Optimal Score Cutoffs and Pricing of Regulatory Capital In Retail Credit Portfolios

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0. Abstract

This paper addresses the risk cutoff policies of a retail bank whose objectives are to maximize return on equity for shareholders and live within regulatory capital requirements, such as those of the Basel Capital Accord, to meet unexpected default losses. It investigates the changes that have to be made in the operating decision of which applicants for loans to accept and which to reject because of the changes in the financial regulations imposed on the bank. It is assumed that portfolios consist entirely of consumer credit accounts (mortgages, auto loans, revolving credit…) for which acquisition risk scores are available to the lender and regulator. The solutions that we obtain not only yield an optimal cutoff score for default risk but also optimal pricing conditions for additional equity capital in the event that the existing level can not satisfy the regulatory requirements. The paper concludes with several numerical examples illustrating the effects of current and proposed Basel regulations. We believe that some important insights are derived from this formulation linking the financial variables such as the lending and borrowing rates, and the debt and equity structure of the lender and the operational decisions of which level of risk to set as the cutoff in the consumer credit portfolios.

Keywords: Regulatory Capital, Equity Capital, Basel Accords, Risk Scoring, Price of Capital

1. Introduction

In 1974 the Bank of International Settlements set up the Basel Committee on Banking Supervision to formulate supervisory standards and guidelines following the failure of several large international banking organisations. The topic that has been highest on the Committee's agenda in the last two decades has been the capital adequacy of banks. Prior to 1988 it imposed no special capital requirements (Basel 0). Between 1988 and 2006, Basel regulations required that banks set aside a fixed percentage (8% in most cases) of equity capital for each dollar that is loaned (Basel 1). Beginning with the year 2008, the Capital Accord proposed by the Basel Committee will introduce an important new set of regulations governing capital adequacy whose terms require that the capital set aside per unit loan be a function of the loss given default and the probability of default (Basel 2). Portfolios of consumer loans such as revolving accounts (credit cards), auto loans, mortgages or personal loans will be affected. The Basel Accord has raised the importance of consumer credit risk assessment methods such as credit and behavioural scoring within banks but it also requires practitioners to be clearer in what such risk assessment systems actually measure, how they should be validated, and how they are used to influence operating policies and strategies. These regulations will have an impact on the acquisition and maintenance policies of retail credit portfolios. In this paper we are primarily interested in the effects the new regulations will have on optimal acquisition policies as well as the shadow prices of constraints on regulatory capital and the tradeoffs between expected profit and volume.

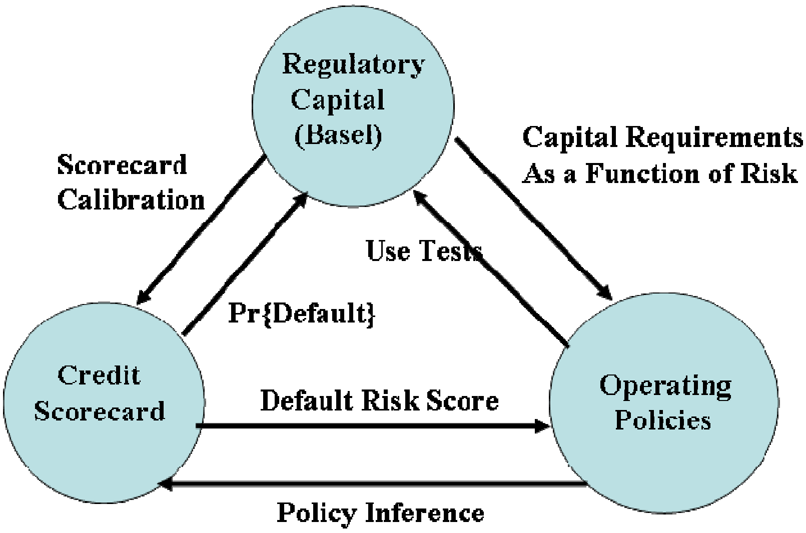

Figure 1: Interactions between Regulatory Capital, Credit Scores and Acquisition Policies

There is a three-way relationship between capital adequacy regulations, the use of credit scoring in consumer lending and the operational decision to acquire and manage retail credit. It has long been recognised that credit scoring provides excellent tools for measuring and controlling credit risk in retail portfolios, by ranking the relative risk of the applicants for credit. Up to now many lenders have had similar products and operating policies which means that operating decisions have had little impact on their default scores and booking rates. As lending becomes more customized this will change and the data that will then be available to build new risk assessment scorecards will have in it both the riskiness of the different customers but also the impact of the different policies that were applied to them. The resulting problem of policy inference - what would have happened under a standard operating policy - will make risk assessment more difficult.

The connection between credit scoring and Basel regulations is clear. Without credit scoring one could not apply the internal ratings based approach (IRB) of the Basel regulations to portfolios of consumer loans – the area where one can make the most savings under IRB. Moreover the Basel regulations are forcing credit scoring technology to continue improvement of its calibration techniques whereas previously the emphasis was on discriminatory power. The use test in the Basel regulations means that the way operating decisions are made is an integral part of the new Basel Accord. In this paper we look at the last link – how the Basel regulations will affect operating decisions and specifically the decision on setting the accept/reject cut-off.

In the mathematical models that follow this introductory section we assume that the retail bank has in place a risk scoring system and appropriate databases; this means that the distribution of account scores, predictive distributions for the probability of default and non-default are available to the lender and that it is possible to estimate statistical measures (such as the ROC Curves) that measure the quality of the discriminatory power of the risk scores and the resulting economic benefits.

The fundamental requirement is that we have the conditional distribution functions (either cumulative or tail distributions for scores given a Good (non-default) or a Bad (default) and estimates of the population odds of being Good or Bad which means that it is possible to obtain predictive risk distributions of business measures defined for individual accounts and portfolios of consumer credit accounts.

We assume throughout that regulatory capital is derived from some or all of the equity capital and provides the legal reserves to withstand large unexpected losses in the higher risk assets. We also assume that some or all funds to source loans are derived from borrowings (debt). The objective of this paper is to understand the consequences of the new Basel Accord on operational policies for controlling risk in retail credit portfolios. We have found it easier to encapsulate the regulatory requirements in terms of an arbitrary function of risk rather than the current detailed Basel formulas provided for mortgage, revolving and other retail risk; by doing so we rely on a continuous risk formulation that allows us to study the effect of changes in future regulations or the mathematical structure of improved designs. Optimal policies are expressed in terms of marginal profitability, shadow prices for regulatory capital and the tradeoffs between risk, profits and market share. Specific numerical results are based on the existing Basel capital formulas. Figure 2 shows the requirement for the minimum regulatory capital that needs to be set aside for each dollar loaned, $f_D\kappa(p)$, if the chance of the loan defaulting is p , and f_D is Loss Given Default (LGD) ,the fraction of the defaulted amount that is actually lost. This is given under the three cases -before there were any requirement (Basel 0), under the 1988 Accord (Basel 1), and under the new Accord (Basel 2).

Basel 0 – (pre 1998)

- No capital needed to be set aside by banks to cover risk

Basel 1 – (1988-2006)

- Basel regulations fix capital set aside as a constant fraction (8%) of loan

Basel2 – (2007 beyond)

- Basel requires capital set aside to be function of default risk, *p*.
- If Loss given default, f_D, is 1 then capital set aside per unit loan is $\kappa(p)$.

So shorthand will be, Basel 0 $\Rightarrow \kappa(p) = 0$

Basel 1 $\Rightarrow f_D\kappa(p) = 0.08$ (or 0.04 for mortgages)

Basel 2 $\Rightarrow f_D\kappa(p)) = f_D N\left(\left(\frac{1}{1\text{-}\rho}\right)^{1/2} N^{-1}(1-p) + \left(\frac{\rho}{1-\rho}\right)^{1/2} N^{-1}(0.999)\right) - f_D(1\text{-}p)$

where ρ=0.15(mortages0; =0.04(credit cards);= $0.03\left(\frac{1-e^{-35(1-p)}}{1-e^{-35}}\right) + 0.16\left(1 - \frac{1-e^{-35(1-p)}}{1-e^{-35}}\right)$(other retail)

Figure 2: Summary of Basel Capital Regulations for Retail Consumer Portfolios

Basel 2 has generated considerable interest in the credit risk of loan portfolios. The publication of the regulatory texts (BCBS 2004(a)(b)) was preceded by research on the models that would be appropriate (Gordy [2002]) and how they connected with existing ideas on Value at Risk (Jackson and Perraudin[2002]). These papers and most subsequent work concentrated on the impact of the Basel regulations on corporate lending. On the consumer (retail) side, there has been far less research and, so far as we know, none of it considered the impact on operating decisions. Allen DeLong and Saunders [2004] give an overview of the Basel accord and credit scoring and examine how corporate credit models are modified to deal with small business lending. Adam et al [2005] suggest an alternative credit risk model for retail lending by focusing on the dependence structure in mortgage lending, but they do not investigate the operational impact of their work or of the Basel Accord. Perli and Najda [2004] also suggest an alternative approach to the Basel capital allocation; although they offer a model for the profitability of a consumer revolving loan they use it to suggest that the regulatory capital should be some percentile of the profitability distribution of the loan, rather than considering the effect on operating decision. Blochlinger and Leopold [2005] discuss profit maximising cut-off decisions and how ROC curves can be used to make such decisions but there is no reference to capital pricing or allocation in their work.

This paper is organised as follows. Section 2 looks at the profitability model of a single loan with the different types of Basel and pre-Basel capital reserve requirements. Section 3 discusses the operating decision associated with a portfolio of such loans that wants to maximise expected profit and ROE. Section 4 extends the model by placing a lower bound on the required regulatory capital and shows how this leads to an optimal price that the firm should pay for additional equity capital to obtain an optimal accept/reject cut-off decision. Section 5 explains how these results can be viewed in terms of ROC curves, while section 6 gives some numerical illustrations of the results when the current Basel requirements are imposed. Section 7 draws some conclusions from this work.

2. Expected Profit, Losses and Expenses for a Single Account

In general, profit includes the revenues and expenses associated with lending and borrowing and the costs of economic capital (including regulatory capital) used to finance lending operations. These contributions can be written as the word equation:

$$\text{Profit} = \text{Revenue from loans} - \text{Default losses} - \text{Debt funding costs} - \text{Fixed costs}$$

We define regulatory capital as the amount of equity (shareholder) capital required to cover a level of unusual or **unexpected** losses arising from defaults.

The conditional expected profit for a single account from the acquisition of a single borrower with risk score *s* can therefore be written as

$$E[R\,|\,s]\text{-}E[L\,|\,s]\text{-}E[C_B\,|\,s]-C_F$$

where the notation *R* and *L* denotes random revenue and default losses. C_B denotes the cost of borrowed debt and C_F denotes the fixed costs for acquiring and operating a portfolio of loan accounts. These have no effect on the derivation of optimal policies but we include them for completeness.

If the loan to a single borrower is one unit (a dollar, pound, Euro…) the gross expected profit on assets, P_A , is conditional on the risk score, s , and has a mean value given by

$$E[P_A \mid s] = r_L p(G \mid s) - f_D p(B \mid s) \qquad p(G \mid s) + p(B \mid s) = 1$$

where the loss given default (LGD) is f_D, the outcome B denotes a default (a Bad), G a non-default (a Good), the lending rate is r_L and the conditional probabilities of defaulting or not depend on the risk score. Of the funds loaned by the bank to the borrowers, let the percentage borrowed from external sources be b, which is related to the leverage ratio. The net expected profit of an account P_E , (the profit on equity) , subtracts the cost of debt (expressed by the borrowing rate, r_B) from the gross expected profit, so

$$E[P_E \mid s] \equiv E[P_A \mid s] - br_B = r_L p(G \mid s) - f_D p(B \mid s) - br_B; \qquad p(G \mid s) + p(B \mid s) = 1 \qquad \textbf{(1a)}$$

The riskiness of the population who apply for such loan facilities is described by the distribution of the risk scores s where $f(s)$ is the density function of the score distribution. It is well known from Bayes' Rule that the posterior probability and odds of non-default are

$$p(G \mid s) = \frac{p_G f(s \mid G)}{f(s)}, \qquad o(s) = \frac{p(G \mid s)}{p(B \mid s)} = \frac{p_G}{p_B} \frac{f(s \mid G)}{f(s \mid B)}$$

where conditional Good and Bad likelihood functions are denoted by $f(s|G)$ and $f(s \mid B)$ respectively. The odds of being a Good is the product of two factors, the first being the population odds (PopOdds) and the second being the ratio of the conditional score densities. The PopOdds represents the a priori odds of a Good for a randomly selected individual from the population of interest and is independent of behavioral data for the individual; the second factor, on the other hand, depends on the economic and behavioral data of the individual and is usually obtained from proprietary financial databases or credit bureaus. Policies that maximize expected profit of a portfolio of accounts are well known in the scoring literature and are expressed in terms of the conditional score densities (Hoadley and Oliver[1998], Oliver and Wells[2001], Thomas et. al. [2002]). Using Bayes' Rule in Equation 1(a) it is easy to rewrite the equation to show that the risk cutoff score that yields zero expected profit for an individual account is the solution of

$$r_L p_G f(s \mid G) - f_D p_B f(s \mid B) - br_B f(s) = 0 \,. \qquad \textbf{(1b)}$$

Larger scores yield positive and smaller scores yield negative expected account profit, which leads to the notion of a cutoff or point of indifference. The first two terms involve the conditional score densities but the final term is the unconditional score density (i.e. the fraction of accounts having score s) as loan funds must be borrowed before it is known whether an individual account will default or not. By rearranging terms in Equation 1(b), we see that an optimal cutoff which accepts borrowers with positive expected profit is one where the odds of being a Good exceeds

$$o^* = \frac{p_G f(s^* \mid G)}{p_B f(s^* \mid B)} = \frac{f_D + br_B}{r_L - br_B} > \frac{f_D}{r_L} \qquad\qquad r_L > r_B \,. \qquad\qquad \textbf{(2)}$$

This cutoff depends only on economic parameters associated with borrowing and lending rates, LGD and leverage but independent of the size of the loan. The last inequality in (2) confirms that the Good:Bad odds at the cut-off point with debt financing must be higher than that without debt financing (b=0).

3. Portfolios of Accounts

Before we consider the influence of regulatory capital it may be useful to review the optimal policies that apply to the unregulated case. The unconditional expected profit derived from a portfolio of loan assets is obtained from the conditional account profit by integrating over the risk profile of booked accounts, i.e. those above the cutoff score, s_c, and subtracting the fixed costs:

$$E[P_A(s_c)] = \int_{s_c}^{\infty} E[P_A \mid s]dF(s) - C_F = r_L p_G F^{(c)}(s_c \mid G) - f_D p_B F^{(c)}(s_c \mid B) - C_F \qquad \textbf{(3)}$$

The difference between expectations

$$E[P_A \mid s] \quad \text{and} \quad E[P_A(s_c)]$$

is that the former refers to the expected profit of a single account conditional on a risk score s while the latter is used to denote the expected revenue minus fixed costs derived from the loan assets in a portfolio of booked accounts with cutoff s_c. If $dF(s)$ ($f(s)ds$ if differentiable) denotes the fraction of accounts with risk score in the interval (s,s+ds]. The tail or complementary score distribution is denoted by $F^{(c)}(\bullet)$.

The net expected profit (after cost of debt) for a portfolio with cutoff s_c is therefore

$$\begin{aligned} E[P_E(s_c)] &\equiv E[P_A(s_c)] - br_B \int_{s_c}^{\infty} dF(s) \qquad\qquad \textbf{(4)} \\ &= (r_L - br_B)p_G F^{(c)}(s_c \mid G) - (f_D + br_B)p_B F^{(c)}(s_c \mid B) - C_F \end{aligned}$$

Given fixed equity capital Q, return on assets, r_A (ROA), and return on equity, r_E (ROE), are defined by

$$r_A \equiv \frac{E[P_A(s_c)]}{Q + bF^{(c)}(s_c)} \qquad\qquad r_E \equiv \frac{E[P_E(s_c)]}{Q} = \frac{E[P_A(s_c)] - br_B F^{(c)}(s_c)}{Q} \qquad\qquad (\mathbf{5a},\mathbf{b})\,.$$

If there is no requirement for regulatory capital and there is no risk-based pricing or adverse selection by borrowers, the expected returns on equity and assets satisfy the fundamental Modigliani-Miller proposition (Ho and Lee[2004]), which requires that the value of a firm is independent of the capital structure or the dividend policy. In a non-deterministic world, the expected return on equity can be calculated from expected return on assets and the cost of debt by the formula:

$$r_A = r_E \frac{Q}{Q+D} + r_D \frac{D}{Q+D} \quad \text{or,} \quad r_E = r_A + (r_A - r_D)\frac{D}{Q} \qquad \textbf{(6a)}$$

In this equation A denotes assets, D denotes debt and Q is the equity of the firm. Obviously, ROE is proportional to the Debt/Equity ratio and is larger than ROA as long as the return on assets is greater than the (borrowing) cost of debt. Equation **(6a)** is easily verified for the retail portfolio by setting $r_D = r_B$ in (5) and noting that for any cutoff

$$r_A + (r_A - r_D)\frac{D}{Q} = \frac{E[P_A(s_c)]}{Q + bF^{(c)}(s_c)} \frac{Q + bF^{(c)}(s_c)}{Q} - r_B \frac{bF^{(c)}(s_c)}{Q} = \frac{E[P_E(s_c)]}{Q} = r_E \,. \qquad \textbf{(6b)}$$

With equity capital Q, ROE is maximized at the same cut-off as that which maximizes the expected net profit of the acquired portfolio and occurs when the lender accepts all accounts above the optimal cutoff in **Eqn. (2)**

$$o^* = o(s_c^*) = \frac{p_G}{p_B} \frac{f(s_c^* \mid G)}{f(s_c^* \mid B)} = \frac{f_D + br_B}{r_L - br_B} > \frac{f_D}{r_L}$$

This cutoff represents the optimal Basel 0 solution when there is no regulatory capital requirement. As we have already discussed, this cutoff is always larger than the case where there is no borrowing and no debt capital. If all the funds are obtained by borrowing, the cut-off is inversely proportional to the difference between the lending and borrowing rate. The larger the gap between r_L and r_B the more risky are the borrowers that the lender can accept, which, of course, is a primary consideration for requiring regulatory capital. If all money loaned by the bank is obtained from borrowed funds, and no equity is required, the return on equity (ROE) is infinite. Though unrealistic for real banking applications, it nevertheless points out the enormous profits that can be made when one uses low cutoffs, accepts high risk loans, and obtains large market share with $r_L >> r_B$.

4. Optimal Policies and Prices of a Regulatory Capital Requirement

Regulatory capital is set aside for booked accounts **before** it is known whether the acquired account does or does not default. If the lender had perfect information there would be no need for regulatory capital as lenders could guarantee that they would only accept Good accounts. One can easily incorporate the required amount of regulatory capital to support unexpected default losses within the decision model by explicitly including a cost for that requirement. With this formulation the total cost of capital required to provide funds for a loan would now include the cost of regulatory capital as well as the borrowing costs for funds that source the loan. If the minimum capital requirement (MCR) per unit loan when LGD=1 is defined to be $\kappa(s) = \kappa(p(s))$ and the loss given default (LGD) is f_D then total regulatory capital for a unit loan is $f_D\ \kappa(s)$. The $\kappa(s)$ function required in the Basel Accord is given in Figure 2.

As each account has a risk-dependent regulatory capital requirement, total regulatory capital, $K(s)$ is proportional to the amount loaned to all booked risky accounts:

$$K(s) \square\ f_D \int_s^\infty \kappa(u) dF(u). \qquad \textbf{(7)}$$

The total regulatory capital set aside must be less than or equal to the total available equity capital, Q, after subtracting any equity used in financing the loans. This means that the decision problem can be restated with a constraint and an associated shadow price

$$\underset{s}{Max}\, r_E(s) = \frac{1}{Q} \underset{s}{Max}\ [r_L p_G F^{(c)}(s \mid G) - f_D p_B F^{(c)}(s \mid B) - br_B F^{(c)}(s) - C_F] \qquad \textbf{(8a)}$$

subject to

$$\lambda \geq 0: \quad K(s) + (1-b)F^{(c)}(s) \leq Q \qquad \textbf{(8b)}$$

An equivalent problem since Q is given, is to find a stationary cutoff s and shadow price λ so as to maximize the Lagrangian $\boldsymbol{L}\ (s,\lambda)$ where

$$L\ (s,\lambda) = E[P_E(s)] - \lambda\left(K(s) + (1-b)F^{(c)}(s) - Q\right).$$

Stationary points yield an equation linking the optimal odds cutoff and shadow price with the borrowing and lending rate and LGD. A necessary condition for optimality of (8) is that

$$o(s^*) = \frac{(f_D + br_B + (1-b)\lambda^*) + \lambda^* f_D \kappa(s^*)}{(r_L - br_B - (1-b)\lambda^*) - \lambda^* f_D \kappa(s^*)} \qquad \lambda^* \geq 0 \qquad \textbf{(9)}$$

As both sides of **(9)** depend on the optimal cutoff, it is not always possible to obtain a closed form solution with $\lambda \geq 0$; however, it is easy to show that the optimal cutoff odds is unique and will be greater than or equal to that obtained with Basel 0. The numerator is larger than or equal to the numerator in the case when no regulatory capital is required and the denominator is less than or equal to the denominator in that case; this means that the odds ratio is greater than or equal to the Basel 0 cutoff. The uniqueness of the optimal solution in **(9)** follows since as the cut-off score s increases increases the odds on the left hand side of Equation (9) is monotone increasing. Also, if the $\kappa(s)$ function is positive and monotone decreasing in the score the numerator of the right hand side is monotone decreasing and the denominator is monotone increasing. Thus, there is a single crossing and the solution is unique.

Two possibilities arise: one corresponds to the situation (i) where the regulatory capital constraint is a strict inequality and the shadow price is zero (the unconstrained or Basel 0 solution). The other case (ii) occurs when there is insufficient equity to meet regulatory requirements and the institution may want to acquire additional equity to grow the business and take on greater profit and risk.

Case (i) corresponds to an optimal shadow price $\lambda^* = 0$ and the familiar odds cutoff in Eqn (2). In this case the expected net profit derived from assets in the retail credit portfolio is

$$E[P_E(s^*)] = (r_L - br_B)p_G F^{(c)}(s^* \mid G) - (f_D + br_B)p_B F^{(c)}(s^* \mid B) - C_F \qquad s^* = \ln\frac{f_D + br_B}{r_L - br_B} \qquad \textbf{(10)}$$

When condition (ii) holds the regulatory constraint in **(8b)** is binding and the optimal cutoff is determined by solving $K(s^*) + (1-\ b)F^{(c)}(s^*) = Q$. Instead of expressing the optimal cutoff odds in terms of borrowing and lending rates, the optimal shadow price is now determined by solving **(9)**:

$$\lambda^* = \frac{E[P_A \mid s^*] - br_B f(s^*)}{1 - b + f_D\kappa(s^*)} = \frac{E[P_E \mid s^*]}{1 - b + f_D\kappa(s^*)} \geq 0\,. \qquad \textbf{(11)}$$

The numerator is the expected profit (net of default and borrowing costs) of the **next** acquisition; the denominator is the additional regulatory (and equity!) capital required for the acquisition so that the optimal price is just the **marginal** ROE, i.e. the ratio of marginal profit to marginal equity at the optimal cutoff. The price of new equity capital is not specified a priori but rather is a result of an optimization problem in which both price and cutoffs are decision variables; obviously LGD, lending, borrowing rates and scoring technology affect the solutions.

To summarize, the optimal solution for the shadow price of regulatory capital is given by

$$\lambda^* = \begin{cases} 0 & \text{with } s^* = \ln\dfrac{f_D + br_B}{r_L - br_B}: \quad \text{sufficient equity} \\[2ex] \dfrac{E[P_E \mid s^*]}{1 - b + f_D\kappa(s^*)} & \text{with } s^* \text{ solution of } K(s^*) + (1-b)F^{(c)}(s^*) = Q: \text{insufficient equity} \end{cases} \qquad \textbf{(12)}$$

which may be larger or smaller than the market price for additional equity. As λ^* is the marginal profit increase for an extra unit of equity this assessment should help management decide whether or not to expand the retail credit portfolio in preference to other investments.

We have assumed that both b and Q are fixed by some previous allocation and policy decisions by the bank and so one only has to determine the cut-off, and hence the size of the portfolio, that maximizes expected profit and ROE. These occur at the same cut-off because ROE equals profit divided by a fixed Q. As it is never optimal to have unused equity capital when the equity available exceeds the required regulatory capital, one can increase the profit and the return on equity by decreasing b, the fraction of borrowed funds. On the other hand when the regulatory constraint is binding **and** the shadow price is greater than the borrowing rate, increased amounts of equity capital can be shifted to meet regulatory needs by borrowing additional funds at the rate r_B.

Consider the extreme case when all funds that source loans are borrowed, b=1 and the fixed cost C_F is positive. **Figure 3(a)** is a plot of maximum expected portfolio profit versus equity capital along the high-profit, low-risk, low-volume efficient frontier. When equity capital is small, optimal cutoffs are large, default losses and portfolio size are small as the optimal portfolio can only include the lowest risk borrowers. As equity (and regulatory) capital increase, expected profits from the portfolio increase as new borrowers are added, eventually reaching the value they would have had under Basel 0. Further increases in equity capital provide no additional increase in expected profits and only serve to decrease ROE.

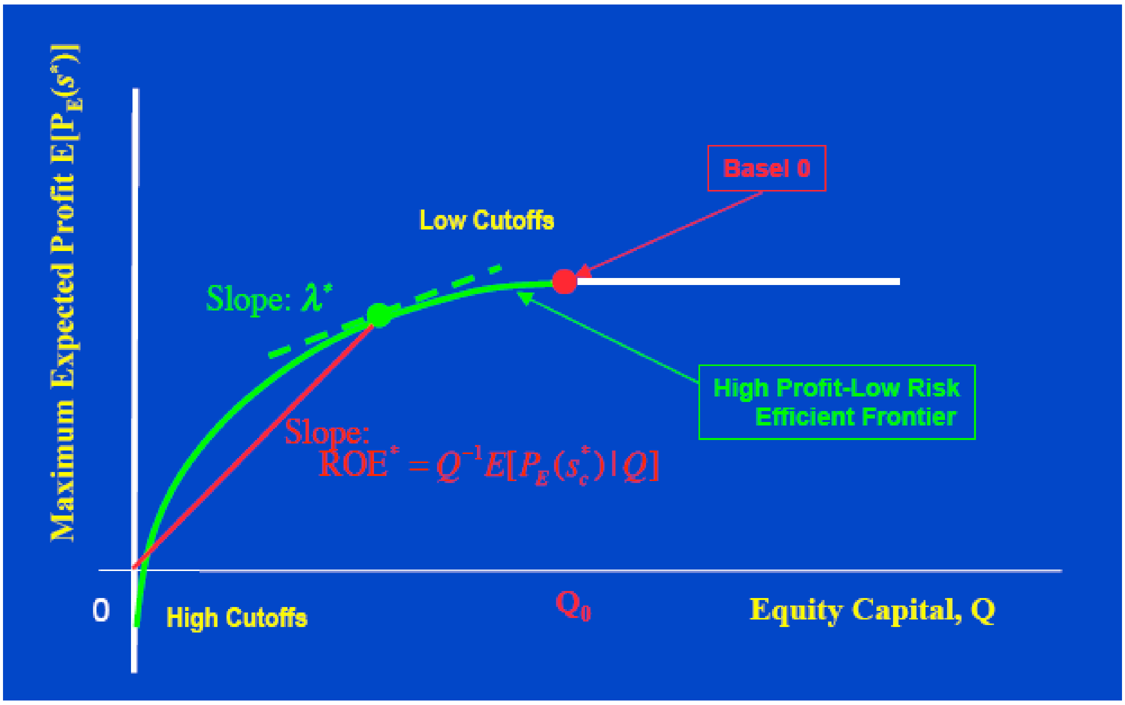

Figure 3(a): Efficient Frontier for Expected Profit versus Equity Capital

In **Figure 3(a)** optimal shadow prices are tangent lines to the efficient frontier and in **Figure 3(b)** they are shown by the dashed line. In **Figure 3(a)** the corresponding ROE is the slope of the line segment connecting the origin to the point of tangency and is shown as the solid line in **Figure 3(b)**. Once sufficient borrower revenue is generated to overcome fixed costs, ROE becomes positive but smaller than the large shadow prices associate with small amounts of regulatory capital and few defaults. Initially, small increases in regulatory capital lead to increases in ROE, decreases in the shadow price with shadow price less than ROE. Eventually, one reaches a cutoff where the slope of the tangent line equals the slope of the line segment from the origin which means that the optimal shadow price equals the ROE.

With further increases in equity capital the optimal ROE and the shadow price both decrease but ROE is always larger than the shadow price. To find the largest value of the optimal ROE as Q is varied, one finds the equity level where the optimal shadow price and optimal ROE are equal.

We should point out that there are at least two variants of the problem that might be of interest to the corporate decision maker. If the equity is given and fixed by considerations external to the regulatory analysis then the maximum profit problem is equivalent to the maximum ROE problem as Q is not part of the decision problem and is unaffected by the maximization operator (**Eqn. (8a)**). On the other hand, if the only equity capital required by the bank is regulatory capital, then the denominator in the ROE ratio (**Equation 5 (a,b)**) is no longer fixed but varies with portfolio size and the risk composition of acquisitions; in this case the determination of equity capital becomes part of the decision problem.

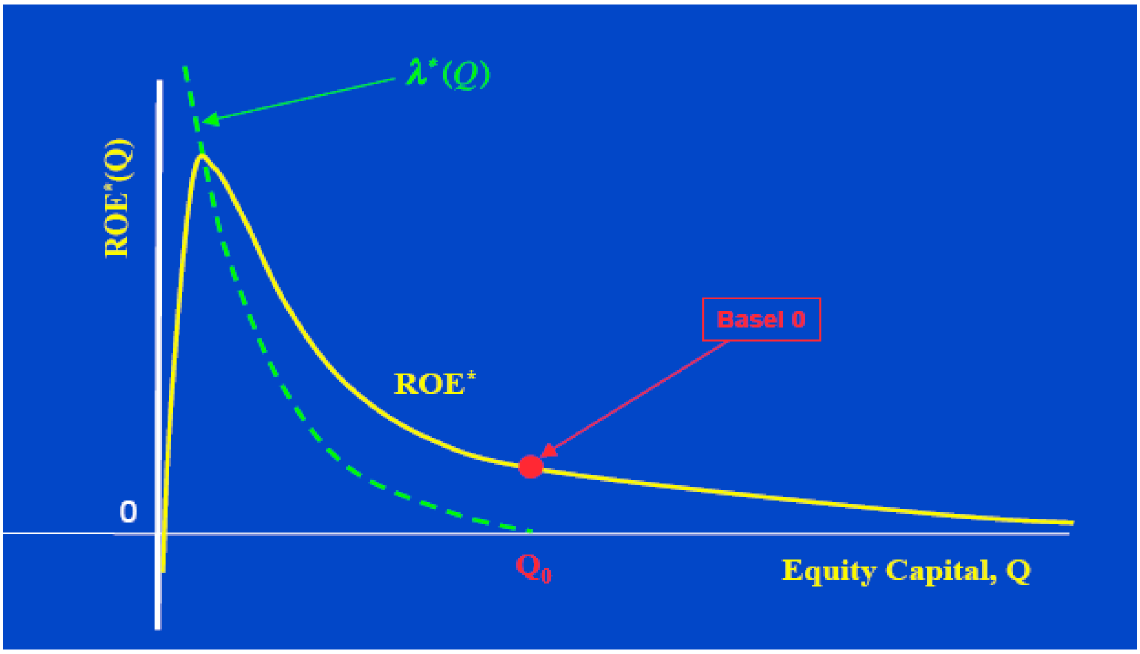

Figure 3(b): Optimal ROC and Shadow Prices on Regulatory Capital

5. Optimal Cutoffs and ROC Curves

It is well known that iso-contours of most business measures can be superimposed on the ROC curve of the score distribution which is a plot of the cumulative fractions of Bads rejected as a function of the cumulative number of Goods rejected for each possible cutoff point (**Figure (4)**). It is easy to show (Beling et. al. [2005]) that without regulatory capital constraints the slope of iso-contours of expected profit in the ROC space is given by

$$\eta_0 = \frac{f(s\,|\,B)}{f(s\,|\,G)} = \frac{p_G}{p_B}\frac{r_L - br_B}{f_D + br_B}$$

In the unconstrained case the optimal cutoff is obtained when an iso-contour with this slope is tangent to the ROC curve, i.e. the Basel 0 cutoff. The slope of iso-contours of expected volume (portfolio size) is always equal to the negative of the PopOdds, i.e. $-p_G/p_B$ and it is easy to see from **Figure 4** how retail credit portfolios must calculate the tradeoffs between increasing volume and increasing profits.

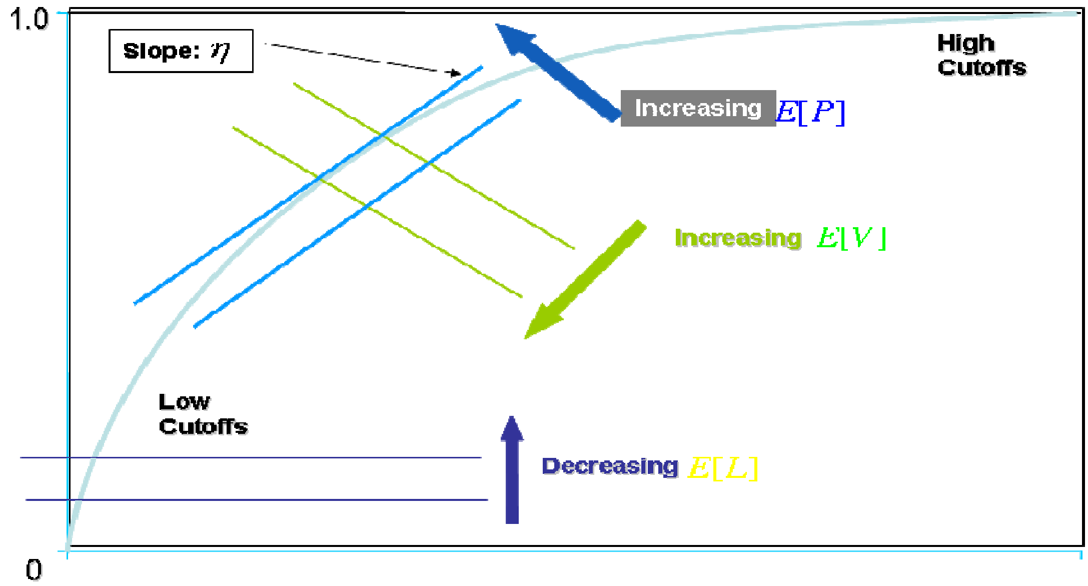

Figure 4: Iso-contours of Expected Profit and Volume in ROC Space

With Basel 1 the optimal solution that maximizes expected return on equity, in the presence of the regulatory capital constraint, has slope

$$\eta_1 = \frac{p_G}{p_B} \frac{(r_L - br_B - (1-b)\lambda^*) - 0.08\lambda^*}{(f_D + br_B + (1-b)\lambda^*) + 0.08\lambda^*} \qquad \lambda^* \geq 0$$

When the shadow price is zero we have the Basel 0 solution and equity capital is sufficiently large to provide the required regulatory capital. When the shadow price is positive the optimal cutoff is determined by the solution of **Equation (12)** which is

$$s_2^* = F^{-1}\left(1 - \frac{Q}{1-b+0.08}\right)$$

This optimal cutoff, in turn, determines the shadow price and the slope of the iso-profit line.

With Basel 2 constraints, the slope of the iso-profit lines is given by

$$\eta_2 = \frac{p_G}{p_B} \frac{(r_L - br_B - (1-b)\lambda^*) - \lambda^* f_D \kappa(s^*)}{(f_D + br_B + (1-b)\lambda^*) + \lambda^* f_D \kappa(s^*)} \qquad \lambda^* \geq 0$$

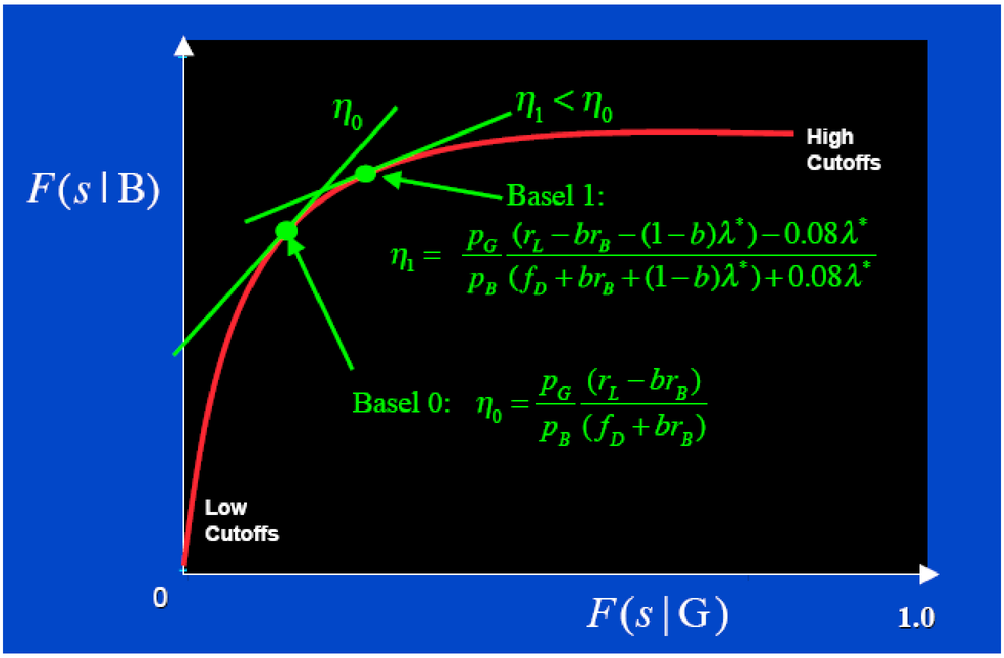

Figure 5: The Effect of Regulatory Capital Requirements on the ROC Operating Point

6. Numerical Examples Comparing Basel 0-1-2

In the numerical results that follow we calculate cutoffs where portfolio acquisition requires use of the Basel capital adequacy formula for Other Retail. For simplicity we assume that all the funds are borrowed (b=1) and all equity capital, if needed, is used to cover the regulatory requirements of Basel. As we require the conditional distributions for the probability of Good (Bad) outcomes, we have assumed that the density of p=p(s) is a Beta distribution with parameters (m,n) subject to the restriction that the PopOdds be given by the ratio m/n. It is easy to show that if the marginal density is Beta (m,n) the conditional densities of p given a Good (Bad) are then Beta(m+1,n) and Beta(m,n+1) respectively. With m=17, n=3, p_G=0.0.850, the default rate for the population as a whole (not the acquired portfolio population) is equal to 0.150 and the PopOdds is o_0=5.667. An examination of the ROC curve produced by this distributional assumption shows that it is consistent with the discriminatory power of many ROC curves obtained in practice and that it represents a moderately discriminatory scoring technology.

In the examples that follow we assume b=1, f_D=0.5, r_B=0.05, r_L=0.10, , i.e. the bank lends at 10%, borrows at 5% in a portfolio where the loss given default is 50%. In the Basel 1 models we assume f_D κ_1=0.08 for revolving and other loans while the Basel 2 models use the regulatory capital formulae for other retail in Figure 2. We denote the three different cases by subscripts 0, 1 and 2. The reader should be careful to note that in the body of the paper we have used o_0 to denote the PopOdds but that, in what follows, o_0^* denotes the optimal cutoff for the Basel 0 case.

Basel 0: When there is no Basle requirement $\kappa=0$, and Q=0.01 we obtain the familiar optimal cutoff:

$$
\begin{aligned}
&o_0^* = \frac{f_D + r_B}{r_L - r_B} = \frac{0.5+0.05}{0.1-0.05} = 11 \Rightarrow s_0^* = \ln o_0^* = 2.398 \\
&p(G \mid s_0^*) = p_0^* = 0.917; \quad p(B \mid s_0^*) = 1 - p_0^* = 0.083 \qquad \mathbf{(13)} \\
&ROE = \frac{0.0032}{0.01} = 0.320; \quad E[V(s_0^*)] = F^{(c)}(s_0^*) = 0.203
\end{aligned}
$$

where $E[V(s_0^*)]$ is the volume of accounts accepted , i.e. the fraction (approximately 20%) of the total original population accepted at the optimal cut-off. Expected portfolio profits and size require no regulatory capital and are independent of Q but ROE decreases inversely with Q.

Basel 1: In Basel 1 the regulatory capital is a constant $f_D\kappa_1$=0.08 independent of risk level of accounts. In this case the optimal cutoff odds is either higher that Basel 0 because there is insufficient regulatory capital or is equal to the Basel 0 case when there is ample capital. This occurs when the capital is at least $Q_1 = 0.08F^c(s_0^*) = 0.08(0.203) = 0.01624$. Using the solutions given in (9), (10), (11) and (12) for a given amount of equity capital Q, the optimal cut-off, and the shadow price of equity satisfy

$$
\begin{aligned}
&s_1^* = \ln o_1^* = \ln \frac{(f_D + r_B) + f_D\kappa_1\lambda_1^*}{(r_L - r_B) - f_D\kappa_1\lambda_1^*} = \ln \frac{0.55 + 0.08\lambda^*}{0.05 - 0.08\lambda^*} \\
&\lambda_1^* = \frac{E[P_E \mid s_1^*]}{f_D\kappa_1} > 0 \Rightarrow Q = K(s^*) = 0.08F^{(c)}(s_1^*) \text{ or } s_1^* = F^{-1}(1-12.5Q) \qquad (14)
\end{aligned}
$$

If Q=0.01 the optimal cutoff is determined by the binding constraint of Basel 1 capital requirements; this means that the cutoff is larger than the point at which Basel 0 ROE is maximized. We have the following solution of these equations:

$$
\begin{aligned}
&Q = 0.01 = .08F^{(c)}(s_1^*); \qquad s_1^* = 2.666 \\
&o_1^* = 14.385 \quad p_1^* = p(G \mid s_1^*) = 0.935; \qquad 1 - p_1^* = 0.065; \\
&\lambda_1^* = \frac{0.1p_1^* - 0.5(1-p_1^*) - .05}{0.08} = \frac{0.011}{0.08} = 0.138 \\
&ROE = 0.275; \qquad E[V(s_1^*)] = F^{(c)}(s_1^*) = 0.125
\end{aligned}
$$

Basel 2: Following Figure 2, the Basel 2 formulas require that the regulatory capital set aside $\kappa(s)$ is given by

$$
\kappa(s)) = f_D N\left(\left(\frac{1}{1\text{-}\rho}\right)^{1/2} N^{-1}(1-p(s)) + \left(\frac{\rho}{1-\rho}\right)^{1/2} N^{-1}(0.999) \right) - f_D(1\text{-}p(s)) \quad \mathbf{(15a)}
$$

($N(x)$ is the cumulative distribution function of the unit normal and, in our notation, probability of default is 1-$p(s)$) Whereas the correlation is assumed to be the constant ρ=0.15 for mortgages and 0.04 for revolving credit, the formula for other forms of retail credit such as personal loans is given by

$$\rho(s) = 0.03\left(\frac{1-e^{-35(1-p(s))}}{1-e^{-35}}\right) + 0.16\left(1-\frac{1-e^{-35(1-p(s))}}{1-e^{-35}}\right) \qquad \textbf{(15b)}$$

Basel formulas for capital requirements are plotted in **Figure (6)**.

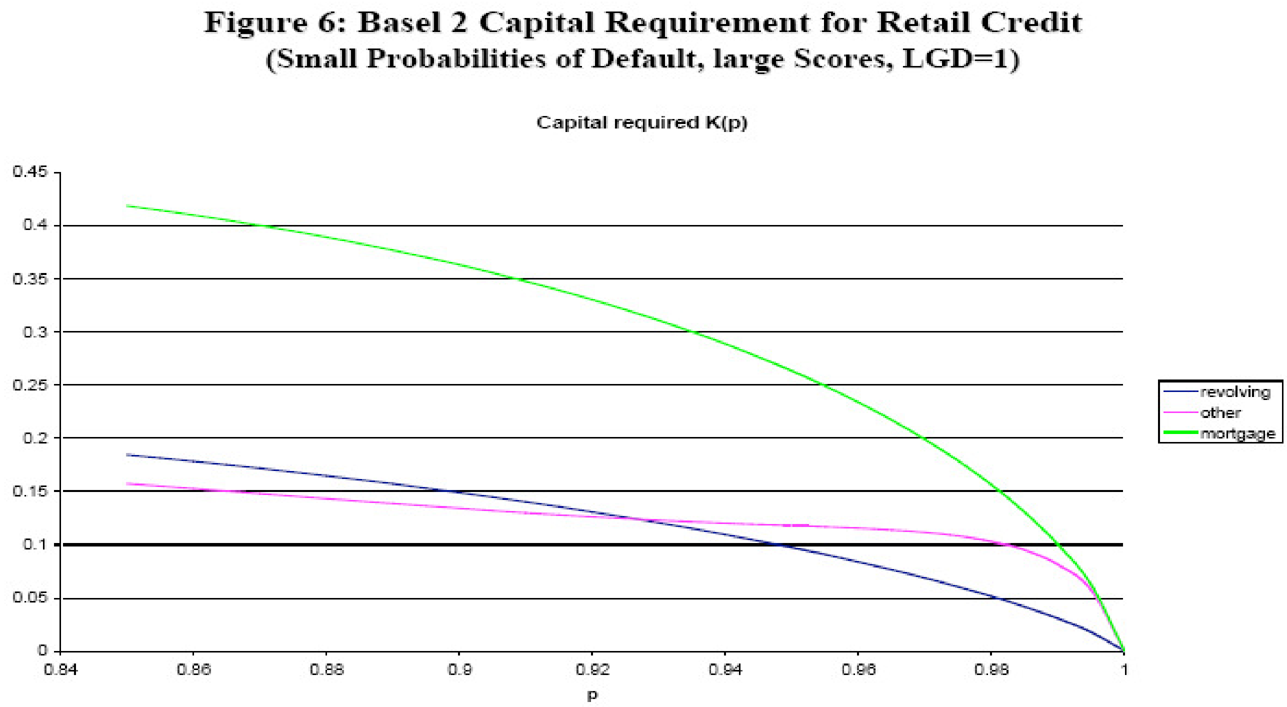

Figure 6: Basel 2 Capital Requirement Formulas for Retail Credit

Solutions for the Basel 2 case are obtained from the following equations

$$o_2^* = \frac{(f_D + r_B) + \lambda_2^* \kappa(s_2^*)}{(r_L - r_B) - \lambda_2^* \kappa(s_2^*)} = \frac{0.55 + \lambda_2^* \kappa(s_2^*)}{0.05 - \lambda_2^* \kappa(s_2^*)}$$

$$\lambda_2^* = \frac{E[P_E \mid s_2^*]}{f_D \kappa(s_2^*)} > 0 \Rightarrow Q = K(s_2^*) = f_D \int_{s_2^*}^{\infty} \kappa(s) dF(s); \qquad \textbf{(16)}$$

With a capital restriction of Q=0.01, the optimal score cutoff for Basel 2is lower than that for Basel 1 because the Basel 2 capital requirement is lower than that of Basel 1. Notice that in this case the equity level at which we can meet the unconstrained Basel 0 requirements is $Q_2 = K(s_0^*) = K(2.398) = 0.012$).

The new cutoffs, shadow prices, and ROE are now given by

$$Q = 0.01 = K(s_2^*); \quad s_2^* = 2.498;$$
$$o_2^* = 12.158 \quad p_2^* = p(G \mid s_2^*) = 0.924; \quad 1 - p_2^* = 0.076;$$
$$\lambda_2^* = \frac{0.1 p_2^* - 0.5(1 - p_2^*) - .05}{0.5(0.128)} = \frac{0.004}{0.064} = 0.063$$
$$ROE = 0.312; \qquad E[V(s_2^*)] = F^{(c)}(s_2^*) = 0.168$$

Because the cutoff lies between the Basel 0 and Basel 1 solutions, the ROE, expected portfolio size and shadow price for new equity capital also lies between their respective Basel 0 and 1 values.

As might be expected, the Basel 2 regulatory capital requirements are larger than those of Basel 1 when the loss given default is large. To illustrate such a case we examine the solutions when r_L=0.15, f_D=0.75 Q=0.02. This example shows that increased capital requirements from Basel 2 leads to further reduction in portfolio size and reduced ROE than would be found under either Basel 0 or Basel 1. The results are summarized in Table 1 and in Figure 7 which compares the relative location of the operating points on the ROC curve. In these and previous examples it should be remembered that calculations of ROE are only based on regulatory capital which may be much smaller than equity capital; hence, ROE numbers may appear to be unrealistically large. Most U.S. banks have equity capital far in excess of the Basel 2 requirements for regulatory capital.

r_L=0.15, r_B=0.15, f_D=0.75, Q=0.02	**Basel 0**	**Basel 1**	**Basel 2**
Score Cutoff, s^*	2.079	2.290	2.364
Non-Default, p^*	0.889	0.908	0.914
Default Rate, 1-p^*	0.111	0.092	0.086
Shadow Price, λ^*	N/A	0.170	0.235
$E[V(s^*)]$	0.375	0.252	0.220

ROE	0.585	0.545	0.515

Table 1: Optimal Solutions with r_L=0.15, f_D=0.75 Q=0.02

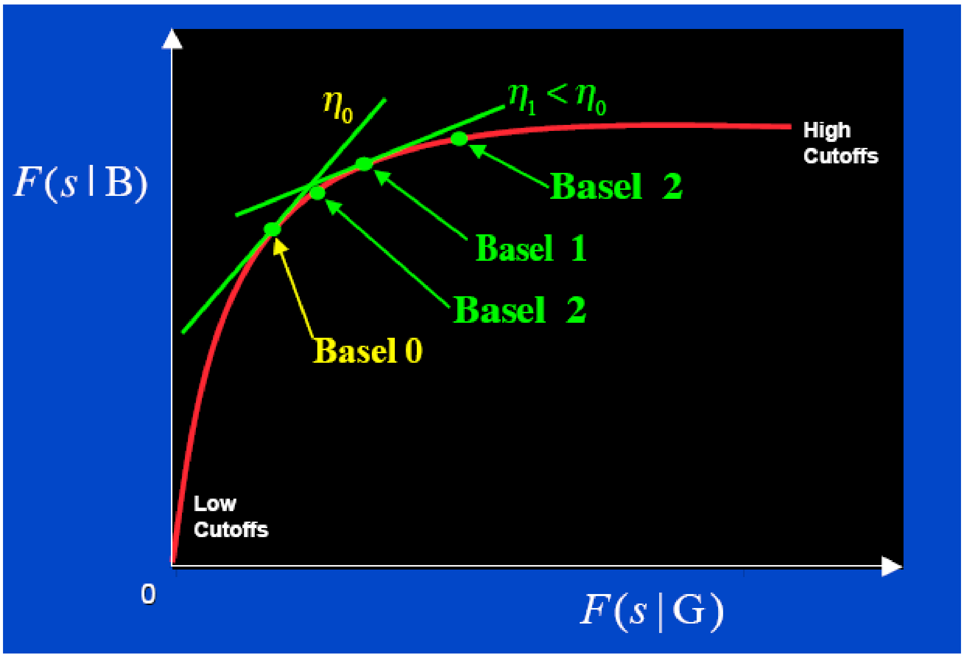

Figure 7: Comparing Basel 0, 1, 2 Operating Points on the ROC Curve

7. **Conclusions**

Our analysis began with a base case, Basel 0, which allows the study of various business measures including expected profit and ROE in the absence of any capital adequacy requirements. The major conclusion one can draw from the mathematical models is that the introduction of regulatory requirements in Basel 1 and 2, will increase score cutoffs, with a consequent decrease of expected profits, portfolio size and ROE. Depending on the particular assumptions about lending and borrowing rates and LGD, optimal cutoffs in Basel 2 will always be greater than or equal to the Basel 0 case but may be smaller or larger than the optimal Basel 1 cut-offs. If there is insufficient equity to provide the regulatory capital needed for the retail portfolio there is a positive shadow price one is willing to pay to obtain additional equity – this price may be different from the market price for new equity at similar risk levels and may suggest to management the need for alternative investments. Eventually, with sufficient regulatory capital the expected profits and ROE will yield the Basel 0 case at which point there is no value in obtaining additional equity capital.

If there are no fixed costs, the shadow price for additional equity will always be less than ROE and will become infinitely large at very low equity levels because capital requirements are negligible when the marginally highest profit accounts are being accepted. If, however, there are substantial fixed costs in the retail operations, at low levels of equity capital, ROE will be smaller than the high shadow price of new equity and the two objectives of increasing ROE and portfolio size are aligned. In these cases management will want to obtain extra equity capital to meet the regulatory requirements of the consumer credit portfolio. Eventually we reach a level of equity capital where the shadow price for additional equity equals ROE; with larger values of equity both the optimal shadow price and ROE decrease. In this region increasing the portfolio size will decrease ROE even though it continues to increase expected profits. Eventually the unconstrained optimal Basel 0 portfolio is attained.

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