where necessary to avoid confusion writing, for example, $f^N_{n,x}$.

Let x and $x' \in \mathbb{Z}^d$ be adjacent vertices and suppose, without loss of generality that x' is closer to the origin than x. It follows that the interval $[O, x']$ is contained in $[O, x]$. Further, for every $y \in [O, x']$ we have $x' \in [y, x]$ so that $d(x, y) = d(x', y) + 1$. We calculate the difference, for $y \in [0, x']$,

$$\begin{aligned} f^N_{n,x'}(y) - f^N_{n,x}(y) &= \binom{n - d(x', y) + \delta(y)}{\delta(y)} - \binom{n - (d(x', y) + 1) + \delta(y)}{\delta(y)} \\ &= \binom{n - d(x', y) + \delta(y) - 1}{\delta(y) - 1} = f^{N-1}_{n,x'}(y), \end{aligned}$$

where the last equality results from the observation that replacing N by $N-1$ has the effect of reducing all deficiencies by one. Note also that $N-1 \geq d-1$ so that $f^{N-1}_{n,x'}$ is non-negative valued. We conclude from Proposition 2.3 that

$$\sum_{y\in[O,x']} \left|f^N_{n,x'}(y) - f^N_{n,x}(y)\right| = \sum_{y\in[O,x']} f^{N-1}_{n,x'}(y) = \left\|f^{N-1}_{n,x'}\right\| = \binom{n+N-1}{N-1}. \tag{5}$$

Recall that $f^N_{n,x'}$ is supported in $[O,x'] \subset [O,x]$, whereas $f^N_{n,x}$ and the difference $f^N_{n,x'} - f^N_{n,x}$ are supported in $[O,x]$. Applying again Proposition 2.3 we obtain

$$\sum_{y\in[O,x]} f^N_{n,x'}(y) = \sum_{y\in[O,x]} f^N_{n,x}(y),$$

which, by rearranging, leads to

$$\sum_{y\in[O,x']} f^N_{n,x'}(y) - f^N_{n,x}(y) = \sum_{y\in[O,x]\setminus[O,x']} f^N_{n,x}(y) - f^N_{n,x'}(y),$$

where all terms in both sums are positive. Thus

$$\sum_{y\in[O,x]} \left|f^N_{n,x'}(y) - f^N_{n,x}(y)\right| = 2\sum_{y\in[O,x']} f^N_{n,x'}(y) - f^N_{n,x}(y) = 2\binom{n+N-1}{N-1}. \qquad \square$$

Theorem 2.5. *The Euclidean space $\mathbb{R}^d$ has Property A for every d.*

Proof. As $\mathbb{R}^d$ and $\mathbb{Z}^d$ are coarsely equivalent, it suffices to show that $\mathbb{Z}^d$ has Property A. To accomplish this we shall show that the sequence of families $f_{n,x}$ defined above, together with the sequence of constants $S_n = n$ satisfy the conditions given in Proposition 1.5. The support condition (a) is immediate: $f_{n,x}$ is supported in $B_n(x)$ since $\binom{n-d(x,y)+\delta(y)}{\delta(y)}$ vanishes if $n - d(x,y) + \delta(y) < \delta(y)$. The variation condition (b) follows directly from Propositions 2.3 and 2.4: if $d(x,x') \le 1$ then

$$\frac{\|f_{n,x} - f_{n,x'}\|}{\|f_{n,x}\|} \le \frac{2\binom{n+N-1}{N-1}}{\binom{n+N}{N}} = \frac{2N}{n+N} \to 0$$

as $n \to \infty$, the convergence being uniform on $\{\,(x,x') : d(x,x') \le 1\,\}$. $\square$

3. Property A for CAT(0) cube complexes

In this section we shall generalise the techniques of the previous section to prove that a finite dimensional CAT(0) cube complex has Property A. The construction of the weight functions $f_{n,x}$ generalises in a fairly straightforward manner. The main obstacle to the analysis of the weight functions is the computation of their norm, as in Proposition 2.3. To accomplish this step we shall develop a fibring technique for intervals in a CAT(0) cube complex. Let X be a CAT(0) cube complex of dimension $d < \infty$. As in the previous section, fix an ambient dimension $N \ge d - 1$.

3.1. **Construction of the weight functions.** The definition of the weight functions is exactly as in the Euclidean case. Fix a basepoint $O \in X$.

Definition 3.1. The deficiency $\delta(y)$ of a vertex $y \in X$ is the ambient dimension minus the number of hyperplanes both adjacent to y and separating it from O:

$$\delta(y) = N - |\mathfrak{N}_O(y)|\,.$$

In the Euclidean case, with basepoint $O = 0$, the cardinality of $\mathfrak{N}_O(y)$ is the number of nonzero coordinates of y. Thus, the definition generalises the one in the previous section.

Definition 3.2. For a vertex $x \in X$ define the weight function $f_{n,x} : X \to \mathbb{N} \cup \{0\}$ by

$$f_{n,x}(y) = \begin{cases} \binom{n-d(x,y)+\delta(y)}{\delta(y)}, & y \in [O,x] \\ 0, & \text{otherwise.} \end{cases}$$

As in the Euclidean case, $f_{n,x}$ is a non-negative integer valued function because $N \geq d - 1$ implies that $\delta(y) \geq -1$ for all y.

3.2. **Fibring intervals.** Let $x \in X$. According to Theorem 1.14 we may embed the interval $[O, x]$ into an interval in $\mathbb{Z}^d$. We denote the image of a vertex y by $\overline{y}$ and assume that the embedding maps the basepoint $O \in X$ to the basepoint $\overline{O} = (0, \ldots, 0) \in \mathbb{Z}^d$; by our convention the coordinates of $\overline{x}$ are non-negative. Our objective is to fibre the interval $I = [\overline{O}, \overline{x}]$ (in $\mathbb{Z}^d$) over the image J of the interval $[O, x]$.

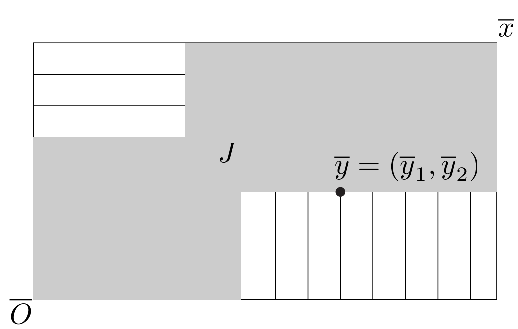

FIGURE 2. An interval embedded in the plane

Definition 3.3. Let $y \in [O, x]$ with image $\overline{y}$. The i-coordinate is y-bound if the vertex in $\mathbb{Z}^d$ with coordinates $(\overline{y}_1, \dots, \overline{y}_i - 1, \dots, \overline{y}_d)$ is in the image of the embedding. The i-coordinate is y-free if it is not y-bound.

In Figure 2 the first coordinate of $\overline{y}$ is y-bound, whereas the second coordinate is y free.

Definition 3.4. Let $y \in [O, x]$. The fibre of I over $\overline{y}$ is the set of vertices $a = (a_1, \dots, a_d) \in \mathbb{Z}^d$ with coordinates satisfying

(a) if i is y-bound then $a_i = \overline{y}_i$,
(b) if i is y-free then $0 \leq a_i \leq \overline{y}_i$.

The fibre of I over $\overline{y}$ is denoted by $\mathfrak{F}_y$.

Remark. For every $y \in [O, x]$ the fibre $\mathfrak{F}_y$ is an interval in $\mathbb{R}^d$; in fact if O_y is defined in coordinates by

$$O_{y,i} = \begin{cases} \overline{y}_i, & i \text{ is } y\text{-bound} \\ 0, & i \text{ is } y\text{-free} \end{cases}$$

then $\mathfrak{F}_y = [O_y, \overline{y}]$. In particular, for every $y \in [O, x]$ we have $\overline{y} \in \mathfrak{F}_y$.

As the terminology suggests we shall show, in a sequence of lemmas, that each fibre contains a *unique* vertex of J, that the fibres of distinct vertices are disjoint, and indeed that they partition I.

Lemma 3.5. *For every $y \neq z \in [O, x]$ the fibres $\mathfrak{F}_y$ and $\mathfrak{F}_z$ are disjoint.*

Proof. Let $y \neq z \in [O, x]$. Since $y \neq z$ it follows that either $y \notin [O, z]$ or $z \notin [O, y]$; exchanging y and z if necessary we may assume that $y \notin [O, z]$. Let m be the median of O, y and z; since m is the unique vertex in $[O, y] \cap [O, z] \cap [y, z]$ it follows that $m \neq y$ and $m \in [O, x]$. Let $H \in \mathfrak{H}(y, m)$ be adjacent to y. See Figure 3.

It follows from the definition of m that $H \in \mathfrak{H}(y, z) \cap \mathfrak{H}(y, O)$ so that also $H \notin \mathfrak{H}(z, O)$. Let i be the coordinate to which H contributes, and suppose that H is the pth hyperplane in the chain. It follows that $\overline{z}_i \leq p - 1$, so that the same inequality holds for every vertex in $\mathfrak{F}_z$. On the other hand, it follows from the definitions that $\overline{y}_i = p$ and that i is y-bound so that every vertex in $\mathfrak{F}_y$ has i-coordinate equal to p. We conclude that $\mathfrak{F}_y$ and $\mathfrak{F}_z$ are disjoint. $\square$

Lemma 3.6. *For every $a \in I$ there exists $y \in [O, x]$ such that $a \in \mathfrak{F}_y$.*

Proof. Let $\overline{y} \in [a, \overline{x}]$ minimise the distance from a to $[a, \overline{x}] \cap J$. We shall show that $a \in \mathfrak{F}_y$. The condition $\overline{y} \in [a, \overline{x}]$ is equivalent to the inequalities $\overline{y}_i \geq a_i$, for all coordinates i.

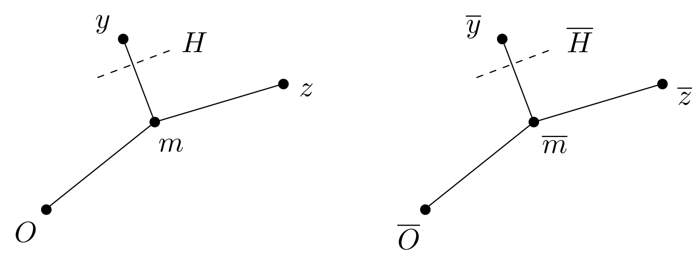

FIGURE 3. Medians

Consequently, it remains to show that for every y-bound coordinate i we have $a_i \geq \overline{y}_i$. But, if the i-coordinate is y-bound and $a_i < \overline{y}_i$ then $(\overline{y}_1, \ldots, \overline{y}_i - 1, \ldots, \overline{y}_d) \in [a, \overline{x}] \cap J$ and is nearer a than $\overline{y}$. This contradicts the choice of $\overline{y}$. $\square$

From these lemmas and the preceeding discussion we obtain:

Proposition 3.7. *The interval I is the disjoint union of the fibres of the vertices in $[O, x]$, and each fibre intersects J in exactly one point.* $\square$

Definition 3.8. For vertices x and z in a CAT(0) cube complex we define $\mathfrak{n}_z(x) = |\mathfrak{N}_z(x)|$. Recall that $\mathfrak{N}_z(x)$ is the set of hyperplanes in $\mathfrak{H}(x, z)$ adjacent to x.

Remark. We shall employ this notation when z is the basepoint of an interval $[z, y]$ containing x. In this case $\mathfrak{N}_z(x) \subset \mathfrak{H}(z, y)$.

We record two special cases of this notation. If $a \in I = [\overline{O}, \overline{x}]$ then $\mathfrak{n}_{\overline{O}}(a)$ is the number of non-zero coordinates of a; further, if y is the unique element of $[O, x]$ such that $a \in \mathfrak{F}_y$, an interval with basepoint O_y, then $\mathfrak{n}_{O_y}(a)$ is the number of non-zero y-free coordinates of a.

Lemma 3.9. *For every $y \in [O, x]$ the number of y-bound coordinates is $\mathfrak{n}_O(y)$; for every $a \in \mathfrak{F}_y$ we have*

$$\mathfrak{n}_{\overline{O}}(a) = \mathfrak{n}_{O_y}(a) + \mathfrak{n}_O(y). \tag{6}$$

Proof. Suppose the i-coordinate is y-bound. Obtain $z \in [O, x]$ such that $\overline{y}$ and $\overline{z}$ agree except in the i-coordinate for which $\overline{z}_i = \overline{y}_i - 1$. Since the embedding $y \mapsto \overline{y}$ is an isometry, we have $d(y, z) = 1$ and $d(O, y) = d(O, z) + 1$. Hence, the unique hyperplane H separating y and z also separates O and y.

We have thus described a function $i \mapsto H$ from the set of y-bound coordinates to the set of hyperplanes adjacent to y and separating y from O. It remains to show it is bijective. For injectivity, we merely observe that the hyperplane H associated to i separates O from x, belongs to the chain $\mathfrak{P}_i$ and the distinct $\mathfrak{P}_i$ are disjoint. For surjectivity, we observe that if H is adjacent to y and separates y from O then H separates O from x and is the image of the i for which H belongs to the chain $\mathfrak{P}_i$.

For the equation we need to count the number of non-zero coordinates of a. Each of these is either y-bound or y-free. By the observation above the number of non-zero y-free coordinates is precisely $\mathfrak{n}_{O_y}(a)$. By definition of the fibre all y-bound coordinates of a are equal to the corresponding coordinates of $\overline{y}$ which are themselves non-zero so the number of these is given by $\mathfrak{n}_O(y)$. □

Remark. It is instructive to examine the case $a = \overline{y}$ of the lemma. The number $\mathfrak{n}_{O_y}(\overline{y})$ of non-zero y-free coordinates of $\overline{y}$ is simply the dimension of the interval $\mathfrak{F}_y$. As a consequence, subtracting both sides of (6) from N, we conclude that this dimension is the difference of the deficiencies of y and $\overline{y}$:

$$\text{dimension of } \mathfrak{F}_y = \delta(y) - \delta(\overline{y}).$$

Figure 4 illustrates the fibring in the case of an interval $[O, x]$ embedded in $\mathbb{R}^3$. The vertex x maps to $\overline{x} = (2, 1, 2)$, while $\overline{O} = (0, 0, 0)$. The fibres of the points w, x, y and z are as indicated:

$$\begin{aligned}
\mathfrak{F}_w &= \{\, \overline{w} \,\} \\
\mathfrak{F}_x &= \{\, (2,0,2), (2,1,2) = \overline{x} \,\} \\
\mathfrak{F}_y &= \{\, (2,0,0), (2,0,1), (2,1,0), (2,1,1) = \overline{y} \,\} \\
\mathfrak{F}_z &= \{\, (0,0,2), (0,1,2), (1,0,2), (1,1,2) = \overline{z} \,\}.
\end{aligned}$$

The vertex x has deficiency one (computed with $N = 3$) while both y and z have deficiency two. However, the corresponding elements $\overline{x}$, $\overline{y}$ and $\overline{z} \in I$ all have deficiency zero. As expected, the fibre $\mathfrak{F}_x$ has dimension one and the fibres $\mathfrak{F}_y$ and $\mathfrak{F}_z$ both have dimension two. The vertex w has deficiency two, as does $\overline{w}$, so the fibre $\mathfrak{F}_w$ has dimension zero and is reduced to the single point $\overline{w}$.

3.3. **Analysis of the weight functions.** We complete our analysis of the weight functions defined for a CAT(0) cube complex following the strategy we used in the Euclidean case. The following analog of Proposition 2.3 provides the crucial step.

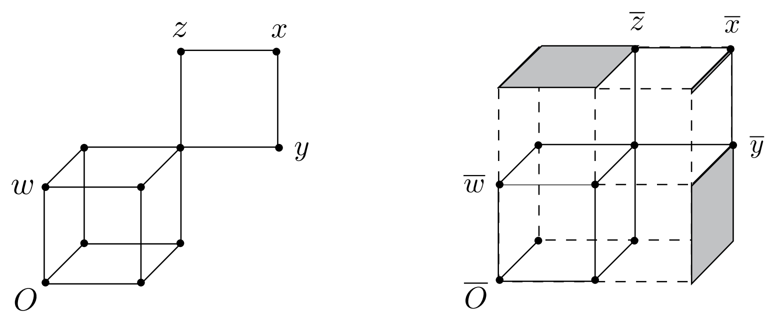

FIGURE 4. Fibring an interval over the embedding

Proposition 3.10. *Let X be a* $\mathrm{CAT}(0)$ *cube complex of dimension at most d, and let $N \geq d-1$. For a vertex of $x \in X$, the ℓ^1-norm of the weight function $f_{n,x}$ is $\binom{n+N}{N}$. In particular the norm does not depend on the vertex x or the complex X.*

The proof rests on a rather remarkable fact: although the construction of the fibres relies heavily on the *non-canonical* embedding of an interval of X into a Euclidean interval the process of summing the weights over each fibre gives a quantity which is independent of all choices. Specifically, summing over the fibre $\mathfrak{F}_y$ one gets the value of $f_{n,x}(y)$, a quantity that is defined intrinsically without reference to an embedding.

Proof. In the proof we shall encounter weight functions for the complex X and Euclidean spaces of various dimensions, as well as for various values of the ambient dimension. To avoid confusion we incorporate these parameters into the notation writing, for example, $f_{n,x}^{N,X}$.

Fix x and an identification of the interval $[O,x]$ with a subset J of an interval $I=[0,\overline{x}]$ in $\mathbb{R}^d$. As described above, we shall prove that for $y \in [O,x]$

$$f_{n,x}^{N,X}(y) = \sum_{a \in \mathfrak{F}_y} f_{n,\overline{x}}^{N,\mathbb{R}^d}(a). \tag{7}$$

Assuming this equality for the moment, we complete the proof of the theorem. Since $f_{n,x}^{N,X}$ is non-negative valued and supported in the interval $[O,x]$ and since the fibres partition I it follows that

$$\left\|f_{n,x}^{N,X}\right\| = \sum_{y \in [O,x]} f_{n,x}^{N,X}(y) = \sum_{a \in I} f_{n,\overline{x}}^{N,\mathbb{R}^d}(a) = \left\|f_{n,\overline{x}}^{N,\mathbb{R}^d}\right\| = \binom{n+N}{N},$$

the last equality being Proposition 2.3.

We turn to the proof of (7). Fix a vertex $y \in [O, x]$. If $d(x, y) > n$ then $d(a, \overline{x}) > n$ for all $a \in \mathfrak{F}_y$ and both sides of (7) are zero. Therefore, we may assume $d(x, y) \leq n$.

The deficiency of y with respect to the basepoint O is denoted $\delta^{N,X}(y)$. A vertex $a \in \mathfrak{F}_y$ has two deficiencies: one with respect to the basepoint $\overline{O} \in I$, which we denote $\delta^{N,I}(a)$ and another with respect to the basepoint O_y of the interval $\mathfrak{F}_y$, which we denote $\delta^{N,\mathfrak{F}_y}(a)$. As one might expect, these are related by a shift in the ambient dimension according to

$$\delta^{N,I}(a) = \delta^{N_y,\mathfrak{F}_y}(a), \quad N_y = N - \mathfrak{n}_O(y). \tag{8}$$

According to our conventions, the deficiency on the right is defined only when the dimension of $\mathfrak{F}_y$ does not exceed $N_y + 1$. Indeed, this is the case: $\mathfrak{F}_y$ has dimension $\mathfrak{n}_{\mathfrak{F}_y}(\overline{y})$ and applying Lemma 3.9 we conclude

$$\mathfrak{n}_{O_y}(\overline{y}) = \mathfrak{n}_{\overline{O}}(\overline{y}) - \mathfrak{n}_O(y) \leq d - \mathfrak{n}_O(y) \leq N + 1 - \mathfrak{n}_O(y) = N_y + 1.$$

The proof of (8) is straightforward. Indeed, directly from the definitions we have

$$\delta^{N,I}(a) = N - \mathfrak{n}_{\overline{O}}(a) \qquad \delta^{N,\mathfrak{F}_y}(a) = N - \mathfrak{n}_{O_y}(a) \qquad \delta^{N,X}(y) = N - \mathfrak{n}_O(y). \tag{9}$$

so that applying Lemma 3.9 we conclude

$$\delta^{N,I}(a) = N - \mathfrak{n}_{\overline{O}}(a) = (N - \mathfrak{n}_O(y)) - \mathfrak{n}_{O_y}(a) = \delta^{N_y,\mathfrak{F}_y}(a).$$

On the basis of (8) we complete the proof of (7). For $a \in \mathfrak{F}_y$ we have the coordinate-wise inequalities $0 \leq a_i \leq \overline{y}_i \leq \overline{x}_i$ so that $d(\overline{x}, a) = d(\overline{x}, \overline{y}) + d(\overline{y}, a)$. Hence

$$\begin{aligned} f_{n,\overline{x}}^{N,\mathbb{R}^d}(a) &= \binom{n - d(\overline{x}, a) + \delta^{N,I}(a)}{\delta^{N,I}(a)} \\ &= \binom{(n - d(\overline{x}, \overline{y})) - d(\overline{y}, a) + \delta^{N_y,\mathfrak{F}_y}(a)}{\delta^{N_y,\mathfrak{F}_y}(a)} \\ &= f_{n-d(\overline{x},\overline{y}),\overline{y}}^{N_y,\mathfrak{F}_y}(a). \end{aligned}$$

Observe that $n - d(\overline{x}, \overline{y}) = n - d(x, y) \geq 0$. Summing over $a \in \mathfrak{F}_y$, applying Proposition 2.3 and using again the fact that $d(\overline{x}, \overline{y}) = d(x, y)$ we get

$$\sum_{a \in \mathfrak{F}_y} f_{n,\overline{x}}^{N,\mathbb{R}^d}(a) = \left\| f_{n-d(\overline{x},\overline{y}),\overline{y}}^{N_y,\mathfrak{F}_y} \right\| = \binom{n - d(x, y) + N_y}{N_y}.$$

Comparing (8) and (9) we see $N_y = \delta^{N,X}(y)$. A glance at the definition of $f_{n,x}^{N,X}(y)$ reveals that (7) is proved. $\square$

The following results are direct analogs of Proposition 2.4 and Theorem 2.5; their proofs are identical to the proofs of their analogs in the Euclidean case, except making use of Proposition 3.10 in place of Proposition 2.3.

Proposition 3.11. *Let X be a* CAT(0) *cube complex of dimension at most d, and let $N \geq d$. For every pair x and x' adjacent vertices in X the ℓ^1-norm of the difference $f_{n,x} - f_{n,x'}$ of weight functions is $2\binom{n+N-1}{N-1}$.* □

Theorem 3.12. *A finite dimensional* CAT(0) *cube complex has Property A.* □

4. Point stabilisers at infinity

An amenable group of isometries of a locally compact Hadamard space is known either to fix a point at infinity, or to preserve a flat subspace [AB98]. Under certain circumstances there is a converse to this result, for example when a group G acts properly on a proper CAT(0) space the stabiliser of a flat is virtually abelian [BH99], and if the space is an Hadamard space, e.g., a building, then the stabiliser of a point in a suitable refinement of the visual boundary is necessarily amenable [Cap07]. We shall adapt our construction from the previous section to prove an analogous result for the combinatorial boundary of a CAT(0) cube complex.

Of the numerous characterizations of amenability for countable groups we select the Reiter condition, which is most convenient for our purposes.

Definition 4.1. A countable discrete group G is amenable if there exists a sequence of finitely supported probability measures $\xi_n \in \ell^1(G)$ such that for every $g \in G$

$$\lim_{n\to\infty} \|\xi_n - g \cdot \xi_n\| = 0.$$

An action of a discrete group G on a CAT(0) cube complex X is understood to be cellular. In particular, G acts on the set of vertices of X and on the sets of hyperplanes and half spaces, and preserves all relevant combinatorics of the complex. In particular, the action on vertices is isometric for the edge-path metric. Further, the action extends to the combinatorial boundary ∂X and to the completion $\overline{X}$. Not having gone into detail concerning the topology on the combinatorial boundary, we remark only that if $z_j \to z$ then $g \cdot z_j \to g \cdot z$.

Theorem 4.2. *Let G be a countable discrete group acting properly on a finite dimensional* CAT(0) *cube complex X and let z be a vertex at infinity of X. The stabiliser of z in G is amenable, and hence virtually abelian.*

Our proof will use the following criterion for amenability.

Proposition 4.3. *Let G be a countable group acting properly on a discrete metric space X. Assume X admits a sequence of families of ℓ^1 functions $f_{n,x} : X \to \mathbb{N} \cup \{0\}$, indexed by $x \in X$, such that:*

(a) *For every pair of points x and $x' \in X$ we have*

$$\frac{\|f_{n,x} - f_{n,x'}\|}{\|f_{n,x}\|} \to 0.$$

(b) *For every $g \in G$, $x \in X$, and $n \in \mathbb{N}$, $f_{n,gx} = g \cdot f_{n,x}$.*

Then G is amenable.

Remark. The properness assumption is equivalent to the action having finite point stabilizers.

Proof. We shall construct a sequence of probability measures as required by Definition 4.1. Fix a base point $x_0 \in X$. Let T be a transversal for the action of G on X; thus T contains precisely one point from each G-orbit. For each $n \in \mathbb{N}$ and $g \in G$ define

$$\phi_n(g) = \sum_{x \in T} \frac{f_{n,x_0}(gx)}{|G_x|},$$

where G_x is the stabilizer of x. Observe that $f_{n,x}$ is finitely supported, being an element of $\ell^1(X)$ with values in $\mathbb{N} \cup \{0\}$. Consequently the sum is finite, as indeed are all sums below. Further, ϕ_n is finitely supported. We compute $\|\phi_n\|$ as follows:

$$\begin{aligned}
\|\phi_n\| = \sum_{g \in G} \phi_n(g) &= \sum_{g \in G, x \in T} \frac{f_{n,x_0}(gx)}{|G_x|} \\
&= \sum_{x \in T} \sum_{y \in G \cdot x} f_{n,x_0}(y) \sum_{g \in G : gx = y} \frac{1}{|G_x|} \\
&= \sum_{x \in T} \sum_{y \in G \cdot x} f_{n,x_0}(y) = \|f_{n,x_0}\| .
\end{aligned}$$

A similar calculation yields the following estimate:

$$\|\phi_n - g \cdot \phi_n\| \leq \|f_{n,x_0} - f_{n,gx_0}\| .$$

We obtain the required probability measure by normalizing: $\xi_n = \phi_n / \|\phi_n\|$. ∎

Proof of Theorem 4.2. Let z be a vertex at infinity. Replacing G by the stabiliser of z, we assume that G stabilises z. Define weight functions as in Definition 3.2, with z playing the role of the base point O:

$$f_{n,x}(y) = \begin{cases} \binom{n-d(x,y)+\delta(y)}{\delta(y)}, & y \in [x,z] \\ 0, & y \notin [x,z], \end{cases} \tag{10}$$

where the deficiency is defined relative to an ambient dimension N by $\delta(y) = N - |\mathfrak{N}_z(y)|$. Choosing N to be at least the dimension of the cube complex we ensure that all deficiencies are non-negative so that $f_{n,x}$ takes its values in the non-negative integers.

We first note that the support of $f_{n,x}$ lies in the intersection of the ball of radius n around x with the interval $[x,z]$. While the ball itself may contain infinitely many vertices, Theorem 1.14 tells us that the interval embeds in $\mathbb{R}^n$ for some (finite) n, so the intersection is in fact finite, and $f_{n,x}$ is finitely supported, and therefore ℓ^1.

The equivariance condition is an immediate consequence of the manner in which G acts on X and the fact that G fixes z. We verify the remaining condition through a limiting process. Let z_j be a sequence of vertices of $[x,z]$ converging to z; this is possible by Lemma 1.17. Define the weight functions as in Definition 3.2 with z_j playing the role of the base point O:

$$f^{z_j}_{n,x}(y) = \begin{cases} \binom{n-d(x,y)+\delta^j(y)}{\delta^j(y)}, & y \in [x,z] \\ 0, & y \notin [x,z], \end{cases} \tag{11}$$

where the deficiency is defined relative to an ambient dimension N by $\delta^j(y) = N - \left|\mathfrak{N}_{z_j}(y)\right|$. We now show that $f^{z_j}_{n,x} = f_{n,x}$, for almost every j. The support of $f_{n,x}$ is contained in $[x,z] \cap B(x,n)$; similarly the support of $f^{z_j}_{n,x}$ is contained in $[x,z_j] \cap B(x,n)$. Applying Lemma 1.8 (with $w = z_j$) we see that the support of $f^{z_j}_{n,x}$ is also contained in $[x,z] \cap B(x,n)$. According to Theorem 1.14 this is a finite set.

It remains to show that for $y \in [x,z] \cap B(x,n)$ we have $f^{z_j}_{n,x}(y) = f_{n,x}(y)$ for almost every j. The only terms in (10) and (11) dependent on j are the deficiencies $\delta(y)$ and $\delta^j(y)$. Applying Lemma 1.11 we see that $y \in [x,z_j]$ for almost every j and applying Lemma 1.9 (with $w = z_j$) we conclude that

$$\mathfrak{N}_{z_j}(y) \subset \mathfrak{N}_z(y), \tag{12}$$

for almost every j. Applying Lemma 1.10 we have

$$\mathfrak{N}_z(y) = \bigcup_k \bigcap_{j \geq k} \mathfrak{N}_{z_j}(y).$$

Since $\mathfrak{N}_z(y)$ is a finite set, and the union on the right is increasing, we conclude that

$$\mathfrak{N}_z(y) \subset \mathfrak{N}_{z_j}(y), \tag{13}$$

for almost every j. Combining (12) and (13) we conclude that $\delta(y) = \delta^j(y)$ for almost every j. Comparing the definitions (10) and (11) we are done.

The almost invariance of the $f_{n,x}$ now follows. Let x and $x' \in X$. Let $m = m(x, x', z)$ so that $m \in [x, z] \cap [x', z]$, hence also $[m, z] \subset [x, z] \cap [x', z]$. Let $z_j \to z$ and $z_j \in [m, z]$. We have shown above that if $z_j \to z$ and $z_j \in [x, z]$ then $f_{n,x}^{z_j} = f_{n,x}$ for almost every j. Applying this to both x and x' we conclude that if x and x' are adjacent then

$$\|f_{n,x} - f_{n,x'}\| = \left\|f_{n,x}^{z_j} - f_{n,x'}^{z_j}\right\| = 2\binom{n+N-1}{N-1}$$

and also

$$\|f_{n,x}\| = \left\|f_{n,x}^{z_j}\right\| = \binom{n+N}{N},$$

where in each case the first equality holds for almost every j and the second for every j by Propositions 3.11 and 3.10, respectively. The argument now follows exactly the same course as that of Theorem 2.5:

$$\frac{\|f_{n,x} - f_{n,x'}\|}{\|f_{n,x}\|} \leq 2d(x, x')\frac{\binom{n+N-1}{N-1}}{\binom{n+N}{N}} = \frac{2d(x, x')N}{n+N},$$

which tends to zero uniformly on $\{(x, x') : d(x, x') \leq R\}$ as $n \to \infty$. □

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