

## **DEVELOPING PEDAGOGIC APPROACHES FOR PROOF: LEARNING FROM TEACHING IN THE EAST AND WEST**

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*In our work we focus on learning from the teaching of proof in geometry at the lower secondary school level across countries in the East and in the West. In this paper we summarize selected findings from a series of classroom-based experiments carried out over an extended period of time. By extracting key findings from our research, we show how we are identifying good models of pedagogy and using these to develop new pedagogic principles that are intended to help secondary school students not only to know ‘how to proceed’ with deductive proof, but also to understand more fully why such formal proof is necessary to verify mathematical statements.*

### **INTRODUCTION**

This paper focuses on how improvements in student capabilities with proof and proving might result from the identification of good models of pedagogy. In our research we address questions of how teachers might foster students’ fuller appreciation of the meaning of proof (including the discovery and explanatory functions of proof) and how teachers might motivate students to prove theorems. We also seek to identify teaching approaches that might inform future research into developing new pedagogic approaches for teaching deductive proof.

Our various studies focus on researching, and comparing, the teaching of proof in geometry at the lower secondary school level in countries in the East and in the West, specifically China, Japan and the UK. For example, in our studies of teaching we show some of the varying ways in which teachers structure their lessons to develop students’ deductive reasoning and in our analysis of curriculum materials (such as school textbooks) we report varying amounts of emphasis on ‘justifying and proving’ across our countries (Ding, Fujita, & Jones, 2005; Ding & Jones, 2007; Fujita and Jones, 2003; Fujita, Jones and Kunimune, 2008). What we are finding is that even when ‘justifying and proving’ is prominent, and principles about how to proceed with mathematical proof are explained for students, there remain students who may be able to construct deductive proofs but who do not necessarily understand *why* such deductive arguments are necessary (Kunimune, 1987; 2000). Such findings point to opportunities to improve the teaching of proof *and* to develop new pedagogic principles.

The results reported in this paper are illustrative of how we are extracting from our research studies some principles for teaching proof, and then testing these in

the classroom, aimed at helping secondary school students not only to know ‘how to proceed’ with deductive proof in geometry, but also to understand more fully *why* such formal proof is necessary to verify mathematical statements.

## **RESEARCH IN THE TEACHING AND LEARNING OF PROOF**

As Mariotti and Balacheff (2008) summarize, current research in the teaching and learning of proof has been making effort across a range of issues. In terms of providing pedagogic principles for the teaching of proof and proving in geometry, it goes without saying that various scholars have worked on this. For example, the van Hiele's proposed five ‘phases’ of geometry teaching that aim to take learners to success in deductive reasoning (van Hiele, 1999), Bartolini Bussi, (1996) has worked on teaching sequences with ‘germ theorems’, and Boero (1999) has provided a view of ‘conjecture production and mathematical proof construction’. In this paper we do not have space to relate in full to all such models. What we do show is how we are deriving some principles for the teaching of proof in geometry through the analysis of classroom-based research carried out over an extended period of time. Where we have space to do so in this paper, we relate our findings to various theoretical models.

## **IDENTIFYING GOOD MODELS OF PEDAGOGY FOR PROOF**

In our research we are sensitive to *how* the ways in which teachers structure their lessons in the countries that we are researching in the East and in the West (specifically China, Japan and the UK) is influenced (as, no doubt, everywhere) by various cultural factors and by specific educational issues such as the specification of the mathematics curriculum, the demands of examinations, and the design of textbooks (see, for example, Ding, Fujita, & Jones, 2005; Jones, Fujita & Ding, 2004, 2005).

In China, for example, one distinctive character of Confucian heritage in respect of learning is to ask questions constantly and to review previous knowledge frequently. This is reflected in the ways teachers teach proof in geometry. In Jones, Fujita & Ding (2004) we provide a case-study of a Grade 7 lesson on angles in parallel lines, showing how the teacher’s questions are carefully sequenced and how the special vocabulary is introduced. In this way the students are gradually involved in investigating the characteristics of each definition associated with angles in parallel lines, and they are expected to articulate their thinking through providing explanations. In later work we report on classroom data showing how teachers in China use sophisticated instructional strategies in explicating the discovery function of proof for lower secondary school students (Ding & Jones, 2008).

In Japan, an influential factor is ‘Lesson study’, one of the most common forms of professional development for teachers that involves them working in small teams in collaboratively crafting lesson plans through a cycle of planning, teaching and reviewing. For example, as we illustrate in Jones, Fujita & Ding (2004), to teach the properties of the parallel lines and ratio in Grade 8, a Japanese teacher might

organize a lesson (of 50 minutes duration) as follows. First, a problem for the day is introduced such as ‘Let us prove that if  $PQ \parallel BC$  in a triangle  $ABC$ , then triangles  $APQ$  and  $ABC$  are similar to each other’. Then, in the ‘development’ stage of the lesson, students would undertake to prove this problem, either individually or in groups. Their ideas are shared in a whole class discussion. Finally, the topic of this lesson is summarized as ‘If  $PQ \parallel BC$  in a triangle  $ABC$ , then triangles  $APQ$  and  $ABC$  are similar to each other, and therefore  $AP:AB=AQ:AC=PQ:BC$ , and if  $PQ \parallel BC$  then  $AP:PB=AQ:QC$ ’.

In the UK (specifically England) teachers have, in recent years, been provided with much guidance through a major Government initiative to improve mathematics teaching. In a Grade 8 case study that we analyze in Jones, Fujita & Ding (2004) the emphasis is on reasoning, with the teaching aiming to encourage greater rigor by re-establishing already familiar definitions and properties into a logical hierarchy. The idea is to apply properties established in earlier lessons to the solution of problems that involve constructing geometrical diagrams and analysing how these are built up. The lesson develops written solutions, where the ‘given’ facts (assumptions) are stated as justification in logically ordered explanations and proofs. The lesson reviews established facts and properties and the connections between them, so that students begin to gain a sense of a logical hierarchy.

Yet it seems that while the above approaches may be relatively successful at teaching ‘how to proceed’ with deductive proof in geometry, it can happen that there are students who do not fully understand *why* such formal proof is necessary to verify mathematical statements (Kunimune, 1987; Kunimune, Fujita and Jones, 2008). Hence our interest in teaching approaches that might inform future research into developing new pedagogic approaches for deductive proof and the reason we now turn to some selected findings from some of our classroom-based experiments.

## **DEVELOPING NEW PEDAGOGICAL PRINCIPLES**

In this section we highlight some principles for lower secondary school (Grades 7-9) extracted from a range of our classroom-based research carried out in Japan (for more details see Kunimune *et al*, 2007). Our aim, in researching these principles, is seeing how, and to what extent, they might help students appreciate the *need* for formal proofs (in addition to the students being able to construct such proofs).

- Grade 7 lessons can start from carefully-selected problem solving situations; *for example, a Grade 7 lesson starting point might be ‘consider how to draw diagonals of a cuboid’ – the classroom research that we have carried out suggests that this starting point can help develop students’ geometrical reasoning and provide experiences of mathematical processes that are useful in studying deductive proofs in Grades 8 and 9;*

- Geometrical constructions can be taught in Grade 8 alongside their proofs; *this might replace the practice of teaching constructions in Grade 7, and then proving these same constructions in Grade 8, as such a gap between the teaching of constructions and their proofs, our classroom research suggests, may not always be helpful;*
- Grade 8 lessons can provide students with explicit opportunities to examine differences between experimental verifications and deductive proof; *this helps students to appreciate such differences;*
- Grade 8 lessons involving the teaching of deductive geometry can be based around a set of ‘already learnt’ properties which are shared and discussed within the classroom, and used as a form of axioms (a similar idea to that of the ‘germ theorems’ of Bartolini Bussi, 1996); *this provides students with known starting points for their proofs.*

While we do not have space to provide data to support all these principles, in what follows we substantiate the principles related to problem solving and to geometrical construction (plus see Kunimune, Fujita & Jones, 2008).

### **Problem solving**

In a series of teaching experiments, we investigated problem-solving lessons that might link typical geometry topics in a way that supports students’ deductive reasoning. As an example, rather than merely showing students the diagonals of a cuboid, in one our Grade 7 teaching experiments the students were asked to investigate ‘how to construct diagonals of a cuboid’. The reason we chosen this approach was that it integrates the properties of 3D shapes and geometrical constructions in a way that emphasizes deductive thinking. From our teaching experiment, we observed the following:

- Students freely explored various ways to draw the diagonals of a cuboid. These ideas were shared in the classroom. The definition of diagonal was then introduced and students understood that ‘there are four diagonals in a cuboid’.
- Some students noticed that ‘the lengths of diagonals of a cuboid are equal’, and this led the class to consider *why*. Students then shared their own ideas such as ‘the diagonals are on a rectangle, and we have learnt that the lengths of diagonals of a rectangle are equal, and therefore the diagonals of the cuboid are equal’.
- Students further explored how to construct the diagonals using ideas that they had already learnt, such as how to construct angle bisectors, perpendicular bisectors, and so on.

We were interested to note that, at the beginning of our classroom experiment, the students just explored the properties of diagonals. As the experiment progressed, the students’ attention progressed towards certain geometrical properties and

deductive arguments. This, we contend, is a kind of ‘conjecture production and proof construction’ proposed by Boero (1999).

### **Constructions and proofs**

In our analyses of curriculum materials we have found that while geometrical constructions (with ruler and compasses) may be taught in Grade 7, these constructions are often not proved until Grade 8 (after students have learnt how to prove simple geometrical statements). In a series of teaching experiments, we investigated the use of more complex geometrical constructions (and their proofs) in Grade 8. As an example, one of our lessons in Grade 8 started from the more challenging construction problem ‘Let us consider how we can trisect a given straight line AB’. After students have worked on this problem, one of ideas from the students was chosen and its proof considered by the students in groups. In the final stage of the lesson the relevant theorem (which students would have noticed during the construction activities) was introduced and summarized: ‘In a triangle ABC, P and Q are on the line AB and AC respectively. If  $PQ \parallel BC$  then  $AP:AB=AQ:AC=PQ:BC$  and  $AP:PB=AQ:QC$ ’.

In our classroom studies, we observed that such lessons are more active for the students. The students could also experience some important processes which bridge between conjecturing and proving. For example, students could first investigate theorems/properties of geometrical figures through construction activities, and this would lead them to consider *why* the construction worked. Following appropriate instructions by the teachers, the students then started proving the construction. Again, this relates to Boero’s (1999) ‘conjecture production and mathematical proof construction’.

### **CONCLUDING COMMENTS**

In our work we focus on learning from the teaching of proof in geometry at the lower secondary school level across countries in the East and in the West. In this paper we summarize some of the results of our research, including the findings of a series of classroom-based experiments carried out over an extended period of time. Through our research we are identifying good models of pedagogy and using these to develop new pedagogic principles. Our aim is not only help students to know ‘how to proceed’ with deductive proof in geometry, but also help students to understand more fully *why* such formal proof is necessary to verify geometrical statements.

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