AFFINE BUILDINGS AND PROPERTY A

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Abstract. Yu's Property A is a non equivariant generalisation of amenability introduced in his study of the coarse Baum Connes Conjecture. In this paper we show that all affine buildings of type $\tilde{A}_2$, $\tilde{B}_2$ and $\tilde{G}_2$ have Property A. Together with results of Guentner, Higson and Weinberger, this completes a programme to show that all affine building have Property A. In passing we use our technique to obtain a new proof for groups acting on $\tilde{A}_n$ buildings.

Introduction

Property A was introduced by Yu in [21] as a non equivariant generalisation of the group theoretic notion of amenability. He used it in his seminal proof that the property implies uniform embedding into a Hilbert space before appealing to a theorem by Higson and Roe to obtain the result that such groups satisfy the coarse Baum Connes conjecture.

It is defined as follows

Definition 0.1. *A discrete metric space X has* Property A *if for all $R, \epsilon > 0$ there exists $S > 0$ and a family of finite non-empty subsets A_x of $X \times \mathbb{N}$, indexed by x in X, such that*

- *for all x, x' with $d(x, x') < R$ we have $\frac{|A_x \Delta A_{x'}|}{|A_x|} < \epsilon$;*
- *for all (x', n) in A_x we have $d(x, x') \leq S$.*

This property is a coarse invariant. We will review the definitions of coarse functions and coarse invariance later on in section 2. There are several generalisations of Property A for non discrete spaces however, as remarked in [3]. The following is the most general:

Definition 0.2. *An arbitrary metric space X is said to have Property A if it contains a discrete coarsely dense subset with property A.*

Property A has been extensively studied [1, 9, 12, 13, 14, 19, 21]. Any group with Property A, (for example any group acting properly on a metric space with Property A), is exact or in other words its reduced C^*-algebra is exact. It is uniformly embeddable in a Hilbert space. Its action on its Stone-Čech compactification is amenable. Furthermore it satisfies both the coarse Baum Connes and the strong Novikov conjectures.

Examples of groups with Property A include free groups and amenable groups [4], word hyperbolic groups [21], discrete subgroups of connected Lie groups [21], and groups acting properly on CAT(0) cube complexes of finite dimension [5]. Examples of metric spaces with Property A include trees [9] and finite dimensional CAT(0) cube complexes [3]. In particular it is known that linear groups and affine buildings

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arising as buildings of linear groups have Property A [11]. However that leaves the cases of so called exotic affine buildings which are only found in dimension 2 and consist of those buildings which are not associated with a linear group. Such buildings are of type $\tilde{A}_2$, $\tilde{B}_2$ and $\tilde{G}_2$ [17].

In this paper we will prove the following results:

Theorem 1. *Affine buildings of type $\tilde{A}_2$, $\tilde{B}_2$ or $\tilde{G}_2$ have Property A.*

Together with results of [11] we obtain:

Corollary 0.3. *All affine buildings have Property A.*

All classical affine buildings admit a natural transitive action. However some exotic buildings also admit interesting actions, including a vertex transitive automorphism group [7, 20] and others with an orbit on two vertices but no transitive action on vertices and apartments [2].

Remark 0.4. *As commented in* [8]*, a non linear group with property T admitting a proper action with compact quotient could only possibly be found as an exotic group in a thick affine building of dimension 2.*

As a further result of this paper, any such group would also have Property A.

In group theoretic circles there is a known method introduced in [14] for showing that groups acting properly and cocompactly on a metric space have Property A. It was further developed in [12] using Gromov's notion of compression in order to give a new property of groups termed Hilbert space compression. This assigns a group a value between 0 and 1, and roughly speaking measures how much distortion occurs when embedded into a Hilbert space. Any group with Hilbert space Compression strictly greater than 1/2 has Property A [12]. In [12] they showed that free groups have Hilbert space compression 1 and thus have Property A and the method was adapted in [5] and extended to all groups acting properly and cocompactly on CAT(0) cube complexes.

The original main theorem from [14] on which all this relies was used in [4] to show that free groups and amenable groups have Property A. We will use this method to give explicit constructions for groups acting on buildings of dimension greater than two, which are partially based on those used in the first part of the paper, thus giving a new proof of the following theorem:

Theorem 2. *$\tilde{A}_n$ groups have Property A.*

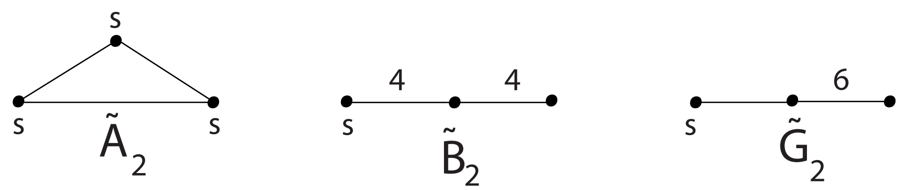
The author would like to thank Guyan Robertson for his guidance and useful talks during the course of this research and Graham Niblo for editorial comments.

1. Background

We will first introduce some necessary background on affine buildings and Property A and the main theorems on which our method is based.

A building Δ is affine when each apartment is an affine Coxeter complex. The Coxeter diagram must therefore be connected and the Coxeter matrix must be positive semi-definite [10]. Each apartment arises from a tiling of Euclidean space [17]. The simplest example of an affine building is a tree.

When the affine buildings are two dimensional, they are of type $\tilde{A}_2$, $\tilde{B}_2$ or $\tilde{G}_2$ and are obtained from the appropriate Coxeter diagrams as illustrated below. The s represent the special vertices which will be defined in a moment.



This gives us the $\tilde{A}_2$, $\tilde{B}_2$ and $\tilde{G}_2$ diagrams. Apartments are isometric to $\mathbb{E}^2$ tessellated by triangles.

When considering the tessellation of an apartment as a Euclidean space, $\tilde{A}_2$ is a tessellation by equilateral triangles, $\tilde{B}_2$ a tessellation by right angled isoceles triangles and $\tilde{G}_2$ a tessellation by right angled triangles with angles $\pi/6$ and $\pi/3$.

A chamber is a maximal simplex. A wall is associated to a reflection s and consists of the simplices which are fixed pointwise by s [10]. In the buildings we consider, the chambers are triangles and walls are lines. Some vertices have particular properties and are called *special*. These are identified as follows. Fix a chamber and let S be the set of reflections in its walls and W the Coxeter group generated by it. Let $\bar{W}$ be the group of linear parts of W. A point x such that $W_x \longrightarrow \bar{W}$ is an isomorphism is special [10].

A wall divides any apartment into two roots. Consider a special vertex s and a base chamber c containing it. The intersection of the roots determined by the panels of c (ie the roots containing c) which contain s is a simplicial cone. We call this a sector. We define a sector face to mean a face of a sector treated as a simplicial cone and say that they are parallel if the distance between them is bounded.

It is easy to see that parallellism is an equivalence relation. Two sectors are parallel if and only if their intersection contains a sector [17].

By using equivalence classes of sectors, we can construct the "building at infinity" Δ^∞. Its chambers are defined to be parallel classes of sectors. Two chambers are adjacent if there are representative sectors having sector panels which are parallel.

Our method will rely on the following lemma:

Lemma 1.1. [17] *If s is a special vertex, then each parallel class of sectors contains a unique sector having vertex s.*

This means that if we fix a chamber in the building at infinity, then each special vertex has a unique sector associated to it. We will use this induced structure to construct our functions and prove that the metric space has Property A.

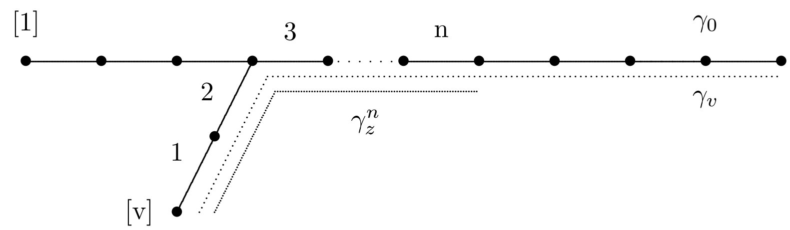
Our method is inspired by a proof by Dranishnikov and Januszkiewicz that trees have Property A [9]. Their proof is as follows.

Proof. Let T be the Cayley graph of a free group (a tree) and V its set of vertices.

Fix a point at infinity and let $\gamma_0 : \mathbb{R} \to T$ be a fixed geodesic ray in T.

For any vertex z, let γ_z be the unique geodesic ray issuing from z and intersecting γ_0 along a geodesic ray.

Let γ_z^n be the initial ray of γ_z of length n as represented in the following diagram:



Dranishnikov and Januszkiewicz then use the following characterisation of Property A for metric spaces of bounded geometry:

Definition 1.2. *A discrete metric space Z has the Property A if and only if there is a sequence of maps $a^n : Z \longrightarrow P(Z)$ such that:*

(1) *for every n there is some $R > 0$ with the property that for every $z \in Z$; supp $a_z^n \subset \{z' \in Z | d(z, z') < R\}$*
(2) *for every $K > 0$, $\lim_{n\longrightarrow\infty} < K \sup_{d(z,w)<K} \|a_z^n - a_w^n\|_1 = 0$*

They define a_z^n as the Dirac measure $\sum_v \frac{1}{n+1}\delta_v$ supported uniformly by vertices lying in the geodesic segment $[z, \gamma_z(n)]$. This obviously satisfies the first condition. To see the second condition, observe that if $d(z, w) < K$, then the overlap of the geodesic segments $[z, \gamma_z(n)]$ and $[w, \gamma_w(n)]$ is at least $n-K$. Then $\sup \|a_z^n - a_w^n\|_1 \le \frac{K}{n+1}$ and condition 2 holds. □

Our method will also rely on constructing weighted functions on the vertices of the metric spaces we consider. However we will use different characterisations of Property A.

In particular we will make use of the following proposition:

Proposition 1.3. *An arbitrary metric space X, has Property A if there exists a sequence of families $f_{n,x}$, indexed by x in X, of finitely supported functions from X to $\mathbb{N} \cup \{0\}$, and a sequence $S_n > 0$, such that for each n, x the function $f_{n,x}$ is supported in $B_{S_n}(x)$, and for any $R > 0$*

$$\frac{\|f_{n,x} - f_{n,x'}\|_1}{\|f_{n,x}\|_1} \to 0$$

uniformly on the set $\{(x, x') : d(x, x') \le R\}$ as $n \to \infty$.

2. Property A for affine buildings

Let S, Δ be metric spaces. Recall that two maps $i, j : S \longrightarrow \Delta$ are said to be "close" if there is $D < \infty$ such that $d(i(s), j(s)) < D$ for all $s \in S$.

A map $i : S \longrightarrow \Delta$ is called coarse if it is both coarsely proper (the inverse image of any bounded set is bounded) and bornologous ($\forall R > 0, \exists S > 0$ such that $d_\Delta(i(x), i(x')) < S$ whenever $d_S(x, x') < R$).

A map i is called a coarse equivalence if it is a coarse map $S \longrightarrow \Delta$ and there exists another coarse map $j : \Delta \longrightarrow S$ such that $i \circ j$ is close to Id_S and $j \circ i$ is close to Id_Δ.

In this case S is coarsely equivalent to Δ.

Let Δ be an affine building. We equip the 1-skeleton path metric where each edge has length 1. Note that this metric is quasi-isometric to the geodesic metric.

In Δ every chamber is a simplex and contains a special vertex. Thus every vertex is either a special vertex or is adjacent to a special vertex. Let $\mathcal{S}$ be the metric space of the set of all special vertices in Δ equipped with the subspace metric. Now consider the two maps i from $\mathcal{S}$ to Δ which maps every special vertex to itself and j from Δ to $\mathcal{S}$ which maps every special vertex to itself and every other vertex to a special vertex distance 1 away.

Lemma 2.1. *Δ and $\mathcal{S}$ are coarsely equivalent.*

Proof. The maps i and j are both coarsely proper since the inverse image of any bounded set is bounded. For any $r > 0$, $d(j(x), j(x')) \leq r + 2$ for all pairs x, x' such that $d(x, x') \leq r$. The maps i and j are both bornologous and hence are coarse maps.

Consider the map $i \circ j$. Every vertex is mapped to itself and then back to itself. This map is exactly the identity map Id_S.

Consider the map $j \circ i$. Every special vertex is mapped to itself and then back to itself. Every non special vertex is mapped to a vertex distance one away and then that vertex is mapped back to itself, ending up distance one away from the original point. The map $j \circ i$ is close to Id_B, with $D = 1$.

□

Note that when Δ has type $\tilde{A}_2$ then $\mathcal{S} = \Delta$.

Let Δ be any affine building of dimension 2 equipped with the 1-skeleton metric. Let Δ^∞ be its building at infinity and fix a chamber $\omega \in \Delta^\infty$. Now let $\mathcal{S}$ be the metric space of special vertices of Δ equipped with the subspace metric.

Given any vertex $x \in \mathcal{S}$ we consider the sector based at x defined by ω and denote it $S^x(\omega)$. Let $B(n, x)$ denote the ball or radius n around x.

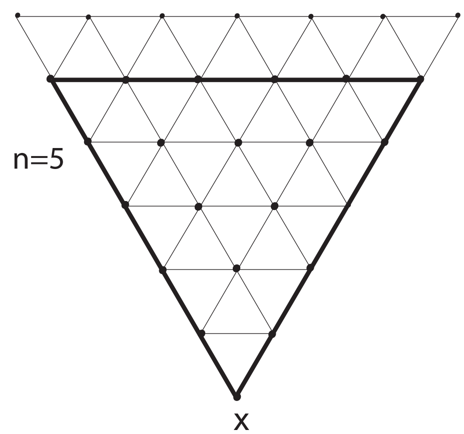
For any vertex $y \in \mathcal{S}$, let $f_{n,x}$ denote the characteristic function of the intersection $S^x(\omega) \cap B(n, x)$, so

$$f_{n,x}(y) = \begin{cases} 1 & \text{if } y \in S^x(\omega) \text{ and } d(x, y) \leq n \\ 0 & \text{if } d(x, y) > n \text{ and/or } y \notin S^x(\omega). \end{cases}$$

Proposition 2.2. *Let Δ be an affine building and $\mathcal{S}$ the metric space of its special vertices equipped with the 1-skeleton path subspace metric. If Δ is of type $\tilde{A}_2$, then $\mathcal{S}$ has Property A.*

Proof. Let Δ be an affine building of type $\tilde{A}_2$ and consider $f_{n,x}(y)$.

The following diagram represents the contributing vertices in the case $n = 5$.



Note that $\|f_{n,x}\|_1$ is the number of vertices in the enclosed black area.

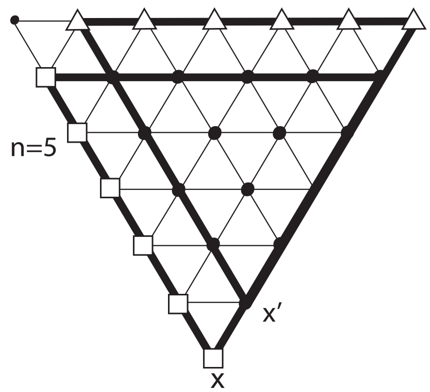
Lemma 2.3. *If Δ is an $\tilde{A}_2$ building and $x \in \Delta$ is a special vertex, then* $\|f_{n,x}\|_1 = \frac{n^2+3n+2}{2}$

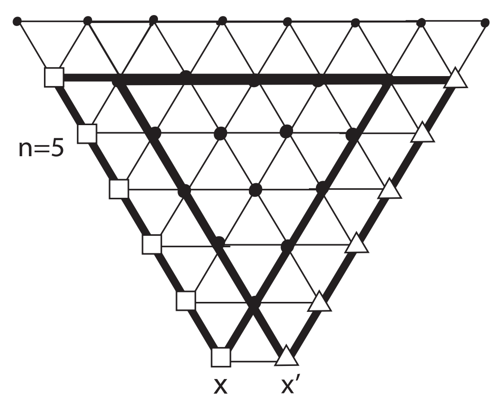
Proof. $\|f_{n,x}\|_1 = \sum_{i=1}^{n+1} i = \frac{n^2+3n+2}{2}$ □

Now consider two adjacent vertices x and x'. Since they are adjacent, x, x' and ω must all lie in a common plane.

Given a vertex, there are only 6 adjacent vertices. By symmetry we need only consider 3 of them. The sector based at the vertex intersects two of the adjacent vertices symmetrically. Thus there are only two possible generic situations. An adjacent vertex can either belong to the other's sector or not.

This is illustrated in the following diagram (here for $n = 5$). The squares represent vertices belonging to S^x but not $S^{x'}$ and the triangles represent vertices belonging to $S^{x'}$ but not S^x:





Since we are working in the Euclidean plane, no distortion or scaling occurs irrespective of the size of n. Note that this will be the case throughout this paper, whichever affine building we are considering.

And so in the more general case, the number of square vertices or the number of triangle vertices is always $n+1$, giving a total of $2(n+1)$.

Thus $\|f_{n,x} - f_{n,x'}\|_1 = 2(n+1)$ and by the triangle inequality,

$$\|f_{n,x} - f_{n,x'}\|_1 \leq 4d(x,x')(n+1)$$

Hence

$$\frac{\|f_{n,x} - f_{n,x'}\|_1}{\|f_{n,x}\|_1} \leq \frac{8d(x,x')(n+1)}{n^2+3n+2}$$

On the set $d(x,x') \leq R$, this tends uniformly to 0 as n tends to infinity as required. The conditions of Proposition 1.3 are satisfied and $\mathcal{S}$ has Property A. □

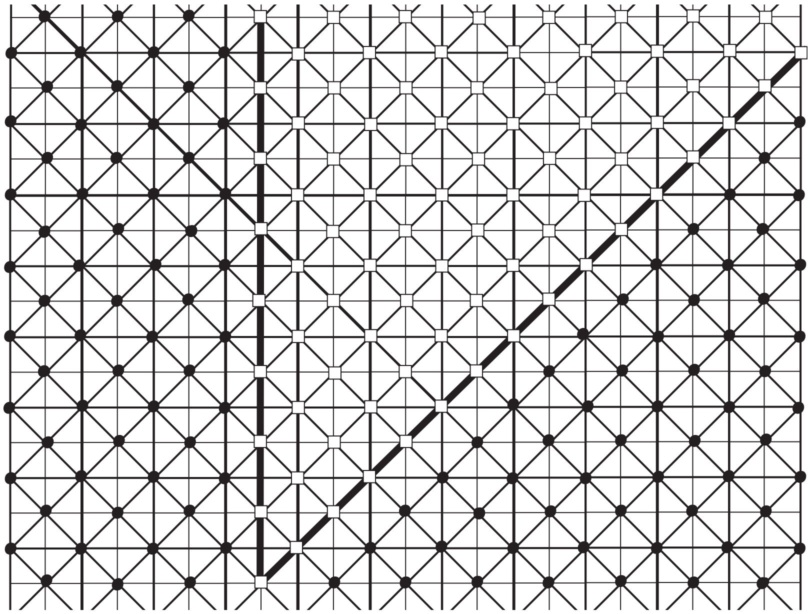
Since in this particular case $\mathcal{S}$ and buildings of type $\tilde{A}_2$ are one and the same, we obtain:

Corollary 2.4. *Affine buildings of type $\tilde{A}_2$ have Property A.*

Proposition 2.5. *Let Δ be an affine building and $\mathcal{S}$ the metric space of its special vertices equipped with the 1-skeleton path subspace metric. If Δ is of type $\tilde{B}_2$, then $\mathcal{S}$ has Property A.*

Proof. Let Δ be an affine building of type $\tilde{B}_2$ equipped with the 1-skeleton path metric. Let Δ^∞ be its building at infinity and fix a chamber $\omega \in \Delta^\infty$. Now let $\mathcal{S}$ be the subspace of Δ consisting of all the special vertices and equipped with the subspace metric.

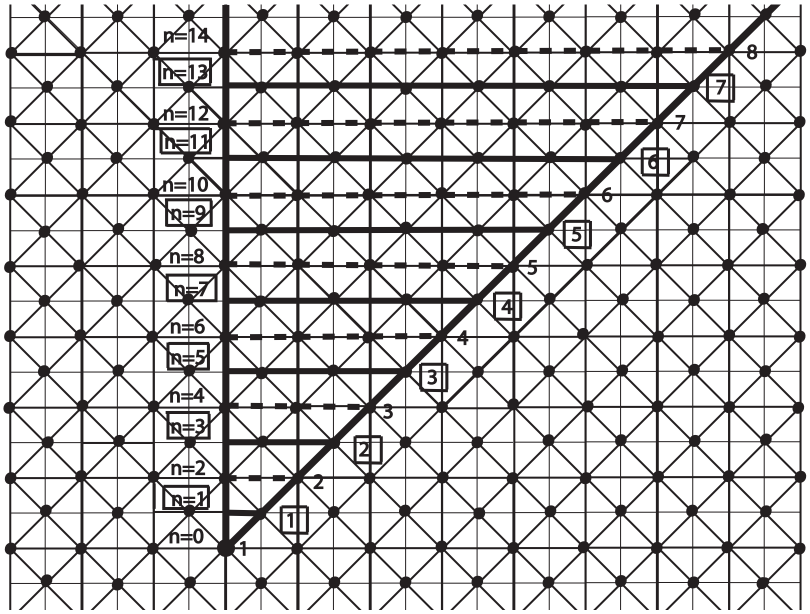
A diagram of an apartment and a sector in $\mathcal{S}$ follows. The thick black lines delimit a sector. The squares represent vertices belonging to it, while the circles are vertices outside it:



Comment: since we are using the subspace metric the distances between vertices remain the same as in $\tilde{B}_2$. For example the distance between two adjacent square vertices on the thick vertical line delimiting the sector is 2.

We consider the function $f_{n,x}$. Note that $\|f_{n,x}\|_1$ is the number of vertices in the sector truncated at n, (ie all vertices in S_x at distance at most n away from the base point x).

Consider the following diagram of a sector in $\mathcal{S}$, delimited by the thick black lines. The horizontal thick and dashed lines represent the points belonging to the sector and distance exactly n away from the base point x, where thick is the case when n is odd and dashed is the case when n is even. The numbers on the left are the distance n. The numbers on the right are the number of such special vertices.



The number of special vertices in the sector at distance exactly n is $(n+1)/2$ if n is odd and $(n/2+1)$ if n is even.

Thus if n is odd, then

$$|\text{special vertices y} \in S^x(\omega), d(y,x) \leq n| = 2 * \sum_{i=1}^{\frac{n+1}{2}} i = \frac{n^2}{4} + n + 3$$

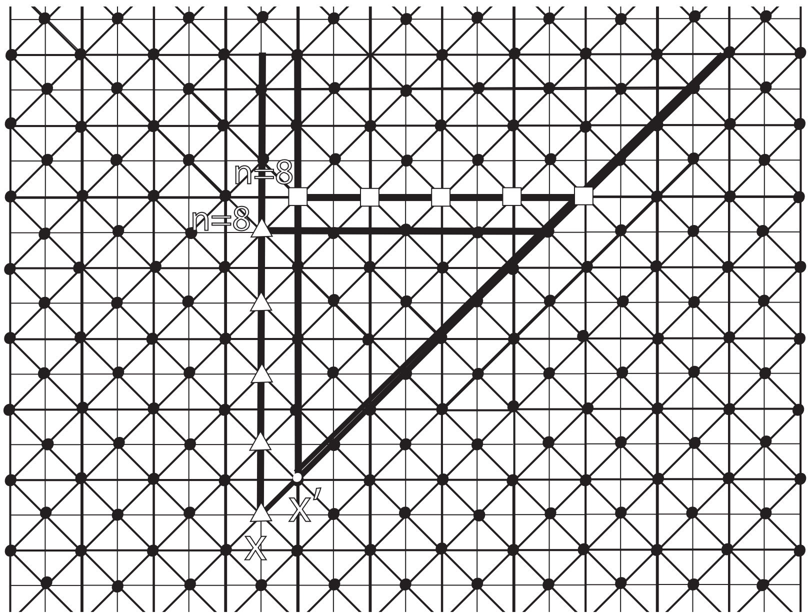
On the other hand, if n is even, then

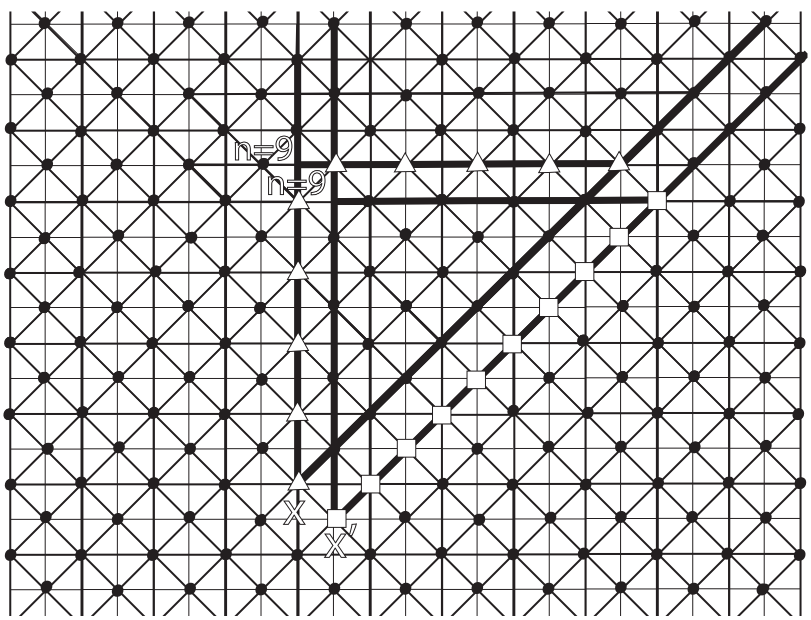
$$|\text{special vertices y} \in S^x(\omega), d(y,x) \leq n| = \sum_{i=1}^{\frac{n}{2}+1} i + \sum_{i=1}^{\frac{n}{2}} i = \frac{n^2}{4} + n + 1$$

Hence $\|f_{n,x}\|_1$ is either $\frac{n^2}{4} + n + 3$ or $\frac{n^2}{4} + n + 1$.

Now consider the case of two adjacent vertices x, x'. Given a vertex x, an adjacent vertex x' can lie in four different positions. By symmmetry we need only consider two of them. There are two general possibilities, either one vertex belongs to the other's sector, or it does not.

This is illustrated in the following diagrams where we have taken the specific examples with $n = 8$ and $n = 9$. The triangles represent vertices which belong to S^x but not $S^{x'}$ and the squares are vertices which belong to $S^{x'}$ but not S^x.





In the first case, in general the number of special vertices which belong to $S^x(\omega)$ or $S^{x'}(\omega)$ but do not lie in the intersection is either $n+1$ or $n+2$. It is the sum of the triangle and square vertices. The number of triangle or square vertices is $\frac{n}{2}+1$ when n is even and $\frac{n+1}{2}$ when n is odd. Hence when n is odd, the result is $n+1$ and when n is even it is $n+2$.

In the second case the number of triangle or square vertices is always $n+1$ regardless of whether n is odd or even and so the general total is $2(n+1)$.

Thus the only possibilities for $\|f_{n,x} - f_{n,x'}\|_1$ are $n+1$, $n+2$ and $2(n+1)$.

Looking at the worse case scenario, the largest $\|f_{n,x} - f_{n,x'}\|_1$ can be is $2(n+1)$, and the smallest $\|f_{n,x}\|_1$ can be is $\frac{n^2}{4} + n + 1$.

In general by the triangle inequality, $\|f_{n,x} - f_{n,x'}\|_1 \leq 2d(x, x')2(n+1)$

Hence

$$\frac{\|f_{n,x} - f_{n,x'}\|_1}{\|f_{n,x}\|_1} \leq \frac{4d(x, x')(n+1)}{\frac{n^2}{4} + n + 1}$$

On the set $d(x, x') \leq R$, this tends uniformly to 0 as n tends to infinity as required.

By proposition 1.3, $\mathcal{S}$ has Property A. □

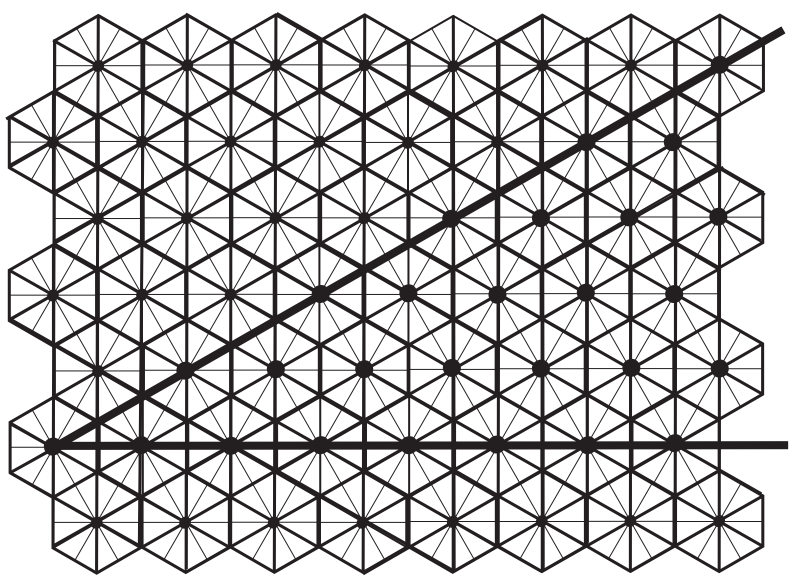
By coarse equivalence we deduce:

Corollary 2.6. *Affine buildings of type $\tilde{B}_2$ have Property A.*

Proposition 2.7. *Let Δ be an affine building and $\mathcal{S}$ the metric space of its special vertices equipped with the 1-skeleton path subspace metric. If Δ is of type $\tilde{G}_2$, then $\mathcal{S}$ has Property A.*

Proof. Let Δ be a an affine building of type $\tilde{G}_2$. Let Δ^∞ be its building at infinity and fix a chamber $\omega \in \Delta^\infty$. Now let $\mathcal{S}$ be the subspace of Δ consisting of all the special vertices and equipped with the 1-skeleton path subspace metric.

A diagram of an apartment and sector in $\mathcal{S}$ is as follows:

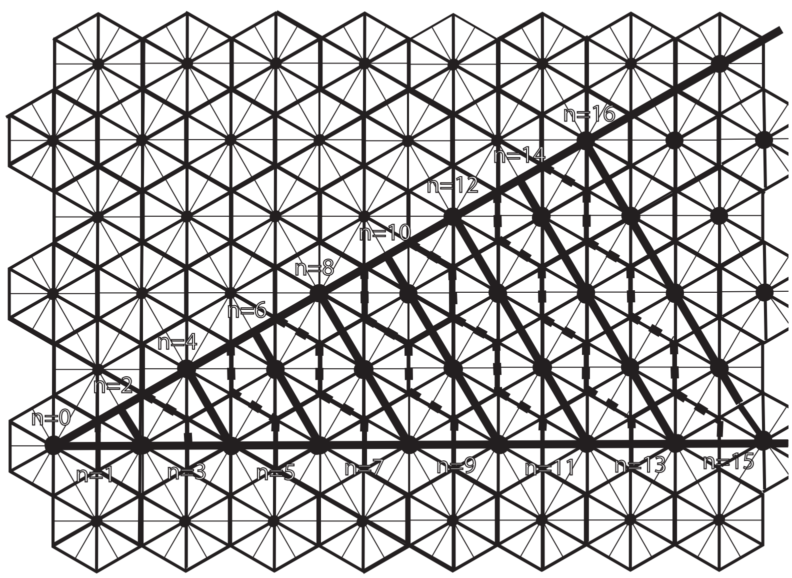


As before, since we are using the subspace metric the distances between special vertices remain the same as in $\tilde{G}_2$. For example the distances between two adjacent vertices on the top and horizontal thick lines delimiting the sector are 4 and 2 respectively.

We consider $f_{n,x}$.

As earlier, $\|f_{n,x}\|_1$ is the number of vertices in the sector truncated at n. We need to calculate the number of vertices contained within a sector x, distance n or less away from the base point x.

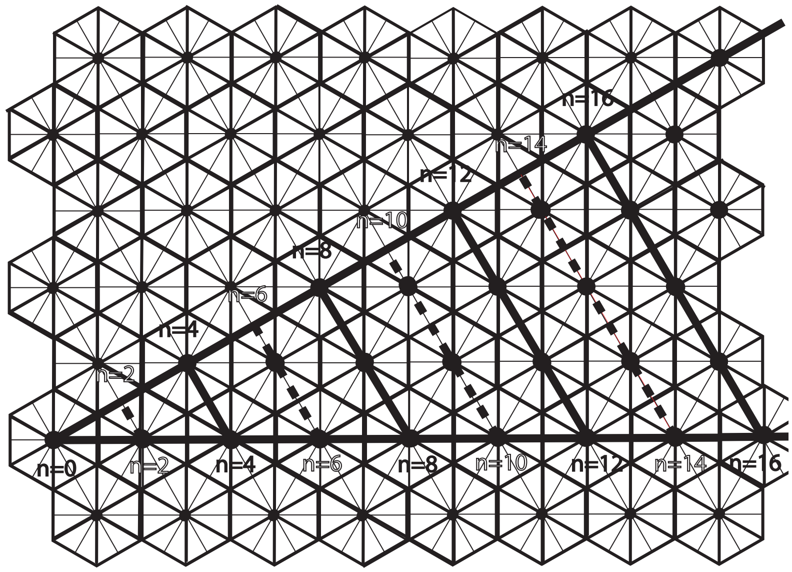
The following diagram represents this. These are all the vertices in the sector from the base point up to and belonging to the appropriate line.



Since adjacent special vertices are distance 2 apart in $\tilde{G}_2$, note that when n is odd, the number of vertices is exactly the same as that for $n-1$.

In view of this, let us concentrate on the case when n is even.

Consider the following diagram.



Both dashed and thick lines follow a pattern of form $\sum_{i=1}^{i=k} i$ for some k.

If n is a multiple of 4 (i.e. $n = 4 * j$) for some integer j, or if $n = 4 * j + 1$ (since we saw that for n odd, the number of vertices is the same as that of $n - 1$), then the number of vertices is the appropriate sum of the dashed vertices and that of the thick vertices. Specifically:

If n is of the form $4 * j$ or $4 * j + 1$, then

$$|\text{vertices y} \in S^x(\omega), d(y, x) \leq n| = \sum_{j=1}^{j+1} j + \sum_{j=1}^{j} j = (j + 1)^2$$

If n is of the form $4 * j + 2$ or $4 * j + 3$, then

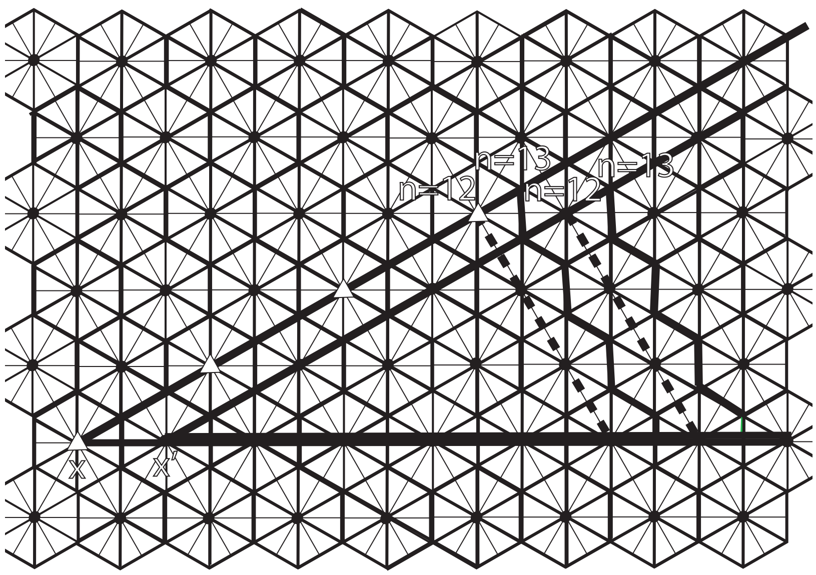
$$|\text{vertices y} \in S^x(\omega), d(y, x) \leq n| = 2 * \sum_{j=1}^{j+1} j = (j + 1)(j + 2)$$

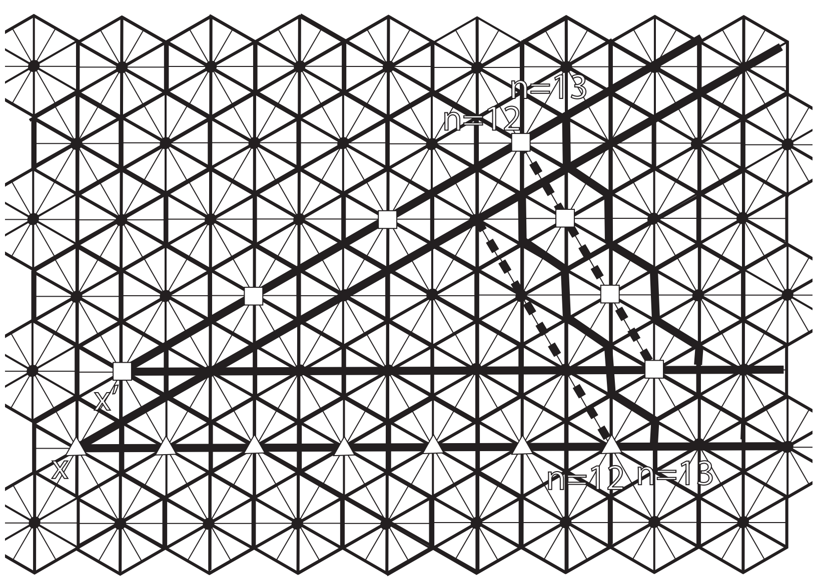
In particular, $|\text{vertices y} \in S^x(\omega), d(y, x) \leq n|$ is at least $\left(\left\lfloor \frac{n}{4} \right\rfloor + 1\right)^2$ and so we have

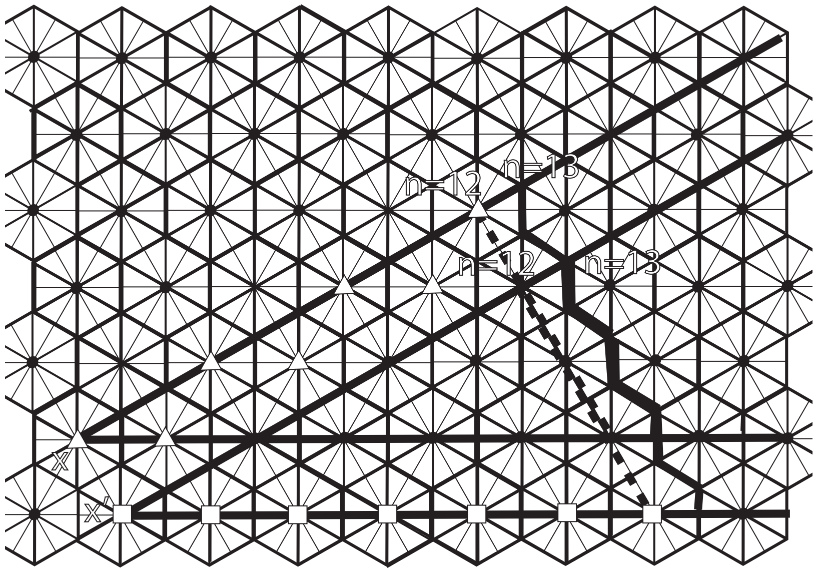
$$\|f_{n,x}\|_1 \geq \left(\left\lfloor \frac{n}{4} \right\rfloor + 1\right)^2$$

Now consider the case of two vertices x, x' distance 2 apart, and their associated sectors. Assume that ω lies in the same plane as x and x'. Given a vertex x, there are six possible positions for x', three of which we can ignore by symmetry. The other three are all distinct and we must consider each one in turn. x' could lie on the sector based at x. In the other two cases it does not and the intersection point of its sector with the sector based at x is either distance 2 or 4 away from x'.

We have illustrated the three possibilities in the following diagrams where we have taken the specific examples with $n = 12, 13$.







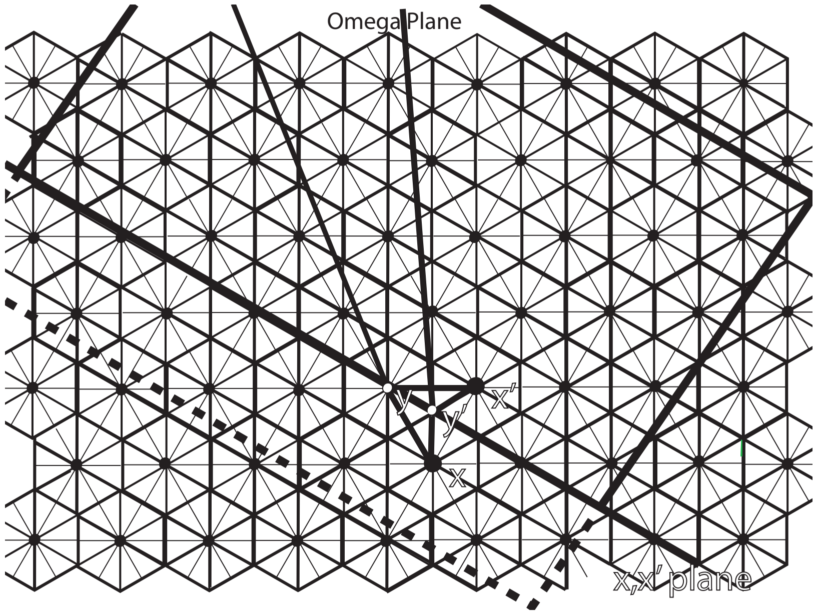
In the first case, in general the number of vertices which belong to $S^x(\omega)$ or $S^{x'}(\omega)$ but do not lie in the intersection is the number of triangle vertices and since the distance between special vertices is 4, is equal to $\lfloor \frac{n}{4} \rfloor + 1$.

In the other cases it is the sum of the triangle and square vertices. The number of triangle or square vertices is $\frac{n}{2}+1$ when n is even and $\frac{n+1}{2}$ when n is odd. Hence when n is odd, the result is $n+1$ and when n is even it is $n+2$.

Unlike the $\tilde{B}_2$ case x,x' and ω need not lie in the same plane, since the distance between x and x' is 2. However the short distance imposes restrictions on the amount of branching which can occur. The plane on which ω lies has to intersect the wall between x and x'. Otherwise there exists a plane on which x, x' and ω all lie, which contradicts our original assumption.

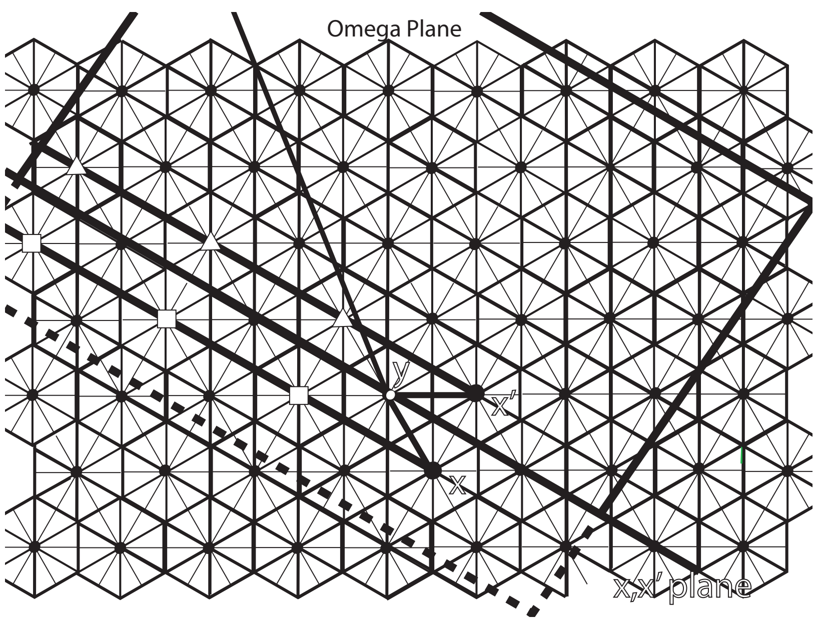
By symmetry we only need to consider two possible outcomes, depending upon whether the delimiting sector lines issuing from x and x' intersect the ω-plane once or twice.

We first look at the case where they intersect the ω-plane twice. Since the ω-plane constitutes a plane of symmetry between x and x', the points of intersection from x and those from x' are the same. We will call them y and y'. In order to obtain two intersection points, y and y' must lie distance 1 or 2 away. Otherwise one of the the delimiting sector lines is parallel to the intersection of the ω-plane with the (x, x')-plane and there is only one interesection point. An example is illustrated below.



The intersection of the two sectors lies entirely within the ω-plane. Thus the only special vertices which do not belong to the intersection but do belong to the sector are x and x' themselves and the number of such vertices is 2.

In the second case, if only one delimiting sector line from each of x and x' intersects the ω-plane, then we get the following generic situation:



The intersection of the sectors based at x and x' is a sector lying entirely in the ω-plane based at the intersection point y. However the union of these sectors includes in addition a strip lying in the (x, x')-plane, which is delimited by the two parallel lines to the intersection of the ω-plane with the (x, x')-plane as shown in the diagram.

The special vertices lie distance 4 apart along these lines. Hence the total number of such vertices is equal to $2\lfloor \frac{n}{4} \rfloor + 2$.

By definition, the sectors based at x and x' must intersect the ω-plane. In any other cases this would not occur, so there are no more cases to consider.

So the possibilities for $\|f_{n,x} - f_{n,x'}\|_1$ are $2, \lfloor \frac{n}{4} \rfloor + 1, 2\lfloor \frac{n}{4} \rfloor + 2, n+1$ or $n+2$.

Considering the worse case scenario, when $d(x, x') = 2$, $\|f_{n,x} - f_{n,x'}\|_1 \leq n + 2$. Now Δ is connected and the distance between two special vertices is always 2. This implies that $\mathcal{S}$ is 2-connected, i.e. any 2 vertices can be connected by a path where the distance at each step is 2.

So in general by the triangle inequality, $\|f_{n,x} - f_{n,x'}\|_1 \leq 2d(x, x')(n + 2)$.

Hence

$$\frac{\|f_{n,x} - f_{n,x'}\|_1}{\|f_{n,x}\|_1} \leq \frac{2d(x, x')(n + 2)}{\left(\lfloor \frac{n}{4} \rfloor + 1\right)^2}$$

On the set $d(x, x') \leq R$, this tends uniformly to 0 as n tends to infinity as required. Thus by proposition 1.3 $\mathcal{S}$ has Property A. □

By coarse equivalence we obtain:

Corollary 2.8. *Affine buildings of type $\tilde{G}_2$ have Property A.*

Putting together propositions 2.2, 2.5 and 2.7 we obtain:

Theorem 3. *Let Δ be an affine building and $\mathcal{S}$ the metric space of its special vertices equipped with the subspace metric. If Δ is of type $\tilde{A}_2$, $\tilde{B}_2$ or $\tilde{G}_2$, then $\mathcal{S}$ has Property A.*

By coarse equivalence we deduce our main theorem:

Theorem (1)**.** *Affine buildings of type $\tilde{A}_2$, $\tilde{B}_2$ or $\tilde{G}_2$ have Property A.*

Since affine buildings in higher dimensions arise from linear groups and have Property A [11], we also deduce:

Corollary (0.3)**.** *All affine buildings have Property A.*

3. A direct construction for buildings of type $\tilde{B}_2$

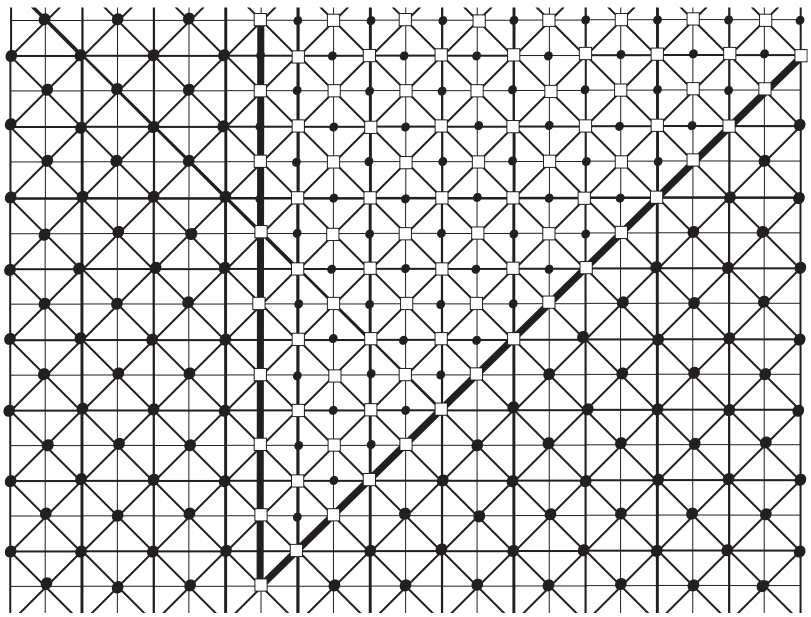
The use of the coarse equivalence obscures the nature of the functions except in the case of the buildings of type $\tilde{A}_2$ where every vertex is special. Here we show that in the case of affine buildings of type $\tilde{B}_2$ the functions can also be constructed explicitly. This may be of value in attempting to generalise these methods to other spaces of non positive curvature.

Theorem 4. *Affine buildings of type $\tilde{B}_2$ have Property A.*

Proof. Let Δ be an affine building of type $\tilde{B}_2$. Let Δ^∞ be its building at infinity and fix a chamber $\omega \in \Delta^\infty$.

The situation here is slightly tricky since not every vertex is a special vertex. Since non special vertices do not admit sectors at infinity, we cannot proceed directly as we did in the $\tilde{A}_2$ case where every vertex is a special vertex.

This is illustrated in the following diagram of an apartment in Δ . The thick black lines delimit a sector. The square vertices represent the special vertices and the small black round ones the non special vertices within this sector. The emphasized black vertices represent the special vertices which do not belong to the sector.



For every vertex $x \in \Delta$, choose s_x to be one of the special vertices closest to it. This means that if x is a special vertex, then s_x is x itself. If on the other hand x is not a special vertex, then s_x is one of the special vertices distance 1 away. The actual choice of special vertex is not important for the purposes of this paper.

3.1. **The function.** Given any vertex $x \in \Delta$ we will consider the sector based at s_x defined by ω and denote it $S^{s_x}(\omega)$.

We now define our function as follows:

$$f_{n,x}(y) = \begin{cases} 1 & \text{if } y \text{ is a special vertex} \in S^{s_x}(\omega) \text{ and } d(s_x, y) \leq n \\ 0 & \text{otherwise} \end{cases}$$

All values of $f_{n,x}$ are positive and $\|f_{n,x}\|_1$ is the number of special vertices in the sector truncated at n, which as seen earlier in this paper in the proof of theorem 4 is either $\frac{n^2}{4} + n + 3$ or $\frac{n^2}{4} + n + 1$.

Now consider the case of two adjacent vertices x, x'. There are only two possibilities. Either both are special vertices, or one is special and one is not.

Let us first assume that both are special and thus s_x and $s_{x'}$ are the points x, x' themselves.

As seen previously in the paper during the proof of theorem 4, the only possibilities for $\|f_{n,x} - f_{n,x'}\|_1$ in this situation are $n+1$, $n+2$ and $2(n+1)$.

Now let us assume that one of x, x' is special but the other is not. Without loss of generality , assume that x is special and thus s_x is x itself.

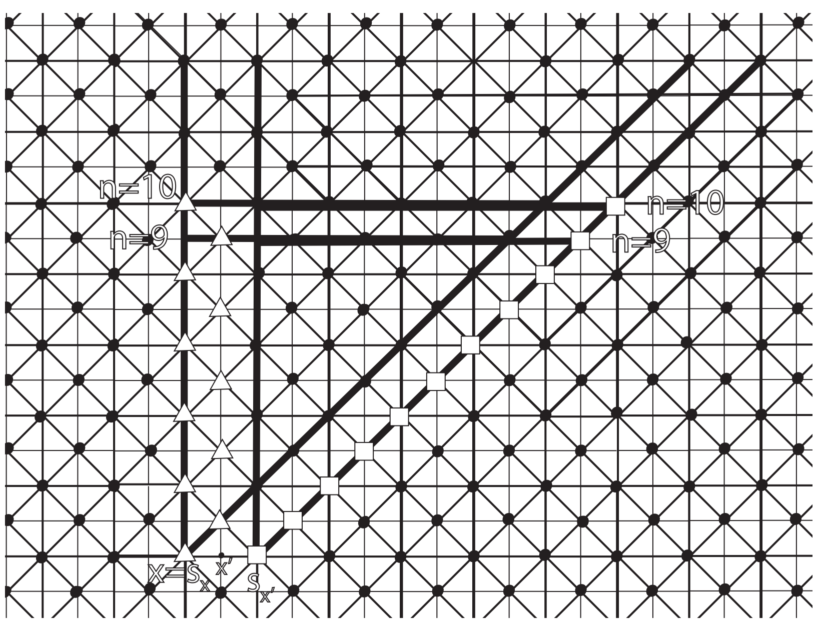
x' can be mapped to several special vertices.

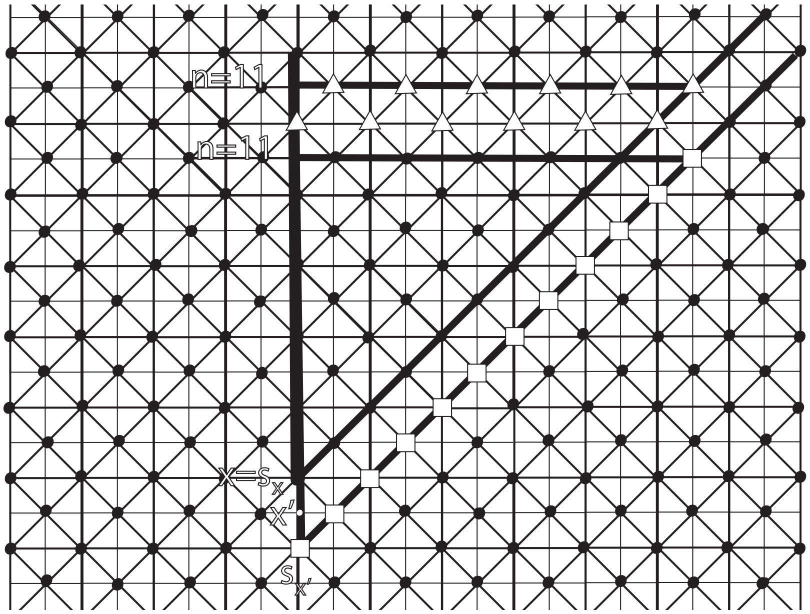
$s_{x'}$ could be s_x itself. In this case $\|f_{n,x} - f_{n,x'}\|_1 = 0$.

x' could be mapped to a special vertex distance 1 away from s_x. In this case s_x and $s_{x'}$ are distance 1 apart and as we have just seen, $\|f_{n,x} - f_{n,x'}\|_1$ is either $n+1$, $n+2$ or $2(n+1)$.

Finally, x' could also be mapped to a special vertex distance 2 away from s_x.

Assume the special vertex remains in the same apartment as the sector based at x. There are four positions it could go, two of which we can ignore by symmetry. We need to consider the remaining two cases. We illustrate them in the following diagrams with the special cases of $n = 9, 10, 11$.

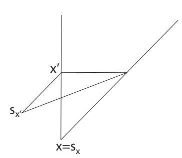




In fact these are essentially the same and the number of special vertices which belong to $S^{s_x}(\omega)$ or $S^{s_{x'}}(\omega)$ but do not lie in the intersection remains identical.

The number of square vertices is once again $n+1$, as is the number of triangle vertices. In both cases $\|f_{n,x} - f_{n,x'}\|_1 = 2(n+1)$.

It is feasible that $s_{x'}$ not lie in the same apartment as the sector based at x, in which case the sector based at $s_{x'}$ intersects that based at $x = s_x$ at the first two vertices along the sector as illustrated in the following diagram and $\|f_{n,x} - f_{n,x'}\|_1 = 2$.



Looking at the worst case scenario, the largest $\|f_{n,x} - f_{n,x'}\|_1$ can be is $2(n+1)$, and the smallest $\|f_{n,x}\|_1$ can be is $\frac{n^2}{4} + n + 1$.

In general by the triangle inequality, $\|f_{n,x} - f_{n,x'}\|_1 \leq 2d(x,x')2(n+1)$

Hence

$$\frac{\|f_{n,x} - f_{n,x'}\|_1}{\|f_{n,x}\|_1} \leq \frac{4d(x,x')(n+1)}{\frac{n^2}{4} + n + 1}$$

On the set $d(x,x') \leq R$, this tends uniformly to 0 as n tends to infinity as required. Thus by proposition 1.3, buildings of type $\tilde{B}_2$ have Property A. $\square$

4. Groups acting on $\tilde{A}_n$ buildings

We turn our attention more specifically to groups acting on affine buildings and look at an example for higher dimensions.

The result for $\tilde{A}_n$ groups had previously been deduced to be true by a different method. If Γ is a closed subgroup of $Aut(\Delta)$ where Δ is a euclidean building of dimension > 2, then Γ acts amenably on its boundary and so as previously discussed has Property A [13, 16].

We will now use the ideas from [14] and [12] to obtain a shorter direct geometric construction proof for $\tilde{A}_n$. In practise this simply involves slightly modifying the general method used in the previous sections of our paper of weighting the vertices in the intersection of sectors.

To do this we use the following group characterisation of Property A[12]:

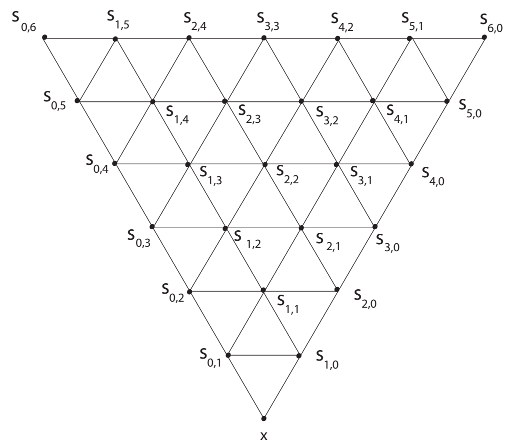
Proposition 4.1. *Let Γ be a finitely generated discrete group equipped with word length and metric associated to a finite symmetric set of generators. Then Γ is exact and has Property A iff there exists a sequence of positive functions $u_k : \Gamma \times \Gamma \longrightarrow \mathbb{R}$ satisfying:*

(1) *For all $C > 0$, $u_k \longrightarrow 1$ uniformly on the strip $x, y : d(x, y) < C$.*
(2) *For all k there exists R such that $u_k(s, t) = 0$ if $d(x, y) \geq R$.*

We first introduce a labelling commonly used in buildings of type $\tilde{A}_2$. This labelling is already known and used in [6] and [16]. During our proof we will extend it to $\tilde{A}_n$ buildings.

Given any vertex $x \in \Delta$ we consider the sector based at x defined by ω and denote it $S^x(\omega)$. A vertex $s^x_{i,j}$ is a vertex $\in S^x(\omega)$ such that its distance from the left side of the sector is i and similarly its distance from the right side of the sector is j.

We define $S^x_k(\omega)$ to be the part of the sector $S^x(\omega)$ such that $\max(i + j) = k$. These are all the vertices distance at most k away from the base point x. This is illustrated in the following diagram for $k = 6$.



Let Δ be an affine building of type $\tilde{A}_n$. Let Ω be its building at infinity and fix a chamber $\omega \in \Omega$.

Given any vertex $x \in \Delta$ we consider the sector based at x defined by ω and denote it $S^x(\omega)$. A vertex $s^x_{k_1,k_2,\ldots,k_n}$ is a vertex $\in S^x(\omega)$ such that its distances from the sides of the sector are $k_1, k_2, \ldots, k_n$.

We define $S^x_k(\omega)$ to be the part of the sector $S^x(\omega)$ such that $\max(k_1+k_2+\ldots+k_n) = k$.

We now need to prove the following technical theorem:

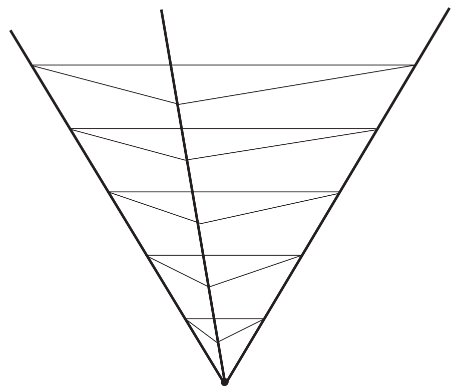
Theorem 5. *Let Δ be an affine building of type $\tilde{A}_n$. Then given any vertex $x \in \Delta$, the number of vertices in $S^x_k(\omega)$ is a polynomial in k of order n.*

Proof. Let $A_n(k) = |S_k(\omega)|$ in an $\tilde{A}_n$ building.

For example, when $n = 1$ (a tree), $A_n(k) = k$.

When $n = 2$ (the case $\tilde{A}_2$ seen previously), $A_n(k) = \sum_{i=1}^{k+1} A_1(k) = \sum_{i=1}^{k+1} i = \frac{1}{2}(k^2 + 3k + 2)$.

More generally since we are considering simplices, then by construction each sector is a stack of sectors of degree one below as illustrated here for $n = 3$:



and

$$A_n(k) = \sum_{i=1}^{k+1} A_{n-1}(i)$$

By [18], $\sum_{i=1}^{k} i^p = c_{p+1}k^{p+1} + \ldots c_1 k$, where $c_1, \ldots, c_{p+1}$ can be found by solving the system of $p+1$ equations $\sum_{i=j+1}^{p+1} (-1)^{i-j+1} \begin{pmatrix} i \\ j \end{pmatrix} c_i = \delta_{j,p}$, where $\delta_{j,p}$ is the kronecker delta. Note that these are independent of k.

The sum of some polynomial of degree n :$\sum_{i=1}^{k}(a_n i^n + a_{n-1}i^{n-1} + \ldots + a_0)$ can be rewritten as $a_n \sum_{i=1}^{k} i^n + a_{n-1} \sum_{i=1}^{k} i^{n-1} + \ldots + \sum_{i=1}^{k} a_0$. This is another polynomial in k of degree $n+1$: $b_{n+1}k^{n+1} + b_n k^n + \ldots + b_0$, with new constants $b_0, \ldots, b_{n+1}$ dependent on $a_0, \ldots, a_n$ and n via the above Schultz equality.

Since $A_n(k)$ is obtained by the repeated $n-1$ summing of polynomials, the first of which is of degree 1, $A_n(k)$ is in fact ultimately a polynomial of order n with coefficients dependent on n and obtained using the Schultz equalities.

So we have $|S_k(\omega)| = b_n k^n + \ldots b_1 k + b_0$. □

Theorem 6. *Given an $\tilde{A}_n$ group, there exists a sequence of positive functions $u_k : \tilde{A}_n \times \tilde{A}_n \longrightarrow \mathbb{R}$ satisfying:*

(1) *For all $C > 0$, $u_k \longrightarrow 1$ uniformly on the strip $x, y : d(x,y) < C$.*
(2) *For all k there exists R such that $u_k(s,t) = 0$ if $d(x,y) \geq R$.*

Proof. Given two vertices $x, y \in \Delta$ and some $k \in \mathbb{N}$ we consider the intersection of the sectors $S_k^x(\omega)$ and $S_k^y(\omega)$. We define

$$u_k(x,y) = \frac{|S_k^x(\omega) \cap S_k^y(\omega)|}{|S_k(\omega)|}$$

where $|S_k(\omega)|$ is the number of vertices in the defined sector at any base point.

$u_k(x, y)$ is a positive function $\forall k$ as required. This could be shown explicitly by calculation by rewriting $|S_k^x(\omega) \cap S_k^y(\omega)|$ as a double sum in terms of x and y and obtaining a sum of squares as seen in [4]. Alternatively note that this intersection is in fact the inner product of two characteristic functions.

1) By definition, the vertex $s_{d(x,y),\ldots,d(x,y)}$ (with n coordinates) relative to the sector based at x and that based at y lies in the intersection of these two sectors. Choose the vertex which maximizes the distance between it and one of those base points and call it z. The distance between z and x or y is at most $(n+1) * d(x, y)$. To simplify notation, we will call this value m. Furthermore, the sector S^z is a common subsector to $S^x(\omega)$ and $S^y(\omega)$ [6].

Fix some $C > 0$. For all pairs of points (x, y) distance less than C apart, $m < (n+1) * C$, which is a fixed number. We thus have the following:

$$|S_{k-m}^z(\omega)| \leq |S_k^x(\omega) \cap S_k^y(\omega)| \leq |S_k(\omega)|$$

By Theorem 5, we have $|S_k(\omega)| = b_n k^n + \ldots + b_1 k + b_0$.

Similarly $|S_{k-m}(\omega)| = b_n(k-m)^n + \ldots + b_1(k-m) + b_0$, where $b_0, \ldots, b_n$ are constants dependent on n, obtained using the Schultz equalities.

$$\frac{|S_{k-m}^z(\omega)|}{|S_k(\omega)|} = \frac{b_n(k-m)^n + \ldots + b_1(k-m) + b_0}{b_n k^n + \ldots + b_1 k + b_0}$$

Hence

$$\lim_{k \longrightarrow \infty} \frac{|S_{k-m}^z(\omega)|}{|S_k(\omega)|} = 1$$

Since we had

$$\frac{|S_{k-m}^z(\omega)|}{|S_k(\omega)|} \leq u_k(x, y) \leq \frac{|S_k(\omega)|}{|S_k(\omega)|}$$

$\lim_{k \longrightarrow \infty} u_k(x, y) = 1$ for all pairs of points (x, y) such that $d(x, y) < C$.

2) Given some fixed $k \in \mathbb{N}$, define R to be $2k$. In that case, when $d(x, y) > R = 2k$, there is no intersection between the sectors $S_k^x(\omega)$ and $S_k^y(\omega)$ and so $u_k(x, y) = 0$. $\square$

By proposition 4.1, we get the following corollary to Theorem 6:

Corollary 4.2. *$\tilde{A}_n$ groups have Property A.*

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