17$^{\rm th}$ UK Conference on Computational Mechanics (ACME-UK)
6$^{\rm th}$–8$^{\rm th}$ April 2009
Nottingham
ISBN 978 085358 255 7

A FORMULATION OF THICKNESS OPTIMIZATION FOR PLANE STRESS

*** Athanasios Makrodimopoulos[1], Atul Bhaskar[2] and Andy J. Keane[3]**

[1] School of Engineering Sciences
University of Southampton
Southampton, SO17 1BJ, UK
http://www.soton.ac.uk/ses
am1a06@soton.ac.uk

[2] School of Engineering Sciences
University of Southampton
Southampton, SO17 1BJ, UK
http://www.soton.ac.uk/ses
atul.bhaskar@soton.ac.uk

[3] School of Engineering Sciences
University of Southampton
Southampton, SO17 1BJ, UK
http://www.soton.ac.uk/ses
ajk@soton.ac.uk

Key Words: *Thickness optimization, Second-order cone programming, Plane stress*

ABSTRACT

Thickness optimization can be considered as a case of sizing optimization for plane structures. It can also be used as an intermediate step for topology problems, i.e. we can eliminate the parts where the thickness tends to be zero. This paper is concerned with the case of plane stress structures coupled with the finite element method. The aim is to present a formulation of this problem as a case of second-order cone programming which is a standard form of mathematical programming. The advantage is that, on the one hand, all that the engineer has to do is to compute elemental data, and on the other, large discretized structures can be optimized accurately due to the efficiency of the proposed formulation. Different types of elements regarding the thickness field are considered.

1 INTRODUCTION

Thickness optimization of a structure means the arrangement of a thickness field in order to get some optimal result which can be minimum weight, compliance or stress and strain related quantities. A review of developments on this topic is given in the book of Bendsøe [1]. A major issue when thickness optimization is coupled with the finite element method is that large nonlinear optimization problems have to be solved. In this paper we consider the minimum compliance method and the problem is formulated as a case of second-order cone programming (SOCP) in a similar way to truss optimization [2]. The advantage of this formulation is that various solvers exist (both free and commercial), hence an engineer has only to compute the element related quantities and solve the arising problem by using an efficient solver. In addition to our recent work [3], elements with both continuous and discontinuous thickness fields are considered.

2 FORMULATION AS AN SOCP PROBLEM

Consider a plane stress stress structure. The goal is, for a given volume V, to find a thickness field $h(x, y)$ such that the compliance (i.e., the work of the applied loads) is a minimum. Taking into account the principle of complementary energy and that this energy is half the compliance, the minimization problem for a continuum takes the form

$$\begin{array}{ll} \min & \frac{1}{2}\int_V \boldsymbol{\sigma}^T \mathbf{C}^{-1} \boldsymbol{\sigma} h \, \mathrm{d}A \\ \text{s.t.} & \boldsymbol{\sigma} \in S_{\text{eq}} \\ & \int_A h \, \mathrm{d}A = V \\ & h(x,y) \geq 0 \end{array} \tag{1}$$

where the first integral in the objective represents the complementary energy and the integral in the constraints represents the total volume. Also $\mathbf{C}$ is material compliance matrix and S_{eq} is the set of the stresses which can carry the loads. Now if the structure is discretized into NE displacement finite elements the problem takes the form

$$\begin{array}{ll} \min & \frac{1}{2}\sum_{i=1}^{NE} \int_{A_e} \boldsymbol{\sigma}^T \mathbf{C}^{-1} \boldsymbol{\sigma} h \, \mathrm{d}A_i \\ \text{s.t.} & \sum_{i=1}^{NE} \int_{A_e} \mathbf{H}^T \boldsymbol{\sigma} h \, \mathrm{d}A_i = \mathbf{p} \\ & \sum_{i=1}^{NE} \int_{A_e} h \, \mathrm{d}A_i = V \\ & h(x,y) \geq 0 \end{array} \tag{2}$$

where $\mathbf{H}$ is the strain displacement matrix and $\mathbf{p}$ is the load vector. Assume now that we use in total NG number of Gauss points. Then the integrals in (2) become summations as

$$\begin{array}{ll} \min & \frac{1}{2}\sum_{i=1}^{NG} \boldsymbol{\sigma}_i^T \mathbf{C}^{-1} \boldsymbol{\sigma}_i h_i w_i \\ \text{s.t.} & \sum_{i=1}^{NG} \mathbf{H}_i^T \boldsymbol{\sigma}_i w_i h_i = \mathbf{p} \\ & \sum_{i=1}^{NG} h_i w_i = V \\ & h_i \geq 0 \end{array} \tag{3}$$

where w_i is the weight associated with the ith Gauss point and the determinant of the Jacobian between the global coordinates and the physical ones. If we set

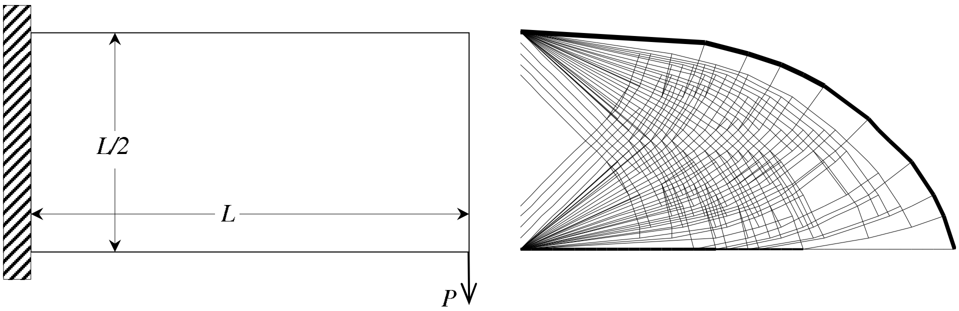
$$\xi_i = w_i h_i, \quad \bar{\boldsymbol{\sigma}}_i = \boldsymbol{\sigma}_i h_i w_i, \quad \mathbf{C}^{-1} = \mathbf{Q}^T \mathbf{Q} \quad \text{and} \quad r_i = \frac{\|\mathbf{Q}\bar{\boldsymbol{\sigma}}\|^2}{2\xi_i} \tag{4}$$

and transform the stresses at the ith Gauss point to a new variable $\mathbf{z}_i = \mathbf{Q}\bar{\boldsymbol{\sigma}}_i$ the problem takes the form

$$\begin{array}{lll}
\min & \sum_{i=1}^{NG} r_i & \\
\text{s.t.} & 2r_i\xi_i \geq \|\mathbf{z}_i\|^2 & \forall i \in \{1,\ldots,NG\} \\
& r_i, \xi_i \geq 0 & \forall i \in \{1,\ldots,NG\} \\
& \sum_{i=1}^{NG} (\mathbf{H}_i^T \mathbf{Q}^{-1})\mathbf{z}_i = \mathbf{p} & \\
& \sum_{i=1}^{NG} \xi_i = V \; . &
\end{array} \tag{5}$$

The first constraint together with the non-negativity of ξ_i, r_i define a rotated quadratic cone and, therefore, the above problem is an SOCP case. Consequently, all we have to do is to construct the matrix data of the above problem and solve it by an efficient SOCP algorithm. Note that regarding thickness, several other different schemes (e.g. continuous thickness) could be employed. However the technique for the formulation as an SOCP problem would be the same.

3 NUMERICAL APPLICATION



(a) (b)

Figure 1: (a) Notation for the structure (b) Optimal layout based on the truss optimization problem [3]

We performed thickness optimization for the cantilever beam shown in Fig. 1a in order to illustrate the procedure in section 2. The SOCP optimizer MOSEK [4] was used for the solution of the numerical optimization problems and GiD [5] was used as pre/post processor. Two types of elements were considered. In the first case six-node triangular elements with six Gauss points were used but nothing was assumed regarding the thickness field. The same type of elements were considered in the second case, however, the thickness field was continuous and linearly interpolated. Two different meshes were applied. Each mesh consisted of $M \times N$ quadrants and each subdivided into four triangles. Results and statistics are given in Table 1. We see that due to the additional constraints the solution of problems with continuous thickness takes significantly longer. Moreover as expected the compliance in this case is greater (i.e. worse). The thickness variation in Fig. 2 compares favourably with the topology obtained in a truss optimization problem [3] as shown in Fig 1b. We see that in general, thickness optimization can give boundaries similar to the truss analogue. We also observe that although a continuous thickness field leads to higher compliance, it produces a clearer structure. Finally it is interesting to note that the truss optimization problem resulted in a compliance of $W = 13.335$ i.e. almost 14% higher than the lowest compliance of the thickness optimization problem.

Table 1: Results and statistics for the numerical optimization problems. The compliance W has been multiplied by EV/PL^2. Gp means Gauss points.

		Thickness evaluated at Gp			Linearly interpolated thickness		
MESH	NE	It	CPU(s)	W	It	CPU(s)	W
20×40	3200	31	8.7	11.630	47	124	12.031
40×80	12800	38	48.7	11.640	45	981	11.856

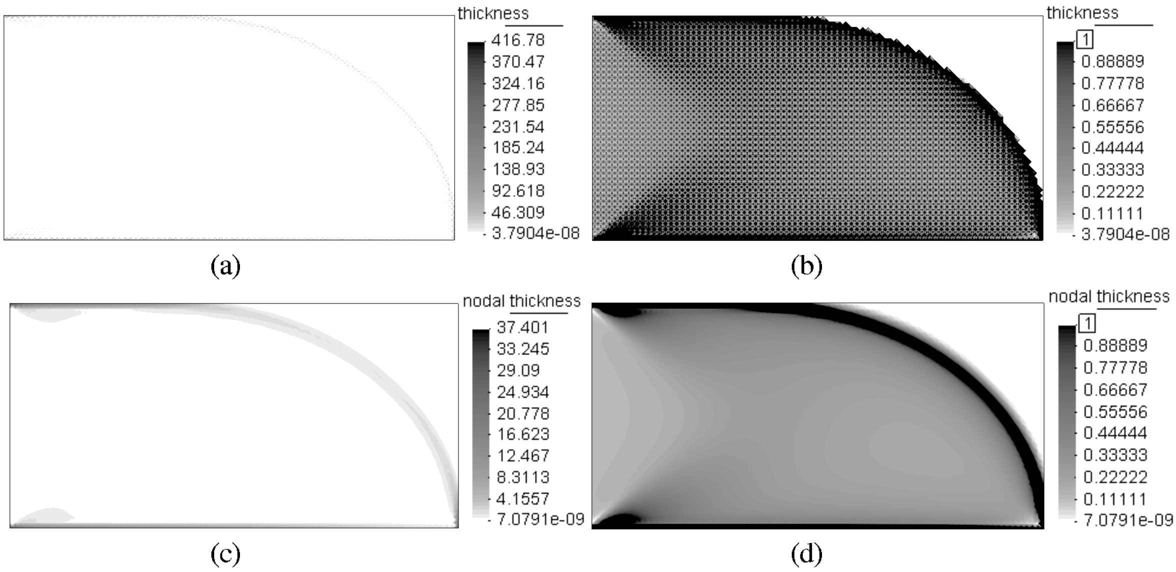


Figure 2: (a) Optimal thickness (thickness evaluated at Gauss points), (b) Scaled optimal thickness (thickness evaluated at Gauss points), (c) Optimal thickness (linearly interpolated), (d) Scaled optimal thickness (linearly interpolated).

4 ACKNOWLEDGEMENT

This work is funded by an EPSRC grant (EP/E004547/1).

REFERENCES

[1] Bendsøe. *Optimization of Structural Topology, Shape and Material*, Springer-Verlag, Berlin, 1995.

[2] A. Makrodimopoulos, A. Bhaskar and A.J. Keane. A formulation for large-scale truss optimization. In *CD-ROM Proceedings of the 6th GRACM International Congress on Computational Mechanics*, Thessaloniki, 19-21 June 2008.

[3] A. Makrodimopoulos, A. Bhaskar and A.J. Keane. Second-order cone programming formulations for a class of problems in structural optimization. *Structural and Multidisciplinary Optimization*, accepted for publication.

[4] MOSEK ApS, The MOSEK optimization tools manual. Version 5.0, Revision 60. Available from http://www.mosek.com.

[5] International Center for Numerical Methods in Engineering (CIMNE). GiD version 8.0. Reference Manual. Available from http://gid.cimne.upc.es