UNIVERSITY OF SOUTHAMPTON

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Aspects of strongly-coupled field theory from gauge-gravity duality

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The issue of calculating at strong coupling is a hard problem in physics. The discovery of gauge-gravity duality at the end of the Twentieth Century provides a novel means of calculating in a large-N gauge theory at strong coupling. In this thesis we apply the method of gauge-gravity duality to a variety of questions. Firstly we review the string theory background material and then introduce the gauge-gravity duality. We discuss the procedure for adding fundamental representation matter to gravity duals. We present a method for calculating the quasinormal frequencies associated to meson-like excitations in non-zero temperature gravity duals and apply it to excitations of bosonic and fermionic type. We study the thermal phase transition in a somewhat QCD-like gravity dual deformed by the presence of a relevant operator and find a plausible transition between a QCD-like confining phase and a high temperature phase which is just the generic black hole geometry. Finally we examine the effect of chemical potential on the behaviour of fundamental representation matter in a gravity dual and look for superconductivity-like behaviour.

Contents

List of Figures

Author's declaration

The work in this thesis was carried out in collaboration with Professor Nick Evans of the University of Southampton (and part of Chapter 3 in collaboration with the co-authors of the papers mentioned below). The material has been previously published as follows:

- Chapter 3: N. J. Evans, J. W. French, K. Jensen and E. J. Threlfall, 'Hadronization at the AdS wall', [arXiv:0908.0407[hep-th]] and J. K. Erdmenger, N. J. Evans, I. Kirsch and E. Threlfall, 'Mesons in gauge/gravity duals - a review', Eur. Phys. J. A **35** (2008) 81 [arXiv:0711.4467[hep-th]].

- Chapter 4: N. J. Evans and E. J. Threlfall, 'Mesonic quasinormal modes of the Sakai-Sugimoto model at high temperature', Phys. Rev. D77: 126008, 2008 [arXiv:0802.0775[hep-th]].

- Chapter 5: N. J. Evans and E. J. Threlfall, 'The thermal phase transition in a QCD-like holographic model', Phys. Rev. D78: 105020, 2008 [arXiv:0805.0956[hep-th]].

- Chapter 6: N. J. Evans and E. J. Threlfall, 'Chemical potential in the gravity dual of a 2+1 dimensional system', Phys. Rev. D79: 066008, 2009 [arXiv:0812.3273[hep-th]].

No claims for originality are made for the material in Chapter 2 and in the remainder of Chapter 3 which was compiled from a variety of sources.

Any significant help received from other sources is explicitly given credit in the relevant chapter.

Some simple calculations performed by me which are somewhat aside from the main text are relegated to appendices.

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Chapter 1

Introduction and overview

Today's fundamental theories of nature are relativistic quantum field theories, possibly converting into 'something else' at vastly high energy. Despite the lack of any direct experimental evidence, the current mainstream view is that the 'something else' is string theory or M-theory. The energy scale associated to the fundamental objects of string/M-theory is inaccessible to the experiments of today (and the forseeable future) but there is a great deal of theoretical richness and consistency in these theories. During the past decade or so string theory has given rise to remarkable new techniques for studying certain quantum field theories at strong coupling, via gauge-gravity duality, and in this thesis we explore some small corners of this methodology.

Quantum field theories arose from the coming together of quantum mechanics and special relativity - generically one seeks a quantum-mechanical theory which is consistent with Lorentz invariance in four dimensions. One especially important class of quantum field theory is that of the gauge theories which started off with the notion of there being a local change in the phase of the wave function of a matter field. This change of phase is a local $U(1)$ gauge invariance and the field A_μ we introduce to maintain the gauge invariance of the Lagrangian is the four-vector potential of classical electrodynamics. On quantization it describes the photon of quantum electrodynamics (QED). This theory provides an excellent perturbative description of the dynamics of charged particles in the weakly-coupled regime at energies below the electroweak scale.

It is not inconsistent with Lorentz invariance to generalize to a nonabelian gauge symmetry. The result is Yang-Mills theory [1] coupled to fundamental representation matter, with the Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{4}Tr\left(F^{\mu\nu}F_{\mu\nu}\right) + i\bar{\Psi}_i\gamma^\mu\left(\nabla_\mu - igA^a_\mu\tau^a\right)\Psi_i - m\bar{\Psi}_i\Psi_i \; . \tag{1.1}$$

Here $F^a_{\mu\nu} = 2\nabla_{[\mu}A^a_{\nu]} + gf^{abc}A^b_\mu A^c_\nu$. The dimensionless coupling constant is g and f^{abc} are the Lie algebra structure constants.

With gauge group $SU(3)$ the quantized theory is an excellent model of the strong nuclear force, the interaction of the fundamental representation quarks Ψ_i and adjoint gluons A^a_μ which we see in nature (at high energies). This theory is known as quantum chromodynamics (QCD).

It is an important feature that the quantum interactions can make the coupling constant g vary with energy scale. The running of the coupling is described by the *beta function* of the theory, which is defined by

$$\beta(g) = \frac{dg(\mu)}{d\ln\mu} \; , \tag{1.2}$$

where μ is the energy scale. For QCD with N_f flavours of fundamental quark the beta function at leading order in the coupling (ie one loop) is given by [2, 3]

$$\beta(g) = \frac{g^3}{(4\pi)^2}\left(-11 + \frac{2}{3}N_f\right) \; . \tag{1.3}$$

For a sufficiently small number of flavours this is clearly negative, showing that QCD is perturbative above some scale and the gauge coupling gets weaker as the scale is taken to infinity - a property known as asymptotic freedom (in this sense the theory is well defined at high energies and is 'UV complete' - in modern language the theory has a scale-invariant UV fixed point). Conversely, in the perturbative regime the coupling grows stronger as we lower the scale until we lose control of perturbation theory (this scale has to be experimentally determined and is known as Λ_{QCD}). Notice that the pure gauge theory has a nonzero beta function - this represents the anomalous breaking of conformal invariance. The quantization preserves only a subgroup (the Poincaré group) of the conformal symmetry group of the classical action. An associated phenomenon is the generation of a mass gap in the theory - any state orthogonal to the QCD vacuum has a finite positive mass - this mass gap is the reason we do not observe classical massless nonlinear Yang-Mills waves in the world around us - the classical wave picture is only appropriate at very high energy ($\gg \Lambda_{QCD}$) [4].

If the interaction strength becomes 'large' (ie the dimensionless coupling constant $g \sim 1$ - referred to as 'strongly-coupled') we have a quantum field theory which cannot be treated perturbatively (ie interpreted as a tree-level theory with 'small' higher-order corrections). A perturbative computation is not useful at strong coupling because the few terms of the diagrammatic asymptotic expansion we can compute do not capture the dynamics (the asymptotic expansion fails so one is not saved even if all diagrams can be summed). A strongly-coupled theory would be dominated by the quantum dynamics and the physical degrees of freedom would not necessarily be anything like the original field degrees of freedom present in the Lagrangian. There is no general procedure for doing calculations in a quantum field theory at strong coupling without resorting to full-blown numerical simulation of the dynamics - today a vast area of research despite the many challenges involved [5]. This inability to calculate at strong coupling is a serious problem because our theory of strong nuclear force, QCD, is a strongly-coupled field theory at everyday energies (below Λ_{QCD}). To test if QCD really is the correct theory at all scales, we would like to be able to calculate low-energy hadron physics (for example hadron masses, decay constants, magnetic moments...) from QCD. Techniques for calculating in field theories at strong coupling are therefore of great interest even if those field theories are somewhat different from actual QCD.

In recent years there has been much progress in the study of supersymmetric generalizations of gauge theories. Instead of seeking a Lorentz-invariant field theory one may extend the Poincaré algebra with the addition of anticommuting spinorial supercharges (one can also seek to extend the conformal algebra to the super-conformal). The point is that the supersymmetry constrains the form of the quantum theory which makes possible certain analytic calculations. In addition some novel dualities have been uncovered in which the strong coupling regime of some theories has a dual description which is perturbative in some different set of degrees of freedom [11].

In addition to the forces of nature which are described by gauge theories, there is also the force of gravity. At low energies, Einstein's theory of General Relativity (GR) [6] provides an excellent model. One significant problem with GR is that, when a perturbative quantization is sought, the resulting theory is not renormalizable. There can be no microscopic understanding of how a gravitational background is built up from elementary graviton particle states (whereas we do understand eg. how the Coulomb potential of the atom arises in a particular limit of QED). Attempts to alleviate the nonrenormalizability problem have been sought using supersymmetric generalizations of gravity known as supergravities. It does not appear that this is enough to solve the problem though at the time of writing it is possible that the maximally-supersymmetric $\mathcal{N} = 8$ supergravity avoids the problem [7]. The leading candidate quantum theory of gravity is (super-) string theory [8] which reduces to supergravity at energies which are small compared to the string scale (the latter provides a natural ultraviolet cut-off which cures the nonrenormalizability malady). There are alternative quantum gravity theories [9, 10] but string theory is by far the most developed today.

At the close of the Twentieth Century, string theory was found to contain non-perturbative objects known as D-branes [12]. D-branes are solitonic objects and the link to perturbative string theory is that open strings have endpoints localized on D-branes. It is natural that N coincident D-branes have a $U(N)$ gauge theory living on their world volume in the low energy limit. It is also the case that D-branes can emit and absorb closed strings which at low energy lead to a bulk supergravity theory. There are therefore two possible ways of describing the brane physics - the open and closed string sectors. Of course, in general neither description contains all of the physics since open and closed strings couple. However, it turns out that a certain limit exists in which the open and closed sectors decouple and there is then the possibility that all the physics is captured by each sector, independently. This opens the possibility of learning about a gauge theory by studying a supergravity, and vice versa.

In 1997 a specific construction of this type was proposed by Maldacena [13], using N coincident D3-branes in ten-dimensional IIB superstring theory. The field theory is the maximally supersymmetric $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory with gauge group $SU(N)$ in the large-N limit - it has superconformal symmetry and is strongly coupled. The bulk description in the large-N limit is weakly-coupled IIB supergravity on a background manifold with the metric of $AdS_5 \times S^5$. This was the first explicit gauge-gravity dual (the original specific construction is known as the AdS-CFT Correspondence). It gave a new method for studying a strongly-coupled field theory analytically because it is a strong-weak duality. Of course the field theory is not QCD (it has $\mathcal{N} = 4$ supersymmetry, a large number of colours and initially contained only adjoint fields).

Over the past ten years or so, gauge-gravity duality has been extensively used to address various problems in strongly-coupled field theories. It has brought new insight into some of the features of QCD, black hole physics, collider physics and also to certain systems in condensed matter physics. In this thesis I present work which is unified by virtue of using gauge-gravity duality to address questions in strongly-coupled quantum field theory. All original work was carried out in collaboration with my supervisor, Professor

Nick Evans.

Chapter Two gives a flavour of the phenomenology of strongly-coupled field theory we are going to study. The basics of string theory are presented and the background to gauge-gravity duality is outlined. I focus on the original proposal for the AdS-CFT Correspondence [13] which is derived from the D3-brane solution of string theory. I present the remarkable relation between quantum field theory operators and classical supergravity degrees of freedom [14, 15]. This chapter contains no original research.

Much of the phenomenology of QCD is dominated by the behaviour of fundamental representation quarks. Gauge-gravity duality can be used to study the behaviour of fundamental matter coupled to gauge theory. In Chapter Three I review also the prescription for adding fundamental matter to the field theory by including additional D-branes in the supergravity background [16]. I review the computation of the masses of mesonic bound states for the D3-D7 construction and I describe a procedure for computing the meson density surrounding an isolated static quark and the radiation of mesons by an accelerated quark. Parts of the latter computation are my own work [18].

There exists an interesting model which describes a non-supersymmetric large-N field theory with the adjoint fermions decoupled by giving them a large Kaluza-Klein mass [64]. Fundamental matter can be added to give the Sakai-Sugimoto model of chiral symmetry breaking [46, 47]. The phase structure of this setup has been studied and a high-temperature, deconfined phase found [50]. This model allows the study of mesonic excitations in a large-N deconfined plasma phase. Chapter Four presents a computation of the masses and decay widths of mesons in this phase, using an ingoing coordinate chart to simplify the numerical method, as published in [19]. Returning to the D3 model, in the high-temperature phase, I compute the masses and decay widths of some spin-half excitations which may be interpreted as the superpartners of the field theory glueballs or as the superpartners to the mesons associated to a probe D7 brane at zero hard quark mass.

Gauge-gravity duality can be used to investigate the phase structure of large-N field theories. Chapter Five presents research concerning the deconfinement phase transition in a QCD-like deformation of the D3 brane construction, beginning with a discussion of the thermal behaviour of the theory on its moduli space. Some of the main results from the literature concerning the features of the high and low temperature phases we find are reviewed. This work was published in [20].

Recently there has been interest in using gauge-gravity duals to model strongly-coupled condensed matter systems - it is known that certain condensed matter systems can become strongly-coupled, 'relativistic' and conformal when tuned to a quantum critical point. Descriptions of superconductivity-like behaviour has emerged from certain models [17] giving the hope of understanding unconventional superconductivity as a strong-coupling effect. Chapter Six presents research concerning adding a chemical potential to the D3 system and examining the effect on fundamental quarks living on a $2+1$-dimensional defect. We attempt to obtain a dual description of a superconductor where the brane construction gives an explicit understanding of what the degrees of freedom in the condensate actually are. This work was published in [21].

Chapter 2

Gauge fields, strings and duality

We review some of the classic features of strongly-coupled QCD and their generalization to a large number of colours (N). We introduce the basics of string theory leading to the AdS/CFT Correspondence.

2.1 QCD and large-N

The problem of strong coupling is ubiquitious in physics and a particularly important case is the low energy regime of QCD. The main features of the theory arise from strongly-coupled quantum dynamics and there is no derivation of the full force between quarks from the underlying QCD Lagrangian. Some of the characteristic features of low-energy QCD are:

Confinement - the only states we see are singlet representations of $SU(3)$ (colour singlets) - either states with three quarks (baryons), quark-antiquark pairs (mesons), along with colourless associations of gluons (glueballs). It is supposed that there is a linear $\bar{q}q$ potential describing a constant force as one tries to separate a quark and an antiquark. At some point enough energy has been put into the colour field between the q and $\bar{q}$

to get pair creation and one ends up not with a separated quark and antiquark but a bunch of colour singlet hadrons.

Chiral symmetry breaking - the key point is to note that, for one flavour of massless quark, the QCD Lagrangian is invariant under the global $U(1)$ vector and axial transformations

$$\psi \to e^{i\alpha}\psi \ , \quad \psi \to e^{i\alpha\gamma^5}\psi \ .$$

The vacuum state of QCD contains a nonzero vacuum expectation value for an operator describing a quark condensate $\langle\bar{\psi}\psi\rangle$. If this is the case, this quantum effect breaks chiral symmetry spontaneously (meaning that the system has to choose a particular vacuum which breaks one of the symmetries present in the Lagrangian) because the variable $z \equiv \bar{\psi}\psi - ii\bar{\psi}\gamma^5\psi$ transforms under the axial transformation as $z \to ze^{2i\alpha}$. There is one spontaneously broken symmetry generator and we expect one massless state from Goldstone's theorem. This is the phase of z which is a pseudoscalar since the tangent of this phase is $\frac{i\bar{\psi}\gamma^5\psi}{\bar{\psi}\psi}$.

In the case of multiple flavours the symmetry breaking generalizes to $U(N_f)\times U(N_f) \to U(N_f)_V$ and N_f^2 Goldstone bosons would be expected. Experimentally the light flavours (u, d) have very small current algebra masses and obey an approximate chiral symmetry. One finds three light pseudoscalar pion states which is actually $N_f^2 - 1$ because one state receives a larger mass due to an anomaly in QCD.

These features are impossible to calculate using the perturbative expansion in Feynman diagrams. Some aspects of the physics associated to chiral symmetry breaking can be derived from the effective theory of the chiral Lagrangian but of course this comes from arguing that the symmetries of the problem survive at strong coupling (which appears to be true because the analysis works, but it cannot be proven from the Lagrangian).

It was theorized some time ago that strongly-coupled gauge theory could be described by a kind of effective string theory [22]. QCD contains string-like objects which are the colour flux tubes or Wilson lines. The correct gauge-invariant operator describing a quark-antiquark pair is

$$\bar{\psi}(x')\mathcal{P}e^{i\int_{x'}^{x}A_\mu dx^\mu}\psi(x)\ . \tag{2.1}$$

The strong dynamics cause the gauge field to form a 'tube' connecting the quark and antiquark (unlike the case of a dipole in QED). In the $SU(3)$ theory there would be a 'thickness' associated to the flux tube but it is possible to imagine that as the number of colours N increases then the flux tube becomes thinner and more like a fundamental string (at large N one also expects the string to become harder and harder to break as the tension will become large). This can also be expressed in diagrammatic language such that the quark and antiquark propagators are connected by a kind of 'weave' of gluon field, which is suggestive of a string worldsheet.

It is also possible to imagine that as N becomes very large, the gauge theory simplifies in some way and contact with QCD can be made via an expansion in $\frac{1}{N}$ (there are no other dimensionless parameters in the theory one can base an expansion on). This argument was successfully applied by 't Hooft [22]. The one-loop beta function for $SU(N)$ pure YM theory is given by

$$\frac{dg}{d\ln\mu} = -\frac{11}{3}\frac{1}{16\pi^2}Ng^3\ . \tag{2.2}$$

It is apparent that both terms are of the same order if we take $N\to\infty$ while keeping $\lambda\equiv g^2N$ fixed. This is known as the 't Hooft limit and λ the 't Hooft coupling. This is the correct parameter to use for the coupling as N is allowed to vary because perturbative interaction vertices receive contributions $\sim N$ from group theory factors as well as factors of g.

One can consider the vacuum diagrams of the theory written in 'double-line' notation where an adjoint field propagator is expressed as a fundamental and antifundamental propagator. A diagram takes the form of a simplicial decomposition of a surface, with each single-line loop viewed as the perimeter of a 'face'. It is clear that each diagram can be drawn without the propagators crossing on a surface which has a topology determined by the number of crossing propagators if the diagram is projected onto a plane. Now schematically the Lagrangian takes the form

$$\mathcal{L} \sim \frac{1}{g_{YM}^2}\left(Tr\, p^2\ \phi_i\phi_j + p\ c^{ijk}\phi_i\phi_j\phi_k + d^{ijkl}\phi_i\phi_j\phi_k\phi_l\right)\ . \tag{2.3}$$

A natural question is what factors of N and λ are associated with a given diagram. From the Lagrangian we see that a propagator will be proportional to $\frac{\lambda}{N}$, an interaction vertex will carry a factor of $\frac{N}{\lambda}$ and each closed loop will contribute a group theory factor of N. Thus a diagram corresponding to a simplex with V vertices, E edges (propagators) and F faces (closed loops) comes with a factor of $N^{V-E+F}\lambda^{E-V}$. The combination $\chi \equiv V - E + F$ is the topological invariant known as the Euler characteristic of the surface.

It turns out that if the diagrammatic expansion is made, diagrams in which the gluon propagators cross over each other are suppressed by a positive multiplicative power of $\frac{1}{N}$ and indeed the expansion in $\frac{1}{N}$ takes the form of a genus expansion. The leading order term is a planar diagram theory. Thus at large N one may expect the interaction of two fundamental degrees of freedom to look like a two-dimensional 'sheet' of gluons. This can be thought of as looking like a tree-level open string worldsheet (the fundamental index forces the inclusion of a boundary) and perhaps the string itself can then be thought of as the basic object. There are some problems in doing this in the four dimensions one might expect the gauge theory to live in - obviously in four dimensions one has a non-critical string theory containing an unwanted anomaly. Also because it seems the string theory must necessarily be supersymmetric in order to work (the purely bosonic theory has a tachyon when quantized in a Minkowski background), one may suspect that supersymmetry may play a part in the field theory.

In recent years there has been much study of the supersymmetric cousins of $SU(N)$ gauge theory. The four-dimensional Yang-Mills action can be made globally supersymmetric by adding one, two or four spinors (four real degrees of freedom per spinor) of supercharges to give $\mathcal{N} = 1, 2, 4$ super-Yang-Mills (SYM) theories. The resulting theories contain adjoint fields of spin 0 (scalars), $\frac{1}{2}$ (gauginos) and 1 (gauge bosons), while adding more than sixteen supercharges results in fields of greater spin.

Of these theories the maximally supersymmetric $\mathcal{N} = 4$ theory has some particularly interesting features. In particular this theory manages to retain its conformal invariance upon quantization - it thus has a zero beta function (true order-by-order in perturbation theory and known to be true non-perturbatively) because the supersymmetry in the theory protects against quantum violations of scale invariance. The coupling will not run and in the large-N limit it is possible to have a theory which is at large λ (strongly-coupled) at all energy scales. We will see later that this exotic cousin of QCD does have an explicit description in terms of a string theory.

2.2 String theory

Let us now turn our attention to the basics of string theory. The discussion is based in part on lectures given by Dr. James Gray during a visit to Southampton University.

2.2.1 Stringy hadron physics

String theory arises from the attempt to quantize the embeddings of a $1+1$-dimensional manifold into a spacetime of larger dimensionality. Historically, string theory arose in the 1960s in attempts to explain the large number of high-spin hadronic resonances and also features of elastic hadron scattering eg. $\pi\pi \to \pi\pi$.

Before the modern theory of strong interactions, the proliferation of high-spin hadronic resonances was a problem. These states seemed to have masses obeying the approximate relation $m^2 = \frac{J}{\alpha'}$ where the Regge slope $\alpha \sim 1GeV^{-2}$. Of course we now have a theory telling us that these resonances are QCD bound states, but at the time they were mysterious. It did not seem they could be new fundamental particles since theories of elementary particles with spin > 1 were not known (and still are not).

We can consider the high-energy properties of $\pi\pi \to \pi\pi$ scattering by the exchange of a σ particle. The σ particle has some spin J, and experiments showed a tower of states of increasing integer J. A fundamental particle of spin > 1 leads to a divergence in the ultraviolet behaviour. It seems there is only one way that the amplitude can be made finite and that is to include an infinite tower of σ states and hope that the full sum is actually convergent (one can compare to the series $e^{-x} = 1 - x + \frac{x^2}{2!} - ...$ where the removal of any term except the first one renders the $x \to \infty$ limit divergent whereas it's finite with all terms in place). There is also the fact that Bose statistics and the trace cyclicity mean that the poles in the s channel must also be present in the t channel, and it was argued [23] that the s and t channel amplitudes should actually be the same (this also requires a sum over an infinite set of poles). In this case one need only sum over s *or* t-channel poles and each description is an alternative or 'dual' version of the other. One should be able to find a formula $\mathcal{A}(s,t)$ which can be expressed as a sum over either s or t channel poles and is symmetric in (s,t). This is given by the formula proposed by Veneziano in 1968 [24]

$$\mathcal{A}(s,t) = \beta\left(-\alpha(s), -\alpha(t)\right) \equiv \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} , \tag{2.4}$$

which works because the Euler beta function obeys the identity

$$\beta(a,b) \equiv \sum_{n=0}^{\infty} \frac{1}{b+n} \frac{(-1)^n}{n!} (a-1)(a-2)...(a-n) . \tag{2.5}$$

The function is symmetric in its arguments so the Veneziano amplitude can be expanded as a sum over s or t-channel poles.

Selecting a linear 'Regge trajectory' $\alpha(s) = \alpha' s - \alpha(0)$ means the n^{th} particle in the tower has mass squared $\frac{n-\alpha(0)}{\alpha'}$. Thus asymptotically the spin J and mass m of the resonances are related by $J \sim \alpha' m^2$. This relation between mass and spin is characteristic of a rotating relativistic string, as shown in Appendix A. What the Veneziano model is really describing is the scattering of hadrons by the exchange of string states - the Veneziano formula can be explicitly derived from a tree-level exchange of string states ((6.4) in [8]).

The description of hadronic states as relativistic strings forms a reasonable description of certain processes (Regge scattering) but it was abandoned as a model for strong interactions when found to be inconsistent with data from deep inelastic scattering (which showed point-like hadron substructure - now interpreted as quarks). There were also a number of 'technical' problems (no obvious way of coupling to external currents, 26 dimensions were required, no fermionic states were present, there was a tachyon and unwanted massless spin-2 states). Of course around 1973 a successful field theory of the strong interaction was found (QCD!) and interest in strings was diminished for a number of years.

2.2.2 Bosonic strings

In this section we review some of the basic features of the bosonic string theory.

The classical Nambu-Goto (NG) action describing the dynamics of a relativistic string

$$S_{NG} = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-Det\left(\frac{\partial X^\mu}{\partial \zeta^a}\frac{\partial X_\mu}{\partial \zeta^b}\right)} \tag{2.6}$$

can be made equivalent to the Polyakov action

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\gamma} \left(\gamma^{ab}\partial_a X^\mu \partial_b X_\mu\right) \; . \tag{2.7}$$

In a D-dimensional spacetime this has the appearance of a set of D massless scalars X^μ living on the string worldsheet.

Varying the induced metric γ_{ab} one obtains 'Einstein's equation' for the induced metric on the worldsheet

$$R_{ab} - \frac{1}{2}R\gamma_{ab} = 8\pi\alpha' T_{ab} \ , \tag{2.8}$$

where T_{ab} is just the stress-energy tensor corresponding to the scalars X^μ. Since in two dimensions the $R_{ab} \equiv \frac{1}{2}R\gamma_{ab}$ identically, one must impose $T_{ab} = 0$ as a constraint on the scalars. This constraint is

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X^\mu \partial_d X_\mu = 0 \ . \tag{2.9}$$

It is simple to show the Polyakov action recovers the NG action by computing the determinants associated to both the original induced metric $g_{ab} = \partial_a X^\mu \partial_b X_\mu$ and the independent world sheet metric γ_{ab} - using the constraint equation one obtains $\sqrt{-g} = \frac{1}{2}\sqrt{-\gamma}\gamma^{cd}\partial_c X^\mu \partial_d X_\mu$.

Classically one is free to choose any parametrization of the world-sheet and the Polyakov action is invariant. There is a special subclass of coordinate transformations which are local rescalings of the worldsheet metric

$$\gamma'_{ab} = e^{2\omega}\gamma_{ab} \ . \tag{2.10}$$

This is known as *Weyl invariance* and it will concern us greatly when we come to quantize the string.

There are a couple of other terms with the same symmetries which one could write down. Firstly, a coupling between the worldsheet Ricci scalar and a bulk scalar field Φ

$$\frac{1}{4\pi\alpha'}\int d^2\sigma\ \sqrt{-\gamma}\alpha' R\Phi\ . \tag{2.11}$$

Secondly, a coupling between the worldsheet induced metric and a bulk antisymmetric tensor field $B_{\mu\nu}$ known as the Kalb-Ramond field (in the same way the worldline of a charged particle couples to a one-form gauge potential)

$$\frac{1}{4\pi\alpha'}\int d^2\sigma\ \sqrt{-\gamma}\epsilon^{ab}B_{\mu\nu}\nabla_a X^\mu \nabla_b X^\nu\ . \tag{2.12}$$

Varying the Polyakov action with respect to the world-sheet scalars yields

$$\delta S_P = \frac{1}{4\pi\alpha'}\int_w d^2\sigma\ \left(-\nabla_a\left(2\sqrt{-\gamma}\gamma^{ab}\nabla_b X^{nu} g_{\mu\nu}\right)\right)\delta X^\mu + \frac{1}{4\pi\alpha'}\int_{\partial_w} d\sigma\ 2\sqrt{-\gamma}\gamma^{ab}\nabla_b X^\nu g_{\mu\nu}\delta X^\mu n_a \tag{2.13}$$

the second term being a world-sheet boundary term. There are two obvious ways to make this term vanish:

Neumann condition $n^b\nabla_b X^\mu = 0$ at the boundary - this means we have an open string with endpoints which move transversely to the string on null geodesics of the bulk spacetime (in the absence of any other extended objects in the theory!).

No boundary - a worldsheet without boundary describes a closed string. All fields on the string must be periodic in the coordinate used to parametrize the spacelike extent of the string.

Note one might also take $X^\mu =$ constant but this actually leads to D-branes as we shall reveal later.

The equations of motion from the Polyakov action are

$$\nabla_a \left(\sqrt{-\gamma}\gamma^{ab}\nabla_b X^\mu g_{\mu\nu}\right) = \frac{1}{2}\gamma^{ab}\nabla_a X^\mu \nabla_b X^\rho \partial_\nu g_{\mu\rho} \, , \tag{2.14}$$

$$T_{ab} = 0 \, , \tag{2.15}$$

plus equations for the fields $B_{\mu\nu}$, Φ if present in the background.

For now we specialize to Minkowski spacetime with no additional background fields present. In this case the world-sheet scalars X^μ form a two-dimensional free field theory.

There is considerable freedom to choose coordinates on the worldsheet. We will choose the conformal gauge such that the worldsheet line element is given by $ds^2 = e^{2\omega(\sigma)}\left(-d\tau^2 + d\sigma^2\right)$. One can use the Weyl symmetry to set ω to zero. Then transforming to lightcone coordinates $\sigma^\pm = \tau \pm \sigma$ one has the equations of motion in the form

$$\partial_+\partial_- X^I = 0 \tag{2.16}$$

$$T_{++} = T_{--} = 0 \, . \tag{2.17}$$

Open string mode expansion

For an open string these equations have the general solution

$$X^\mu = X_0^\mu + l^2 p^\mu \tau + il \sum \frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma \, . \tag{2.18}$$

This describes a string with spacetime momentum p_μ and which carries around excitations with the amplitudes given by the Fourier coefficient α_n^μ (note for the spacetime coordinates to be real, one must have $\alpha_{-n}^\mu \equiv \alpha_n^\mu$). Importantly, the mass squared of the string $M^2 = -p_\mu p^\mu$ is given by a Parseval's theorem-like sum over these amplitudes. The Hamiltonian related to the Lagrangian can easily be seen to be

$$H = \frac{1}{4\pi\alpha'} \int d\sigma \left(\dot{X}^2 + X'^2 \right) . \tag{2.19}$$

Because of the constraint equation ($T_{++} = T_{--} = 0$) one finds that this Hamiltonian needs to vanish. Evaluating the Hamiltonian on the solution and setting the result to zero, one finds

$$M^2 = \frac{1}{\alpha'} \sum \alpha_{-n} \cdot \alpha_n = \frac{1}{\alpha'} \sum \alpha_n \cdot \alpha_n . \tag{2.20}$$

This is clearly positive definite and so a string carrying any excited modes has a positive mass squared.

In fact the Hamiltonian is only one of an infinite set of conserved quantities associated to the string. In the lightcone coordinates the law of energy-momentum conservation $\nabla_a T^{ab} = 0$ gives the two equations $\partial_- T_{++} = \partial_+ T_{--} = 0$, each of these separately implying the existence of an infinite set of conserved charges.

The constraint equations imply the vanishing of the Fourier components

$$L_m = \frac{1}{16\pi\alpha'} \int_{-\pi}^{\pi} d\sigma \; e^{-m\sigma} \left(\dot{X} + X' \right)^2 = \frac{1}{2} \sum \alpha_{m-n} \cdot \alpha_n . \tag{2.21}$$

The constraint from the Hamiltonian is just the 'zero mode' of these ($m = 0$). In fact the solution of the equation of motion (ie the classical dynamics of the Hamiltonian) shows there are relations between the Poisson brackets of the α_m

$$[\alpha_m^\mu, \alpha_n^\nu]_{PB} = im \; \delta_{m+n,0} \; \eta^{\mu\nu} , \tag{2.22}$$

which lead to the Poisson brackets for the L_m as

$$[L_m, L_n]_{PB} = i(m-n) L_{m+n} . \tag{2.23}$$

This is known as the Virasoro algebra. Note it has a representation in terms of the simple operators $L_m \equiv \frac{d}{d(e^{imx})}$ which actually generate diffeomorphisms on a circle.

Closed string mode expansion

The general solution to the 'X' equation is a combination of left and right moving waves

$$X^I = X^I_L(\sigma^+) + X^I_R(\sigma^-)\ . \tag{2.24}$$

The expansions consistent with the periodicity of the closed string are

$$X^I_L = \frac{1}{2}x^I + \frac{\alpha'}{2}p^I\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}^I_n e^{-in\sigma^+} \tag{2.25}$$

$$X^I_R = \frac{1}{2}x^I + \frac{\alpha'}{2}p^I\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^I_n e^{-in\sigma^-}\ . \tag{2.26}$$

To ensure the reality of X^I one requires $\alpha^I_{-n} = \left(\alpha^I_n\right)^*$ and similarly for the barred coefficients.

To see what the constraint equations look like here let us compute $T_{++} = \partial_+ X^I_L \partial_+ X^J_L \eta_{IJ}$ which is zero by the constraint equation. Noting that $\partial_+ X^I_L = \sqrt{\frac{\alpha'}{2}}\sum_n \bar{\alpha}^I_n e^{-in\sigma^+}$ one computes

$$T_{++} = \frac{\alpha'}{2}\sum_{m,n}\bar{\alpha}^I_n\bar{\alpha}_{m\ I}\ e^{-i(m+n)\sigma^+} = 0\ . \tag{2.27}$$

One can set $m+n$ to any integer q obtaining the infinite set of conserved quantities

$$\bar{L}_q \equiv \frac{1}{2}\sum_n \bar{\alpha}^I_n\bar{\alpha}_{q-n\ I} = 0\ . \tag{2.28}$$

There is another set for the unbarred right movers. Both sets obey the classical Virasoro algebra.

Quantization

We now consider quantization of the degrees of freedom associated to the string. There are only a handful of background spacetimes in which it is known how to quantize strings. Again we stick to the simplest case which is flat D-dimensional Minkowski spacetime with no background fields and the task is just to canonically quantize a set of free scalar fields in 1+1 dimensions.

In order to quantize the theory canonically we replace the Poisson brackets by quantum commutators

$$[\ ,\ ]_{PB} \rightarrow \frac{i}{\hbar}[\ ,\ ]\ . \tag{2.29}$$

We replace real-valued functions on the classical phase space with operators acting on a set of quantum states $|\psi\rangle$.

In order to quantize canonically the open string we would like to obtain $[X^I, \pi^J] = i\eta^{IJ}\delta(\sigma - \sigma')$. This can be successfully implemented by promoting the Fourier coefficients in our mode expansions to operators satisfying

$$[\alpha^I_m, \alpha^J_n] = m\ \delta_{m+n}\ \eta^{IJ}\ . \tag{2.30}$$

By changing the normalization we can make these into a set of bosonic creation and annihilation operators

$$
\begin{aligned}
a_m^I &= \frac{1}{\sqrt{m}} \alpha_m^I & (2.31) \\
(a_m^I)^\dagger &= \frac{1}{\sqrt{m}} \alpha_{-m}^I \ . & (2.32)
\end{aligned}
$$

For the closed string there are two commuting oscillator algebras, barred and unbarred. Naively there is now a Fock space of states, the vacuum $|0, p^I\rangle$ annihilated by the lowering operators and some excited states such as $\alpha_{-1}^I |0, p^J\rangle$. There is a problem with states associated to the 'wrong sign' kinetic term for the timelike world-sheet scalar X^0 - there are states of negative norm (ghosts), for example one can show the first excited state of this scalar has a norm of opposite sign to the vacuum norm

$$
\langle 0, p^J | (\alpha_{-m}^0)^\dagger \alpha_{-m}^0 | 0, p^J \rangle = -m \langle 0, p^J | 0, p^J \rangle \ . \tag{2.33}
$$

This is the same problem that is fixed in the Gupta-Bleuler quantization of electrodynamics. We will use the constraints in order to eliminate the unphysical states. The classical constraints

$$
L_m = \frac{1}{2} \sum_{n=0}^{\infty} \alpha_n^I \alpha_{I\ m-n} = 0 \tag{2.34}
$$

can be unambiguously turned into operator equations for $m \neq n$ since the quantum algebra is of the form $[\alpha_{-m}^I, \alpha_{-n}^J] \sim \eta^{IJ} \delta_{m+n}$. For the $m = 0$ charge we define the normal ordered form to have the lowering operators on the right of the raising operators. The conserved quantum charge is modified to

$$
L_0 = \frac{1}{2} \sum_{n=0}^{\infty} : \alpha_{-n}^I \alpha_{I\ n} : \ . \tag{2.35}
$$

In the quantum theory the Hamiltonian constraint that $L_0 |\psi\rangle = 0$ is not a consistent physical state condition since $[L_0, L_0] |\psi\rangle$ fails to vanish due to the presence of the anomaly and we must modify $L_0 = 0$ to $(L_0 - a)|\psi\rangle = 0$ for a constant a. Evaluating the commutator $[L_m, L_n]$ using the quantum commutators in place of the classical Poisson brackets shows that the Virasoro algebra is no longer satisfied, rather one has in D spacetime dimensions

$$[L_m, L_n] = i(m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\,\delta_{m,-n}\ . \tag{2.36}$$

The new term is known as the central charge of the algebra, which becomes the centrally-extended Virasoro algebra. Taking the constant $a = 0$ would lead to the conclusion that $D = 0$ which is clearly not right. In fact only setting $a = 1$ gives the correct number of positive norm states and allows interactions.

All physical states are now required to satisfy all the constraints. One can show the ghost states do not satisfy these and are thus removed from the physical spectrum.

With the quantum version of the Hamiltonian constraint the formula for the mass of a state is modified to

$$M^2 = \frac{1}{\alpha'}\left(\sum\ \alpha_n \cdot \alpha_n - a\right)\ . \tag{2.37}$$

In the quantum theory the sum over oscillators gives a number $N = 0, 1, 2, ...$ and so the quantum states are arranged into levels. The lowest level with $N = 0$ is clearly tachyonic for $a > 0$. The next level is a state with a single Lorentz index and one finds this state has negative norm unless $a = 1$. This is the only choice which gives a correct number of degrees of freedon for this state to be a massless vector. With this choice of a the ground state of the bosonic string theory is tachyonic when treated in flat vacuum spacetime and so it does not seem a good candidate for a fundamental theory.

Strings in curved backgrounds

In the case of a string embedded in a background spacetime with some nonzero curvature one obtains an interacting two dimensional field theory on the worldsheet. This can again be quantized and one could compute the beta function. Of course, there are still ghost states and we need the Virasoro constraints in order to eliminate these. This means that we require the Weyl symmetry to be preserved by the quantization, which in turn implies we must have a two dimensional conformal field theory with vanishing beta function. The beta function involves the worldsheet scalars and takes the form of an infinite expansion in powers of α'. If the background contains a metric, a dilaton and a Kalb-Ramond field strength $H = dB$, setting the one-loop beta function to zero gives the conditions

$$
\begin{aligned}
R_{ab} + \frac{1}{4}H_a^{cd}H_{bcd} - 2\nabla_a\Phi\nabla_b\Phi &= 0 && (2.38)\\
\nabla_c H_{ab}^c - 2H_{ab}^c\nabla_c\Phi &= 0 && (2.39)\\
4\nabla_c\Phi\nabla^c\Phi - 4\nabla^2\Phi + R + \frac{1}{12}H_{cde}H^{cde} + \frac{D-26}{3\alpha'} &= 0\ . && (2.40)
\end{aligned}
$$

These consistency conditions are the Einstein, ‘Maxwell’ and Klein-Gordon equations for the bulk fields, showing that low energy perturbative string theory recovers a General Relativity - esque limit. The $D - 26$ term in the Klein-Gordon equation implies that $D = 26$ for the bosonic string. If the two-dimensional beta function is calculated to higher loop order one obtains something like (for vanishing dilaton and Kalb-Ramond fields)

$$
\beta \sim R_{ab} + \frac{\alpha'}{2}R_{acde}R_b^{cde} + \mathcal{O}(\alpha'^2)\ . \qquad (2.41)
$$

Setting this to zero gives a higher-derivative theory of gravity, with the higher terms interpreted as stringy corrections to General Relativity.

2.2.3 Supersymmetric strings and supergravity

The bosonic string theory contains a tachyon state when expanded about a Minkowski background. It is logical to seek a form of the theory which avoids this problem and one thing which can be tried is to seek a supersymmetric string theory (as done in the 1980s' 'First Superstring Revolution'). The Polyakov action can be made supersymmetric by adding fermionic degrees of freedom living on the worldsheet

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{1}{4\pi} \int d^2\sigma \sqrt{-\gamma}\, \bar{\psi}^\mu \Gamma^a \nabla_a \psi_\mu \, . \tag{2.42}$$

There is of course the usual constraint that the two-dimensional stress tensor vanishes, and in the supersymmetric case the two-dimensional supercurrent also must vanish.

The fermionic fields can obey one of two boundary conditions on the periodic worldsheet of a closed string - they can be periodic (known as the Ramond, R, sector) or antiperiodic (Neveu-Schwarz, NS, sector). The space of states consists of pairings of left and right moving degrees of freedom giving the bosonic spacetime degrees of freedom in the NS-NS and R-R sectors and spacetime fermions in the NS-R and R-NS sectors. The NS-NS sector contains massless bosonic fields corresponding to the metric, a two-form and the scalar dilaton. The R-R sector contains various form fields.

It is found that the Weyl anomaly is cancelled only if the theory lives in ten spacetime dimensions. The tachyon is removed by the GSO projection ((10.6) of [8]) which also imposes spacetime supersymmetry.

During the 1980's several consistent supersymmetric string theories were found.

Type I - a theory of open and closed un-oriented superstrings with gauge group $SO(32)$.

Type II - closed oriented theories. In IIA theory the right movers and left movers have opposite chiralities whereas in IIB theory the chiralities are the same.

Heterotic - strings with different constraint algebras for the left and right moving sectors (possible because one can select different normal-ordering constants for the left and right movers). Only one set of excitations is required to be supersymmetric and lives in ten dimensions, the other living in 26. The gauge group is either $SO(32)$ or $E8 \times E8$.

During the 1990's with the advent of D-brane physics some of these theories were shown to be connected. Further, in 1995, Witten proposed (see for example [25]) that all the consistent ten-dimensional string theories, as well as the eleven-dimensional supergravity, are various limits of a single theory which is known as M-theory.

It is possible to truncate the string theory to a low energy limit by taking $\alpha' \to 0$. One is left with the massless states and the first 'stringy' correction has mass $\mathcal{O}\left(\alpha'^{-\frac{1}{2}}\right)$. If one considers the closed oriented Type II theories, the low energy action is required to live in ten dimensions and have the same amount of supersymmetry as the string theory, thus $\mathcal{N} = 2$ supersymmetry since there is one generator for the left moving modes and one for the right movers. Depending on the relative chirality of the generators one has either type IIA or type IIB theory. We focus on the IIB theory here. There is no covariant action describing the self-dual five-form field strength so the field equations must be supplemented by a constraint to impose the self-duality. With this in mind the action can be taken to be

$$S = \frac{1}{4\kappa^2} \int d^{10}x \sqrt{-g} e^{-2\Phi} \left(2R + 8\nabla_a \Phi \nabla^a \Phi - H^2\right) - \frac{1}{4\kappa^2} \int d^{10}x \sqrt{-g} \left(F_1^2 + F_3^2 + \frac{1}{2} F_5^2\right) \tag{2.43}$$

where the latter term contains the various R-R fluxes (F_n) of the IIB theory.

There is also a Chern-Simons (topological) term and of course the fermions.

2.3 D-branes

As mentioned the R-R sector of closed string theory contains various antisymmetric tensor fields (the IIA theory contains forms C_{p+1} with p even and the IIB theory p odd). These were a puzzle since it was not clear what the sources of these fields were in perturbative string theory where there are no suitable extended objects. It was argued that the sources were the 'black' p-brane solutions of supergravity since a p-brane (an object with a $p+1$-dimensional worldvolume) would naturally couple to a $p+1$-form in a generally-covariant way. These solutions resemble higher dimensional black holes coupled to the field strength F_{p+2} from the form - the prototype is the four-dimensional Reissner-Nordström (RN) black hole. Like the RN solution the black p-brane solutions have an extremal limit where the charge saturates the BPS bound. The extremal cases interpolate between flat spacetime (away from the brane) and an anti-de Sitter spacetime (near the horizon).

This can be extended to the full string theory by identifying the extremal black p-branes with the solitonic Dp-branes which were required in string theory [12]. These D-branes are objects in string theory that allow open strings to exist within the closed string type II theories - perturbatively they are hypersurfaces where string endpoints reside. In a dimension orthogonal to the D-brane worldvolume, one can impose a Dirichlet boundary condition (the 'D' is for Dirichlet) for a string worldsheet scalar, $X^\mu = constant$, so the endpoint really is stuck at the brane position. The D-branes can also act as a source of closed strings which in the supergravity limit means they source all the massless fields eg. the metric, dilaton and the appropriate R-R forms.

Strictly the low-energy supergravity limit of a string theory D-brane is the extremal p-brane - the non-extremal case is not supersymmetric and might be expected to decay via the emission of Hawking radiation. Also the supersymmetry is needed in order to argue that the properties of the low-energy supergravity p-brane solution are valid as the coupling is increased.

2.3.1 Toroidal compactification and T-duality

One can consider the effect of compactifying one dimension of a $D+1$-dimensional theory. As an example, consider a bosonic string in background with the X^{25} direction compactified on a circle of radius R. A new feature is that the string may wind round the compact dimension a number of times w.

The closed string mode expansion can be written

$$X^\mu = x_0^\mu + \bar{x}_0^\mu - i\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \bar{\alpha}_0^\mu)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \bar{\alpha}_0^\mu)\sigma + \ldots \, . \tag{2.44}$$

The additional terms are oscillators which are automatically periodic when going round the compactified dimension. Say the string has n units of momentum on the compactified dimension. Then one clearly has

$$\alpha_0^{25} + \bar{\alpha}_0^{25} = \frac{2n}{R}\sqrt{\frac{\alpha'}{2}} \tag{2.45}$$

$$\alpha_0^{25} - \bar{\alpha}_0^{25} = wR\sqrt{\frac{\alpha}{2}} \, . \tag{2.46}$$

Inserting these into the L_0 and $\bar{L}_0$ constraints gives the mass and level matching formulae

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}\left(N + \bar{N} - 2\right) \tag{2.47}$$

$$nw + N - \bar{N} = 0 \, . \tag{2.48}$$

These equations are invariant under the exchange (n, w) and $\left(R, \frac{\alpha'}{R}\right)$. Exchanging the momentum and winding number and changing the compactification radius $R \to \frac{\alpha'}{R}$

is an invariance known as T-duality. The remarkable feature is that compactifying a dimension on a very small radius does not make the degrees of freedom associated to it go away (as they would in field theory since they would get an enormous Kaluza-Klein mass). In the string case the momentum states get large masses but winding state masses are correspondingly lowered. In fact all the properties of the world sheet quantum field theory are invariant under T-duality (though the coupling has to be shifted $g \rightarrow \frac{\sqrt{\alpha'}R}{g}$) so it is an exact symmetry of closed string perturbation theory.

One should also consider what happens in the open string case. These cannot wind around the compact dimensions so there is no winding number. The states have some associated momentum which becomes large and decouples states with momentum on the compact dimension, so one is left with a $D-1$ dimensional open string theory. In fact the string can still oscillate in the compact dimension but the endpoints are restricted to a $D-1$ dimensional hyperplane.

With supersymmetry included, T-duality interchanges the two type II theories since it reversed the relative chiralities of the left and right movers.

2.3.2 D-brane action

The dynamics of a single Dp-brane in a given background is given by the Dirac-Born-Infeld (DBI) action

$$S_{DBI} = -T_p \int d^{p+1}\xi \; e^{-\Phi} \sqrt{Det\left(\mathcal{P}[g_{ab} + B_{ab}] + 2\pi\alpha' F_{ab}\right)} \, . \tag{2.49}$$

With the metric only this is the action describing the embedding of a relativistic extended object in a background spacetime developed by Dirac in 1962 as a model for a spatially-extended electron [26]. The same action with nonzero F_{ab} was proposed by Born and Infeld in 1934 [27] as a non-linear extension of electromagnetism - in this case the action would describe a brane filling all of Minkowski spacetime.

The tension parameter is given by $T_p = \frac{1}{(2\pi)^p g_s \alpha'^{\frac{p+1}{2}}}$ - it's apparent that at weak string coupling, D-brane states are heavy compared to f-strings. The $\frac{1}{g_s}$ dependence of the tension is typical for a solitonic object.

In a background containing the correct R-R form there would also be a coupling term $Q_p \int d^{p+1}\xi \; A_{p+1}$.

In the case of a multi-brane state with N branes the brane worldvolume degrees of freedom (coordinates, gauge field and superpartners) become $N \times N$ Chan-Paton matrices and the action involves a trace over these.

If the Yang-Mills field on the brane is small this can be expanded as

$$S_{DBI} = -T_p \int d^{p+1}\xi \; \mathrm{Tr}\left(e^{-\Phi}\sqrt{Det\left(\mathcal{P}[g_{ab} + B_{ab}]\right)}\right)\left(1 + \frac{1}{4}(2\pi\alpha')^2 \mathrm{Tr}\left(F^2\right) + \mathcal{O}(\alpha'^4)\right). \tag{2.50}$$

In this limit there is clearly a Yang-Mills field on the brane worldvolume with a gauge coupling given by $g_{YM}^2 = (2\pi)^{p-2} g_s \alpha'^{\frac{p-3}{2}}$ which is obviously dimensionless for a D3 brane.

There is a bound on how strong an electromagnetic field one can have on the brane before it becomes unstable - indeed one of the ingredients of Born-Infeld theory was that there should be a maximum value for electric fields so the self-energy of a point charge would be finite. The 'stringy' interpretation of this is that the open string endpoints, which are charged, repel each other more strongly than the string tension can counter.

D-branes are BPS states. A state in one of the Type II theories with a number of (parallel) D-branes breaks half the supersymmetries of the string theory. This is because the presence of the brane means that the open strings couple to the closed strings so the separate supersymmetry of the left and right movers are no longer preserved. This is also true of the supergravity vacua which are low-energy string theory states.

D-brane technology can be used to compute the entropy of certain black holes giving agreement with the results of Hawking and Bekenstein from the 1970s [30]. An explicit computation of the entropy of an extremal black hole was presented for the first time in 1996 by Strominger and Vafa [28] and was subsequently extended to some non-extremal cases [31]. The computation can only be performed for certain black holes in string theory and is only really on a firm footing for the extremal (supersymmetric) cases (nobody can perform the calculation for the four-dimensional Schwarzschild black hole, for example). Also in the 1990s it was realized that the absorption cross section ('grey body factor') of certain black branes could be successfully computed using the field theory on the brane surface, giving a hint that this field theory had something to do with the properties of gravity away from the brane. It was these considerations that led to gauge-gravity duality.

2.4 AdS-CFT Correspondence / gauge-gravity duality

Gauge-gravity duality refers to a method for studying a large-N gauge theory using supergravity. The basic idea is to take a vacuum of string (or M-) theory containing some number N of branes. The physics of these branes is then described by the dynamics of open strings ending on the branes and the closed strings sourced by the branes. One seeks a limit in which these two sectors are decoupled and then either description is a complete description of the brane physics. Generically, the low-energy excitations of the open strings for N branes create a field theory which for coincident branes is a $U(N)$ theory. The closed strings naturally have a low-energy description in terms of supergravity in the bulk spacetime and the idea of gauge-gravity duality is to compute using this gravity description and interpret the result in terms of the open-string field theory.

The original gauge-gravity duality was proposed by Maldacena in 1997 and involved a superconformal field theory, hence the appellation Anti-de-Sitter / Conformal Field Theory (AdS/CFT) Correspondence. Nowadays a wide range of field theory / gravity duals (which are not all conformal) are known and hence 'gauge-gravity duality' is a more appropriate handle.

At first it seemed that the details of a gauge-gravity dual were strongly dependent on the microscopic theory from which it originated (string or M-theory). More recently the idea has become accepted in its own right as an effective description of a quantum field theory and is known as holography. The original ideas that a gauge theory might be described by string theory were due to 't Hooft [22], though his string theory was thought to live in the same four dimensional spacetime as the field theory. Ideas that a field theory system may be described by something higher dimensional were given by Susskind [29], inspired by the Bekenstein-Hawking results which showed that the entropy associated to a black hole is proportional to the horizon area only, not the volume.

2.4.1 The D3 brane construction

The original construction proposed by Maldacena is to start with type IIB string theory in its critical dimension $D = 10$. One then considers adding a number N of parallel D3 branes into the ten-dimensional bulk. Because the world volume where the field theory is going to live is four-dimensional and possesses Poincaré symmetry this is a natural starting point for the dual description of a 'realistic' field theory (at least it exists in the same dimensionality as real QCD!).

Open string description

Let us first consider the field theory associated to the lightest open string degrees of freedom, which will live on the four-dimensional D3 world volume. If the D3s are taken to be coincident these degrees of freedom are massless and constitute a Lorentz vector and six real scalars, plus their superpartners comprising four Weyl fermions, all in the adjoint of the $U(N)$. There is $\mathcal{N}=4$ supersymmetry which is descended from the ten-dimensional supersymmetry of the string theory. The field theory is uniquely determined by supersymmetry (of course it also comes from the $\alpha' \to 0$ limit of the nonabelian DBI action of the D3s) and is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{2g_{YM}^2} Tr\left(F_{\mu\nu}F^{\mu\nu} + 2\nabla_\mu X_i \nabla^\mu X_i - [X_i, X_j]^2\right) - \frac{i}{g_{YM}^2} Tr\left(\bar{\lambda}\gamma^\mu \nabla_\mu \lambda + i\bar{\lambda}\gamma_i [X_i, \lambda]\right) . \tag{2.51}$$

This is a remarkable field theory in that the classical conformal invariance is preserved at the quantum level, both order by order in perturbation theory and also non-perturbatively.

The gauge coupling g_{YM} is derived from the string coupling g_s by $g_{YM}^2 = 2\pi g_s$. For the low-energy string description to be valid we need the strings to be weakly coupled. This clearly means the gauge coupling is small also. However, we can consider the case where N is a free parameter and in this case the appropriate coupling to use is the 't Hooft coupling $\lambda \equiv g_{YM}^2 N$ mentioned in the previous section. In a perturbative description the group theory factors at interaction vertices become larger for increased N and increase the stringth of the coupling. Thus as we increase N we can keep the theory at strong 't Hooft coupling even as $g_s \to 0$.

Closed string description

There exists a ten-dimensional supergravity vacuum corresponding to the spacetime surrounding the N coincident D3 branes. It is constructed by considering the fields of IIB supergravity that couple to a massive object carrying D3 brane charge. The D3 charge is related to the D3 tension by a supersymmetry preservation condition which is equivalent to making the resulting solution extremal (it is a BPS state and preserves

16 of the 32 supersymmetry charges of the IIB theory). In addition the self-duality condition means that the total charge is quantized (by a Dirac quantization condition) and so one can only have an integer number of branes.

It turns out the fields which take non-trivial values are the metric and the four-form, (the two-form, dilaton and axion being zero). The vacuum solution is the extremal p-brane solution of IIB supergravity which is explicitly

$$ds^2 = f^{-\frac{1}{2}} dx_4^2 + f^{\frac{1}{2}} \left(dr^2 + r^2 d\Omega_5^2 \right) \tag{2.52}$$

$$C_{(4)} = f^{-1} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \ , \tag{2.53}$$

where

$$f = 1 + \frac{R^4}{r^4} \ . \tag{2.54}$$

In the above, $R^4 \equiv 4\pi g_s N \alpha'^2$. This parameter is not an arbitrary mass (as in the Schwarzschild solution) but is fixed by the requirement of extremality.

Decoupling limit

As things stand one might worry about the gravity description being affected when we excite some modes of the open string sector on the branes (the above solution assumes the D3s are in the ground state). We can ask if there is a limit in which we do not have to worry about this. Conceptually it may arise that the gravitational red shift factor between the surface of the branes and the exterior spacetime is so large that the excitations of the open strings are not a significant perturbation to the exterior spacetime.

It is apparent that this can indeed be made the case by making R very large in relation to the 'radius' variable r. In this limit the supergravity metric becomes

$$
\begin{aligned}
ds^2 &= \frac{r^2}{R^2}dx_4^2 + R^2\left(\frac{dr^2}{r^2} + d\Omega_5^2\right) && (2.55)\\
C_{(4)} &= \frac{r^2}{R^2}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3\ . && (2.56)
\end{aligned}
$$

This corresponds to being trapped at small $\frac{r}{R}$, the 'near-horizon' limit of the D3 brane solution. The metric is exactly the product of five-dimensional anti-de-Sitter spacetime with a five-sphere. In this limit, 16 supersymmetries that were removed by the presence of the D3 branes are actually recovered. On the supergravity side this corresponds to the fact that the spacetime isometry group is actually the superconformal group rather than the conformal group. It can be understood in terms of the string theory in that the large redshift factor prevents closed strings in the bulk from coupling to the open strings.

This region is a good description of the whole spacetime provided g_sN can be considered large. This is also the regime in which string theoretic corrections to the supergravity description become small since the curvature of the above geometry is proportional to $\frac{1}{R}$. Considering again the field theory, one recognizes that in the decoupling limit the 't Hooft coupling is large and the field theory is strongly coupled.

Now the remarkable thing is that because we have decoupled the open and closed string descriptions, we can pick either description and it must contain all information about the physics of the D branes. This physics is seen to take the form of either a strongly-coupled field theory or a weakly-coupled classical supergravity theory in the decoupling limit. It is much easier to compute in the supergravity description so this immediately becomes an immensely useful tool for studying the field theory at strong coupling.

It turns out that the D3 model has (super)conformal symmetry. This is again unlike QCD where the gauge coupling is expected to become zero at very high energy (asymptotic freedom). In all of the deformations of AdS-CFT, the UV limit always recovers the conformal symmetry and supersymmetry.

The field theory can be thought of as residing on the boundary of AdS space. The 'radius' parameter can be thought of as representing the energy scale of the theory. The $\mathcal{N}=4$ theory of the original theory is invariant under changes of scale and this is reflected in the fact that the gravity dual is self-similar under translations in the r direction.

It can be seen that the global symmetries of the two theories are the same. Both IIB supergravity and the $\mathcal{N}=4$ theory have an $SL(2,\mathbf{Z})$ symmetry group. Both have 32 supersymmetries. The $SO(6)$ symmetry in the geometry is matched by the $SO(6)\sim SU(4)$ R-symmetry group of the field theory. The spacetime isometry group of AdS_5, $SO(2,4)$, is precisely the field theory conformal symmetry group.

The AdS-CFT Correspondence is yet to be proven mathematically, though work is currently being done with the aim of showing that $\mathcal{N}=4$ SYM is integrable (ie completely solvable) [32].

2.4.2 Operator-field matching

Let us first consider a scalar field on an $AdS_5\times S^5$ background. There is no coupling to the four-form. The metric is

$$ds^2=\frac{r^2}{R^2}dx_4^2+R^2\left(\frac{dr^2}{r^2}+d\Omega_5^2\right)\ . \tag{2.57}$$

We are going to solve this by doing a harmonic decomposition on AdS_5 to find the effective mass on the S^5 (so we'll end up with a compactified theory with $\nabla^2_{(S^5)}\Phi+\mu^2R^2\Phi=0$). To this end let us solve the massive Klein-Gordon equation, $\nabla^2\Phi=m^2R^2\Phi$ on AdS_5. This takes the form, for a function with only r-dependence

$$\frac{1}{r^3}\left(r^5\Phi'\right)' = m^2R^2\Phi = \mu^2R^2\Phi \ . \tag{2.58}$$

The equation is solved by $\Phi \propto r^{\lambda_\pm}$ where $\lambda_\pm = \frac{-4\pm\sqrt{16+4\mu^2}}{2}$ with both roots obeying the quadratic $\lambda\left(\lambda+4\right) = \mu^2$.

The authors of [14, 15] proposed using the asymptotic behaviour of supergravity fields in AdS to propose an explicit formula connecting the supergravity partition function and observables in the dual field theory. Given a supergravity source ϕ_0 at the boundary of AdS and the corresponding operator $\mathcal{O}$ of the conformal field theory, one has

$$e^{-S_{SUGRA}[\phi_0]} = \langle e^{\int d^4x\ \phi_0\mathcal{O}}\rangle_{CFT} \ . \tag{2.59}$$

The great thing is that for each supergravity field on $AdS_5\times S^5$ there is an operator with the correct superconformal algebra quantum numbers (mass dimension, Lorentz, $SU(4)_R$) in the set of chiral primary operators of the $\mathcal{N}=4$ field theory. This is how one picks the correct operator $\mathcal{O}$ to put with a given SUGRA source ϕ_0. This formula is very remarkable in that one can extract n-point functions at strong coupling easily simply by finding the appropriate Green's function corresponding to a source in AdS.

It is clear that for a supergravity field scaling like $\sim r^{\lambda_+}$ near the boundary of AdS, Witten's formula gives $\Delta = \lambda_+ + 4$ as the operator dimension which therefore satisfies $\Delta\left(\Delta-4\right) = \mu^2$. Now one has

$$\Delta = \frac{4+\sqrt{16+4\mu^2}}{2} \ , \tag{2.60}$$

and so for a relevant operator $\Delta < 4$ and one immediately sees this corresponds to a tachyonic scalar in AdS_5 ($\mu^2 < 0$). For a marginal operator one has $\Delta = 4$ corresponding to a massless scalar and for an irrelevant operator $\Delta > 4$. Intuitively $\Delta > 4$ corresponds to $\lambda_+ > 0$ which implies this field diverges toward the boundary of AdS_5 ($r\to\infty$). This is in keeping with the picture that the AdS_5 radius r represents the field theory energy scale.

The most negative value of μ^2 is in fact $\mu^2 = -4$. This is the most negative mass squared which is stable. One can see how this might arise by considering the action for a massless scalar in AdS_5 (note actually the Klein-Gordon scalar does not have a tachyonic mode). The action is

$$S = \int d^5x \, \sqrt{-g} \left(\nabla_a \phi \nabla^a \phi + m^2 \phi^2 \right) \, . \tag{2.61}$$

The solution is $\phi \propto r^\lambda$ where $\lambda(\lambda+4) = m^2$. Plugging that in one has an action density

$$S \sim \int dr \, r^{3+\lambda} \left(\lambda^2 + m^2 \right) \, . \tag{2.62}$$

The quantity $\lambda^2 + m^2$ equates to $8 + 2m^2 - \pm 4\sqrt{4+m^2}$. It is easy to see that this is a positive quantity provided $m^2 > -4$ which is actually the correct bound for AdS_5 (this is known as the Breitenlohner-Freedman bound [33]. In short tachyons are allowed because the vacuum configuration has some variation in the radial direction which prevents the action from becoming a runaway negative.

The operator dimension also ought to tell you what representation of $SO(6)$ or $SU(4)$ the SUGRA field needs to live in. For this one needs to understand spherical harmonics on the S^5. For simple fields the operator dimension Δ is an integer (due to the supersymmetry) from which it follows that μ^2 is a positive or negative integer and we need it to satisfy $\nabla^2_{(S^5)}\Phi + \mu^2\Phi = 0$. In fact, for spherical harmonics on S^5 one has always $\nabla^2_{(S^5)}\Phi + l(l+4)\Phi = 0$ for S^5 spin l and so one deduces $l = \lambda$. Note this is only true for operators described by the complex SUGRA scalar in the AdS_5 dual, otherwise the harmonics are clearly not scalar harmonics.

It turns out that the Kaluza-Klein harmonics of the IIB supergravity can be uniquely matched to the chiral primary operators of the $\mathcal{N} = 4$ theory. These operators are in small representations (short multiplets) of the supersymmetry algebra and so have their operator dimensions protected by supersymmetry. The supergravity masses are also protected by supersymmetry. Operators not in short multiplets are the duals of excited 'string' states in $AdS_5 \times S^5$ - their dimensions and supergravity masses become large at large λ ($\Delta \sim \lambda^{\frac{1}{4}}$).

2.5 Non-zero temperature

The prescription for placing a gravitational dual at non zero temperature is to add a black hole into the geometry. The thermodynamic properties of the black hole are then associated to the thermodynamics of the degrees of freedom in the dual field theory. The black horizon cuts off all energy scales below some scale in the radial holographic direction, which is naturally associated to the Hawking temperature of the horizon.

The gravitational dual of the $\mathcal{N} = 4$ theory is obtained by replacing the AdS_5 with the metric known as AdS-Schwarzschild

$$ds^2 = \frac{r^2}{R^2}\left(-f dt^2 + dx_3^2\right) + R^2 \frac{dr^2}{r^2 f} \ , \tag{2.63}$$

where $f = 1 - \frac{r_H^4}{r^4}$ giving a plane four dimensional black horizon at $r = r_H$.

There is a Hawking temperature which is easily computed by going to the Euclidean section and demanding the geometry be non-singular. Putting $r = r_H\left(1 + \frac{x^2}{R^2}\right)$ the line element in the Euclidean time τ and radial direction plane is

$$ds^2 = \frac{4r_H^2}{R^4} d\tau^2 + dx^2 \ . \tag{2.64}$$

In order for there not to be a conical surfeit/deficit at $x = 0$, the periodicity of Euclidean time needs to be $\frac{\pi R^2}{r_H}$ which is the reciprocal of the Hawking temperature. Thus $T_H = \frac{r_H}{\pi R^2}$. Note in AdS, the larger the horizon radius the larger the temperature, which is unlike the four-dimensional Schwarzschild solution ($k_B T_H = \frac{\hbar c^3}{8\pi GM}$ with all physical constants explicit).

It is possible to imagine another way of putting the theory at a nonzero temperature and that is to go to the Euclidean section and compactify the Euclidean time with a period of $\beta = \frac{1}{T}$. One would then consider comparing the free energy of this geometry to that of the Euclidean black hole to find which 'phase' is favoured at a given temperature. This is known as the Hawking-Page phase transition. For uncompactified spatial dimensions one finds that for any $T > 0$ the black hole phase has the lower free energy and is the thermodynamically-favoured phase. The free energy of the periodic AdS phase is of order unity while that of the black hole phase scales as N^2 - the interpretation is that in the black hole phase the gauge degrees of freedom are deconfined as was argued by Witten in [64].

Chapter 3

Adding fundamental matter to gauge-gravity duals

In this chapter we will review the construction of the gravity dual with fundamental matter. This consists of the near-horizon D3 brane solution with added probe D7 branes. We outline a procedure for computing the cloud of meson density surrounding a quark in the $\mathcal{N} = 2$ theory and also compute the meson-strahlung by an accelerated quark.

3.1 Fundamental flavoured matter from D-branes

In the AdS/CFT Correspondence described so far we have a gravitational dual of a field theory in which all fields transform in the adjoint representation of the gauge group. In order to better make contact with the gauge theories thought to describe nature, we would like to be able to include fields transforming in the fundamental representation of the gauge group. Almost all experimentally-known hadronic states contain valence quarks and, in addition, at strong coupling, the effect of quark-antiquark loops on the dynamics of the gauge field itself is expected to be strong in QCD (though not in a large-N field theory).

The fields in the gravity dual to $\mathcal{N} = 4$ SYM transform in the adjoint because there are two string endpoints in contact with the D3-branes, each carrying a fundamental index. In order to add fundamental matter, there needs to be one fundamental index and hence only one string endpoint in contact with the D3s. This can be accomplished by allowing a different type of D-brane to be present in the geometry as was done in [16]. The IIB string theory contains Dp branes where $p = 1, 3, 5, 7, 9$. In order that the field theory fundamental matter lives in four dimensions, the Minkowski directions of the spacetime should be filled by the brane. A D3 brane away from the stack of N coincident ones is known to describe a spontaneously broken $SU(N)$ theory which is just $\mathcal{N} = 4$ SYM on its moduli space (although a probe D3 brane at *infinite* separation behaves as effectively a quark of infinite mass and can be used to study the response of the theory to an external source in the fundamental representation [34]). It is hard to see how a D9 brane would describe any meaningful dynamics since it would fill the ten-dimensional spacetime entirely. Thus if we wish to add four-dimensional fundamental matter we are led to consider adding D5 or D7 branes. It turns out that the D7 is easier to understand since the D3-D7 intersection preserves half of the supersymmetry of $\mathcal{N} = 4$ SYM whereas the D5 breaks all supersymmetry. In the supersymmetric case, the anomalous dimension of operators describing the quark mass and condensate are zero and thus in the ultraviolet of the theory have the same value as we expect for the ultraviolet of QCD (also zero due to asymptotic freedom).

In order to add extra branes into the gravity dual, we need to find a solution to the IIB supergravity equations consistent with the presence of D3 and D7 branes. The reaction of the D7 brane to the geometry will tell about the effect of the gauge theory vacuum on the fundamental matter and the backreaction effect of the D7 on the geometry will tell how the gauge theory is affected by the presence of fundamental loops. It is actually difficult to construct the full solution [37, 39] but progress can be made by neglecting the backreaction effects (known as the quenched approximation). This can be justified if the number of flavour branes we add is much smaller than the number of D3 branes, that is $N_f \ll N$. In this regime, the procedure is simply to extremize the D7 brane action when embedded into the background.

Under this procedure a quark should be thought of as a fundamental string with one end point on one of the D3's and the other on a D7 brane. The mass of the quark is given by the product of the string tension and the length of this stretched string.

Let us search for a regular probe D7 brane embedding in the $AdS_5 \times S^5$ geometry. It is convenient to take the D7 branes to wrap an S^3 of the S^5 (this is part of the statement that some of the supersymmetry is preserved). We will look for a solution in which the D7 fills the Minkowski directions, the radial direction down to some particular value, and three out of the five angles on the S^5. The set-up is as follows

	0	1	2	3	4	5	6	7	8	9
D3	-	-	-	-	•	•	•	•	•	•
D7	-	-	-	-	-	-	-	-	•	•

and the geometry for $AdS_5 \times S^5$ is then parametrized as in [40],

$$ds^2 = \frac{\rho^2 + r^2}{R^2} dx_4^2 + \frac{R^2}{\rho^2 + r^2}\left(d\rho^2 + \rho^2 d\Omega_3^2 + dr^2 + r^2 d\varphi^2\right) \ . \tag{3.1}$$

The embedding can be set to constant φ and $r = r(\rho)$. Note that this embedding function encodes the source and vev of the bilinear quark condensate operator, via the leading asymptotic terms near $\rho \to \infty$, $r \sim d + \frac{c}{\rho^3}$. These clearly have the correct radial dependence to represent a dimension one and dimension three operator and in fact

$$\begin{aligned} d &= \frac{m_q}{2\pi\alpha'} && (3.2)\\ c &= \frac{\langle\bar{\psi}\psi\rangle}{(2\pi\alpha')^3}\;. && (3.3)\end{aligned}$$

The constant d is a current algebra quark mass and the constant c is a quark condensate.

The dynamics of the brane in the curved background are governed by the Dirac-Born-Infeld (DBI) action (there is no coupling between the D7 and the self-dual five-form field strength). For our D7 brane embedding the bosonic part takes the form (where the tension is, in string theory constants, $T_7 = \frac{1}{(2\pi)^7\alpha'^4 g_s}$):

$$S_{D7} = -T_7 \int d^8\zeta \sqrt{-Det\left(g_{MN}\frac{\partial x^M}{\partial\zeta^a}\frac{\partial x^N}{\partial\zeta^b}\right)}\;. \qquad (3.4)$$

This is

$$S_{D7} = -T_7 \int d\rho\, \rho^3 \sqrt{1+r'^2}\;. \qquad (3.5)$$

The equation of motion is

$$\partial_\rho\left(\frac{\rho^3 r'}{\sqrt{1+r'^2}}\right) = 0\;. \qquad (3.6)$$

Clearly the quantity in the brackets is a constant (J) and so the equation is just $r' = \pm\frac{J}{\sqrt{\rho^6-J^2}}$. Since we expect the D7 to occupy the range $\rho \in [-\infty, +\infty]$ we must have $J = 0$ for a regular embedding (else the gradient clearly blows up at a finite value of ρ). This shows that all regular solutions have $r' = 0$.

The regular solutions are therefore

$$r(\rho) = d\;. \qquad (3.7)$$

This is just a trivial constant. These solutions preserve $\mathcal{N} = 2$ supersymmetry. By the above asymptotic, these solutions describe a quark mass m_q and zero quark vev $\langle\bar{\psi}\psi\rangle$. The supersymmetry ensures that the quark mass does not run (so there is simply a current algebra mass) and also means a non-zero bifundamental condensate is forbidden. We have added in a scale d to the problem so for $d \neq 0$ there is no longer conformal symmetry in the theory coupled to fundamental matter - we therefore anticipate the presence of a discrete set of fluctuations of the D7 brane with a scale set by d. This is indeed the case and the normalizable fluctuations have the interpretation as the mesons of the theory ($\bar{\psi}\psi$ bound states) whose spectrum can be found analytically.

Note we have ignored back-reaction effects in the above, justified because at large 't Hooft coupling there are a large number of colours N_C and we added only one flavour brane. In fact one can consider adding N_f flavour branes and if $N_f \sim N_C$ one would have to allow for the effect of the D7 brane reaction on the background. Since the D3-D7 intersection is still $\mathcal{N} = 2$ supersymmetric we expect the embedding solutions we have found will survive even if back-reaction is fully allowed for. An example of a simple method for including back-reaction in a simple physical system can be found in Appendix B.

3.2 Field theory of the D3/D7 intersection

By adding a number N_f D7 branes we have added additional field content to the dual field theory. This takes the form of N_f $\mathcal{N} = 2$ hypermultiplets transforming in the fundamental representation of the gauge group. These modes come from the lightest modes of the 3-7 strings. One finds that the 7-7 string modes (describing mesons) are decoupled from the string sectors (3-3, 3-7, 7-3) because the eight-dimensional 't Hooft coupling for the D7 branes is given by $\lambda' = \lambda(2\pi l_s)^2 \frac{N_f}{N}$ which vanishes in the supergravity $\alpha' \to 0$ limit. This makes the $U(N_f)$ group a global flavour symmetry group in the gauge theory.

The actual degrees of freedom associated to the 3-7 strings are in the fundamental representation of $SU(N)$ and comprise scalars and spin-$\frac{1}{2}$ superpartners. One notable result is that the one-loop beta function is $\beta \propto \lambda \frac{N_f}{N}$ which is positive. This is actually the exact all-order pertubative beta function - it is possible that it is modified by nonperturbative effects.

3.3 Meson spectra from the D7 brane

We will assay a fluctuation which is a plane wave on the Lorentz-symmetric 'Minkowski' part of the geometry and which is a harmonic function on the S^3 with quantum number l (this gives the R-charge of the state). The equation of motion from linearizing the DBI action is

$$\frac{1}{\rho^3}\left(\rho^3\Phi'\right)' + \frac{R^4}{(\rho^2+d^2)^2}M^2\Phi - \frac{l(l+2)}{\rho^2}\Phi = 0 \ . \tag{3.8}$$

Making the rescalings $x \equiv \frac{\rho}{d}$ and $\mu^2 \equiv \frac{M^2R^4}{d^2}$ one obtains the equation

$$\Phi'' + \frac{3}{x}\Phi' + \left(\frac{\mu^2}{(1+x^2)^2} - \frac{l(l+2)}{x^2}\right)\Phi = 0 \ . \tag{3.9}$$

Taking the equation for the spin-zero mode ($l = 0$) one can define $z = 1+r^2$ to transform to

$$(z-1)\Phi'' + 2\Phi' + \frac{\mu^2}{4z^2}\Phi = 0 \ . \tag{3.10}$$

Defining $\Phi \equiv z^p u$ with $p(p-1) = \frac{\mu^2}{4}$ one has

$$z(1-z)u'' + (2p - 2(p+1)z)\,u' - p(p+1)u = 0 \ . \tag{3.11}$$

This is the hypergeometric equation with $a = p$, $b = p+1$, $c = 2p$. There is a globally-regular, polynomial solution if either a or b is zero or a negative integer and c is not a negative integer. Thus we will set $b = 0, -1, -2, \ldots$. This clearly implies $4(n+1)(n+2) = \lambda$ for $n = 0, 1, 2, \ldots$ which is the result from [40]. Note our solution looks slightly different to theirs but this is simply the fact that the singularities at 0 and 1 can be swapped without changing the hypergeometric function. So we've found that the regular solutions are

$$\Phi = (1+r^2)^p \; {}_2F_1\left(p, p+1, 2p; 1+r^2\right) \; . \tag{3.12}$$

Thus we have found the discrete meson mass spectrum for scalar mesons

$$M = \frac{2d}{R^2}\sqrt{(n+l+1)(n+l+2)}\,. \tag{3.13}$$

Since the current algebra quark mass is given by $m_q = \frac{d}{2\pi\alpha'}$ this can be expressed as

$$M = \frac{2\sqrt{\pi} m_q}{\sqrt{g_s N}}\sqrt{(n+l+1)(n+l+2)}\,. \tag{3.14}$$

It is worth noting that this supersymmetric field theory is quite unlike QCD. The meson masses are suppressed relative to the quark mass by a factor of $\sim \frac{1}{\sqrt{\lambda}}$ so at strong coupling they are extremely light compared to the hard quark masses. In QCD the scale of light meson masses is typically of order Λ_{QCD} (much greater than the hard quark masses!) and the majority of the meson mass comes from chromodynamic energy. We will see in later work that certain deformations of AdS-CFT allow us to approach QCD-like meson physics much more closely. Another problem to note is that states of non-zero R-charge have masses of the same order as R singlet - so this model is unlike realistic models of supersymmetric QCD which need to have R-charged states more massive by order of the SUSY breaking scale.

Of course there are also vector mesons coming from the Maxwell field associated to the D7 brane DBI action, and fermionic 'mesino' states which are superpartners to the integer-spin degrees of freedom [59]. The pattern of all of these masses is essentially the same as we have shown for the scalar case. Naturally, a one-to-one matching can be done for bosonic and fermionic degrees of freedom since the theory is supersymmetric.

Here AdS/CFT has given the full solution to bound states in a field theory which is strongly coupled - a remarkable achievement since the result is non-perturbative in the field theory. There have been many extensions to the method described above and a review of meson physics in gauge-gravity duals can be found in [35].

The solutions are basically of the 'particle in a box' type ($m^2 \propto n^2$) - this could have been anticipated from plotting the effective Schrödinger potential corresponding to the eigenvalue equation. Transforming eq.(3.8) into a Schrödinger equation by changing variable to $d \tan dy = \rho$ where the new variable y is defined on $[0, \frac{\pi}{2d}]$ one sees the potential is

$$V_S(y) = d^2 \left(\frac{5 + 8l(l+2) + \cos 4dy}{2 \sin^2 2dy} \right) . \tag{3.15}$$

These potentials look very much like a 'box'. One sees that the quark mass parameter d sets the depth of the 'box' and the inverse width.

Note that if we take the limit $d \to 0$ one obtains $V_s = \frac{\frac{3}{4} + l(l+2)}{y^2}$ with $y \geq 0$ this is clearly a semi-infinite box which does not support standing waves - what has happened is that in this limit the theory is conformal and there are no discrete states.

In the massless case it is easy to treat the spinor excitation which is a superpartner to the scalar fluctuation (because the D7-brane world volume has a simple product structure 4×4 gamma matrices can be used). The massive Dirac equation on AdS_5 takes the form

$$\left(i\Gamma^a \partial_a - 2i\gamma_4 + m\right)\psi = 0\,. \tag{3.16}$$

The mass comes from the spin on the sphere but shifted down one unit by coupling to the five-form field strength.

Using the gamma matrices

$$\gamma_0 = \begin{pmatrix} \cdot & I \\ I & \cdot \end{pmatrix} \quad \gamma_4 = \begin{pmatrix} -iI & \cdot \\ \cdot & iI \end{pmatrix}\,,$$

and assuming a Lorentz-invariant state in which u_1 and u_2 multiply constant two-spinors one has

$$\begin{pmatrix} -x\partial_x - 2 + m & \frac{\omega}{x} \\ \frac{\omega}{x} & x\partial_x + 2 + m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0\,.$$

One can eliminate one of the functions to obtain, for example,

$$u_1'' + \frac{6}{x}u_1' - \frac{(m-2)(m+3)}{x^2}u_1 + \frac{\omega^2}{x^4}u_1 = 0\,. \tag{3.17}$$

Transforming to a Schrödinger problem and inserting the mass $m = \frac{1}{2} + l$ for $l = 0, 1, 2, ..$ from the spin one finds a potential

$$V_S(y) = \frac{(\frac{1}{2}+l)(\frac{3}{2}+l)}{y^2}\,. \tag{3.18}$$

This is clearly the same potential as for the scalar (obvious since the setup is supersymmetric).

3.4 Green's functions on the brane: dressed quarks and radiation

The prototype 'quark' can be visualized as a string of minimum length stretching between the D7 brane and the D3 brane. Since the string endpoint carries $U(1)$ charge with respect to the D7 brane worldvolume gauge field (which describes baryon number chemical potential) one expects the quark to act as a source of baryon number density. This 'back reaction' effect can be computed by evaluating to electric potential sourced by a point electric charge on the D7 worldvolume ie. by computing the holographic Green's function. We can think of the holographic Green's functions as encoding information about the way a probe quark will 'dress' itself in a cloud of meson density. We will find that, for a non-accelerating quark, there is a Yukawa potential for each meson mode, with the weighting for each mode given by the amplitude of the partial wave at the position of the string endpoint, $\rho = 0$. One can also calculate the radiation into rho mesons by an accelerated string endpoint (one imagines the endpoint is moved by the string pulling on it as the quark is accelerated by the gauge field dynamics). This work forms part of [18].

Rho meson production

To understand how the string solutions above radiate energy into hadronic modes, one must study the electromagnetic theory on the surface of the D7 brane. We take a D7 embedding $r = d$ with induced metric

$$P[G] \equiv g = \frac{\rho^2 + d^2}{R^2} dx_4^2 + \frac{R^2}{\rho^2 + d^2}(d\rho^2 + \rho^2 d\Omega_3^2) \ . \tag{3.19}$$

Here, ρ is the radial direction on the world volume of the D7 so that $\rho^2 = r_1^2 + r_2^2 + r_3^2 + r_4^2$. Although the analysis will be done for this simple setup we will see that the ρ dependence of the problem enters essentially just through the mass of the mesonic states which can therefore be easily altered to any more complicated set up (for example the QCD-like deformed geometries studied in Chapter 5).

The end points of the string act as electrically charged sources for the gauge field that lives on the world volume of the D7 brane. The equation of motion for that gauge field follows from the variation of the electromagnetic action on the brane,

$$S_{\mathrm{EM}} = -\frac{1}{4}\int d^8x\sqrt{-g}F_{ab}F^{ab} + \int d^8x\sqrt{-g}j^a A_a \ . \qquad (3.20)$$

The variation of the gauge field gives both the equation of motion and the boundary action,

$$\delta S_{\mathrm{EM}} = \int d^8x\sqrt{-g}\delta A_b\left[\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}F^{ab}) + j^b\right] + \delta S_{\mathrm{bdy}} \ , \qquad (3.21)$$

so that the equations of motion are just Maxwell's equations,

$$\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}F^{ba}) = j^b \ , \qquad (3.22)$$

and there is a boundary action (at $\rho = 1/\epsilon \to \infty$)

$$\delta S_{\mathrm{bdy}} = -\int_{\rho=\frac{1}{\epsilon}} d^4x d\Omega_3\sqrt{-g}\delta A_a F^{\rho a} \ . \qquad (3.23)$$

Let us first understand the solutions in the absence of a source.

Rho mesons

The propagating modes of the gauge field arrange themselves into multiplets of the $SO(4)$ isometry group of the S^3. The resulting Kaluza-Klein fields each map to different operators of the gauge theory; in particular, the singlet on the S^3 maps to a conserved baryon current [40]. To study this current, we therefore give the solutions of Eq. (3.22) without sources for the modes that only have non-zero A_μ and are singlets on the S^3. We impose the gauge choice $\nabla_\mu A^\mu = 0$. The equation of motion in the absence of sources is then [40]

$$\mathcal{D}A_\mu = -\frac{1}{\rho^3}\partial_\rho(\rho^3\partial_\rho A_\mu) - \frac{R^4}{(\rho^2+d^2)^2}\nabla^2_{(4)}A_\mu = 0\ , \tag{3.24}$$

where $\nabla^2_{(4)} = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the scalar Laplacian in Minkowski space and $\mathcal{D}$ is a second-order differential operator. Fourier transforming in the Minkowski directions, we can write the equation above as an eigenvalue equation in the radial coordinate with eigenfunctions $f_n(\rho)$ given by

$$f_n(\rho) = I_n \frac{{}_2F_1(-n\ , -n+1, 2; -\rho^2/r_0^2)}{(\rho^2+d^2)^{n+1}}\ , \tag{3.25}$$

where the I_n are normalization constants, $n = 0, 1, 2...$, and eigenvalues

$$M_n^2 = 4(n+1)(n+2)\frac{d^2}{R^4}\ . \tag{3.26}$$

The solutions to Eq. (3.24) are therefore given by the modes $A_{\mu,n} = \epsilon_\mu f_n(\rho) e^{ik_n\cdot x}$ with $k_n^2 = -M_n^2$.

These states have a discrete mass spectrum and are identified with the rho mesons of the dual gauge theory. The factor of $d^2/R^4 \sim m_q^2/\lambda$ indicates that the meson masses are much smaller than the quark mass at large 't Hooft coupling. Moreover, the f_n (with appropriate choices of the I_n normalizations) are orthonormal functions (subject to the weight factor w)

$$\int d\rho\ w\ f_n f_m = \delta_{mn}\ , \quad w = \frac{\rho^3 R^4}{(\rho^3+d^2)^2}\ . \tag{3.27}$$

We are using the solutions appropriate for the $\mathcal{N}=2$ D3-D7 configuration. However, as we will see, the holographic directions only enter into our final radiation computation through the masses they endow the four dimensional rho mesons and the value of I_n, the normalization of the wave functions. In the more complicated case, one could simply switch the spectrum and normalizations as appropriate.

Green's functions

To observe the emission of rho mesons by the string end points we will solve (3.22) by means of a Green's function for the field A_μ. For the minimum-energy configuration quark (a string connecting the D3 and D7 at their point of closest approach) the endpoint lies at $\rho = 0$ where the volume of the three-sphere is zero and hence the Green's function is a constant on the three-sphere (the equivalent of the f_n functions for R-charged states fall to zero at $\rho = 0$). This implies that there is no production of R-charged rho mesons associated with non-trivial spherical harmonics on the S^3 (this is quite interesting given that it is a generic problem with gravitational duals that R-charged meson states tend to have masses comparable to the R-singlets). For a more generic string motion such states would be produced. Moreover, since the sources do not move in the r^{1-4} four-plane, both the radial and angular components of the source current j^a vanish and thus A_ρ and A_i (the components of the gauge field along the S^3) also vanish in this gauge. We can therefore consider only the Minkowski components of the Green's function.

Having chosen the Lorenz gauge, that Green's function satisfies

$$\mathcal{D}G^{\mu'}_{\mu} = \frac{1}{\rho^3}\delta^{\mu'}_{\mu}\delta(\rho - \rho')\delta(x^\nu - x^{\nu'}) \ , \tag{3.28}$$

where $\mathcal{D}$ is the differential operator defined in Eq. (3.24). Since the equation of motion for the gauge field in the presence of our source is given by

$$\mathcal{D}A_\mu = \eta_{\mu\nu}j^\nu, \tag{3.29}$$

the full solution for an arbitrary current distribution j^μ follows from the convolution integral

$$A_\mu(x) = \int d^8x' \sqrt{-g}\, G^{\mu'}_{\mu}(x, x')\eta_{\mu'\nu'}j^{\nu'}(x') \ . \tag{3.30}$$

The actual current distribution will be localized on the worldline of the string endpoint and will take the form $j^\mu = q\int d\tau \dot{x}^\mu \delta^8(x)$ where the dot represents differentiation with respect to proper time.

In order to obtain the Green's function, let us expand in the basis of eigenfunctions describing the rho mesons used in Eq.(3.25) so that $G^{\mu'}_\mu(\rho, x^\nu; \rho', x'^\nu) = \sum_n f_n(\rho) f_n(\rho') \bar{G}^{\mu'}_{n,\mu}(x^\nu, x'^\nu)$. Inserting this form into Eq.(3.28), multiplying by $\rho^3 f_m$ and integrating over all space we find that the four-dimensional functions $\bar{G}_n$ are just the Green's functions for massive vectors in Minkowski spacetime with masses corresponding to the rho meson masses.

Boundary data

The near-boundary behaviour of the gauge field is related to the one-point function of the dual conserved baryon current in the field theory. In particular, that one-point function is given as

$$\langle J^\mu(x^\nu)\rangle = \lim_{\epsilon\to 0} \frac{\delta S_{\mathrm{SUGRA}}}{\delta A_\mu(x^\nu, 1/\epsilon)} , \tag{3.31}$$

where S_{SUGRA} is the on-shell bulk gravity action and the bulk gauge field A_μ is the singlet mode on the S^3. Using the variation of the bulk action in Eq. (3.21), the boundary current is simply

$$\langle J^\mu(x^\nu)\rangle = -\lim_{\epsilon\to 0} \rho^3 \eta^{\mu\nu} \partial_\rho A_\nu(x^\nu, \rho)|_{\rho=1/\epsilon} . \tag{3.32}$$

We can therefore write a bulk-to-boundary Green's function that relates the bulk source to the boundary current. In particular, we write

$$\langle J^\mu(x^\nu)\rangle = \int d^8x' \sqrt{-g}\, \mathcal{G}^\mu_{\ \mu'}(x^\nu; x') j^{\mu'}(x') , \tag{3.33}$$

where we define the bulk-to-boundary Green's function $\mathcal{G}$ as

$$\mathcal{G}^\mu_{\ \mu'}(x^\nu; x'^\nu, \rho') \equiv \sum_n 2(-1)^n I_n f_n(\rho') \bar{G}^\mu_{n,\mu'}(x^\nu, x'^\nu) , \tag{3.34}$$

where the f_n are the eigenfunctions in Eq. (3.25) and $\bar{G}_n$ is the 4d Green's function for a massive vector as before. The factor of $2(-1)^n I_n$ comes from the insertion of the near-boundary expansion of the f_n's,

$$f_n(\rho) = (-1)^{(n+1)} I_n \frac{1}{\rho^2} + O(\rho)^{-3} , \tag{3.35}$$

into the form of the boundary current in Eq. (3.32).

Retarded potential

We have now reduced the problem to solving for each mode G_n the retarded potential for a massive field in flat space. The retarded potential takes the form [36]

$$\bar{G}^{\mu}_{\ \mu'} = \frac{1}{4\pi}\theta(t-t')\left(\delta(\sigma) + V(\sigma)\theta(-\sigma)\right)\delta^{\mu}_{\ \mu'} . \tag{3.36}$$

Here we use the Synge world-function $\sigma = \frac{1}{2}\eta_{\mu\nu}(x-x')^{\mu}(x-x')^{\nu}$. The non-singular part of the solution is given by $V(\sigma) = -\frac{M_n}{\sqrt{-2\sigma}} J_1(M_n\sqrt{-2\sigma})$ where J_1 is the Bessel function of order 1.

Static string endpoint: a dressed quark

As a first example of using this formalism we will compute the baryon density around a static quark. Consider such a charge at $x = 0$ and at $\rho = 0$ ($r = r_0$), the point of closest approach on the D7 brane. We will concentrate on the temporal component of the gauge field A^0 which is dual to the operator $\bar{\psi}\gamma^0\psi$, the quark density.

One seeks to evaluate the integral over the past trajectory of a point source moving with a constant speed in a 'static gauge' given by $x' = \beta t'$. Doing the spatial integral using the fact that the source is located at a point in the space-like dimensions leaves one with the integral

$$\begin{aligned} \left\langle J^0(x^{\nu})\right\rangle_n = {} & \tfrac{2(-1)^n I_n^2 q}{4\pi}\times \\ & \textstyle\int dt'\, \theta(t-t')\left(\delta(\sigma) + V(\sigma)\theta(-\sigma)\right)\frac{d\tau}{dt'}\cdot\frac{dt'}{d\tau} \end{aligned} \tag{3.37}$$

where we have used the fact that $f_n(\rho = 0) = I_n$.

The time component of the four-velocity is actually cancelled by a Lorentz factor coming from the splitting of Minkowski spacetime into space-like sections when integrating along the particle worldline. The two contributions to the integral are easily computed (for the non-singular piece due to the massive field using the integration variable $u \equiv M_n\sqrt{-2\sigma}$)) giving the correctly Lorentz-covariant expression (γ is the usual boost factor)

$$\left\langle J^0(x^\nu)\right\rangle_n = \frac{2(-1)^n I_n^2 q}{4\pi}\frac{\gamma\, e^{-M_n\left(\sqrt{\gamma^2(x-\beta t)^2+y^2+z^2}\right)}}{\sqrt{\gamma^2(x-\beta t)^2+y^2+z^2}}\,. \tag{3.38}$$

In the rest frame of the point source this reduces to the usual Yukawa form. The full solution is a sum over modes weighted by the f_n normalizations I_n^2 - these factors are plotted in Figure 8. There is a rapid rise in these normalizing factors with n which is due to the end point of the string being a delta function (in fact the formula $I_n^2 = 2(n+1)(n+2)(3+2n)$ was found by one of my co-authors of [18](KJ)). Away from $N \to \infty$ one would expect the string to have some width and the expansion to truncate at some intermediate n. In any case this rise is not faster than the exponential fall off of the solutions so the physics away from the source is still dominated by the lightest modes. The Green's function converges for all $|x| > 0$ due to the exponential factor in the Yukawa potential of each partial wave. The behaviour is dominated by the lighter modes at distances comparable to the Compton wavelength of the lightest mode. We interpret this Green's function as the 'dressing' of an isolated quark by a cloud of mesons. Holography gives the relative amounts of each of the excited states in the cloud.

Radiation from accelerated string endpoints

We now have a framework in which the emission of mesons can be modelled using the techniques of classical relativistic wave equations. The retarded Green's function is straightforwardly integrated over the past worldline of an accelerating endpoint, giving, for a particle moving in the x-direction (the u variable is as defined in the preceding

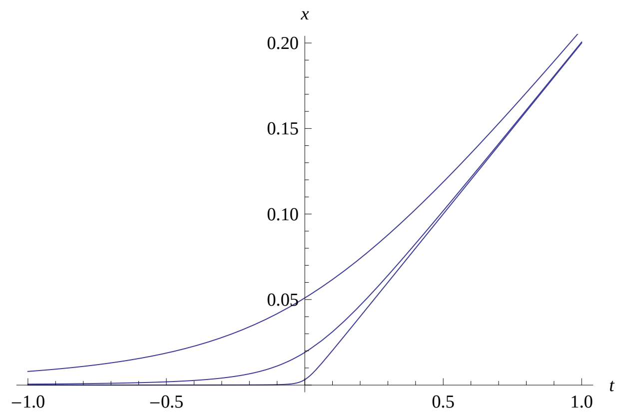


Figure 3.1: Plots of the function $x'(t')$ in (3.41) used to describe the motion of an accelerating point source. The parameter a controls the final speed and is set to $a = 0.2$ here. b controls the time scale of the acceleration and the plots show $b = 0.8$ (top), $b = 0.3$ (middle) and $b = 0.05$ (bottom).

section)

$$\left\langle J^0(x^\mu)\right\rangle_n = \frac{2(-1)^n I_n^2 q}{4\pi}\left[\frac{1}{t-t'(\sigma=0)-\frac{dx'}{dt'}(x-x'(\sigma=0))}+ \int_0^\infty du \frac{J_1(u)}{t-t'-\frac{dx'}{dt'}(x-x')}\right] . \tag{3.39}$$

We will again plot the baryon number density which is holographically encoded by the sum over the J_n^0. It may be noted that the plots we obtain give the superposition of radiated baryon density and the static baryon density associated with the probe quark. An elementary prescription is available for computing the reaction force on the probe quark due to the radiation (by differencing the advanced and retarded potentials) but this is not what we are interested in here (it involves a negative counting of the non-causal advanced potential and so would not produce a plot resembling meson *emission*). In holographic scenarios (large N) the force exerted on the quark by the dynamics of the colour flux tube far exceeds the reaction force from meson emission anyway. In the case of an *instantaneous* acceleration it is possible to subtract the appropriate static and boosted solutions inside and outside of the particle's light cone but we do not apply this here.

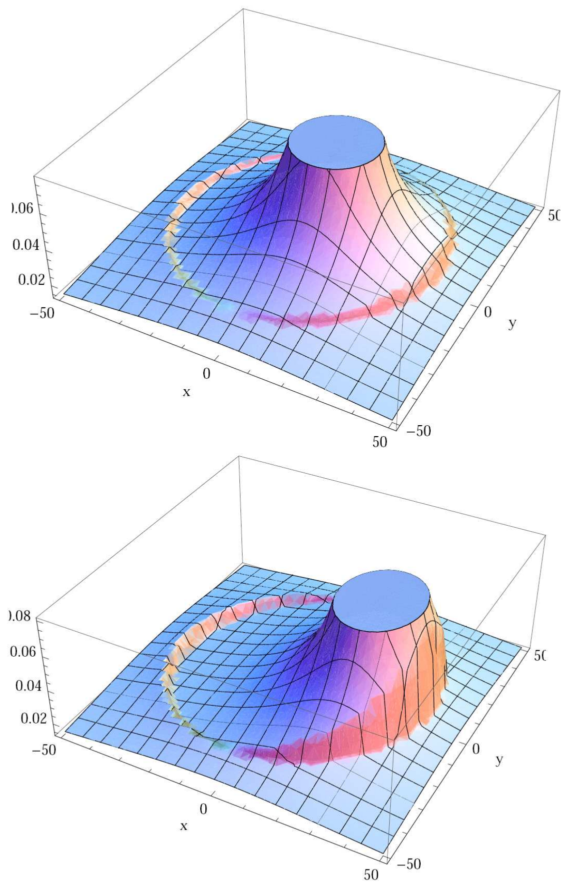


Figure 3.2: The radiation of light mesons by a quark given an impulse in the positive x-direction, shown in the $z = 0$ plane. The plot shows the density of the emitted (radiative part of field) and bound (boosted static part) mesons. The top plot is for a terminal velocity of $0.2c$ and below is $0.6c$ (in both plots the parameter $b = 0.2$).

Massless meson limit

In the strict $\lambda \to \infty$ limit the meson masses are very small relative to the string mass (see (3.26)). At least for the lightest members of the tower, it is therefore interesting to compute the radiation into a massless gauge field on the D7. For this case $\sigma = \frac{1}{2}\left(-(t-t')^2 + (x - x'(t'))^2\right)$. The first term in (3.39) then gives

$$\begin{aligned}\left\langle J^0(x^\nu)\right\rangle_n &= \tfrac{2(-1)^n I_n^2 q}{4\pi} \int_{-\infty}^{t} dt' \delta(\sigma) \\ &= \tfrac{2(-1)^n I_n^2 q}{4\pi} \int \tfrac{d\sigma \delta(\sigma)}{\left((t'-t)+(x'-x)\frac{dx'}{dt'}\right)} \\ &= \tfrac{2(-1)^n I_n^2}{4\pi} \tfrac{1}{t-t'(\sigma=0)-\dot{x}_0(x-x'(\sigma=0))} \; .\end{aligned} \tag{3.40}$$

This is straightforward to evaluate for accelerations of the string endpoint. For example, the static end point is accelerated quickly to a constant speed. As an example form for the function $x'(t')$ that describes a stationary particle accelerating to a final speed a we take

$$x'(t') = \frac{ab}{\pi} + at'\left(\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{t'}{b}\right)\right) \; , \tag{3.41}$$

where the b controls the time interval over which the acceleration occurs - we plot some sample trajectories in Fig.(3.1).

It is a simple matter to plot the resulting wave induced. Emission is typically a spherical shell radiating from the point of acceleration - there is an SO(2) symmetry in the y, z coordinates so we shall plot the intensity of the wave in the x, y plane at $z = 0$. Examples of the gauge field produced are shown in Fig.(3.2). The radiative piece is visible along with the 'hill' of the boosted static potential. A clear, narrow emission wave is observable. For larger values of the final speed a the forward emission is typically enhanced relative to the backwards emission, and the overall emission is greater. For smaller acceleration times (smaller b) the wave front simply becomes narrower. For the accelerations of the string end points in given by the trajectories of (3.41) we expect precisely such emission of the lower mass members of the mesonic tower.

Massive meson limit

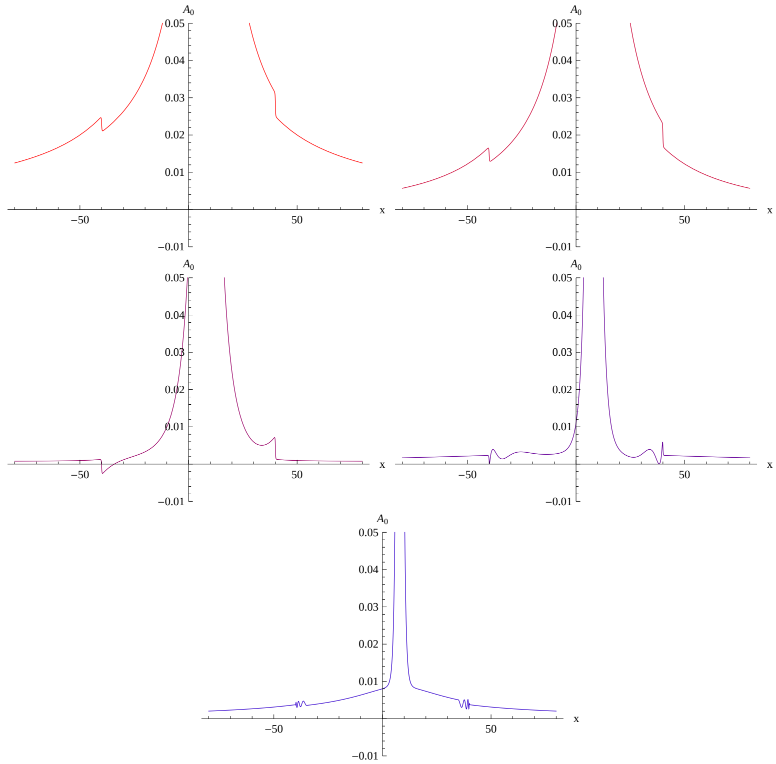


Figure 3.3: The radiation of massive mesons by a quark given an impulse in the positive x-direction, plotted along the x-axis. The plot shows the density of the emitted mesons. The meson masses increase through the plots as $0, 10^{-2}, 10^{-1}, \frac{1}{3}, 1$. Note the background static field peak becomes narrower as the mass is increased.

For members of the meson tower with masses close to the quark mass (very high n at large 'tHooft coupling) we must compute the non-singular term in (3.39) which involves numerical integration of the Bessel function. In Figs.(3.3) and (3.4) we show the effect of increased meson mass on the radiated mesons (the meson mass should be compared to the inverse time over which the string end point is accelerated). As the meson mass is increased we find emission of the more massive states are suppressed.

In the massive case the waves are dispersive and produce an interesting pattern which is not just a wave localized on the light-front. The 'wavy' emission of meson density can be considered to be arising from quantum-mechanical interference effects.

In principle we could sum over the emission of all of the meson states. At large 'tHooft coupling though there are many states lighter than the quark mass so the result would be unilluminating. The precise form of the meson masses and the coefficients I_n^2 are also model dependent. However, we believe that the computations we have made show how in principle the radiation could be computed and give a good understanding of the generic features of that meson radiation.

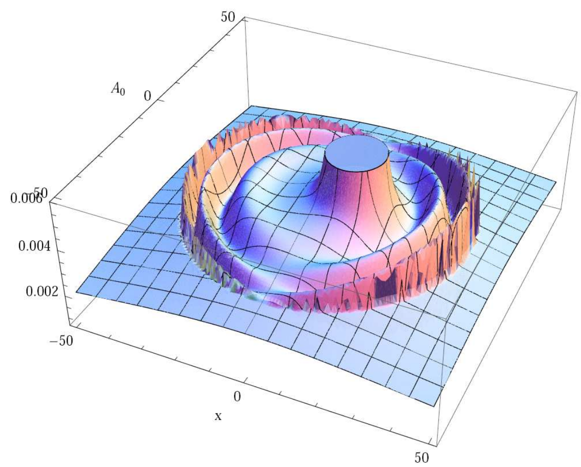


Figure 3.4: Emission of massive vector mesons ($m = \frac{1}{2}$). The parameters $a = b = 0.2$.

3.5 The meson spectrum with D7 brane backreaction

We note that including the D7 brane in the probe limit allowed us to study the behaviour of quarks coupled to the $\mathcal{N} = 4$ theory. What this limit does not capture is the effect of fundamental matter loops in intermediate states. For example, the fundamental matter would be expected to give the gauge theory a running coupling by contributing to the beta function (as discussed in section). To include such effects, the gravity dual prescription is to allow for the back reaction of the D7 branes on the D3 geometry. This is called the D3-D7 brane intersection geometry. It still preserves the $\mathcal{N} = 2$ supersymmetry (at zero temperature). Of course, in the large-N limit one expects the D3 geometry to be little affected by the presence of a small number of flavours ($N_f \ll N$). However in QCD the number of colours and the number of flavours are comparable and we would like to study the effects of virtual quark loops on the gauge dynamics.

In the literature an approximate solution for the D3-D7 intersection geometry is available (under the assumption of a logarithmic dilaton profile). The result is an $\mathcal{N} = 2$ supersymmetric theory with a running coupling (obviously there is going to be a positive beta function once we include the effect of virtual quark loops) and a non-trivial theta angle.

One can attempt to add a probe D7 into this system and compute the meson spectrum, which would allow for the presence of say N_f flavours of sea quark. It is most convenient to take the sea quarks to be massless.

The back-reacted D3-D7 supergravity solution is given by

$$ds^2 = h^{-\frac{1}{2}} dx_4^2 + h^{\frac{1}{2}} \left(d\rho^2 + \rho^2 d\Omega_3^2 + e^{-\phi}(dw^2 + w^2 d\theta^2) \right) . \qquad (3.42)$$

Here $h = 1 + \frac{R^4}{(\rho^2 + e^{-\phi} w^2)^2}$, where R is the usual $R^4 = 4\pi g_s N \alpha'^2$ and $e^{-\phi} = \frac{N_f}{4\pi} \ln\left(\frac{w_\Lambda^2}{w^2}\right)$ which also specifies the nontrivial dilaton in the backreacted geometry. Note this solution is the 'near core' (small w limit) of the actual solution which is given in terms of a series expansion in [39].

One can add a single quenched flavour into this background. The action is

$$S \sim \int d\rho\, e^{\phi} \left(\sqrt{1 + e^{-\phi} (\partial_\rho w)^2} - 1 \right) . \tag{3.43}$$

The latter term comes from the coupling of the D7 to the C_8 field which is the Hodge dual of the dilaton. This action is clearly extremized for w a constant, just like in the unquenched background. The fact that the probe D7 still lies flat is because there is still $\mathcal{N} = 2$ supersymmetry.

We choose to add the probe brane at $w = d$. Then the Lagrangian for a small fluctuation φ (ie leading order) of the brane in the geometry is

$$\mathcal{L} = \frac{1}{2} \rho^3 e^{\phi(w=d)} \left(\frac{R^4}{(\rho^2 + e^{-\phi(w=d)} d^2)^2} (\partial_x \varphi)^2 + (\partial_\rho \varphi)^2 \right) . \tag{3.44}$$

This is the same as for the calculation in the unquenched background except for the replacement $d \to d \left(\frac{N_f}{4\pi} \ln \left(\frac{w_\Lambda^2}{d^2} \right) \right)$. Thus the meson spectrum is given by

$$M^2 = \frac{8\pi}{g_s N} \left(\frac{N_f}{4\pi} \ln \left(\frac{\Lambda^2}{m_q^2} \right) \right) m_q^2 (n+1)(n+2) , \tag{3.45}$$

for $n = 0, 1, 2, ...$

The effect of unquenching the sea quarks is just to replace the gauge coupling with the appropriate renormalized value at the scale of the probe quark mass.

Chapter 4

Quasinormal modes of gauge-gravity duals

Generically, the eigenfrequencies of classical wave equations on manifolds containing horizons (ie black hole spacetimes) admit spectra of complex eigenvalues ω which are interpreted as field states with mass $Re(\omega)$ and decay width $Im(\omega)$ as the field is absorbed by the horizon. In this chapter we will examine some applications of this physics to strongly-coupled gauge theory at finite temperature where the gravity dual contains a black hole. In this chapter we examine the mesonic thermal spectrum of the Sakai-Sugimoto model of holographic QCD by finding the quasinormal frequencies of the supergravity dual. If flavour is added using D8-$\bar{D}8$ branes there exist embeddings where the D-brane worldvolume contains a black hole. For these embeddings (the high-temperature phase of the Sakai-Sugimoto model) we determine the quasinormal spectra of scalar and vector mesons arising from the worldvolume DBI action of the D-brane. We stress the importance of a coordinate change that makes the in-falling quasinormal modes regular at the horizon allowing a simple numerical shooting technique. Finally we examine the effect of finite spatial momentum on quasinormal spectra. We also briefly examine *the quasinormal modes associated to a spin-$\frac{1}{2}$ field in the AdS-Schwarzschild background which is the dual of nonzero temperature $\mathcal{N} = 4$ super-Yang-Mills theory.*

4.1 Introduction

Gravity dual descriptions [13, 14, 15] of strongly coupled gauge theories with quarks have recently shed light on the physics of mesons and chiral symmetry breaking [35]. There has also been considerable interest in studying the finite temperature behaviour of these systems.

The simplest such dual is the near horizon geometry of a D3-D7 brane system which describes an $\mathcal{N} = 2$ gauge theory. The D7 branes can be treated as probes [16] in the limit where the number of flavours is much less than the number of colours, $N_f \ll N_c$, or the full back reacted geometry can be found [37, 38, 39]. The meson spectrum in the probe limit has been computed in [40]. Finite temperature manifests itself as the presence of a black hole in the dual space-time [14]. In the infinite volume limit the black hole geometry is energetically preferred for any temperature greater than zero. The transition from an AdS space corresponds to the analogue of the deconfinement transition in the pure glue gauge theory (which in a conformal theory occurs as soon as the dimensionful parameter, T, is introduced). A further first order phase transition has also been found in this system [41, 42, 43, 44, 93] when the temperature passes through the scale of the mass of the mesonic bound states - this corresponds to when the horizon of the black hole grows to swallow the D7 probe in the interior of the space. This transition has an associated small jump in the chiral condensate's value but the main physics of the transition appears to be the meson fields melting into the thermal background. Once the D7 brane enters the horizon there are no longer normalizable fuctuations of the D7 brane that generate a discrete set of meson bound states. Instead there are quasi-normal modes of the black hole corresponding to fluctuations of the

D7 brane that are pure in-falling at the horizon. These fluctuations correspond in the dual gauge theory to excitations of the plasma with a complex mass parameter - the excitations have both a mass and a decay time. The spectrum of these quasi-normal modes has been explicitly computed in [51].

Another interesting model is the Sakai-Sugimoto model [46, 47] which is based on a (wrapped) D4 D8 $\bar{D}8$ system. It is a gravity dual of a non-supersymmetric gauge theory (which is four dimensional in the IR but five dimensional in the UV) that dynamically breaks a non-abelian chiral symmetry of its quark fields. The high temperature phase again corresponds to a transition to a black hole geometry. The transition occurs when the black hole's radius becomes of order the wrapped circumference of the D4 brane which is also the parameter that determines the mass gap of the theory. This behaviour is more akin to what one would expect in QCD than that of the conformal theory discussed above.

Massless chiral quarks can be introduced by placing the probe D8 and $\bar{D}8$ branes at antipodal points on the circle the D4 brane is wrapped on. In the near horizon limit of the D4 branes these D8 branes choose to join at the scale of the mass gap breaking the chiral symmetries on their world volumes to the diagonal sub-group and generating a mass gap for the mesonic fluctuations of the D8s. When the geometry makes the transition at finite temperature to the black hole background the D8 and $\bar{D}8$ disconnect and instead lie straight and fall into the horizon [50]. Chiral symmetry breaking is therefore restored along with deconfinement.

There is a larger class of embeddings in which the D8 and $\bar{D}8$ join at a larger radius in the space so there is a bigger mass gap for the quarks. In [48] we have argued that these embeddings describe a quark mass in the theory although it has been also argued in the literature [49] that the chiral symmetry breaking scale is being enhanced in these cases by higher dimension operators. The distinction is not important for what we discuss here - in these cases there is a further first order transition as the temperature (horizon) grows through the mass scale of the mesons. The transition is very much like that of the D3-D7 system in that the mesonic fluctuations of the D8 branes are replaced by quasinormal modes of the black hole. The mesons of the theory have melted into the plasma.

The mesonic fluctuations of the D8 branes above the phase transition have been studied in [53]. Here we will concentrate on the very high temperature phase where the D8 branes lie straight and fall into the black hole horizon. We will explicitly compute the quasinormal mode spectrum corresponding to the scalar and vector mesons of the theory.

As a prelude to this we compute the quasinormal spectrum of a Klein-Gordon scalar living on the D8 brane worldvolume. We apply the idea of regularizing the coordinates for ingoing modes, which has previously been used in asymptotically-flat spacetimes (see for example [54]) and in the context of AdS-CFT in [55]. The result of this is that the ingoing mode is described by a regular Taylor series at the black hole horizon. We use this as the initial condition and obtain the quasinormal spectra by shooting out from the horizon. We wish to stress that this is a much cleaner numerical process than trying to match on to oscillating solutions at the horizon. We use the same method to examine the spectra of modes arising from the DBI action of an embedded D8 brane. We treat a scalar fluctuation of the brane in the geometry and a Lorentz vector arising from the Maxwell field on the D-brane. Finally we briefly discuss the effect of nonzero momentum on the spectra and extract the diffusion coefficient from the lowest quasinormal modes of longitudinal vector excitations in the small k 'hydrodynamic' limit.

4.2 The Sakai-Sugimoto model

The aim of the Sakai-Sugimoto model is to provide a dual description of large-N QCD coupled to N_f species of fundamental matter in the probe limit. The background geometry is that surrounding N D4-branes in IIA string theory, compactified on a circle.

One can choose antiperiodic boundary conditions for the fermions on the circle so they all get masses of order the inverse compactification radius. Thus the low energy field theory content is just four-dimensional large-N QCD with fermions and KK excitations suppressed at energies below the inverse compactification radius. Of course, the field theory is five-dimensional in the UV and therefore requires a high-energy completion - presumably this is just the open string theory.

The metric and dilaton describing the ten-dimensional bulk geometry is

$$ds^2 = \left(\frac{u}{R}\right)^{\frac{3}{2}}\left(dx_4^2 + f d\tau^2\right) + \left(\frac{R}{u}\right)^{\frac{3}{2}}\left(\frac{du^2}{f} + u^2 d\Omega_4^2\right) \tag{4.1}$$

$$e^{\phi} = g_s\left(\frac{u}{R}\right)^{\frac{3}{4}} . \tag{4.2}$$

Here $f = 1 - \left(\frac{u_{KK}}{u}\right)^3$. The non-trivial dilaton is an indication that the theory is not conformal and does in fact exhibit running coupling.

Into this geometry Sakai and Sugimoto introduced a probe D8 / $\bar{D}8$ brane pair with the set up as follows (the fifth dimension (τ) is the compactified one)

	0	1	2	3	(4)	5	6	7	8	9
D4	-	-	-	-	-	•	•	•	•	•
D8	-	-	-	-	•	-	-	-	-	-

The key insight is that the D8 / $\bar{D}8$ brane pair likes to join together, breaking the $U(N_f) \times U(N_f)$ flavour symmetry group spontaneously to the vector subgroup - this is the geometrical manifestation of chiral symmetry breaking.

The brane system exhibits a rich spectrum of mesonic excitations which are found from the spectrum of the D8 brane DBI action in a similar way as for the D3-D7 system.

There is also a rich phenomenology associated to the Sakai-Sugimoto model at non-zero temperature, as explored in [50]. Just as there is a non extremal of the near-horizon D3 geometry (AdS-Schwarzschild), there is a non-extremal version of the compactified D4 background which is a candidate for the D4 field theory at non zero temperature. One can seek a phase transition between the black hole phase and the Euclidean time phase. One finds the black hole phase has the lower free energy above some finite temperature and thus is the high-temperature phase. Now one can ask about the behaviour of the probe fundamental matter in the high-temperature background - the free energy can be computed by evaluating the DBI action associated to a single probe D8.

We will begin with the (non-Euclideanized) metric for the finite temperature Sakai-Sugimoto model in the form

$$ds^2 = \left(\frac{u}{R}\right)^{\frac{3}{2}} \left(-f dt^2 + dx_3^2 + d\tau^2\right) + \left(\frac{R}{u}\right)^{\frac{3}{2}} \left(\frac{du^2}{f} + u^2 d\Omega_4^2\right) . \tag{4.3}$$

Here $f = 1 - \left(\frac{u_T}{u}\right)^3$ and the dilaton is $e^{\phi} = g_s \left(\frac{u}{R}\right)^{\frac{3}{4}}$.

Parameterizing the D8-embedding by $\tau(u)$ and calculating the DBI action one finds

$$S_{DBI} = \int du g_s^{-1} R^{\frac{3}{2}} u^{\frac{5}{2}} \sqrt{1 + \left(\frac{u}{R}\right)^3 f\, \tau'^2} \, . \tag{4.4}$$

From this we obtain the equation describing the D8 embedding. One sees the first integral $\frac{\partial \mathcal{L}}{\partial \tau'} = J$. The equation is then

$$\tau' = \left(\frac{R}{u}\right)^{\frac{3}{2}} \frac{J}{\sqrt{g_s^{-2} u^8 f^2 - J^2 f}} \, . \tag{4.5}$$

Insisting the D8 turns over at $u = u_0$ gives us $J = g_s^{-1} u_0^4 \sqrt{f(u_0)}$.

Inserting this solution into the action we obtain the expression

$$S_{DBI} = g_s^{-1} R^{\frac{3}{2}} \int_{u_0}^{\infty} du\, u^{\frac{5}{2}} \frac{1}{\sqrt{1 - \frac{u_0^8 f(u_0)}{u^8 f(u)}}} \, . \tag{4.6}$$

Now the thing to note is the equation for the D8-embedding also admits a solution $\tau'(u) = 0$ with $J = 0$. This 'flat' configuration would have an action

$$S_{DBI}^{flat} = g_s^{-1} R^{\frac{3}{2}} \int_{u_T}^{\infty} du\, u^{\frac{5}{2}} \ . \tag{4.7}$$

Now although the two actions above are infinite their difference is finite and may be evaluated. Changing variables to $x \equiv \frac{u}{u_0}$ (and now the temperature enters as $x_T \equiv \frac{u_T}{u_0}$, a parameter in the range [0,1])

$$\Delta S \equiv \frac{S_{DBI} - S_{DBI}^{flat}}{g_s^{-1} R^{\frac{3}{2}} u_0^{\frac{7}{2}}} = \left(\int_1^{\infty} dx\, x^{\frac{5}{2}} \frac{1}{\sqrt{1 - \frac{1}{x^8} \frac{1 - x_T^3}{1 - \frac{x_T^3}{x^3}}}} \right) + \frac{2}{7} \left(x_T^{\frac{7}{2}} - 1 \right) \ . \tag{4.8}$$

The extra term in from the fact that the curved embedding goes down to $u = u_0$ whereas the flat embedding goes all the way down to the horizon at $u = u_T$.

Plotting this (Fig.(4.1)) reproduces Fig.(6) of [50]. Clearly for larger x_T the flat embedding has the lower action, so embeddings that would have had turnarounds near the horizon become flat. In u_0 is large compared to u_T then x_T is close to zero and the curved brane embedding is favoured.

This analysis indicates that in the limit of high temperature, the D8 branes lie flat and have a black hole horizon on their world volume. As a result of the latter, there is not a disrete spectrum of mesons with real masses. Instead one expects mesons to have a quasinormal spectrum and we now turn our attention to finding the characteristic complex frequencies of these modes.

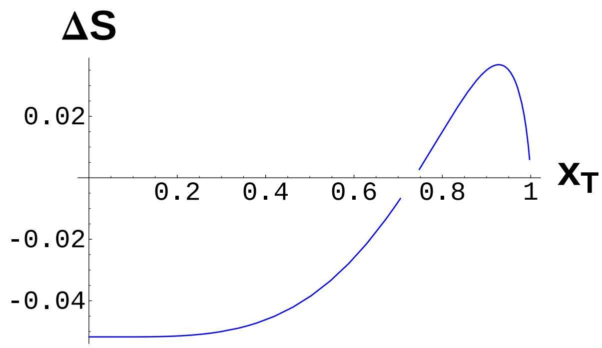


Figure 4.1: Plot illustrating which phase has lower action. For $\Delta S > 0$ the 'flat' D8 embedding is favoured (corresponding to the higher temperature phase). For $\Delta S < 0$ the curved D8 embedding is favoured and the flavour branes do not intersect the black hole horizon.

4.3 The geometry

The metric of the high temperature Sakai-Sugimoto model is

$$\begin{aligned} ds^2 &= \left(\frac{u}{R}\right)^{\frac{3}{2}} \left(-f(u)dt^2 + dx_3^2 + d\tau^2\right) \\ &\quad + \left(\frac{R}{u}\right)^{\frac{3}{2}} \left(\frac{du^2}{f(u)} + u^2 d\Omega_4^2\right) . \end{aligned} \tag{4.9}$$

Here $f(u) = 1 - \left(\frac{u_T}{u}\right)^3$ and the dilaton is $e^{-\phi} = g_s^{-1}\left(\frac{R}{u}\right)^{\frac{3}{4}}$.

The parameter u_T, representing the position of the horizon in the geometry, gives the temperature in the dual field theory by the relation $T = \frac{3}{4\pi}\frac{u_T^{\frac{1}{2}}}{R^{\frac{3}{2}}}$. This is the Hawking temperature of the black hole.

Let us work in the dimensionless radial coordinate $x \equiv \frac{u}{u_T}$ and measure the Minkowski and τ dimensions in units of $\sqrt{\frac{R^3}{u_T}}$. This corresponds to measuring frequencies and momenta in units $\propto T$.

The metric becomes

$$
\begin{aligned}
\frac{ds^2}{R^{\frac{3}{2}}\sqrt{u_T}} &= x^{\frac{3}{2}}\left(-f(x)dt^2 + dx_3^2 + d\tau^2\right) \\
&\quad + x^{-\frac{3}{2}}\left(\frac{dx^2}{f(x)} + x^2 d\Omega_4^2\right) .
\end{aligned}
\tag{4.10}
$$

We will consider the spectrum of modes associated to a D8 brane. The D8 branes fill the space except for the τ direction in which they live at a single value of $\tau = \tau_0$ - this is the energetically preferred high temperature configuration [50]. The induced metric on the D8 brane worldvolume is just (4.10) with $d\tau = 0$.

4.4 Regular horizon coordinates for quasinormal modes

As a warm-up we shall first consider linear fluctuations of a Klein-Gordon scalar restricted to the worldvolume of the D8 brane. This is not a physical mesonic state of the field theory but it exhibits similar quasinormal modes in the supergravity dual and we use it to illustrate our calculational technique. The 9D action describing the fluctuation is

$$
\mathcal{S}_{SF} = \frac{1}{2}\int d^9x\sqrt{-g}g^{ab}\nabla_a\Phi\nabla_b\Phi . \tag{4.11}
$$

Here the geometry is given by the induced metric on the D8 brane.

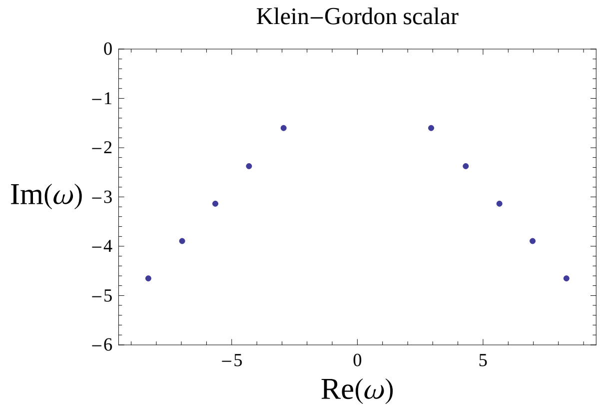
The equation of motion for the fluctuation is

$$
\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}g^{ab}\partial_b\Phi\right) = 0 . \tag{4.12}
$$

Writing out the equation for a scalar fluctuation with spatial momentum k with respect to the plasma rest frame and zero S_4 spin as $\Phi \propto e^{-i\omega t + ik\cdot x_3}$ one obtains the equation

$$
\left(x^{\frac{19}{4}}f(x)\Phi'\right)' + x^{\frac{7}{4}}\left(\frac{\omega^2}{f(x)} - k^2\right)\Phi = 0 . \tag{4.13}
$$

Here the prime indicates an x derivative. The large-x asymptotic of the equation is $\left(x^{\frac{19}{4}}\Phi'\right)' = 0$ with solution $\Phi \sim c_1 + c_2\, x^{-\frac{15}{4}}$. For a normalizable solution we clearly want the decaying power.



n	Ω_n
1	$\pm$ 2.9348 - 1.6032 i
2	$\pm$ 4.3133 - 2.3760 i
3	$\pm$ 5.6509 - 3.1367 i
4	$\pm$ 6.9725 - 3.8928 i
5	$\pm$ 8.3151 - 4.6531 i

Figure 4.2: The lowest five $k = 0$ Klein-Gordon scalar quasinormal frequencies in the complex ω plane.

Taking the near horizon limit ($x \to 1$) one finds $\Phi \sim (x-1)^{\pm i\frac{\omega}{3}}$. Since the whole solution is $\exp(-i\omega t \pm i\frac{\omega}{3}\ln(x-1))$ the solutions are ingoing ($-$) and outcoming ($+$) waves which oscillate infinitely many times before reaching the horizon.

In these coordinates it will be hard to find the full solution numerically - one needs to shoot out from, or on to, a highly oscillatory solution near the horizon. A better way to proceed is to change coordinates to make the infalling solution regular at the horizon so numerical methods can be more easily used. In particular we will shift coordinates so

$$t = h - \alpha(x) \; . \tag{4.14}$$

To make the infalling solution regular we require

$$\alpha(x) = \frac{1}{3}\ln(x-1) + \; ... \; , \tag{4.15}$$

where the additional terms are regular at the horizon. One way of satisfying this is to define

$$\frac{\partial \alpha}{\partial x} = \frac{1}{x^3 - 1} \; . \tag{4.16}$$

We then have

$$dh = dt + \frac{1}{x^3 f(x)}\, dx \ , \tag{4.17}$$

leaving the metric in the new coordinates as

$$\begin{aligned} ds^2 \quad &= \quad -x^{\frac{3}{2}} f(x) dh^2 + 2x^{-\frac{3}{2}} dh\ dx + x^{-\frac{3}{2}} dx^2 \\ &\quad + x^{\frac{3}{2}} \left(dx_3^2 + d\tau^2 \right) + \sqrt{x} d\Omega_4^2 \ . \end{aligned} \tag{4.18}$$

The induced metric is the above with $d\tau = 0$.

The coordinates we have chosen here make the spatial sections look as if there is no black hole present - in this sense they are analogous to the Painléve-Gullstrand coordinates used in Appendix C. There are infinitely many other coordinate systems which preserve the regularity of the ingoing wave at the horizon. Coordinates corresponding to ingoing photon trajectories are an example - these were tried but did not behave as well numerically as the ones above.

Note that the near horizon solution of (4.13) only depended on the powers of f in the function not the powers of x (which becomes one in the near horizon limit). These powers of f will turn out to be the same for all of the modes we consider below and so this change of coordinates will suffice to make all infalling modes we look at regular.

We can now recompute the scalar equation of motion and we find

$$\left(x^{\frac{19}{4}} f \Phi' \right)' - i\omega \left(2x^{\frac{7}{4}} \Phi' + \frac{7}{4} x^{\frac{3}{4}} \Phi \right) + \left(\omega^2 - k^2 \right) x^{\frac{7}{4}} \Phi = 0 \ . \tag{4.19}$$

We note that this is an equation with five singular points in the complex x-plane. The general solution of such equations is not given in terms of well-known functions, nor can one immediately apply the continued fraction method as used in [56]. Accordingly we use a purely numerical approach.

There are two criteria for a good solution. Firstly the solution should be a purely ingoing wave at the black hole horizon since classically a black hole can absorb but not emit particles. Secondly the solution must be normalizable when integrated over the D-brane worldvolume.

The large-x asymptotic of the equation is the same as we found in the usual 'Schwarzschild' coordinates and so we choose the decaying power which is normalizable as $x \to \infty$.

At the horizon we will seek a solution in the form of a Frobenius series $\Phi = \Sigma_{n=0}^{\infty} a_n z^{n+s}$ in $z \equiv x - 1$. Now substituting this into the differential equation yields the indicial equation

$$s\left(s - \frac{2}{3}i\omega\right) = 0 \ . \tag{4.20}$$

The solution with $s = 0$ is regular at the horizon and corresponds to a purely infalling solution.

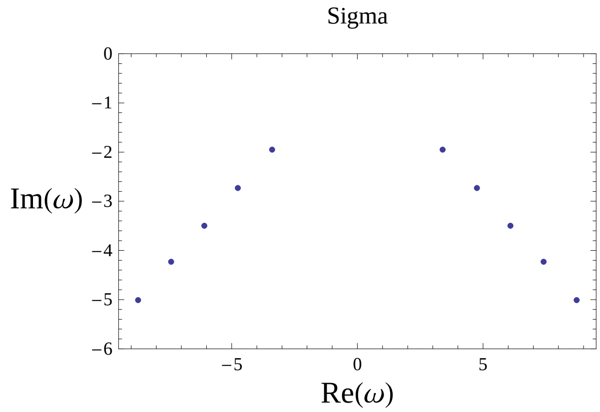
Using the regular Taylor series for the infalling solution as the initial condition we shoot out from the horizon. By requiring our solution to vanish as $x \to \infty$ we can find the quasinormal frequencies. They are displayed in Fig.(4.2).

4.5 Quasinormal modes from flavour branes

We will now consider the physical quasinormal modes of D8 branes, coming from the DBI action.

The presence of the brane corresponds to the inclusion of a chiral quark field in the gauge theory. Anomaly cancellation requires quarks to occur in vector like pairs so there must naturally be a partner $\bar{D}8$. The quasi-normal mode spectrum we compute below will therefore be parity doubled in the gauge theory.

We will again use the coordinates (10) to make infalling solutions regular at the horizon.



n	Ω_n
1	$\pm$ 3.3896 - 1.9498 i
2	$\pm$ 4.7560 - 2.7320 i
3	$\pm$ 6.0900 - 3.4970 i
4	$\pm$ 7.4110 - 4.2297 i
5	$\pm$ 8.7234 - 5.0105 i

Figure 4.3: The lowest five $k = 0$ sigma meson quasinormal frequencies in the complex ω plane.

4.5.1 Scalar mesons

We first analyze the scalar mode corresponding to a geometric fluctuation of the D8 embedding. The DBI Lagrangian for this is

$$\mathcal{L} = e^{-\phi}\sqrt{-Det\left(g_{MN}\frac{\partial x^M}{\partial\xi^a}\frac{\partial x^N}{\partial\xi^b}\right)} \,. \tag{4.21}$$

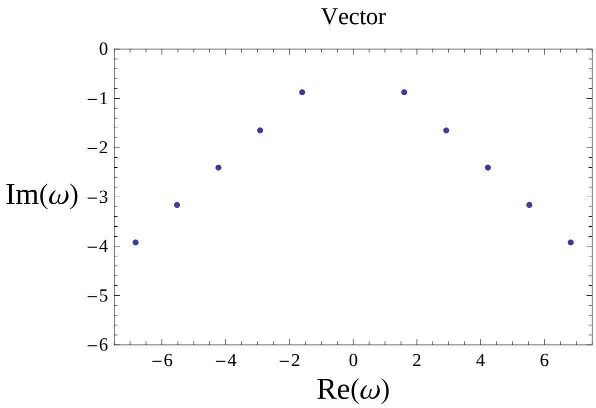
We will parameterize the fluctuation as $\tau = \tau_0 + \phi(x)e^{-i\omega h + ik\cdot x_3}$ representing a mesonic excitation with spatial momentum k relative to the plasma rest frame with zero S_4 spin. The Lagrangian is (a dot indicates an h-derivative and a prime an x-derivative)

$$\mathcal{L} = x^{\frac{5}{2}}\sqrt{1 + x^3 f(x)\phi'^2 + 2\dot{\phi}\phi' - \dot{\phi}^2} \,. \tag{4.22}$$

Expanding the square root to quadratic order the equation of motion is

$$\left(x^{\frac{11}{2}} f\phi'\right)' - i\omega\left(2x^{\frac{5}{2}}\phi' + \frac{5}{2}x^{\frac{3}{2}}\phi\right) + \left(\omega^2 - k^2\right)x^{\frac{5}{2}}\phi = 0 \,. \tag{4.23}$$

Using the regular Taylor series as initial condition we shoot out from the horizon and requiring our solution to vanish as $x \to \infty$ we can find the quasinormal frequencies.



n	Ω_n
1	$\pm$ 1.6016 - 0.8957 i
2	$\pm$ 2.9217 - 1.6512 i
3	$\pm$ 4.2265 - 2.4054 i
4	$\pm$ 5.5269 - 3.1611 i
5	$\pm$ 6.8264 - 3.9217 i

Figure 4.4: The lowest five static vector meson quasinormal frequencies in the complex ω plane.

They are shown in Fig.(4.3).

4.5.2 Vector mesons - transverse

We obtain the quasinormal spectrum for a Maxwell field on the D8 brane worldvolume, which is dual to the quasinormal spectrum of vector mesons. We use the ansatz $A_\mu = \xi_\mu A(x) e^{ik\cdot x_4}$ which is a Lorentz vector with zero S_4 spin. The equation of motion is

$$\partial_a \left(e^{-\phi} \sqrt{-g} F^{ab} \right) = 0 \; . \tag{4.24}$$

Fixing the gauge $k^\mu \xi_\mu = 0$ one obtains the equation of motion for a transversely-polarized Lorentz vector. With this ansatz the only nontrivial equation is (for example choosing k_2 nonzero and the vector in the 1-direction) for $b = 1$, ie $\partial_a(e^{-\phi}\sqrt{-g}F^{a1}) = 0$, giving

$$(x^{\frac{5}{2}} f A')' - i\omega \left(2x^{-\frac{1}{2}} A' - \frac{1}{2} x^{-\frac{3}{2}} A \right) + \left(\omega^2 - k^2\right) x^{-\frac{1}{2}} A = 0 \; . \tag{4.25}$$

Using the regular Taylor series as initial condition we shoot out from the horizon and requiring our solution to vanish as $x \to \infty$ we can find the quasinormal frequencies. They are displayed in Fig.(4.4).

4.5.3 Vector mesons - longitudinal

In the zero-temperature case the transverse mesons exhaust the vector spectrum. For finite temperature however one can also identify a purely electric longitudinal solution of the supergravity dual Maxwell field. The longitudinal electric field we deal with is $E_1 = kA_0 + \omega A_1$ (k points along the 1-axis making the vector potential curl-free).

There are three relevant equations of motion: first $\partial_a \left(e^{-\phi}\sqrt{-g}F^{ax}\right) = 0$. Writing this out for our choice of metric one obtains

$$i\omega A_0' + ikg^{11}g^{xx}A_1' = -g^{11}g^{x0}\left(k^2 A_0 + \omega k A_1\right) \ . \tag{4.26}$$

The second equation is $\partial_a \left(e^{-\phi}\sqrt{-g}F^{a0}\right) = 0$. Writing this out for our choice of metric one obtains

$$\begin{aligned} 0 \ &= \ -\left(e^{-\phi}\sqrt{-g}A_0'\right)' - ike^{-\phi}\sqrt{-g}g^{11}g^{0x}A_1' \\ &\quad -e^{-\phi}\sqrt{-g}g^{11}g^{00}\left(k^2A_0 + k\omega A_1\right) \ . \end{aligned} \tag{4.27}$$

Finally from $\partial_a \left(e^{-\phi}\sqrt{-g}F^{a1}\right) = 0$ one obtains

$$\begin{aligned} 0 \ &= \ \left(e^{-\phi}\sqrt{-g}g^{xx}g^{11}A_1'\right)' - i\omega e^{-\phi}\sqrt{-g}g^{0x}g^{11}A_1' \\ &\quad - \left(e^{-\phi}\sqrt{-g}g^{x0}g^{11}\left(i\omega A_1 + ikA_0\right)\right)' \\ &\quad -e^{-\phi}\sqrt{-g}g^{00}g^{11}\left(\omega^2A_1 + \omega kA_0\right) \ . \end{aligned} \tag{4.28}$$

The trick is to form a second order ODE for the gauge invariant combination $E_1 = kA_0 + \omega A_1$. We do this by putting the differential equations into such form as the coefficients of A_0'' and A_1'' in the second and third equations are unity, then adding k times the second equation to ω times the third equation. We patch up the first derivative terms by adding zero in the form given by (4.26),

$$\left(i\omega A_0' + ikg^{11}g^{xx}A_1' + g^{11}g^{x0}\left(k^2A_0 + \omega kA_1\right)\right) \equiv 0 \ . \tag{4.29}$$

We add this term with coefficient such that the first derivative terms add up to a multiple of $kA_0 + \omega A_1$.

The equation we finally obtain is

$$E_1'' + f_1 E_1' + f_2 E_1 = 0 \ . \tag{4.30}$$

Here one has

$$\begin{aligned} f_1 &= \tfrac{5}{2x} - \tfrac{i\omega}{x^3 f} + \\ &\quad \frac{\frac{ik^2}{x^3} + \omega\left(\frac{5}{2xf} + \frac{1}{2x^4 f}\right) - \frac{2i\omega^2}{x^3 f} + \omega\left(\frac{i\omega}{x^3 f} - \frac{5}{2x}\right)}{\omega - \frac{k^2}{\omega} f} \ , \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} f_2 &= -\tfrac{k^2}{x^3} + \tfrac{i\omega}{2x^4 f} + \tfrac{\omega^2}{x^3 f} + \\ &\quad \frac{k}{x^3}\left(\frac{k\left(\frac{ik^2}{x^3} + \omega\left(\frac{5}{2xf} + \frac{1}{2x^4 f}\right) - \frac{2i\omega^2}{x^3 f}\right) + \omega\left(\frac{i\omega k}{x^3 f} - \frac{5k}{2x}\right)}{i(\omega^2 - k^2 f)}\right) \ . \end{aligned} \tag{4.32}$$

After all this work, it is easy to check that this equation has the same $k \to 0$ limit as the transverse mode - the quasinormal frequencies are degenerate in this limit. We also note that by rescaling $\omega \to \lambda^2 \omega$ and $k \to \lambda k$ and taking $\lambda \to 0$ one finds that a normalizable, regular solution to the longitudinal equation exists for $k = \omega = 0$, which is just $E_1 = x^{-\frac{3}{2}}$. This additional mode is not present in any of the other spectra, and is related to the hydrodynamic behaviour of the field theory.

4.6 Finite spatial momentum

We can obtain an effective dispersion relation for our modes ie a function $\omega(k)$. This is done by solving the wave equations obtained in the previous sections for general complex k. Similar computations in the D3/D7 system can be found in [58].

Real momentum k corresponds to a state which is a travelling plane wave on the Minkowski spacetime of the dual field theory. Switching on a finite real k in the equation

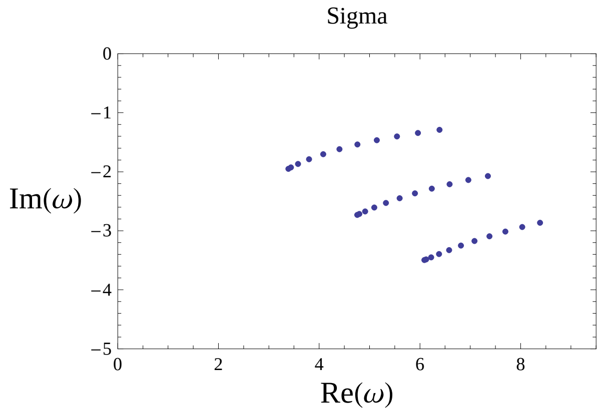


Figure 4.5: Lowest quasinormal frequencies at finite real spatial momentum k in the complex ω plane for the scalar $\bar{q}q$ bound state. The momentum k ranges from 0 to 5.0 in steps of 0.5 (in the same units as ω) as one moves to the right in the plot.

and using shooting we have found the behaviour of the quasinormal frequencies in the complex ω plane. It is found that the states become more massive and more stable as k is increased.

The results for the first three quasinormal modes for the scalar $\bar{q}q$ bound state are plotted in Fig.(4.5). In Fig.(4.6) we show the first three vector quasinormal modes for the transverse and longitudinal modes. These are degenerate for $k = 0$ but behave differently as k is increased - the main difference is that the longitudinal states become more stable but less massive relative to the transverse states for the same k as k is increased.

Finally we note that, as mentioned above, there is an additional quasinormal mode for the longitudinal electric field component of the Maxwell field on the flavour brane. For small k this lies close to the origin and on the imaginary axis. As shown in for example [57] the diffusion coefficient D for flavoured fundamental matter can be computed from this state. For small spatial momentum k it obeys the ‘hydrodynamic’ relation $\omega = -iDk^2$. Here we test whether this relation can be obtained using our ingoing co-

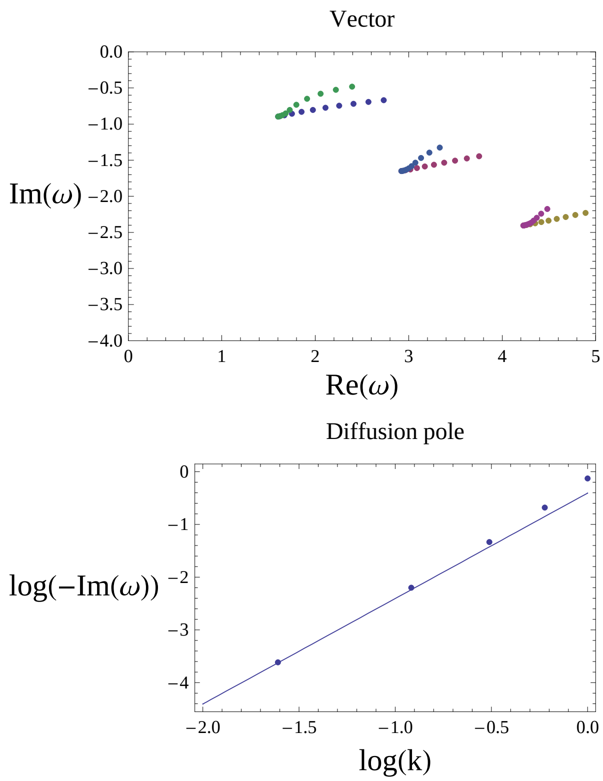


Figure 4.6: Above we show the lowest vector $\bar{q}q$ bound state quasinormal frequencies at finite real spatial momentum k in the complex ω plane (top). The momentum ranges from 0 to 2.0 in steps of 0.2 as one moves to the right (the lowest mode in the longitudinal spectrum, corresponding to the diffusion pole in the hydrodynamic limit, is excluded). For each mode the flatter trajectory corresponds to the transverse species and the steeper to the longitudinal. Below this we show the momentum dependence of the lowest quasinormal mode ('diffusion pole') for the longitudinal vector meson. The equation of the line is $y = \log \frac{2}{3} + 2x$ showing the validity of the relation $\omega = -iDk^2$ for small k, with $D = \frac{2}{3}$.

ordinates. The calculation we do is then the generalization of the calculation done for the vector meson, including a nonzero spatial momentum, and examining the gauge-invariant longitudinal electric field component. In Fig.(4.6) we plot the position of this pole on the imaginary axis as a function of k. We extract $D = \frac{2}{3}$.

We note our result for the diffusion coefficient is related to the value obtained in [57] which is found to be $D = \frac{1}{2\pi T}$. We are measuring in units of $\sqrt{\frac{R^3}{u_T}}$ so we obtain the numerical value

$$D = \frac{1}{2\pi}\frac{4\pi}{3} \equiv \frac{2}{3}\ . \tag{4.33}$$

Our result therefore matches that of [57] providing a check on our numerics (albeit for small ω and k).

4.7 Conclusion

We have found the quasinormal frequencies for a variety of different species (the Klein-Gordon scalar, scalar quark bound states and vector mesons) in the Sakai-Sugimoto model at high temperature. A crucial part of the analysis was to change coordinates so that the infalling quasi-normal modes become regular at the horizon so numerical shooting becomes straightforward. It is noteworthy that in these coordinates the equations for the Klein-Gordon scalar, the scalar $\bar{q}q$ and the transverse vector $\bar{q}q$ all have the form

$$\begin{aligned}(x^n f\phi')' - i\omega\left(2x^{n-3}\phi' + (n-3)x^{n-4}\phi\right) \\ + \left(\omega^2 - k^2\right)x^{n-3}\phi = 0\ .\end{aligned} \tag{4.34}$$

For the Klein-Gordon scalar $n = \frac{19}{4}$, for the sigma $n = \frac{11}{2}$ and for the vector $n = \frac{5}{2}$. This means the quasinormal spectra look extremely similar. The only thing making the frequencies different is the value of n. The effect of increasing the value of n is to move the quasinormal modes out from the origin in the complex frequency plane.

We also computed these states at finite real momenta where the modes become more massive and more stable. We have obtained the numerical value for the diffusion coefficient for fundamental flavoured matter and our result is consistent with the calculation of [57].

4.8 Fermionic excitations of the AdS-Schwarzschild background

A discrete spectrum of fermionic 'mesino' states was found for probe D7 branes embedded in the background geometry surrounding a stack of D3 branes in [59] and it was found that the fermion masses were degenerate with the boson masses (a consequence of the $\mathcal{N} = 2$ supersymmetry). In this section we extend the analysis to the non-extremal (finite temperature) background and find a discrete spectrum of quasinormal modes. Since the finite-temperature background is non-supersymmetric we find that the spectra of bosonic and fermionic excitations of the probe D7 are no longer degenerate. Our result can be used for excitations of a space-filling probe D7 or for the full $AdS_5 \times S^5$ geometry depending on the choice of Dirac mass in AdS_5. One can thus compare to the spectrum of a minimally-coupled scalar on the nonextremal $AdS_5 \times S^5$ background studied in [60], in which the quasinormal spectrum was evaluated using Leaver's continued fraction method [56]. We use shooting methods here.

4.8.1 The Dirac equation on curved spacetime

Firstly we note that the massless Dirac equation on a manifold of the form $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ (the direct product of two lower-dimensional manifolds) takes the form

$$i\Gamma^a\nabla_a\psi \equiv i\Gamma^a\nabla_{1,a}\psi + i\Gamma^a\nabla_{2,a}\psi \ . \tag{4.35}$$

In our case the manifold is the direct product of five-dimensional AdS-Schwarzschild and a sphere. The Dirac operator on for example the five-sphere can be replaced with its eigenvalues which are well-known and given by $i\Gamma^a\nabla_{S_5,a}\psi = \pm\left(\frac{5}{2}+l\right)\psi$ for $l = 0, 1, 2, ...$ (in the field theory this means states of increasing R-charge). In the absence of other couplings this means we can solve our problem simply by solving the Dirac equation on five-dimensional AdS-Schwarzschild with a Dirac mass $m = \frac{5}{2}$ for the ground state on the five-sphere.

An expression for the covariant derivative of a spinor can be straightforwardly obtained (see for example [61]). The derivative will take the form

$$\nabla_\mu\psi_A = \partial_\mu\psi_A - \Sigma_{A\ \mu}^{\ B}\psi_B \ , \tag{4.36}$$

along with a similar expression for a complex conjugate spinor, $\nabla_\mu\bar{\psi}_{A'} = \partial_\mu\bar{\psi}_{A'} - \bar{\Sigma}_{A'\ \mu}^{\ B'}\bar{\psi}_{B'}$.

The curved spacetime gamma matrices satisfy $\Gamma_{(\mu}\Gamma_{\nu)} = g_{\mu\nu}I$ and since the covariant derivative preserves the metric one has

$$\nabla_\mu\Gamma^\mu_{AB'} = 0 \ . \tag{4.37}$$

Writing this out explicitly one obtains

$$\partial_\mu\Gamma^\nu_{AB'} + \Gamma^\nu_{\mu\rho}\Gamma^\rho_{AB'} - \Sigma^C_{A\mu}\Gamma^\nu_{CB'} - \bar{\Sigma}^{D'}_{B'\mu}\Gamma^\nu_{AD'} \ . \tag{4.38}$$

Multiplying on the left with $\Gamma^{EF'}_\nu$ and noting eg. $\Gamma^{EF'}_\nu\Gamma^\nu_{CB'} \equiv \delta^E_C\delta^{F'}_{B'}$ leads to the expression

$$\Gamma_{\nu}^{EF'}\left(\partial_{\mu}\Gamma^{\nu}_{AB'}+\Gamma^{\nu}_{\mu\rho}\Gamma^{\rho}_{AB'}\right)-\Sigma^{E}_{A\mu}\delta^{F'}_{B'}-\bar{\Sigma}^{F'}_{B'\mu}\delta^{E}_{A}=0\,. \qquad (4.39)$$

This expression can be contracted over F' and B' indices to give, for a $n\times n$ gamma matrix representation (we will be using $n=4$)

$$\Gamma_{\nu}^{EB'}\partial_{\mu}\Gamma^{\nu}_{AB'}+\Gamma^{\nu}_{\mu\rho}\Gamma_{\nu}^{EB'}\Gamma^{\rho}_{AB'}=n\,\Sigma^{E}_{A\mu}\,. \qquad (4.40)$$

The expression for the conjugate spin coefficient follows from contracting (4.39) over the A and E indices. From now we will suppress the spinor indices of the gamma matrices. Note that it is easy to extend the above prescription to higher half-integer spin and so one could calculate quasinormal frequencies associated to spin-$\frac{3}{2}$ excitations and so forth.

The Dirac equation therefore takes the explicit form

$$i\Gamma^{\mu}\left(\partial_{\mu}+\frac{1}{4}\Gamma_{\nu}\left(\partial_{\mu}\Gamma^{\nu}+\Gamma^{\nu}_{\lambda\mu}\Gamma^{\lambda}\right)\right)\psi+m\,\psi=0\,. \qquad (4.41)$$

Here the 3-index symbol $\Gamma^{\nu}_{\lambda\mu}$ is the usual Christoffel symbol. The meaning of the Γ^{μ} symbols will become clear in the next section.

This we take in the form

$$\left(i\Gamma^{\mu}\partial_{\mu}+\frac{i}{4}\left(\alpha+\beta\right)\right)\psi+m\,\psi=0\,. \qquad (4.42)$$

Here one has

$$\alpha=\Gamma^{\mu}\Gamma_{\nu}\partial_{\mu}\Gamma^{\nu}\,, \qquad (4.43)$$

and

$$\beta = \Gamma^{\nu}_{\lambda\mu}\Gamma^{\mu}\Gamma_{\nu}\Gamma^{\lambda} \; . \tag{4.44}$$

4.8.2 Coordinate system and calculation

The metric is $(+----)$ having done rescaling so that energies are measured in units of the Hawking temperature and translated the metric to an analogue of the infalling Painleve-Gullstrand coordinates (ie we make the spatial sections look like there is no black hole - this is known as the Newtonian gauge in some of the literature)

$$ds^2 = x^2 f dh^2 - \frac{2}{x^2} dh dx - \frac{dx^2}{x^2} - x^2 \left(dy_1^2 + dy_2^2 + dy_3^2\right) \; . \tag{4.45}$$

Here $f = 1 - \frac{1}{x^4}$.

With the flat space little gammas defined in the obvious sense ($\gamma_0^2 = 1$ all others square to -1 and all anticommute) we now want a basis (Γ_μ) such that

$$\Gamma_0 = x\left(\gamma_0 + \frac{1}{x^2}\gamma_4\right) \tag{4.46}$$

$$\Gamma_i = x\gamma_i \tag{4.47}$$

$$\Gamma_4 = \frac{1}{x}\gamma_4 \; . \tag{4.48}$$

These satisfy $\Gamma_{(\mu}\Gamma_{\nu)} = g_{\mu\nu} I$ for our five-dimensional metric.

The reciprocal basis (Γ^μ) satisfying $\Gamma^{(\mu}\Gamma_{\nu)} = \delta^{\mu}_{\nu} I$ (easily checked using the standard gamma matrix algebra) is clearly

$$\Gamma^0 = \frac{1}{x}\gamma_0 \tag{4.49}$$
$$\Gamma^i = -\frac{1}{x}\gamma_i \tag{4.50}$$
$$\Gamma^4 = -x\left(\gamma_4 + \frac{1}{x^2}\gamma_0\right) . \tag{4.51}$$

Evaluating the matrix-valued quantities needed for the spinor covariant derivative (expressions (4.43) and (4.44) for α and β) for the coordinate basis we have chosen, one obtains

$$\alpha = 3\left(\gamma_4 + \frac{1}{x^2}\gamma_0\right) , \tag{4.52}$$

and

$$\beta = -\frac{7}{x^2}\gamma_0 - 11\gamma_4 . \tag{4.53}$$

Thus $\alpha + \beta = -\frac{4}{x^2}\gamma_0 - 8\gamma_4$. Note that if the same calculation is done in the background with no black hole the term in γ_0 vanishes.

The Dirac equation takes the Hamiltonian form (setting $\partial_h = i\omega$ and $\partial_i = -ik$, leaving m general for now)

$$-\frac{\omega}{x}\gamma_0\psi - \frac{1}{x}k_i\gamma_i\psi - i\left(x\gamma_4 + \frac{1}{x}\gamma_0\right)\psi' - i\left(\frac{1}{x^2}\gamma_0 + 2\gamma_4\right)\psi + m\psi = 0 . \tag{4.54}$$

Choosing the gamma matrices to be the analogue of the chiral basis (but with $i\gamma_5$ playing the role of γ_4) we set

$$\gamma_0 = \begin{pmatrix} \cdot & I \\ I & \cdot \end{pmatrix} \quad \gamma_i = \begin{pmatrix} \cdot & \sigma_i \\ -\sigma_i & \cdot \end{pmatrix} \quad \gamma_4 = \begin{pmatrix} iI & \cdot \\ \cdot & -iI \end{pmatrix} .$$

Taking the two functions $u_1(x)$ and $u_2(x)$ to multiply constant 2-spinors ξ^1, ξ^2 of the 4D Lorentz subgroup of the 5D Lorentz group one obtains the system of equations

$$\begin{pmatrix} x\partial_x + 2 + m & -\frac{\omega + k\cdot\sigma}{x} - \frac{i}{x}\partial_x - \frac{i}{x^2} \\ -\frac{\omega - k\cdot\sigma}{x} - \frac{i}{x}\partial_x - \frac{i}{x^2} & -x\partial_x - 2 + m \end{pmatrix} \begin{pmatrix} u_1\,\xi^1 \\ u_2\,\xi^2 \end{pmatrix} = 0\ .$$

The two-spinor polarization can be resolved into helicity eigenstates $(\sigma \cdot k)\,\xi^{\pm} = \pm|k|\xi^{\pm}$ as and for the both positive helicity case one has

$$xu_1' + (2+m)u_1 - \left(\frac{\omega + k}{x} + \frac{i}{x^2}\right) u_2 - \frac{i}{x}u_2' \quad = \quad 0 \tag{4.55}$$

$$xu_2' + (2-m)u_2 + \left(\frac{\omega - k}{x} + \frac{i}{x^2}\right) u_1 + \frac{i}{x}u_1' \quad = \quad 0\ . \tag{4.56}$$

Including $|k| \neq 0$ allows the calculation of the dispersion relation for the spinor wave (the equations are straightforward but cumbersome). Here we will focus on the case $|k| = 0$ only.

Note that if $|k| = 0$ and if there is no Dirac mass m the two equations are not linearly independent - one can add i times the second equation to the first, obtaining the equation for $v \equiv u_1 + iu_2$ (with $m = 0$)

$$xv' + 2v + \left(\frac{i\omega}{x} - \frac{1}{x^2}\right) v - \frac{1}{x}v' = 0\ . \tag{4.57}$$

This first order equation can be solved exactly to something which clearly does not admit a discrete spectrum and is not regular at the horizon

$$v = \frac{v_0}{x\sqrt{1-x^2}} \left(\frac{1+x}{1-x}\right)^{\frac{i\omega}{2}}\ . \tag{4.58}$$

A naive attempt to solve this equation by shooting out from the horizon (using the

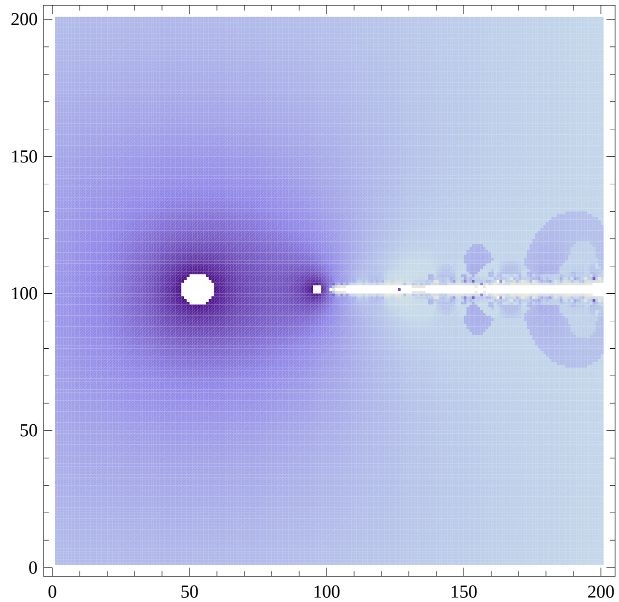


Figure 4.7: What you get in the complex ω-plane for the asymptotic value of $\ln|v|$ if you fail to include the Dirac mass term from the sphere (ie. $m = 0$ in the equations in the text).

Taylor series for the ingoing mode as initial data) produces a curious-looking plot which is exhibited in Fig.(4.7).

Of course the problem is we need a Dirac mass coming from the spin of the fermion on the sphere part of the geometry (the spectrum of the Dirac operator on spheres of various dimension can be found in [62]). For an n-sphere one should have $m = \pm\frac{n}{2}$. In the case of a fermionic excitation of the $\mathcal{N} = 4$ theory one would expect to use $m = \pm\frac{5}{2}$ and in the case of a fermionic excitation of a zero mass 'quark' (on the worldvolume of a space-filling probe D7 brane) one expects $m = \pm\frac{3}{2}$. In fact the degeneracy of the particle and antiparticle spectrum is broken by the coupling to the background five-form field strength (for the D7 case see [63]) which has the effect of shifting the 'particle' and 'antiparticle' Dirac masses up one unit. So for the $\mathcal{N} = 4$ excitation one wants Dirac masses of $m = +\frac{7}{2}, -\frac{3}{2}$ and for the probe brane excitation $m = +\frac{5}{2}, -\frac{1}{2}$.

To obtain the quasinormal spectrum one could choose to solve either the equation for u_1 or that for u_2 - the spectra will be identical. It is an easy exercise to eliminate either u_1 or u_2 in order to obtain a linear second order ODE in the remaining variable. Eliminating u_1 gives the equation

$$u_2'' + f_1 u_2' + f_2 u_2 = 0 \, , \tag{4.59}$$

were

$$f_1 = \frac{1+m-2i\omega(1+m)x+2\omega^2x^2-7(1+m)x^4-6i\omega x^5}{x(1-x^4)(1+m+i\omega x)} \, , \tag{4.60}$$

and

$$f_2 = \frac{-(1+m)-(3+m)i\omega x-(1+m)\omega^2x^2-i\omega^3x^3-(1+m)(2-m)(m+4)x^4+i(m^2-m+6)\omega x^5}{x^2(1-x^4)(1+m+i\omega x)} \, . \tag{4.61}$$

Note this differential equation seems to have an additional singular point compared to the scalar case - this would appear to preclude the use of Leaver's method.

In the near-horizon limit the solution for either u_1 or u_2 is $(x-1)^s$ times a regular Taylor series in $x-1$. The solutions for the exponent are $s=0$ which is the ingoing mode and $s=-\frac{1}{2}-\frac{i\omega}{2}$ which is the outgoing mode. Note this does not depend on the mass m.

In the large-x limit close to the boundary of AdS the solution tends to a linear combination of two powers of x. The exponents are given by $-\frac{5}{2} \pm \left(m+\frac{1}{2}\right)$. For this case we have $m=\frac{5}{2}$ and so the exponents are $+\frac{1}{2}$ and $-\frac{11}{2}$, the latter being the normalizable mode.

Using the regular Taylor series solution at a distance ϵ from the horizon as initial data for the numerical solution and tuning to the normalizable mode at large x we can find the quasinormal spectrum.

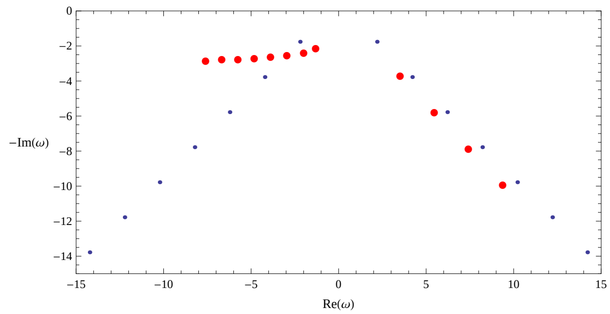


Figure 4.8: Plot showing the quasinormal modes of excitations of a probe D7 at $k = 0$. The blue points are the scalar modes found in [51]. The first four 'particle' fermionic excitations ($m = +\frac{5}{2}$) are shown on the right and the first eight 'antiparticle' excitations ($m = -\frac{1}{2}$) on the left.

A plot of the solution corresponding to the lowest modes appropriate to the spin-$\frac{1}{2}$ equation on the worldvolume of a zero current algebra quark mass D7 brane is shown in Fig.(4.8). Our results look qualitatively different from the scalar cases analyzed in [60] and [51] because the scalar particle and antiparticle spectra are degenerate, as there is no coupling to the five-form.

4.8.3 Finite momentum

We could solve the same problem for a state with $k \neq 0$ to obtain the dispersion relation for fermions in the plasma background. There presumably are not any hydrodynamic poles for the fermions due to the fact that the fermions always have a minimum mass of order the supersymmetry breaking scale (the temperature), hence no poles in the $\omega, k \to 0$ limit.

Acknowledgements: I would like to thank one of the authors of [54] (SD) for helpful comments regarding the transformation to ingoing coordinates, and also Professor Tim Morris for pointing out the representation of the five-dimensional Clifford algebra.

Chapter 5

The thermal phase transition in a QCD-like holographic model

We can attempt to apply gauge-gravity duality to study the high-temperature phase structure of strongly-coupled field theory. In this context we investigate the high temperature phase of a dilaton flow deformation of the AdS/CFT Correspondence. We argue that these geometries should be interpreted as the $\mathcal{N} = 4$ gauge theory perturbed by a SO(6) invariant scalar mass and that the high-temperature phase is just the well-known AdS-Schwarzschild solution. We compute, within supergravity, the resulting Hawking-Page phase transition which in this model can be interpreted as a deconfining transition in which the vev for the operator TrF^2 dissolves. In the presence of quarks the model also displays a simultaneous chiral symmetry restoring transition with the Goldstone mode and other quark bound states dissolving into the thermal bath.

5.1 Introduction

The simplest examples of non-supersymmetric deformations of the AdS/CFT Correspondence [13, 15] are those in which the dilaton has some non-trivial profile in the radial direction of the space [65]-[71]. Since the dilaton carries no R-charge the five-sphere is left intact. The presence in the dilaton profile at large AdS radius, r, of a term of the form $1/r^4$ indicates the presence of a vev for the operator TrF^2 in the $\mathcal{N} = 4$ gauge theory. This operator is the F-term of a chiral superfield so supersymmetry is manifestly broken.

Simple gravitational theories of this type have been shown to generate confining behaviour in Wilson loops in the dual gauge theory and a discrete glueball spectrum [66, 70]. Quarks have also been introduced through D7 brane probes [16]-[35] and chiral symmetry breaking in the pattern of QCD is induced [41, 73]. These models are therefore a nice toy model of a gauge theory that behaves in many respects like QCD.

Given the successes of this model at zero temperature it is interesting to investigate the finite temperature behaviour of the solution. One expects a Hawking-Page type phase transition [76] where, when the temperature grows greater than the perturbation, the vev for TrF^2, a first order transition would occur from the zero temperature solution with a compact time dimension to a black hole geometry. The former has free energy of order one whilst that of the latter is of order N^2 - we would be seeing deconfinement of the gauge degrees of freedom [14, 64]. It would be interesting in addition to understand how chiral symmetry breaking behaves through this transition.

Here we will seek dilaton flow black hole geometries that might describe the high temperature phase of the gauge theory. In fact though we will analytically show that there are no black hole geometries with a non-trivial dilaton in five dimensional supergravity. The only candidate for the high temperature phase of the dilaton flow geometry, at the supergravity level, is in fact the normal AdS-Schwarzschild geometry. This is of course identical to that describing the high T phase of the $\mathcal{N} = 4$ gauge theory. We conclude that the vev for TrF^2 is an induced operator which dissolves at finite temperature. Taking this as our assumption we compute the Hawking-Page type transition and show that the behaviour matches the usual intuition discussed above, though we must caveat our analysis because we ignore potential string theory corrections to the highly-curved region of the low temperature phase.

We learn that in the low temperature phase the vevs of the glue and quark fields are temperature independent as are the glueball and meson masses. This matches large N field theory arguments made in [77]. The finite temperature solution of the $\mathcal{N} = 4$ theory has also been studied vigorously [83] including in the presence of quarks [35]- all of those results can now be seen to apply to the high temperature phase of the dilaton flow geometry too. In particular the vev of TrF^2 and the chiral symmetry breaking quark condensate (at zero quark mass) both switch off in the high T phase. The Goldstone boson of the symmetry breaking becomes massive and indeed melts into the thermal bath.

The above account though requires some additional explanation. In the low T phase if the vev for TrF^2 is an induced operator what is the perturbation to the $\mathcal{N} = 4$ gauge theory that is breaking supersymmetry? We propose a story that might explain this. Since supersymmetry is broken but SO$(6)_R$ preserved we expect all other SO$(6)_R$ invariant operators to switch on. As was argued in [66] and we will discuss more below, amongst these operators, at large N, is an SO(6) preserving scalar mass term. This mass term is not described by a supergravity field so is essentially invisible in the solutions - there are examples of supersymmetric [78, 79] and non-supersymmetric [81] flows where this operator is present yet invisible in terms of a supergravity field. In those geometries the mass appears to be generated through the RG flow by a fermion mass term that *is* described by a supergravity mode. Here a simple D3 brane probe shows there is no explicit mass in the dilaton flow geometry though and it is not in this class.

The SO(6)$_R$ invariant scalar mass is described by a string rather than supergravity mode so one naively expects it to describe a super-irrelevant source (if nothing else is there to regenerate it). That is, if present, the source would be invisible in the IR before growing sharply in the UV and dominating the physics. Such a source term would show up as a sharp UV cut off, so the only impact of that cut off in the low energy theory would be the symmetries it imprinted. We propose that the dilaton flow geometry describes the IR physics below such a UV cut off.

When the dual gauge theory is viewed as the $\mathcal{N} = 4$ theory with a scalar mass perturbation it seems a very natural theory to study as a toy for QCD - it is a sensible non-supersymmetric, strongly coupled gauge theory. Of course none of the gaugino super-partners are decoupled from the strong dynamics so it is not QCD.

Our results are also interesting as ten dimensional realizations of the "hard-wall" transitions explored in [77] (the introduction of a hard infra-red wall has long been used as a very simplistic way of introducing a mass gap into the gauge theory [84]). To make that connection stronger we will begin by describing briefly the thermal transition in the $\mathcal{N} = 4$ gauge theory with an SO(6) invariant scalar vev. This should be the ultra-relevant operator that is described by the string mode discussed above. The gravity dual of this theory is precisely AdS$_5 \times S^5$ with a hard IR cut off at some finite radius - the cut off corresponds to the surface of a five-sphere that the D3 branes have been spread evenly over. This case serves as an example of how these scalar operators are invisible in the supergravity solution.

Our main computation here is to show that there is no dilaton flow black hole in five dimensional supergravity and to compute the Hawking-Page transition to a pure AdS-like black hole in that model. Finally we will briefly review the phase structure of the non-supersymmetric gauge theory collating results from elsewhere in the literature.

5.2 A Hard wall - $\mathcal{N} = 4$ SYM On Moduli Space

We begin by creating a true AdS-dual of a hard wall model. The set up is simple - one spreads the D3 branes at the origin ($r = 0$) of the usual AdS/CFT construction onto the surface of a five-sphere centred at the origin. If a finite number of D3 branes are evenly distributed then the configuration will preserve a discrete sub-group of the SO(6) symmetry group. In the infinite N limit where the distribution becomes smooth the SO(6) group is preserved. In the gauge theory this configuration corresponds to one where

$$Tr\phi_i\phi_j \propto \delta_{ij} \; , \tag{5.1}$$

which is an SO(6) singlet. The U(N) gauge theory is broken to a $U(1)^N$ theory that is non-interacting (all matter is in the adjoint so uncharged) in the case of a finite number of D3 branes. As the density increases the scalar vevs connecting adjacent sites become small and one has a deconstructed model of a five dimensional U(1) gauge theory living on the five sphere surface - again it is non-interacting.

On the gravity side it is clear from a Gauss' law argument that the geometry does not change (as in the case of a planet being described by the Schwarzschild black hole metric). The dual is just $AdS_5 \times S^5$ with the usual four form (here $L^4 = 4\pi g_s N \alpha'^2$)

$$ds^2 = \frac{u^2}{L^2}dx_4^2 + \frac{L^2}{u^2}du^2 + L^2 d\Omega_5^2 \tag{5.2}$$

$$C_{(4)} = \frac{u^4}{L^4}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \; , \tag{5.3}$$

except that the surface of the five sphere of D3 branes acts as a cut off on the space (formally within there is flat space since there are no sources).

The obvious candidate for the finite temperature version of the theory is just AdS-Schwarzschild [64]. Restricting ourselves to a black hole background with temperature equal to $\frac{u_h}{\pi L^2}$ and working with the usual Poincaré coordinates

$$ds^2 = \frac{K(u)}{L^2}d\tau^2 + L^2\frac{du^2}{K(u)} + \frac{u^2}{L^2}dx_3^2 + L^2 d\Omega_5^2 \; , \tag{5.4}$$

with

$$K(u) = u^2 - \frac{u_h^4}{u^2} \, . \tag{5.5}$$

We can seek a Hawking-Page transition by comparing the free energy of AdS cut off at the radius of the D3 five sphere, u_0, and with a compact time dimension versus that of the AdS Schwarzschild solution cut off by either the radius of the five sphere or the horizon whichever is largest.

This computation can be performed within the five dimensional truncation of IIB supergravity on this space. The two five dimensional metrics are then simply

$$ds^2 = \frac{u^2}{L^2} d\tau^2 + L^2 \frac{du^2}{u^2} + \frac{u^2}{L^2} dx_3^2 \tag{5.6}$$

$$ds^2 = \frac{K(u)}{L^2} d\tau^2 + L^2 \frac{du^2}{K(u)} + \frac{u^2}{L^2} dx_3^2 \, , \tag{5.7}$$

with no four-forms. The comparison at this level is naively precisely the computation of Herzog [77] which we briefly review. We will see shortly that in the full theory there is an extra contribution to the computation.

The Euclidean action for either cut-off AdS or AdS-Schwarzschild is

$$S = -\frac{1}{4\pi G_5} \int d^5x \sqrt{g} \left(\frac{1}{4} R + \frac{3}{L^2} \right) \, . \tag{5.8}$$

On-shell $R = -\frac{20}{L^2}$ for both backgrounds.

One must rescale the time coordinates so as to ensure that the period of the time directions match at the cut off Λ [64]. One then finds the action difference

$$S_{BH} - S_{AdS} = \frac{1}{2G_5 u_h L^3} \left(\int_{u_h}^{\Lambda} u^3 du - \sqrt{\frac{K(\Lambda)}{\Lambda^2}} \int_{u_0}^{\Lambda} u^3 dr \right) \, . \tag{5.9}$$

Taking the limit $\Lambda \to \infty$ one obtains $S_{BH} - S_{AdS} = \frac{1}{8G_5 u_h L^3} \left(u_0^4 - \frac{1}{2} u_h^4 \right)$. This is Herzog's result.

This computation suggests that if $u_0 > \sqrt[4]{\frac{1}{2}} u_h$ the black hole action is larger and a hard wall solution is favoured - giving Herzog's transition temperature $T_c = \frac{\sqrt[4]{2} u_0}{\pi L^2}$.

In fact the above computation can be interpreted as a valid result in the AdS-QCD approach, in which one constructs phenomenological models without insisting that the holographic physics is an exact realization of IIB SUGRA. To study the actual behaviour of the dual $\mathcal{N} = 4$ gauge theory we must solve the SUGRA problem in its entirety. There is an additional term in the action from the boundary between the flat spacetime within the shell of D3 branes and the AdS geometry outside, which is the difference between the Gibbons-Hawking term from each 'side' of the boundary (we thank Andreas Karch and Steve Paik for pointing this out to us). It is simplest to perform the calculation in the full $10D$ geometry. The Gibbons-Hawking term is $\frac{1}{8\pi G_5}\int_\Sigma K d\Sigma$ where the integration is over the boundary Σ and K is the trace of the second fundamental form on the boundary. This is easily evaluated using the relation [82]

$$\int_\Sigma K d\Sigma = \frac{\partial}{\partial n}\int_\Sigma d\Sigma \, . \tag{5.10}$$

The normal derivative is evaluated by setting the metric coefficient of the radial holographic direction to unity (by means of a coordinate transformation) and then differentiating with respect to the radial direction.

The ten-dimensional flat space within the D3 brane shell has line element $ds^2 = L^2(d\tau^2 + dx_3^2) + \frac{1}{L^2}\left(du^2 + u^2 d\Omega_5^2\right)$ and thus $\int_\Sigma d\Sigma = u^5$ up to constants, a multiple of the five-sphere volume and four-space volume. Performing the normal derivative one obtains a contribution of $5u_0^4$.

The $AdS_5 \times S^5$ geometry outside the D3 brane shell has line element $ds^2 = L^2 e^{2r}(d\tau^2 + dx_3^2) + \frac{1}{L^2}\left(dr^2 + d\Omega_5^2\right)$ and thus $\int_\Sigma d\Sigma = e^{4r}$ up to volume factors. Performing the normal derivative and transforming back to the 'u'-type coordinates one finds a contribution of $4u_0^4$ times the overall factors.

Including this in our computation we see that it cancels out the total action for the cut-off AdS spacetime entirely leaving

$$S_{BH} - S_{AdS} = \frac{1}{8G_5 u_h L^3}\left(-\frac{1}{2}u_h^4\right) \ . \tag{5.11}$$

This implies that the Hawking-Page transition actually takes place at any finite temperature for the theory on its moduli space, which is the same result as for the theory at the superconformal point. Field theoretically this is because the temperature generates a potential for the adjoint scalars of the gauge theory which forces their vev to zero. The transition then naturally occurs immediately above T=0.

5.3 Dilaton Flow Geometries

We now turn to constructing solutions of the supergravity equations of motion with non-trivial dilaton flows.

5.3.1 Five-dimensional action and equations of motion

We will work in $\mathcal{N} = 8$ SUGRA in five dimensions [85]-[87] which is a truncation of IIB string theory on $AdS_5 \times S^5$ and it is known that any solution can be lifted to a complete ten dimensional geometry. The 40 scalars which participate in the superpotential will be set to zero (leaving a constant superpotential which acts as a negative cosmological constant) and we consider a solution with nontrivial dilaton and zero axion.

The effective five-dimensional action is (we use the normalization conventions of [66] and set $L \equiv 1$ for this section)

$$S = \frac{1}{4\pi G_5}\int d^5x\sqrt{-g}\left(\frac{1}{4}R - \frac{1}{8}g^{ab}\nabla_a\phi\nabla_b\phi + 3\right) \ . \tag{5.12}$$

The non-extremal background and the background with a nontrivial scalar are both non-supersymmetric and we cannot apply the technique of Killing spinor equations. Instead we use symmetry to constrain the form of the solutions.

We will make an ansatz, following the analysis of the $\mathcal{N}=2^*$ gauge theory in [88], of the form

$$ds_5^2 = e^{2A}\left(-e^{2B}dt^2 + dx_3^3\right) + dr^2\ . \tag{5.13}$$

The presence of A allows the dilaton to have a non-trivial r dependence and that of B allows for non-zero temperature.

The field equations are

$$\begin{aligned} \frac{1}{4}R_{ab} &= \frac{1}{8}\partial_a\phi\partial_b\phi - g_{ab} && (5.14)\\ \nabla^2\phi &= 0\ . && (5.15)\end{aligned}$$

Using the linear combinations $\bar{A}\equiv A+\frac{1}{4}B$ and $\bar{B}\equiv\frac{\sqrt{3}}{4}B$ as in [88] these equations take the form

$$\begin{aligned} \phi'' + 4\bar{A}'\phi' &= 0 && (5.16)\\ \bar{B}'' + 4\bar{A}'\bar{B}' &= 0 && (5.17)\\ 6(\bar{A}')^2 &= \frac{1}{4}(\phi')^2 + 2(\bar{B}')^2 + 6\ . && (5.18)\end{aligned}$$

The first two equations can be integrated to yield $\phi' = c_1 e^{-4\bar{A}}$ and $\bar{B}' = c_2 e^{-4\bar{A}}$. We will see later that the two solutions we concentrate on correspond to setting either one or the other of these constants zero. Defining $6c_3^2 \equiv (\frac{1}{4}c_1^2 + 2c_2^2)$ we obtain

$$(\bar{A}')^2 = c_3^2 e^{-8\bar{A}} + 1\ . \tag{5.19}$$

These equations are analytically solvable with solutions

$$e^{4\bar{A}} = \frac{c_4^2 e^{8r} - c_3^2}{2c_4 e^{4r}} \tag{5.20}$$

$$\bar{B} = \frac{c_2}{4c_3}\ln\left(\frac{c_4 e^{4r} - c_3}{c_4 e^{4r} + c_3}\right) + B_0 \tag{5.21}$$

$$\phi = \frac{c_1}{4c_3}\ln\left(\frac{c_4 e^{4r} - c_3}{c_4 e^{4r} + c_3}\right) + \phi_0\ . \tag{5.22}$$

For any solution that returns to AdS asymptotically $B_0 = 0$ and ϕ_0 is the dilaton value in the AdS limit.

5.3.2 Solution with no event horizon

Let us first take the solution above and set $B \equiv 0$ to find a zero temperature dilaton flow with manifest 4D Lorentz invariance. The solution can be recast (by setting $c_3 = c_4\zeta$) in the form

$$e^{2A} = \sqrt{\frac{c_4}{2}}\,\sqrt{e^{4r} - \zeta^2 e^{-4r}} \tag{5.23}$$

$$\phi = \sqrt{\frac{3}{2}}\ln\left(\frac{e^{4r} - \zeta}{e^{4r} + \zeta}\right) + \phi_0\,. \tag{5.24}$$

To match to other results in the literature [73] one can rescale the x_4 coordinates to effectively set $c_4 = 1/2$, set $2u^2 = e^{2r}$ and $\zeta = -4u_0^4$. Reinstating the five sphere and moving to string frame one arrives at the 10D metric

$$ds^2 = e^{\phi/2}\left(\frac{u^2}{L^2}\mathcal{A}^2(u)dx_4^2 + \frac{L^2}{u^2}du^2 + L^2 d\Omega_5^2\right)\,, \tag{5.25}$$

with

$$\mathcal{A}(u) = \left(1 - \left(\frac{u_0}{u}\right)^8\right)^{\frac{1}{4}}, \qquad e^{\phi} = \left(\frac{(u/u_0)^4 + 1}{(u/u_0)^4 - 1}\right)^{\sqrt{3/2}}\,. \tag{5.26}$$

The four form is just that in pure AdS. This metric clearly becomes $AdS_5 \times S^5$ at large u and has a deformation parameter u_0^4 which has dimension four and no R-charge - this parameter is naturally identified with TrF^2 in the gauge theory. Since TrF^2 is the F-term of a chiral superfield supersymmetry is therefore broken in this gauge theory.

A crucial aspect of the geometry is that it is singular at $u = u_0$ with the dilaton blowing up. A singularity should be a source of unease and we do not have a full explanation of it but we wish to argue there are number of ideas that suggest such geometries are worthy of study none the less. The presence of D3 branes in the geometry do provide sources that, in some non-supersymmetric configuration, might complete the geometry (compare to the hard wall model where they account for the discontinuity between AdS and flat space). The N=2* geometry [79] is also singular at a point where the effective gauge coupling diverges - this geometry has been matched to the expected field theory solution at a particular point on moduli space [80]. This model provides evidence that a divergent gauge coupling can show up as a divergence in the geometry.

Our real motivation for the continued study here though is the phenomenological successes of the geometry. It has been shown to be confining in [66, 70] and to break chiral symmetries when quarks are introduced [73] as we will discuss below. In this sense we can think of it as a back reacted hard wall with the correct properties to describe QCD-like physics.

5.3.3 Black hole geometry

Let us now turn to finding the high temperature phase of the dilaton flow geometry just explored.

To have a solution with a horizon we will choose constants such that the function B goes as a constant plus $\ln r$ near $r = 0$ ($\bar{B} \sim \sqrt{3}/4 \ln r$). From (5.21) this gives the constraints

$$c_2 = \sqrt{3} c_3 \, , \qquad c_4 = c_3 \, . \tag{5.27}$$

At this point we note that, with the definition of c_3 ($6c_3^2 \equiv (\frac{1}{4}c_1^2 + 2c_2^2)$), c_1 vanishes so the dilaton profile in the non-extremal solution is just a constant. This result is a simple example of a 'no hair' theorem. For another example in a four-dimensional context see Appendix D.

We have learnt that there is no black hole solution with a radially dependent dilaton. This means there is not a gravity dual of a high temperature theory with TrF^2 switched on. One's immediate response is to become worried that if the $\mathcal{N} = 4$ gauge theory cannot be perturbed by a vev for TrF^2 at finite temperature then the zero temperature model is suspect. In fact though we believe it is telling us that the vev for TrF^2 is not the fundamental perturbation but an operator induced by the dynamics. We will discuss what the true perturbation might be in the next section.

We are left with a unique black hole solution which in the unbarred quantities is thus

$$e^{2A} = \frac{c_4}{2}\left(e^{2r} + e^{-2r}\right) \tag{5.28}$$

$$B = \ln\left(\frac{e^{4r}-1}{e^{4r}+1}\right) \tag{5.29}$$

$$\phi = \phi_0 \, . \tag{5.30}$$

This geometry should describe the high temperature phase of the dilaton flow theory. In fact by rescaling x_4 to set $c_4 = 2u_h^2$ and defining $e^{2r} = \frac{u^2+\sqrt{u^4-u_h^4}}{u_h^2}$ (which are asymptotically the same choices as for the dilaton flow geometry) this solution can be reduced to the usual Poincaré coordinate form of five-dimensional AdS-Schwarzschild (5.4) with Hawking temperature $T_H = \frac{u_h}{\pi}$. We conclude that the scalar mass deformed gauge theory shares the same supergravity description at finite temperature as the unperturbed $\mathcal{N} = 4$ gauge theory! A similar conclusion was reached in [74] in the context of adding a dilaton to the hard wall model.

5.4 The Origin Of Supersymmetry Breaking

We now turn to the question of the origin of supersymmetry breaking in the dilaton flow geometry if TrF^2 is an induced operator as it appears it must be from the above analysis.

Since supersymmetry is broken, yet SO(6)$_R$ preserved, in the T=0 geometry, we expect all SO(6) invariant operators to be present. Amongst these SO(6) invariant operators is an equal mass for each of the six scalar fields - one would expect the scalar masses to rise to the scale of the supersymmetry breaking scale. Such an SO(6) invariant mass is invisible in the supergravity solution for the same reasons as the SO(6) scalar vev operator discussed above. A frequently argued interpretation of the fact that this source is not described by a supergravity mode is that it is a super-irrelevant operator. The vev for the scalar operator discussed in the hard wall model above would then be super-relevant; that is it would have no impact on the UV of the theory until suddenly at some point in the IR it would dominate it - this can certainly be matched to its appearance as a sharp IR cut off on the geometry. In this language one would expect the mass term to show up as a sharp cut off on the space at some large radius or UV scale. Below that scale it would naively have no impact on the dynamics. This is not quite true though because it would define the symmetries of the theory below that UV cut off and in particular leave a non-supersymmetric but SO(6) invariant flow at lower energies. The dilaton flow geometry is the natural candidate for this flow. In this interpretation one should cut off the dilaton flow at some point in the UV, although this point could presumably be set at an arbitrarily high scale. In analogy with the hard wall model the point where the cut off appears would be undetectable in the low energy flow.

The above seems a consistent interpretation but the supersymmetric N=1* [78] and N=2* [79] theories suggest a more complicated story is also possible. In those theories precisely the scalar mass discussed here is present in the Lagrangian of the gauge theory yet no supergravity mode directly represents it in the supergravity duals. The scalar mass must, by supersymmetry, be present and tied to the fermion masses (in superspace there is only the one mass parameter) which are described by supergravity fields. The flows for these fields show the mass term to be relevant. These are therefore examples of theories with a relevant scalar mass present but no explicit dual operator to indicate it. The Yang Mills* geometry [81] is a non-supersymmetric example - a fermion mass term is introduced there yet the potential for a D3 probe shows there to be a scalar mass present too. This is the simple way for us to test whether there is a scalar mass present here - we look at the potential for moving a D3 probe in the transverse directions of the dilaton flow geometry. The result is clear on simple dimensional grounds - the asymptotic potential for the D3 motion must go like u_0^4 (note the fourth power is the lowest to occur in the metric) to some positive power so that it vanishes when u_0 does. We can only have a u dependent term of the form $V \sim u_0^8/u^4$ - there is no m^2u^2 term and so no explicit mass. The full D3 potential is given by

$$V \sim u^4\mathcal{A}^4 - u^4 \sim -\frac{u_0^8}{u^4}\,, \qquad (5.31)$$

which is stable.

Presumably in the theories with fermion masses the scalar mass is continually regenerated through the RG flow whilst if only the scalar mass is present it flows to zero in the IR. We conclude that in the dilaton flow geometry the scalar mass would indeed only be visible through a sharp UV cut off as discussed above. This seems a consistent interpretation to us and with this in mind we will go on to analyze the high temperature transition in the dilaton flow geometry.

Note this geometry is also closely related to those of Constable and Myers [71] which have in addition non-trivial u dependence in the four form - the existence of this larger class of geometries suggest that there are multiple SO(6) invariant string modes that are invisible in the supergravity and that determine the dynamics. In the field theory one can imagine higher dimension operators and so forth that could play a role. These geometries typically also show confinement and chiral symmetry breaking though [41].

5.5 Thermodynamic computation

One way to test our assertion that AdS-Schwarzschild is the high temperature phase of the non-supersymmetric deformation of the $\mathcal{N} = 4$ gauge theory is to check the Hawking-Page phase transition makes sense. We will compute the Euclidean action for both solutions, specifying a black hole horizon at $u = u_h$ and a dilaton flow singularity at $u = u_0$.

To make the comparison fair we must set the parameter c_4 equal in the two geometries so they have the same large-r AdS limit. We will perform the calculation in the Schwarzschild-type coordinates, rescaling the Euclidean time coordinate for the dilaton flow geometry as for our hard-wall calculation. Both geometries asymptote to AdS_5 so we can set the same UV cut-off Λ in both cases, before taking the limit $\Lambda \to \infty$.

Our interpretation above that the dilaton flow geometry is the IR theory below some UV cut off associated with the presence of a scalar mass means that formally we should keep the UV cut off fixed. We can though imagine that that scale is arbitrarily high. In any case we will give the result for arbitrary Λ below.

The Euclidean action density per unit spatial volume for the black hole solution is

$$S_{BH} = -\frac{1}{4\pi G_5} \int_0^{\frac{\pi L^2}{u_h}} d\tau \int_{u_h}^{\Lambda} \sqrt{-g} \left(\frac{1}{4} R + \frac{3}{L^2} \right) dr \; . \tag{5.32}$$

The trace of the Einstein equation gives $R = -\frac{20}{L^2}$ so

$$S_{BH} = \frac{1}{2G_5 u_h L^3} \int_{u_h}^{\Lambda} u^3 du \;=\; \frac{1}{8G_5 u_h L^3} \left(\Lambda^4 - u_h^4 \right) \; . \tag{5.33}$$

The Euclidean action density per unit spatial volume for the dilaton flow solution is, having used the trace of the equation of motion to remove the scalar gradient term and allowing for the rescaling of Euclidean time, simply

$$S_{DF} = \frac{1}{2G_5 L^2} \int_0^{\frac{\pi L^2}{u_h}\sqrt{1-\frac{u_h^4}{\Lambda^4}}} d\tau \int_{u_0}^{\Lambda} \sqrt{-g}\; dr \; . \tag{5.34}$$

This is

$$\begin{aligned} S_{DF} &= \frac{1}{2G_5 u_h L^3} \sqrt{1 - \frac{u_h^4}{\Lambda^4}} \int_{u_0}^{\Lambda} \left(u^3 - \frac{u_0^8}{u^5} \right) dr \\ &= \frac{1}{8G_5 u_h L^3} \sqrt{1 - \frac{u_h^4}{\Lambda^4}} \left(\Lambda^4 + \frac{u_0^8}{\Lambda^4} - 2u_0^4 \right) \; . \end{aligned} \tag{5.35}$$

Hence, in the $\Lambda \to \infty$ limit, the difference in the actions is simply

$$S_{BH} - S_{DF} = \frac{1}{16 G_5 u_h L^3} \left(4u_0^4 - u_h^4 \right) \; . \tag{5.36}$$

For a deformation scale $u_0 > \frac{u_h}{\sqrt{2}}$ the dilaton flow solution is thermodynamically favoured whereas for a deformation scale $u_0 < \frac{u_h}{\sqrt{2}}$ the AdS-Schwarzschild solution is favoured. The transition temperature is $T_c = \frac{\sqrt{2}u_0}{\pi L^2}$.

The phase transition appears to make complete sense with our interpretation of the high and low temperature phases. For temperatures below the value of the supersymmetry breaking scale u_0 (up to a factor of order unity) the non-supersymmetric SO(6) invariant scalar mass deformed $\mathcal{N} = 4$ gauge theory is described by the dilaton flow geometry with an induced vev for TrF^2. As the temperature passes through the supersymmetry breaking scale there is a transition to the deconfined plasma described by AdS-Schwarzschild - here the vev of TrF^2 is zero.

It is important to stress though that the computation above may not be complete. We saw above that when the hard wall model was converted into a full string geometry a Gibbons-Hawking term appeared at the IR singularity that played a crucial role in the thermodynamics of the $\mathcal{N} = 4$ theory on moduli space. One must worry that a similar surface term might appear if the dilaton flow geometry were completed to a fuller string theoretic understanding. In addition the negative term in our action computation is dominated near the singularity and might also change were the singularity resolved in someway. Within supergravity, our only available tool, it is hard to see how to address these complications - it is encouraging though that the calculation as is does agree with expected field theory intuition and the naive hard wall geometry as applied to QCD. It is interesting to compare this case to the AdS-QCD models of [72] in which the action computation is dominated away from the IR singularity - those theories may have better control.

Another interesting point is that the supergravity computation naively suggests the dilaton flow geometry's action is lower than that of AdS! Our discussion of the origin of supersymmetry breaking though suggests this is not a correct comparison - we have argued that the geometry only applies below some UV cut off corresponding to the scale where a super-relevant scalar mass becomes important. It is an artefact of supergravity that this cut off is invisible in the IR. Above that cut off the presence of a non-normalizable mode would make the dilaton flow geometry's action much greater than the case of AdS extended to infinity. We do not expect the $\mathcal{N} = 4$ theory to spontaneously break supersymmetry. This does not affect our computation since both the AdS Schwarzschild black hole and the dilaton flow geometry share the same UV (at least if the cut off is far enough into the UV) and hence would see the same UV cut off physics.

5.6 Aspects of the Phase Transition

The identification of AdS-Schwarzschild as the high T phase of a non-supersymmetric theory and the dilaton flow as the low temperature phase of that same theory is quite remarkable. Our findings go some way towards explaining why the AdS-Schwarzschild gravity dual is a reasonable toy model of high-temperature QCD whereas the zero-temperature supersymmetric D3-D7 model is quite unlike low-temperature QCD (it is conformal, for example, when the quarks are massless). Both of the geometries have been studied in detail already in the literature, including in the presence of quarks, and we can look at a number of properties of the transition.

5.6.1 Glueballs

Below the critical temperature, the core of the dilaton flow geometry is repulsive to strings. The result is that a Wilson line calculation shows there to be a linear quark-antiquark potential as the string falls towards $r = 0$ before settling a little away from the singularity at u_0 [66, 73]. In keeping with the implied confinement there is a discrete spectrum of glueballs which are the eigenmodes of the Klein-Gordon equation on the geometry for the usual plane-wave ($\propto e^{ik\cdot x_4}$) ansätze. Table I shows the masses of the lowest five scalar glueball states, in units of u_0/L^2 (we can reproduce the field equations for these fluctuations in [66] but disagree on the numerical values for the masses) [1]

n	1	2	3	4	5
M_n	4.1	7.2	10.2	13.2	16.2

Table I: Lowest five glueball masses in the zero temperature dilaton flow geometry in units of the deformation scale u_0/L^2.

n	ω_n
1	± 3.119452 - 2.746676 i
2	± 5.169521 - 4.763570 i
3	± 7.187931 - 6.769565 i
4	± 9.197199 - 8.772481 i
5	± 11.202676 - 10.774162 i

Table II: Lowest five glueball quasi-normal modes in the AdS-Schwarzschild geometry in units of $\frac{u_h}{L^2}$.

Above the critical temperature, the theory is in a deconfined phase. There is no longer

[1]The dilaton equation of motion is given by $\partial_u(u^5\mathcal{A}^4\partial_u\phi) + u\mathcal{A}^2M^2\phi = 0$ (here we write u in units of u_0 and rescale x_4 so that factors of L, u_0 are common to the metric). The UV solutions take the form $\phi \sim c_1 + c_2/u^4$ with the latter being required for a glueball fluctuation. In the IR the equation can be recast in Schrödinger form - we write $u = 1 + z$, then change coordinates to y such that $dy/dz = 1/(8z)^{1/4}$, and finally write $\phi = uv$ with $\frac{1}{v}\frac{dv}{dy} = -3/(8z)^{3/4}$. The potential is then of the form $V = -\frac{1}{4y^2}$. Such a potential is of the limiting form that possesses a discrete spectrum bounded from below (see [94] or chapter 5 of [95]). The IR solutions, written in the original u coordinates are of the form $\phi \sim c_3 + c_4 \ln(u-1)$ - the former are the physical solutions, the latter blow up and are therefore inconsistent with linearization. All of this is in complete agreement with the analytic discussion in [66] and we can numerically shoot from both the IR and UV solutions to find the values of M for which solutions match to the required UV and IR boundary conditions. We disagree with [66] on the numerical values of these solutions though.

a spectrum of glueball normal modes, rather the gravity dual admits a spectrum of unstable quasinormal modes which was calculated in [60]. The field theory interpretation of the quasinormal spectrum is to give the mass and decay width for a glueball excitation embedded in a thermal bath of SYM plasma. The finite decay timescale can be viewed as the timescale for the 'melting' of the glueball state. The breaking of Lorentz symmetry means there is a nontrivial dispersion relation $\omega(k)$ for a scalar glueball excitation [60]. Table II shows the lowest five quasinormal frequencies, which are measured in units of $\frac{u_h}{L^2}$ which is $\propto T$ - the natural scale of the lowest quasinormal frequency is the temperature.

We would of course expect the same features for the fermionic superpartners of glueballs, with a discrete spectrum of masses in the confining phase and a quasinormal spectrum in the deconfined phase as we found in the previous chapter. In both cases bosonic and fermionic masses will not be degenerate as neither background is supersymmetric.

5.6.2 Quarks and mesons

Now that we have a candidate model which may describe a more QCD-like theory, a natural thing to do is to study the behaviour of probe quarks in the theory. Following the method outlined in chapter 3 let us consider adding a D7 brane into the geometry. The action in the isotropic coordinates is

$$S_{D7} = \int d^8\xi \; e^{\phi}\sqrt{-g} = \frac{(w^4+1)^{\sqrt{\frac{3}{2}}+1}}{(w^4-1)^{\sqrt{\frac{3}{2}}-1}}\frac{\rho^3}{w^8}\sqrt{1+r'^2}\,. \tag{5.37}$$

The equation for the embedding can be solved and the crucial point is that the singluar region of the geometry repels the D7 brane.

There is stable D7 brane embedding which describes a theory with zero current algebra quark mass, shown in Fig.(5.1).

One can seek the spectra of mesonic excitations associated to this brane. One finds the

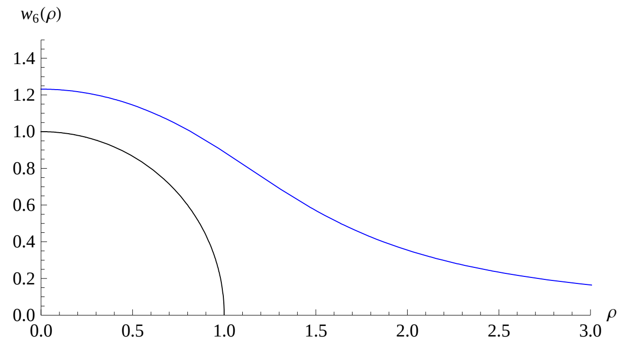


Figure 5.1: Embedding with zero current algebra quark mass. Note the singularity is shown as a quarter-circle here (the singularity is located on the circular locus $\rho^2+w_6^2 = 1$) - this is an artifact of the isotropic coordinates as it actually has zero metric surface area.

degeneracy of masses lifted due to the fact that there is no supersymmetry.

Flavour degrees of freedom can be included by embedding D7 branes in each of the geometries discussed [37, 38, 35] - the results to date in these geometries use the probe or quenched limit [16]. The asymptotic profile of the D7 encodes the relationship between the hard quark mass m_q and the expectation value of the quark condensate $\langle\bar{q}q\rangle$ [41]. Mesonic modes are dual to fluctuations derived from the DBI action of the D7 [40].

Below the critical temperature we find the probe D7 always wants to lie outside the deformation scale of the dilaton flow geometry for the embeddings of physical interest (usefully avoiding the singular region of the geometry). One finds there is a nonzero value of $\langle\bar{q}q\rangle = 1.51u_0^3$ for zero m_q, indicating spontaneous breaking of a $\mathrm{U}(1)_R$ symmetry of the model (this symmetry is analogous to the axial $\mathrm{U}(1)_A$ of QCD) - this is a nice model of QCD-like behaviour since the dynamics of the quark condensate generation is included even if the full non-abelian chiral symmetry breaking is not present. There are discrete spectra corresponding to pion-like and sigma-like scalar excitations [70]. The lowest pion-like state is massless for $m_q = 0$ and its mass grows in accordance with the Gell-Mann-Oakes-Renner relation for small m_q. In addition there is a tower of massive vector meson excitations dual to the Maxwell field on the D7 worldvolume. The numerical values for all these meson masses tend to the no-deformation result (equation (3.19) in [40]) for large m_q, that is $M \sim 2\sqrt{2}m_q/L^2$ - we list them in Table III and display them in Fig.(5.4) in units of u_0.

It is possible to obtain the effective Schrödinger potentials for the mesons. Because the embedding is known only numerically we must proceed using numerical methods only.

The equation one (linearized 'pion' fluctuation) has is of the form

$$\left(f\phi'\right)' = g\phi - \lambda h\phi \ . \tag{5.38}$$

We change variables to $y(x)$ where $\frac{dy}{dx} = \sqrt{\frac{h}{f}}$ to yield

$$\lambda\phi = -\frac{d^2\phi}{dy^2} + p(x)\frac{d\phi}{dy} + \frac{g}{h}\phi \ , \tag{5.39}$$

where $p(x) = -\left(\frac{f}{h}\frac{d^2y}{dx^2} + \frac{f'}{h}\frac{dy}{dx}\right)$ (here f' is an x-derivative). Now writing $\phi \equiv uv$ with $2\frac{d}{dy}\left(\ln v\right) = p$ and noting $\frac{v''}{v} \equiv \left(\frac{v'}{v}\right)' + \left(\frac{v'}{v}\right)^2$ one obtains the Schrödinger potential as

$$V(x) = \frac{p^2}{4} - \sqrt{\frac{f}{h}}\frac{p'}{2} + \frac{g}{h} \ , \tag{5.40}$$

where $p = -\left(\frac{f}{h}\left(\sqrt{\frac{h}{f}}\right)' + \frac{f'}{h}\sqrt{\frac{h}{f}}\right)$ (the primes are x-derivatives). This can be plotted to give the potential as a function of the untransformed x-coordinate. A parametric plot can give the potential as a function of the Schrödinger y-coordinate where $\frac{dy}{dx} = \sqrt{\frac{h}{f}}$ is solved numerically with $y(0) = 0$. The resulting potential should be compared with the potential in the undeformed geometry for the same asymptotic quark mass (Figs.(5.2) and (5.3)).

The reason you can have a massless pion ground state is the fact that the potential goes below the axis for small quark masses in the deformed geometry.

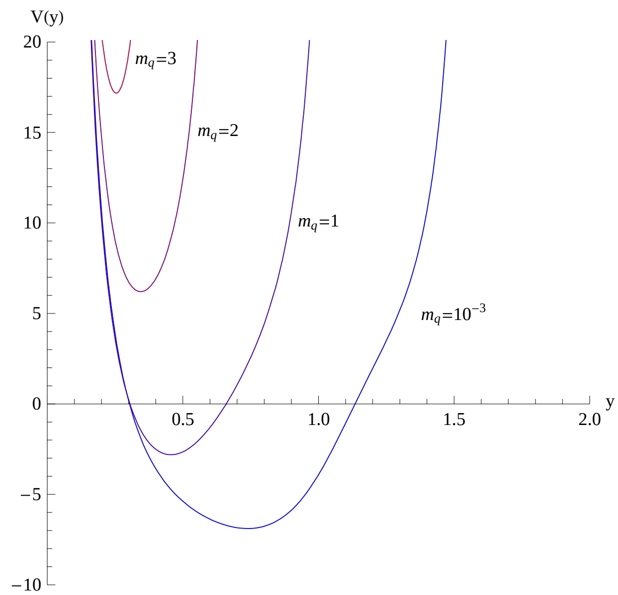


Figure 5.2: Schrödinger potentials for the pion in Ghoroku geometry for different quark masses. The presence of the deformation stops the box becoming infinitely wide as the hard quark mass is dialled to zero since an additional energy scale is introduced (the 'QCD bag'). It also makes the potential dip down below zero, which is what gives the zero-mass ground state.

It is possible to show that the potential returns to the 'particle in a box'-like form for D7 embeddings with current algebra mass much greater than the deformation scale.

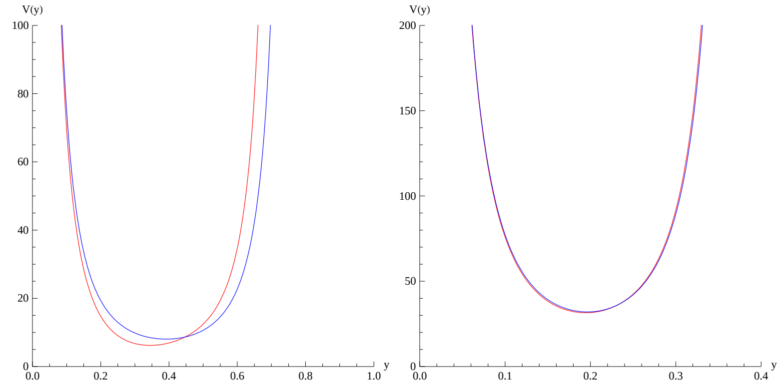


Figure 5.3: Here we show that the Schrödinger potential in the Ghoroku geometry (the non-symmetric curve) tends back to the undeformed box-like (symmetric) case as the quark mass becomes large relative to the deformation scale (unity for our numerics). The left hand plot shows the potentials for $m_q = 2$ and the right hand plot for $m_q = 4$.

m_q	$M_\pi L^2$	$M_\sigma L^2$	$M_{vector} L^2$
0.10	0.7	3.1	2.9
0.50	1.9	3.5	3.3
1.00	3.1	4.1	3.9
2.00	5.7	6.0	6.0
3.00	8.5	8.6	8.6
4.00	11.3	11.4	11.4

Table III: the mass of the pion, sigma and rho meson modes as a function of the quark mass in the low temperature dilaton flow geometry - all in units of u_0.

Above the critical temperature the physically relevant D7 embeddings in the black hole geometry [41, 92, 42, 43, 44, 93] give $\langle \bar{q}q \rangle = 0$ for $m_q = 0$ - there is no chiral symmetry breaking and hence no pion-like meson. The D7 can either end on the black hole horizon

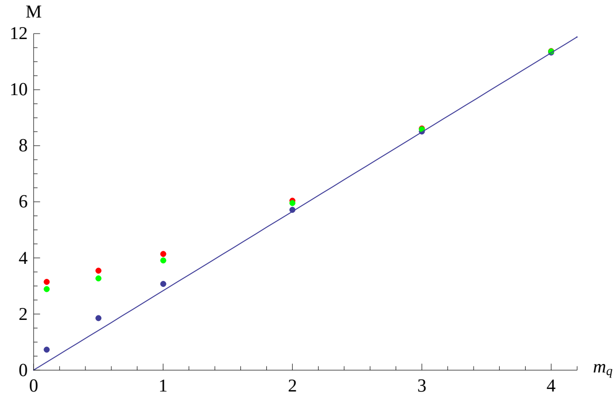


Figure 5.4: Pion (lowest mass state for given quark mass), sigma (highest) and vector (intermediate) masses as a function of quark mass - all in units of u_0. The line shows the large-m_q limit.

(small m_q) or for large enough m_q it has sufficient tension to support itself away from the black hole (Fig.(5.5))- there is a first-order phase transition in the behaviour of (quenched) quark matter as one raises the quark mass in the plasma background. In the former case there are quasinormal spectra representing the melting of scalar and vector mesonic excitations in the hot background [53, 51, 57]. In the latter case there are discrete spectra of scalar and vector meson masses with scale set by m_q which tend to the values in [40] as $m_q \gg T$. Our transition behaves in the same way as long as the quarks are sufficiently light ($m_q < 0.92\frac{\sqrt{\lambda}T}{2}$) [51]. In the undeformed theory this bound implies that the mesons melt once the temperature of the background becomes of order the meson mass since the meson masses are $\sim \frac{m_q}{\sqrt{\lambda}}$. In our case the pion-like meson is an exception to this - one can have a massless pion that does not 'melt' until the background reaches some finite temperature.

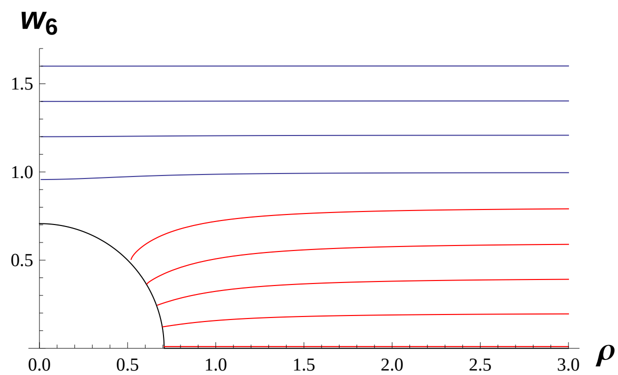


Figure 5.5: D7 brane embeddings in the AdS-Schwarzschild geometry in isotropic coordinates. They may either terminate on the horizon (shown as the quarter-circle at $\rho^2 + w_6^2 = \frac{1}{2}$) or wrap a contractible cycle that closes outside the horizon.

n	ω_n
1	$\pm$ 2.1988 - 1.7595 i
2	$\pm$ 4.2119 - 3.7749 i
3	$\pm$ 6.2155 - 5.7773 i
4	$\pm$ 8.2172 - 7.7781 i
5	$\pm$ 10.2181- 9.7785 i

Table IV: the scalar mesonic quasinormal frequencies in the high T phase ($m_q = 0$) - in units of $\frac{u_h}{L^2}$.

There are recent results concerning a lower bound for the ratio of viscosity to entropy density of a strongly-coupled field theory [91], $\frac{\eta}{s} \geq \frac{1}{4\pi}$, where the equality is for the deconfined phase of $\mathcal{N} = 4$ SYM theory (for a review see [83]). Our findings show that strict equality also applies to certain non-supersymmetric theories in their deconfined phase, physically due to the universality of this phase in the large-N limit.

We can perform an estimate of the deconfinement temperature in our model. The mass of the lowest-lying vector state for zero quark mass can be compared to the mass of the

ρ meson, experimentally 776 MeV. The vector mass is $2.80u_0/L^2$ and the deconfinement temperature is $T_c = \frac{\sqrt{2}u_0}{\pi L^2}$. This gives an estimate for the deconfinement temperature of $T_c \sim 124$ MeV. This is very similar to the estimate produced by the 'hard-wall' model [77] and is somewhat low compared to real QCD.

Note that in addition to the well-understood operator deformations of the AdS/CFT Correspondence one can also try to construct the gravity dual of real QCD in a bottom-up fashion. An extremely simple example calculation can be found in Appendix E.

5.7 Conclusion

We have found the finite-temperature version of a chiral symmetry breaking dilaton flow. Performing a calculation at the level of classical supergravity we found that for a temperature sufficiently high that the black hole radius is greater than the deformation scale, the geometry undergoes a first-order phase transition to the AdS-Schwarzschild geometry. From the gauge-theory perspective, we have argued that the dilaton flow solution results from turning on a $SO(6)$ invariant scalar mass deformation. This deformation is not described by a SUGRA mode and is probably present only as a UV cut off that determines the symmetries of the IR theory, but one does observe that the non-trivial dilaton profile describes a running coupling and the operator TrF^2 is 'induced'. We have found that at high temperature this does not happen and TrF^2 remains zero. Incorporating other results already in the literature one can see that the transition corresponds not just to a deconfinement transition but also a simultaneous chiral symmetry restoration transition if (quenched) quarks are introduced into the theory. We believe this is the simplest, four dimensional, AdS/CFT derived caricature of a QCD-like theory.

It may also be noted that essentially the same results apply to the system in one dimension fewer, with a low-temperature deformed AdS_4 dilaton flow background and a corresponding AdS_4-Schwarzschild phase This presumably defines the phase transition of a $2+1$-dimensional field theory. It is of somewhat less interest than the $3+1$ dimensional field theory, for example there is no chiral symmetry for $2+1$ dimensional quarks.

Acknowledgements: During the production of the paper on which this chapter is based, the authors had the benefit of comments from Johanna Erdmenger, Ingo Kirsch and Kazuo Ghoroku.

Chapter 6

Chemical potential in the $D3-D5$ intersection

Recently there has been much interest in applying gauge-gravity duality to understand problems in strongly-coupled condensed matter particularly non-BCS superconductivity. Here we study probe D5 branes in D3 brane AdS_5 and AdS_5-Schwarzschild backgrounds as a prototype dual description of strongly coupled 2+1 dimensional quasi-particles. We introduce a chemical potential through the $U(1)_R$ symmetry group, U(1) baryon number and a U(1) of isospin in the multi-flavour case. We find the appropriate D5 embeddings in each case - the embeddings do not exhibit the spontaneous symmetry breaking that would be needed for a superconductor. The isospin chemical potential does induce the condensation of charged meson states.

6.1 Introduction

Recently there has been interest in whether the AdS/CFT Correspondence [13, 14, 15] can be used to understand 2+1 dimensional condensed matter systems (for example

[96, 17, 97, 98, 99, 100, 101]). The typical UV degrees of freedom in these systems are electrons in the presence of a Fermi surface and a gauged U(1), QED. When brought together in certain 2d states they can become relativistic and strongly coupled - possibly such systems might induce superconductivity too by breaking the gauge symmetry. The philosophy, which may be overly naive, is to find relativistic strongly coupled systems that show these behaviours and hope they share some universality with the physical systems. Whether or not that linkage becomes strong, it is interesting to study the AdS duals of 2+1d systems.

In this chapter we will study the dynamics of the theory on the world volume of a mixed D3 and D5 brane construction with a 2+1 dimensional intersection, which has previously been studied at zero temperature in the absence of chemical potential in [102, 103, 104]. The gravity dual of the D3s, at zero temperature, is $AdS_5 \times S^5$, which is dual to the 3+1 dimensional $\mathcal{N} = 4$ super Yang Mills theory. Here these interactions will be used to loosely represent strongly coupled "phonons". We will introduce 2+1d "quasi-particles" via D5 branes (with a 2+1d intersection with the D3s). Strings connecting the D3 and D5 branes should be expected to carry quantum numbers that interact with the D3 brane background and flavour quantum numbers associated with the D5 branes - the full field theory can be found in [104]. We will work in the probe approximation for the D5 branes which corresponds to quenching quasi-particle loops in the phonon background [16]. At zero chemical potential the theory has $\mathcal{N} = 4$ supersymmetry and at zero quasi-particle mass is conformal [102, 103, 104]. The system is related to the higher dimensional D3-D7 intersection where the $\mathcal{N} = 4$ gauge theory on the D3 branes has been used to describe gluon dynamics and the D3-D7 strings quarks - some progress in the study of the properties of mesons in 3+1d strongly coupled gauge theories has been achieved [35]. The D3-D5 defect system seems a natural starting point therefore for 2+1 dimensional systems.

To attempt to mimic a solid state system one must weakly gauge a U(1) symmetry of the system and introduce an associated chemical potential (by setting $A_t = \mu$). There are a number of possible U(1)s that can play this role.

Firstly the D3-D5 world volume theory has an unbroken SO(3) global symmetry corresponding to rotations in the 4-plane transverse to the D5 brane. We will introduce a chemical potential for the quasi-particles with respect to a U(1) subgroup of the SO(3)- this can be done by simply spinning the D5 branes in an SO(2) plane [105]. The embedding of the D5 brane is described by a scalar that is charged under this U(1) symmetry so one naively expects to trigger superconductivity in the spirit described in [17] - but here we would have an explicit understanding of the UV degrees of freedom the scalar describes. Naively one expects the scalar describing the D5 embedding to be destabilized by the presence of a chemical potential which gives the scalar a negative mass squared. An equivalent statement is that one expects the centrifugal force associated with the rotational motion of the brane to force it off the spin axis. In fact though, by finding the minimum-area embedding for such spinning probe D5 branes, we show that this is not the case.

The crucial physics is that the speed of light decreases as one moves into the centre of AdS - eventually it becomes less than the rotation speed of the D5 brane. We show, following the higher dimensional analysis in [105, 106, 108] that there are regular D5 embeddings into the interior which have a more complicated embedding structure. The branes bend in the direction of the rotation so that there are two linked scalar fields describing the embedding - this richer theory turns out to not include superconductivity, a subtlety on top of the arguments in [17].

We can introduce mass terms for the quasi-particles that explicitly break the U(1) symmetry and we discuss the embeddings in these cases. There is a first order phase transition when the R-charge chemical potential grows above the mass of the quasi-particle bound states - below the transition the quasi-particles exist as deconfined particles whilst above it they are confined into bound states. This transition is analogous to the meson melting transition seen in this system and the D3-D7 system at finite temperature [41, 53, 51, 43]. We also analyze the finite temperature behaviour of these solutions by using the AdS_5 Schwarzschild geometry as the background.

Next we study a chemical potential for the U(1) associated with baryon number for the quark fields - this seems the most natural candidate for how QED would manifest in the effective relativistic theory of a solid state system. The U(1) appears in the gravity dual as the U(1) gauge symmetry on the surface of the D5 branes - we allow configurations with non-zero profiles for these fields on the D5. Here we are again led by results in the D3-D7 system [109].

At zero temperature the presence of the gauge field on the brane naively adds in an additional constraint on the solutions that that gauge field should be regular as the brane passes from positive to negative values of the effective radial coordinate on the brane. This criterion rules out small perturbations of the standard flat embeddings of the D5 brane - instead the true solutions become ones where the D5 brane kinks through the origin of the space. The kink, which for large quark mass is rather sharp, has been interpreted in [109] as a tube of strings connecting the asymptotic branes. In [111] it has been argued that an external charge could be responsible for the irregularity of the gauge field and that the flat embeddings should be retained. In either interpretation, amongst these configurations is one for zero quark mass which turns out to simply lie straight through the origin of the space - the chemical potential does not induce any R-charged operators to condense. The other fields on the D5 world volume carry no net baryon number and so do not couple to the chemical potential - there is no condensation. The system does not therefore act like a superconductor.

The finite temperature behaviour of these solutions with non-zero baryon number chemical potential is also explored. The D5 brane embeddings kink on to the black hole event horizon in this case. As in the D3-D7 case there is a first order phase transition between two different sorts of in-falling solutions which is analogous to the meson melting transition at finite temperature but zero chemical potential [41, 53, 51, 43].

Finally we turn to isospin chemical potential in the case of multiple (but still probe) D5 branes. This theory seems less relevant to solid state physics because the quasi-particles have a U(2) flavour symmetry - electrons don't! On the other hand it is natural to discuss in this context and as advocated in [99, 97] may provide some lessons for p-wave superconductors. The embeddings of any individual brane is simply the same as for an equal magnitude baryonic chemical potential and there is no induced R-charge breaking at zero quark mass. Where the theories differ is that there are vector bosons ('$W^{\pm}$') on the branes' world volume that couple to the chemical potential - we work in a truncated version of the DBI action that is just Yang Mills theory on the D5 world volume. We show, with an analysis very similar to that of [99, 97] and recent work in the D3-D7 system [112, 113], that below some critical value of the temperature the W-bosons condense at a second order transition. This is dual to the formation of a spin one condensate which is charged under U(1) isospin number in the gauge theory. Were one to identify that U(1) with QED we would have a superconductor.

6.2 The D3 Theory

We will represent the strong interaction dynamics with the large N $\mathcal{N} = 4$ super Yang Mills theory on the surface of a stack of D3 branes. It is described at zero temperature by $\mathrm{AdS}_5 \times S^5$

$$\begin{aligned} ds^2 &= \tfrac{(\rho^2+r^2)}{L^2} dx_{3+1}^2 \\ &\quad + \tfrac{L^2}{(\rho^2+r^2)}(d\rho^2 + \rho^2 d\Omega_2^2 + dr^2 + r^2 d\tilde{\Omega}_2^2) \,, \end{aligned} \tag{6.1}$$

where we have written the geometry to display the directions the D3 lie in (x_{3+1}), those we will embed the D5 on (x_{2+1}, ρ and Ω_2) and those transverse (r and $\tilde{\Omega}_2$). L is the AdS radius.

At finite temperature the description is given by the AdS-Schwarzschild black hole

$$ds^2 = \frac{u^2}{L^2}(-h(u)dt^2 + dx_3^2) + \frac{L^2}{u^2 h(u)} du^2 + L^2 d\Omega_5^2 \tag{6.2}$$

$$h(u) = 1 - \frac{u_0^4}{u^4} \, . \tag{6.3}$$

It is helpful to make the change of variables to isotropic coordinates

$$\frac{u\, du}{\sqrt{u^4 - u_0^4}} = \frac{dw}{w} \, , \tag{6.4}$$

and choose the integration constant such that if $u_0 = 0$ the zero-temperature geometry is recovered

$$2w^2 = u^2 + \sqrt{u^4 - u_0^4} \, . \tag{6.5}$$

The metric can now be written as

$$\begin{aligned} ds^2 \;\; &= \;\; \tfrac{1}{L^2}\left(w^2 + \tfrac{u_0^4}{4w^2}\right)\left(-\left(\frac{w^4 - \frac{u_0^4}{4}}{w^4 + \frac{u_0^4}{4}}\right)^2 dt^2 + dx_3^2\right) \\ &\quad + \tfrac{L^2}{w^2}\left(d\rho^2 + \rho^2 d\Omega_2^2 + dr^2 + r^2 d\tilde{\Omega}_2^2\right) \, , \end{aligned} \tag{6.6}$$

with $w^2 = \rho^2 + r^2$, which shares the coordinate structure of (6.1).

6.3 Quenched Matter from a D5 Probe At T=0

We will introduce quenched matter via a probe D5 brane. The underlying brane configuration is as follows:

	0	1	2	3	4	5	6	7	8	9
D3	-	-	-	-	•	•	•	•	•	•
D5	-	-	-	•	-	-	-	•	•	•

In polar coordinates the D5 fills the radial direction of AdS_5 and is wrapped on a two sphere.

The action for the D5 is just its world volume

$$S \sim T \int d^6\xi \sqrt{-\det G} \sim \int d\rho \, \rho^2 \sqrt{1 + r'^2} \, , \tag{6.7}$$

where T is the tension and we have dropped angular factors on the two-sphere.

This is clearly minimized when r is constant so the D5 lies straight. The value of the constant is the size of the mass gap for the quasi-particles. We will mainly be interested in the conformal case where the constant is zero. Note the general large ρ solution is of the form

$$r = m + \frac{c}{\rho} + \dots \, . \tag{6.8}$$

Here m is an explicit mass term for the quasi-particles in the Lagrangian and c the expectation value for a bi-quasi-particle operator - note m has dimension one and c dimension two adding to three as required for a Lagrangian term in 2+1d. The solution with non-zero c is not normalizable in pure AdS_5. Note that when $m = c = 0$ the theory is conformal. Including a non-zero m or c breaks the SO(3) symmetry ie it breaks one transverse SO(2) symmetry. From this it is apparent that m and c carry charge under that U(1). Were c to be non-zero when $m = 0$ it would be an order parameter for the spontaneous breaking of the U(1) symmetry.

6.4 R-Charge Chemical Potential/Spin

Our theory as yet lacks the relevant perturbation of the Fermi surface and the U(1) of QED. We will associate the U(1) with a subgroup of the SO(3) of the $\tilde{\Omega}_2$ - for concreteness we will use the angle in the $x_7 - x_8$ directions.

To include a chemical potential we will spin the D5 brane in the angular direction ϕ of this U(1) with angular speed μ.

The spinning of the D5 branes implies that the quasi-particles see a chemical potential. This is in fact a little bit of a peculiar limit since the background D3 theory also has fields, including scalars, charged under the U(1). We are not allowing that geometry to backreact to the chemical potential. In fact we had better not - the pure D3 theory has a moduli space for separating the D3s in the transverse 6-plane. Were we to set them spinning they would scatter to infinity since there is no central force to support rotation. In the theory on the D3 surface there is a run away Bose-Einstein condensation. We simply wish to switch off this physics - it is not what we are interested in - so we forbid such backreaction. The D3 theory is in an unstable state but will nevertheless provide some strongly coupled interactions for the quasi-particles that do see the chemical potential.

6.4.1 An Overly Naive Ansatz

We first look for solutions where the D5 embedding has $\phi = \mu t$ and we will allow the position r (the radial distance in $x_7 - x_8$) to be a function of ρ. The action is

$$S \sim \int d\rho\, \rho^2 \sqrt{(1+r'^2)(1-\frac{L^4}{(\rho^2+r^2)^2} r^2\mu^2)}\,. \tag{6.9}$$

Naively one is expecting the centrifugal force from the spinning to eject the brane from the axis at all but the end points where the boundary conditions hold the brane. This would lead to a spontaneous symmetry breaking or superconducting state. We will see that this is what this naive system tries to achieve.

The equation of motion for r as a function of ρ is easily computed but unrevealing. At large ρ the solutions tend to the no-rotation limit $r \sim m + \frac{c}{\rho}$.

The (pair of) circle(s) in the (ρ, r) plane described by $L^4\mu^2 r^2 = (\rho^2+r^2)^2$ is clearly a zero of the action so branes wrapped there provide a solution to the equation of motion. Anything going within the locus described by the two circles is moving faster than the local speed of light and is presumably not physical. This locus is a stationary limit surface - we call it the ergosurface below.

There exist "Karch-Katz" type solutions [16] for D5-branes that do not encounter the stationary limit surface - these solutions essentially lie flat above everything plotted in Fig.1. We want to know what happens to those which have a close encounter with the ergosurface. It turns out that the curves which minimize the action like to hit the surface at a right angle. They then kink onto the surface where they can have zero action.

The relation between m and c for curves impacting on the stationary limit surface in this way is shown in Fig.(6.1) - the presence of non-zero c at $m = 0$ appears to indicate spontaneous breaking of the U(1) symmetry ie superconductivity. Note there is a first order phase transition between the Karch-Katz embeddings and those hitting the ergosurface - we will discuss this transition further below.

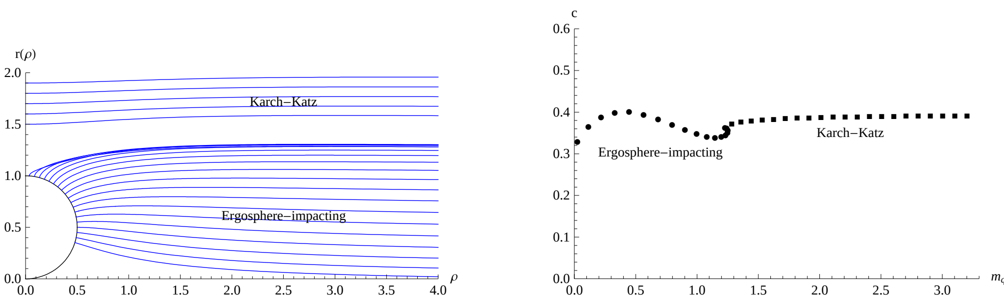


Figure 6.1: Embeddings of D5 branes impacting on the ergosurface and some of the Karch-Katz type embeddings (note these actually exist down to the top of the ergosurface) (top). At the bottom is a plot of c vs m for embeddings impacting on the ergosurface (circles) and Karch-Katz embeddings (squares). The solutions oscillate around the value for the lowest Karch-Katz solution as the D5 approaches the very top of the ergosurface.

The problem of course here is that the solutions are singular at the ergosurface where they kink. This is a sign that our ansatz is wrong - none of this is the right physics.

6.4.2 A More Sophisticated Ansatz

We will now try a more sophisticated ansatz where the brane has in addition some profile $\phi(w)$ where ϕ is the angle on which they spin (ie $\phi = \mu t + \phi(w)$). The ansatz is inspired by the work in [106] where similar issues are encountered when a magnetic field is switched on on the brane's world-volume.

We find it numerically convenient to switch coordinates and write the AdS geometry as

$$\begin{aligned} ds^2 &= \tfrac{w^2}{L^2} dx_{3+1}^2 + \tfrac{L^2}{w^2}\left(dw^2\right. \\ &\left. +w^2\left(d\theta^2 + \sin^2\theta d\Omega_2^2 + \cos^2\theta d\tilde{\Omega}_2^2\right)\right) \,. \end{aligned} \tag{6.10}$$

The D5 will now be embedded in the x_{2+1}, w and Ω_2 directions - the naive solutions above are recovered by looking for solutions that have $\theta(w)$ and $\phi = \mu t$ where ϕ is the 'first' angle of the $\tilde{\Omega}_2$.

In these coordinates the Lagrangian for our more ambitious ansatz for the rotating D5 embedding is

$$\begin{aligned} \mathcal{L} &= w^2 \sin^2\theta \times \\ &\sqrt{\left(1 - \frac{L^4\mu^2\cos^2\theta}{w^2}\right)(1 + w^2\theta'^2) + w^2\cos^2\theta\ \phi'^2} \,. \end{aligned} \tag{6.11}$$

If $\phi' \sim \mu$ the two μ^2 terms compete against each other removing the naive intuition about centrifugal force.

Since the action only depends on ϕ' and not ϕ one can integrate the equation of motion for ϕ'. One could then substitute back in for ϕ' in terms of the integration constant - this though gives an action with a "zero over zero" form at the ergosurface that is hard to work with. Instead, following [106, 107], we Legendre transform to $\mathcal{L}' \equiv \mathcal{L} - \phi' \frac{\partial \mathcal{L}}{\partial \phi'}$.

This gives (setting $\frac{\partial \mathcal{L}}{\partial \phi'} = J$)

$$\begin{aligned} \mathcal{L}' &= \tfrac{1}{w \cos\theta} \sqrt{\left(1 - \tfrac{L^4 \mu^2 \cos^2\theta}{w^2}\right)} \sqrt{(1 + w^2 \theta'^2)} \\ &\quad \times \sqrt{\left(w^6 \sin^4\theta \cos^2\theta - J^2\right)} \,. \end{aligned} \tag{6.12}$$

This has a "zero times zero" form at the ergosurface which is simpler to work with numerically.

For a solution that crosses the ergosurface we demand that the action be positive everywhere and this fixes J - the two terms must pass through zero and switch signs together. Having fixed J in this way one can then look at the θ equation of motion near the ergosurface. Expanding near the surface, and after some algebra, one finds the following consistency equation for the θ derivative

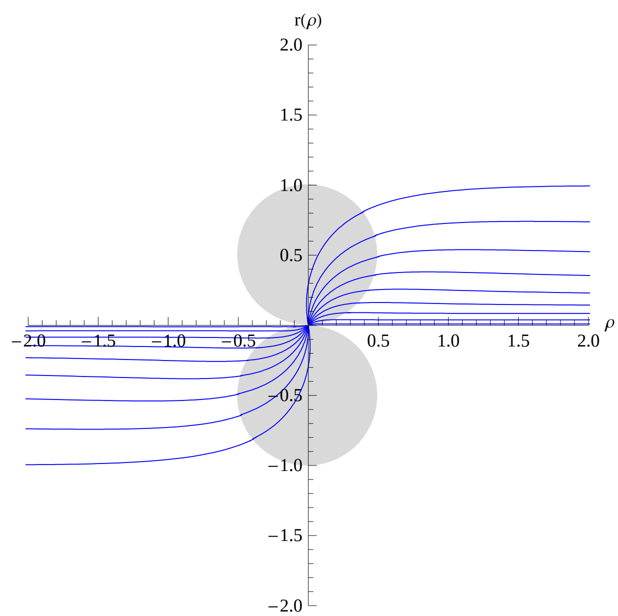


Figure 6.2: A selection of solution curves for D5 embeddings. The grey region is the interior of the ergosurface.

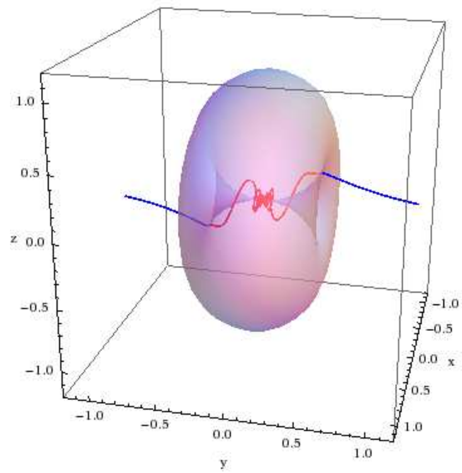


Figure 6.3: A particular solution curve in the three dimensional (w, θ, ϕ) subspace. The torus represents the ergosurface. Note the D5 rotates at speed μ in the ϕ direction (around the symmetry axis).

$$w^2\ \theta'^2 + \tan\theta\ w\ \theta' - 1 = 0 \tag{6.13}$$

There are thus two allowed gradients at the ergosurface. In fact numerically we find choosing any gradient focuses on to the same flow both within and outside the ergosurface. We can numerically shoot in and out from a point near the ergosurface in order to generate regular embeddings.

In the three-dimensional (w, θ, ϕ) subspace the ergosurface is the torus given by $L^2\mu\cos\theta = \pm w$, which in a plane of constant ϕ gives two adjacent circles of radius $\frac{\mu L^2}{2}$. Fig.(6.2) shows a sequence of regular solutions in the $(\rho, r(\rho))$ coordinates of the previous section. To obtain regular solutions one should make an odd continuation to the negative quadrant as shown. We show a full D5 embedding in Fig.(6.3) with both the $\theta(w)$ and $\phi(w)$ dependence plotted - note the D5 rotates at speed μ in the ϕ direction (around the axis of the torus).

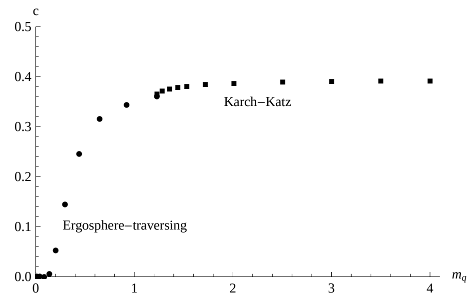


Figure 6.4: A plot of c vs m for embeddings smoothly penetrating the ergosurface (circles) and Karch-Katz embeddings (squares). The solutions again oscillate around the value for the lowest Karch-Katz solution as the D5 approaches the very top of the ergosurface indicating a first order transition.

Clearly there is no spontaneous symmetry breaking in these solutions - the solutions smoothly map onto the solution which lies along the axis as the mass parameter m is taken to zero. In the field theory presumably the conformal symmetry breaking parameter (μ) which might trigger symmetry breaking is the same parameter as that telling us there's a plasma density cutting off the theory - there's no room for dynamics. This model turns out not to be an example of the behaviour studied in [17].

The presence of a non-trivial profile $\phi(w)$ for the embeddings that penetrate the ergosurface indicates on the field theory side of the duality that there is a vev for the scalar field associated with the phase of the condensate c - this would be the Goldstone mode if there were spontaneous symmetry breaking. Note that the regular Karch Katz embeddings, away from the ergosurface, have $\phi(w)$ constant so there is no such vev.

Again we see there is a first order transition between the Karch-Katz type solutions and those that enter the ergosurface region. We plot the values of c vs m for these solutions in Fig.(6.4) - it shows the same spiral structure around the first order transition as we saw with the naive ansatz. We will discuss the meaning of this transition below in the thermal context.

6.4.3 Thermal behaviour

One can perform the same analysis in the thermal background. Writing $b^4 \equiv \frac{u_0^4}{4}$, there is again a torus-like ergosurface given by the equation

$$L^2\mu\cos\theta = \pm\frac{1}{w}\frac{w^4-b^4}{\sqrt{w^4+b^4}}\ , \tag{6.14}$$

and also a spherical horizon at $w = b$. One finds the horizon always lies within the ergosurface because the local speed of light is zero at the horizon. Note, below we find no phase transition when raising the temperature through the scale of the chemical potential. There would be a transition from a runaway Bose-Einstein condensation to a stable theory were we to allow the chemical potential to backreact on the geometry.

One can form the Legendre-transformed Lagrangian (which recovers the $T = 0$ case for $b = 0$)

$$\begin{aligned} \mathcal{L} \ &= \ \tfrac{1}{wc_\theta}\tfrac{w^4+b^4}{w^4-b^4}\sqrt{\tfrac{(w^4-b^4)^2}{w^4(w^4+b^4)}\left(1-L^4\mu^2c_\theta^2w^2\tfrac{(w^4+b^4)}{(w^4-b^4)^2)}\right)} \\ &\sqrt{(1+w^2\theta'^2)}\sqrt{\left(s_\theta^4c_\theta^2\tfrac{(w^4-b^4)^2}{w^2}-J^2\tfrac{w^4}{w^4+b^4}\right)}\ . \end{aligned} \tag{6.15}$$

The embeddings which extremize the action fall into two types - Karch-Katz type embeddings and those which hit the ergosurface. Fluctuations of the former would reveal a bound state spectrum. The latter embeddings inevitably fall onto the event horizon (a selection of these is plotted in Fig.(6.5) for $u_0 = 1$). In addition for these embeddings that pass through the ergosurface g_{tt} switches sign on the world volume - the ergosurface acts like a horizon for the world volume fields [108]. Here fluctuations would have a quasinormal spectrum along the lines of [51].

There is therefore a first order transition in the behaviour of the theory as the quasiparticle mass goes through the scale of the chemical potential or temperature. This transformation is explored in detail in [108]. Note here it seems the transition is always a meson melting transition at finite temperature. At zero temperature the transition is

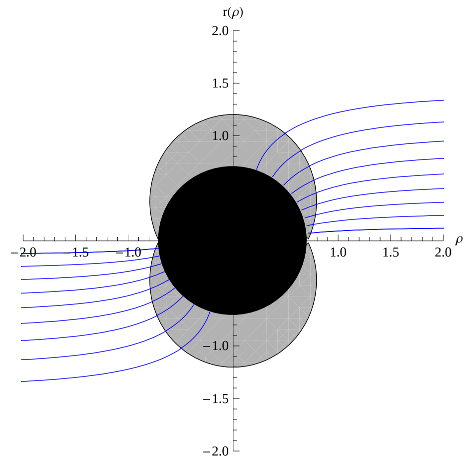


Figure 6.5: A selection of solution curves for D5 branes in the thermal geometry (with $u_0 = 1$). The grey region is the interior of the ergosurface and the black region is the interior of the event horizon.

driven by quantum rather than thermal fluctuations and has been described in terms of a metal-insulator transition in [110].

6.4.4 The D3-D7 System

Much of the above parallels results already found in the D3-D7 system [105, 108]. That system describes an $\mathcal{N} = 2$ 3+1d gauge theory with fundamental matter hypermultiplets in the gauge background of $\mathcal{N} = 4$ super Yang-Mills theory. In [105] an analysis similar to our "naive ansatz" was performed suggesting spontaneous symmetry breaking. Those authors have since refined their analysis in a related system with a background electric field [106] and concluded that if regular embeddings are insisted upon the symmetry breaking is not present (see also [107]). Were they to update [105] they would find embeddings analogous to our D5 embeddings above as they indicate in [108].

6.5 Baryon number chemical potential

Another, and perhaps the most likely, way in which to embed the U(1) symmetry of QED into the brane set up is through the quasi-particle number global symmetry (essentially baryon number). The conserved vector current and its source, which is effectively a background gauge field configuration for this symmetry, manifest holographically as the U(1) gauge symmetry living on the world volume of the D5 brane. We can introduce a chemical potential for baryon number by switching on a constant A_t component for this U(1) gauge field. We will study the embeddings of such a configuration at zero and non-zero temperature. Much of this analysis again mirrors that for the D3/D7 system which can be found in [109].

6.5.1 Zero temperature

The DBI action for the D5 brane including the surface gauge field is

$$S \sim T_5 \int d^6\xi \sqrt{\det(P[G_{ab}] + 2\pi\alpha' F_{ab})} \, . \tag{6.16}$$

We consider embeddings of the D5 brane in the $\rho - r$ plane with in addition $2\pi\alpha' A_0(\rho) = A(\rho)$ to represent the chemical potential. The action is then of the form

$$\mathcal{L} \sim \rho^2 \sqrt{1 + r'^2 - A'^2} \, . \tag{6.17}$$

Since the action is independent of A the equation of motion for A implies $\frac{\partial \mathcal{L}}{\partial A'}$ is a constant, Q. We find

$$A'^2 = Q^2 \frac{1 + r'^2}{\rho^4 + Q^2} \, . \tag{6.18}$$

It is useful to perform a Legendre transform again ($\mathcal{L}' = \mathcal{L} - A' \frac{\partial \mathcal{L}}{\partial A'}$) and work with

$$\mathcal{L}' = \sqrt{(1 + r'^2)(\rho^4 + Q^2)} \, . \tag{6.19}$$

The r-independence again gives a simple equation for the embedding that is

$$r' = \frac{c_1}{\sqrt{\rho^4 + Q^2 - c_1^2}} \; , \tag{6.20}$$

with c_1 a constant.

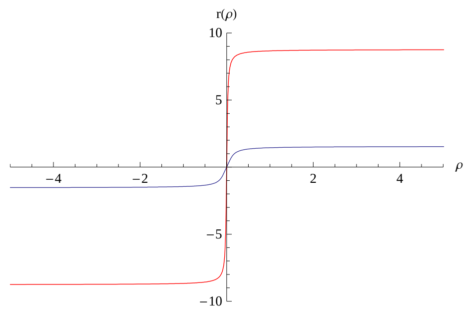


Figure 6.6: Embeddings with a baryon number chemical potential for $Q = 0.1$ at $T = 0$ with with nonzero c_1. As c_1 approaches the numerical value of the charge Q the quark mass can be made arbitrarily large.

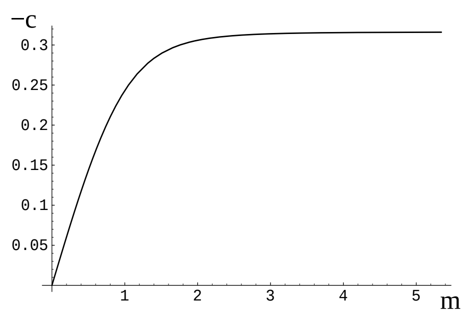


Figure 6.7: Plot of parameters c, proportional to the condensate, versus m proportional to the quark mass in the case of baryon chemical potential for $Q = 0.1$.

To interpret this equation it is helpful to initially turn off the chemical potential, $Q = 0$. It is then clear that the solutions are singular at $\rho = \sqrt{c_1}$ and the only regular case is $c_1 = 0$ so that $r' = 0$ - we recover the usual flat embeddings.

When we allow non-zero Q there become a bigger set of regular solutions - those with $c_1 \leq Q$. These solutions are plotted in Fig.(6.6) for varying c_1 and provide an alternative embedding, that crosses through the origin, for each value of m in the large ρ asymptotics of the embedding. In [109] it was argued (in the D3-D7 case) that these are the true embeddings when there is a surface gauge field on the brane. We can see that the flat embeddings ($c_1 = 0$) are not regular from (6.18)- they have a none zero gradient A' at $\rho = 0$ so there will be a kink in the A field as it crosses over the r axis. For the solutions that pass through the origin though the A field is regular. In [111] it has been argued that the irregular solutions should be maintained with the irregularity interpreted as the presence of an external source.

As can be seen from Fig.(6.6), whichever interpretation is taken, the embedding for the case of massless quarks is unchanged from the usual flat embedding. The embedding does not therefore spontaneously break any symmetry with the introduction of a baryon chemical potential. Away from $m = 0$ for the case of the regular embeddings the embeddings do change and there is a condensate present. From (6.20) we can see that up to a sign c_1 is just the asymptotic parameter c that determines the condensate. For small c_1 the quark mass grows linearly but as c_1 approaches Q m rises sharply - the resulting plot of c vs m therefore shows that the condensate asymptotes to a constant value for large mass - see Fig.(6.7).

Another possible source of spontaneous breaking would be if the gauge field vev on the D5 led to other fields in the D5 brane world volume condensing. In fact though all the fields on the D5 are in the adjoint representation of, generically, a U(N_f) flavour symmetry. Adjoint fields of the U(1) of baryon number are chargeless and hence have no interaction with the gauge field. There is no possibility for such condensation and the system is not superconducting.

6.5.2 Finite temperature

We can also study the theory with baryon number chemical potential at finite temperature by finding D5 embeddings with a non-zero surface A_t gauge field in the black hole geometry (6.6). We again set $b^4 \equiv \frac{u_0^4}{4}$.

The DBI Lagrangian for such an embedding is

$$\mathcal{L} = \rho^2 \frac{w^4 + b^4}{w^2} \sqrt{\frac{1}{w^4} \frac{(w^4 - b^4)^2}{(w^4 + b^4)} (1 + r'^2) - A'^2} \, . \tag{6.21}$$

This time we have the gauge field

$$A'^2 = \frac{Q^2 \frac{1}{w^4} \left(\frac{(w^4 - b^4)^2}{(w^4 + b^4)} \right) (1 + r'^2)}{\rho^4 \left(\frac{w^4 + b^4}{w^4} \right)^2 + Q^2} \, . \tag{6.22}$$

The Legendre-transformed version of the Lagrangian is

$$\mathcal{L}' = \sqrt{\frac{1}{w^4} \frac{(w^4 - b^4)^2}{w^4 + b^4} (1 + r'^2) \left(\rho^4 \left(\frac{w^4 + b^4}{w^4} \right)^2 + Q^2 \right)} \, . \tag{6.23}$$

One can shoot out from the horizon attempting to fill out the m parameter space asymptotically. The solutions are very similar to those in the D3-D7 case as outlined in [109] and see also [110]. There is no spontaneous symmetry breaking. There is a first order phase transition between the large m embeddings, that are essentially flat except for a spike down onto the black hole, and smoother embeddings that fall into the black hole at lower m (some embeddings are shown in Fig.(6.8)). This transition persists until the chemical potential becomes large, where there is no transition and the two phases coexist. In Fig.(6.9) we plot the quark condensate, c versus quark mass, m, for varying values of Q (which determines the chemical potential) around the critical value of Q where the phase transition between "spike" embeddings and smooth horizon entering embeddings ends. The disappearance of the phase transition is evident - the physics closely resembles that in the D3-D7 case discussed in detail in [109].

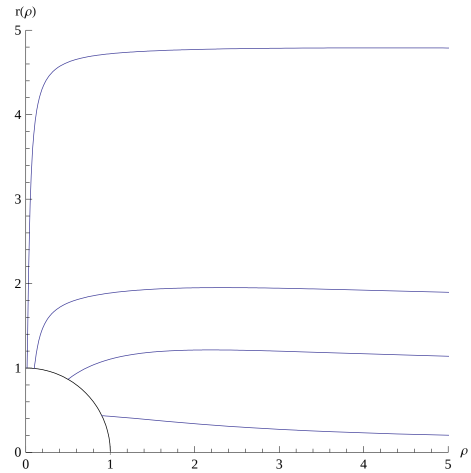


Figure 6.8: Embeddings of the D5 branes for $Q = 0.1$ baryon chemical potential in units of the black hole temperature.

6.6 Isospin chemical potential

The final possible source of a chemical potential in the D3-D5 set up is from the isospin symmetry present when there are two or more flavours of quasi-particle (D5) present. In contrast to the discussion of baryon number above, there are clearly operators which carry isospin charge eg. $\langle\bar{\psi}\gamma_0\tau_3\psi\rangle$. These can be expected to condense at zero isospin chemical potential and break the symmetry spontaneously in the spirit of the zero temperature work in [114] and the phenomenological holographic model of p-wave condensation in [99, 97]. Here we will work first at finite temperature and only consider the case of zero quark mass - the embeddings of the branes are identical to those above for baryon number, where one uses the modulus of the isospin as the chemical potential, so at zero quark mass the D5s lie straight along the axis as shown in Figure 6. The mesons of the theory are therefore melted by the thermal bath but the operators can nevertheless condense. One is perhaps making a departure from any obvious connection to a solid state system at this point since one would require a system with a U(2) or greater flavour symmetry on the quasi-particles - presumably there is only one sort of

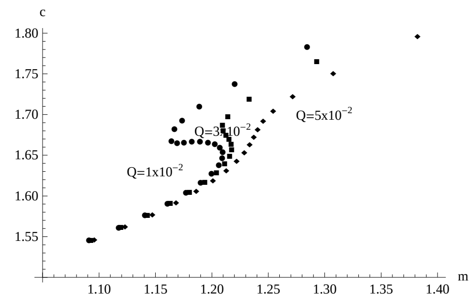


Figure 6.9: Plot of the quark condensate, c versus quark mass, m, for varying values of Q (which determines the baryon number chemical potential) around the critical value of Q where the phase transition between "spike" embeddings (the top of the s-shape) and smooth horizon entering embeddings ends (lower part of the s-shape).

electron in a solid state system!

The full theory of flavours on the D5 brane is expected to be unstable in the presence of a chemical potential at zero temperature. The situation is analogous to the D3-D7 system discussed in [115] - the squarks have a moduli space in the gauge theory at zero temperature and chemcial potential which shows up on the gravity side as a moduli space for the size of instanton configurations on the D7 (here D5s) world-volume [116]. An isospin chemical potential will induce a negative mass squared for the scalars forcing the vev or instanton size to infinity. We will simply neglect this runaway behaviour here, fix the scalar vevs to zero and study the fermionic operators of the theory - hopefully this tells us about the behaviour of a theory with fermions but no scalars.

The full DBI action to all orders in the surface gauge field is not fully known - an attempt to use the full DBI on a similar problem in the D3-D7 set up has recently appeared [112]. We though will follow the path of the D3-D7 analysis in [113] and just use the first order expansion of the action

$$S \sim T_5 \int d^6\xi \sqrt{-\text{det}G} \left(1 - \frac{1}{4} Tr\left(F^2\right)\right) \ . \tag{6.24}$$

We expect this action to represent the dynamics well.

We will write the ansatz for the gauge fields as

$$A = \Phi(\rho)\tau^3 dt + w(\rho)\tau^1 dx_1 \ , \tag{6.25}$$

as in [99]. The coordinate representation of the Yang-Mills equation is

$$\partial_\rho \left(\sqrt{-g} F^{\rho\nu i}\right) = -g_{YM} \left(\delta^{im}\delta^{jl} - \delta^{il}\delta^{jm}\right) A^j_\mu A^{\mu l} A^{\nu m} \ . \tag{6.26}$$

For our ansatz there are two equations of motion

$$\left(\sqrt{-g} g^{00} g^{rr} \Phi'\right)' \;=\; g_{YM}\sqrt{-g} g^{00} g^{11} w^2 \Phi \tag{6.27}$$

$$\left(\sqrt{-g} g^{11} g^{rr} w'\right)' \;=\; g_{YM}\sqrt{-g} g^{00} g^{11} \Phi^2 w \ . \tag{6.28}$$

Restricting ourselves to a massless quark D5 brane so the induced brane metric is given by isotropic AdS-Schwarzschild, the equations are (having absorbed factors of g_{YM} and L into the definitions of the fields)

$$\left(\frac{(r^4+1)^{\frac{3}{2}}}{r^4-1}\Phi'\right)' \;=\; +\frac{\sqrt{r^4+1}}{r^4-1} w^2 \Phi \tag{6.29}$$

$$\left(\frac{r^4-1}{\sqrt{r^4+1}} w'\right)' \;=\; -\frac{\sqrt{r^4+1}}{r^4-1} \Phi^2 w \ . \tag{6.30}$$

In the isotropic coordinates one should shoot out from a small displacement x from the horizon using the initial condition $w = w_0$ and $\Phi = \Phi_2 x^2$ for constants w_0 and Φ_2. Near the boundary of AdS the solutions behave as

$$\Phi \;\sim\; \mu - \frac{\rho}{r} \tag{6.31}$$

$$w \;\sim\; \mu' + \frac{c}{r} \ . \tag{6.32}$$

We search for solutions which have $\mu' = 0$ because these are solutions which are normalizable and hence describe a condensate of the charged mesonic operator. Requiring this to be the case one can solve the coupled nonlinear equations to yield multiple branches of solutions. The lowest, monotonic branch is presumably the stable solution and for these solutions one can plot the dependence of the condensate on the chemical potential. Since the quarks we put in were massless, the only two scales are the chemical potential and the temperature and so large chemical potential can be equivalently viewed as low-temperature.

The results are plotted in Fig.(6.10) - for $\mu \ll T$ there is no condensation. For large μ though there is a second order phase transition to a phase with a charged vector condensate. The behaviour is similar to that previously observed in [99, 97, 113, 112]. As μ/T goes to infinity the condensate c tends to a finite constant (≈ 0.3) when measured in units of μ^2, which is similar to the behaviour found in [113] for a pair of D7 probe branes - in their 3+1 dimensional theory the condensate tends to a constant in units of μ^3.

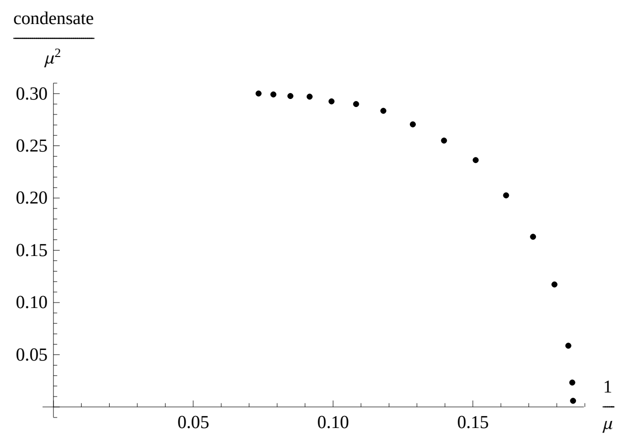


Figure 6.10: Plot of the charged vector condensate (the parameter c from (6.31)) in units of chemical potential squared versus $1/\mu$ in the case of isospin chemical potential at zero quark mass.

6.7 Summary

We have proposed probe D5 branes in D3 brane backgrounds as a plausible dual for a strongly coupled quasi-particle theory in 2+1 dimensions - at zero temperature and chemical potential the theory is supersymmetric and conformal. We introduced a chemical potential with respect to the global U(1) symmetries associated with R-charge and baryon number and found the resulting regular D5 embeddings. These embeddings do not display spontaneous symmetry breaking and, indeed, at zero temperature and zero intrinsic mass the theory is essentially indifferent to the chemical potential remaining as a state of conformal quasi-particles.

For the R-charge case we show a first order phase transition in the massive theory as the quasi-particle mass crosses the value of the chemical potential - on one side the quasi-particles are confined on the other they are not. At finite temperature the transition is between solutions that fall into the black hole and those that don't.

At finite temperature in the baryon number case there is also a phase transition between two different black hole embeddings which is the equivalent of the meson melting transition at finite temperature but zero chemical potential. If the chemical potential becomes too large then the transition ceases to occur for any quark mass.

Finally we introduced isospin chemical potential in the case of two probe D5 branes and reproduced the second order transition to a phase with a charged vector condensate previously seen in other systems in [99, 97, 113, 112]

We hope that these explorations will form a useful platform from which to find a holographic model of some real solid state system. We note that many of the transport properties of this system have also been recently explored in [117].

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Appendix A

A rotating string in flat spacetime

Consider the static-gauge ansatz

$$X^0 = l\,\tau \tag{A.1}$$

$$X^1 = l\cos\sigma\ \cos\tau \tag{A.2}$$

$$X^2 = l\cos\sigma\ \sin\tau \tag{A.3}$$

$$X^3 = 0\ . \tag{A.4}$$

We will fix the gauge by demanding the independent world-sheet metric to be flat two-dimensional Minkowski space-time. In order to show the above ansatz extremizes the Polyakov action note that the coordinates satisfy the Klein-Gordon equation

$$\nabla^2 X^\mu \equiv \left(-\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\sigma^2}\right) X^\mu = 0\ . \tag{A.5}$$

The condition $\frac{\partial X^\mu}{\partial\sigma} = 0$ is satisfied at the endpoints $\sigma = 0, \pi$ and the σ coordinate covers the string once.

The constraint equation can be verified by noting that the induced metric on the worldsheet, $h_{ab} \equiv \partial_a X^\mu \partial_b X_\mu$ is given by

$$h_{ab} = l^2 \sin^2 \sigma \begin{pmatrix} -1 & \cdot \\ \cdot & 1 \end{pmatrix} . \tag{A.6}$$

Noting that the independent worldsheet metric is just

$$\gamma_{ab} = \begin{pmatrix} -1 & \cdot \\ \cdot & 1 \end{pmatrix} \tag{A.7}$$

allows one to quickly verify that the equation $h_{ab} - \frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd} = 0$ is satisfied.

The energy of the string can be calculated as $\int d\sigma \mathcal{P}_0 = \frac{l}{2\alpha}$ which has the interpretation of length times tension. The angular momentum evaluates to $J_3 = \int d\sigma\,(X_1\mathcal{P}_2 - X_2\mathcal{P}_1) = \frac{l^2}{4\alpha'^2}$ giving the relation $J_3 = \alpha' E^2$. The parameter α' is therefore referred to as the Regge slope.

The solution has of course all the features of Lorentz invariance (for example if boosted in the X^3 direction the speed of spinning slows down since end points must move at light speed!).

Appendix B

Back-reaction in the hydrogenic atom

One can wonder about the effect of the charge of the electron in the hydrogen atom solution - if treated as a classical wave the electron ought to feel its own field, which would be expected to result in a very loosely-bound electron (much less well-bound than in the usual Schrödinger model). In a classical treatment the field equations are to be derived from the Lagrangian (for simplicity our electron is a charged scalar here)

$$\mathcal{L} = \left(\tilde{\nabla}_a \phi\right)^* \tilde{\nabla}^a \phi + m^2 \phi^* \phi - \frac{1}{4} F^2 \ . \tag{B.1}$$

The derivative operator in the above equation is the gauge covariant derivative ($\tilde{\nabla}_a \equiv \nabla_a - ieA_a$). The equation of motion for the scalar is

$$\nabla_a \left(\nabla^a - ieA^a\right) \phi = m^2 \phi + ieA_a \left(\nabla^a - ieA^a\right) \phi \ . \tag{B.2}$$

If we allow for back reaction we must compensate the gauge potential for the presence of the oppositely charged scalar field. For a spherically-symmetric state and harmonic time dependence, $\phi(r)e^{-i\omega t}$ one has

$$\nabla^2 A_0 = -\frac{Ze}{\epsilon_0 c}\left(\delta(\underline{x}) - \frac{1}{Z}\phi^2\right) . \tag{B.3}$$

One solves the problem perturbatively by means of the expansion in $\frac{1}{Z}$ (this will be good for highly-charged nuclei)

$$\begin{aligned} \phi &\rightarrow \phi + \frac{1}{Z}\tilde{\phi} && \text{(B.4)} \\ \omega &\rightarrow \omega + \frac{1}{Z}\tilde{\omega} && \text{(B.5)} \\ \frac{e}{\hbar}A_0 &\rightarrow -\frac{\Upsilon}{r} + \frac{1}{Z}\tilde{A}_0 . && \text{(B.6)} \end{aligned}$$

Here $\Upsilon \equiv Z\alpha$ is a parameter we keep fixed. Perturbation theory gives us a consistent way to implement the computation of finding how the electron field behaves in a given potential, then how that electron field changes the potential, then how that change to the potential affects the electron field and so forth. Inserting into the Maxwell equation one has

$$\nabla^2\left(-\frac{\Upsilon}{r} + \frac{1}{Z}\tilde{A}_0\right) = -4\pi\Upsilon\left(\delta(\underline{x}) - \frac{1}{Z}\phi^2\right) . \tag{B.7}$$

Since $\nabla^2\left(\frac{1}{r}\right) = 4\pi\delta(\underline{x})$ the zero-order equation is clearly satisfied. At the next order in $\frac{1}{Z}$ one has

$$\nabla^2\tilde{A}_0 = 4\pi\Upsilon\phi^2 . \tag{B.8}$$

The physical interpretation of this is that is a correction to the electrostatic potential sourced by the zero-order electron field ϕ. Since the zero-order wave function is known this can be solved by the method of Green's functions to obtain $\tilde{A}_0$ (particularly easy as here spherically-symmetric and Gauss' law can be applied).

Putting the perturbative expansion into the Klein-Gordon equation one obtains at zero-order

$$\omega^2\phi + \frac{1}{r^2}(r^2\phi')' = m^2\phi - \frac{2\Upsilon}{r}\omega\phi - \frac{\Upsilon^2}{r^2}\phi \, , \tag{B.9}$$

and at first order

$$\omega^2\tilde{\phi} + \frac{1}{r^2}(r^2\tilde{\phi}')' - m^2\tilde{\phi} + \frac{2\Upsilon}{r}\omega\tilde{\phi} + \frac{\Upsilon^2}{r^2}\tilde{\phi} = -2\left(\omega + \frac{2\Upsilon}{r}\right)\left(\tilde{\omega} - \tilde{A}_0\right)\phi \, . \tag{B.10}$$

Note setting $\tilde{\phi}$ to the original ϕ makes the left hand side vanish (in that case the tilde quantities on the right hand side would make that side vanish also).

At least in the limit of a non-relativistic electron moving in the background field of a much heavier particle these classical backreaction effects actually do not matter since the effective description is a particle moving in the background Coulomb potential with no backreaction. The argument that the electron's own field is as strong as the proton in hydrogen is irrelevant since the true physics is whatever limit of QED one is in, and not classical electrodynamics. The removal of backreaction in atoms is therefore an extremely important *quantum* effect!

Appendix C

Scalar quasinormal modes of a Schwarzschild black hole

The dumbest 'quantum gravity' problem imaginable is to consider a Schrödinger atom with the $\frac{1}{r^2}$ electrostatic force replaced by the Newtonian $\frac{1}{r^2}$ gravitational force. By solving the Klein-Gordon equation on the Schwarzschild background, it is possible to address the relativistic version of this problem.

Start with the Schwarzschild metric

$$ds^2 = -f(r)c^2dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ , \tag{C.1}$$

where $f(r) \equiv 1 - \frac{2GM}{c^2r}$.

Now doing the transformation $cdt = cd\tau - \beta(r)dr$ and requiring the $d\tau = 0$ spacelike sections look like flat spacetime (thus giving $\beta = \sqrt{\frac{2GM}{c^2r}}\frac{1}{1-\frac{2GM}{c^2r}}$) one obtains the metric in ingoing Painleve-Gullstrand coordinates [52] (τ is the time as measured by an infalling massive observer starting at rest at infinity)

$$ds^2 = -fc^2 d\tau^2 + \sqrt{\frac{8GM}{c^2 r}} c\,d\tau\,dr + dr^2 + r^2 d\Omega_2^2 \;. \tag{C.2}$$

The equation we consider is $\left(\nabla^2 - \frac{m^2c^2}{\hbar^2}\right)\phi = 0$ which has the coordinate representation

$$\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}g^{ab}\partial_b\phi\right) - \frac{m^2c^2}{\hbar^2}\phi = 0 \;. \tag{C.3}$$

For time dependence $\phi \propto e^{-i\omega\tau}$ and assuming a spherically-symmetric state (it is easy to put in nonzero angular momentum l, m) one obtains

$$\left(r^2 f\phi'\right)' - i\frac{\omega}{c}\sqrt{\frac{2GM}{c^2}}\left(2r^{\frac{3}{2}}\phi' + \frac{3}{2}\sqrt{r}\phi\right) + r^2\left(\frac{\omega^2}{c^2} - \frac{m^2c^2}{\hbar^2}\right)\phi = 0 \;. \tag{C.4}$$

We rescale to $x \equiv \frac{r}{r_0}$ with $r_0 = \frac{2GM}{c^2}$ and set $\omega = \frac{mc^2}{\hbar}\omega_0$ so ω_0 is a dimensionless frequency. Then defining the dimensionless gravitational coupling $\alpha_G \equiv \frac{GMm}{\hbar c} \equiv \frac{Mm}{M_{Pl}^2}$ one obtains the equation

$$\left(x^2 f\phi'\right)' - 2i\omega_0\,\alpha_G\left(2x^{\frac{3}{2}}\phi' + \frac{3}{2}\sqrt{x}\phi\right) + 4\alpha_G^2 x^2\left(\omega_0^2 - 1\right)\phi = 0 \;. \tag{C.5}$$

As usual when using the ingoing coordinates, ingoing modes at the horizon are described by a regular Taylor series which we use as initial condition and shoot out from the horizon. At large x the state is a spherical wave modulated by either a growing exponential (non normalizable) or, for a discrete set of complex frequencies, a decaying exponential. The latter are the 'bound states'. The tightest-bound states are shown in Fig.(C.1).

It is easy to obtain a picture of the Schrödinger potential for the black hole bound states problem. In the usual Schwarzschild coordinates the Klein-Gordon equation takes the form

$$\left(x(x-1)\phi'\right)' = 4\alpha_G^2 x^2\left(1 - \frac{\omega_0^2 x}{x-1}\right)\phi \;. \tag{C.6}$$

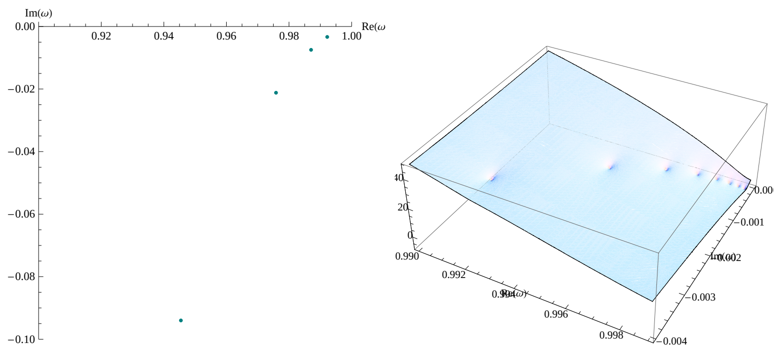

Figure C.1: Left hand figure shows the lowest four quasinormal frequencies in the complex ω plane for $\alpha = 0.5$, converging on $\frac{mc^2}{\hbar}$ (= 1 in the units used). On the right is a scan of the ω plane where the logarithm of the field at large radius is plotted - the 'pits' show normalizable solutions where the field vanishes at infinity.

To bring this into the form of a canonical Schrödinger equation one should change variables such that $\frac{dy}{dx} = \frac{2\alpha_G x}{x-1}$. This is satisfied by $\frac{y}{2\alpha_G} = x + \ln(x-1)$ which is the 'Regge-Wheeler tortoise coordinate'. This can be inverted to give $x = 1 + PL\left(e^{\frac{y}{2\alpha_G}-1}\right)$ where PL is the product-log function.

One obtains the following equation in the transformed variable

$$\frac{d^2\phi}{dy^2} + \frac{1}{\alpha_G}\frac{x-1}{x^2}\frac{d\phi}{dy} = \left(\frac{x}{x-1} - \omega_0^2\right)\phi\,. \tag{C.7}$$

To complete the transformation we put $\phi = uv$ and set the term $\propto u'$ zero. This determines $v = \frac{1}{x}$ and the Schrödinger potential is

$$V(x) = 1 - \frac{1}{x} + \frac{1}{4\alpha_G^2}\frac{x-1}{x^4}\,. \tag{C.8}$$

The variable y is on the interval $(-\infty, \infty)$ where the negative infinity corresponds to the black hole horizon and the positive infinity to the asymptotically-flat region. In Fig.(C.2) the potential is shown for $\alpha_G = 0.1$ (top curve) to 0.35 (bottom curve) in the

Schrödinger variable y. The 'bound states' sit in the dip but decay by tunnelling into the horizon which is located at $y = -\infty$.

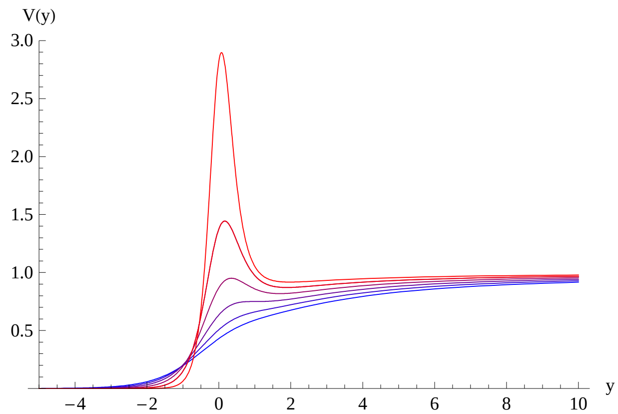

Figure C.2: Schrödinger potentials for the scalar in a Schwarzschild black hole background.

The problem we have looked at here does actually have a known solution - in the Schwarzschild coordinates the equation obtained is technically a singly-confluent Heun equation. In (C.6) put $\phi = (x-1)^q e^{-kx}\zeta$ and choose $k^2 = 4\alpha_G^2(1-\omega_0^2)$ and $q = -2i\alpha_G\omega_0$ to reduce to the canonical form of the singly-confluent Heun equation (two regular singular points and one irregular)

$$\zeta'' + \left(-2k + \frac{1}{x} + \frac{2q+1}{x-1}\right)\zeta' + \left(-\frac{k+q}{x} + \frac{q - k(2q+1) - 4\alpha_G^2 + 8\alpha_G^2\omega_0^2}{x-1}\right)\zeta = 0 \quad \text{(C.9)}$$

which should be compared to the canonical form for the singly-confluent Heun equation

$$H'' + \left(\alpha + \frac{\beta+1}{x} + \frac{\gamma+1}{x-1}\right)H' + \left(\frac{\mu}{x} + \frac{\nu}{x-1}\right)H = 0\,. \quad \text{(C.10)}$$

There does not seem to be a closed form solution for the eigenvalues though one can obtain a solution via the continued-fraction method. There is a closed form solution for the eigenvalues of the confluent hypergeometric equation (which has one fewer regular singular point) which gives for the energy levels of a charged scalar in a Coulomb background, with $p(p-1)+Z^2\alpha^2=0$ and $n=0,1,2,...$

$$\hbar\omega = \frac{mc^2}{\sqrt{1+\frac{p^2\alpha^2}{(p+n)^2}}}\,. \tag{C.11}$$

Appendix D

Dilaton flows in asymptotically-flat geometry

The Schwarzschild solution in four dimensions is very well-known. In fact there is a very simple generalization to Einstein gravity coupled to a massless scalar field. Let us seek static, spherically-symmetric solutions to the Einstein and Klein-Gordon equations in four dimensions with zero cosmological term

$$
\begin{aligned}
R_{ab} &= l_P^2 \left(\nabla_a\phi\nabla_b\phi + m^2 g_{ab}\phi^2\right) &\quad \text{(D.1)}\\
\nabla^2\phi &= m^2\phi\,. &\quad \text{(D.2)}
\end{aligned}
$$

Using the ansatz $ds^2 = -f dt^2 + \frac{1}{f}dr^2 + R^2 d\Omega_2^2$ one has, setting the scalar field mass to zero

$$\begin{aligned}
\frac{1}{2}f\left(\frac{2f'R'}{R}+f''\right) &= 0 && \text{(D.3)}\\
-\frac{2f'R'+Rf''+4fR''}{2fR} &= l_P^2\,\phi'^2 && \text{(D.4)}\\
1-fR'^2-R(f'R'+fR'') &= 0 && \text{(D.5)}\\
\left(R^2f\phi'\right)' &= 0\,. && \text{(D.6)}
\end{aligned}$$

A simple solution can be found for $f=1$ which is just

$$\begin{aligned}
R &= \sqrt{(r-r_+)(r-r_-)} && \text{(D.7)}\\
\phi &= \phi_0+\frac{1}{\sqrt{2}l_P}\ln\left(\frac{r-r_+}{r-r_-}\right)\,. && \text{(D.8)}
\end{aligned}$$

This is a solution with no mass, but non-zero scalar charge (for $r_+\neq r_-$). If instead we keep f general it is possible to obtain the equation (with δ an arbitrary constant and l_P set to unity)

$$2f''+4f'-f\left(\frac{f''}{f'}\right)^2=-2\delta^2\frac{f'^2}{f}\,. \qquad \text{(D.9)}$$

This looks nasty but can be easily solved with $f\equiv e^y$ to yield (using dimensional analysis to restore the Planck length)

$$\begin{aligned}
f &= \left(1-\frac{2l_P^2M\Delta}{r}\right)^{\frac{1}{\Delta}} && \text{(D.10)}\\
R &= r\left(1-\frac{2l_P^2M\Delta}{r}\right)^{\frac{\Delta-1}{2\Delta}} && \text{(D.11)}\\
\phi &= \phi_0+\frac{\delta}{\Delta\,l_P}\ln\left(1-\frac{2l_P^2M\Delta}{r}\right)\,. && \text{(D.12)}
\end{aligned}$$

Here $\Delta \equiv \sqrt{1+2\delta^2}$. This is clearly the generalization of the Schwarzschild solution ($\delta = 0$) to include a massless scalar - the (Komar) mass is M and the scalar charge $q = 2l_P M\delta$ (in fact the above $f = 1$ solution can be recovered by setting $M\Delta$ to a constant and taking $\Delta \to \infty$, equivalent to maintaining constant scalar charge while letting the mass go to zero). If there is a nontrivial scalar present ($\delta \neq 0$), instead of a horizon of finite area it has a true singularity of apparently zero metric area - this is the same sort of thing we found in AdS in the main text. This solution was actually found long ago [75].

Appendix E

Bottom-up models - cooking the dilaton

It's not difficult to cook up models which return whatever spectrum you like - these can then be interpreted as the glueballs or mesons of the theory. For example, suppose we wanted to put in a dilaton giving a linear Regge trajectory with a massless mode at the bottom (inviting the interpretation as a trajectory with the pion at the bottom and then a tower of excited states).

We take the Klein-Gordon equation with background dilaton Φ to be

$$\partial_a \left(\sqrt{-g} g^{ab} e^{\Phi} \partial_b \phi \right) = 0 \,. \tag{E.1}$$

In AdS_5 the metric is

$$ds^2 = \frac{r^2}{L^2} dx_4^2 + L^2 \frac{dr^2}{r^2} \,. \tag{E.2}$$

Transforming to $\Omega^2 = \omega^2 L^4$ one has the Klein-Gordon equation with background dilaton on this background

$$\phi'' + (5 \ln r + \Phi)' \phi' + \frac{\Omega^2}{r^4} \phi = 0 \; . \tag{E.3}$$

Changing variables to $x = \frac{1}{r}$ one has

$$\phi'' + \left(-\frac{3}{x} + \Phi' \right) \phi' + \Omega^2 \phi = 0 \; . \tag{E.4}$$

Now setting $\phi = uv$ with $v = x^{\frac{3}{2}} e^{-\frac{\Phi}{2}}$ one obtains the Schrödinger potential for the problem

$$V_S = \frac{15}{4x^2} + \frac{1}{2}\Phi'' - \frac{3}{2x}\Phi' + \frac{1}{4}\Phi'^2 \; . \tag{E.5}$$

One the half-space $x \geq 0$ we want a quadratic potential with a constant subtracted (assume if we're on the half-space one loses the even modes... odd ones have a node at end of the half-space) so it has a massless lowest mode (pion...) so $V_S = \frac{1}{2}\omega^2 x^2 - \omega$. So we must solve

$$\Phi'' - \frac{3}{x}\Phi' + \frac{1}{2}\Phi'^2 = 2 \left(-\frac{15}{4x^2} + \frac{1}{2}\omega^2 x^2 - \omega \right) \; . \tag{E.6}$$

We can reduce to a linear equation using $\Phi = 2 \ln y$ yielding

$$y'' - \frac{3}{x} y' = \left(-\frac{15}{4x^2} + \frac{1}{2}\omega^2 x^2 - \omega \right) y \; . \tag{E.7}$$

Now put $y = x^p e^q f$ where $p^2 - 4p + \frac{15}{4} = 0$ and $q = -\frac{\omega}{\sqrt{8}} x^2$. One obtains

$$f'' + \left(-\sqrt{2}\omega x + \frac{2p-3}{x} \right) f' + \left(q'' + \frac{2p-3}{x} q' + \omega \right) f = 0 \; . \tag{E.8}$$

One of the roots of the quadratic for p is $p = \frac{3}{2}$ - it seems obvious to choose this root. Then changing variable to $u = \frac{\omega}{\sqrt{2}}x^2$ the equation is

$$u\frac{d^2f}{du^2} + \left(\frac{1}{2} - u\right)\frac{df}{du} - \frac{1-\sqrt{2}}{4}f = 0 \; . \tag{E.9}$$

This is the confluent hypergeometric equation $zf'' + (c-z)y' - ay = 0$ and one solution is ${}_1F_1\left(\frac{1-\sqrt{2}}{4}, \frac{1}{2}; \frac{\omega}{\sqrt{2}}x^2\right)$. So the dilaton needed is

$$\Phi = 3\ln x - \frac{\omega}{\sqrt{2}}x^2 + 2\ln\left({}_1F_1\left(\frac{1-\sqrt{2}}{4}, \frac{1}{2}; \frac{\omega}{\sqrt{2}}x^2\right)\right) \; , \tag{E.10}$$

or in terms of the original AdS r-coordinate (plotted in Fig.(E.1)),

$$\Phi = -3\ln r - \frac{\omega}{\sqrt{2}}\frac{1}{r^2} + 2\ln\left({}_1F_1\left(\frac{1-\sqrt{2}}{4}, \frac{1}{2}; \frac{\omega}{\sqrt{2}}\frac{1}{r^2}\right)\right) \; . \tag{E.11}$$

Some far more sophisticated versions of this approach have been used in order to construct models of the dual to real QCD in a bottom-up fashion [90]. These models fall short of the brane models in that there is no understanding of what the UV degrees of freedom are and also that they are not actually solutions to the supergravity equations. There is a ‘hand of god’ keeping the background dilaton profile and geometry stable.

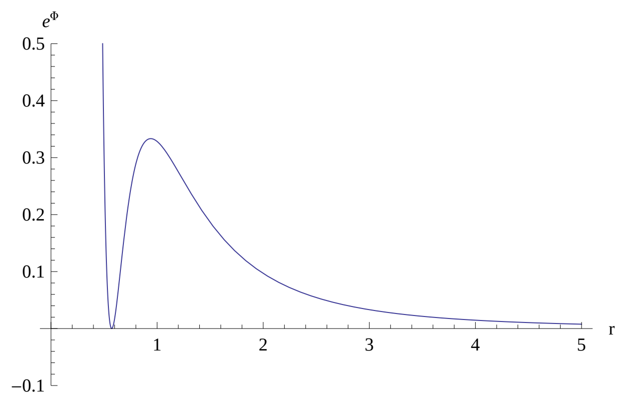

Figure E.1: Plot of background dilaton required to give a linearly-spaced tower of glueball states in AdS with the lowest state being massless. Here the spacing parameter ω has been set to unity.

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