Effects of Non-linear Stiffness on Performance of an Energy Harvesting Device

by

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Vibration-based energy harvesting devices have received much attention over the past few years due to the need to power wireless devices in remote or hostile environments. To date, resonant linear generators have been the most common type of generators used in harvesting energy for such devices. Simple tuning and modelling methods make it a more favourable solution theoretically if not practically. This thesis considers the limitations of resonant linear devices and investigates two non-linear generators to see if they can outperform the linear devices in certain situations. So far, in most of the literature, the energy harvester is assumed to be very small dynamically compared to the source so the source is not affected by the presence of the device. This thesis considers how the dynamics of the source is affected by the device if its impedance is significant compared to that of the source. A tuning condition for maximum power transfer from the source to the device is derived. This tuning condition converges to the one presented in most of the literature when the impedance of the device is assumed to be very small compared to that of the source i.e. tuned so that the natural frequency of the device equals the excitation frequency. For the case when the impedance of the device has a negligible effect on the source, the performance of the device is only limited to a narrow frequency band and drops off rapidly if mistuned. To accommodate the mistuning limitations, new types of generators are proposed mainly by using a non-linear mechanism. These mechanisms are made up of a non-linear spring connected together with a mass and a linear viscous damper i.e. the energy harvesting component. The analysis of the fundamental performance limit of any non-linear device compared to that of a tuned linear device is carried out using the principal of conservation of energy. The analysis reveals that the performance of a non-linear device in terms of the power harvested is at most $\frac{4}{\pi}$ greater than that of a tuned linear system and is strongly dependent upon the type of the non-linearity used. Two types of non-linear mechanisms are studied in this thesis. The first one is a non-linear bi-stable mechanism termed a snap-through mechanism which rapidly moves the mass between two stable states. The aim is to steepen the displacement response curve as a function of time which results in the increase of velocity for a given excitation, thus increasing the amount of power harvested. This study reveals that the performance of the mechanism is better than a linear system when the natural frequency of the system is much higher than the excitation frequency. The study also shows that the power harvested by this mechanism rolls off at a slower rate compared to that of the linear system. Another non-linear mechanism described in this thesis uses a hardening-type spring. The aim of this mechanism is to provide a wider bandwidth over which the power can be harvested. This mechanism demonstrates numerical solutions and approximate analytical solutions for the bandwidth and effective viscous damping of a non-linear device employing a hardening-type stiffness. Unlike the linear system, in which the bandwidth is only dependent on the damping ratio, it is found that the bandwidth of the non-linear device depends on both the strength of the nonlinearity and the damping ratio. Experimental results are presented to validate the theoretical results. This thesis also investigates the benefits of the non-linear device for a low frequency and high amplitude application using the measured vibration inputs from human motion such as walking and running. The effect of harmonics on the power harvested is also studied. Numerical simulations are carried out using measured input vibrations from human motion to study the best placement of the natural frequency of the device across the range of harmonics.
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Nomenclature

Roman Symbols

$A$ Area

$a(t)$ Measured acceleration signal

$A_i = \omega^2 Y$

$a_n, b_n$ Fourier coefficient of the acceleration signals

$a(t)$ Measured acceleration signal

$B$ Magnetic field strength

$C$ Capacitance

$c_{cd}$ Critical damping (device)

$c_{cs}$ Critical damping (source)

$c_d$ Damping coefficient (device)

$C_n = \sqrt{a_n^2 + b_n^2}$

$c_s$ Damping coefficient (source)

$D$ Charge density

$d$ gap

$\Delta T$ Temperature difference

$d_n, e_n$ Fourier coefficient of the velocity signals

$d_p$ Piezoelectric coefficient that transduces a charge in an electric field into electric displacement

$E_e$ Electric field strength

$\hat{E}_p$ Non-dimensional elastic potential energy

$E_p$ Elastic potential energy

$F$ Force

$F_b$ Blocked force

$\hat{F}_c$ Non-dimensional critical static force before snap-through occurs
$F_d$  Force applied to the device
$\hat{F}$  Non-dimensional force
$f(s)$  Non-linear spring function
$F_s$  Force applied to the source
$I$  Second moment of area
$k_1$  Linear stiffness of the hardening mechanism (Nm$^{-1}$)
$k_3$  Non-linear stiffness of the hardening mechanism (Nm$^{-3}$)
$k_d$  Linear stiffness (device)
$k_{eff}$  Effective stiffness of the beam ($3Y_{ym}I/L^3$)
$k_s$  Linear stiffness (source)
$k_{sn}$  Stiffness
$l$  Length
$l_o$  Original length of the spring
$m_d$  mass (device)
$m, n$  The order of harmonic
$m_s$  mass (source)
$N$  Number of turns
$P_d$  Power harvested by the device
$\hat{P}$  Non-dimensional power harvested
$P_{in}$  Power input to the device
$P_n$  Power harvested by a non-linear device
$P_r$  Ratio of the harvested by a non-linear device to the tuned linear device
$Q$  Charge
$R_d$  Real part of the impedance (device)
$R_{int}$  Internal resistance of the device
$R_{load}$  Load resistance
$R_s$  Real part of the impedance (source)
$S$  Amplitude of relative displacement
$s, \dot{s}, \ddot{s}$  Relative displacement, velocity and acceleration
$S_l$  Maximum relative displacement
$T$  Period of oscillation
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<td>$\hat{\alpha}$</td>
<td>Estimate of the non-linearity of the hardening system</td>
</tr>
<tr>
<td>$\hat{\Omega}_{dwn}$</td>
<td>Non-dimensional estimate of jump-down frequency</td>
</tr>
<tr>
<td>$\hat{\omega}_{dwn}$</td>
<td>Estimate jump-down frequency</td>
</tr>
<tr>
<td>$\hat{\Omega}_{up}$</td>
<td>Non-dimensional estimate of jump-up frequency</td>
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<tr>
<td>$\hat{\omega}_{up}$</td>
<td>Estimate jump-up frequency</td>
</tr>
<tr>
<td>$\hat{\zeta}$</td>
<td>Estimate of the damping ratio of the hardening system</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Stiffness ratio $k_s / k_d$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Mass ratio $m_s / m_d$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$\Omega_{dwn} - \hat{\Omega}_{dwn}$</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
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<tr>
<td>--------</td>
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</tr>
<tr>
<td>Ω</td>
<td>Non-dimensional frequency</td>
</tr>
<tr>
<td>ω</td>
<td>Excitation frequency</td>
</tr>
<tr>
<td>ω</td>
<td>Excitation frequency</td>
</tr>
<tr>
<td>Ω_{d1,d2}</td>
<td>Non-dimensional half power point frequencies</td>
</tr>
<tr>
<td>Ω_{d}</td>
<td>Non-dimensional frequency ω/ω_{d} (device)</td>
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<tr>
<td>Ω_{down}</td>
<td>Non-dimensional jump-down frequency</td>
</tr>
<tr>
<td>ω_{down}</td>
<td>Jump-down frequency</td>
</tr>
<tr>
<td>Ω_{hp}</td>
<td>Non-dimensional half-power point frequency of the hardening system</td>
</tr>
<tr>
<td>Ω</td>
<td>Non-dimensional frequency at maximum relative transmissibility of the hardening system</td>
</tr>
<tr>
<td>ω_{n}</td>
<td>Undamped natural frequency $\sqrt{k_1/m}$</td>
</tr>
<tr>
<td>Ω_{s}</td>
<td>Non-dimensional frequency ω/ω_{s} (source)</td>
</tr>
<tr>
<td>ω_{s}</td>
<td>Undamped natural frequency $\omega_s = \sqrt{k_s/m_s}$ (source)</td>
</tr>
<tr>
<td>Ω_{up}</td>
<td>Non-dimensional jump-up frequency</td>
</tr>
<tr>
<td>ω_{up}</td>
<td>Jump-up frequency</td>
</tr>
<tr>
<td>ω_{d}</td>
<td>Undamped natural frequency $\omega_d = \sqrt{k_d/m_d}$ (device)</td>
</tr>
<tr>
<td>φ</td>
<td>Phase angle between the input and the response</td>
</tr>
<tr>
<td>Φ_{B}</td>
<td>Magnetic flux</td>
</tr>
<tr>
<td>σ</td>
<td>Mechanical stress</td>
</tr>
<tr>
<td>τ</td>
<td>$\omega_n t$</td>
</tr>
<tr>
<td>θ</td>
<td>Inclination angle</td>
</tr>
<tr>
<td>ξ</td>
<td>$\Omega_{up} - \hat{\Omega}_{up}$</td>
</tr>
<tr>
<td>ζ_{d}</td>
<td>Damping ratio $c_d/c_{cd}$ (device)</td>
</tr>
<tr>
<td>ζ_{dl}</td>
<td>Optimum damping ratio for a given maximum relative displacement</td>
</tr>
<tr>
<td>ζ_{elect}</td>
<td>Electrical damping</td>
</tr>
<tr>
<td>ζ_{eload}</td>
<td>Electrical load damping</td>
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<tr>
<td>ζ_{eloss}</td>
<td>Electrical damping loss</td>
</tr>
<tr>
<td>ζ_{mech}</td>
<td>Mechanical damping loss</td>
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<tr>
<td>ζ_{s}</td>
<td>Damping ratio $c_s/c_{cs}$ (source)</td>
</tr>
</tbody>
</table>

**Other Symbols**

$(\bullet)'$ $d/d\tau$
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\bullet)^\prime\prime$</td>
<td>$d^2/d\tau^2$</td>
</tr>
<tr>
<td>*</td>
<td>Complex conjugate</td>
</tr>
<tr>
<td>$j$</td>
<td>$\sqrt{-1}$</td>
</tr>
<tr>
<td>$\Re{\ldots}$</td>
<td>Real part</td>
</tr>
<tr>
<td>rms</td>
<td>Root mean square</td>
</tr>
</tbody>
</table>
Declaration of authorship

I, Roszaidi Ramlan, declare that the thesis entitled *Effects of Non-linear Stiffness on Performance of an Energy Harvesting Device* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research.

I confirm that this work was done wholly or mainly while in candidature for a research degree at this University; where I have consulted the published work of others, this is always clearly attributed; I have acknowledged all main sources of help.

Roszaidi Ramlan

September 2009
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By three methods we may learn the wisdom:
First, by reflection, which is noblest;
Second, by imitation, which is easiest;
and third by experience, which is the bitterest.

⋯ CONFUCIUS ⋯
Chapter 1

Introduction

1.1 Energy harvesting from ambient sources

The use of ambient energy sources has been around for many years. Among common examples in ancient times was the windmill which was first designed in the 9th century [1]. The windmill uses ambient energy by capturing the wind energy using rotating blades. Back then the windmill was used to grind grain and pump water. The windmill has advanced since then and is now used to generate electricity. This is commonly known as the wind turbine. There are other common macro power generators which are used to generate electricity. Among these are hydropower and the solar panel. These are often referred as renewable energy sources. In this thesis they are referred to as macro generators which are of large size and have the ability to generate power of the order of megawatts. The primary purpose of the macro generator is to reduce the dependency on fossil fuel to power equipment in generating electricity. The reduction in the dependency on fossil fuel is required because of two main factors. The first one is to reduce global warming and the second is that it is an alternative source which is needed due to the depletion of the natural resources.

In recent years, the concepts behind “renewable energy” have also been applied to “energy harvesting”. While, in principle, both terms have the same meaning, the latter term is primarily referred to a micro power generator like those used in wireless sensors.
and wearable electronics, which are small and the power requirement is of the order of milliwatts. The objective of the micro power generator is quite different from the macro power generator. It is designed mainly as a perpetual power source so that maintenance can be reduced, which provides a clear advantage for applications in remote environments.

Traditionally, a battery is used as the power source. Thus one of the main aims of energy harvesting research is to find an alternative but reliable source to replace or recharge the battery by using an ambient source. There are a lot of ambient sources that can be used as an energy source. In this chapter, the literature review of sources is restricted to the most common sources, namely solar, thermal and vibrational sources. First, however, the advantages and the disadvantages of the battery as a reliable power source are considered.

1.1.1 Batteries

There are numerous types of battery available, the specifications and performance varies with manufacturer and the type of battery. To restrict this study, consider a D alkaline cell produced by Duracell. The alkaline or alkaline-manganese dioxide cells comprise zinc/potassium hydroxide/manganese dioxide cells. This cell typically can provide around 22500 mWhr. Thus if we consider the type of application which requires 1 mW, this means that the cell can last just over 2.5 years. Thus if a particular application falls within this life span, then such a battery may be an appropriate power source.

Another advantage of a battery is that it does not couple with the environment. The decoupling from the environment means that it can provide a mobile power source irrespective of the environment and is able to perform in either dark or cool environments. However, due to limited life span, the battery has to be replaced regularly if the duration of the application is more than its life span. Although this type of battery can perform well at low temperature, its performance at high temperature is poor since its optimum temperature range lies between -20°C to 54°C \cite{2}. The discharge rate increases rapidly with temperature as shown in Figure 1.1. It can also be seen in the figure that the zinc-carbon and the silver oxide batteries fail to perform at 60°C.
1.1.2 Solar source

Solar energy has been used as a large scale energy source for many years. Generally, two types of energy can be extracted from the solar source. The first one is the heat from the sun and the second is the energy from the sunlight. In the case of a micro power generator, energy from sunlight is more practical.

Up to now, using the photovoltaic effect, solar cells have been used to power street lights and traffic signs as well as calculators and wrist watches. In one example, it has been used from the 1980s to power a mobile fridge to transport medicine across African villages. An illustration of this is shown in Figure 1.2 [3]. For obvious reasons, the power requirement varies accordingly with the type of application. The specification also differs from one manufacturer to another. However, for the amount of power for pervasive computing or wearable electronics, a credit card size of solar cell is generally sufficient for a power source [4].

The effectiveness of solar cells relates directly to the amount of sunlight received by the cell. Thus the cell should be installed so that it receives sunlight continuously. This means that the user must avoid shady areas and the cell must not be placed indoors. The performance of the solar cell is good during daylight and reduces significantly at dawn or dusk. This means that in many cases a secondary storage source is required to store the excess energy harvested during the daylight, so that the device can be used when sunlight is not available. The maintenance is only minor, mainly involving clearing the debris on the cell. However, this has to be done regularly, thus restricting the usage of solar cells in the area where debris is high. Apart from these negative points, solar energy has many advantages. It is relatively durable, its cost is low and its size is generally small. The voltage generated by the cell is DC, thus it does not require rectification.

1.1.3 Thermal source

Thermoelectric energy harvesting exploits the Seebeck effect which is that a voltage can be produced if a temperature gradient exists between two different metals or semicondu-
tors [5]. The Seiko Thermic wristwatch shown in Figure 1.3 [6], utilizes the temperature difference between the body and the ambient environment to power a mechanical clock.

The voltage $V_{\text{out}}$ generated by two different materials between which there is a temperature difference $\Delta T$ is given by [7]

$$V_{\text{out}} = \alpha_s \Delta T,$$

where $\alpha_s$ is the Seebeck coefficient. Fundamentally, the performance of this generator is limited by the efficiency of the Carnot cycle which is given by [6]

$$\frac{\Delta T}{T_H} \equiv \frac{T_H - T_L}{T_H},$$

where $T_H$ and $T_L$ are the high and the low temperatures over which the device is operating. In addition, the effectiveness of the thermoelectric device relies heavily on the properties of thermoelectric materials. An ideal thermoelectric material has a high Seebeck coefficient, high electrical conductivity and low thermal conductivity. Low thermal conductivity is crucial to maintain a high temperature difference between the surfaces. Commonly, bismuth telluride is used as the thermoelectric element due to a high operating temperature.

According to Equation 1.2, the temperature gradient across the thermoelectric element will always be less than the maximum. Thus, in order for the device to extract the maximum power possible, a good heat sink is needed. Like photovoltaic materials, a thermoelectric device also produces DC power. However, since the potential across the thermoelectric element depends strongly on the temperature gradient, a DC-DC conversion is needed to provide a stable potential.

1.1.4 Vibration source

Harvesting energy from ambient vibration has become a popular research topic within the past decade. For quite some time, this invaluable source has been ignored and in many cases solutions are sought to reduce vibrations by dissipating energy. Nowadays,
instead of dissipating the vibrational energy by converting it to heat, ways are sought to convert the energy into useful electrical energy.

The first mechanical model for energy harvesting was proposed by Williams and Yates \cite{8}. This mechanical model consists of a single-degree-of-freedom (SDOF) mass-spring-damper system. In this generator, a mass is suspended on an inertial frame by a spring element. Placing this frame on the vibrating source results in a relative displacement between the mass and the frame. By using a suitable transduction mechanism, energy can be harvested. The energy harvested is represented by the energy dissipated in the dashpot, as the conversion of mechanical energy into electrical energy has the same effect as the energy which is extracted from the system by the damper. Three common transduction methods are used to convert vibration energy into electrical energy, namely electromagnetic, piezoelectric and electrostatic. A detailed discussion on energy harvesting from vibration sources can be found in two review papers \cite{9,10}.

1.1.5 Electromagnetic Technology

One of the earliest electromagnetic generators was proposed by Amirtharajah et. al. \cite{11} based on the general model proposed by Williams and Yates \cite{8}. The schematic of the generator is shown in Figure 1.4. The system consists of a mass $m$ attached to a spring of stiffness $k$. The other end of the spring is attached to top of the housing. The coil is fixed to the mass while the magnetic core is attached at the lower end of the housing which gives direct interaction with the vibration source. This type of arrangement has the advantage of producing extra space for the mass to move. As the device is put on a vibrating structure, the case will move up and down. As the housing moves, it generates motion of the mass relative to the housing. A wire coil is attached to the mass and moves through the magnetic field setup by the permanent magnet as the mass vibrates. The motion of the coil cuts through the flux formed by the permanent magnet which then induces voltage in the coil.

The voltage induced in the coil is determined by Faraday’s Law \cite{12}

$$V_{\text{emf}} = \frac{d\Phi_B}{dt},$$

(1.3)
where $V_{\text{emf}}$ is the induced emf and $\Phi_B$ is the magnetic flux. For the case where the coil and the magnetic field are perpendicular to each other, the maximum open circuit voltage $V_{\text{oc}}$ is given by

$$V_{\text{oc}} = NBl \frac{ds}{dt},$$

(1.4)

where $N$ is the number of turns in the coil, $B$ is the magnetic field strength, $l$ is the length of one turn and $s$ is the distance the coil moves through the magnetic field.

Electromagnetic transduction offers a clear advantage in terms of the operating point of view, since it does not require any external source to initiate the process. A device with this transduction mechanism can be designed without having mechanical contact between the parts. This improves reliability and decreases mechanical damping as a result of less contact between the moving parts. However, the voltage that can be produced by this technology is far too low for practical applications at the moment [12].

The integration with the microelectronics is another issue, because with this system, it is not so easy to integrate it with microelectronics. Besides, the amount of power that can be harvested depends on the strength of the magnetic field, thus requiring a stronger magnetic field. From the micro-electronics point of view, the use of a strong magnet may interfere with and affect the performance of the electronics. A detailed review on devices with electromagnetic transduction is given by Arnold [13].

### 1.1.6 Piezoelectric Technology

Piezoelectric materials are materials which become electrically polarized when subjected to mechanical stress (direct piezoelectric effect), or conversely, the materials will physically deform (change in mechanical dimensions of strain) under the presence of an electrical field (indirect piezoelectric effect). There are several ceramic materials which exhibit these characteristics including lead-zirconate-titanate (PZT), lead-titanate (PbTiO$_2$), lead-zirconate (PbZrO$_3$), and barium-titanate (BaTiO$_3$).

The interaction between the mechanical and electrical behaviour of the piezoelectric
material is given by \[14\]

\[
\begin{bmatrix}
\varepsilon \\
D
\end{bmatrix} =
\begin{bmatrix}
1/Y_{ym} & d_p \\
d_p & \epsilon
\end{bmatrix}
\begin{bmatrix}
\sigma \\
E_e
\end{bmatrix},
\] (1.5)

where \(\varepsilon\) is the mechanical strain (dimensionless), \(Y_{ym}\) is the Young’s Modulus of the material, \(\sigma\) is the mechanical stress, \(D\) is the charge density, \(\epsilon\) is the absolute permittivity, \(d_p\) is the piezoelectric coefficient that transduces a change in an elastic field into electric displacement and \(E_e\) is the electric field strength. The first equation represents the direct effect of the piezoelectric material. As can be seen, without the piezoelectric coupling \(d_pE_e\), the equation is just Hooke’s law.

There are two coupling modes that are commonly used to convert mechanical energy into electrical energy using the direct piezoelectric effect. The first coupling mode is called the 31 mode and the second coupling mode is called the 33 mode. In the 31 mode, the force applied to the piezoelectric material is perpendicular to the poling direction whereas in the 33 mode, the force acts in the same direction as the poling direction. It was found that the 33 mode has a better coupling coefficient than the 31 mode \[15\]. Based on their study, Baker et. al. also concluded that that the 31 mode is suitable for applications with a low acceleration (or force) level and the 33 mode is suitable in the application which involve large accelerations (or force) levels typically involving large machinery. Table 1.1 shows the values for the electro-mechanical coupling coefficient \(k\), piezoelectric strain constant \(d\) and piezoelectric voltage constant \(g\) of common piezoelectric materials.

In most of the energy harvesting devices employing piezoelectric material, the 31 mode is the most popular choice due to the low environmental frequency. The 31 and 33 modes are illustrated in Figure 1.5.

A schematic diagram of the generator is shown in Figure 1.6. As the generator is put on a vibrating structure, the mass will vibrate, thus causing mechanical strain of the piezoelectric shim which produces an electrical voltage that can be harvested with the use of an electrical circuit. A simple electrical equivalent circuit of the piezoelectric generator is shown in Figure 1.7. If the mass vibrates sinusoidally due to the vibration source, an AC voltage is developed within the material and the power harvested from
the vibrating structure can simply be calculated by connecting a resistive load to the generator. The power dissipated in the resistive load, $R_{\text{load}}$ is simply given by

$$P_{\text{load}} = \frac{1}{2} \frac{V_{\text{load}}^2}{R_{\text{load}}},$$

(1.6)

where $V_{\text{load}}$ is the voltage across the resistive load.

Piezoelectric transduction provides a fairly high voltage around 3 - 10 V. Similar to electromagnetic transduction, this method does not require an external energy source to work. Depending on the coupling mode, the mechanism can result in very small mechanical damping because no mechanical limit stops are needed. However, the operation of the piezoelectric material is limited to certain temperature, voltage and applied stress levels.

Every piezoelectric material has its own Curie temperature. This temperature depends on the chemical composition of the material. Above this point, the material loses its piezoelectric properties and the material can be fully de-polarized. Further increase in temperature will accelerate the ageing process which decreases the performance, thus lowering the maximum allowable stress that the material can withstand. Reviews of devices using piezoelectric materials are given in [16, 17].

### 1.1.7 Electrostatic Technology

Electrostatic transduction is capacitive in nature, similar to the piezoelectric transduction. A common capacitor consists of two conductor plates separated by a vacuum, air or electrically insulated material. The capacitor needs to be charged to deposit equal but opposite charge on the plates. For a given charge $Q$ and voltage $V$, the capacitance is given by

$$C = \frac{Q}{V}.$$  

(1.7)

In general, the capacitance is also proportional to the overlapping area between the two plates and inversely proportional to the gap that separates the two conductor plates. The coefficient of proportionality between the capacitance, the area and the gap relates
directly to the type of material (or medium) between the two plates. The constant of proportionality is obtained by taking the ratio between the electric flux density in vacuum and the electric field strength [18]. This constant is termed the permittivity of free space, $\varepsilon_0$. Thus the capacitance of a capacitor can also be written as

$$C = \frac{\varepsilon_0 A}{d}. \quad (1.8)$$

In the case when the space between the plates is filled with another dielectric material, the absolute permittivity, $\varepsilon_a$ is used instead of the permittivity of free space and is given by [18]

$$\varepsilon_a = \varepsilon_0 \varepsilon_r, \quad (1.9)$$

where $\varepsilon_r$ is the relative permittivity of the dielectric. Thus, a capacitance of a capacitor with a dielectric material between the plates is given by

$$C = \frac{\varepsilon_a A}{d}. \quad (1.10)$$

The voltage across a parallel plate capacitor can be determined by combining Equations 1.7 and 1.10 to give

$$V = \frac{Qd}{\varepsilon_a A}. \quad (1.11)$$

The energy stored in the capacitor is given by

$$E = \frac{1}{2} CV^2 = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} QV. \quad (1.12)$$

For the case when the voltage across the two plates is constant, the attraction force between the two plates is given by [9]

$$F = \frac{1}{2} \varepsilon_a A \left(\frac{V}{d}\right)^2, \quad (1.13)$$

and in the case when the charge on the plates is constant, the attraction force is given by [9]

$$F = \frac{1}{2} \frac{Q^2 d}{\varepsilon_a A}. \quad (1.14)$$
The energy harvested for electrostatic transduction can be obtained by the work done against these attractive forces.

The electrostatic generator can operate in three different modes as illustrated in Figures 1.8(a) - 1.8(c). These three modes are called in-plane overlap, in-plane gap closing and out-of-plane gap closing, respectively. In the in-plane overlap case, the change in capacitance results from the change in the overlapping area between the fingers while in the in-plane gap closing case, the capacitance changes as the gap between the fingers varies. The out-of-plane gap closing system works using the same principle as the in-plane gap closing case, only now the capacitance changes with the change in gap between the two plates.

Electrostatic transduction offers the most potential for integration with microelectronics. It is reported that around 2 - 10 V can be easily generated using this method [14]. Unlike piezoelectric technology, electrostatic technology does not require smart materials to operate. However, electrostatic transduction requires an initial external source to charge the plates. In addition, mechanical limits must be included in the device to prevent the two plates coming into contact with each other so as to prevent short circuit. This can cause a reliability problem and the addition of a mechanical limit can increase the mechanical damping of the system.

Tables 1.2 - 1.4 show a summary of the current performance of the device using electromagnetic, piezoelectric and electrostatic materials, respectively. The summary presents comprehensive information on the device ranging from environmental input (frequency and amplitude) to the device dimension (size, weight a proof mass) as well as the power output. These three tables show that electromagnetic devices are the popular choices among the energy harvesting researchers compared to piezoelectric and electrostatic devices. This summary also shows that many devices operate at a frequency greater than 50 Hz (on average), and quite few of them have an excitation frequency which is less than 10 Hz. This opens a new area of energy harvesting research to focus on low frequency applications.
1.2 Performance limit of the resonant generator

Compared to solar and thermoelectric power harvesting, vibration harvesting provides a clear advantage in the absence of illumination of sunlight and high temperatures. However, the optimum performance of a linear generator requires close coupling between the device and the environment in which for resonant generators, the undamped natural frequency of the device must be tuned to match the dominant environmental frequency. A slight mistune can result in a large reduction in the power harvested. The amount of power harvested for a resonant generator is also proportional to the cube of the excitation frequency, which limits the use of resonant generators for low frequency applications such as wearable electronics, portable devices and implantable medical devices. In these applications, the excitation frequencies are from human motion and are generally less than 10 Hz. The limit of the performance of a resonant generator is discussed in detail in Chapter 2. In this chapter, several methods proposed by researchers to overcome the performance limit of resonant generators mentioned before are briefly reviewed. However, before investigating the solutions provided by previous research, the power consumption of some wearable electronics, portable devices and implantable medical devices, is first reviewed.

1.3 Power requirement for wearable, portable and implantable medical devices

The evolution of technology for wearable electronics has been rapid. Advancements in the circuitry technology has reduced the power requirement to operate the devices. However the improvement in battery energy densities compared to the advances in computer technology has been poor [6]. This has led many researchers to seek an alternative energy source.

The advances in electrical and electronics in wearable electronics has also shrunk the physical size of the device. This means that the resonant generator may not be appropriate for such use, since, as the size of the devices shrink, so does the maximum allowable
displacement that the proof mass can move. However, the reduction in the size means a reduction in the power requirement to operate the device. According to Moore’s law, for a scale reduction by a factor $\varrho$ ($\varrho > 1$), the power consumption is approximately reduced by a factor of $1/\varrho^3$ \cite{19}.

Among the most popular applications for wearable electronics are portable devices such as the mobile phone, MP3 players and implantable medical devices. In this section, a review of the power consumption by several wearable electronic devices is conducted to set a target for energy harvesting devices. For extensive reviews of the power requirement for mobile devices readers are referred to \cite{5, 19, 20}.

Implanted medical devices such as a cochlear implant, artificial heart and a pacemaker are first discussed. Currently, batteries are used as the energy source. Although a battery can provide good performance, periodic replacement makes it a less favoured choice. In addition, the replacement of a battery will cost money and most importantly they require a surgical operation. In general, implantable medical devices can be categorized into two types \cite{5}. The first one is a passive device which includes artificial joints and valves. Generally these devices do not require an external power source to operate. The second category is active devices which includes the implantable pacemaker and the neural stimulator. Active devices require an external power source to operate because they involve constant measurement and actuation. The amount of power required depends very much upon the function of the device. Table 1.5 summarizes the voltage and power consumption of three common implantable medical devices. Among the three devices, implantable pacemakers require the least power while the insulin pump consumes the most power.

Other fashionable wearable electronics devices are wireless communication devices. Wearable wireless communication devices utilizing bluetooth and GSM technology have evolved rapidly within the last decade. This rapid evolution has decreased the required power consumption by the devices. Table 1.6 shows the power consumption of the two communication devices. As can be seen the power consumption for these devices reduced considerably from 1998 to 2005. Most importantly, efficient electronics have reduced the gap between the power requirement for maximum and normal usage in 2005.
The power consumption of several portable devices based on the different functions of the device is shown in Table 1.7. Apart from laptops, handheld devices and high-end MP3 players, the power requirement for other portable device is only of the order of milliwatts. For an immediate goal, one can just consider the idle power consumption of the device rather than other power-hungry functions. In this case, if an alternative power source can provide enough power during the standby mode, this will help to reduce the charging frequency of the device to a certain extent.

Table 1.8 shows power consumption of several portable radios with the same functionality [21]. Based on this information, it can be seen that the portable radio does not require high power to operate, just of the order of a few milliwatts. In the table, the first two radios, BayGen Freeplay and Dynamo & Solar, are powered by alternative sources while the last two are powered by batteries. The BayGen radio was manufactured in South Africa to be used in a remote area where it is difficult to obtain batteries. This radio is charged by winding a torque spring, which can be wound up to a maximum of 60 revolutions. When fully charged, the radio can operate for around 30 minutes. On the other hand, the Dynamo & Solar is more versatile since it can be powered using several methods. This radio can be powered using batteries or by a built-in nickel cadmium (NiCd) battery with 280 mAh capacity. This battery can be charged by a solar panel or by winding the hand powered dynamo.

1.4 Solutions to the low frequency limitation

As can be seen in the previous section, some of the portable devices require a small amount of power to operate. This is positive news for the resonant vibration generator. However, the dependency of the device on the cube of the excitation frequency sets the limits for resonant generator. In this section, a brief review of the methods proposed by researchers to overcome this limitation is presented. Many of the methods proposed involve the conversion of low frequency excitation to a higher frequency.

The device discussed by Renaud et. al. [22] has a moving mass which impacts a piezoelectric cantilever beam at either end of the frame which contains the device. The energy
is harvested due to the impact force as the frame vibrates. However, actual testing of the device was not conducted. It was reported that, using real dimensions and mechanical properties of the piezoelectric cantilever, the device is capable of generating around 40 μW of useful electrical power when placed on the wrist of a walking person.

Kulah and Najafi [23] proposed a method to convert a low excitation frequency to a higher frequency. The system contained two resonating structures. The first resonating structure is a plate suspended on a soft spring representing a low resonance frequency structure. The second resonating structure consists of an array of cantilever beams. These cantilever beams have much higher resonance frequencies. An illustration of the mechanism is shown in Figure 1.9. As the device is excited, the magnet on the low frequency resonator moves and at some point, it catches the magnets on the high frequency resonators. Further motion of the low frequency resonator magnet will bend the cantilever beams. At some point, the high frequency resonator magnets are released from the low frequency resonator thus causing the cantilever beams to vibrate at a much higher frequency. It was reported that this mechanism can convert ambient frequencies in range of 1-100 Hz into 1-20 kHz frequencies.

Chung et. al. [24] employed a magnetic force to convert a low excitation frequency to a higher frequency by using a repulsive and attractive force between magnets. The system consists of a piezoelectric cantilever beam with an NdFeB magnet attached to its end. A glider consists of an array of magnets arranged in such a way that the poles of each magnet alternately changes between south and north poles as shown in Figure 1.10. As the device experiences a mechanical vibration, the glider oscillates horizontally. As the glider oscillates, the cantilever beam experiences a repulsive and attractive force, alternately. The alternate change between the repulsive and attractive force causes the beam to oscillate vertically. A higher frequency can be achieved by having a larger number of magnets in the glider. It was reported that the increase in frequency is proportional to the number of magnets on the glider. The prototype was tested and provided an output voltage between 8 V to 12 V. The frequency was also increased by a factor of 2, from 10 Hz to 20 Hz.

Another frequency-up conversion mechanism was proposed by Lee et. al. [25] employing
piezoelectric technology. A schematic concept of the mechanism is shown in Figure 1.11. This device is made of a sharp probe, micro ridges, a micro slider and a piezoelectric cantilever. The principle of operation was not clearly described by the authors. However, based on Figure 1.11, it was thought that as the slider oscillates, the piezoelectric cantilever bends due to contact between the probe and the side of the ridges. Above a certain displacement, the cantilever is released and starts to oscillate with its higher natural frequency thus harvesting more power. Based on this study, they concluded that the generated voltage depends on the depth of the ridges while the up-conversion of the frequency is a function of the spacing of the ridges. In terms of the capability of the device, they discovered that the optimized system could generate up to 4.2 $\mu$W/cm$^2$/Hz. They also claimed that the device can generate 225 $\mu$W/cm$^2$ using 7 rectifications at 60 Hz excitation frequency and 2.25 mW/cm$^2$ with 70 rectifications.

The reviews so far have described methods to convert a low forcing frequency to a much higher frequency based on a high natural frequency of the device. These methods have improved the performance of the generator to some extent. However, it is thought these methods are more useful when dealing with MEMS applications. Some other research groups have proposed non-resonant methods which are much more suitable for applications related to human motion. First, Saha et. al. [26] used a magnetic spring instead of a normal mechanical spring. It was reported that the magnetic spring is more advantageous than a normal planar spring (e.g. cantilever) in terms the ease of construction, tunability and resistance to fatigue. The prototype generated between 0.3 - 2.46 mW of power when placed in a rucksack which was worn during walking and slow running. Another non-resonant generator was proposed by Bower and Arnold [27]. Their device contained a magnetic ball which was placed in a spherical cage. They implemented two designs of coil winding. In the first design, the coil was wrapped around the equator of the spherical cage. The second design consists of two-series connected coils which were offset on the northern and the southern hemispheres. The device was tested under walking and running conditions during which it was placed in the hand and in a pocket. Based on the prototype, it was reported that this device produced a voltage output of 80 - 700 mV with power densities ranging from 0.03 - 0.6 mW/cm$^3$. 
1.5 Solutions to the problem of mistune

Another limitation for a resonant generator is that it requires a close coupling with the environment which means that the device has to be tuned so that its undamped natural frequency matches the dominant environmental excitation frequency. A slight mistune will cause a large reduction in output. This is true when the damping in the system is small. Thus, to keep the device at its optimal performance, the undamped natural frequency of the device must be constantly tuned to match a particular environmental frequency.

For the resonant generator, the natural frequency can be altered by changing the stiffness or the mass, although in many cases, changing the stiffness is more feasible. There are two main ways to do this. The first is by tuning the device passively. Passive tuning requires power at the beginning of the process to match the natural frequency to the forcing frequency and once this matching is achieved, the power can be turned off. Active tuning, on the other hand, requires continuous power to provide the frequency matching. Based on this, passive tuning is mainly suitable when the ambient frequency does not vary much over time while the active tuning method suits applications where the excitation frequency varies with time. In this section, a brief review on passive and active tuning mechanisms is presented.

Leland and Wright [28] proposed a design of a tunable-resonance vibration energy scavenger by axially compressing a piezoelectric bimorph to lower its resonance frequency as shown in Figure 1.12. Their work showed that the resonance frequency can be lowered by up to 24% below its unloaded resonance frequency. They also concluded that the power output to a resistive load can be between 65% - 90% of the nominal value at frequencies 19% - 24% below its unloaded resonance frequency. On a particular prototype with a 7.1 g proof mass, their device was able to vary the resonance frequency from 200 Hz to 250 Hz while producing power output between 300 $\mu$W and 400 $\mu$ W.

Challa et. al. [29] used a magnetic force to alter the resonance frequency of the system. An illustration of the mechanism is shown in Figure 1.13. The system consists of a piezoelectric cantilever beam with a proof mass placed on the tip. They put a magnet
on either side of the beam as well as at a distance away from the beam. The magnets were placed in such a way that they will produce repulsive and attractive forces as shown. However, only one of the modes (repulsive or attractive) can be used at a particular time. When operating in attractive mode, the resulting stiffness is lower than that of the original cantilever stiffness, thus producing a lower resonance frequency. In the repulsive mode, the stiffness is larger thus resulting in a higher resonance frequency. The prototype reported by them has a nominal resonance frequency of 26.2 Hz which was due to the cantilever alone. When applying the tuning method, they were able to alter the resonance frequency between 22 Hz to 32 Hz while harvesting power in the range of 240 $\mu$W to 280 $\mu$W.

Morris et. al. [30] used the same concept of altering the resonance frequency but in a slightly different way. The first two methods described before use the bending mode to extract the energy, in such a way that the cantilever is not stretched in the longitudinal direction. Instead of using the bending mode, they used pre-tensioned piezoelectric sheet. This device is called the extensional mode resonator (XMR). This resonator is formed by suspending a proof mass with two piezoelectric sheets. The tuning is done by using a adjustable link which pre-tensions the two piezoelectric elements symmetrically. Using this method, it was reported that this prototype was able to vary the resonance frequency from 80 Hz to 235 Hz. The proposed mechanism of the device is shown in Figure 1.14.

Roundy and Zhang [31] proposed an active tuning method to alter the resonance frequency of the device. A PZT generator was developed and the active tuning electrode was created in which the stiffness of the device can be altered by providing varying voltage to the electrode. This device was able to alter its resonance frequency. However, the power required during the tuning process was larger than the power that can be produced by the device.

Zhu et. al. [32] proposed a closed-loop frequency tuning mechanism for an electromagnetic generator. The basic concept of this type of mechanism is to alter the effective stiffness of the cantilever using a compressive and tensile load. The variation of these loads was achieved by providing a repulsive and attractive force provided by the mag-
The distance between the magnets was altered by the linear actuator which was controlled by a microcontroller. It was reported that they managed to vary the resonance frequency between 67.6 Hz to 98 Hz while producing power of 61.6 $\mu$W to 156.6 $\mu$W, respectively, when exciting the device at 0.588 ms$^{-2}$. However, this was a proof-of-concept experiment. This may not be the case in reality since in this work, an external power source was used to power the microcontroller and the linear actuator.

Apart from the SDOF frequency tuning mechanisms, some researchers have studied multiple-degree-of-freedom (MDOF) systems to capture a wider range of excitation frequency. Sari et. al. [33] proposed a method to widen the operating frequency band of the device by having several cantilever beams connected in series with different lengths resulting in various resonance frequencies (Figure 1.16). The device was implemented using electromagnetic transduction. It was reported that this device was capable of harvesting 0.4 $\mu$W of continuous power with 10 mV voltage across an 800 HZ frequency band, from 4.2 kHz to 5 kHz. Some other researchers [34, 35] utilized the same concept but employed piezoelectric technology. A general discussion on the device which consists of several cantilever beams is given in [36] without referring to any specific transduction technology.

1.6 Use of non-linear mechanisms

With the problems of the resonant generators discussed before, researchers are finding new ways to overcome the limitations. Among these are the use of a non-linear mechanism. Mitcheson et. al. [37] studied several architectures for an energy harvesting device. One of the architectures, a coulomb-force parametric generator (CFPG), used a non-linear mechanism. This model is idealized by having a mass connected to a coulomb damper which acts like a holding mechanism. Using the mechanism, the motion of the mass is controlled by the holding force. In general the holding force can not be greater than the maximum applied force, thus the ratio between the holding force and the maximum applied force from the input motion will always be less than one.
Recently, the use of a non-linear mechanism consisting of a non-linear spring in the form of hardening spring has been proposed in [38, 39]. Both papers describe the benefits of using a hardening spring instead of a linear spring in terms of the bandwidth over which the device can perform well. The first paper provides a discussion mainly based on simulation results while the second paper provides both theoretical and experimental results of the proposed device.

1.7 Other ways of harvesting energy from human motion

So far, the literature review has concentrated on devices that are excited by an oscillating displacement source. In this section, another source of energy, from human motions, is investigated. Starner [40] provides an extensive review on the power that can be harvested from the human body ranging from body heat, exhalation, breathing, finger motion, blood pressure, arm motion and footfall. The summary of the maximum amount of energy that can be harvested by each category mentioned before is shown in Figure 1.17. Compared to others, the power that can potentially be harvested from arm motion and footfall seems promising. As a result of this, Shenck and Paradiso [41] developed a generator using piezoelectric technology utilizing the energy from footfall. They developed two methods to capture the energy. The first was by bending the ball of the foot using a multilaminar polyvinylidene fluoride (PVDF) bimorph stave which was placed under the insole of a normal Nike athletic sneaker. When tested at a walking frequency of 0.9 Hz, the method was able to harvest 1.3 mW across a 250 kΩ load. The second method was designed so that the energy could be harvested by flattening curved, pre-stressed metal spring strips laminated with a piezoelectric lead zirconate (PZT) which was placed under the heel of a US navy work boot. When tested under a similar excitation as before, this method produced a much higher power of 8.3 mW across a 500 kΩ load. The implementation of these two methods is shown in Figure 1.18.

Another group of researchers from SRI international led by Ron Pelrine developed a heelstrike generator employing electrostatic technology. The materials used in this device are called electroactive polymers (EAP) or dielectric elastomers. EAP is very flexible
and can provide more strain than piezoelectric material. The excellent strain properties of this material allows more energy to be stored during compression. It was reported that a displacement of 2-6 mm can result in 50-100 percent area strain [6]. Due to its softness and flexibility, this material can easily be used as the shoe heel. This implementation of the device is shown in Figure 1.19. This type of device performed very well since it can produce 0.8 J/step energy output when the heel was compressed by only 3 mm, producing 800 mW of power output when the subject was walking at a speed of 2 steps/s [42]. However, like other electrostatic generators, EAP requires a voltage to initially charge the electrode.

Recently, a bio-mechanical energy harvester was developed by utilizing knee motion [43]. This device consists of mechanical power transmission, electrical power generation and control as shown in Figure 1.20. A gear train was used to convert low velocity and high torque at the knee into a high velocity and low torque. The chassis of the device was mounted on an orthopaedic knee brace which was designed so that the knee motion drives the gear train through a one way clutch. From the energy harvesting point of view, the device can operate in two modes, namely continuous generation mode and the generative braking mode. In the continuous mode, the generator provides some resistance to the acceleration of the knee while assisting the deceleration. However, in the generative braking mode, the harvester is programmed to engage during the end of the swing phase only thus providing no resistance during the acceleration of the knee. Like the continuous-generation mode, this generative-braking mode also assists the deceleration of the knee. The power generated by the continuous-generation mode was higher than the generative braking mode but requires extra metabolic cost. They tested male subjects walking at 1.5 m/s on a treadmill. From this test, the average electrical power generated by the generative braking mode was 4.8 W and 7.0 W for the continuous generation mode. However, the generative braking mode only requires 0.7 cost of harvesting (COH) while the continuous-generation mode requires 2.3 COH.

1 metabolic power in watts required to generate 1 W of electrical power
1.8 Motivation for the thesis

This thesis concerns various aspects of energy harvesting from ambient vibration. It centres on the benefits of using a non-linear mechanism in terms of stiffness non-linearity which is aimed to improve the performance of the device. Apart from that, it also considers the loading of the device on the vibrating source and the performance of the linear and non-linear devices when dealing with an excitation which is composed of a number of harmonics.

The performance of a device in this case refers to the maximum power that can be generated by the device and the frequency range over which the device can perform reasonably well, i.e. the bandwidth. Both linear and non-linear mechanisms are investigated and compared. The aim is to investigate the amount of power that can be generated by the non-linear mechanism compared to the linear mechanism. A mechanism consisting of a non-linear spring is proposed so as to achieve the theoretical maximum power that can be harvested by a non-linear device. In terms of the bandwidth of the device, a different but readily available type of spring called a hardening spring is used. Unlike other documents which discuss the same type of spring element, this thesis provides extra knowledge by deriving an analytical expression for the bandwidth. This thesis also provides a way to estimate the bandwidth using analytical expressions with a novel derivation of the viscous damping expression in the Duffing oscillator based on measurable jump-up and jump-down frequencies.

In terms of the loading effect of the device on the vibrating source, this thesis provides a study to show the importance of recognizing that the impedance of the device which needs to be considered rather than the physical size of the device. In most of the literature, a tonal frequency vibration source was used. This thesis takes this a further step by investigating the effect of harmonics on the performance of both the linear and non-linear devices. The material presented in this thesis centres on the mechanical part of the device rather than the electrical part of the device.
1.9 Objectives of the thesis

The objectives of the thesis are to:

1. Study the effect of an energy harvesting device on a vibrating source.

2. Investigate the condition for maximum power transfer from a vibrating source to an energy harvesting device.

3. Investigate the benefits of having a certain type of nonlinear stiffness mechanism in an energy harvesting device.

4. Investigate the effect of harmonics of excitation on the power harvested.

1.10 Overview of the thesis

This thesis consists of 7 chapters which cover both linear and non-linear devices as well as giving theoretical and experimental results.

In Chapter 1 an overview of ambient energy sources stating the limit of performance of each source is given. An overview of the resonant generator is also described with three main transduction technologies, namely electromagnetic, piezoelectric and electrostatic, being discussed. This chapter also briefly discusses the limitations of the linear generator and solutions that have been provided to overcome those limitations.

Chapter 2 presents an analytical study of the device with a linear mechanism. The study starts by considering a 2-DOF mass-spring-damper system representing the device which is attached to a vibrating source. This study is conducted to investigate the loading effect of the device on the source. In this case, the source is assumed to operate within 3 different regions i.e. stiffness, damping and mass controlled regions. A tuning criterion for all three regions is derived in the chapter and the effect of tuning the device to resonate at the particular tuned frequency is also investigated. This chapter also shows that the tuning conditions derived converge to the one in most of the literature for the case where the impedance of the device is much smaller than that of the source so that
the source is not affected by the presence of the device. In this case, the source can be regarded as the velocity source. The limitations of the performance of the linear device are also discussed using the analytical expressions for the power derived. A potential benefit of having a non-linear spring instead of linear spring to improve the device performance is studied by comparing the maximum power that can be harvested by the device with both types of spring using the principle of conservation of energy.

Chapter 3 discusses the behaviour of an energy harvesting device which has a non-linear spring. This non-linear spring is designed to achieve the performance limit laid out in previous chapter. This type of mechanism is called a snap-through or a bi-stable mechanism. The main objective of the mechanism is to speed up the motion of the mass so that the response approximates a square wave. The chapter starts by investigating the static properties of such a spring which involves the force and stiffness characteristics as well as the potential energy wells. The dynamic characteristics of the device are investigated by using a bifurcation diagram, the phase portrait and the steady-state time history of the displacement response. The power harvested by this device is compared with its equivalent linear mechanism for various degrees of non-linearity, input level and damping ratios.

Another type of device with a different non-linear spring is described in Chapter 4. This type of spring is commonly called a hardening spring. The intention of having this type of spring is to widen the bandwidth over which the device can harvest a reasonable amount of power. This chapter provides an analytical expression for the bandwidth with this type of spring. This chapter also presents an alternative way to estimate the linear viscous damping in such systems by using the distinctive properties of the device with this non-linear spring, i.e. the jump-up and the jump-down frequencies for time harmonic excitation. The analytical expression for the damping ratio is derived and presented in this chapter. With the knowledge of the jump-up frequency, the damping ratio and the jump-down frequency, the bandwidth of the hardening mechanism can be estimated.

The results from the experimental investigation are presented in Chapter 5. These experimental results cover both the hardening and the snap-through mechanisms although
more discussion is presented for the hardening mechanism. The experimental results for
the hardening mechanism mainly show the effect of the non-linearity, input level and
the damping ratio on the responses, i.e. the acceleration, displacement and voltage.
The analytical expression for the damping ratio derived in the previous chapter is also
validated. This chapter also presents the comparison in terms of the bandwidth, power
harvested and the optimal resistance, between the hardening mechanism and its equiva-
 lent linear mechanism. The results for the snap-through mechanism are fundamentally
aimed at the behaviour of the system with frequency by using different non-linearity,
input level and damping ratio.

In the previous 4 chapters, the study is mainly conducted using a tonal excitation
frequency. In Chapter 6, the performance of the device is studied using an excitation
which is composed of several harmonics. The input vibration is the measured vibration
input on the human waist while the subject was walking or running on the treadmill.
In this case the source is regarded as the velocity source. The main aim of this chapter
is to investigate the effect of higher harmonics on the device. For the linear system,
the power harvested by the device is calculated using three methods, two of which use
the time integration method and one uses the frequency domain method. However,
due to invalidity of the superposition assumption in the non-linear system, the power
harvested by the hardening and the snap-through mechanism are computed using the
time integration method.

The conclusions from the work presented in this thesis are presented in Chapter 7.
Suggestions for future improvement and study on the device with a non-linear spring
are also included.

1.11 Contributions of the thesis

In the course of the work, this thesis provides some original contributions to the field of
energy harvesting and non-linear dynamics as follows:

1. It is shown that the maximum power transfer from a source to the device cannot be
achieved simply by tuning the impedance of the device to be equal to the complex conjugate of the source impedance due to several constraints on the mass and the size of the device. This is because the real and imaginary parts of the impedance of the device are mutually interdependent.

2. A tuning condition is introduced for maximum power transfer from a vibrating source to the energy harvesting device without assuming that the impedance of the device is very small compared to that of the source. The tuning condition converges to the tuning condition described in most of the literature, if the impedance of the device is much smaller than that of the source, i.e. tuning so that the natural frequency of the device equals the excitation frequency.

3. It is shown that the maximum power harvested by any device with stiffness non-linearity is $\frac{4}{\pi}$ times that of the optimally tuned linear system. This ratio can be achieved if an approximation to a square wave response for displacement is produced, for a sinusoidal input.

4. A model of a new mechanism for an energy harvesting device is proposed containing two oblique springs, a mass and a damper. The two oblique springs steepen the displacement response to approximate a square wave. However, it is found that the mechanism only outperforms the linear system when the natural frequency of the device is much higher than the excitation frequency.

5. Approximate expressions are derived for the bandwidth of the hardening system. It is found that the bandwidth of the hardening mechanism depends on the damping ratio, the input excitation and the degree of nonlinearity.

6. Simple analytical expressions are derived for the viscous damping in a non-linear system which has hardening spring characteristics. This derivation leads to the possibility of determining the bandwidth of the hardening mechanism by just using knowledge of the jump-up and the jump-down frequencies.

7. It is experimentally verified that the jump-up frequency in a hardening Duffing oscillator is dependent on the degree of the non-linearity and not highly dependent
on the damping while the jump-down frequency is dependent on both the damping and the degree of non-linearity.

8. When a linear device is excited with an input which is composed of a number of harmonics, a comparable amount of power can be harvested by tuning the undamped natural frequency of the device to match the harmonic for which \( \omega A_i \) is maximum, where \( \omega \) is the frequency at the tuned harmonic and \( A_i \) is the amplitude of the acceleration of the tuned harmonic. However, for a given maximum displacement constraint, much less damping is needed so that the oscillation of the proof mass of the device fully utilizes the maximum allowable relative displacement.

9. When a device with a hardening mechanism is excited with an input with several harmonics, if the device is driven strongly enough so that there is a significant hardening characteristic in the response, the harmonics higher than the low amplitude natural frequency (i.e. the natural frequency when the mass oscillates within the linear regime of the system) may contribute substantially to the power harvested. For the snap-through mechanism, the low amplitude natural frequency should be made as high as possible but not too high such that the snap-through does not occurs. If snap-through occurs, the harmonics below the low amplitude natural frequency may contribute substantially to the power harvested.

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Table 1.1: Coefficients of several piezoelectric materials (source: [9]).
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Table 1.2: A summary of current performance of electromagnetic devices [10].
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Table 1.3: A summary of current performance of piezoelectric devices [10].
### Table 1.4: A summary of current performance of electrostatic devices [10].

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<th>Author</th>
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<th>Generator Volume [cm³]</th>
<th>Proof Mass [g]</th>
<th>Input Amp. [µm]</th>
<th>Input Freq [Hz]</th>
<th>S [µm]</th>
<th>Power (unprocessed) [µW]</th>
<th>Power (processed) [µW/cm³]</th>
<th>Density [µW]</th>
<th>Effec- tiveness [%]</th>
<th>Figure of Merit [%]</th>
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Table 1.5: Power consumption of several implantable medical devices (source: [5]).

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<td>12.5 mW</td>
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<td>8 mW</td>
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Table 1.6: Power consumption of instant communication methods, GSM and Bluetooth, between 1998 and 2005 (source: [19]).

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<th>Browse</th>
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<th>Messaging</th>
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<td>Reply</td>
<td>Speaker</td>
<td>Headset</td>
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Table 1.7: Power consumption of several portable device undertaking several tasks (source: [20]).

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<th>Weight of the energy system [g]</th>
<th>Stored amount of electrical energy [J]</th>
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<td>33</td>
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Table 1.8: Power consumption, weight and stored energy for several portable radios (source: [21]).
Figure 1.1: Effect of temperature on capacity retention for various DURACELL zinc anode systems [2].

Figure 1.2: Mobile fridge using solar technology which uses photovoltaic cells mounted on the camel developed by NAPS Systems (picture taken from [3]).
Figure 1.3: Seiko thermic wrist watch using the temperature difference between the body and the ambient (a) the wristwatch, (b) a cross-sectional diagram, (c) thermoelectric modules and (d) an array of thermopile (picture taken from [6]).

Figure 1.4: Schematic of the electromagnetic generator (source: [11]).

Figure 1.5: Coupling mode of the piezoelectric energy harvester (a) 33 mode and (b) 31 mode (source: [14]).
Figure 1.6: Schematic of the piezoelectric generator (source: [12]).

Figure 1.7: Electrical circuit representation of the piezoelectric generator where $V_{oc}$ is the open circuit voltage when the electrical displacement is zero, $C$ is the capacitance and $R_{int}$ is the internal resistance of the piezoelectric material (source: [12]).
Figure 1.8: Three modes of operations of the electrostatic generator (a) in-plane overlapped area varying, (b) in-plane gap closing and (c) out-of-plane gap closing (source: [14]).
Figure 1.9: Frequency up conversion design proposed in [23] (a) schematic of the cross-sectional area and (b) microscale implementation.

Figure 1.10: Proposed design in [24] (a) overall mechanism of the device and (b) arrangement of the magnets.
Figure 1.11: Frequency up conversion mechanism proposed in [25].

Figure 1.12: Schematic of the frequency tuning mechanism proposed by Leland and Wright [28].

Figure 1.13: Schematic of the frequency tuning mechanism proposed by Challa et. al. [29].
Figure 1.14: Schematic of the frequency tuning mechanism proposed by Morris et. al. [30].

Figure 1.15: Schematic of the frequency tuning mechanism proposed by Zhu et. al. [32].

Figure 1.16: Schematic of the method proposed by Sari et. al. [33] to capture wider excitation frequency.
Figure 1.17: The power that can be harvested from human body and motion. The power available is given in parentheses (source: [40]).

Figure 1.18: Foot strike generator utilizing piezoelectric material (a) PZT dimorph heel insert, (b) PVDF insole stave, (c) the insole of a running shoe employing 31-mode and (d) Shoes energy scavenger with mounted electronics (source: [6]).
Figure 1.19: Foot strike electrostatic generator based on compression of a charged electroactive polymer (source: [6]).

Figure 1.20: Bio-mechanical energy harvesting utilizing knee motion proposed by Donelan et. al. [43].
Chapter 2

Linear inertial vibration generator

2.1 Introduction

As illustrated in the previous chapter, a single-degree-of-freedom (SDOF) mass-spring-damper mechanical model of the energy harvesting device was first proposed by Williams and Yates [8], and since then researchers [11, 37, 82, 84] have been using their model mostly to study electromagnetic, piezoelectric and electrostatic generators. However, one main assumption that was made in the model by Williams and Yates was that the host structure will not be affected by the motion of the device, i.e. the device was small dynamically compared to the host structure. Using this assumption, it is commonly concluded that the maximum power harvested is obtained by tuning the undamped natural frequency of the device to coincide with the excitation frequency of the vibration source.

This chapter presents a discussion on how the dynamics of the source are affected by the device if its impedance is significant compared to that of the source. A 2-DOF mass-spring-damper system shown in Figure 2.1 is used to represent the case when the energy harvesting device is placed on the vibrating source. The amount of energy harvested is represented by the energy dissipated by the dashpot, $c_d$, because the conversion of
mechanical energy into electrical energy is assumed to have the same effect as mechanical
damping. This model is necessary to study the effect of the device on the source. The
important issue to consider is the impedance of the device which can be very large
especially when the device resonates even though it has a relatively small mass.

A tuning condition for maximum power transfer from the source to the device is pre-
sented later in the chapter. This tuning condition converges to the one generally pre-
sented in the literature when the impedance of the device is assumed to be very small
compared to that of the source, i.e. tuning so that the undamped natural frequency of
the device equals the excitation frequency. The performance of the resonant linear gen-
erator is also discussed, including the maximum power harvested as well as the frequency
bandwidth over which the device can still perform reasonably well. This chapter also
presents a discussion on the limit of the increase in performance of an energy harvesting
device with a non-linear spring compared to that of a tuned linear device, using the
principle of conservation of energy.

2.2 Mechanical impedance

In this section, a brief discussion of the concept of mechanical impedance is given,
focussing particularly on the mass-spring-damper model of the source and the device.
The interconnection of the two models is represented later using the Thévenin equivalent
system. Details on the mechanical impedance concept is given by Hixson in [85].

The mechanical impedance of an element is defined as the ratio of the amplitude of the
driving force acting on that element to the resulting velocity of that element for the time
dependent harmonic excitation and in the steady state. On the other hand, the mobility
of a mechanical element is defined as the ratio of the amplitude of the velocity to that
of the driving force, i.e. mobility is the reciprocal of the impedance. The mechanical
impedance and mobility of a spring, damper and mass are given in Table 2.1

From Table 2.1 it can be seen that the mechanical impedance of a damper is real while
the mechanical impedances of a spring and mass are imaginary and are in anti-phase
with each other. Simplification of the complex impedance diagram of connected elements
can be performed depending upon the type of connection between elements, i.e. whether the elements are in series or parallel. Adopting the concept from electrical engineering but in an opposite manner, the series and parallel connected elements can be combined as follows;

1. Element in series

\[
\frac{1}{Z_{total}} = \sum_{i=1}^{N} \frac{1}{Z_i}, \tag{2.1}
\]

2. Element in parallel

\[
Z_{total} = \sum_{i=1}^{N} Z_i. \tag{2.2}
\]

where \( Z_i \) represents the mechanical impedance of an element.

### 2.2.1 Mechanical impedance of the source

Figure 2.2 shows the mass-spring-damper model of the source as well as its impedance diagram. The mass \( m_s \) is connected to a spring \( k_s \) and a damper \( c_s \) while the other ends of spring and damper are connected to a rigid support. The force \( F_s \) is applied to the mass. The three elements are in parallel due to fact that all elements experience the same velocity. Hence, the total impedance of the source is simply the addition of the mechanical impedance of each element.

The total impedance of the source is given by

\[
Z_s = Z_{ks} + Z_{cs} + Z_{ms}. \tag{2.3}
\]

Substituting the impedance of each element into Equation 2.3 gives

\[
Z_s = c_s + j(\omega m_s - \frac{k_s}{\omega}), \tag{2.4}
\]

or in dimensionless form

\[
\frac{Z_s}{c_{cs}} = \frac{2\zeta_s\Omega_s + j(\Omega_s^2 - 1)}{2\Omega_s}, \tag{2.5}
\]

where \( \omega \) is the excitation frequency, \( \omega_s = \sqrt{k_s/m_s} \) is the source undamped natural
frequency, $\Omega_s = \omega/\omega_s$, $c_{cs} = 2m_s\omega_s$ is the critical damping and $\zeta_s = c_s/c_{cs}$ is the damping ratio of the source.

The impedance of the source is large both at low and high frequencies while it is small when the excitation frequency $\omega$ equals its undamped natural frequency $\omega_s$. At low frequencies, the motion is controlled by the spring while at high frequencies it is controlled by the mass. Around the resonance region, it is dominated by the damping. Therefore, increasing the damping ratio will only affect the motion around the resonance i.e. increasing the damping ratio will increase the impedance. Figure 2.3 illustrates the non-dimensional impedance of the source as a function of the non-dimensional frequency plotted for a value of $\zeta_s = 0.01$.

2.2.2 Mechanical impedance of the energy harvesting device

Figure 2.4 shows the energy harvesting device as a mass-spring-damper system and its corresponding impedance diagram. The connections between elements are the same as for the source described earlier. However, the transmitted force from the source acts on the base rather than on the mass making the impedance of the device differ from the source. In this case, the spring and the damper have the same velocity while the mass experiences a different velocity. The impedance diagram of the device shows that the mass is in series with the parallel combination of the spring and damper. The total impedance of the system is given by

$$Z_d = \frac{1}{Z_{md}} + \frac{1}{Z_{kd} + Z_{cd}}.$$ (2.6)

Substituting the impedance of each element into Equation 2.6 and rearranging gives

$$Z_d = \frac{\omega m_d \left[ \omega m_d c_d + j \left( \frac{c_d^2}{\omega} - \frac{k_d}{\omega} \right) \left( \omega m_d - \frac{k_d}{\omega} \right) \right]}{c_d^2 + \left( \omega m_d - \frac{k_d}{\omega} \right)^2}.$$ (2.7)
This can be represented in a dimensionless form by normalizing with respect to the critical damping \( c_{cd} \) as follows

\[
Z_d = \frac{2\zeta_d\Omega_d^2 + j\Omega_d \left[ (2\zeta_d\Omega_d)^2 - \Omega_d^2 + 1 \right]}{2 \left( (2\zeta_d\Omega_d)^2 + (\Omega_d^2 - 1)^2 \right)},
\]  

(2.8)

where \( \omega_d = \sqrt{k_d/m_d} \) is the device undamped natural frequency, \( \Omega_d = \frac{\omega}{\omega_d} \) and \( \zeta_d = \frac{c}{c_{cd}} \) is the damping ratio of the device.

When \( \omega = \omega_d \) and if \( \zeta_d \ll 1 \) the impedance of the device is maximum and is given by

\[
Z_{dres} = \frac{\omega_d m_d}{2\zeta_d}.
\]

(2.9)

The mobility of the device \( Y_d \) is given by

\[
Y_d = \frac{c_d + \frac{1}{j\omega m_d} \left[ c_d^2 - \frac{k_d}{\omega} \left( \omega m_d - \frac{k_d}{\omega} \right) \right]}{c_d^2 + \left( \frac{k_d}{\omega} \right)^2}.
\]

(2.10)

The impedance of the system is very small at low frequencies while it is constant at high frequencies. However, the impedance is maximum at its resonance i.e. when \( \omega = \omega_d \) for \( \zeta_d \ll 1 \). The motion is controlled by the mass at low frequencies and by the damper at high frequencies. At resonance, it is controlled by the combination of all the elements that make up the system as given by Equation (2.9). Based on Equation (2.9) the maximum impedance of the load i) increases with frequency, ii) increases with mass, iii) decreases with damping ratio, and iv) is real, thus behaving like a damper. The impedance is plotted with respect to the frequency ratio \( \Omega_d \) in Figure 2.5 for \( \zeta_d = 0.01 \) to illustrate the characteristics of the impedance as frequency changes.

### 2.2.3 Thévenin Equivalent System

The vibration source can be characterised by a blocked force \( F_b \) in parallel with the source impedance \( Z_s \) and this can be connected to load impedance \( Z_d \) as shown in Figure 2.6. This type of source representation is called a Thévenin equivalent system. The blocked
force $F_b$ is the force generated by the source when it is connected to a rigid load and is, in principle, a measurable quantity. The Thévenin equivalent system will be used later in determining the amount of power transmitted to the load.

The time averaged power transmitted to the load is proportional to the real part of the product of the amplitudes of the transmitted force and the velocity. Therefore, it is desirable to have a large in-phase component of force and velocity in order for a large amount of power to be transmitted to the load. However, this is not possible and the reason why is examined by studying the behaviour of the force and velocity using the Thévenin equivalent system shown in Figure 2.6. For simplicity, the source impedance $Z_s$ is assumed constant and the load impedance $Z_d$ is varied.

The velocity $V_d$ is given by

$$V_d = \frac{F_b}{Z_s + Z_d}. \tag{2.11}$$

The force applied to the source is given by

$$F_s = \frac{Z_s}{Z_s + Z_d} F_b = \frac{1}{1 + \frac{Z_d}{Z_s}} F_b, \tag{2.12}$$

and the force transmitted to the load is given by

$$F_d = \frac{Z_d}{Z_s + Z_d} F_b = \frac{1}{1 + \frac{Z_s}{Z_d}} F_b. \tag{2.13}$$

The power that is supplied to the device is simply given by

$$P_d = \frac{1}{2} \Re\{F_d V_d^*\}, \tag{2.14}$$

where $\Re\{\ldots\}$ denotes the real part of a complex quantity and $\{\ast\}$ represents the complex conjugate. This can also be written as

$$P_d = \frac{1}{2} |F_d|^2 \Re\{Y_d\}. \tag{2.15}$$

According to Equation 2.11 to obtain a maximum velocity $V_d$, the load impedance $Z_d$ has to be as small as possible such that $Z_d \to 0$. On the other hand, Equation 2.13
reveals that if $Z_d \to 0$, the force applied to the load $F_d$ will be zero and all the force is applied to the source $F_s$, thus resulting in zero power being transmitted to the load. The optimum value of the force and the velocity must be determined to obtain maximum power.

### 2.3 Conditions for maximum power transfer for different types of sources

As mentioned earlier, the models considered by previous researchers assumed that the device is small compared to that of the source, so that for a given amount of damping and harmonic excitation, the power is maximum when the undamped natural frequency of the device coincides with the excitation frequency. In this study, the impedance of the device is assumed to have some effect on the source. In order to consider the size of the impedance of the source, the source is assumed to operate in 3 different frequency regions namely low frequency, high frequency and around resonance in which its impedance is dominated by the stiffness, mass and the damping, respectively as discussed earlier in Section 2.2.1.

Consider the general form of mechanical impedances of the device and the source respectively given by

$$Z_d = R_d + jX_d, \quad (2.16)$$

$$Z_s = R_s + jX_s, \quad (2.17)$$

where $R$ represents the real part and $X$ represents the imaginary part of the impedance. The subscripts $d$ and $s$ denote the device and source respectively.

Noting that the mobility is the inverse of the impedance, substituting the expression for $Z_d$ and $Z_s$ into Equation 2.15, the expression for the power transmitted to the load becomes

$$P_d = \frac{1}{2} F_b^2 \frac{R_d}{(R_d + R_s)^2 + (X_d + X_s)^2}. \quad (2.18)$$

As can be seen from Equation 2.18, the expression for power transmitted to the device
consists of the dynamics of both the source and the device. According to Equation 2.18, the maximum power will be transferred from a vibrating source to the device when the device impedance is the complex conjugate of the source impedance, i.e.

\[ R_d + jX_d = R_s - jX_s. \]  

(2.19)

This can be verified by partial differentiation of the expression with respect to the terms \( R_d \) and \( X_d \) separately, and solving the resulting expressions simultaneously. Equation 2.19 ensures the force and the velocity are in-phase thus resulting in maximum power transfer to the device.

The tuning condition derived in Equation 2.19 is easily achieved if the real part and the imaginary part of the device impedance are independent. This may not be easily achievable when the real part and the imaginary part are dependent, as in the case for the energy harvesting device here. This is because, in practice, there may be further design constraints such as the mass, damping and maximum relative displacement, thus it may not be possible to achieve the condition in Equation 2.19 due the real and imaginary parts of the device impedance being dependent on each other.

### 2.3.1 Stiffness controlled source

Below the source frequency resonance, the impedance of the source is controlled by the stiffness and is approximately given by

\[ Z_{sk} = -j \frac{k_s}{\omega}. \]  

(2.20)

Normalizing \( Z_{sk} \) with respect to the critical damping of the device, \( c_{cd} \) gives

\[ \hat{Z}_{sk} = -j \frac{1}{2} \frac{\kappa}{\Omega_d}, \]  

(2.21)

where \( \hat{Z}_{sk} = Z_{sk}/c_{cd} \) and \( \kappa = k_s/k_d \) is the ratio between the stiffness of the source and the stiffness of the device. Since the impedance of the source is purely imaginary, Equa-
tion 2.18 becomes

\[ P_d = \frac{1}{2} F_b^2 \frac{R_d}{(R_d)^2 + (X_d + X_s)^2} \]  

(2.22)

Figure 2.7 shows the maximum power harvested by the device with different stiffness ratios across a range of the damping ratio. It can be seen that more power is harvested when the device is put on the flexible source and the power harvested decreases as the source becomes stiffer. This is true since, for a given blocked force, a larger force is transmitted to the device and increases the velocity of the device as well. Figure 2.8 shows the effect of the damping in the device on the power harvested by the device while the stiffness ratio, \( \kappa \) is kept constant at 10. This figure reveals that the power harvested by the device decreases with the damping in the device. Notice that for this particular value of \( \kappa \), the maximum power harvested does not occur at \( \Omega_d = 1 \) but occurs at \( \Omega_d \) slightly less than 1. The frequency at which the maximum power harvested occurs will be investigated in Section 2.3.3.

### 2.3.2 Mass controlled source

Above the resonance frequency of the source, the source impedance is controlled by the mass and is approximately given by

\[ Z_{sm} = j \omega m_s. \]  

(2.23)

Normalizing with respect to the critical damping of the device, \( c_{cd} \) gives

\[ \hat{Z}_{sm} = j \frac{1}{2} \mu \Omega_d, \]  

(2.24)

where \( \hat{Z}_{sm} = \frac{Z_{sm}}{c_{cd}} \) and \( \mu = \frac{m_s}{m_d} \) is the ratio between the mass of the source and the mass of the device. Similar to the stiffness controlled source as before, Equation 2.18 becomes

\[ P_d = \frac{1}{2} F_b^2 \frac{R_d}{(R_d)^2 + (X_d + X_s)^2}. \]  

(2.25)

The only difference in the equation is that the imaginary part of the source is now positive. The effect of the mass ratio, \( \mu \) on the maximum power harvested is plotted
across a range of the damping ratio is shown in Figure 2.9. Referring to this figure, more power is harvested when the device is put on a light source and the power harvested decreases as the source becomes heavier. Figure 2.10 shows the power harvested by the device for various damping ratios while the mass ratio is kept constant at 10. This figure shows that the power harvested decreases with the damping in the device. Similar to the stiffness controlled source, for this particular mass ratio, the maximum power harvested does not occur at $\Omega_d = 1$ but occurs at $\Omega_d$ slightly more than 1. The frequency when the power is maximum will be investigated in the next section.

2.3.3 Tuning condition for a purely reactive source

For a given damping and stiffness ratio (or mass ratio), the frequency when the power harvested is maximum can be determined by differentiating the expression for power harvested with respect to the frequency as

$$\frac{dP_d}{d\Omega_d} = 0.$$  \hfill (2.26)

Another way of finding the frequency when the power is maximum, is by examining the frequency when the relative quantities i.e. displacement, velocity and acceleration, are maximum. The frequencies when the relative quantities are maximum are then compared to that when the power harvested is maximum.

Figures 2.11 and 2.12 show the plots of frequencies for maximum power, relative displacement, relative velocity and maximum relative acceleration as a function of the damping ratio for the stiffness and mass controlled sources, respectively. From these figures, it can be seen that the frequency when the relative displacement is maximum coincides with the one when power is maximum especially for systems with a small damping. Thus instead of differentiating a more complicated power expression with respect to the frequency ratio, one can also differentiate the expression for the relative displacement with respect to the frequency ratio in order to obtain the expression for the frequency ratio when the power is maximum.

The non-dimensional relative displacement for a stiffness and a mass controlled source
are respectively given by

\[ \left| \frac{X_o - X_i}{\frac{f_k}{\kappa_d}} \right|^2 = \frac{\Omega_d^4}{(\kappa (1 - \Omega_d^2) - \Omega_d^2)^2 + 4\zeta_d^2 (\kappa - \Omega_d^2)^2}, \quad (2.27) \]

\[ \left| \frac{X_o - X_i}{\frac{f_m}{\kappa_d}} \right|^2 = \frac{\Omega_d^4}{\left[ \Omega_d^2 (\kappa \mu - 1) \right]^2 + 4\zeta_d^2 \Omega_d^2 (\kappa \mu - 1)^2}, \quad (2.28) \]

where \(X_o\) and \(X_i\) represent the amplitude of the absolute displacement of the mass of the device and the source, respectively. The subscript \(k\) denotes the stiffness controlled source and the subscript \(m\) denotes the mass controlled source.

Differentiating Equation 2.27 with respect to \(\Omega_d\) and equating the resulting expression to zero gives

\[ \Omega_d = \left( \frac{\kappa (1 + 4\zeta_d^2)}{\kappa + (1 + 4\zeta_d^2)} \right)^{\frac{1}{2}}. \quad (2.29) \]

If \(\zeta_d \ll 1\) Equation 2.29 reduces to

\[ \Omega_d \approx \left( \frac{\kappa}{1 + \kappa} \right)^{\frac{1}{2}}. \quad (2.30) \]

Equation 2.30 shows that for a stiff source, the power is maximum when \(\Omega_d = 1\) while for a flexible source, the power is maximum when \(\Omega_d < 1\), depending on the stiffness ratio as shown in Figure 2.13.

For a mass controlled source, differentiating Equation 2.28 with respect to \(\Omega_d\) and equating the resulting expression to zero gives

\[ \Omega_d = \left( \frac{1 + \mu}{\mu} \right)^{\frac{1}{2}}. \quad (2.31) \]

Note that this is independent of damping. Thus for a mass controlled source, it follows that if the source is heavy, the maximum power can be achieved by setting \(\Omega_d = 1\) and if the source is light, the maximum power occurs when \(\Omega_d > 1\), depending on the mass ratio as shown in Figure 2.14.

Figures 2.15 and 2.16 show the ratio between the power harvested by the device when
the frequency is tuned at $\Omega_d = 1$ and the one which is tuned using the criteria derived in Equations 2.30 and 2.31. According to these figures, tuning the natural frequency of the device to coincide with the excitation frequency without carefully considering the relative size of the impedance between the source and the device results in a huge trade-off in the power harvested especially when dealing with a very flexible source (or very light) with a small amount of damping. However, the trade-off reduces as the impedance of the source increases and the damping in the device gets larger.

### 2.3.4 Purely resistive source

At the resonance frequency of the source, the impedance of the source is controlled by damping while the spring and the mass have negligible effect. Thus, the impedance of the source at resonance is written as

$$Z_{sr} = R_s = c_s. \quad (2.32)$$

Since the source is now purely resistive, the expression for power in Equation 2.18 becomes

$$P_d = \frac{1}{2} F_b^2 \frac{R_d}{(R_d + R_s)^2} \frac{X_d^2}{R_d}.$$

(2.33)

From Equation 2.33, the power is maximum when $X_d = 0$ thus making the device purely resistive and this can be achieved by tuning the device to resonate. By setting $X_d = 0$, Equation 2.33 becomes

$$P_d = \frac{1}{2} F_b^2 \frac{R_d}{(R_d + R_s)^2}.$$

(2.34)

Differentiating $P_d$ with respect to $R_d$ and equating to zero yields

$$R_d = R_s.$$

(2.35)

Equation 2.35 gives the condition for maximum power transfer from a vibrating source to the device when the source is purely resistive. $R_d$ is the mechanical impedance of the
load at resonance given by Equation 2.9 and is repeated here for convenience

\[ Z_{d_{res}} = R_d = \frac{\omega_d m_d}{2\zeta_d}. \]  

Equation 2.36

Equating Equations 2.32 and 2.36 with some rearrangements yields

\[ \zeta_s = \left(\frac{\mu}{4}\right) \frac{1}{\zeta_d}. \]  

Equation 2.37

According to Equation 2.37, if the source has a very small damping ratio \( \zeta_s \), the maximum power transfer to the device can be achieved provided that the device has a small mass \( m_d \) and a relatively high damping ratio \( \zeta_d \).

### 2.4 Modelling of an inertial generator with infinitely large source

Referring to the Norton equivalent system in Figure 2.17, the velocity at the base of the device \( V_d \) can be obtained using

\[ V_d = \frac{Z_s}{Z_d + Z_s} V_f, \]  

Equation 2.38

where \( V_f \) is the free velocity i.e. velocity at the interface point when disconnected. If \( |Z_d| \ll |Z_s| \), then

\[ V_d \approx V_f. \]  

Equation 2.39

Thus when the impedance of the source is infinitely larger than the impedance of the device, the velocity of the source \( V_f \) is unaffected by changes in the device. The source can be approximated by a constant velocity known as velocity excitation. The power harvested by the device is given by

\[ P_d = \frac{1}{2} |V_f|^2 \Re \{Z_d\}, \]  

Equation 2.40
where $V_f = \omega Y$ is the amplitude of the free velocity at the base and $Y$ is the amplitude of the base displacement. Substituting the real part of the device impedance, $\Re\{Z_d\}$ from Equation 2.6 into Equation 2.40 yields

$$P_d = \frac{m_d Y^2 \omega^3 \zeta_d \Omega_d^3}{(2\zeta_d \Omega_d)^2 + (\Omega_d^2 - 1)^2}. \quad (2.41)$$

### 2.4.1 Existing model of the device

In most of the literature in which the device is assumed to have no effect on the source, the energy harvesting device is modelled using a single-degree-of-freedom (SDOF) mass-spring-damper shown in Figure 2.18, which has the same dynamics as the model shown in Figure 2.4. The equation of motion of such system is given by

$$m_d \ddot{s} + c_d \dot{s} + k_d s = -m_d \ddot{y}, \quad (2.42)$$

where $s = x - y$ is the relative displacement between the mass and the housing. Assuming that $y = Y \cos(\omega t)$, the amplitude of the relative displacement in the frequency domain is given by

$$S = \frac{\Omega_d^2 Y}{(1 - \Omega_d^2) + j (2\zeta_d \Omega_d)}. \quad (2.43)$$

Multiplying both sides of Equation 2.42 by the relative velocity $\dot{s}$ gives

$$m_d \ddot{s} \dot{s} + c_d \dot{s} \dot{s} + k_d s \dot{s} = m_d \omega^2 Y (\cos \omega t) \dot{s}. \quad (2.44)$$

Integrating each term in Equation 2.44 over one period of the excitation and assuming that the power harvested in one cycle is the same as the power dissipated by the damper results in

$$P_d = P_{in}, \quad (2.45)$$

where $P_d$ is the harvested power given by

$$P_d = \frac{\omega}{2\pi} \int_0^{2\pi} c_d \dot{s}^2 \, dt, \quad (2.46)$$
and $P_m$ is the input power given by

$$P_m = \frac{\omega^2}{2\pi} \int_0^{2\pi} m_d\omega^2 Y (\cos \omega t) \dot{s} dt. \tag{2.47}$$

Equation 2.46 can be solved to yield

$$P_d = \frac{c_d\omega^2 |S|^2}{2}. \tag{2.48}$$

Substituting $c_d = 2m_d\omega_d\zeta_d$ and Equation 2.43 for $S$ into Equation 2.48 yields the same expression as in Equation 2.41. This shows that the energy harvesting device can be modelled as a single-degree-of-freedom mass-spring-damper system excited by a velocity source provided that the impedance of the source is much larger than the impedance of the device such that the presence of the device will not affect the motion of the source.

### 2.4.2 Maximum power harvested by a linear device

There are two key performance factors in energy harvesting. The first is the maximum power that the device can harvest. The second key performance factor is the bandwidth over which the device can be useful so that a reasonable amount of power can be harvested for a wider range of excitation frequency. Referring to Equation 2.41, the maximum power is obtained at resonance i.e. when the undamped natural frequency of the device matches with the excitation frequency, and is given by

$$P_d = \frac{m_d Y^2 \omega_d^3}{4\zeta_d}. \tag{2.49}$$

Equation 2.49 can also be expressed in terms of the excitation acceleration levels, $A_i$ instead of the excitation displacement levels, $Y$ as

$$P_d = \frac{m_d A_i^2}{4\omega_d\zeta_d}. \tag{2.50}$$

where $A_i = \omega^2 Y$.

According to Equations 2.49 and 2.50 it is quite inviting to conclude that the power
harvested $P_d$ is infinite by having zero damping in the system. However, this conclusion is wrong as described by [37, 83, 86]. When $\zeta_d = 0$, the motion of the mass is unconstrained and becomes infinite. In terms of the dynamics of the device, the impedance also becomes infinite as described by Equation 2.36. In addition, this also means that the system will never reach the steady state and most importantly, the system is purely conservative and energy cannot be harvested due the absence of the dissipative damping element.

The damping in the system must be adjusted to accommodate the physical constraints of the device so that the amplitude of the oscillation is finite and within the size limit of the device. Assuming that the maximum relative displacement of an energy harvesting device is $S_l$, the minimum damping ratio, $\zeta_{dl}$, can be computed from Equation 2.43 to yield

$$\zeta_{dl} = \frac{Y}{2S_l}. \quad (2.51)$$

Substituting Equation 2.51 into Equation 2.49 gives

$$P_d = \frac{1}{2} m_d \omega_d^3 Y S_l. \quad (2.52)$$

Equation 2.52 shows that the maximum harvested power depends on the size of the mass, the amplitude of the input displacement, the amplitude of the relative displacement between the seismic mass and the housing and the cube of the excitation frequency. Based also on the equation, the performance of such a device can be very poor for low frequency applications due to the dependency of the power harvested on the cube of the excitation frequency.

To improve this, Elliott et. al. [86] suggested that for a vibration source which is composed of several harmonics, more power can be harvested by tuning the device to coincide with a higher harmonic even though the level of acceleration, $\omega_d^3 Y$, is slightly lower at higher harmonics, due to the linear dependency on the $\omega_d$. In fact, more power can be harvested by tuning the natural frequency of the device to match with a harmonic where $\omega_d^3 Y$ is maximum. A more detailed discussion on the performance of the linear device due to input with a number of harmonics is presented in Chapter 6.
In the case where the level of excitation acceleration, $A_i$ is large and constant, Equation [2.49] suggests that power can still be harvested at low frequency. However, the amount of damping in the system needs to be altered so that the motion of the mass satisfies with the size constraint of the device as well as to maintain the linear behaviour of the device.

### 2.4.3 Bandwidth of a linear system

For the light damping ($\zeta_d^2 \ll 1$), the maximum relative transmissibility, $T_R$ for the linear device occurs at $\Omega_d = 1$. Setting this into Equation [2.43] yields

$$T_R \equiv \left| \frac{S}{Y} \right| = \frac{1}{2\zeta_d}. \quad (2.53)$$

The half-power point amplitude of the relative transmissibility is given by

$$T_{R1,2} = \frac{1}{2\sqrt{2}\zeta_d}. \quad (2.54)$$

Substituting this expression in Equation [2.43] with some re-arranging gives

$$\Omega_d^4 + (4\zeta_d^2 - 2) \Omega_d^2 + 1 - 8\zeta_d^2 = 0. \quad (2.55)$$

Solving for $\Omega_d$ results in half-power frequencies

$$\Omega_{d1,d2} = \sqrt{(1 - 2\zeta_d^2) \pm 2\sqrt{\zeta_d^2 (1 + \zeta_d^2)}}. \quad (2.56)$$

If $\zeta_d^2 \ll 1$ Equation [2.56] can be simplified to become

$$\Omega_{d1,d2} \approx (1 \pm 2\zeta_d)^{1/2}. \quad (2.57)$$

Applying a binomial expansion to Equation [2.57] gives

$$\Omega_{d1} \approx 1 - \zeta_d, \quad (2.58a)$$

$$\Omega_{d2} \approx 1 + \zeta_d. \quad (2.58b)$$
The bandwidth for the linear system $\Delta \Omega_d$ is given by [87]

$$\Delta \Omega_d \approx \Omega_{d2} - \Omega_{d1} \approx 2\zeta_d.$$  \hfill (2.59)

The graphical representation of the maximum relative transmissibility, the half-power point and the bandwidth is shown in Figure 2.19. As investigated in the previous section, the linear device performs very well when the device is optimally tuned so that its undamped natural frequency equals the excitation frequency. However, performance of the linear device is limited to a narrow frequency band only and drops off rapidly if mistuned. The device can be made useful for a wider frequency range by increasing the damping in the system as indicated by Equation 2.59. However, increasing the damping will trade off the amount of power harvested at resonance as shown in Figure 2.20, assuming that the power harvested is proportional to the maximum relative displacement.

To overcome these limitations, a new type of generator needs to be investigated. Some research groups have proposed ways to improve the performance of a linear device. Among these are using a mechanism to amplify the frequency [22–25], to provide a wider bandwidth [33–36] and to provide a tuning mechanism so that the device is more robust over a wider frequency range [28–32]. In addition, studies have also been conducted on a non-resonant type of generator especially when dealing with low frequency applications [26, 27]. Details on these methods have been discussed in Chapter 1. In this thesis, the performance of an energy harvesting device in which a linear spring is replaced with a non-linear spring is investigated. However, before deciding which type of non-linear spring to use, it is prudent to investigate the limit of the performance of a non-linear device comprising a non-linear spring compared to that of a linear device.

### 2.5 Available power from a non-linear device

In this section, the limit of the performance of a device consisting of a non-linear spring compared to one with a linear device is studied using the principle of conservation of
energy. Consider a general non-linear system with the equation of motion given by

\[ m \ddot{s} + c \dot{s} + f(s) = -m \ddot{y}, \quad (2.60) \]

where \( f(s) \) is a conservative non-linear spring force. If a similar analysis to that in Section 2.4 for the linear device is conducted and each term in Equation (2.60) is integrated over a cycle, the result is the same as that in Equation (2.45). Note that the integrands involving the mass and the stiffness vanish since the net change in the kinetic and potential energies of the system over one period of motion are zero. This occurs for any non-linear mechanism due to the conservative properties of the mass and spring elements.

For a device containing a non-linear spring, \( \dot{s} \) is not harmonic. However, an upper bound to the integral can be found for the power harvested by the non-linear device by noting that

\[ P_n \leq \frac{\omega}{2\pi} \int_0^{2\pi} m_d \omega^2 Y |\cos(\omega t)| ||\dot{s}| dt, \quad (2.61) \]

where the subscript \( n \) denotes the non-linear system. Since \( |\cos(\omega t)| \leq 1 \), the inequality in Equation (2.61) can be further simplified to

\[ P_n \leq \frac{m_d \omega^3 Y}{2\pi} \int_0^{2\pi} |\dot{s}| \, dt. \quad (2.62) \]

Because \( \dot{s} \) is a periodic function, the integral in Equation (2.62) can be rewritten as

\[ \int_0^{2\pi} |\dot{s}| \, dt = 4 \int_0^{\pi} \dot{s} \, dt, \quad (2.63) \]

which can also be written in terms of the relative displacement \( s \) rather than \( t \) as

\[ 4 \int_0^{\pi} \dot{s} \, dt = 4 \int_0^{S_i} ds = 4S_i. \quad (2.64) \]

Combining Equations (2.62), (2.63) and (2.64) gives the upper bound for the power harvested by a device containing a non-linear spring

\[ P_n \leq \frac{2m_d \omega^3 Y S_i}{\pi}. \quad (2.65) \]
For the non-linear case here, $\dot{s}$ can also be expanded as a sum of the time harmonics since the motion is assumed to be periodic. However, note that only the first harmonic of the response contributes to the harvested power since the harmonics are orthogonal to each other.

The ratio of the power harvested by a non-linear device to that by a tuned linear device can be found by dividing Equation 2.65 by Equation 2.52 to give

$$P_r \leq \frac{4}{\pi}.$$  \hspace{1cm} (2.66)

The actual power harvested by a non-linear device depends on the form of the non-linearity $f(s)$. Equation 2.66 shows that, if optimally tuned, the maximum amount of power harvested by a non-linear mechanism, is $4/\pi$ times larger than the power harvested by the tuned linear system. In the most favourable non-linear system, the displacement response approximates a square wave, so that $\dot{s}$ is non-zero only when $|\cos \omega t| \approx 1$.

However, it should be emphasized that the ratio of the power harvested by a device with a non-linear mechanism to that of a linear system may be greater than $4/\pi$ when the linear system is not tuned and hence the non-linear system may be able to cope better with mistune than a linear system.

2.6 Conclusions

A 2-DOF mass-spring-damper system has been used in this chapter to model an energy harvesting device which is placed on a vibration source. This model was later represented using the mechanical impedance approach. A Thévenin equivalent system was used to study the effect of the device impedance on the force and power transmitted to the device.

This study revealed that it may not be possible to obtain maximum power transfer to the device by matching the impedance of the device to the complex conjugate of the impedance of the source. This method is easily applicable if the real and imaginary parts of the impedance of the device are independent. For the mass-spring-damper energy
harvesting model used here, this method may not be easily applied due to the constraints of the device such as the mass, damping and the maximum relative displacement, because a change in the real part will cause a change in the imaginary part as well.

The loading of the device on the source was also investigated in this chapter. This study was conducted by considering that the source operates in three frequency regions namely the damping, stiffness and the mass controlled regions. When the source operates at resonance, maximum power to the device can be obtained by matching the real impedance of the device at resonance to the real impedance of the source. For the purely reactive source, i.e. stiffness and mass controlled regions, the relative size between the device and the source is compared by using the stiffness and the mass ratio between the device and the source. Tuning frequencies for maximum power transfer to the device were also derived. These frequencies show the dependency on the stiffness and the mass ratio. The study also revealed that if the impedance of the source is not large enough, tuning the device to resonate at that operating frequency will result in a large reduction in the amount of power harvested. The most important point in this study is that the physical size is not the only thing that needs to be considered when considering the loading of the device on the source. The impedance of the device is the factor that needs to be considered because, in particular, the impedance of the device is large at resonance and the device can hence provide a large opposing force on the source. The tuning frequency for both the stiffness and the mass controlled sources converges to the tuning frequency conventionally assumed in the literature, i.e. tuning so that the undamped natural frequency of the device matches the excitation frequency, in which case when the device has no effect on the source.

For the case when the impedance of the device is much smaller than the impedance of the source, the source can be represented as a velocity source in which the presence of the device will not affect the source. In this case, the power harvested by the device can be computed in the frequency domain by using the product of the square of the velocity and the real part of the impedance of the device. The expression for the power harvested is similar to the one obtained using the time integration method which has been used in the literature. For a given amount of damping, the power harvested by
the device depends on the seismic mass, the amplitude of the base displacement, the
cube of the excitation frequency and the motion of the mass within the housing of the
device which is represented by the maximum relative displacement. The performance
of the linear device is limited to a narrow frequency band only and drops off rapidly if
mistuned. The performance of the device can be made useful for a wider frequency band
by adding a larger amount of damping in the system. However, increasing the damping
will reduce the amount of power harvested at resonance.

The dependence on the cube of the excitation frequency limits the usefulness of the
resonant generator for low frequency applications. However, some researchers have sug-
gested that the resonant generator can be used for low frequency applications when the
input vibration is composed of a number of harmonics. It was suggested that tuning
the device to resonate at the frequency of a higher harmonic harvests more power than
tuning the device to resonate at the fundamental harmonic due to the linear dependency
on the frequency even though the level of acceleration is relatively small.

This is not true for all cases. In the case where the level of excitation acceleration is large
and constant, power can still be harvested even at low excitation frequency. However,
the amount of damping in the system needs to be altered so that system behaves linearly
and satisfies with the maximum allowable displacement that the mass can move inside
the housing of the device.

To overcome the limitations of the linear device, this chapter proposed the use of a
non-linear spring instead of the linear spring. The limit of the performance of the device
with a non-linear spring over a tuned linear device with a linear spring was studied using
the principle of conservation of energy. The study revealed that the non-linear device
can outperform the tuned linear device by a factor of only $4/\pi$. This can be achieved
provided that, for a given sinusoidal input displacement, the response is a square wave.
In the next chapter, a device with a non-linear spring will be investigated so as to
produce an approximate square wave output displacement for a given sinusoidal input
displacement. The mechanism used is called a non-linear bistable mechanism.

Starting from the next chapter, the mechanical impedance of the device is assumed
to be much smaller than that of the source such that the source can be regarded as
the velocity source. In the next chapters, the notations for the device parameters are changed. In particular, $\omega_n$, $\zeta$ and $m$ represent the undamped natural frequency, the damping ratio and the mass of the device, respectively. Note that these parameters are expressed without the subscript $d$. 
### Table 2.1: Mechanical impedance and mobility of spring, damper and mass.

<table>
<thead>
<tr>
<th>Mechanical element</th>
<th>Mechanical impedance, $Z$</th>
<th>Impedance phase angle (degree)</th>
<th>Mechanical mobility, $Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spring</td>
<td>$Z_k = \frac{k}{j\omega}$</td>
<td>$-90^\circ$</td>
<td>$Y_k = \frac{j\omega}{k}$</td>
</tr>
<tr>
<td>Damper</td>
<td>$Z_c = c$</td>
<td>$0^\circ$</td>
<td>$Y_c = \frac{1}{c}$</td>
</tr>
<tr>
<td>Mass</td>
<td>$Z_m = j\omega m$</td>
<td>$+90^\circ$</td>
<td>$Y_m = \frac{1}{j\omega m}$</td>
</tr>
</tbody>
</table>

**Figure 2.1:** 2-DOF model representation of the source of vibration and the harvesting device (load).
Figure 2.2: Mass-spring-damper model (left) and the impedance diagram (right) of the source.

Figure 2.3: Magnitude of the source impedance normalized with respect to the critical damping ratio $c_{cs}$ as a function of the frequency ratio $\Omega_s$. 
Figure 2.4: Mass-spring-damper model (left) and the impedance diagram (right) of the device.

Figure 2.5: Magnitude of the device impedance normalized with respect to the critical damping ratio $c_{cd}$ as a function of the frequency ratio $\Omega_d$. 
Figure 2.6: Thévenin equivalent system consists of a source with impedance $Z_s$ connected in parallel with a device of impedance $Z_d$.

Figure 2.7: Normalized maximum power harvested by the device plotted against the damping ratio, $\zeta_d$ for a stiffness controlled source for different $\kappa$. 

Figure 2.8: Normalized power harvested by the device for different damping ratio for $\kappa = 10$: $\zeta_d = 0.01$ (—), $\zeta_d = 0.05$ (--) and $\zeta_d = 0.1$ (···).

Figure 2.9: Normalized maximum power harvested by the device plotted against the damping ratio, $\zeta_d$ for a mass controlled source for different $\mu$.
Figure 2.10: Normalized power harvested by the device for a mass controlled source when $\mu = 10$: $\zeta_d = 0.01$ (—), $\zeta_d = 0.05$ (−−) and $\zeta_d = 0.1$ (⋯).

Figure 2.11: Frequency at which the power, relative displacement, relative velocity and relative acceleration is maximum as a function of the damping ratio for a stiffness ratio $\kappa = 10$: power (—), relative displacement (⋯), relative velocity (−−) and relative acceleration (−·−).
Figure 2.12: Frequency at which the power, relative displacement, relative velocity and relative acceleration is maximum as a function of the damping ratio for a mass ratio $\mu = 10$: power ($\ast$), relative displacement ($\cdots$), relative velocity ($\cdots\cdots$) and relative acceleration ($\cdot\cdot\cdot$).

Figure 2.13: The frequency when the power, $P_d$ is maximum against $\zeta_d$ simulated for different stiffness ratio, $\kappa$. 
Figure 2.14: The frequency when the power, $P_d$ is maximum against $\zeta_d$ simulated for different mass ratio, $\mu$: $\mu = 1$ (−·−), $\mu = 10$ (−−), $\mu = 20$ (−) and $\mu = 1000$ (···).

Figure 2.15: Ratio of the power harvested by the load when $\Omega_d = 1$ ($P_{\text{dres}}$) to the maximum power harvested ($P_{\text{dmax}}$): $\kappa = 2.0$ (thin −), $\kappa = 10$ (−−), $\kappa = 20$ (−·−), $\kappa = 50$ (−·−) and $\kappa = 100$ (thick −).
Figure 2.16: Ratio of the power harvested by the load when $\Omega_d = 1$ to the maximum power harvested for different mass ratio $\mu$: $\mu = 2.0$ (thin --), $\mu = 10$ (---), $\mu = 20$ (-----), $\mu = 50$ (-----) and $\mu = 100$ (thick ---).

Figure 2.17: Norton equivalent system consists of a source with mobility $Y_s$ connected in parallel with a device of mobility $Y_d$. 
Figure 2.18: A mass-spring-damper mechanical model for energy harvesting device.

Figure 2.19: Relative transmissibility $T_R$ and bandwidth of the linear system.
Figure 2.20: Relative transmissibility $T_R$ for different damping ratios: $\zeta = 0.01$ (—), $\zeta = 0.05$ (--) and $\zeta = 0.1$ (···).
Chapter 3

Non-linear snap-through mechanism

3.1 Introduction

The results from the previous chapter showed that a system with a non-linear spring can outperform the tuned linear mechanism by a factor of only $4/\pi$. However, this is very much dependent upon the type of non-linear spring used in the device. The $4/\pi$ performance ratio can only be achieved provided that for a sinusoidal displacement input, the response is a square wave. Furthermore, the bandwidth of the tuned linear device is narrow, and there is the potential for a non-linear device to have a large bandwidth, and hence accommodate mistune better than a linear device.

In order to produce a square displacement response, a new type of spring mechanism needs to be designed. The linear spring described by Hooke’s law shown in Figure 3.1 always provides a restoring force against the motion of the mass which produces a sinusoidal response to a sinusoidal input. An approximate square wave response can be produced by steepening the gradient of the sinusoidal response between the two maxima. In order to achieve this, a new spring mechanism is needed in such a way that the restoring force is not acting against the motion of the mass, but acting along the motion of mass, instead. This can be done by using a non-linear spring which has
negative stiffness characteristic in the displacement range of interest so that the travel of the mass between the two maxima is speeded up.

The type of mechanism chosen has been studied by Brennan et. al. [88] to model the flight mechanism of an insect. It has the ability to steepen the gradient of the displacement response which suits the purpose described in the previous paragraph. This provides the motivation to study a general non-linear spring arrangement using two linear springs which can produce a negative stiffness. The static properties of such a spring arrangement are studied in the next section. The dynamic characteristics of such a system are studied using a bifurcation diagram. The power harvested by the device is also investigated and compared with the equivalent linear system in a later section of this chapter.

### 3.2 Static analysis of the oblique spring system

This section focuses on a static analysis of the system comprising two oblique springs and shows how the arrangement results in a negative stiffness mechanism. Figure 3.2 shows a possible arrangement of the mass-spring-damper for the snap-through mechanism. It consists of two linear oblique springs connected to a mass and a damper. Unlike a linear system [8], when the springs are unextended, they are inclined at an angle $\pm \theta$ to the line $x = 0$. Although the springs provide a linear restoring force along their axes, this particular arrangement yields a non-linear restoring force in the $x$-direction.

#### 3.2.1 Restoring force and stiffness of the spring

Figure 3.3 shows two inclined springs attached together with an axial force acting on both springs. The total axial component of the spring force $F$ at any displacement $x$ is such that

$$F = 2F_k,$$  \hspace{1cm} (3.1)
where \( F_k \) is the axial component of each spring force. The total axial restoring force as a function of \( x \) is given by

\[
F = 2k_{sn} \left( \sqrt{x^2 + l^2} - l_o \right) \sin \theta,
\]

where \( k_{sn} \) is the stiffness of the spring, \( \sqrt{x^2 + l^2} \) is the length of the spring, \( l_o \) is the original length of the spring and \( \theta \) is the inclination of the spring with respect to the origin. Referring to Figure 3.3

\[
\sin \theta = \frac{x}{\sqrt{x^2 + l^2}}.
\]

Substituting this into Equation 3.2 gives

\[
F = 2k_{sn} \left( 1 - \frac{l_o}{\sqrt{x^2 + l^2}} \right) x,
\]

which can be expressed in dimensionless form as

\[
\hat{F} = \left( 1 - \frac{1}{\sqrt{x^2 + \gamma^2}} \right) \hat{x},
\]

where \( \hat{F} = \frac{F}{2k_{sn}l_o} \), \( \hat{x} = \frac{x}{l_o} \) and \( \gamma = \frac{l}{l_o} \). Figure 3.4 shows the non-dimensional axial restoring force of the system as a function of \( \hat{x} \) for various \( \gamma \).

The non-dimensional stiffness, \( \hat{k} = \frac{d\hat{F}}{d\hat{x}} \) of the system is

\[
\hat{k} = 1 - \frac{\gamma^2}{(\hat{x}^2 + \gamma^2)^{\frac{3}{2}}},
\]

where \( \hat{k} = \frac{k_n}{2k_{sn}} \) and \( k_n = \frac{dF}{dx} \) is the physical stiffness of the system in Figure 3.3. The non-dimensional stiffness of the spring is shown in Figure 3.5 as a function of \( \hat{x} \). In the case when \( \gamma = 0 \), \( l = 0 \) and the two springs are vertical as shown in Figure 3.6 the system behaves linearly since the force is acting along the axis of deformation of the spring. When \( 0 < \gamma < 1 \), the stiffness of the spring changes from negative to positive when displaced away from \( x = 0 \). The strength of the negative stiffness depends on \( \gamma \). The smaller the value of \( \gamma \), the more negative the stiffness will be. Finally when
\( \gamma = 1 \), the spring behaves like a hardening spring and the region for negative stiffness disappears.

The critical static force \( \dot{F}_c = \frac{d\dot{F}}{dx} \) that the system can withstand before snap-through occurs is given by

\[
\dot{F}_c = \left(1 - \gamma^2\right)^{\frac{3}{2}}. \tag{3.7}
\]

This shows that the smaller the value of \( \gamma \), the larger the force needed to snap the spring from one equilibrium position to another. A smaller value of \( \gamma \) also results in a smaller range of the negative stiffness. Substituting \( \dot{F}_c \) into Equation 3.5 and solving for \( \dot{x} \) gives the position at which the snap-through occurs

\[
\dot{x}_c = \gamma^2 \sqrt{1 - \gamma^2}. \tag{3.8}
\]

### 3.2.2 Potential energy

The total elastic potential energy, \( E_p \) is given by

\[
E_p = 2 \left[ \frac{1}{2} k_{sn} \left( \sqrt{x^2 + l^2} - l_o \right)^2 \right], \tag{3.9}
\]

which can also be represented in dimensionless form as

\[
\dot{E}_p = \left( \sqrt{\dot{x}^2 + \gamma^2} - 1 \right)^2, \tag{3.10}
\]

where \( \dot{E}_p = \frac{E_p}{k_{sn} l_o^2} \). The non-dimensional potential energy is plotted in Figure 3.7, which shows that the system has a double-well potential \[89, 90\]. The equilibrium positions, \( \dot{x}_{1,2,3} \) of the system are given by

\[
\dot{x}_1 = \sqrt{1 - \gamma^2} \quad \{ \text{stable} \}, \tag{3.11}
\]

\[
\dot{x}_2 = -\sqrt{1 - \gamma^2} \quad \{ \text{stable} \}, \tag{3.12}
\]

\[
\dot{x}_3 = 0 \quad \{ \text{unstable} \}. \tag{3.13}
\]
The figure also shows the arrangement of the springs at each equilibrium position given Equations 3.11 - 3.13.

The motion of the of such system (if $0 < \gamma < 1$) can be categorized into two types. The first one is local oscillation about one of the two stable equilibria positions. Another type is the global oscillation about the three equilibria positions. The local oscillation occurs when an input is small in such a way that the potential barrier (i.e. the height of the potential energy hill at the unstable equilibrium position) is too large for the mass to snap from one stable equilibrium position to another. The global oscillation is induced when the input increases to a much larger value in such a way that the oscillation can overcome the potential barrier. This global oscillation can be either periodic or chaotic. The type of response the system will produce depends on the values of the parameters. The dynamic behaviour of the device will be investigated in the next section.

3.3 Dynamic analysis

The equation of motion of the snap-through mechanism can also be written in dimensionless form where the non-linear stiffness force $f(s)$ of the form given in Equation 3.5 is used to give

$$u'' + 2\zeta u' + \left(1 - \frac{1}{\sqrt{u^2 + \gamma^2}}\right) u = \Omega^2 \hat{Y} \cos (\Omega \tau),$$

(3.14)

where $0 < \gamma < 1$, $s = x - y$ is the relative displacement between the housing, $y$ and the mass, $x$, $u = \frac{s}{l_o}$, $\omega_n^2 = \frac{k}{m}$, $k = 2k_{sn}$, $\tau = \omega_n t$, $\Omega = \frac{\omega}{\omega_n}$, $\zeta = \frac{c}{2m\omega_n}$, $\hat{Y} = \frac{\hat{Y}_{lo}}{l_o}$, $(\bullet)' = \frac{d}{d\tau}$ and $(\bullet)'' = \frac{d^2}{d\tau^2}$.

The dynamic analysis of this system given in Equation 3.14 under harmonic excitation has already been widely studied [91, 92]. However, more often the dynamic analysis of such a system is conducted by using the approximate version of the spring force in Equation 3.5 so that the spring force can be represented in the form of the Duffing equation, $-k_1 \dot{x} + k_3 \dot{x}^3$. This type of representation of the non-linear spring can be obtained from Equation 3.5 using a series expansion in $\dot{x}$ about the unstable equilibrium.
position. However, this has to be done carefully since the series is only valid when the motion of the mass is small and comparable to the position of the stable equilibria. The list of references for this type of non-linear system is huge, although most of them involved the study of a force-excited rather than base-excited system. For example, readers are referred to textbooks [90, 93–98] and some papers [89, 99–104]. In this chapter, it is not the objective to discuss the dynamics of the system in a great detail. The objective is to study how the system dynamics affects the performance of an energy harvesting device.

3.3.1 Chaotic response

There are several techniques to obtain an approximate analytical solution to the system with a non-linear spring. In general, these techniques assume that the response is dominated by the fundamental harmonic and the non-linearity is weak. However, an approximate analytical solution to the system with the type of non-linear spring studied here, may not be feasible due to the possibility of a chaotic response. Although, an approximate analytical solution may be possible when the response is not chaotic.

The are many ways to investigate the chaotic behaviour of the response. Among these are the phase portrait, Poincaré map and the bifurcation diagram. The first two methods are useful in providing information based on one excitation frequency only. On the other hand, the bifurcation diagram can describe the behaviour of the system across a range of frequencies, while other parameters can also be varied.

In this chapter, the bifurcation diagram is chosen to study the response of the system given in Equation 3.14 since it is more relevant to energy harvesting i.e. the ability to describe the behaviour of the response across a range of frequencies. The bifurcation diagrams are plotted using 99001 iteration points from of $\Omega = 0.1$ up to $\Omega = 10$ at a step of 0.0001 increment. Initially, the frequency is fixed and Equation 3.14 is solved using the 4th order Runge-Kutta method at every $\Omega \tau = 2n\pi$ up to 300 cycles, where $n$ is the number of cycles. The transient parts of the response are ignored and only the last 50 values of the solution are plotted. The frequency $\Omega$ is then increased to a desired step and the process is repeated. The vertical axis labeled $r$ represents the distance of
the point in the Poincaré map from the origin \[105\]

\[ r = \sqrt{u'^2 + u''^2} \]  

(3.15)

Figures 3.8 - 3.15 show the bifurcation diagrams for the system shown in Figure 3.2. These figures are plotted using different values for \( \hat{Y}, \gamma \), and \( \zeta \). A periodic response of the system is represented by a single point and a chaotic response is represented by dispersed points. As can be seen in these figures, the response changes between periodic and chaotic especially when the input is small i.e. \( \hat{Y} = 0.5 \) and \( 1.0 \), as shown in Figures 3.8 - 3.11. In these cases the input is too small to throw the mass periodically across the three equilibria yet large large enough to overcome the central potential barrier occasionally. However, the response of the system is more settled when the input increases, e.g. \( \hat{Y} = 10 \) and \( 30 \), as the response changes from chaotic to periodic and maintains almost periodic motion for the rest of the frequency range as the input is large enough to oscillate the mass from one stable equilibrium position to another in an almost periodic manner, if not perfectly periodic.

From the system with low and large inputs, it can be seen that the chaos in the response is more settled when the level of damping is higher. Adding more damping, restricts the oscillation of the mass about one of the stable equilibrium positions. This restriction settles the oscillation of the mass about that stable equilibrium position quicker which in effect reduces the tendency of the response to behave chaotically.

When the input is large i.e. \( \hat{Y} \) is 10 and 30, the frequency range when the response is chaotic increases with the decrease in the non-linearity especially at low frequency and when the damping is small as shown by Figures 3.12(a), 3.13(a) and Figures 3.14(a), 3.15(a). The decrease in the non-linearity has the effect of lowering the potential energy barrier. Thus for a given input, the tendency for a cross-well motion to occur is higher for the system with a smaller non-linearity. The small amount of damping in the system contributes to the tendency of the response to become chaotic as the oscillation of the mass about one of the stable equilibria takes longer to settle.
3.3.2 Numerical simulation of the power harvested

The power harvested by the device containing either linear or snap-through mechanisms as a function of frequency is discussed in this section. A numerical simulation is conducted to compare the performance of both systems. The equation of motion of a linear system described in Chapter 2 (Equation 2.4) can be written in dimensionless form as

\[ u'' + 2\zeta u' + u = \Omega^2 \hat{Y} \cos(\Omega \tau), \quad (3.16) \]

The non-dimensional power harvested by the device for both linear and snap-through mechanisms is obtained numerically using the equivalent non-dimensional expression for power dissipated by the damper as described in Equation 2.46

\[ \hat{P} = \frac{\Omega \zeta}{\pi} \int_0^{2\pi} (u')^2 d\tau, \quad (3.17) \]

where \( \hat{P} = \frac{P}{m\omega_0^2 l^2} \). Equations 3.14 and 3.16 are solved numerically using the 4th order Runge-Kutta method and the power harvested is calculated using Equation 3.17. It should be noted that the system parameters are fixed, rather than being optimized to produce maximum harvested power at any frequency \( \Omega \). The main objective of the numerical simulation is to investigate the general trend of the power harvested by the device with the snap-through mechanism and to compare it with that for the linear mechanism.

3.3.2.1 Effect of damping

The effect of damping on the power harvested is investigated in this section. In this case, the input, \( \hat{Y} \) is fixed at 30. The power harvested is plotted in Figures 3.16 and 3.17 for \( \gamma = 0.1 \) and 0.5, respectively, with different values of damping. In general, the trend of the power harvested is similar to that with a linear mechanism investigated in Chapter 2. An increase in the damping will reduce the peak of the power harvested around \( \Omega = 1 \) and increase the power harvested in the region below and above the resonance. For the case when \( \gamma = 0.1 \), it can be seen that the jump-up point (i.e. cross-well motion) occurs
3.3.2.2 Effect of the non-linearity and the input level

Figures 3.18 - 3.21 shows the power harvested by the snap-through and linear mechanisms for different input levels, $\hat{Y}$. For each input level, the power harvested by the linear and the snap-through mechanisms with $\gamma = 0.1$ and 0.5 for a damping ratio of 10% are compared. It can be seen in Figure 3.18 for the case when $\hat{Y} = 0.5$, the power harvested by the system with $\gamma = 0.5$ is slightly larger than the one when $\gamma = 0.1$ and that of the linear mechanism in the region when $\Omega < 1$. This is due to a slightly larger oscillation of the mass about one stable equilibrium position by the system with $\gamma = 0.5$, for a given input. Generally, the power harvested by the snap-through mechanisms for both configurations are approximately equal to that from the linear system. The small amount of harvested power is due to the inability of the mass to oscillate across the three equilibria. There is a scattered region of the power harvested by the snap-through mechanisms around $0.6 < \Omega < 2$. This corresponds to the response in the system as has been described by the bifurcation diagram previously. Figure 3.22 shows the steady state time history displacement response and the phase portraits of the system for the case when $\gamma = 0.1$. It can be seen that at low frequency, the mass the just oscillates about a stable equilibrium position as shown in Figures 3.22(a) - 3.22(d). A further increase in the frequency introduces a cross-well motion which results in the response being chaotic (Figures 3.22(e) - 3.22(f)). The cross-well oscillation continues until a periodic response is produced with a further increase in frequency (Figures 3.22(g) - 3.22(h)). As the frequency increases further, the cross-well motion vanishes as the mass returns to oscillate about one of the stable equilibria (Figures 3.22(i) - 3.22(j)). A similar trend can also be observed for the system with $\gamma = 0.5$ as shown in Figure 3.23.

For the case when the input is doubled, i.e. $\hat{Y} = 1.0$, a slightly different behaviour
of the characteristics of the power harvested can be observed in Figure 3.19. It can be seen that there is a sudden increase in the power harvested by both snap-through mechanisms for different $\gamma$. However, one notices that the jump occurs at different frequencies. Equation 3.7 shows that, the smaller the value of $\gamma$, the larger the static force needed to move the mass from one stable equilibrium position to another. Due to this, the system with $\gamma = 0.5$ jumps up at a lower frequency at around $\Omega \approx 0.35$ and at a slightly higher frequency around $\Omega \approx 0.45$ when $\gamma = 0.1$. Thus when the frequency is smaller than these two frequencies, the mass just oscillates about one of its stable equilibrium positions resulting in a small amount of power being harvested. When $\Omega$ is increased to about these two frequencies, for each case, the excitation is large enough to oscillate the mass between the two stable equilibrium positions which reflects the sudden increase in the harvested power. The steady state time history of the displacement response and the phase portraits of these two configurations are shown in Figures 3.24 and 3.25.

The power harvested by the system with a larger input, $\hat{Y} = 10$ and 30, are shown in Figures 3.20 and 3.21, respectively. The trend of the power harvested at low frequencies is similar to that when $\hat{Y} = 1$. The obvious difference in the power harvested by these two input levels is the disappearance of the chaotic regions in the response around the resonance as shown by the smooth power harvested curve. This is because the input is large enough to oscillate the mass in a periodic manner from one stable position to another. The power harvested by both configurations converges to that of the linear mechanism at around $\Omega \approx 0.6$ due to the large response. However, in the case when $\hat{Y} = 30$ and $\gamma = 0.5$, as shown in Figure 3.21, there is no sudden increase in the power harvested since the input is large enough to throw the mass between the two stable equilibrium positions even when $\Omega = 0.1$. The steady state time history displacement response and the phase portraits for $\hat{Y} = 30$ are shown in Figure 3.26 and 3.27.

It can also be seen in Figures 3.20 and 3.21 that the snap-through mechanism outperforms the linear mechanism at low frequencies. At low frequencies, the rate at which the normalized power increases with frequency is 60 dB per decade for the linear case but only about 30 dB per decade for the non-linear case. Thus considerably more power
can be harvested from the non-linear system if $\Omega$ is much less than unity.

### 3.4 Practical constraints

Equation 2.66 suggests that the maximum amount of power harvested by a non-linear mechanism can be at most $4/\pi$ greater than the tuned linear mechanism assuming that the output displacement, for a given sinusoidal input, is a square wave. However, Figure 3.21 shows that the power harvested by the snap-through mechanism is much greater than that harvested by the linear system at low frequencies. This is because the linear system is tuned to only one excitation frequency and the power harvested drops off rapidly for excitation away from the natural frequency of the device. The ideal response of the snap-through mechanism is a square wave for the displacement. However, the response has to satisfy the equation of motion given in Equation 3.14, thus an approximation to the square wave is produced, instead. This approximation is good at low frequencies and becomes worse at high frequencies.

To produce a good approximation to a square wave, the rise time $\delta$ (time to travel from $u = -U$ to $u = U$) has to be much shorter than a quarter of a period i.e. $\delta \ll T/4$, where $T = 2\pi/\Omega$ is the period of oscillation (see Figure 3.28). For a given physical system, $\delta$ is further limited by the presence of the mass and the damping. As the frequency increases, the system is unable to produce a square wave like response since the period becomes shorter and the condition $\delta \ll T/4$ can no longer be achieved. As a result, an approximately sinusoidal response is produced at high frequencies rather than a square wave which results in a similar amount of power harvested as the linear device. As shown in Figures 3.20 and 3.21, the power harvested at low frequencies by the linear system rolls off at a higher rate than the system with the snap-through mechanism. Thus a potential benefit of this mechanism is that it can accommodate mistune better. Also the figure shows that the real benefit of the snap-through mechanism is by having a system with a much higher natural frequency compared to the excitation frequency.
3.5 Conclusions

A non-linear mechanism has been investigated in this chapter. This mechanism consists of two springs inclined at an angle to the horizontal. The mechanism is called a snap-through mechanism and makes use of positive stiffness and the negative stiffness of the spring created by the geometry of the spring configuration. The negative stiffness is designed to snap the mass from one position to another at much faster rate so that for a given sinusoidal input, the displacement response approximates a square wave.

In terms of the dynamic properties of the device with this mechanism, the approximate analytical solution to the equation of motion may not be feasible due to the unpredictable response. The response may change between periodic and chaotic. The type of the response is highly dependent upon several parameters such as the normalized frequency, $\Omega$, input level, $\hat{Y}$, damping ratio, $\zeta$ and the non-linearity, $\gamma$. Due to these, the results presented in this chapter are based on numerical simulations.

The bifurcation diagram has been used to study the type of response the system produces with particular values of damping, the non-linearity and the input levels across a range of frequencies. This diagram showed that for a small input, the response was unsettled and kept changing between periodic and chaotic. The response of the system was much more settled when the input becomes larger since the system has enough energy to throw the mass between the two stable equilibria. Apart from this, adding more damping to the system helped to reduce the chaos in the response since the addition of damping helped to settle the oscillation of the mass about a stable equilibrium more quickly.

In terms of the energy harvesting point of view, this mechanism is more beneficial when the input is large such that a cross-well motion can be produced. However, the frequency range where it is beneficial depends very much upon the level of input, the damping ratio and the non-linearity. The study also shows that it is not possible to produce a square wave response for all frequencies. However, a good approximation to a square wave can be obtained when the excitation frequency is much smaller than the natural frequency of the device. Due to this capability of producing a square wave response, the power harvested by the snap-through mechanism is much greater than the linear mechanism.
at frequencies much lower than the natural frequency. In addition, the roll off rate of the power harvested by the snap-through mechanism is much less than that of the linear mechanism. Thus the device has capability to cope better with mistune.

Another type of a non-linear mechanism is discussed in the next chapter. The aim of the mechanism is quite different from the snap-through mechanism, hence the difference in the type of spring used. The ability of that mechanism to cope with the mistune will be studied.
Chapter 3 Non-linear snap-through mechanism

**Figure 3.1:** Input and out displacement response of a system with a linear and a non-linear spring.

**Figure 3.2:** Arrangement of the mass-spring-damper for the snap-through mechanism.
Figure 3.3: The force acting on the oblique spring.

Figure 3.4: Non-linear dimensionless restoring force $\hat{F}$ as a function of $\hat{x}$: $\gamma = 0$ (thick solid line), $\gamma = 0.1$ (---), $\gamma = 0.5$ (· · ·), $\gamma = 0.7$ (− · −) and $\gamma = 1.0$ (thin solid line).
Figure 3.5: Dimensionless non-linear stiffness $\hat{k}$ as a function of $\hat{x}$: $\gamma = 0$ (thick solid line), $\gamma = 0.1$ (−−), $\gamma = 0.5$ (···), $\gamma = 0.7$ (−·−) and $\gamma = 1.0$ (thin solid line).

Figure 3.6: Position of the spring when $\gamma = 0$. 
Figure 3.7: Dimensionless elastic potential energy $\hat{E}_p$ of the springs as a function of position $\hat{x}$: $\gamma = 0.1$ (--) and $\gamma = 0.5$ (\cdots).
Figure 3.8: Bifurcation diagram for $\hat{Y} = 0.5$ and $\gamma = 0.1$: (a) $\zeta = 0.01$, (b) $\zeta = 0.05$ and (c) $\zeta = 0.1$. 
Figure 3.9: Bifurcation diagram for $\tilde{Y} = 0.5$ and $\gamma = 0.5$: (a) $\zeta = 0.01$, (b) $\zeta = 0.05$ and (c) $\zeta = 0.1$. 
Figure 3.10: Bifurcation diagram for $\hat{Y} = 1.0$ and $\gamma = 0.1$: (a) $\zeta = 0.01$, (b) $\zeta = 0.05$ and (c) $\zeta = 0.1$. 
Figure 3.11: Bifurcation diagram for $\hat{Y} = 1.0$ and $\gamma = 0.5$: (a) $\zeta = 0.01$, (b) $\zeta = 0.05$ and (c) $\zeta = 0.1$. 
Figure 3.12: Bifurcation diagram for $\hat{Y} = 10.0$ and $\gamma = 0.1$: (a) $\zeta = 0.01$, (b) $\zeta = 0.05$ and (c) $\zeta = 0.1$. 
Figure 3.13: Bifurcation diagram for $\hat{Y} = 10.0$ and $\gamma = 0.5$: (a) $\zeta = 0.01$, (b) $\zeta = 0.05$ and (c) $\zeta = 0.1$. 
Figure 3.14: Bifurcation diagram for $\hat{Y} = 30.0$ and $\gamma = 0.1$: (a) $\zeta = 0.01$, (b) $\zeta = 0.05$ and (c) $\zeta = 0.1$. 
Figure 3.15: Bifurcation diagram for \( \dot{Y} = 30.0 \) and \( \gamma = 0.5 \): (a) \( \zeta = 0.01 \), (b) \( \zeta = 0.05 \) and (c) \( \zeta = 0.1 \).
Figure 3.16: Power harvested by the snap-through mechanism when $\gamma = 0.1$ and $\hat{Y} = 30$ for several damping ratios: $\zeta = 0.01$ (black), $\zeta = 0.05$ (blue) and $\zeta = 0.1$ (red).

Figure 3.17: Power harvested by the snap-through mechanism when $\gamma = 0.5$ and $\hat{Y} = 30$ for several damping ratios: $\zeta = 0.01$ (black), $\zeta = 0.05$ (blue) and $\zeta = 0.1$ (red).
**Figure 3.18:** Power harvested in the damper when $\zeta = 0.1$ and $\dot{Y} = 0.5$: linear mechanism (---), snap-through mechanism $\gamma = 0.1$ (blue) and snap-through mechanism $\gamma = 0.5$ (red).

**Figure 3.19:** Power harvested in the damper when $\zeta = 0.1$ and $\dot{Y} = 1.0$: linear mechanism (---), snap-through mechanism $\gamma = 0.1$ (blue) and snap-through mechanism $\gamma = 0.5$ (red).
Figure 3.20: Power harvested in the damper when $\zeta = 0.1$ and $\hat{Y} = 10$: linear mechanism (---), snap-through mechanism $\gamma = 0.1$ (blue) and snap-through mechanism $\gamma = 0.5$ (red).

Figure 3.21: Power harvested in the damper when $\zeta = 0.1$ and $\hat{Y} = 30$: linear mechanism (---), snap-through mechanism $\gamma = 0.1$ (blue) and snap-through mechanism $\gamma = 0.5$ (red).
Figure 3.22: Steady state time history of the displacement response (1st column) and the phase portraits (2nd column) for \( \hat{Y} = 0.5, \zeta = 0.1 \) and \( \gamma = 0.1 \) corresponding to Figures 3.8(c) and 3.18: \( \Omega = 0.1 \) (a, b), \( \Omega = 0.5 \) (c,d), \( \Omega = 0.9 \) (e, f), \( \Omega = 1.5 \) (g, h) and \( \Omega = 5 \) (i, j).
Figure 3.23: Steady state time history of the displacement response (1st column) and the phase portraits (2nd column) for $\hat{Y} = 0.5$, $\zeta = 0.1$ and $\gamma = 0.5$ corresponding to Figures 3.9(c) and 3.18: $\Omega = 0.1$ (a, b), $\Omega = 0.5$ (c,d), $\Omega = 0.9$ (e, f), $\Omega = 1.5$ (g, h) and $\Omega = 5$ (i, j) .
Figure 3.24: Steady state time history of the displacement response (1st column) and the phase portraits (2nd column) for $\hat{Y} = 1.0$, $\zeta = 0.1$ and $\gamma = 0.1$ corresponding to Figures 3.10(c) and 3.19: $\Omega = 0.1$ (a, b), $\Omega = 0.5$ (c, d), $\Omega = 0.9$ (e, f), $\Omega = 1.5$ (g, h) and $\Omega = 5$ (i, j).
Figure 3.25: Steady state time history of the displacement response (1st column) and the phase portraits (2nd column) for $\hat{Y} = 1.0$, $\zeta = 0.1$ and $\gamma = 0.5$ corresponding to Figures 3.11(c) and 3.19: $\Omega = 0.1$ (a, b), $\Omega = 0.5$ (c, d), $\Omega = 0.9$ (e, f), $\Omega = 1.5$ (g, h) and $\Omega = 5$ (i, j).
Figure 3.26: Steady state time history of the displacement response (1st column) and the phase portraits (2nd column) for $\hat{Y} = 30$, $\zeta = 0.1$ and $\gamma = 0.1$ corresponding to Figures 3.14(c) and 3.21: $\Omega = 0.1$ (a, b), $\Omega = 0.1282$ (c,d), $\Omega = 0.1807$ (e, f), $\Omega = 0.2269$ (g, h) and $\Omega = 0.9$ (i, j).
Figure 3.27: Steady state time history of the displacement response (1st column) and the phase portraits (2nd column) for $\hat{Y} = 30$, $\zeta = 0.1$ and $\gamma = 0.5$ corresponding to Figures 3.15(c) and 3.21: $\Omega = 0.1$ (a, b), $\Omega = 0.1282$ (c, d), $\Omega = 0.1807$ (e, f), $\Omega = 0.2269$ (g, h) and $\Omega = 0.9$ (i, j).
Figure 3.28: Square wave displacement response and its velocity.
Chapter 4

Non-linear hardening mechanism

4.1 Introduction

It has been shown in Chapter 2 that an energy harvesting device with a non-linear spring can only outperform the device with a linear spring, which is tuned so that its natural frequency matches the excitation, by a factor of only $4/\pi$. For a given sinusoidal input displacement, this can be achieved provided that the response is a square wave. To produce a response that tends towards a square wave, a snap-through mechanism described in Chapter 3 has been proposed. The spring is arranged in such a way that the mass snaps through from one stable position to another by employing negative stiffness so as to reduce the rise time from one stable position to another. This mechanism decreases the rise time by steepening the gradient of the displacement response. However, it is not possible to move the mass from one stable position to another instantaneously due to the presence of the mass and the damper.

In this chapter, another mechanism using a non-linear spring is proposed. This mechanism consists of a hardening spring which has the force-deflection and stiffness-deflection characteristics shown in Figure 4.1. A system with this type of spring has the capability of shifting the resonance frequency to the right of the linear resonance frequency. Thus, the main idea of this mechanism is to widen the frequency band over which the device can harvest a reasonable amount of power without adding a further degree-of-
freedom to the system. Several papers in the literature, for example \[38, 39, 106, 107\], describe the hardening characteristics of the device. The last two papers describe the benefit of the hardening mechanism in the energy harvesting device, i.e. causing an increase in the bandwidth. However, no analytical expressions have been derived to describe the bandwidth of the system with a hardening spring.

An expression for the bandwidth of the system with a hardening spring is derived in this chapter. In addition, this chapter also presents an alternative way to estimate the amount of linear viscous damping in the system containing a hardening mechanism. It follows on previous work conducted by Carrella \[108\] which determined simple expressions for the jump-up and the jump-down frequencies for a base- and a force-excited Duffing oscillator. Some of the expressions are repeated here for convenience. However, some of them are rearranged for better interpretation of the behaviour of the system and to facilitate further simplification.

### 4.2 Equation of motion

The equation of motion for a base-excited hardening Duffing oscillator is given by \[108\]

\[
m\ddot{s} + c\dot{s} + k_1 s \left(1 + \frac{k_3}{k_1} s^2\right) = -m\ddot{y}, \tag{4.1}
\]

where \(s = x - y\) is the relative displacement between the seismic mass, \(x\), and the housing, \(y\), and \(y = Y \cos(\omega t)\). The system consists of a mass connected in series with a parallel combination of a damper and a non-linear spring whose spring force is of the form \(k_1 x + k_3 x^3\) where \(k_1\) is the linear spring constant and \(k_3\) is the non-linear spring constant. The system is base-excited with \(k_3 > 0\) denoting a hardening system. Equation 4.1 can be expressed in non-dimensional form as

\[
u'' + 2\zeta u' + u + \alpha u^3 = \Omega^2 \cos(\Omega \tau + \phi), \tag{4.2}
\]

where \(\omega_n = \sqrt{\frac{k_1}{m}}, \quad \Omega = \frac{\omega}{\omega_n}, \quad u = \frac{s}{Y}, \quad \alpha = \frac{k_3 Y^2}{k_1}, \quad \tau = \omega_n t, \quad \zeta = \frac{c}{2m\omega_n}, \quad (\bullet)' = \frac{d}{d\tau}, \quad (\bullet)'' = \frac{d^2}{d\tau^2}\) and \(\phi\) is the phase angle between the excitation and the response.
Note that \( u \) is the ratio of the relative displacement \( s \) and the amplitude of the input displacement \( Y \). In the linear system, this is regarded as the relative transmissibility. The degree of non-linearity of the device \( \alpha \) depends on both the physical properties of the device, \( k_1 \) and \( k_3 \), and the level of the input displacement, \( Y \). Thus when the level of excitation is small, the system behaves almost linearly.

### 4.3 Harmonic balance method

There are several methods to obtain an approximate solution to Equation 4.2. For a system with a weak non-linearity the equation can be solved by assuming a small perturbation of the simple harmonic oscillator. Normally the method involves the series expansion of a small perturbation parameter. These methods include the Linstedt-Poincaré, multiple scale and averaging methods which are described by Nayfeh and Mook [109], Nayfeh [110] and Meirovitch [111]. The averaging and multiple scales methods have the advantage because they allow the checking of the local stability of the steady state solution [112]. Another technique to obtain the approximate solution to the equation of motion is the method of harmonic balance [113]. This method is not restricted to a small non-linearity which gives an advantage of this method over other methods [114]. This is true provided that the fundamental harmonic of the response is dominant over the higher harmonics. However, the main disadvantage of this method is it does not provide information on stability of the system [108].

Due to its simplicity over other methods, the harmonic balance method is used to find the approximate solution to Equation 4.2. It is assumed that higher order harmonics are negligible and that the steady state solution of Equation 4.2 is of the form

\[
u = U \cos (\Omega \tau),
\]

so that the velocity and the acceleration are respectively given by

\[
u' = -\Omega U \sin (\Omega \tau),
\]
and

\[ u'' = -\Omega^2 U \cos (\Omega \tau). \quad (4.5) \]

Substituting Equations 4.3, 4.4 and 4.5 into Equation 4.2 and expanding the term on the right hand side yields

\[- \Omega^2 U \cos (\Omega \tau) - 2\zeta \Omega U \sin (\Omega \tau) + U \cos (\Omega \tau) + \frac{3}{4} \alpha U^3 \cos (\Omega \tau) + \frac{1}{4} \alpha U^3 \cos (3\Omega \tau)\]

\[= \Omega^2 \cos (\phi) \cos (\Omega \tau) - \Omega^2 \sin (\phi) \sin (\Omega \tau). \quad (4.6)\]

Collecting coefficients with similar expression (\cos (\Omega \tau) and \sin (\Omega \tau)) and neglecting the higher order harmonic term, \cos (3\Omega \tau), gives

\[U + \frac{3}{4} \alpha U^3 - \Omega^2 U = \Omega^2 \cos (\phi), \quad (4.7a)\]

\[2\zeta \Omega U = \Omega^2 \sin (\phi). \quad (4.7b)\]

Squaring Equations 4.7a and 4.7b and adding them together yields

\[\left( U + \frac{3}{4} \alpha U^3 - \Omega^2 U \right)^2 + (2\zeta \Omega U)^2 = \Omega^4. \quad (4.8)\]

This equation represents the frequency-amplitude relationship. The relationship between the frequency and the amplitude of the response can be solved in terms of the cube of \(U^2\) as given by

\[\frac{9}{16} \alpha^2 U^6 + \frac{3}{2} \alpha (1 - \Omega^2) U^4 + \left[ (1 - \Omega^2) + (2\zeta \Omega)^2 \right] U^2 - \Omega^4 = 0. \quad (4.9)\]

However, this can be complicated. An alternative way is to solve for \(\Omega^2\) for a given \(U\) by re-arranging Equation 4.9 so that it can be represented as a polynomial in \(\Omega^2\) squared, which can be written as

\[(U^2 - 1) \Omega^4 + \left[ (4\zeta^2 - 2) U^2 - \frac{3}{2} \alpha U^4 \right] \Omega^2 + \left( U + \frac{3}{4} \alpha U^3 \right)^2 = 0. \quad (4.10)\]
Solving Equation 4.10 for positive $\Omega$ gives the frequency-amplitude relationship as

$$\Omega_1 = \sqrt{\frac{3\alpha U^4 + 4U^2 (1 - 2\zeta^2) - U\sqrt{3\alpha U^2 + 4)^2 - 64U^2\zeta^2 (1 - \zeta^2) - 48\alpha\zeta^2 U^4}}{4(U^2 - 1)},} \tag{4.11}$$

and

$$\Omega_2 = \sqrt{\frac{3\alpha U^4 + 4U^2 (1 - 2\zeta^2) + U\sqrt{(3\alpha U^2 + 4)^2 - 64U^2\zeta^2 (1 - \zeta^2) - 48\alpha\zeta^2 U^4}}{4(U^2 - 1)}}. \tag{4.12}$$

If $\zeta^2 \ll 1$, with some re-arranging, Equations 4.11 and 4.12 reduce to

$$\Omega_1 \approx \sqrt{U^2 (1 + \frac{3}{4}\alpha U^2) - U\sqrt{(1 + \frac{3}{4}\alpha U^2) \left[1 + \left(2\zeta U\right)^2 \left(\frac{3\alpha}{16\zeta^2} - 1\right)\right]}} \tag{4.13},$$

which gives the amplitude of the response on the resonant branch (steady state response with a larger amplitude at a given frequency), and

$$\Omega_2 \approx \sqrt{U^2 (1 + \frac{3}{4}\alpha U^2) + U\sqrt{(1 + \frac{3}{4}\alpha U^2) \left[1 + \left(2\zeta U\right)^2 \left(\frac{3\alpha}{16\zeta^2} - 1\right)\right]}} \tag{4.14},$$

which describes the amplitude of the response on the non-resonant branch (steady state response with a smaller amplitude) of the system, in the region where multiple stable solutions exist.

### 4.4 Frequency response curves

Figure 4.2 shows a typical frequency response curve (FRC) for a hardening system plotted using Equations 4.13 and 4.14 for arbitrary system parameters. With increasing frequency, the response will be along the curve A-F-B. An infinitesimal change in frequency at point B will cause the response to jump down to point C. With further increase in frequency, the response continues to be on curve C-D. When the frequency is decreased, the response follows the curve D-E. At point E, an infinitesimal decrease in
frequency will cause the response to jump up to point F. The response continues along
the curve F-A with further decrease in frequency. The response between B and E is not
visible in practice as it represents unstable solutions.

### 4.4.1 Jump-down frequency

Like the linear system, it is essential to locate the point where the amplitude of the re-
sponse is maximum just to study the how much the response is amplified or attenuated
by the input. This point can be regarded as the resonance point of the non-linear system
at it gives the maximum response. When the damping is small such that \( \zeta^2 \ll 1 \), the
jump-down point is approximately equal to the point where the response is maximum.

The obvious technique to find this amplitude of the response and the frequency is by
using differentiation and finding the point where \( \frac{dU}{d\Omega} = \frac{d\Omega}{dU} = 0 \). However, this
way of computing the parameters can be complicated. Carrella [108] described a simple
way to compute these parameters by investigating the real and imaginary parts of Equation
4.13 or 4.14. In order for the Equations 4.13 and 4.14 to yield a real solution, the
internal radicands have to be positive, i.e.

\[
(1 + \frac{3}{4\alpha U^2}) \left[ 1 + (2\zeta U)^2 \left( \frac{3\alpha}{16\zeta^2 - 1} \right) \right] > 0.
\] (4.15)

The equations also yield a real solution if and only if

\[
U < U_m,
\] (4.16)

where \( U_m \) is the maximum amplitude of the response. Thus the maximum amplitude
can be obtained by equating Equation 4.15 to zero. Equating the first term to zero

\[
1 + \frac{3}{4\alpha U_m^2} = 0,
\] (4.17)

and rearranging this expression yields

\[
U_m = \sqrt{\frac{-4}{3|\alpha|}},
\] (4.18)
which is not real since $|\alpha| > 0$. However, equating the second term to zero

$$1 + (2\zeta U)^2 \left( \frac{3\alpha}{16\zeta^2} - 1 \right) = 0,$$

(4.19)

and rearranging this expression yields

$$U_m = \frac{1}{2\zeta} \sqrt{\frac{1}{1 - \frac{3\alpha}{16\zeta^2}}}.$$  

(4.20)

This expression is real if

$$\alpha < \frac{16\zeta^2}{3},$$

(4.21)

which sets the limit for $\alpha$. When $\alpha = 0$, Equation 4.20 becomes

$$U_{m(\text{linear})} = \frac{1}{2\zeta},$$

(4.22)

which is the peak relative transmissibility of the linear system. The frequency, $\Omega_m$ when the maximum response occurs can computed by substituting Equation 4.20 into either Equations 4.13 or 4.14 to yield

$$\Omega_m = \sqrt{\frac{1}{1 - \frac{3\alpha}{16\zeta^2} \left[ 1 - \left( 2\zeta \right)^2 \left( 1 - \frac{3\alpha}{16\zeta^2} \right) \right]}}.$$  

(4.23)

Rearranging Equation 4.23 gives

$$\Omega_m = \sqrt{\frac{1}{1 - \left( \frac{3\alpha}{16\zeta^2} \right) \left( 1 - \frac{3\alpha}{4} - 4\zeta^2 \right)}}.$$  

(4.24)

For the case when $\zeta^2 \ll 1$ and the non-linearity is weak, such that $\alpha \ll \frac{\zeta}{3}$, Equation 4.24 can be simplified to become

$$\Omega_m \approx \sqrt{\frac{1}{1 - \frac{3\alpha}{16\zeta^2}}}.$$  

(4.25)

In the case when $\alpha = 0$, Equation 4.25 reduces to

$$\Omega_m = 1,$$

(4.26)
which is the frequency for the peak relative transmissibility in the linear system. Equations [4.20] and [4.25] also imply that the jump-down point in the non-linear system depends on both the damping and the non-linearity.

### 4.4.2 Jump-up frequency

The method of computing the jump-up frequency is slightly different from the one for the jump-down frequency. As mentioned before, the jump-up frequency occurs at \( \frac{dU}{d\Omega^2} = \infty \), or \( \frac{d\Omega^2}{dU} = 0 \). However, this way of determining of the jump-up frequency can result in a very complicated intermediate expression.

Figure 4.3 shows the frequency response curve for different damping ratios \( \zeta \) while the non-linearity \( \alpha \) is kept constant. It can be seen that for the case with a small damping ratio, the jump-up frequency is not strongly dependent upon the damping ratio. Thus, the jump-up frequency can be computed by setting \( \zeta = 0 \) into Equation 4.9 and solving the polynomial equation for \( U^2 \) in terms of \( \Omega^2 \) as follows

\[
\frac{9}{16} \alpha^2 U^6 + \frac{3}{2} \alpha (1 - \Omega^2) U^4 + (1 - \Omega^2)^2 U^2 - \Omega^4 = 0.
\]  

(4.27)

This equation can either have one real and two complex conjugate roots or three real roots [115]. At the jump-up point, the imaginary part of the complex root becomes zero, yielding 3 real roots, two of which are equal [108]. Equating the imaginary part of the complex conjugate roots to zero and solving for \( \Omega \) yields

\[
\Omega_{up} = \frac{1}{12} \left[ 27 \left( 729 \alpha^3 + 1296 \alpha^2 + 384 \alpha + 192 \sqrt{3} \alpha^3 + 4 \alpha^2 \right)^{1/3} - \ldots \right. \\
\left. \frac{768 \left( -\frac{27}{8} \alpha - \frac{729}{256} \alpha^2 \right)}{\left( 729 \alpha^3 + 1296 \alpha^2 + 384 \alpha + 192 \sqrt{3} \alpha^3 + 4 \alpha^2 \right)^{1/3}} + 144 + 243 \alpha \right]^{1/2}.
\]  

(4.28)

If \( \alpha \) is small such that \( \alpha \ll 4/3 \) and \( \alpha \ll (16/3) \zeta^2 \), Equation 4.28 can be simplified and approximated to become [108]

\[
\Omega_{up} \approx 1 + \frac{1}{22} \left( \frac{3}{2} \right)^3 \alpha^3.
\]  

(4.29)
Equation 4.29 shows that the jump-up frequency depends on the non-linearity only due to the approximation stated before i.e. in the case where the non-linearity is small, the jump-up frequency is independent of the damping.

### 4.4.3 Absolute Transmissibility

The ratio between the amplitude of the absolute displacement of the mass and the amplitude of the input displacement is considered in this section. In the linear system, this is called the absolute transmissibility. In this section, the absolute transmissibility of the hardening system is derived and expressed in terms of the the relative transmissibility. As in previous section, the relative displacement between the seismic mass and the housing is given by

\[
s = x - y. \tag{4.30}
\]

Dividing Equation 4.30 by the amplitude of the input displacement, \(Y\) and re-arranging yields

\[
v_a = u + \cos (\Omega \tau + \phi), \tag{4.31}
\]

where \(v_a = \frac{x}{Y}\), \(u = \frac{s}{Y}\) and \(y = Y \cos (\Omega \tau + \phi)\). Noting that \(u = U \cos (\Omega \tau)\), Equation 4.31 can be expanded and rearranged to become

\[
v_a = [U + \cos (\phi)] \cos (\Omega \tau) - [\sin (\phi)] \sin (\Omega \tau). \tag{4.32}
\]

The amplitude of the absolute transmissibility can be computed from the resultant of the components given by the coefficient of \(\cos (\Omega \tau)\) and \(\sin (\Omega \tau)\). This can be expressed as

\[
V_a = \sqrt{[U + \cos (\phi)]^2 + [\sin (\phi)]^2}, \tag{4.33}
\]

which can be further simplified to become

\[
V_a = \sqrt{2U \cos (\phi) + U^2} + 1, \tag{4.34}
\]

where

\[
\cos (\phi) = \frac{U + \frac{3}{2} \alpha U^3 - \Omega^2 U}{\Omega^2}. \tag{4.35}
\]
Chapter 4 Non-linear hardening mechanism

The absolute transmissibility is used later to compare the theoretical and experimental results. Although the absolute transmissibility is used rather than the relative transmissibility to compare the theoretical and experimental results, the parameters of interest here, i.e. the jump-up and jump-down frequencies, are approximately the same as those derived in previous sections. This is because the absolute transmissibility only differs from the relative transmissibility at low and high frequencies as shown in Figure 4.4.

4.5 Bandwidth of the hardening system

Referring to Figure 4.5, the bandwidth of the hardening system $\Delta \Omega_h$ is defined as

$$\Delta \Omega_h = (\Omega_{\text{dwn}} - \Omega_m) + (\Omega_m - \Omega_{\text{hpb}}). \quad (4.36)$$

As mentioned before in Section 4.4.1, if $\zeta^2 \ll 1$, the jump-down frequency $\Omega_{\text{dwn}}$ is approximately equal to the maximum response frequency $\Omega_m$, and Equation 4.36 reduces to

$$\Delta \Omega_h \approx \Omega_m - \Omega_{\text{hpb}}. \quad (4.37)$$

The half-power point amplitude $U_{\text{hpb}}$ is obtained by dividing the maximum response, $U_m$ in Equation 4.20 by $\sqrt{2}$ to yield

$$U_{\text{hpb}} = \frac{\sqrt{2}}{4\zeta} \sqrt{\frac{1}{1 - 2\epsilon}}. \quad (4.38)$$

where $\epsilon = \frac{3\alpha}{23^2}$. Substituting $U_{\text{hpb}}$ into Equation 4.14 gives the half-power point frequency $\Omega_{\text{hpb}}$ such that

$$\Omega_{\text{hpb}} = \left[1 + \frac{\epsilon}{1 - 2\epsilon} - 2\zeta \left(1 - \epsilon^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}}. \quad (4.39)$$

The approximate expression for the bandwidth of the hardening system if $\zeta^2 \ll 1$ is thus given by substituting Equations 4.25 and 4.39 into Equation 4.37 to give

$$\Delta \Omega_h \approx \left[\frac{1}{1 - 2\epsilon}\right]^{\frac{1}{2}} - \left[1 + \frac{\epsilon}{1 - 2\epsilon} - 2\zeta \left(1 - \epsilon^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}}. \quad (4.40)$$
For the case when $\alpha = 0$ or $\frac{\alpha}{\zeta^2} \approx \frac{16}{3}$, Equation 4.40 reduces to

$$\Delta \Omega_h \approx \zeta,$$  \hspace{1cm} (4.41)

which is half of the bandwidth of the linear system. This is due to the approximation $\Omega_{dwn} = \Omega_m$ which is only valid when the damping ratio is small. Unlike the linear system in which the bandwidth is $2\zeta$, the bandwidth of the hardening system is dependent both on the damping ratio $\zeta$ and the non-linearity $\alpha$.

Figure 4.6 shows the variation of the bandwidth with respect to the damping ratio. Assuming that the amount of damping in a system with a hardening spring is similar to that without the hardening spring, the bandwidth for the hardening mechanism is wider than the linear system, depending on the strength of the non-linearity, especially when the damping is small. When the damping is large such that $\frac{\alpha}{\zeta^2} \approx \frac{16}{3}$, the bandwidth is approximately half of the bandwidth of the linear system. This may not represent the actual bandwidth of such a system due to the approximation mentioned before. If this is the case, the bandwidth of the system is considered to be equal to the bandwidth of the linear system i.e. approximately twice the plotted values due to the disappearance of the jump phenomenon.

### 4.6 Estimation of linear viscous damping

As has been investigated in the previous section, the bandwidth of the hardening system depends on both the damping and the non-linearity. The value of the non-linearity $\alpha$ can be estimated provided the information on the physical stiffness ($k_1$ and $k_3$) and the level of input displacement $Y$ is known. On the other hand, the value of the damping ratio is often not readily available in practice. Thus a measurement, normally a dynamic measurement, is conducted to measure the amount of damping in such a system. Assuming that the fundamental harmonic is dominant over the higher harmonics, the amount of damping can be estimated in both the time and the frequency domain [117]. In the time domain, it can be estimated using the logarithmic decrement method while in the frequency domain, for a base excited system it can be estimated using the peak of the
transmissibility. Both time and frequency domain methods provide a simple analytical expression for the linear viscous damping in a linear system.

However, so far there is no simple analytical expressions to estimate the amount of damping in a non-linear system particularly involving a hardening Duffing oscillator. Parzygnat and Pao [118] have studied a method to estimate the amount of damping. It is thought that they plotted the amplitude of the free vibration as a function of time. Assuming the damping is linear, the amount of damping can be estimated using the gradient of that plot. Another method to estimate the amount of damping in a Duffing oscillator is proposed in [119] using the Lyapunov spectrum. However, they studied a different type of Duffing oscillator in which the linear stiffness is negative and the system behaves like a bi-stable mechanism. In the case of energy harvesting employing electromagnetic technology, the results described in [107] showed the hardening characteristic of the device with clear jump-up and jump-down frequencies. In determining the Q-factor of the device, the damping in the system was estimated using a log-decrement method of voltage induced across the coil.

In this section, an alternative way to estimate the linear viscous damping in the frequency domain is investigated using the distinctive properties of the hardening mechanism i.e. the jump-down, $\Omega_{\text{dwn}}$, and the jump-up, $\Omega_{\text{up}}$, frequencies [120], since these frequencies are dependent upon the strength of non-linearity and the damping. Apart from the two jump frequencies, a prior knowledge of the linear natural frequency of the system is also essential. This is the natural frequency when the amplitude of oscillation of the mass is small and is within the linear regime of the force-stiffness characteristic. This frequency can be estimated easily provided that the seismic mass and the effective linear stiffness are known.

The non-linearity, $\alpha$ and the damping ratio, $\zeta$ can be estimated by rearranging Equations 4.29 and 4.25 with some substitution, to yield

$$\alpha \approx 2^6 \left( \frac{2}{3} \right)^9 \left( \Omega_{\text{up}} - 1 \right)^3,$$  

(4.42)
and
\[ \zeta \approx 2^\frac{2}{3} \left( \frac{2}{3} \right)^4 \frac{\Omega_{\text{down}} (\Omega_{\text{up}} - 1)^{\frac{3}{2}}}{(\Omega_{\text{down}}^2 - 1)^{\frac{3}{2}}}. \tag{4.43} \]

Equations 4.42 and 4.43 implies that the non-linearity in a hardening mechanism is only dependent on the jump-up frequency while the damping ratio depends on both the jump-up and jump-down frequencies.

### 4.6.1 Sensitivities of \( \alpha \) and \( \zeta \) on the jump frequencies

In practice, the jump frequencies are estimated from measurement. Thus the exact values of the jump frequencies are not known in advance. Failure to identify the exact jump frequencies will result in errors in the estimates of the non-linearity and the damping ratio. In this section, a study to investigate the sensitivity of the non-linearity and the damping ratio on the jump frequencies is conducted.

The jump-up frequency is obtained by sweeping the excitation frequency from a high down to a low frequency, and vice versa for the jump-down frequency. Due to this, in practice, the jump-up and the jump-down frequencies tend to be underestimated. Figure 4.7 shows the true and the estimated jump frequencies. Consider the estimate of the non-dimensional jump-up frequency given by
\[ \hat{\Omega}_{\text{up}} = \Omega_{\text{up}} - \xi, \tag{4.44} \]

where \( \hat{\Omega}_{\text{up}} = \frac{\hat{\omega}_{\text{up}}}{\omega_n} \), \( \hat{\omega}_{\text{up}} \) is the estimated jump-up frequency, \( \omega_n \) is the linear natural frequency of the device, \( \Omega_{\text{up}} = \frac{\omega_{\text{up}}}{\omega_n} \), \( \omega_{\text{up}} \) is the true value of the jump-up frequency and \( \xi \) is the difference between the estimated and the true value. The estimate of the non-linearity, \( \hat{\alpha} \), is given by
\[ \hat{\alpha} = 2^6 \left( \frac{2}{3} \right)^9 \left( \hat{\Omega}_{\text{up}} - 1 \right)^3. \tag{4.45} \]

Substituting Equation 4.44 into Equation 4.45 for \( \hat{\Omega}_{\text{up}} \) gives
\[ \hat{\alpha} = 2^6 \left( \frac{2}{3} \right)^9 \left[ (\Omega_{\text{up}} - 1) - \xi \right]^3. \tag{4.46} \]
Expanding this equation and re-arranging yields

$$\hat{\alpha} = \alpha \left[ 1 - \frac{3\xi}{(\Omega_{up} - 1)} + \frac{3\xi^2}{(\Omega_{up} - 1)^2} - \frac{\xi^3}{(\Omega_{up} - 1)^3} \right], \quad (4.47)$$

where $\alpha$ is the true value of the non-linearity given in Equation (4.42). It can be seen in Equation (4.47) that higher powers of $\xi$ are included. These higher terms cannot necessarily be ignored because the denominator, $\Omega_{up} - 1$, may be small compared to 1. Thus the estimate of the non-linearity is very sensitive to the errors in the estimated jump-up frequency. A slight deviation from the true value may result in a relatively large error.

To investigate the effects of errors in the estimates of the jump frequencies on the estimate of the damping ratio, consider the estimated jump-down frequency given by

$$\hat{\Omega}_{dwn} = \Omega_{dwn} - \nu, \quad (4.48)$$

where $\hat{\Omega}_{dwn} = \frac{\hat{\omega}_{dwn}}{\omega_n}$, $\omega_{dwn}$ is the estimate of the jump-down frequency, $\Omega_{dwn} = \frac{\omega_{dwn}}{\omega_n}$, $\omega_{dwn}$ is the true value of the jump-down frequency and $\nu$ is the difference between the true and the estimated value. The estimated damping ratio, $\hat{\zeta}$ is given by

$$\hat{\zeta} = 2^\frac{3}{2} \left( \frac{2}{3} \right)^4 \frac{\Omega_{dwn} \left( \hat{\Omega}_{dwn} - 1 \right)^{\frac{3}{2}}}{\left( \hat{\Omega}_{dwn}^2 - 1 \right)^{\frac{1}{2}}}. \quad (4.49)$$

Substituting Equations (4.44) and (4.48) into Equation (4.49) for $\hat{\Omega}_{up}$ and $\hat{\Omega}_{dwn}$ yields

$$\hat{\zeta} = 2^\frac{3}{2} \left( \frac{2}{3} \right)^4 \frac{(\Omega_{dwn} - \nu) [(\Omega_{up} - 1) - \xi]^\frac{3}{2}}{[(\Omega_{dwn} - \nu)^2 - 1]^\frac{1}{2}}. \quad (4.50)$$

Factorizing and re-arranging this equation gives

$$\hat{\zeta} = \zeta \left( \frac{\hat{\Omega}_{dwn}}{\Omega_{dwn}} \right) \left( 1 - \frac{\Omega_{up} - \hat{\Omega}_{up}}{\Omega_{up} - 1} \right)^{\frac{3}{2}} \frac{1 + \frac{\hat{\Omega}_{dwn}^2 - \Omega_{dwn}^2}{\Omega_{dwn}^2 - 1}}{\left( \frac{\hat{\Omega}_{dwn}^2 - \Omega_{dwn}^2}{\Omega_{dwn}^2 - 1} \right)^{\frac{1}{2}}}, \quad (4.51)$$
where $\zeta$ is the true damping ratio. Expressing this equation in terms of physical frequencies gives

$$
\hat{\zeta} = \zeta \frac{\left(\hat{\omega}_{\text{dwn}}/\omega_{\text{dwn}}\right) \left(1 + \omega_{up}/\omega_{up} - \omega_{n}/\omega_{dwn} - \omega_{n}^2 / \left(1 + \omega_{dwn}^2 - \omega_{dwn}^2\right)\right)^{\frac{3}{2}}}{\left(1 + \hat{\omega}_{\text{dwn}}^2 - \omega_{dwn}^2\right)^{\frac{3}{2}}}.
$$

Equation 4.52 implies that the damping ratio $\zeta$ depends on the jump-up and the jump-down frequencies. Based on this equation, errors in the estimates of the jump-up frequency and the jump-down frequency result in errors in the estimate of the damping ratio.

### 4.7 Practical constraints

One of the main differences between a linear and a non-linear system is that a non-linear system can produce multiple stable solutions which are dependent on a set of initial conditions. A linear system however always produces a unique stable solution for all initial conditions.

In the frequency range where multiple stable solutions exist, different initial conditions give different steady state solutions. If an inappropriate initial condition is applied to the system, the steady state solution converges to the lower of the two possible values. To study the effect of the initial conditions on the steady state response of the hardening system, the basins of attraction are produced for each case shown in Figure 4.8. The basins of attraction are shown in Figures 4.9 - 4.11. The basin of attraction is shown in such a way that the

- white region gives the initial conditions that result in the response being on the resonant branch (larger-amplitude),
- dark region gives the initial conditions that result in the response being on the nonresonant branch (smaller-amplitude).

For the case where $\alpha = 0.0025$ (small non-linearity), at a frequency reasonably close to the jump-down frequency ($\Omega_{\text{dwn}} \approx 1.12$), there is a fairly large set of initial conditions
for which the steady-state solution is on the resonant branch as shown in Figure 4.9. For the case with a larger non-linearity i.e. $\alpha = 0.005$, at a frequency reasonably far from the jump-down frequency ($\Omega_{dwn} \approx 1.26$), there is a fairly large set of initial conditions for which the steady-state solution is on the resonant branch as shown in Figure 4.10(a). As the frequency gets closer to the jump-down frequency, the set of initial conditions for which the steady-state solution is on the resonant branch reduces as shown in Figure 4.10(b).

When the non-linearity gets larger, e.g. $\alpha = 0.0075$, the change in the area of the basin for the resonant branch decreases more dramatically from frequencies far from and close to the jump-down frequency as shown in Figure 4.11. Thus in practice, the bandwidth of the hardening mechanism may be reduced due to the dependency on the initial conditions.

### 4.8 Conclusions

This chapter investigated another mechanism for an energy harvesting device consisting a non-linear hardening spring. This type of spring has the capability of shifting the resonance frequency to a higher frequency by bending the FRC. The bandwidth of the system increases due to this.

The harmonic balance method was used to obtain an approximate analytical solution to the equation of motion. An analytical expression for the bandwidth of the system with a hardening spring was derived. Unlike the linear system in which the bandwidth varies linearly with the damping ratio, the bandwidth of the hardening mechanism varies with both the damping and the non-linearity. Based on the results of simulation, the bandwidth of the hardening mechanism is wider than the linear system. This bandwidth also increases with the non-linearity.

This chapter also described a new frequency domain method to estimate the amount of linear viscous damping in the hardening system. A simple analytical expression for the damping ratio was derived using the jump-up and the jump-down frequencies. This expression is reasonably accurate if the fundamental harmonic is more dominant than the
higher harmonics. This derivation of the damping ratio ensures that the estimate of the bandwidth can be made if the two jump frequencies are known, since the non-linearity itself can be computed from the jump-up frequency.

Although the bandwidth of the hardening mechanism was found theoretically to be larger, it may not be achievable in practice. This is due to there being multiple steady-state solutions for the non-linear system in which each solution depends on a set on initial conditions. This set of initial conditions is easily applied when the frequency region of interest is far from the jump-down point. However, it is not easy to apply an appropriate set of initial conditions as the frequency region of interest gets closer to the jump down point which deteriorates the performance of the hardening mechanism to a certain extent.

Experimental investigations of the device with snap-through and hardening mechanism are carried out in the next chapter. This experimental investigation mainly investigates the effect of damping, input level and the non-linearity on the response on the system. For the hardening mechanism, in particular, the estimation of the damping is validated and the bandwidth of such system is compared with the one with a linear mechanism.
Figure 4.1: (a) Force-deflection and (b) stiffness-deflection characteristics of the hardening spring.
Figure 4.2: Typical frequency response curve for a hardening system. The solid line represents the stable branch, the dashed line represents the unstable branch and the dashed-dotted curve represents the backbone curve. The red arrows shows the amplitude of the response for sweeping-up frequency and the blue arrow shows the amplitude of the response for sweeping-down frequency. The jump-down occurs at point B and the jump-up point occurs at point E.

Figure 4.3: Frequency response curve when $\alpha = 0.5 \times 10^{-4}$ plotted for different damping ratios: $\zeta = 0.005$ (black), $\zeta = 0.0075$ (red) and $\zeta = 0.01$ (blue).
Figure 4.4: The relative transmissibility (black) and the absolute transmissibility (red) plotted for $\zeta = 0.0075$ and $\alpha = 0.5 \times 10^{-4}$.

Figure 4.5: Typical frequency response curve for a hardening system with; $U_m$ and $\Omega_m$ are the maximum response and frequency at which the maximum response occurs respectively, $U_{dwn}$ and $\Omega_{dwn}$ are the jump-down amplitude and the jump-down frequency respectively, $U_{hpb}$ and $\Omega_{hpb}$ are the half-power point amplitude and frequency respectively (not to scale).
Figure 4.6: Bandwidth of a linear (—) and hardening systems; $\alpha = 1.0 \times 10^{-4}$ (—), $\alpha = 2.5 \times 10^{-4}$ (···) and $\alpha = 4.0 \times 10^{-4}$ (−·−).

Figure 4.7: The true and estimated values of the jump frequencies where $\Omega_{up}$ is the true value of the jump-up frequency, $\hat{\Omega}_{up}$ is the estimate of the jump-up frequency, $\Omega_{down}$ is the true jump-down frequency, $\hat{\Omega}_{down}$ is the estimate of the jump-down frequency, $\xi$ is the difference between the true and estimated value of the jump-up frequency and $\nu$ is the difference between the true and the estimated jump-down frequency.
Figure 4.8: Frequency response curve for the hardening system with $\zeta = 0.05$ for (a) $\alpha = 0.0025$, (b) $\alpha = 0.0050$ and (c) $\alpha = 0.0075$. 
Figure 4.9: Basin of attraction for the hardening system with $\alpha = 0.0025$ and $\zeta = 0.05$ corresponding to the FRC in Figure 4.8(a) when $\Omega = 1.10$ (jump-down frequency, $\Omega_{\text{down}} \approx 1.12$).
Figure 4.10: Basin of attraction for the hardening system with $\alpha = 0.005$ and $\zeta = 0.05$ corresponding to the FRC in Figure 4.8(b) when (a) $\Omega = 1.17$ and (b) $\Omega = 1.25$ (jump-down frequency, $\Omega_{\text{down}} \approx 1.26$).
Figure 4.11: Basin of attraction for the hardening system with $\alpha = 0.0075$ and $\zeta = 0.05$ corresponding to the FRC in Figure 4.8(c) when (a) $\Omega = 1.20$, (b) $\Omega = 1.40$ and (c) $\Omega = 1.50$ (jump-down frequency, $\Omega_{\text{dwn}} \approx 1.52$).
Chapter 5

Experimental investigation of the non-linear mechanisms

5.1 Introduction

Theoretical analysis in Chapters 3 and 4 has shown some benefits of having a stiffness non-linearity in an energy harvesting device. In particular, it has been shown that the bandwidth over which the device can operate around the maximum response can be improved by having a hardening type spring. A system incorporating a negative stiffness (bi-stable mechanism) also promised some benefits as it is relatively insensitive to the excitation frequency especially when dealing with low frequency applications.

This chapter presents experimental results for both hardening and bi-stable mechanisms, although there is more focus on the hardening mechanism. The experiments were conducted in Bristol University by the author, as the device was readily available there and some studies on this device had previously been published by Burrow et. al. [38, 121].

For the hardening configuration, the experimental studies focussed on two main parts. The first part was to investigate the correlation of the dynamics between the theoretical and the measured responses of the device. This included investigating the effects of the non-linearity and the damping on the response of the system. The dynamic behaviour of the device under different levels of random excitation was also investigated. The second
part focused on the energy harvesting potential of the system which includes the optimal power harvested and the estimation of the bandwidth.

The bi-stable mechanism is discussed in the final part of the chapter. However, the experimental studies on this mechanism were not as detailed as the one with the hardening mechanism. The aim is only to show that the mechanism is realizable and the effect of the non-linearity, damping and the level of input on the response of such a system.

5.2 Device Configuration

Photographs of the device are shown in Figure 5.1. Figure 5.1(a) shows a full view of the device which comprises two main parts. The first part consists of a beam fixed at one end and, with a mass and four magnets attached at the other end. The second part is made up of an iron core wrapped around by a copper coil. The arrangement of the magnets on the moving part is shown in Figure 5.1(b). This arrangement allows the continuous flow of flux between the magnets and the iron core. As the beam oscillates, a voltage is induced in the coil and energy can be harvested by attaching a suitable electrical circuit.

The total stiffness of the system is the combination of the positive stiffness from the beam and the negative stiffness from the magnets resulting in a non-linear stiffness characteristic. The two main parts are separated by a gap, \( d \) which controls the degree of non-linearity. When the gap is large the system behaves as a hardening system and when the gap is small the system behaves as a bi-stable system. The word ‘large’ and ‘small’ are used for relative comparison between the two systems only. In practice, the total stiffness is very sensitive to the gap \( d \) as the system can change from being a hardening system to a bi-stable system when the gap is decreased by only 1 mm.

The system behaves like a linear system for a small or moderate deflections when the coil is removed as the stiffness of the system is now due to the stiffness of the beam only. The force-deflection characteristics for linear, hardening and bi-stable systems are illustrated in Figure 5.2 to illustrate the differences between the three systems.
5.3 Experimental Setup

The experiment may be categorized into two parts. The first part was the quasi-static measurement of the deflection of the beam and the restoring force of the beam. The second part of the experiment was the dynamic measurement of the output acceleration of the mass, the output voltage across the coil and the input displacement of the base.

5.3.1 Static Measurement

In the quasi-static measurements, the force-deflection relation of the system was measured to determine the stiffness characteristics together with the amount of non-linearity. A force gauge was used to measure the restoring force of the beam as the mass attached to the beam was driven by a shaker with a high amplitude displacement at a very low frequency (< 1 Hz). The force gauge was connected to the beam using a connecting arm and the deflection of the tip of the beam was measured using a linear variable differential transformer (LVDT). The experimental setup for the quasi-static measurement is shown in Figure 5.3.

5.3.2 Dynamic Measurements

For the dynamic measurements, the whole device was placed onto a shaker so that the system was base-excited rather than force excited. The frequency was increased from 10 Hz to 40 Hz in 1 Hz steps and then decreased from 40 Hz to 10 Hz with the same frequency increment. Since the system is non-linear, the amplitude of the input displacement had to be kept constant for all frequencies. It was kept approximately constant at a desired value by a feedback controller. The damping in the system could be altered by changing the external electrical resistance connected to the coil using a resistor box. The electrical damping is inversely proportional to the resistance, so the larger the resistance, the smaller the damping. A PCB accelerometer was used to measure the acceleration of the tip mass, which was recorded together with the input displacement and the voltage across the resistor at each frequency.
For the measurement of the transfer functions between the input and the outputs (i.e. the ratio between the absolute acceleration of the mass and the input displacement, and the ratio between the voltage induced in the coil and the input displacement) when the system operates in the linear region, the shaker was excited with a low level random displacement input using a Data Physics signal analyzer as the signal generator. The experimental setup for the dynamic measurement is shown in Figure 5.4 and a photograph of the arrangement is shown in Figure 5.5.

5.4 Results and discussion

The results from the measurement with the hardening and bi-stable configurations are presented in this section. The aim of the measurement was mainly to validate the theoretical studies described in Chapters 3 and 4. Some of the results are used to compare qualitatively with those of the theoretical studies while others are used for quantitative comparison.

5.4.1 Hardening configuration

The dynamics of the system were first observed by investigating the response of the system for differing degrees of damping and non-linearity. The capability of the device to perform under random excitation is also presented. The accuracy of the analytical expressions for the damping and bandwidth is also examined. The last part of the experimental studies on the hardening mechanism were conducted by investigating the optimal resistance for harvested power; the comparison with the equivalent linear system is demonstrated.

The configuration of the device used in the experiment can be categorized into two categories. The first category is mainly used to study the effect of the length of the beam and the gap on the non-linearity of the system. The configuration can be further divided into two types, each is made up of different length and gap. In the first type, the length of the beam was fixed at 41.30 mm and the gap, $d$ was maintained at 1.30 mm.
In the second type, the beam was made shorter by 0.15 mm resulting in an increase in the gap by the same amount.

The second category, on the other hand, is used mainly to investigate the non-linearity based on the change in the length of the beam only. In this category, the gap was maintained at 1.50 mm. The second category of the device configuration is made up of four different types. The first two types have the same length, i.e. \( L = 42.70 \) mm. The only difference between these two is the level of input displacement used during the dynamic measurement where the input level of the first type was kept at 0.1 mm while the input for the second type was 0.2 mm.

The last two types of the device configuration have a slightly longer beam, i.e \( L = 47.30 \) mm, while the gap \( d \), was maintained at 1.50 mm, similar to the previous two types. The input displacement of type 3 was 0.1 mm and 0.15 mm for type 4. Table 5.1 summarizes the configurations of the device used in the experiment. To simplify the process of describing each configuration, the system is named by its main category followed by its sub-category. For example, \( A1 \) refers to a system with \( L = 41.30 \) mm and \( d = 1.30 \) mm.

### 5.4.1.1 Effect of the non-linearity on the dynamics of the system

In this section, the effect of the non-linearity on the response of the device is investigated. It has been shown in Chapter 4 that the non-linearity in the system depends on the stiffness \( k_1 \) and \( k_3 \), as well as on the amplitude of input displacement to the system, \( Y \), i.e. \( \alpha = \frac{k_3}{k_1} Y^2 \). The study on the effect of the stiffness on the non-linearity can be divided into two parts. In the first part, the total stiffness is varied by changing both the length \( L \) and the gap \( d \), as shown by configurations \( A1 \) and \( A2 \) in Table 5.1. In the second part, the total stiffness is varied by just altering the length of the beam \( L \) while the gap \( d \) is kept constant at 1.50 mm as shown by configurations \( B1 \) and \( B3 \). On the other hand, the effect of the amplitude of the input displacement on the non-linearity is investigated by comparing systems with configurations \( B1 \) and \( B2 \).

**Non-linearity due to change in both length \( L \) and gap \( d \)** Figure 5.6 shows the measured force-deflection plot for the \( A1 \) system. The solid curve shows the measured
data from the experiment. It can be seen that the plot is not symmetric. This was thought to be due to slight rotation and bending between the connecting arm and the beam. In fitting a polynomial to this curve, symmetry was assumed in which the half-cycle where the deflection is positive is mirrored to give a symmetrical curve. This curve was then fitted using the least square method with a cubic polynomial of the form

\[ F(x) = k_1 x + k_3 x^3, \quad (5.1) \]

which is shown by the dashed curve in the figure, where \( k_1 \) and \( k_3 \) are constants.

The effect of having different lengths and gaps on the degree of the non-linearity can be investigated by plotting the force-deflection for \( A_1 \) and \( A_2 \) as shown in Figure 5.7. These curves are the cubic polynomial fitted curves using the symmetry assumption which gives the expression for the spring force as a function of the tip deflection as in Equation (5.1) as

\[ F(x) \approx 1221 x + 5.82 \times 10^7 x^3, \quad A_1, \quad (5.2a) \]
\[ F(x) \approx 1408 x + 4.52 \times 10^7 x^3, \quad A_2. \quad (5.2b) \]

Based on these equations, the system with configuration \( A_1 \) has a smaller linear stiffness and a larger non-linearity. The difference in the linear stiffness in both systems is due to a slight change in the effective length of the beam, \( L \), as the gap \( d \) changes and the difference in the non-linearity is due to the change in the softening effect of the negative stiffness of the magnets as the gap changes. This can be clearly observed in Figure 5.8 which shows the stiffness for both systems. The stiffness is obtained by differentiating Equation (5.2) with respect to \( x \). It can be seen that, although the \( A_2 \) configuration has a larger linear stiffness, the \( A_1 \) configuration becomes harder as the deflection increases due to the larger non-linearity.

Figure 5.9 shows the acceleration, displacement and the voltage responses of the \( A_1 \) and \( A_2 \) configurations. As has been investigated theoretically in Chapter 4, the system with a larger degree of the non-linearity jumps up and jumps down at higher frequencies. However, in this case, it is not a fair comparison to look at the jump-up and jump-down frequencies by just considering only the degree of the non-linearity. The comparison
is valid when both systems have the same linear stiffness, which is not the case here. Due to this, the response starts to increase at a different frequency which decreases or increases the jump-up and jump-down frequencies depending on the size of the linear stiffness. This explains the reason why the A1 system with $d = 1.3$ mm has a much lower jump-up and jump-down frequencies even though the degree of the non-linearity is larger.

**Non-linearity due to change in length $L$ only** In this section, the change in non-linearity due to the length of the beam is studied. This is achieved using $B1$ and $B3$ configurations, in which the gap, $d$ for both cases is maintained at 1.50 mm. The force-deflection and the stiffness-deflection plots for the two systems are shown in Figures 5.10 and 5.11 respectively. These force-deflection curves are the cubic polynomial fitted curves using the assumption that the system is symmetrical. The quantitative relationship between the the force and the deflection for both configurations are respectively given by

$$F(x) \approx 1495x + 4.26 \times 10^7x^3, \quad B1,$$  \hspace{1cm} (5.3a)

$$F(x) \approx 946x + 3.30 \times 10^7x^3, \quad B3.$$  \hspace{1cm} (5.3b)

In this case, the negative stiffness due to the magnets is similar for both cases due to the gap being the same. Thus, the total stiffness is strongly determined by the linear stiffness from the beam, which is given by

$$k_{eff} = \frac{3Y_{ym}I}{L^3},$$  \hspace{1cm} (5.4)

where $k_{eff}$ is the effective stiffness of the beam, $Y_{ym}$ is the Young’s modulus and $I$ is the second moment of area. This equation shows that the effective stiffness of the beam decreases with the length. Due to a shorter length of the beam, the system with $B1$ configuration has a larger linear stiffness as well as a larger non-linearity. The dynamic responses for $B1$ and $B3$ configurations are shown in Figure 5.12. This figure shows that system $B1$ jumps up and jumps down at a higher frequency because of the larger linear stiffness and the non-linearity.
Non-linearity due to the input displacement  The dependency of the non-linearity on the amplitude of the input displacement is investigated in this section. Systems with $B1$ and $B2$ configurations are used to compare the effect of the amplitude of the input displacement on the non-linearity. The amplitude of excitation for $B1$ is 0.1 mm while the amplitude of excitation for $B2$ is doubled, i.e. 0.2 mm. The measured responses of both configurations are shown in Figure [5.13]. It can be seen that the $B2$ configuration has a larger non-linearity due to having a larger input because $\alpha = k_3/k_1 Y^2$, which is shown by the higher jump-up and jump-down frequencies.

5.4.1.2 Effect of damping

The analytical expressions for the jump-up and jump-down frequencies presented in Chapter 4 shows that the jump-up frequency depends only on the non-linearity while the jump-down frequency depends on both the damping and the non-linearity. Figure [5.14] shows the measured response of the open-circuit system and that with a 200 Ohm resistance for different configurations. From these figures, it can be seen that the jump-up frequency for all configurations depends strongly on the non-linearity and not on the damping as the response for each configuration jumps up at the same frequency for both the open-circuit and 200 Ohm systems. These figures also show that the jump-down frequency depends on both the damping and the non-linearity. For a given non-linearity, the jump-down frequency is inversely proportional to the damping. In all cases, the jump-down frequency for the open-circuit system is much larger than that with the 200 Ohm resistance attached.

5.4.1.3 Dynamics of the system under random excitation

Up to now, all the responses of the device were due to tonal excitation only. In this section, the performance of the hardening system under random excitation is investigated. The device was excited under different levels of random displacement from 10 Hz up to 40 Hz with a frequency resolution of 50 mHz. The random signals used in the test were white noise which had a uniform spectral density across the range of frequency selected. The transfer function between the input displacement and the output displacement was
calculated using 5 averages with 50% overlap. Figures 5.15 - 5.17 show the response of the hardening system with different amounts of damping and under different level of random input displacement. It can be seen in all three figures that the system behaves linearly when the excitation is small. When the excitation gets larger, the response is unable to be on the resonant branch and starts to be distorted as shown in Figure 5.15(d). When the excitation level is further increased, the system behaves non-linearly even at low and high frequencies i.e. outside the multiple stable solutions region. However, the responses become more settled for the system with a larger amount of damping as shown by Figure 5.17. These figures show that since the system is non-linear and the excitation is random, the response of the system is unable to produce a resonant and non-resonant as in the case with tonal excitation. The random excitation also means that the response at a particular frequency depends also on the excitation at other frequencies.

5.4.1.4 Estimation of the linear viscous damping

In this section, the validity of the simple expressions for the non-linearity and the damping ratio presented in Chapter 4 is investigated using the measured jump-up and jump-down frequencies. The comparison between the measured and simulated responses are conducted for the $B_1$ system. The accuracy of the estimation of both quantities using those analytical expressions is important so that the estimation of the bandwidth discussed in Section 5.4.1.6 is reliable.

Comparison between the simulated and measured responses Figure 5.14(b) shows the measured responses for the open-circuit system and that with a 200 Ohm resistance. As can be seen in the figure, the jump-down frequencies of both configuration lie between 29 Hz - 30 Hz and 27 Hz - 28 Hz for the open-circuit system and that with 200 Ohm resistance, respectively. However, the jump-up frequency for both configurations lies between 26 Hz - 27 Hz.

For comparison between the simulated and the measured responses, the assumed true values for the jump-up and jump-down frequencies are chosen based on the vertical tangent of the jump-up and jump-down points as shown in Figure 4.2 in Chapter 4. Thus,
the jump-down frequency for the open-circuit system is chosen to be 30 Hz rather than 29 Hz, and 28 Hz instead of 27 Hz for the one with 200 Ohm resistance. The jump-up frequency for both systems is chosen as 27 Hz rather than 26 Hz. These of course are only upper bounds. The linear natural frequency was determined experimentally from the transmissibility. The transmissibility was obtained by exciting the device with a low amplitude random input displacement (0.0054 mm (rms)) so that the device was still operating in the linear region. The transmissibility and coherence are plotted in Figure 5.18. The non-linearity and the damping ratio were estimated using Equations 4.42 and 4.43 respectively. These parameters were used to compare the simulated and the measured responses.

Figures 5.19(a) and 5.19(b) show the plot of the simulated and measured responses for the open-circuit system and that with a 200 Ohm resistance, respectively. The estimated value of the non-linearity and the damping ratio using measured jump and linear natural frequencies are shown in Table 5.2. Generally, using the values for the non-linearity and the damping ratio estimated from the measured values of the jump and the linear natural frequencies shown in Table 5.2 provides good estimates of the responses, simulated and measured, which agree quite well. The largest error between the simulated and the measured responses occurs between 24 Hz to 26 Hz for both configurations. This error is maybe due to the error in estimating the jump-up and jump-down frequencies used to estimate the non-linearity and the damping ratio. Another possible source of error is due to the assumption that only the fundamental harmonic dominates the response with the higher harmonics being neglected. Figure 5.20 shows the harmonic content of the acceleration response at each frequency. It can be seen that the responses are composed of the second and third harmonics as well as the first harmonic, and in some frequency regions, their contributions to the responses are significant. It is thought that this was because of the asymmetry of the system because of gravity acting on the mass rather than asymmetry of the stiffness.

The accuracy of the estimate of the non-linearity is also examined by comparing the values determined using the dynamic (i.e. jump-up and jump-down frequencies) and
the quasi-static measurements, which is given by

\[
\alpha = \frac{k_3}{k_1} Y^2 = 2.85 \times 10^{-4},
\]  

(5.5)

where \( k_1 = 1495 \text{ Nm}^{-1} \), \( k_3 = 4.260 \times 10^7 \text{ Nm}^{-3} \) and \( Y = 0.1 \text{ mm} \). Comparing this value to the one reported in Table 5.2, it can be seen that the difference between these two quantities is approximately 5%.

The estimate of the damping ratio using the jump frequencies is shown to be accurate as well. This is shown in Figure 5.21 by comparing the values of the damping ratio estimated using this method with those from the log-decrement method and from the peak value of the transmissibility. Generally, the amount of damping estimated using the jump frequencies is comparable to those determined using the log-decrement and the peak transmissibility methods provided that the response is dominated by the fundamental harmonic. In the case where the contributions of higher harmonics are significant, this method may be of limited value.

The accuracy of the estimate of the non-linearity and the damping provides a reliable estimate of the bandwidth of the of the hardening system using the jump-up and the jump-down frequencies and this will be discussed in Section 5.4.1.6.

**Sensitivity of the non-linearity and the damping ratio estimates on the jump frequencies**  The sensitivity of the estimated non-linearity and the damping ratio on the jump frequencies is studied in this section using Equations 4.47 and 4.51 derived in Chapter 4. As has been discussed in the previous section, the jump-down frequencies of the open-circuit system and that with a 200 Ohm resistance may lie between 29 Hz - 30 Hz and 27 Hz - 28 Hz, respectively. The jump-up frequency for both systems may lie between 26 Hz and 27 Hz.

Due to the vertical tangency mentioned before, the true values for the jump-down frequencies are assumed to be respectively given by 30 Hz and 28 Hz, while the true values of the jump-up frequencies for systems are assumed to be 27 Hz. Figure 5.22 shows the relative error which may exist when the jump-up frequency deviates away from the
assumed true value. For this particular configuration, this figure shows that the error in estimation of the non-linearity can be as much as 90% with only 1 Hz deviation from the assumed true value of the jump-up frequency. The effect of error in measuring the jump-up frequency is further illustrated in Figure 5.23 for the system with 200 Ohm resistance. This graph is plotted by using three different possible jump-up frequencies while the jump-down frequency is fixed at 28 Hz. This figure shows how the frequency response curve is affected by the erroneous estimate of the jump-up frequency. This figure shows that the assumed true value of the jump-up frequency, i.e. 27 Hz gives the closest agreement with the measured response compared with other two values of the jump-up frequencies, i.e. 26 Hz and 26.5 Hz. Having a jump-up frequency which is less than the assumed true value results in a weaker non-linearity in the system. The weaker the non-linearity the larger the response of the system around resonance.

Equation 4.43 in Chapter 4 implies that the estimation of the damping ratio in the system depends on both the jump-up and the jump-down frequencies. Figures 5.24 and 5.25 show the relative error in the estimate of the damping ratio with respect to both the jump-up and the jump-down frequencies for the system with 200 Ohm resistance and the open-circuit system, respectively. In general, two conclusions can be made from these figures. The estimate of the damping ratio is much less sensitive to changes in the jump-down frequency than in the jump-up frequency. This can be observed by inspecting the error at constant $\hat{\omega}_{up}$ or at constant $\hat{\omega}_{dwn}$. For a fixed value of the jump-up frequency, i.e. 27 Hz, for example, the error is only around 30% for the system with 200 Ohm resistance while in the open-circuit, the error is less, at around 10% as $\hat{\omega}_{dwn}$ changes from 27 Hz to 28 Hz and 29 Hz to 30 Hz, respectively. However, for a fixed jump-down frequency, the error in the estimate is much more sensitive to variations in the jump-up frequency. This can be observed in these two figures by fixing the assumed true values of the jump-down frequencies at 28 Hz (200 Ohm) and 30 Hz (open-circuit) while varying the jump-up frequencies for both cases from 27 Hz down to 26 Hz. In both cases, the error in measuring the jump-up frequency may lead up to 85% error in the prediction of the damping ratio.
5.4.1.5 Power harvested

The maximum potential power harvested by the hardening mechanism is now compared to that harvested using the linear mechanism. The maximum power harvested by the hardening mechanism is determined by measuring the voltage across the resistance at a frequency of 1 Hz below the jump-down frequency. This is because it is not possible to measure a large steady state response at the jump down point since the change in the response, from large amplitude to low amplitude, is very rapid and the response is only in steady state when it converges to the lower amplitude oscillation. The maximum power harvested by the equivalent linear mechanism could not be measured exactly using the same input as the hardening mechanism. This was because the region in which the device behaves linearly was small and limited to a small input only.

For a fair comparison, the linear mechanism was assumed to have the same stiffness as the linear stiffness in the hardening mechanism. This was done by assuming that there is only the linear stiffness component of the hardening mechanism acting across the whole range of the deflection, -5 mm to 5 mm, as shown by the dashed line in Figure 5.26. The voltage generated across the resistance is first determined by using the transmissibility plot for a very small random input displacement (e.g. 0.0054 mm (rms)), so that the system still operates in the linear region. Since the system is linear, the ratio between the input displacement and the generated voltage is always constant. Thus the generated voltage due to the same input as applied to the hardening system can be computed using this ratio.

Figures 5.27 and 5.28 show the power harvested by both hardening and linear mechanisms for different resistances when the amplitude of the input displacement is 0.1 mm (B1) and 0.2 mm (B2), respectively. Figure 5.29 shows the maximum powers harvested by both mechanisms by the load resistance under different excitation levels. It can be seen that, for the linear system, the optimal resistance for maximum power transfer is around 200 Ohm and 90 Ohm for the hardening system.

The condition for maximum power transfer for the linear system employing electromagnetic transduction has already been established [122, 123], while the study on the
maximum power transfer of the hardening system is not yet available. However, the interpretation of this optimum power harvested by the hardening system has to be done carefully. For the linear system, the maximum power occurs at approximately the same frequency for all resistances. For a given non-linearity, the maximum power harvested by the hardening mechanism occurs at different frequency (i.e. the jump-down frequency) depending on the amount of damping (i.e. the resistance) in the system. Thus the interpretation of the optimal resistance and frequency is not straightforward as the linear system and the tuning maybe more complicated. In the experiment, the jump-down point can be reached easily by sweeping the frequency up. In practical application with a tonal input frequency, it may not be easy to drive the system on the resonant branch in the region between the jump-up and jump-down points without applying the appropriate initial conditions.

5.4.1.6 Bandwidth estimation

Referring to Chapter 4, the most significant advantage of the non-linear hardening mechanism over the linear mechanism is that the non-linear mechanism can provide a wider bandwidth over which the device can be useful. This is due to the capability of the mechanism to bend the frequency response curve (FRC) to the right. The maximum shifting of the resonance frequency (i.e. the jump-down frequency) depends on the non-linearity because of the existence of the cubic stiffness, \( k_3 \), in addition to the linear stiffness, \( k_1 \). The amount of damping in the system also plays an important role in shifting this resonance frequency; for a given non-linearity, the smaller the damping the higher the jump-down frequency.

The accuracy of the estimates of the non-linearity and the damping ratio discussed in Section 5.4.1.4 is proven to be quite good. This allows a reliable estimate of the bandwidth to be made using the expression derived in Chapter 4 by just using the jump-up and the jump-down frequencies. Figure 5.30 shows the comparison between the bandwidth of the linear and the hardening mechanisms as a function of load resistance (i.e. damping). It is shown that bandwidth of the hardening system is better than the linear system even when the non-linearity is relatively small, as shown by the dashed line.
It can also be seen that the bandwidth is better when the damping in the system is small (i.e. large resistance). The bandwidth of the hardening system also depends strongly on the non-linearity. The bandwidth of the system with a stronger non-linearity is shown by the dotted line which represents the same system when the input displacement is doubled at 0.2 mm.

This study shows that the device can be made useful over a wide range of frequency by having a large non-linearity and a small damping in the system, although, in real applications, the performance of the hardening mechanism may be reduced due to the response not being on the resonant branch in the region between the jump-up and jump-down points if inappropriate initial conditions are applied to the system.

5.4.2 Bi-stable configuration

The device is configured to behave as a bi-stable mechanism by using a smaller gap, \( d \). By having this smaller gap, the mechanism has two stable positions and one unstable position. Depending on the input level, non-linearity, damping and the initial conditions, the mass will either oscillate about one of the stable positions or across the two potential wells overcoming the central potential barrier. The comparison of the performance between the hardening and bi-stable mechanism is not possible because the input level used in the hardening mechanism before was not strong enough to throw the mass between the two stable positions in the bi-stable mechanism.

The experimental results for the bi-stable mechanism are presented in this section. However, the discussion of the results is not as detailed as for the hardening mechanism. The main objective is to show that the device which has this kind of mechanism is physically realizable. Technically, the aim of the experiment was to investigate how the dynamics of such system is affected by the non-linearity, input level and the damping. The behaviour of the device under different levels of random input displacement is also presented.

For the bi-stable mechanism, the smaller the gap \( d \) means the higher the non-linearity in the system. The larger the non-linearity, the more difficult it is to snap the mass from one stable position to another. Figure 5.31 shows the measured force-deflection curve.
when the gap was 0.6 mm. It can seen that the system behaves completely differently by having a small change in length $L$ and the gap $d$. This figure also shows that the stiffness of the system changes from positive to negative between the two turning points of this curve.

5.4.2.1 Effect of input displacement, damping and the gap on the response

Figure 5.32 shows the measured peak acceleration response of the bi-stable mechanism. It can be seen that the response is initially very small at low frequencies. This shows that the mass oscillates in one of the potential wells only. The reason for this has already been discussed in Chapter 3. When the frequency is further increased the amplitude of the response increases suddenly which shows that the oscillation of the mass is between the two potential wells. The frequency where the mass starts to oscillate between the two potential wells depends on the level of input, the amount of damping and the gap i.e. the non-linearity. It is shown in Figure 5.32 that the cross-well motion is initiated in the system with a larger input at a lower frequency than the one with a smaller input since the one with the larger input has more energy to overcome the potential barrier even at low frequency.

The effect of damping (i.e. resistance) on the response is shown in Figure 5.33. With the same input level, the one with a smaller damping jumps between the two equilibrium positions at a lower frequency than the one with a larger damping. This due to the less resistance to motion provided by the system with smaller damping.

Referring to Chapter 3, the larger the non-linearity the more difficult it is for the mass to oscillate across the two potential wells. This effect is shown in Figure 5.34 which compares the response for the system with different gaps under the same level of input displacement. As has been mentioned previously, the smaller the gap the larger the non-linearity. This explains why the response for the system with $d = 0.8$ mm starts to cross between the two potential wells at a lower frequency than the one when the gap is 0.6 mm. It can also be seen that after the jump occurs, the responses start to oscillate at an approximately constant level for a certain frequency region and then start to vibrate about one of the equilibrium position again when the frequency is further increased.
This behaviour is may be due to the increase in the impedance of the mass which makes it more difficult for the mass to move at high frequencies.

5.4.2.2 Response of the device under random excitation

The response of the bi-stable mechanism under different levels of random displacement input are very similar to that of the hardening mechanism. When the level of excitation is small, the system behaves linearly as the mass only vibrates about one of the equilibrium position with a very small amplitude. The transmissibility of the system with 0.6 mm gap are illustrated in Figures 5.35 and 5.36 for the open-circuit system and that with 200 Ohm resistance, respectively. The transmissibility when the gap is 0.8 mm are shown in Figures 5.37 and 5.38. The linear behaviour of the system starts to become non-linear when the level of the input increases. The responses of the system are more controlled when the damping in the system is larger as shown by Figures 5.36 and 5.38. However, it can be seen that response of the system is much larger at low frequencies when the input is large, especially when the damping is small as shown by the open-circuit system for both configurations (i.e. for both gaps).

5.5 Conclusions

In this chapter, the experimental results involving the hardening and the bi-stable mechanisms have been presented. For the hardening mechanism, the quasi-static measurement was conducted before the dynamic measurement to study the force-deflection characteristic of the system which could be used to estimate the non-linearity. For the dynamic measurement, the effect of the non-linearity and damping on the response has been investigated to validate the theoretical results. The non-linearity in the system was varied in two ways. Firstly, the non-linearity was varied by changing the stiffness of the system. Secondly, the non-linearity was varied by changing the level of input to the system. In both cases, the stronger the non-linearity the higher the jump-up and jump-down frequencies are.

The amount of damping in the system was altered by changing the resistance across
the coil. The experiment showed that for the same hardening mechanism, altering the damping would only vary the jump-down point as the jump-up point was proven not to depend significantly on the damping. The performance of the system under random input has also been investigated by applying different levels of random input displacement. It was found that under large level of random input displacement, the hardening mechanism was not able perform as well as with the tonal excitation.

This chapter also presented a study to compare the measured and simulated responses of the hardening system. This comparison was made in order to validate the estimates of the non-linearity and the damping ratio using the jump frequencies derived in Chapter 4. Based on the study, it was shown that the estimates of non-linearity and the damping ratio are quite accurate due to the good agreement between the measured and the simulated response. The accuracy in the estimate of the non-linearity was confirmed by comparing the value determined using the force-deflection characteristic from the quasi-static measurement and that obtained using the jump-up frequency from the dynamic measurement.

On the hand, the accuracy in the estimate of the damping ratio was validated by comparing the values determined from the jump frequencies with that computed using the log-decrement method and the peak of the transmissibility. The good accuracy in the estimates of the non-linearity and the damping ratio allows the bandwidth to be predicted by just using the jump frequencies. Based on the comparison with the linear mechanism, the hardening mechanism has been proven to provide a wider bandwidth. This was the case with the tonal excitation frequency provided that appropriate initial conditions were applied to the system so that the steady state response converged to the resonant branch. Note that this bandwidth is the maximum value that can be obtained by that particular configuration. In practice, this performance could easily deteriorate due to the difficulty to be on the resonant branch. The bandwidth of the hardening mechanism increased with the non-linearity which was the case when the input was doubled to provide a stronger non-linearity.

The resistance for optimal power harvested has also been investigated. From the measurement, the study of the condition for the maximum power transfer for the device
containing a hardening spring is certainly needed especially under tonal excitation frequency. Unlike the device with the linear mechanism, the complication to tune the device for the optimal power harvested may arise since the maximum power for different load resistances occurs at different frequencies, i.e. the jump-down frequency.

The experimental results for the bi-stable mechanism have also been presented in this chapter. The experimental results investigated the effect of the non-linearity, the damping and the input on the response. In general, the study found that the weaker the non-linearity, the weaker the excitation level at which there was a jump in the response level, describing the cross-well motion. Similar behaviour was observed for the system with small damping. Apart from that, an increase in the level of the input would allow the cross-well motion to occur even at low frequencies. The dynamics of the system under different levels of random input displacement has also been studied to investigate the performance of such a device under random excitation. Like the hardening system, the system was not able to perform well as in the case for tonal excitation.

Apart from the random excitation measurements, all the measurements conducted in this chapter were made using tonal frequencies. In the next chapter, the performance of the linear and non-linear mechanisms is investigated by using the measured vibration from human walking. Unlike the tonal frequency input, the vibration input from human walking is composed of higher harmonics. The effect of on the response of the device by tuning the linear natural frequency with respect to the harmonics is investigated in the next chapter.
Chapter 5 Experimental investigation of the non-linear mechanisms

Table 5.1: Different configurations of the device categorized based on the length, $L$ and the gap, $d$; width = 38.21 mm, thickness = 0.5 mm and the Young’s Modulus $Y_{ym} = 200$ GPa.

<table>
<thead>
<tr>
<th>Category</th>
<th>Sub-category</th>
<th>Length $L$ (mm)</th>
<th>Gap $d$ (mm)</th>
<th>Input disp. (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>1</td>
<td>41.30</td>
<td>1.30</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>41.25</td>
<td>1.45</td>
<td>0.1</td>
</tr>
<tr>
<td>$B$</td>
<td>1</td>
<td>42.70</td>
<td>1.50</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>42.70</td>
<td>1.50</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>47.30</td>
<td>1.50</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>47.30</td>
<td>1.50</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Table 5.2: The values of the non-linearity and the damping ratio predicted using measured jump and linear natural frequencies.

<table>
<thead>
<tr>
<th>Resistance (Ohm)</th>
<th>Jump-up freq., (Hz)</th>
<th>Jump-down freq., (Hz)</th>
<th>Natural freq., (Hz)</th>
<th>Non-linearity $\alpha$</th>
<th>Damping ratio, $\zeta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$ (open-circuit)</td>
<td>27.0</td>
<td>30.0</td>
<td>25.6</td>
<td>$2.72 \times 10^{-4}$</td>
<td>0.014</td>
</tr>
<tr>
<td>200</td>
<td>27.0</td>
<td>28.0</td>
<td>25.6</td>
<td>$2.72 \times 10^{-4}$</td>
<td>0.018</td>
</tr>
</tbody>
</table>
Figure 5.1: Photographs showing the device: (a) full view and (b) the arrangement of the magnets.

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Chapter 6

Effect of harmonics on the power harvested

6.1 Introduction

In the previous chapters, the performance of the linear and non-linear devices has been studied by using a tonal frequency excitation apart from the section on random excitation in Chapter 5. Based on these studies, the performances of the non-linear snap-through and hardening mechanisms, appear to be quite promising in terms of the bandwidth. In particular, the bandwidth of the device with the hardening mechanism occurs in the region between the jump-up and the jump-down frequencies. Conversely, the snap-through mechanism is less sensitive to excitation frequency, given by the slow roll-off rate at low frequencies.

In this chapter, the performance of linear and non-linear devices is investigated for the case when the input is composed of several harmonics. For the linear device, the aim of the study is to determine the amount of power that can be harvested when the natural frequency is tuned to a higher harmonic of the excitation frequency. For the non-linear mechanism, the study focuses more on the distribution of the frequency content of the power harvested. The input used in the simulations presented in this chapter was measured from human motion, specifically walking and running, using an accelerometer.
placed on the subject’s waist.

6.2 Input measurement

The measurement of the input vibration was conducted at the Jubilee Sport Centre at Southampton University. Seven healthy male subjects of different age, weight and height [age: 28.7 ± 1.6 years, weight: 82.1 ± 7.8 kg, height: 170 ± 5.2 cm (mean ± standard deviation)] were used during the measurement. Each subject was required to walk at 5 km/hr and run at 9 km/hr on the treadmill (Star Trac Elite model 6632SOCEB4). The treadmill was used so that the experiment imitated the real situation. The walking speed was chosen based on the range of comfortable walking speeds described in [124, 125]. The running speed was chosen based on the average jogging speed among the subjects. For each task, the subjects were asked to walk and run for 60s.

A 3-axis accelerometer (Freescale Semiconductor KIT3109 MMA7260QT-ND) was glued to a belt which was tightly strapped around the subject’s waist. The data from the accelerometer was fed through a 4-channel Data Physics portable analyzer before being recorded onto a computer. Figure 6.1 shows the photograph of a subject wearing a belt which was tightly strapped around his waist. The small photograph shows the 3-axis accelerometer that was glued to the belt.

6.3 Frequency contents of the measured input

The measured signal was processed to study the frequency content of the signal. The DC component of the signal due to gravitational acceleration was first removed. The signals were assumed to be periodic. The frequency components of the acceleration signal were computed using the Fourier series [126] in the form of

\[ a(t) = \sum_{n=1}^{\infty} \{a_n \cos n\omega t + b_n \sin n\omega t\}, \]

(6.1)
where $a_n$ and $b_n$ are the Fourier coefficients, $n$ is the number of the harmonics, $a(t)$ is the measured acceleration signal and $\omega$ is the fundamental frequency of the signal. The Fourier coefficients $a_n$ and $b_n$ can be determined using

$$a_n = \frac{2}{T} \int_0^T a(t) \cos n\omega t \, dt, \quad (6.2)$$

and

$$b_n = \frac{2}{T} \int_0^T a(t) \sin n\omega t \, dt, \quad (6.3)$$

where $T$ is the period. The amplitude of the Fourier coefficients at frequency $n\omega$ can be represented as

$$C_n = \sqrt{a_n^2 + b_n^2}. \quad (6.4)$$

### 6.3.1 Walking at 5 km/hr

Figure 6.2 shows an example of the acceleration waveforms and the amplitude of their Fourier coefficients when a subject walks at 5 km/hr. It can be seen that the acceleration in the $x$-direction (vertical) is slightly larger than that in the $y$-direction (fore and aft), while the acceleration in the $z$-direction (lateral) is the smallest. Noticeable acceleration peaks occur at harmonics of 0.91 Hz up to around 20 Hz. The fundamental harmonic of the response, 0.91 Hz, is due to the foot strike frequency i.e. left to right, which is represented by the peaks in the time history plot.

### 6.3.2 Running at 9 km/hr

Similar to the signal from walking, the strength of the acceleration signal for running is most dominant in the $x$-direction as shown in Figure 6.3. The fundamental harmonic of the signal is about 1.37 Hz which again represents the foot strike pace. The peaks of the acceleration are also noticeable at increment of 1.37 Hz up to approximately 28 Hz.
6.3.3 Average signals from walking and running

Figures 6.4 and 6.5 show the mean of the frequency and amplitude of each harmonic in all 3 directions for walking and running, respectively. The signals from walking show a significant deviation from the mean value, especially for the system in the $x$- and $y$-directions for the first 10 harmonics. As mentioned before, the strength of the signals from walking in the $x$- and $y$-directions are relatively comparable as shown by the mean values in Figures 6.4(b) and 6.4(c). The mean values of the signals from running, on the other hand, are more dominant in the $x$-direction than in the other two directions.

6.3.4 Reconstruction of the signal

The measured signals from walking and running are used in the simulations presented later in this chapter to predict the amount of power that can be harvested by the linear and non-linear devices. In order for them to be used in the simulation, the measured signals need to be re-constructed. Figures 6.6 and 6.7 show the re-constructed signals from one of the subjects using the first 20 dominant harmonics for walking and running, respectively. Based on these figures, it can be seen that the measured signal can be approximately represented by the first 20 harmonics.

6.4 Methods to calculate power from an input with several harmonics

In general, there are two methods to compute the power harvested by using an input acceleration which is composed of several harmonics. They are the frequency and the time domain methods. In the simulations presented in this chapter, only the acceleration signals in the $x$-direction are considered since they are the largest. It is also assumed that the $x$-axis of the harvesting device stays in the vertical direction and the rotation in the axis is negligible.
6.4.1 Frequency domain method

The equation of motion of the linear device given in Equation 2.42 of Chapter 2 can be written as

\[ \ddot{s} + 2\zeta \omega_n \dot{s} + \omega_n^2 s = -\ddot{y}, \]  

(6.5)

where \( \ddot{y} \) is the measured signal, \( s \) is the relative displacement and \( \omega_n \) is the undamped natural frequency of the device. Representing the relative displacement and the input displacement in complex quantities as \( s = S e^{j\omega t} \) and \( y = Y e^{j\omega t} \) gives the transfer function between the input acceleration and the relative velocity as

\[ \frac{V}{A_i} = \frac{\omega}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta \omega \omega_n)^2}}, \]  

(6.6)

where \( V = \omega S \) and \( A_i = \omega^2 Y \). The power harvested by the damper is given in Equation 2.46 and also repeated here for convenience as

\[ P = \frac{1}{T} \int_0^T \dot{s}^2 dt, \]  

(6.7)

where \( T \) is the period of the excitation. Assuming that the relative velocity \( \dot{s} \) is made up of several harmonics as

\[ \dot{s}(t) = \sum_{n=1}^{N} d_n \cos(n\omega t) + e_n \sin(n\omega t), \]  

(6.8)

then squaring the signal \( \dot{s} \) gives

\[ \dot{s}^2 = \sum_{m=1}^{N} \sum_{n=1}^{N} (d_md_n \cos(m\omega t \cos n\omega t) + e_me_n \sin(m\omega t \sin n\omega t) \]  

\[ + d_m e_n \cos(m\omega t \sin n\omega t) + d_n e_m \cos(n\omega t \sin m\omega t)). \]  

(6.9)

Integrating the first two terms in the right hand side of Equation 6.9 over a period of the fundamental harmonic results in

\[ \int_0^T d_md_n \cos(m\omega t) \cos n\omega t dt = 0 \quad \text{for} \quad (n \neq m), \]  

(6.10a)
∫_0^T (d_m d_n \cos m\omega t \cos n\omega t) dt = \frac{1}{2} T d_n^2 \text{ for } (n = m), \quad (6.10b)

and

∫_0^T (e_m e_n \sin m\omega t \sin n\omega t) dt = 0 \text{ for } (n \neq m), \quad (6.11a)

∫_0^T (e_m e_n \sin m\omega t \sin n\omega t) dt = \frac{1}{2} T e_n^2 \text{ for } (n = m). \quad (6.11b)

The last two terms in Equation 6.9 are composed of orthogonal components of the signal, thus integrating these terms over a period of the excitation frequency yields

∫_0^T (d_n e_m \cos n\omega t \sin m\omega t) dt = 0, \quad (6.12a)

∫_0^T (d_m e_n \cos m\omega t \sin n\omega t) dt = 0. \quad (6.12b)

Substituting Equations 6.10b and 6.11b into Equation 6.7 gives the power harvested by a system due to the input which is composed of several harmonics as

\[ P = m\omega_n \zeta \sum_{n=1}^{N} G_n^2, \quad (6.13) \]

where \( G_n^2 = d_n^2 + e_n^2 \). Equation 6.13 implies that the power harvested by the device when excited by an input with several harmonics depends on the summation of the squared velocity components in the frequency domain and the product of the mass, undamped natural frequency and the damping ratio.

This method has the advantage of providing information about the contribution of each harmonic to the power harvested. Prior to that it also describes the contribution of each harmonic to the transfer function and the response velocity spectrum. An example is illustrated in Figure 6.8. In this particular example, the eighth harmonic is chosen as the natural frequency of the device.

Figure 6.8 shows the power harvested due to each harmonic when the natural frequency of the device is tuned to match the eighth harmonic of the input acceleration. The total power harvested by the device in this particular case can be obtained by summing up all the powers due to each harmonic, as given in Equation 6.13.
6.4.2 Time integration method

There are two ways to calculate the power harvested by the system using the time integration method. In the first method, the input signal can be decomposed into several harmonics and each harmonic is treated separately. Thus the power harvested due to each harmonic is also calculated separately using Equation 6.7. The total power harvested by the system is obtained by summing up the powers harvested by each harmonic. Using this method, the system experiences a sinusoidal input at various frequencies as shown in Figure 6.9. This way of calculating the power harvested is valid for a linear system only. It cannot be applied to the non-linear system due to the violation of the superposition theory.

In the second method, instead of decomposing the input signal into its harmonics, all harmonics are used simultaneously during the simulation to solve Equation 6.5. This way, the input signal represents the actual measured acceleration signal shown in Figures 6.2 and 6.3. The power harvested is computed using Equation 6.7. This method is used to compute the power harvested by the non-linear device later. In both frequency and time domain methods, the power harvested are normalized with respect to the mass.

6.5 Determination of the power harvested by the linear device

The determination of the power harvested by the linear system is divided into two cases. In the first case, the damping in the system is fixed while in the second case, the optimum amount of damping is used to satisfy the constraint of maximum relative displacement of the device.

6.5.1 Fixed amount of damping

Figures 6.10 - 6.13 show the simulated results of the frequency content of the relative velocity and the power harvested by the linear system under walking and running condition with 1% and 5% damping ratios. These figures are plotted in such a way that the
undamped natural frequency of the system is tuned to match the second, fifth, tenth, fifteenth and twentieth harmonics, respectively. It can be seen from these four figures that, when the damping is small (i.e. $\zeta = 0.01$), the power harvested by the system is dominated by the harmonic which is tuned to coincide with the natural frequency. As the damping gets larger ($\zeta = 0.05$), the power harvested is not only contributed to by the tuned harmonic but also has significant contributions from the neighbouring harmonics. However less power is harvested due to the decrease in the amplitude of the relative displacement.

Figure 6.14 shows the total power harvested when the natural frequency of the device is tuned to resonate at each harmonic for walking and running conditions with 1% and 5% damping ratios. The power plotted in the figure is the total power harvested. This total power is calculated by first tuning the natural frequency to match the first harmonic. The total power is obtained by summing up the power contributed by all harmonics for this particular natural frequency. This process is repeated by tuning the natural frequency to match the next harmonic.

From the figure, it can be seen that the total power harvested rolls off quickly when the natural frequency of the device is tuned to match a higher harmonic. Note that in the simulation, the damping is fixed. Thus the power calculated in Figure 6.14 is not the optimum power harvested for a given size constraint of the device.

By referring to Equation 6.6, the amplitude of the relative displacement when the natural frequency is tuned to match the excitation frequency, $\omega = \omega_n$ is given by

$$S = \frac{1}{2\zeta\omega_n^2}A_i,$$

(6.14)

where $A_i = \omega^2Y$. Thus for a fixed damping ratio, the maximum power harvested by the device decreases with $\omega_n^2$, since the maximum power harvested is directly proportional to the maximum relative displacement, $S$. 
6.5.2 Optimum damping

For a fair comparison, the power harvested is now calculated by assuming that the motion of the mass is limited by the maximum relative displacement, $S_l$. One way to ensure that the device is operating within this limit, is to vary the amount of damping in the system, as discussed in Chapter 2. As before, the optimum damping ratio can be obtained by rearranging Equation 6.14 to give

$$\zeta_l = \frac{1}{2S_l\omega_n^2}A_i, \quad (6.15)$$

where $\zeta_l$ is the optimum damping ratio and $S_l$ is the maximum relative displacement of the device and $\omega_n$ is the undamped natural frequency of the device. Substituting Equation 6.15 into Equation 6.7 gives the maximum power for a given maximum relative displacement constraint as

$$P = \frac{1}{2}m\omega_nA_iS_l. \quad (6.16)$$

Figures 6.15 and 6.16 show the optimum power harvested and the optimum damping ratio for two different maximum relative displacements, 0.5 cm and 1.0 cm. The two values of the maximum relative displacement are chosen to illustrate the condition with different limits. In general, the power harvested by the device with $S_l = 1$ cm is higher than that of 0.5 cm due to a larger maximum relative displacement as shown in Equation 6.16.

It has been claimed before in [86], that tuning the device to the harmonic in which $\omega A_i$ is maximum can generate more power than tuning to the first harmonic. This claim may be valid in terms of the amount of power that can be harvested by the device at resonance. As has been discussed in Chapter 2 before, this maximum amount of power harvested at resonance is due to the size limitation of the device. In the case where the input consists of several harmonics, one will also need to consider another design parameter, i.e. damping.

When the excitation is composed of several harmonics, the optimum damping ratio needed to comply with the maximum relative displacement limitation, differs from one
 harmonic to another. These figures also show that, although a reasonable amount of power can be harvested by tuning the device to resonate at higher harmonics compared to the first few harmonics, the damping ratio needed so that the mass oscillates within the housing of the device without the mass hitting the end of the housing decreases.

Note that up to now the term *power harvested* is the power harvested by the damper in the mechanical domain and is actually the total *available* power. A fraction of this power will be dissipated and the rest will be converted into useful electrical energy depending on the amount of useful and unwanted damping in the system.

The total damping in the physical device consists of both mechanical and electrical damping as \[ \zeta = \zeta_{\text{mech}} + \zeta_{\text{elect}}, \] (6.17)

where \( \zeta_{\text{mech}} \) is the mechanical damping loss and \( \zeta_{\text{elect}} \) is the electrical damping. The electrical damping can be further divided into useful electrical damping and the electrical damping loss as

\[ \zeta_{\text{elect}} = \zeta_{\text{eload}} + \zeta_{\text{eloss}}, \] (6.18)

where \( \zeta_{\text{eload}} \) is the electrical damping load and \( \zeta_{\text{eloss}} \) is the electrical damping loss. For the case of electromagnetic transduction, the electrical damping \( \zeta_{\text{elect}} \) can be be further represented as \[ \zeta_{\text{elect}} = \frac{(BLN)^2}{2m\omega_n (R_{\text{load}} + R_{\text{int}})}, \] (6.19)

where \( BLN \) is the transformation factor, \( B \) is the average flux density, \( Nl \) is the effective length of the coil, \( R_{\text{load}} \) is the load resistance and \( R_{\text{int}} \) is the resistance of the coil.

The need for small damping when tuning the device with a higher harmonic creates another limitation in practice as the total damping in the system is composed of both mechanical and electrical damping described in Equation (6.17). In general, the amount of damping in the system can be altered by changing both the mechanical and electrical damping. However, for a given device, it is much more feasible to alter the damping in the system by changing the electrical damping in the device.

The change in the electrical damping can be done by adjusting \( R_{\text{int}} \) and \( R_{\text{load}} \). Changing
$R_{\text{int}}$ may require re-configuration of the device since the resistance of the coil is directly proportional to the length and inversely proportional to the area. Thus a more feasible way to alter the damping in a physical device, perhaps, by changing the load resistance $R_{\text{load}}$. However, changing the load resistance in the system may affect the useful power harvested by the device.

This study shows that tuning the undamped natural frequency of the device to match with a higher harmonic may generate more power than the first few harmonics. However, a careful consideration in terms of which harmonic to use is important as it affects the amount of the useful power harvested due to different amount of damping needed to satisfy with the size constraint of the device.

### 6.6 Prediction of the power harvested by the non-linear mechanisms

In this section, the power harvested by the non-linear mechanisms is investigated. Due to the non-linearity in the system, setting the maximum limit of the maximum relative displacement is not as straightforward as in the linear mechanism. Thus simulation studies in the next two sections just involve the contribution of each of the harmonics on the power harvested and the study involving the maximum relative displacement and the optimum damping ratio is not conducted here. The study of the contribution of each harmonic to the power harvested is more important for the non-linear mechanism since the aim of the non-linear mechanism is to investigate how the system can cope with mistune.

#### 6.6.1 Hardening mechanism

The equation of motion for the system with a hardening spring has already been given in Chapter 4. The equation is repeated here and expressed in a slightly different form for convenience

$$\ddot{s} + 2\zeta \dot{s} + \omega_n^2 s \left(1 + \alpha_k s^2\right) = -\ddot{y},$$  \hspace{1cm} (6.20)
where $\alpha_k = \frac{k_3}{k_1}$. In Chapter 4, the non-linearity in the system is given by $\alpha = k_3/k_1 Y^2$. This expression shows that the strength of the non-linearity is constant when the input is sinusoidal and of amplitude $Y$. In this study, the strength of the non-linearity can change due to the variation in the input displacement at each harmonic. However, in order to see the frequency content of the power harvested, it is thought that the trend can be studied by fixing the value of $\alpha_k$. In this case, $\alpha_k$ is kept constant at

$$\alpha_k = \frac{k_3}{k_1} = \frac{4.26 \times 10^7}{1495} = 2.85 \times 10^4 \text{m}^{-2},$$

(6.21)

where the values for $k_3$ and $k_1$ are obtained from the force-deflection characteristic of the B1 configuration described in Chapter 5. The power harvested by the mechanism is studied by tuning the natural frequency of the device to match the second, fifth, tenth, fifteenth and twentieth harmonics, respectively. The power harvested by the mechanism is calculated using the second time integration method. At first, the solution to the equation of motion is sought using the 4th order Runge-Kutta method and the power harvested is computed using Equation 6.7. The frequency content of the velocity and the power harvested are computed using Fourier series described before.

Figures 6.17 and 6.18 show the frequency content of the relative velocity and power harvested by the hardening mechanism for walking and running conditions with 1% damping ratio. For the parameters used in the simulation, these two figures show that tuning the natural frequency to match the second, fifth and the tenth harmonics result in a wide band power spectrum. This can be seen in Figures 6.17(a) - 6.17(e) and Figures 6.18(a) - 6.18(e). Tuning the natural frequency to coincide with higher harmonics (fifteenth and twentieth) results in the system behaving periodically and having a discrete spectrum. This behaviour is due to the increase in the stiffness as the natural frequency increases (assuming the mass is fixed) and the decrease in the level of excitation at these particular harmonics. The response is the almost linear. The small level of input and stiffer linear stiffness cause the system to oscillate within the quasi-linear regime of the force-stiffness characteristic.

Another interesting point to see from Figures 6.17 and 6.18 is that the wide frequency band of the contribution of the harmonics occurs at a frequency larger than the natural
frequency. This makes sense since the bend in the frequency response curve of the hardening mechanism occurs at a frequency larger than the low amplitude linear natural frequency, $\omega_n$ of the system. Thus from this simple study, it can be concluded that for an excitation which is composed of several harmonics, if the device is driven strongly enough so that the response exhibits a significant hardening characteristics, the harmonics higher than the low amplitude natural frequency may contribute to the power harvested.

### 6.6.2 Snap-through mechanism

The equation of motion for the snap-through mechanism given in Chapter 3 is repeated here

$$\ddot{s} + 2\zeta \dot{s} + \omega_n^2 s \left( 1 - \frac{1}{\sqrt{\left( \frac{s}{l_o} \right)^2 + (\gamma)^2}} \right) = -\ddot{y}, \quad (6.22)$$

where $\gamma$ is the non-linearity in the system and $l_o$ is the original length of the spring. In this study, two values of non-linearity, $\gamma = 0.1$ and 0.5, are used. The damping ratio is fixed at 0.01 and the original length of the spring is 0.5 cm. The simulation study is conducted by tuning the linear undamped natural frequency, $\omega_n$, of the device to match the second, fifth, tenth, fifteenth and twentieth harmonics, respectively.

Figures 6.19 and 6.20 show the frequency content of the relative velocity and the power harvested under walking condition when $\gamma$ is 0.1 and 0.5, respectively. Figure 6.19 shows that tuning the natural frequency of the device to the fifth and tenth harmonics results in the wide band of the frequency content. Tuning the natural frequency to a much higher harmonic (fifteenth and twentieth) results in the system behaving more like a linear system. The reason for this is similar to that discussed in the previous section, although in this case the level of the input is not enough to oscillate the system between the two stable equilibrium positions.

However, the performance is better when a smaller non-linearity is used in the system. It has been seen in Chapter 3 that the smaller the non-linearity the easier it is for the system to initiate a cross-well motion and overcome the central potential barrier. This effect is shown in Figures 6.20(g) and 6.20(h) where even tuning the natural frequency
to the fifteenth harmonic still results in a wide frequency band of the frequency content of the velocity and the power harvested. A similar behaviour can also be observed in Figures 6.21 and 6.22 under running conditions with a similar amount of damping and degree of non-linearity.

Unlike the hardening mechanism in the previous section, the wide frequency band for the snap-through mechanism occurs in the region below the tuned natural frequency. This validates the claim in Chapter 3 to a certain extent, that the bandwidth of the system can be improved by tuning the natural frequency of the device to be higher than the excitation frequency, as in the case of tonal excitation. Thus when the input is composed of a number of harmonics, the natural frequency of the system should be tuned to the highest harmonic as possible provided there is enough input to snap the mass from one stable position to another so that the harmonics lower than the low amplitude natural frequency may contribute significantly to the power harvested.

6.7 Conclusions

This chapter presented a study on the prediction of power harvested by linear and non-linear mechanisms when the excitation signal is composed of several harmonics. The input used in the simulation was obtained by measuring vibration signals from humans walking and running. The reconstruction of the signal shows that the signal can be approximately represented by the first 20 harmonics. Post-processing of the data revealed that the range of the harmonics for the acceleration during walking was as low as 0.91 Hz and ranged up to around 20 Hz, while it was between 1.37 Hz and 28 Hz for the running data.

The effect of tuning the natural frequency to different harmonics was also investigated in this chapter by using a constant damping ratio for a linear device. For a small damping ratio, this study showed that tuning the natural frequency to match with a particular harmonic result in a peak in the power harvested at that particular frequency only, since that particular harmonic contributes strongly to the power. With an increase in the amount of damping, the frequency content of the relative velocity and the power
harvested was also significant for harmonics below and above the tuned frequency. Although the power was less than that with small damping, the frequency band over which other harmonics contributed to the power harvested was wider. For a fixed amount of damping, the total power harvested when the device was tuned to match each harmonic decreased with $\omega_n^2$. This is because the damping ratio is not optimized for a particular input with a particular natural frequency which results in the decrease in the relative displacement by a factor of $\omega_n^2$, hence the decrease in the power harvested.

This chapter also focused on the optimum power harvested for a given constraint of the maximum relative displacement. By using the optimum damping ratio for each case, the study revealed that a comparable amount of power can be harvested when the natural frequency is tuned to match a higher harmonic. Note that the level of excitation decreases as the frequency gets higher, thus harvesting power at higher harmonic requires small damping so that the system makes full use of the maximum relative displacement. This raises another limitation in practice since producing a device with small damping is also an issue due to the mechanical and electrical losses as well as the useful electrical load resistance.

For the cases with the non-linear mechanisms, the study of the hardening mechanism showed that if the device is driven strongly enough so that there are significant hardening characteristics to the frequency response curve, harmonics higher than the low amplitude natural frequency can contribute significantly to the harvested power. For the snap-through mechanism, on the other hand, it is desirable that the system oscillates so that snap-through occurs. To maximize the harvested power, the low amplitude natural frequency should be chosen as high as possible, but not so high that the system is so stiff that snap-through does not occur.
Figure 6.1: An accelerometer is glued to the belt and tightly strapped around the subject's waist.
Figure 6.2: Typical time history (a, c, e) and the Fourier coefficients (b, d, f) of the acceleration in 3 directions resulting from walking at 5 km/hr: x-direction (a, b), y-direction (c, d) and z-direction (e, f).
Figure 6.3: Typical time history (a, c, e) and the Fourier coefficients (b, d, f) of the acceleration in 3 directions resulting from running at 9 km/hr: x-direction (a, b), y-direction (c, d) and z-direction (e, f).
Figure 6.4: The average of the frequency and the amplitude of each harmonic for walking signal from 7 subjects (error bar represents the standard deviation); (a) average frequency at each harmonic, (b) average amplitude in $x$-direction, (c) average amplitude in $y$-direction and (d) average amplitude in $z$-direction.
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Figure 6.5: The average of the frequency and the amplitude of each harmonic for running signal from 7 subjects (error bar represents the standard deviation); (a) average frequency at each harmonic, (b) average amplitude in $x$-direction, (c) average amplitude in $y$-direction and (d) average amplitude in $z$-direction.
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Figure 6.6: The time histories of the measured data (solid) and the re-constructed data using the first 20 harmonics (dashed) (a, c, e), and the Fourier coefficients of the signal showing the first 20 harmonics (b, d, f) for walking at 5 km/hr: \( x \)-direction (a, b), \( y \)-direction (c, d) and \( z \)-direction (d, e).
Figure 6.7: The time histories of the measured data (solid) and the re-constructed data using the first 20 harmonics (dashed) (a, c, e), and the Fourier coefficients of the signal showing the first 20 harmonics (b, d, f) for running at 9 km/hr: x-direction (a, b), y-direction (c, d) and z-direction (d, e).
Figure 6.8: An example to illustrate the contribution of each harmonic on the power harvested using the frequency domain method (a) frequency content of the input signal, (b) transfer function between the relative velocity and the acceleration when $\omega_n$ is tuned to eighth harmonic, (c) frequency content of the relative velocity and (d) frequency content of the power harvested.
Figure 6.9: Example of the first three harmonic components of the velocity signal (a) $\dot{s}$ and (b) $|\dot{s}|^2$. 
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Figure 6.10: Frequency content of the relative velocity (a, c, e, g, i) and the power harvested (b, d, f, h, j) by the linear mechanism under walking conditions when the natural frequency is tuned to match different harmonics with 1% damping ratio: 2\textsuperscript{nd} – 1.82 Hz (a,b), 5\textsuperscript{th} – 4.55 Hz (c,d), 10\textsuperscript{th} – 9.10 Hz (e,f), 15\textsuperscript{th} – 13.65 Hz (g,h) and 20\textsuperscript{th} – 18.20 Hz (i,j).
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Figure 6.11: Frequency content of the relative velocity (a, c, e, g, i) and the power harvested (b, d, f, h, j) by the linear mechanism under walking conditions when the natural frequency is tuned to match different harmonics with 5% damping ratio: $2^{nd}$ – 1.82 Hz (a,b), $5^{th}$ – 4.55 Hz (c,d), $10^{th}$ – 9.10 Hz (e,f), $15^{th}$ – 13.65 Hz (g,h) and $20^{th}$ – 18.20 Hz (i,j).
Figure 6.12: Frequency content of the relative velocity (a, c, e, g, i) and the power harvested (b, d, f, h, j) by the linear mechanism under running conditions when the natural frequency is tuned to match different harmonics with 1% damping ratio: 2nd − 2.74 Hz (a,b), 5th − 6.85 Hz (c,d), 10th − 13.70 Hz (e,f), 15th − 20.55 Hz (g,h) and 20th − 27.40 Hz (i,j).
Figure 6.13: Frequency content of the relative velocity (a, c, e, g, i) and the power harvested (b, d, f, h, j) by the linear mechanism under running conditions when the natural frequency is tuned to match different harmonics with 5% damping ratio: 2nd - 2.74 Hz (a,b), 5th - 6.85 Hz (c,d), 10th - 13.70 Hz (e,f), 15th - 20.55 Hz (g,h) and 20th - 27.40 Hz (i,j).
Figure 6.14: The total power harvested by the device when the natural frequency is tuned to match each harmonic for (a) walking and (b) running when $\zeta = 0.01$ (dashed) and $\zeta = 0.05$ (solid). The power plotted here is the total power harvested. This total power is calculated by first tuning the natural frequency to match the first harmonic. The total power is obtained by summing up the power due to all harmonics. This process moves on by tuning the natural frequency to match the next harmonic.
Figure 6.15: (a) The optimum power harvested by the linear mechanism and (b) the optimum damping ratio for two displacement constraints $S_l = 0.5$ cm (solid) and $S_l = 1.0$ cm (dashed), under walking condition.
Figure 6.16: (a) The optimum power harvested by the linear mechanism and (b) the optimum damping ratio for two displacement constraints \( S_l = 0.5 \) cm (solid) and \( S_l = 1.0 \) cm (dashed), under running condition.
Figure 6.17: Frequency content of the relative velocity (a, c, e, g, i) and the power harvested (b, d, f, h, j) by the hardening mechanism under walking conditions when the natural frequency is tuned to match different harmonics with 1% damping ratio: $2^{nd} - 1.82$ Hz (a,b), $5^{th} - 4.55$ Hz (c,d), $10^{th} - 9.10$ Hz (e,f), $15^{th} - 13.65$ Hz (g,h) and $20^{th} - 18.20$ Hz (i,j).
Figure 6.18: Frequency content of the relative velocity (a, c, e, g, i) and the power harvested (b, d, f, h, j) by the hardening mechanism under running conditions when the natural frequency is tuned to match different harmonics with 1% damping ratio: 2\textsuperscript{nd} – 2.74 Hz (a,b), 5\textsuperscript{th} – 6.85 Hz (c,d), 10\textsuperscript{th} – 13.70 Hz (e,f), 15\textsuperscript{th} – 20.55 Hz (g,h) and 20\textsuperscript{th} – 27.40 Hz (i,j).
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Figure 6.19: Frequency content of the relative velocity (a, c, e, g, i) and the power harvested (b, d, f, h, j) by the snap-through mechanism under walking conditions when the natural frequency is tuned to match different harmonics with 1% damping ratio and 10% non-linearity: $2^{nd} - 1.82$ Hz (a,b), $5^{th} - 4.55$ Hz (c,d), $10^{th} - 9.10$ Hz (e,f), $15^{th} - 13.65$ Hz (g,h) and $20^{th} - 18.20$ Hz (i,j).
Figure 6.20: Frequency content of the relative velocity (a, c, e, g, i) and the power harvested (b, d, f, h, j) by the snap-through mechanism under walking conditions when the natural frequency is tuned to match different harmonics with 1% damping ratio and 50% non-linearity: 2nd – 1.82 Hz (a,b), 5th – 4.55 Hz (c,d), 10th – 9.10 Hz (e,f), 15th – 13.65 Hz (g,h) and 20th – 18.20 Hz (i,j).
Figure 6.21: Frequency content of the relative velocity (a, c, e, g, i) and the power harvested (b, d, f, h, j) by the snap-through mechanism under running conditions when the natural frequency is tuned to match different harmonics with 1% damping ratio and 10% non-linearity: $2^{nd} - 2.74$ Hz (a,b), $5^{th} - 6.85$ Hz (c,d), $10^{th} - 13.70$ Hz (e,f), $15^{th} - 20.55$ Hz (g,h) and $20^{th} - 27.40$ Hz (i,j).
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**Figure 6.22:** Frequency content of the relative velocity (a, c, e, g, i) and the power harvested (b, d, f, h, j) by the snap-through mechanism under running conditions when the natural frequency is tuned to match different harmonics with 1% damping ratio and 50% non-linearity: 2nd – 2.74 Hz (a,b), 5th – 6.85 Hz (c,d), 10th – 13.70 Hz (e,f), 15th – 20.55 Hz (g,h) and 20th – 27.40 Hz (i,j).
Chapter 7

Conclusions

Research of energy harvesting from ambient vibration has become a hot topic among researchers within the last decade. The main idea of this research is in finding ways for the devices to power themselves by using ambient source. A self-powered device becomes more important especially when dealing with applications in remote environments. Up to now, resonant generators have dominated the architecture of the energy harvesting device.

This thesis focused on various aspects on energy harvesting from ambient vibration. In particular, this thesis has concentrated on the benefit of having a stiffness non-linearity in an energy harvesting device. Besides, it also considered the loading effect of the device on the vibrating source as well as the performance of a linear and non-linear device under excitation with a number of harmonics.

The purposes of using the non-linear spring are to provide more power and a wider bandwidth of the power harvested compared to the linear device. However, before investigating the non-linear mechanism, the limitations and characteristics of a linear energy harvesting device were studied in Chapter 2. The study started by investigating the tuning condition for the maximum power harvested by the device by considering the loading effect of the device on the vibration source.

A 2-DOF mass-spring-damper system was used to model an energy harvesting device which was attached to the vibration source. Using this model, it was found that, due
to the constraints of the device, e.g. size and mass, it may not be possible to transfer maximum power from the source by tuning the device so that the impedance of the device could be equal to the complex conjugate of the impedance of the source. This is because the real and imaginary part of the impedance of the device are not independent of each other. In order to study the power harvested by the device, the source was assumed to operate in three different regions namely stiffness (low frequencies), mass (high frequencies) and damping (around resonance) controlled regions.

It was found that the impedance of the device can affect the source even though the physical size of the source is much larger compared to that of the device, especially with small damping. This is because the impedance of the device can be very high, especially at resonance, and can provide a large opposing force on the source, even though the mass is small compared to that of the source. Tuning conditions to harvest a maximum amount of power from stiffness and mass controlled sources were also derived. It was shown that, tuning the device to resonate at the excitation frequency could result in a large reduction in the power harvested especially when the damping is small. These tuning conditions converge to the one which is found in most of the literature, i.e. matching the undamped natural frequency of the device to the excitation frequency, if the impedance of the source is much larger than the device, such that the source can be regarded as a velocity source.

The upper bound of the power harvested by the device with a non-linear spring compared to that of tuned linear device was derived using the principle of conservation of energy. This simple analysis showed that a device with a non-linear spring is able to outperform the linear device provided that the displacement response is a square wave, for a sinusoidal input. This finding leads to the subject of Chapter 3. In this chapter, a spring arrangement was proposed so as to produce a more square-wave-like displacement response. This spring consists of two oblique springs inclined at an angle to the horizontal. It was shown that, in practice it is not possible to produce a square wave displacement response, which requires instantaneous transitions between the two maxima of the time history displacement response, due to the presence of the mass and the damping. However, it was discovered that the power harvested by the device with
this mechanism is greater than that of the linear device at low frequencies due to the ability to steepen the displacement response. An analytical study of this mechanism is, however, difficult due to the possibility of the response of the system becoming chaotic.

Since the introduction of the snap-through spring arrangement in Chapter 3 was not able to achieve the performance limit derived in Chapter 2, a new spring arrangement was proposed in Chapter 4. This was not aimed at producing a square wave displacement response, but rather to provide a wider bandwidth. This type of spring is called a hardening spring. An analytical expression for the bandwidth was derived which showed that a device with a hardening spring can provide a wider bandwidth compared to that of a linear device. This chapter also presented a way to estimate the linear viscous damping in a system with a hardening spring by using knowledge of the jump-up and the jump-down frequencies. A method to estimate the non-linearity in a hardening system was also proposed using the jump-up frequency. Experimental studies on the snap-through and hardening mechanism were presented in Chapter 5. The main aim of the experimental work was to study the behaviour of the non-linear mechanism with respect to design parameters such as non-linearity, damping and the level of input displacement. In particular, the estimation of the damping in a Duffing oscillator was found to be accurate due to the good agreement between the amount of damping determined using established methods, i.e. the log-decrement and the peak of the transmissibility.

The performances of the linear and non-linear devices with an input consisting of a number of harmonics was presented in Chapter 6. The input used was measured from human motion, i.e. walking and running. From these measurements, it was found that the frequencies for the case of walking lie between 1 Hz and 20 Hz and between 1.3 Hz and 30 Hz for running. For the linear device, the contribution of each harmonic to the power harvested depends on the amount of damping. Small damping leads to less contribution from the neighbouring harmonics, i.e. away from the tuned harmonic, but depending on the level of input at that particular harmonic, may result in a large amount of power being harvested. Large damping, on the other hand, leads to significant contributions from the neighbouring harmonics, although less power is harvested. For a given maximum relative displacement constraint, a comparable amount of power can
be harvested by tuning the device to the frequency of a higher harmonic. However, tuning the undamped natural frequency of the device to match a different harmonic requires a different amount of damping in order to satisfy with the maximum allowable relative displacement. In the case of the non-linear devices, tuning the undamped natural frequency of the device to a higher harmonic produces a wide band of frequency in the power harvested. This means that the non-linear device has some benefits if the input has a wide frequency band.

7.1 Main conclusions from the thesis

In general, the following conclusions can be drawn from this thesis:

- It may not be possible for an energy harvesting device to harvest maximum energy from a vibrating source by tuning so that the impedance of the device is equal to the complex conjugate of the impedance of the source, due to dependence between the real and imaginary parts of the device impedance.

- When the source operates in the stiffness controlled region, the maximum power is harvested by the device when \( \Omega_d = \sqrt{\frac{\kappa}{1+\kappa}} \).

- When the source operates in the mass controlled region, the maximum power is harvested by the device when \( \Omega_d = \sqrt{\frac{1+\mu}{\mu}} \).

- When the source operates around resonance, the maximum power is harvested by matching the real part of the impedance of the device with the real part of the source. This is done by tuning the device to resonate at that frequency.

- For a harmonic input, a device with a non-linear spring can outperform the tuned linear device in terms of power harvested by a factor of only \( \frac{4}{\pi} \). This can be achieved provided that a square wave displacement response is produced for a given sinusoidal input.

- A snap-through mechanism is not able to produce a square-wave response so as to achieved the \( \frac{4}{\pi} \) performance ratio. However, the mechanism can perform better
when the natural frequency of the device is set to be higher than the excitation frequency.

- An expression for the bandwidth of the hardening mechanism shows that the bandwidth of the hardening mechanism is wider than its equivalent linear mechanism. Unlike the linear mechanism where the bandwidth only depends on the damping, the bandwidth of the hardening mechanism depends on both the non-linearity and the damping. However, this is the maximum possible value of the bandwidth that can be obtained from this mechanism. The performance of a device with this mechanism in a real application may be less because the response may not be on the resonant branch, especially in the region close to the jump-down frequency.

- The non-linearity of the hardening system depends strongly on the jump-up frequency only, while the damping ratio depends on both the jump-up and jump-down frequencies.

- For the case in which the input is composed of several harmonics, the most power can be harvested from that harmonic for which $\omega^3 Y$ is maximum, where $\omega$ is the frequency of the harmonic and $Y$ is the amplitude of the input displacement of the harmonic. However, for a given maximum allowable relative displacement, less damping is required if the device is tuned to a higher harmonic. If there is no constraint on the maximum relative displacement and the damping is fixed, the power harvested by the device reduces by a factor of $\omega_n^2$ where $\omega_n^2$ is the undamped natural frequency of the device.

- For the cases with the non-linear mechanisms, the study of the hardening mechanism showed that if the device is driven strongly enough so that there is significant hardening characteristic in the frequency response, harmonics higher than the natural frequency can contribute significantly to the harvested power. For the snap-through mechanism, on the other hand, it is desirable that the system oscillates so that snap-through occurs. To maximize the harvested power, the natural frequency should be chosen as high as possible, but not so high that the system is so stiff that snap-through does not occur.
7.2 Suggestions for future work

The research in this thesis has shown that there are some potential benefits in having a non-linear spring in an energy harvesting device. Based on the material presented in this thesis, a device with a non-linear spring has been shown to have the ability to increase the bandwidth of the system rather than increasing the amount of power harvested relative to that of a tuned linear device. Some recommendations for future research are suggested here for studying the benefits of a non-linear spring in an energy harvesting device:

- This thesis only considers two types of non-linear spring. A design of a new non-linear spring, e.g. softening spring, will broaden the choice of spring to use for different applications.

- The length of the beam and the gap of device used in the experimental work in Chapter 5 were measured using a ruler and a feeler gauge, respectively. A more systematic way to measure these two important parameters are mostly needed for a better accuracy in the measurement.

- The device used in this thesis uses a linear stiffness and a non-linear stiffness from magnets to produce a snap-through action. Other mechanisms to produce a snap-through action could be studied to see the real potential of this mechanism.

- A different method to produce a hardening mechanism could be designed. Among these are the use of a repelling magnetic spring, a combination of a linear spring and a cantilever beam clamped with a shaped block. The latter mechanism is illustrated in Figure 7.1.

- In this thesis, especially for the hardening mechanism, the non-linearity is considered to be weak. When the non-linearity is strong, there is a need to study the detailed coupling between the mechanical and electrical parts of the device and how this contributes to the amount of damping which is needed in order to compute the optimum power harvested.
• The tuning of the device is normally conducted in the mechanical part of the
device, i.e. normally by changing the stiffness. The ability to alter the dynamics
of the device by tuning it electronically is another issue to be studied. If this is
possible, this could provide a great advantage since electronic tuning may not affect
the physical size and complexity of the device as much as tuning it mechanically.

• The study presented in Chapter 6 shows some benefits of the non-linear device
when dealing with a wide frequency band excitation. Thus two small scale devices
which employ each of these two mechanisms needs to be fabricated so as to reveal
the real performance of these two devices.

\[ \text{Figure 7.1: A cantilever beam with a tip mass and clamped between two shaped blocks.} \]
References


REFERENCES


