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Generalized cylindrical coordinates for characteristic boundary conditions and characteristic interface conditions

by

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The aim of this report is to derive generalized coordinates for the specific case of mapping only the streamwise and radial coordinate of a cylindrical coordinate system, while leaving the azimuthal coordinate unchanged. The characteristic equations and the required matrices for the transformation from conservative to characteristic form are presented for this specific case. All equations and procedures are based on previous work on generalized characteristic boundary conditions (Kim & Lee, 2000) and characteristic interface conditions (Kim & Lee, 2003).

1 Governing equations in cylindrical coordinates

The starting point are the compressible Navier–Stokes equations in cylindrical coordinates, given as The flow is assumed to be an ideal gas with constant specific heat coefficients and all quantities were made dimensionless using the flow quantities at a reference location in the flow (the free-stream/inflow location was used). The radius of the body is chosen as the reference length. The non-dimensionalization results in the following dimensionless parameters:

$$Re = \frac{\rho^*_\infty u^*_\infty r^*}{\mu^*_\infty} \quad , \quad M = \frac{u^*_\infty}{a^*_\infty} \quad , \quad Pr = \frac{\mu^*_\infty c^*_p}{\kappa^*_\infty} \quad .$$

With z, r, θ denoting the streamwise, radial and azimuthal directions, respectively, and u, v, w denoting the velocity components in z, r, θ directions, respectively, the non-dimensional compressible Navier–Stokes equations in cylindrical coordinates are

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{E}}{\partial z} + \frac{\partial \mathbf{F}}{\partial r} + \frac{1}{r}\frac{\partial \mathbf{G}}{\partial \theta} = \mathbf{S_v} \, , \tag{1}$$

where the conservative variable and the inviscid flux vectors are

$$\begin{aligned}
\mathbf{Q} &= [\rho,\, \rho u,\, \rho v,\, \rho w,\, \rho E]^T \\
\mathbf{E} &= [\rho u,\, \rho u u + p,\, \rho u v,\, \rho u w,\, u(\rho E + p)]^T \\
\mathbf{F} &= [\rho v,\, \rho u v,\, \rho v v + p,\, \rho v w,\, v(\rho E + p)]^T \\
\mathbf{G} &= [\rho w,\, \rho u w,\, \rho v w,\, \rho w w + p,\, w(\rho E + p)]^T \, ,
\end{aligned}$$

with the total energy defined as $E = T/[\gamma(\gamma-1)M^2] + 1/2 u_i u_i$ and $\gamma = 1.4$. The source term S_v consists of the viscous flux derivatives and the additional terms obtained in the transformation to cylindrical coordinates (H)

$$\mathbf{S_v} = \frac{\partial \mathbf{E_v}}{\partial z} + \frac{\partial \mathbf{F_v}}{\partial r} + \frac{1}{r}\frac{\partial \mathbf{G_v}}{\partial \theta} - \frac{1}{r}\mathbf{H} \, , \tag{2}$$

with

$$\begin{aligned}
\mathbf{E_v} &= [0,\, \tau_{zz},\, \tau_{zr},\, \tau_{z\theta},\, -q_z + u\tau_{zz} + v\tau_{zr} + w\tau_{z\theta}]^T \\
\mathbf{F_v} &= [0,\, \tau_{rz},\, \tau_{rr},\, \tau_{r\theta},\, -q_r + u\tau_{rz} + v\tau_{rr} + w\tau_{r\theta}]^T \\
\mathbf{G_v} &= [0,\, \tau_{\theta z},\, \tau_{\theta r},\, \tau_{\theta\theta},\, -q_\theta + u\tau_{\theta z} + v\tau_{\theta r} + w\tau_{\theta\theta}]^T \\
\mathbf{H} &= [\rho v,\, \rho u v - \tau_{zr},\, \rho v v - \rho w w - \tau_{rr} + \tau_{\theta\theta},\, 2\rho v w - 2\tau_{\theta r}, \\
&\quad\; v(\rho E + p) + q_r - u\tau_{rz} - v\tau_{rr} - w\tau_{r\theta}]^T \, .
\end{aligned}$$

The molecular stress tensor components are

$$
\begin{aligned}
\tau_{zz} &= \frac{2\mu}{3Re}\left[2\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} - \frac{1}{r}\left(\frac{\partial w}{\partial \theta} + v\right)\right] \\
\tau_{rr} &= \frac{2\mu}{3Re}\left[-\frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r} - \frac{1}{r}\left(\frac{\partial w}{\partial \theta} + v\right)\right] \\
\tau_{\theta\theta} &= \frac{2\mu}{3Re}\left[-\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} + 2\frac{1}{r}\left(\frac{\partial w}{\partial \theta} + v\right)\right] \\
\tau_{rz} &= \frac{\mu}{Re}\left[\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right] \\
\tau_{\theta z} &= \frac{\mu}{Re}\left[\frac{\partial w}{\partial z} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right] \\
\tau_{\theta r} &= \frac{\mu}{Re}\left[\frac{1}{r}\left(\frac{\partial v}{\partial \theta} - w\right) + \frac{\partial w}{\partial r}\right] .
\end{aligned}
$$

The heat-flux vector components are

$$
\begin{aligned}
q_z &= \frac{-\mu}{Pr(\gamma-1)M^2 Re}\frac{\partial T}{\partial z} \\
q_r &= \frac{-\mu}{Pr(\gamma-1)M^2 Re}\frac{\partial T}{\partial r} \\
q_\theta &= \frac{-\mu}{Pr(\gamma-1)M^2 Re}\frac{1}{r}\frac{\partial T}{\partial \theta} ,
\end{aligned}
$$

where the Prandtl number is assumed to be constant at $Pr = 0.72$. The molecular viscosity μ is computed using Sutherland's law (c.f. White, 1991), setting the ratio of the Sutherland constant over freestream temperature to 0.36867. To close the system of equations, the pressure is obtained from the non-dimensional equation of state $p = (\rho T)/(\gamma M^2)$.

2 Transformation to general cylindrical coordinates

In order to allow for complex geometries, the (z, r) coordinate system is mapped to general coordinates (ξ, η). By deliberately restricting the coordinate mapping to two-dimensions, the azimuthal grid is required to be equidistant. What might seem as a profound constraint actually is advantageous in two ways: firstly an efficient Fourier spectral method easily can be employed for the discretization of the θ-direction, and secondly, a smaller number of metric terms is required than for the most general case, and all metric terms needed in the current case are two-dimensional (versus three-dimensional if a mapping was performed in all coordinate directions). Therefore much less memory is required for the simulations. The streamwise and radial derivatives need to be expressed in terms of the new variables. By using the chain rule, the following expressions can be derived

$$
\frac{\partial}{\partial z} = \frac{1}{J}\left[\frac{\partial r}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial r}{\partial \xi}\frac{\partial}{\partial \eta}\right] = r^*_\eta \frac{\partial}{\partial \xi} - r^*_\xi \frac{\partial}{\partial \eta} , \tag{3}
$$

$$\frac{\partial}{\partial r} = \frac{1}{J}\left[-\frac{\partial z}{\partial \eta}\frac{\partial}{\partial \xi} + \frac{\partial z}{\partial \xi}\frac{\partial}{\partial \eta}\right] = z^*_\xi \frac{\partial}{\partial \eta} - z^*_\eta \frac{\partial}{\partial \xi} \ , \tag{4}$$

where $J = z_\xi r_\eta - z_\eta r_\xi$ is the determinant of the coordinate transformation. For conciseness, the metric terms are abbreviated as, e.g., $r_\eta = \frac{\partial r}{\partial \eta}$, and the asterisk denotes a metric term already divided by J.

Transforming the original equation (1) to a non-conservation form in generalized coordinates using the chain rule, we obtain

$$\frac{\partial \mathbf{Q}}{\partial t} + \left(\xi_z \frac{\partial \mathbf{E}}{\partial \xi} + \xi_r \frac{\partial \mathbf{F}}{\partial \xi}\right) + \left(\eta_z \frac{\partial \mathbf{E}}{\partial \eta} + \eta_r \frac{\partial \mathbf{F}}{\partial \eta}\right) + \frac{1}{r}\frac{\partial \mathbf{G}}{\partial \theta} = \mathbf{S_v} \ . \tag{5}$$

This non-conservative form can be transformed into a characteristic form in the direction normal to a computational boundary (either the boundaries of the domain or interfaces between blocks or sub-domains). Considering the ξ direction, the quasi-linear characteristic wave equation is

$$\frac{\partial \mathbf{R}}{\partial t} + \underline{\Lambda}\frac{\partial \mathbf{R}}{\partial \xi} = -\underline{\mathbf{P}}^{-\mathbf{1}}\mathbf{S}^*_\mathbf{v} \ , \tag{6}$$

where an underscore denotes a matrix. Only the fluxes in the ξ direction are considered on the right-hand-side of equation (6), while all fluxes in the other directions are contained in the modified source term

$$\mathbf{S}^*_\mathbf{v} = \mathbf{S_v} - \left(\eta_z \frac{\partial \mathbf{E}}{\partial \eta} + \eta_r \frac{\partial \mathbf{F}}{\partial \eta}\right) - \frac{1}{r}\frac{\partial \mathbf{G}}{\partial \theta} = \mathbf{S_v} - \left(-r^*_\xi \frac{\partial \mathbf{E}}{\partial \eta} + z^*_\xi \frac{\partial \mathbf{F}}{\partial \eta}\right) - \frac{1}{r}\frac{\partial \mathbf{G}}{\partial \theta} \ . \tag{7}$$

For a definition of the characteristic $\mathbf{R}$ variables and the transformation matrix from the conservative to the characteristic variables, the reader is referred to (Kim & Lee, 2000) and the Appendix. The eigenvalues in $\underline{\Lambda}$ are

$$\underline{\Lambda} = \begin{pmatrix} U \\ U \\ U \\ U + c\sqrt{\xi_z^2 + \xi_r^2} \\ U - c\sqrt{\xi_z^2 + \xi_r^2} \end{pmatrix} = \begin{pmatrix} U \\ U \\ U \\ U + c\sqrt{\left(r^*_\eta\right)^2 + \left(z^*_\eta\right)^2} \\ U - c\sqrt{\left(r^*_\eta\right)^2 + \left(z^*_\eta\right)^2} \end{pmatrix} , \tag{8}$$

where c is the speed of sound and the contravariant velocity is given by

$$U = \xi_z u + \xi_r v = r^*_\eta u - z^*_\eta v \ . \tag{9}$$

Following (Kim & Lee, 2000), the convection terms in (6) are transformed to the flux derivatives of (5) as

$$\xi_z \frac{\partial \mathbf{E}}{\partial \xi} + \xi_r \frac{\partial \mathbf{F}}{\partial \xi} = r^*_\eta \frac{\partial \mathbf{E}}{\partial \xi} - z^*_\eta \frac{\partial \mathbf{F}}{\partial \xi} = \underline{\mathbf{P}}\ \underline{\Lambda}\frac{\partial \mathbf{R}}{\partial \xi} = \underline{\mathbf{P}}\mathbf{L} \ , \tag{10}$$

thus the convection terms are defined as

$$L_i = \lambda_i \frac{\partial R_i}{\partial \xi} \qquad (i = 1, 2, \ldots, 5) \ . \tag{11}$$

The transformation matrix from characteristic to conservative variables is given in (Kim & Lee, 2000), and in modified form for the current case in the Appendix.
The final step is to derive the conservation form of the Navier–Stokes equations in generalized coordinates. Following Anderson (1995), the r and z derivatives of equation (1) can be transformed as follows

$$\frac{\partial \hat{\mathbf{Q}}}{\partial t} + \frac{\partial \hat{\mathbf{E}}}{\partial \xi} + \frac{\partial \hat{\mathbf{F}}}{\partial \eta} + \frac{J}{r}\frac{\partial \mathbf{G}}{\partial \theta} = \hat{\mathbf{S}}_{\mathbf{v}} \ , \tag{12}$$

where the inviscid flux vectors in generalized coordinates are now given as

$$\hat{\mathbf{Q}} = J\mathbf{Q} \ , \qquad \hat{\mathbf{E}} = r_\eta \mathbf{E} - z_\eta \mathbf{F} \ , \qquad \hat{\mathbf{F}} = -r_\xi \mathbf{E} + z_\xi \mathbf{F} \ ,$$

and $\hat{\mathbf{S}}_{\mathbf{v}}$ is the source vector that contains the viscous fluxes in generalized coordinates and all non-conservative terms due to the cylindrical coordinate system

$$\hat{\mathbf{S}}_{\mathbf{v}} = \frac{\partial \hat{\mathbf{E}}_{\mathbf{v}}}{\partial \xi} + \frac{\partial \hat{\mathbf{F}}_{\mathbf{v}}}{\partial \eta} + \frac{J}{r}\frac{\partial \mathbf{G}_{\mathbf{v}}}{\partial \theta} - \frac{J}{r}\mathbf{H} \ , \tag{13}$$

with

$$\hat{\mathbf{E}}_{\mathbf{v}} = r_\eta \mathbf{E}_{\mathbf{v}} - z_\eta \mathbf{F}_{\mathbf{v}} \ , \qquad \hat{\mathbf{F}}_{\mathbf{v}} = -r_\xi \mathbf{E}_{\mathbf{v}} + z_\xi \mathbf{F}_{\mathbf{v}} \ .$$

3 Characteristic equations in ξ-direction

When using the conservative formulation of the Navier–Stokes equations in generalized coordinates as basis, the quasi-linear characteristic wave equation (for the ξ-direction) can now be written as

$$\frac{\partial \mathbf{R}}{\partial t} + \underline{\Lambda}\frac{\partial \mathbf{R}}{\partial \xi} = \mathbf{S}_{\mathbf{c}} \ , \tag{14}$$

where the modified source term is

$$\begin{aligned} \mathbf{S}_{\mathbf{c}} &= \frac{1}{J}\underline{\mathbf{P}}^{-1}\left\{\hat{\mathbf{S}}_{\mathbf{v}} - \left[\mathbf{E}\frac{\partial r_\eta}{\partial \xi} - \mathbf{F}\frac{\partial z_\eta}{\partial \xi} + \frac{\partial \hat{\mathbf{F}}}{\partial \eta} + \frac{J}{r}\frac{\partial G}{\partial \theta}\right]\right\} \\ &= \frac{1}{J}\underline{\mathbf{P}}^{-1}\left\{\frac{\partial \hat{\mathbf{Q}}}{\partial t} + \frac{\partial \hat{\mathbf{E}}}{\partial \xi} - \mathbf{E}\frac{\partial r_\eta}{\partial \xi} + \mathbf{F}\frac{\partial z_\eta}{\partial \xi}\right\} \ . \end{aligned} \tag{15}$$

The characteristic convection term on a boundary or interface can now be computed as

$$\mathbf{L} = \frac{1}{J}\underline{\mathbf{P}}^{-1}\left\{\frac{\partial \hat{\mathbf{E}}}{\partial \xi} - \mathbf{E}\frac{\partial r_\eta}{\partial \xi} + \mathbf{F}\frac{\partial z_\eta}{\partial \xi}\right\} \ . \tag{16}$$

The characteristic convection term is then either corrected by imposing interface conditions according to (Kim & Lee, 2003), using the modified source term from (15), or using the appropriate LODI relations given in (Kim & Lee, 2000). The normal-flux terms are finally recalculated using the corrected convection terms L_i^* according to

$$\left(\frac{\partial \hat{\mathbf{E}}}{\partial \xi}\right)^* = J\underline{\mathbf{P}}\mathbf{L}^* + \mathbf{E}\frac{\partial r_\eta}{\partial \xi} - \mathbf{F}\frac{\partial z_\eta}{\partial \xi} \ . \tag{17}$$

4 Characteristic equations in η-direction

The quasi-linear characteristic wave equation for the η-direction can be written as

$$\frac{\partial \mathbf{R}_\eta}{\partial t} + \underline{\Lambda} \frac{\partial \mathbf{R}_\eta}{\partial \eta} = \mathbf{S}_{\mathbf{c}_\eta} , \tag{18}$$

where the modified source term is

$$\begin{aligned} \mathbf{S}_{\mathbf{c}_\eta} &= \frac{1}{J} \underline{\mathbf{P}}_\eta^{-\mathbf{1}} \left\{ \hat{\mathbf{S}}_\mathbf{v} - \left[-\mathbf{E} \frac{\partial r_\xi}{\partial \eta} + \mathbf{F} \frac{\partial z_\xi}{\partial \eta} + \frac{\partial \hat{\mathbf{E}}}{\partial \xi} + \frac{J}{r} \frac{\partial G}{\partial \theta} \right] \right\} \\ &= \frac{1}{J} \underline{\mathbf{P}}_\eta^{-\mathbf{1}} \left\{ \frac{\partial \hat{\mathbf{Q}}}{\partial t} + \frac{\partial \hat{\mathbf{F}}}{\partial \eta} + \mathbf{E} \frac{\partial r_\xi}{\partial \eta} - \mathbf{F} \frac{\partial z_\xi}{\partial \eta} \right\} . \end{aligned} \tag{19}$$

The characteristic convection term on a boundary or interface can now be computed as

$$\mathbf{L} = \frac{1}{J} \underline{\mathbf{P}}_\eta^{-\mathbf{1}} \left\{ \frac{\partial \hat{\mathbf{F}}}{\partial \eta} + \mathbf{E} \frac{\partial r_\xi}{\partial \eta} - \mathbf{F} \frac{\partial z_\xi}{\partial \eta} \right\} . \tag{20}$$

The characteristic convection term is then either corrected by imposing interface conditions according to (Kim & Lee, 2003), using the modified source term from (15), or using the appropriate LODI relations given in (Kim & Lee, 2000). The normal-flux terms are finally recalculated using the corrected convection terms L_i^* according to

$$\left(\frac{\partial \hat{\mathbf{F}}}{\partial \eta} \right)^* = J \underline{\mathbf{P}}_\eta \mathbf{L}^* - \mathbf{E} \frac{\partial r_\xi}{\partial \eta} + \mathbf{F} \frac{\partial z_\xi}{\partial \eta} . \tag{21}$$

5 Implementation

In order to advance the solution in time, the right-hand-side of equation 12 needs to be solved

$$RHS = \frac{\partial \hat{\mathbf{Q}}}{\partial t} = - \left[\frac{\partial \hat{\mathbf{E}}}{\partial \xi} + \frac{\partial \hat{\mathbf{F}}}{\partial \eta} + \frac{J}{r} \frac{\partial \mathbf{G}}{\partial \theta} + \frac{J}{r} \mathbf{H} - \frac{\partial \hat{\mathbf{E}}_\mathbf{v}}{\partial \xi} - \frac{\partial \hat{\mathbf{F}}_\mathbf{v}}{\partial \eta} - \frac{J}{r} \frac{\partial \mathbf{G}_\mathbf{v}}{\partial \theta} \right] . \tag{22}$$

The advantage of solving for $\hat{\mathbf{Q}}$ is that all conservative variables are known in (the equidistant) computational space. Thus, if filtering is required, filters derived for equidistantly spaced grids can be employed without loss of accuracy.
To make the algorithm of the novel code most efficient, the number of three-dimensional arrays to be stored needs to be minimized. At the same time, the number of arithmetic operations needs to be kept at bay, hence a good compromise between the number of stored quantities and variables to be recomputed needs to be found.

References

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Appendix

The transformation matrices between the conservative and the characteristic variables, and vice versa, are given as

$$\underline{\mathbf{P}}^{-1} = \begin{pmatrix} \mathbf{B}_0\cdot\mathbf{l}_z & (\gamma-1)\frac{u}{c^2}\tilde{\xi}_z & (\gamma-1)\frac{v}{c^2}\tilde{\xi}_z & (\gamma-1)\frac{w}{c^2}\tilde{\xi}_z-\frac{\tilde{\xi}_r}{\rho} & -\frac{\gamma-1}{c^2}\tilde{\xi}_z \\ \mathbf{B}_0\cdot\mathbf{l}_r & (\gamma-1)\frac{u}{c^2}\tilde{\xi}_r & (\gamma-1)\frac{v}{c^2}\tilde{\xi}_r & (\gamma-1)\frac{w}{c^2}\tilde{\xi}_r+\frac{\tilde{\xi}_z}{\rho} & -\frac{\gamma-1}{c^2}\tilde{\xi}_r \\ \mathbf{B}_0\cdot\mathbf{l}_\theta & \frac{\tilde{\xi}_r}{\rho} & -\frac{\tilde{\xi}_z}{\rho} & 0 & 0 \\ \frac{c}{\rho}\left(\frac{\gamma-1}{2}M^2-\frac{\mathbf{v}\cdot\mathbf{l}_\xi}{c}\right) & \mathbf{C}_+\cdot\mathbf{l}_z & \mathbf{C}_+\cdot\mathbf{l}_r & \mathbf{C}_+\cdot\mathbf{l}_\theta & \frac{\gamma-1}{\rho c} \\ \frac{c}{\rho}\left(\frac{\gamma-1}{2}M^2+\frac{\mathbf{v}\cdot\mathbf{l}_\xi}{c}\right) & \mathbf{C}_-\cdot\mathbf{l}_z & \mathbf{C}_-\cdot\mathbf{l}_r & \mathbf{C}_-\cdot\mathbf{l}_\theta & \frac{\gamma-1}{\rho c} \end{pmatrix} , \tag{23}$$

with

$$\mathbf{B}_0 = \{1-[(\gamma-1)/2]M^2\}\mathbf{l}_\xi - (1/\rho)(\mathbf{v}\times\mathbf{l}_\xi) \ ,$$
$$\mathbf{C}_\pm = \pm(\mathbf{l}_\xi/\rho) - [(\gamma-1)/\rho c]\mathbf{v} \ , \qquad \mathbf{v} = (u,v,w)^T$$

$$\begin{aligned} \mathbf{l}_\xi &= (\tilde{\xi}_z,\tilde{\xi}_r,0) = \left(1/\sqrt{\xi_z^2+\xi_r^2}\right)(\xi_z,\xi_r,0) \\ &= \left(1/\sqrt{\left(r^*_\eta\right)^2+\left(z^*_\eta\right)^2}\right)\left(r^*_\eta,-z^*_\eta,0\right) \ , \end{aligned}$$

$$\mathbf{l}_z = (1,0,0) \ , \qquad \mathbf{l}_r = (0,1,0) \ , \qquad \mathbf{l}_\theta = (0,0,1) \ .$$

$$\underline{\mathbf{P}} = \begin{pmatrix} \tilde{\xi}_z & \tilde{\xi}_r & 0 & \frac{\rho}{2c} & \frac{\rho}{2c} \\ u\tilde{\xi}_z & u\tilde{\xi}_r & \rho\tilde{\xi}_r & \frac{\rho}{2c}(u+\tilde{\xi}_z c) & \frac{\rho}{2c}(u-\tilde{\xi}_z c) \\ v\tilde{\xi}_z & v\tilde{\xi}_r & -\rho\tilde{\xi}_z & \frac{\rho}{2c}(v+\tilde{\xi}_r c) & \frac{\rho}{2c}(v-\tilde{\xi}_r c) \\ w\tilde{\xi}_z-\rho\tilde{\xi}_r & w\tilde{\xi}_r+\rho\tilde{\xi}_z & 0 & \frac{\rho}{2c}w & \frac{\rho}{2c}w \\ \mathbf{b}\cdot\mathbf{l}_z & \mathbf{b}\cdot\mathbf{l}_r & \mathbf{b}\cdot\mathbf{l}_\theta & \frac{\rho}{2c}(H+c\mathbf{v}\cdot\mathbf{l}_\xi) & \frac{\rho}{2c}(H-c\mathbf{v}\cdot\mathbf{l}_\xi) \end{pmatrix} , \tag{24}$$

with

$$\mathbf{b} = \frac{|\mathbf{v}|^2}{2}\mathbf{l}_\xi + \rho(\mathbf{v}\times\mathbf{l}_\xi) \ , \qquad H = \frac{|\mathbf{v}|^2}{2} + \frac{c^2}{\gamma-1} \ .$$

For the current case, where mapping is only performed in the $z-r$-plane and with $\tilde{\xi}_z = lr^*_\eta$ and $\tilde{\xi}_r = -lz^*_\eta$ with

$$l = \frac{1}{\sqrt{\left(r^*_\eta\right)^2+\left(z^*_\eta\right)^2}} \ , \tag{25}$$

and using $\gamma_1 = \gamma - 1$ and $q = \frac{u^2+v^2+w^2}{2}$, the transformation matrices become

$$\underline{\mathbf{P}}^{-1} = \begin{pmatrix} \left(1 - \frac{\gamma_1}{2}M^2\right) lr_\eta^* - \frac{l}{\rho} wz_\eta^* & \frac{\gamma_1 u}{c^2} lr_\eta^* & \frac{\gamma_1 v}{c^2} lr_\eta^* & \frac{\gamma_1 w}{c^2} lr_\eta^* + \frac{lz_\eta^*}{\rho} & -\frac{\gamma_1}{c^2} lr_\eta^* \\ \left(\frac{\gamma_1}{2}M^2 - 1\right) lz_\eta^* - \frac{l}{\rho} wr_\eta^* & -\frac{\gamma_1 u}{c^2} lz_\eta^* & -\frac{\gamma_1 v}{c^2} lz_\eta^* & -\frac{\gamma_1 w}{c^2} lz_\eta^* + \frac{lr_\eta^*}{\rho} & \frac{\gamma_1}{c^2} lz_\eta^* \\ \frac{l}{\rho}\left(uz_\eta^* + vr_\eta^*\right) & -\frac{lz_\eta^*}{\rho} & -\frac{lr_\eta^*}{\rho} & 0 & 0 \\ \frac{c}{\rho}\left(\frac{\gamma_1}{2}M^2 - \frac{lU}{c}\right) & \frac{l}{\rho} r_\eta^* - \frac{\gamma_1}{\rho c} u & -\frac{l}{\rho} z_\eta^* - \frac{\gamma_1}{\rho c} v & -\frac{\gamma_1}{\rho c} w & \frac{\gamma_1}{\rho c} \\ \frac{c}{\rho}\left(\frac{\gamma_1}{2}M^2 + \frac{lU}{c}\right) & -\frac{l}{\rho} r_\eta^* - \frac{\gamma_1}{\rho c} u & \frac{l}{\rho} z_\eta^* - \frac{\gamma_1}{\rho c} v & -\frac{\gamma_1}{\rho c} w & \frac{\gamma_1}{\rho c} \end{pmatrix}, \tag{26}$$

$$\underline{\mathbf{P}} = \begin{pmatrix} lr_\eta^* & -lz_\eta^* & 0 & \frac{\rho}{2c} & \frac{\rho}{2c} \\ ulr_\eta^* & -ulz_\eta^* & -\rho lz_\eta^* & \frac{\rho}{2c}(u + lr_\eta^* c) & \frac{\rho}{2c}(u - lr_\eta^* c) \\ vlr_\eta^* & -vlz_\eta^* & -\rho lr_\eta^* & \frac{\rho}{2c}(v - lz_\eta^* c) & \frac{\rho}{2c}(v + lz_\eta^* c) \\ wlr_\eta^* + \rho lz_\eta^* & -wlz_\eta^* + \rho lr_\eta^* & 0 & \frac{\rho}{2c} w & \frac{\rho}{2c} w \\ qlr_\eta^* + \rho lwz_\eta^* & -qlz_\eta^* + \rho lwr_\eta^* & -\rho l\left(uz_\eta^* + vr_\eta^*\right) & \frac{\rho q}{2c} + \frac{\rho c}{2\gamma_1} + \frac{\rho l}{2} U & \frac{\rho q}{2c} + \frac{\rho c}{2\gamma_1} - \frac{\rho l}{2} U \end{pmatrix}, \tag{27}$$

$$\underline{\mathbf{P}}_\eta^{-1} = \begin{pmatrix} \left(\frac{\gamma_1}{2}M^2 - 1\right) l_\eta r_\xi^* + \frac{l_\eta}{\rho} wz_\xi^* & -\frac{\gamma_1 u}{c^2} l_\eta r_\xi^* & -\frac{\gamma_1 v}{c^2} l_\eta r_\xi^* & -\frac{\gamma_1 w}{c^2} l_\eta r_\xi^* - \frac{l_\eta z_\xi^*}{\rho} & \frac{\gamma_1}{c^2} l_\eta r_\xi^* \\ \left(1 - \frac{\gamma_1}{2}M^2\right) l_\eta z_\xi^* + \frac{l_\eta}{\rho} wr_\xi^* & \frac{\gamma_1 u}{c^2} l_\eta z_\xi^* & \frac{\gamma_1 v}{c^2} l_\eta z_\xi^* & \frac{\gamma_1 w}{c^2} l_\eta z_\xi^* - \frac{l_\eta r_\xi^*}{\rho} & -\frac{\gamma_1}{c^2} l_\eta z_\xi^* \\ -\frac{l_\eta}{\rho}\left(uz_\xi^* + vr_\xi^*\right) & \frac{l_\eta z_\xi^*}{\rho} & \frac{l_\eta r_\xi^*}{\rho} & 0 & 0 \\ \frac{c}{\rho}\left(\frac{\gamma_1}{2}M^2 - \frac{l_\eta V}{c}\right) & -\frac{l_\eta}{\rho} r_\xi^* - \frac{\gamma_1}{\rho c} u & \frac{l_\eta}{\rho} z_\xi^* - \frac{\gamma_1}{\rho c} v & -\frac{\gamma_1}{\rho c} w & \frac{\gamma_1}{\rho c} \\ \frac{c}{\rho}\left(\frac{\gamma_1}{2}M^2 + \frac{l_\eta V}{c}\right) & \frac{l_\eta}{\rho} r_\xi^* - \frac{\gamma_1}{\rho c} u & -\frac{l_\eta}{\rho} z_\xi^* - \frac{\gamma_1}{\rho c} v & -\frac{\gamma_1}{\rho c} w & \frac{\gamma_1}{\rho c} \end{pmatrix}, \tag{28}$$

$$\underline{\mathbf{P}}_\eta = \begin{pmatrix} -l_\eta r_\xi^* & l_\eta z_\xi^* & 0 & \frac{\rho}{2c} & \frac{\rho}{2c} \\ -ul_\eta r_\xi^* & ul_\eta z_\xi^* & \rho l_\eta z_\xi^* & \frac{\rho}{2c}(u - l_\eta r_\xi^* c) & \frac{\rho}{2c}(u + l_\eta r_\xi^* c) \\ -vl_\eta r_\xi^* & vl_\eta z_\xi^* & \rho l_\eta r_\xi^* & \frac{\rho}{2c}(v + l_\eta z_\xi^* c) & \frac{\rho}{2c}(v - l_\eta z_\xi^* c) \\ -wl_\eta r_\xi^* - \rho l_\eta z_\xi^* & wl_\eta z_\xi^* - \rho l_\eta r_\xi^* & 0 & \frac{\rho}{2c} w & \frac{\rho}{2c} w \\ -ql_\eta r_\xi^* - \rho l_\eta wz_\xi^* & ql_\eta z_\xi^* - \rho l_\eta wr_\xi^* & \rho l_\eta\left(uz_\xi^* + vr_\xi^*\right) & \frac{\rho q}{2c} + \frac{\rho c}{2\gamma_1} + \frac{\rho l_\eta}{2} V & \frac{\rho q}{2c} + \frac{\rho c}{2\gamma_1} - \frac{\rho l_\eta}{2} V \end{pmatrix}, \tag{29}$$

where $M^2 = \frac{2q}{c^2}$, $U = ur_\eta^* - vz_\eta^*$, $V = vz_\xi^* - ur_\xi^*$, and

$$l_\eta = \frac{1}{\sqrt{\left(r_\xi^*\right)^2 + \left(z_\xi^*\right)^2}}\ , \tag{30}$$

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