Analysis of single-mode fused tapered fibre couplers

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Indexing terms: *Optical fibres, Couplers*

Abstract: A fabrication technique for fused taper couplers is described. Coupling coefficients are calculated for fibres with raised, depressed and matched refractive-index profiles, and optimum V-values for coupling are found. These suggest a shorter coupler geometry differing from the biconical taper. The high taper loss in depressed cladding fibres is shown to correspond to the presence of the leaky LP_{01}-mode. The wavelength response has been measured for various couplers and fitted theoretically. Coupling periods as small as 12 nm demonstrate potential use for couplers as wavelength-division multiplexers. The dependence of coupling on the refractive index of the surrounding medium has been measured. Thus couplers have also demonstrated potential as temperature sensors and optical switches.

1 Introduction

In many single-mode fibre sensors, in particular those based on fibre interferometers, a four-port directional coupler is needed to couple light from one fibre to another. Although it is possible to use discrete components for this purpose, a more stable system with a lower loss can be obtained if a fibre coupler is used.

In this paper a detailed experimental and theoretical analysis of single-mode fibre couplers made by the fused-taper technique [1–3] is presented. The fused-taper method of making single-mode fibre couplers has the advantage of being both simple and quick, unlike polishing [4, 5] and etching [6, 7] which require considerably more time and effort. First, a fused-taper coupler fabrication technique is described. The coupling efficiency between two fibres having matched, depressed or raised cladding indices in an index-matching potting medium is treated theoretically for different shapes of the tapered region. It is found that there is a V-value for which the coupling coefficient is a maximum. Thus the commonly used biconical taper structure can be modified to optimise the coupler design.

The increase in the insertion loss of the coupler arising from the taper in the fibre has been measured for different types of fibre. It is shown that depressed cladding fibres have a very high taper loss when the V-value is below the fundamental-mode cutoff. On the other hand, fibres having matched and raised claddings can be tapered with negligible loss. The coupling is wavelength dependent. The wavelength response of several couplers is measured in order, first to determine the magnitude of the undesirable variations for a standard coupler and, secondly, to find a way of making a coupler very sensitive to wavelength for wavelength-division multiplexing applications.

Finally it is shown that the sensitivity of the coupled power to the refractive index of the surrounding medium is dependent on the geometry of the coupler. Thus, while couplers can be made which are relatively insensitive to refractive index variations, it is possible to have couplers which are very sensitive to refractive index and which can be used as temperature sensors or as optical switches.

Paper 3192H (E13), first received 21st November 1983 and in revised form 8th March 1984

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2 Fabrication technique

The experimental arrangement used for making couplers is shown in Fig. 1. The two fibres which are to form the couplers are twisted together and secured to two movable

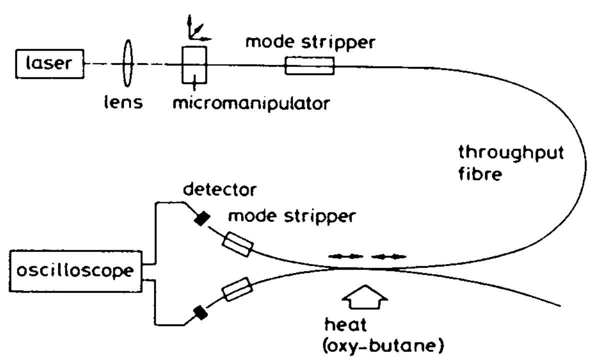
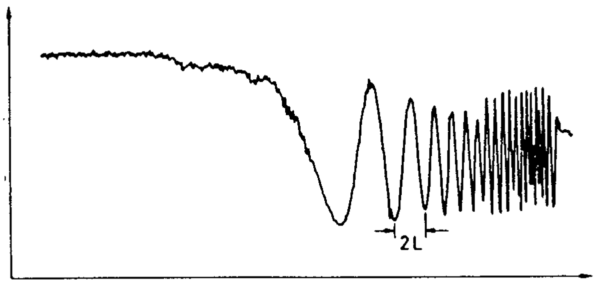


Fig. 1 *Experimental arrangement*

platforms. The fibres are then fused and tapered in an oxy-butane flame. The direction and speed of each moving platform can be varied in order to tailor the shape of the tapered region. As the fibres are tapered, the fundamental mode in each fibre spreads out and interacts with its neighbour; thus the power couples backwards and forwards between the two fibres along the length of the coupler. The amount of power coupled between the two fibres depends on the fibre NA, the core radius, the degree to which the fibre sizes are reduced, their distance apart, the length of the coupling region and the wavelength. Thus, by changing the fibre diameter in the tapered region and the length of the coupler, any amount of coupling can be achieved. In practice, the throughput power and coupled power are monitored as the coupler is made and tapering is stopped when the required ratio of coupled to throughput power is obtained. A typical measurement of the change in the throughput power P_t as the coupler is tapered is shown in Fig. 2, where it is seen that the power is exchanged 13 times between the two fibres during fabrication. Total power transfer between the two fibres takes place over a coupling length L [8], which is determined by the fibre parameters and the V-plane. Since the fibre diameter varies along the length of the coupler, L will also change along the coupler, as seen in Fig. 2. Although it is not necessary to taper the coupler beyond the first complete exchange of power to achieve any desired coupling ratio, multiple-order couplers, in which the power has been xchanged many times between the two fibres, exhibit ıteresting properties as will be shown in later Sections.



g. 2 *Throughput power as a function of pulling time for a step index re*

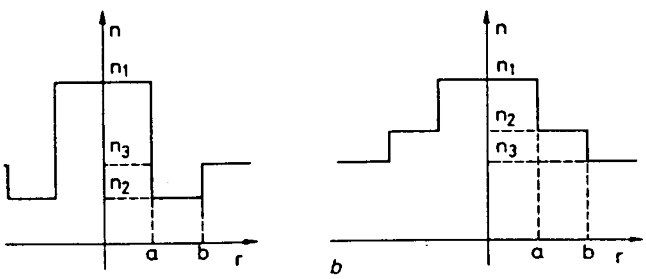
Coupling theory for circular fibres with three layers of refractive index

many single-mode optical fibres made by the MCVD ocess the refractive index of the deposited cladding is not ways matched to the index of the silica substrate and can either depressed or raised relative to that of silica. One portant example not directly treated here of a fibre with depressed cladding is the high birefringence fibre [9, 10], ıich is needed to make polarisation maintaining couplers · sensor applications, including some designs of the fibre roscope. The coupling coefficient has therefore been calculated between two fibres having matched, depressed or sed cladding indices, and the results for each type of re are compared. We consider the case of the potting dium index being matched to that of silica.

' *Field distribution*

e index profile of the three-layer fibre considered is en by (see Fig. 3)

$$n(r) = n_1 \qquad r < a$$
$$n(r) = n_2 \qquad a \leqslant r \leqslant b \tag{1}$$
$$n(r) = n_3 \qquad r > b$$



. 3 *Index profiles*
ır a depressed cladding fibre
ır a raised cladding fibre

analysis will be restricted to small differences in the active index, i.e. $(n_1 - n_2)/n_1$ and $|(n_2 - n_3)|/n_2 \ll 1$; the weak guidance approximation can be used to ain the field distribution. The transverse field coments of the fundamental mode are then given by [11,

$$E_t = A\psi(r)e^{j(\omega t - \beta z)}$$
$$H_t = B\psi(r)e^{j(\omega t - \beta z)} \tag{2}$$

re β is the propagation constant of the mode and ω is angular frequency. Only coupling between guided modes is considered. $\psi(r)$ is given by

$$\psi(r) = \begin{cases} A_0 J_0(Ur/a) & r \leqslant a \\ A_1 J_0(U'r/b) + A_2 Y_0(U'r/b) & a \leqslant r \leqslant b \\ A_3 K_0(Wr/b) & r \geqslant b \end{cases} \tag{3}$$

when $k_0 n_3 < \beta < k_0 n_2$, and

$$\psi(r) = \begin{cases} A'_0 J_0(Ur/a) & r \leqslant a \\ A'_1 I_0(W'r/b) + A'_2 K_0(W'r/b) & a \leqslant r \leqslant b \\ A'_3 K_0(Wr/b) & r \geqslant b \end{cases} \tag{4}$$

when $k_0 n_2 < \beta < k_0 n_1$, where $k_0 = 2\pi/\lambda_0$ is the wave number, with

$$\begin{aligned} U &= a(k_0^2 n_1^2 - \beta^2)^{1/2} = V(1 - B)^{1/2} \\ U' &= b(k_0^2 n_2^2 - \beta^2)^{1/2} \simeq VS\left(\frac{R}{1+R} - B\right)^{1/2} \\ W' &= b(\beta^2 - k_0^2 n_2^2)^{1/2} \simeq VS\left(B - \frac{R}{R+1}\right)^{1/2} \\ W &= b(\beta^2 - k_0^2 n_3^2)^{1/2} = VS\sqrt{B} \end{aligned} \tag{5}$$

V, S, R, B are defined by

$$V = \frac{2\pi a}{\lambda_0}(n_1^2 - n_3^2)^{1/2}$$

$$S = \frac{b}{a}$$

$$R = \frac{n_2 - n_3}{n_1 - n_2}$$

and

$$B = \frac{\beta^2 - k_0^2 n_3^2}{k_0^2(n_1^2 - n_3^2)}$$

The expressions for A_1, A_2, A_3, A'_1, A'_2, and A'_3 are given in Reference 12. A_0 and A'_0 are normalised to unity. For depressed or raised cladding fibres the propagation constant β can be calculated either by solution of the eigenvalue equation or by a perturbation method in which the inner cladding is considered to be a perturbation of a matched cladding fibre. In both cases the results are identical.

3.2 *Determination of the coupling coefficient*

The coupling coefficient between two fibres having a three-layered refractive-index profile is now considered. The power propagating in each fibre is given by $|a_1^2(z)|$ and $|a_2^2(z)|$, respectively, where $a_1(z)$ and $a_2(z)$ are the length-dependent mode amplitude coefficients and satisfy the following coupled-mode equations [8, 13, 14]:

$$\begin{aligned} \frac{da_1(z)}{dz} + i\beta(z)a_1(z) &= iC(z)a_2(z) \\ \frac{da_2(z)}{dz} + i\beta(z)a_2(z) &= iC(z)a_1(z) \end{aligned} \tag{6}$$

where $C(z)$ is the coupling coefficient between the two fibres. Assuming no mode conversion [13] $C(z)$ is simply given by

$$C(z) = -\frac{\omega}{4P}\int_{S_2} (n^2(r_2) - n_3^2)\psi(r_1)\psi^*(r_2)r_2\,dr_2\,d\theta \tag{7}$$

where S_2 is the cross-sectional area of fibre 2, r_1 and r_2 are the radii of fibres 1 and 2, respectively, and P is the power (*c*) *Exponential taper:* It is found that most of the couplers made in our laboratory have an exponential taper in

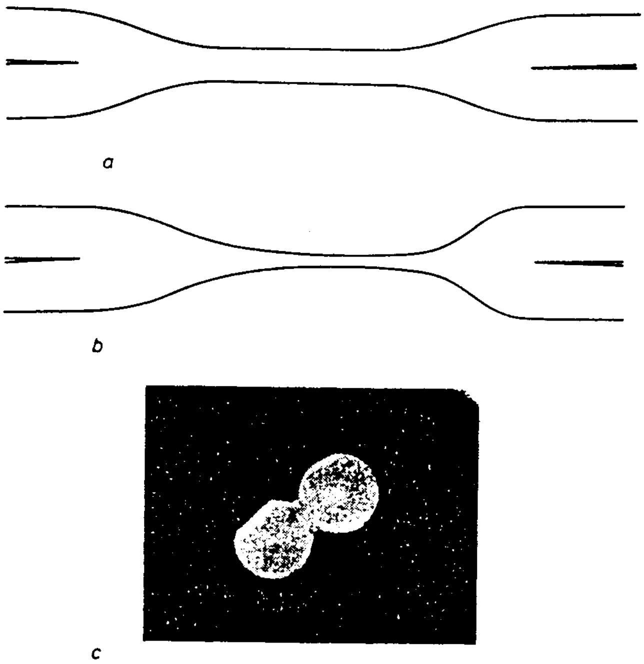


Fig. 5 *Coupler geometry*
a Optimum
b Experimental
c Cross-section of coupler waist ($T_R \simeq 10$)

the fibre. If the fibre diameter decreases exponentially with z, the coupling coefficient is

$$\bar{C}(V_f) = -\log\left(\frac{V_i}{V_f}\right)\int_{V_i}^{V_f}\frac{C(V)}{V}\,dV \tag{16}$$

$\bar{C}(V_f)$ is shown in Fig. 6*b* for the same depressed cladding fibres considered above.

4 Power losses in a fibre taper

It is found experimentally that the insertion losses of couplers made from depressed cladding fibres are very high, whereas couplers made from matched and raised cladding fibres have losses as low as 0.1 dB. In each case, the angle of the taper is very small, and, hence, it can be assumed that the high loss of the depressed cladding fibre is not due to coupling between the guided mode and radiation modes along the taper [15]. However, it is possible for the fundamental mode of a depressed cladding fibre to have a nonzero cutoff [11, 12], below which the mode becomes leaky. It is shown in this Section that it is the loss of the leaky LP_{01}-mode on a taper which gives rise to the high insertion loss of depressed cladding fibre couplers.

4.1 Experimental results

The output power from several different fibres was measured as a single fibre was tapered. The taper in the fibre was exponential, and the radius R is given by

$$R = R_0 \exp\left[-\left(\frac{l - l_0}{2l_0}\right)\right] \tag{17}$$

where l is the taper length, and l_0 is the length of the burner. The results are shown in Fig. 7; curves *a*, *b* and *c* are for depressed cladding fibres, and curve *d* is for a matched cladding fibre. The output from the matched cladding fibre remains constant as the tapering increases until the fibre breaks. At this point, the taper ratio T_R (= initial fibre diameter/diameter at the centre of taper)

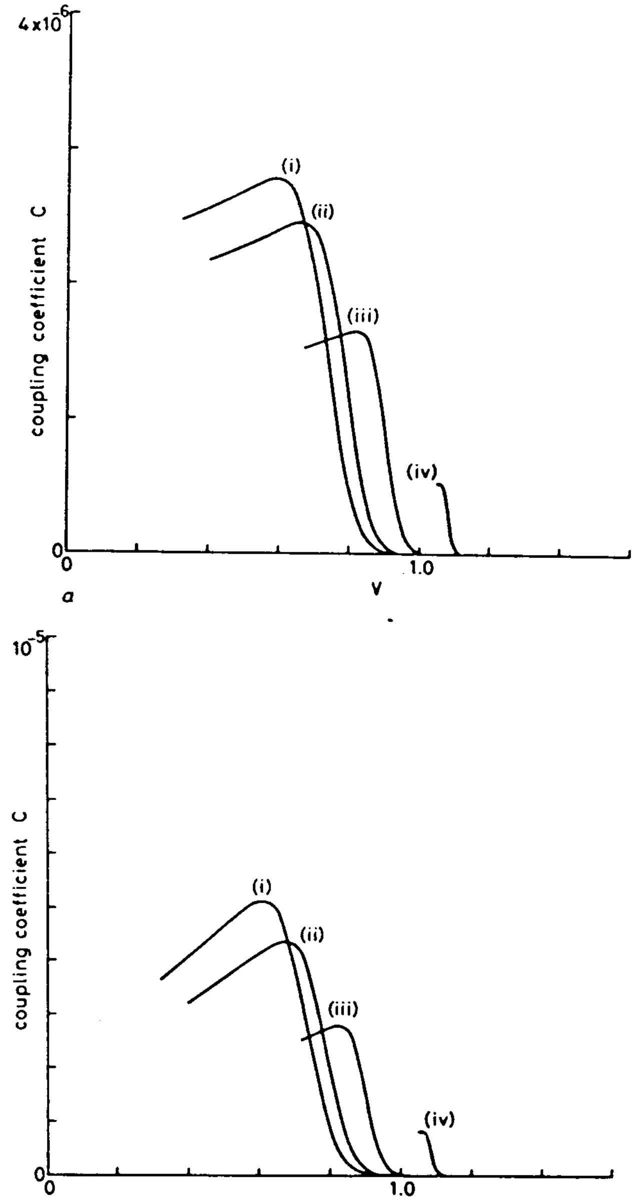


Fig. 6 *Coupling coefficient as a function of V between two fibres*
$n_1 - n_3 = 0.007$, $S = 5$, $D/a = 48$
a For a linear taper
b For an exponential taper
(i) $R = -0.0125$, $V_{cutoff} = 0.32$
(ii) $R = -0.0025$, $V_{cutoff} = 0.40$
(iii) $R = -0.05$, $V_{cutoff} = 0.68$
(iv) $R = -0.1$, $V_{cutoff} = 1.05$

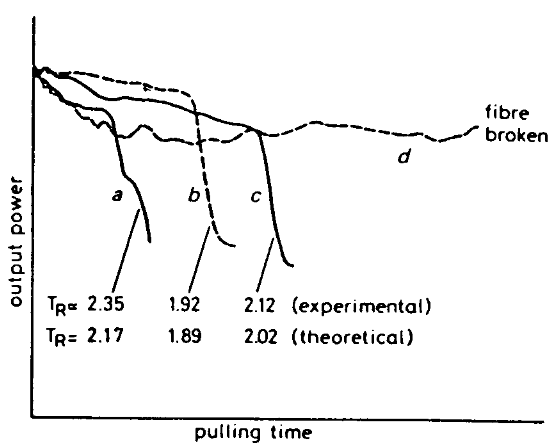


Fig. 7 *Output power of different fibres as a function of pulling time*

as the wavelength changes, $\bar{C}(V_f)$ will change by different amounts for different couplers.

When the taper ratio is high, then around the centre of the coupler the core becomes insignificant, and coupling between the two fibres can be considered as coupling between the two silica claddings [16] surrounded by a medium of index n_4. In Fig. 9 the throughput power on such a structure has been calculated as a function of wavelength for $n_4 = 1.42$ (corresponding to the index of the potting medium used). The fibres are considered to be of constant cross-section, with a diameter corresponding to a taper ratio of 40. The coupling length $L = 0.8$ mm, and the total length of the coupling region L_T is 15 mm. The calculated curve resembles the measured wavelength response of multiple-order couplers. Similar calculations for couplers of different orders also show that the period depends on the order of the coupler.

6 Variation of coupled power with surrounding refractive index

After fabrication of the fibre coupler it is necessary to surround the tapered region with a protective coating to provide a robust package. It was found that the refractive index of the coating could change the coupling ratio, since the field distribution in the taper changes, and thus the amount of coupling changes. The extent of this effect depends on the refractive index of the medium, the taper ratio of the fibres and the shape of the taper.

The sensitivity of the coupled power P_c to the index of the surrounding medium has been measured for several couplers made from matched cladding fibre and tapered by different amounts. The uncoated coupler was immersed in an oil having a refractive index less than that of the cladding index, and the change in the coupled power was measured as the refractive index of the oil was reduced by heating.

Fig. 10 shows the ratio of the coupled power P_c to the

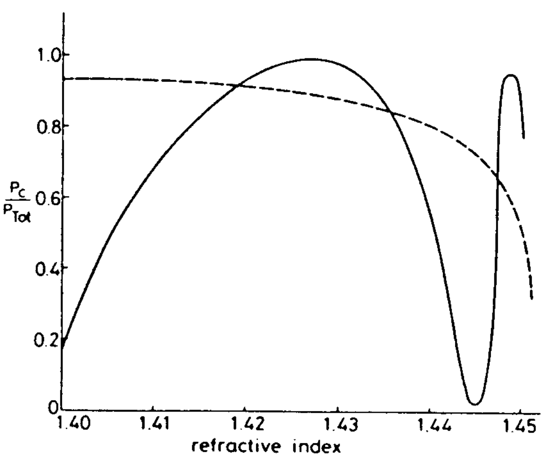


Fig. 10 *Variation of normalised coupled power P_c/P_{tot} with refractive index*

NA = 0.09, D/a = 20
P_c = coupled power
P_{tot} = total power (constant)
- - - - T_R = 10
—— T_R = 20

total power P_T as a function of index for two couplers having taper ratios of 10 (dotted curve) and 20 (solid curve); the shape of the taper was close to exponential. The matched cladding fibre used had NA = 0.09, and the cutoff wavelength of the second mode was 600 nm. The change in the coupled power can be very large, as seen for the coupler with $T_R = 20$ for which the power oscillates between the two output ports as the index is changed. For a smaller value of T_R the change in the coupled power is insignificant. For example, in a coupler with $T_R \simeq 4$ the coupled power changes by only 1.5% of its value over the range of refractive index 1 to 1.45.

The above effect should not pose any problems in coating couplers used for routine purposes, since any required coupling ratio can be obtained for a relatively small taper. Any small change in the coupling ratio resulting from coating the fibre can then be compensated by making the uncoated coupler have a somewhat different coupling ratio from that finally required. However, Fig. 10 shows that couplers made with a large taper ratio are very sensitive to refractive index, and thus to temperature. Hence a single-mode fibre coupler may form the basis of an interesting sensor, with one output port monitoring the variation in either refractive index or temperature and the other output port providing a reference signal. Alternatively the effect can be used to make either a tunable coupler or a switching device.

7 Summary

A fabrication technique used for making fused fibre couplers is described. The coupling coefficient between two fibres has been calculated as a function of V for fibres having depressed, raised or matched cladding in a matched potting medium. It is shown that there is a V-value at which coupling is maximum, and thus a biconical taper is not the shortest geometry for a coupler.

The loss arising from tapering fibres having a depressed cladding has been measured. It is found that the leaky LP_{01}-mode has a very large taper loss, which results in a high insertion loss for couplers made from depressed cladding fibres. However, the theoretical analysis shows that the maximum coupling for a depressed cladding fibre occurs at a V-value greater than the fundamental-mode cutoff. Thus, in special cases, for example, with high birefringence fibres (for example, bow-tie fibre [9]), it should be possible to make low-loss couplers from depressed cladding fibres by carefully controlling the tapering process.

The wavelength response of various couplers made from matched cladding fibres has been measured. The coupled power is found to vary periodically with wavelength with a constant period, which is determined by the order of the coupler and the taper ratio. For a large-order coupler with a large taper ratio it is possible to have a wavelength period as small as 12 nm. Thus single-mode fibre couplers can be used as the basis of a wavelength-division multiplexer or demultiplexer. Such a device would be more stable and have lower loss than an equivalent bulk-optics device. The results also show that low-order couplers can be made which are relatively insensitive to wavelength.

For low V-values in the tapered region, the coupling between the two fibres can be considered as coupling between the claddings alone [16]. The coupling between two fibres has been calculated as a function of wavelength for this case, and the results are seen to closely resemble the wavelength response measured for a large-order coupler.

Finally, the sensitivity of the coupled power to the refractive index of the medium surrounding the coupler has been measured. It was found that the coupled power varied as the index was changed, by an amount which depended on the taper ratio. In some cases the power was exchanged completely from one fibre to another. Although in standard couplers this effect will be small, it is possible to make couplers which are very sensitive to index, and which can therefore form the basis of either a temperature or refractive-index sensor or an optical switch.

8 Acknowledgments

The authors would like to acknowledge S.B. Poole for help with the measurements, R.D. Birch and E.J. Tarbox for making the fibres, and D.N. Payne and J.D. Love for useful discussions. The work was supported by the UK Science & Engineering Research Council and the Centre National de la Recherche Scientifique.

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10 Appendix

The coefficients used in the expressions for the coupling coefficient, eqns. 11 to 13 of Section 3, are defined as follows:

(*a*) In eqn. 11

$$X_1 = \frac{a}{\left(\frac{U}{a}\right)^2 + \left(\frac{W}{b}\right)^2}\left\{\left(\frac{W}{b}\right)J_0(U)I_1\left(\frac{aW}{b}\right) + \left(\frac{U}{a}\right)J_1(U)I_0\left(\frac{aW}{b}\right)\right\}$$

$$X_2 = \frac{b}{\left(\frac{W}{b}\right)^2 - \left(\frac{W'}{b}\right)^2}\left\{\left(\frac{W}{b}\right)I_0(W')I_1(W) - \left(\frac{W'}{b}\right)I_0(W)I_1(W')\right\}$$
$$- \frac{a}{\left(\frac{W}{b}\right)^2 - \left(\frac{W'}{b}\right)^2}\left\{\left(\frac{W}{b}\right)I_0\left(\frac{W'a}{b}\right)I_1\left(\frac{Wa}{b}\right) - \left(\frac{W'}{b}\right)I_0\left(\frac{Wa}{b}\right)I_1\left(\frac{W'a}{b}\right)\right\}$$

$$X_3 = \frac{b}{\left(\frac{W'}{b}\right)^2 - \left(\frac{W}{b}\right)^2}\left\{-\left(\frac{W'}{b}\right)K_1(W')I_0(W) - \left(\frac{W}{b}\right)K_0(W')I_1(W)\right\}$$
$$- \frac{a}{\left(\frac{W'}{b}\right)^2 - \left(\frac{W}{b}\right)^2}\left\{-\left(\frac{W'}{b}\right)K_1\left(\frac{W'a}{b}\right)I_0\left(\frac{Wa}{b}\right) - \left(\frac{W}{b}\right)K_0\left(\frac{W'a}{b}\right)I_1\left(\frac{Wa}{b}\right)\right\}$$

$$Q = \frac{a^2}{2}\{J_0^2(U) + J_1^2(U)\} + \frac{b^2}{2}\{A_1 I_0(W') + A_2 K_0(W')\}^2 - \frac{b^2}{2}\{A_1 I_1(W') - A_2 K_1(W')\}^2$$
$$- \frac{a^2}{2}\left\{A_1 I_0\left(\frac{W'a}{b}\right) + A_2 K_0\left(\frac{W'a}{b}\right)\right\}^2 + \frac{a^2}{2}\left\{A_1 I_1\left(\frac{W'a}{b}\right) - A_2 K_1\left(\frac{W'a}{b}\right)\right\}^2$$
$$+ \frac{b^2}{2}A_3^2\{K_1^2(W) - K_0^2(W)\}$$

(*b*) In eqn. 12

$$X_1' = \frac{a}{\left(\frac{U}{a}\right)^2 + \left(\frac{W}{b}\right)^2}\left\{\left(\frac{W}{b}\right)J_0(U)I_1\left(\frac{aW}{b}\right) + \left(\frac{U}{a}\right)J_1(U)I_0\left(\frac{aW}{b}\right)\right\}$$

$$X_2' = \frac{-a}{\left(\frac{U'}{b}\right)^2 + \left(\frac{W}{b}\right)^2}\left\{\left(\frac{W}{b}\right)J_0\left(\frac{U'a}{b}\right)I_1\left(\frac{aW}{b}\right) + \left(\frac{U'}{a}\right)J_1\left(\frac{U'a}{b}\right)I_0\left(\frac{aW}{b}\right)\right\} + \frac{b}{\left(\frac{U'}{b}\right)^2 + \left(\frac{W}{b}\right)^2}$$
$$\times\left\{\left(\frac{W}{b}\right)J_0(U')I_1(W) + \left(\frac{U'}{a}\right)J_1(U')I_0(W)\right\}$$

$$X_3' = \frac{-b}{\left(\frac{W}{b}\right)^2 + \left(\frac{U'}{b}\right)^2}\left\{-\left(\frac{W}{b}\right)I_1(W)Y_0(U') - \left(\frac{U'}{b}\right)I_0(W)Y_1(U')\right\} + \frac{a}{\left(\frac{W}{b}\right)^2 + \left(\frac{U'}{b}\right)^2}$$
$$\times\left\{-\left(\frac{W}{b}\right)I_1\left(\frac{Wa}{b}\right)Y_0\left(\frac{U'a}{b}\right)\right.$$

$$-\left(\frac{U'}{b}\right)I_0\left(\frac{Wa}{b}\right)Y_1\left(\frac{U'a}{b}\right)\Bigg\}$$

$$Q' = \frac{a^2}{2}\{J_0^2(U) + J_1^2(U)\} + \frac{b^2}{2}\{(A_1J_0(U') + A_2Y_0(U'))^2 + (A_1J_1(U') + A_2Y_1(U'))^2\}$$
$$-\frac{a^2}{2}\left\{\left(A_1J_0\left(\frac{U'a}{b}\right) + A_2Y_0\left(\frac{U'a}{b}\right)\right)^2 + \left(A_1J_1\left(\frac{U'a}{b}\right) + A_2Y_1\left(\frac{U'a}{b}\right)\right)^2\right\} + A_3^2\frac{b^2}{2}\{K_1^2(W) - K_0^2(W)\}$$

(*c*) In eqn. 13

$$X_1'' = \sum_{n=0}^{\infty}\prod_{m=1}^{n}\frac{(2m-1)}{m}\left(\frac{1}{n}\right)\int_0^a r^{2n+1}J_0\left(\frac{Ur}{a}\right)dr$$

$$X_2'' = \sum_{n=0}^{\infty}\prod_{m=1}^{n}\frac{(2m-1)}{2m}\left(\frac{1}{n}\right)\int_a^b r^{2n+1}\left\{A_1J_0\left(\frac{U'r}{b}\right) + A_2Y_0\left(\frac{U'r}{b}\right)\right\}$$

$$X_3'' = 2\left\{\log\frac{WD}{b}\right\}\left\{\frac{a^2}{U}J_1(U)(n_1^2 - n_3^2) + (n_2^2 - n_3^2)\frac{b^2}{U'}\left[A_1J_1(U') + A_2Y_1(U') - A_1J_1\left(\frac{U'a}{b}\right) - A_2Y_1\left(\frac{U'a}{b}\right)\right]\right.$$

when $V \rightarrow 0$