COUPLED-MODE THEORY FOR HELICAL FIBRES

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ABSTRACT

A set of coupled-mode equations for helical fibres are formulated. These equations are then used to analyse the helical single-mode fibres with intrinsic linear bire-fringence. The eigen-modes and the resultant birefringence of the whole waveguides are obtained. The effect of the external perturbation on these wageguide are also considered.

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I. INTRODUCTION

The fact that the polarisation state of the light is rotated if the light advances along a helical fibre path [1,2] was successfully applied to making circular-birefringence fibres by using a helical core in the fibre [3,4]. Theoretical work on the helical fibre was done by Fang et al [5] to prove the axiom presented in Ref.[2]. Although the linear birefringence in the helical fibre was mentioned in these papers, but no detail has been given. Although the formula for the beat-length of the fibres with helical cores was given [3], but it has not been derived rigorously, and the influence of the linear birefringence on the resulting birefringence of the helical-core fibres is still not well known.

In this paper, starting from the Maxwell equations in Tang's coordinate system [6], we transform them to a set of coupled-mode equations which are able to describe the behaviour of all modes in a helical fibre clearly and precisely.

The fields in the helical fibres are expressed by superposition of a set of local normal modes, which are normal in a straight fibre, but are coupled to each other in the helical fibres. If only two orthogonal fundamental modes are considered in the coupled-mode equations (as in the single-mode case), it is simple to obtain the eigen-modes in the helical fibres and the resultant birefringence of the whole waveguides. The intrinsic and external perturbations of the fibres can be taken into consideraton easily in these equations.

II. FORMULATION

The geometry of a helical optical fibre is shown in Fig.1. It is not only for the helically wound optical fibres, but also for the helical-core fibres [3] or the spiral fibres [7]. In the latter case only the core is in a helical form, the cladding surrounding the helical core forms a uniform cylindrical fibre. If the cladding layers are much thicker than the core diameters, there are no differences between them optically.

Before fomulation we have to distinguish two different axes, the core axis s and the helix axis ζ. In the coupled-mode analysis usually we use the s-axis as the reference axis, along which the light propagates. However, it is not convenient to use s-axis in practical use, when the helical fibre is taken as a whole waveguide (see Sec.IV). Thus we prefer ζ-axis as the reference axis and the whole waveguide will be called dielectric helical waveguide.

Thus when different titles, the helical fibre and the dielectric helical waveguide are used below, it implies that different axes, s- and ζ-axis are being used, respectively. Propagation constant of a mode will be changed for different reference axes.

The coordinate system (n, b, s) shown in Fig.1 is the Serret-Frenet frame, which is non-orthogonal so far as the torsion is not equal to zero. To form an orthogonal coordinate system (m, p, s), Tang [6] rotated the Serret-Frenet frame with an angle $\varphi=-\tau s$ in the plane transverse to the s-axis, where τ is the torsion, which is positive in case of right-handed helical system and is given by

$$\tau = \frac{2\pi P}{P^2 + (2\pi R_0)^2} = \frac{2\pi P}{S^2} \tag{1}$$

where P is the pitch, R_o is the radius of the helix, S is the arc length for one turn.

Tang also gives the metric coefficients for the new coordinate system

$$h_m = h_p = 1$$
$$h_s = 1 - \chi(m\cos\varphi - p\sin\varphi) \tag{2}$$

By changing from the transverse coordinate (m,p) to the polar coordinates (r, θ), we have

$$m = r\cos\theta \qquad p = r\sin\theta \tag{3}$$

the corresponding metric coefficients become

$$h_r = h_\theta = 1$$
$$h_s = 1 - \xi = 1 - \chi r\cos(\varphi + \theta) \tag{4}$$

where

$$\chi = \frac{(2\pi)^2 R_0}{(2\pi R_0)^2 + P^2} \tag{5}$$

Maxwell's equation in (m, p, s) coordinate system can be rewritten as

$$\nabla_t \cdot (\underset{\sim}{i_s} \times \underset{\sim}{E_t}) = j\omega\mu_0 H_s \tag{6a}$$

$$\nabla_t \cdot (\underset{\sim}{H_t} \times \underset{\sim}{i_s}) = j\omega\varepsilon E_s \tag{6b}$$

$$\nabla_t (h_s E_s) - \frac{\partial}{\partial s} E_t = j\omega\mu_0 h_s (\underset{\sim}{H_t} \times i_s) \tag{7a}$$

$$\nabla(h_s H_s) - \frac{\partial}{\partial s} H_t = j\omega\varepsilon h_s (i_s \times E_t) \tag{7b}$$

where E_t, H_t and E_s, H_s are the field components in the plane transverse to s-axis and along the s-axis, respectively, i_s is the unit vector in the s directions. $\varepsilon = n^2 \varepsilon_0$ and n is the refractive index of the fibres and is a function of coordinates (p, n), but does not vary along the s-axis. Since we assume a lossless fibre, the n is a real number.

Any transverse fields E_t and H_t along the helical fibres can be represented by a superposition of a set of local normal modes of a reference fibre, which is straight and has the same direction as the local point on the helical fibre (see Fig.1). The refractive index profile n_0 for the reference fibre is circularly symmetrical. Usually we choose the step-index profile, so it is easy to find the modal fields of the normal modes for the reference fibre. Thus we have

$$E_t = \sum_i A_i e_{ti}$$

$$H_t = \sum_i A_i h_{ti} \tag{8}$$

where e_{ti} and h_{ti} denote the transverse components of the modal fields which were described in details elsewhere [8]. The summations in (8) are over both the guided modes and the radiation modes. The latter are so discretised that we can consider them as guided modes [9]. The total modal fields can be expressed as

$$e_i = e_{ti} + i_z e_{zi}$$

$$h_i = h_{ti} + i_z h_{zi} \tag{9}$$

where i_z is the unit vector in the z-direction, which coincides with the s-direction only at local points on the helical fibre.

Thus the z-direction is not changed along a certain reference fibre, but is changed periodically from one reference fibre to another, as the local points move along the helical fibre. The modal fields are orthonormallyized as follows:-

$$\int_{A_\infty} \left[(\underset{\sim}{e}_i \times \underset{\sim}{h}_k^*) \cdot \underset{\sim}{i}_z + (\underset{\sim}{e}_k^* \times \underset{\sim}{h}_i) \cdot \underset{\sim}{i}_z \right] d\Omega = \delta_{ik} \tag{10}$$

where δ_{ki} is the kronecker delta and A_∞ is the infinite cross-sectional area.

The normal modes in the reference fibre which is a straight fibre tangent to the helical fibre at a local point are local normal modes and they are coupled to each other in the helical fibre. Now we proceed to formulate the coupled-mode equations for those local normal modes in the helical fibres.

We first dot-multiply (7a) and (7b) by ($\underset{\sim}{h}_{tk}^*$ x $\underset{\sim}{i}_z$) and ($\underset{\sim}{i}_z$ x $\underset{\sim}{e}_{tk}^*$), respectively, and then integrate them over the entire cross-section, obtaining

$$\begin{aligned} \frac{\partial}{\partial s}\int_{A_\infty} \underset{\sim}{H}_t \cdot (\underset{\sim}{i}_z \times \underset{\sim}{e}_{tk}^*)\, d\Omega = {} & \int_{A_\infty} \underset{\sim}{H}_t \cdot \frac{\partial}{\partial s}(\underset{\sim}{i}_z \times \underset{\sim}{e}_{tk}^*)\, d\Omega - \\ & - j\omega\varepsilon_0 \int_{A_\infty} h_s (n^2 - n_0^2)(\underset{\sim}{i}_s \times \underset{\sim}{E}_t)\cdot(\underset{\sim}{i}_z \times \underset{\sim}{e}_{tk}^*)\, d\Omega \\ & - j\beta_k \int_{A_\infty} h_s (\underset{\sim}{i}_s \times \underset{\sim}{E}_t)\cdot \underset{\sim}{h}_{tk}^*\, d\Omega - \int_{A_\infty} h_{zk}^* \left[(\underset{\sim}{i}_s \times \underset{\sim}{E}_t)\cdot \nabla_t h_s \right] d\Omega \end{aligned} \tag{11a}$$

$$\begin{aligned} \frac{\partial}{\partial s}\int_{A_\infty} \underset{\sim}{E}_t \cdot (\underset{\sim}{h}_{tk}^* \times \underset{\sim}{i}_z)\, d\Omega = {} & \int_{A_\infty} \underset{\sim}{E}_t \cdot \frac{\partial}{\partial s}(\underset{\sim}{h}_{tk}^* \times \underset{\sim}{i}_z)\, d\Omega - \\ & - j\omega\varepsilon_0 \int_{A_\infty} h_s (n^2 - n_0^2) E_s\, e_{zk}^*\, d\Omega \\ & - j\beta_k \int_{A_\infty} h_s (\underset{\sim}{H}_t \times \underset{\sim}{i}_s)\cdot \underset{\sim}{e}_{tk}^*\, d\Omega - \int_{A_\infty} e_{zk}^* \left[(\underset{\sim}{H}_t \times \underset{\sim}{i}_s)\cdot \nabla_t h_s \right] d\Omega \end{aligned} \tag{11b}$$

where β_k is the propagation constant of the k^{th} mode in reference fibre.

Adding (11a) to (11b) and then using (8) and (10), it is ready to have the following coupled-mode equations for all modes including discretized radiation modes:

$$\frac{d}{ds} A_k = -j\beta_k A_k - \sum_i K_{ki} A_i \tag{12}$$

where the coupling coefficient can be expressed as

$$\begin{aligned} K_{ki} = & -\int_{A_\infty} \left[h_{ti} \cdot \frac{\partial}{\partial s}(i_z \times e^*_{tk}) + e_{ti} \cdot \frac{\partial}{\partial s}(h^*_{tk} \times i_z) \right] d\Omega \\ & + j\beta_k \int_{A_\infty} \xi \left[(e_{ti} \times h^*_{tk}) \cdot i_z + (e^*_{tk} \times h_{ti}) \cdot i_z \right] d\Omega \\ & + \int_{A_\infty} \left[h^*_{zk} (i_z \times e_{ti}) + e^*_{zk} (h_{ti} \times i_z) \right] \cdot \nabla \xi \, d\Omega \\ & + j\omega\varepsilon_0 \int_{A_\infty} h_s (n^2 - n_s^2)(e^*_k \cdot e_i) \, d\Omega \end{aligned} \tag{13}$$

Equation (13) also can be used for backward-propagating modes, in that case, $\beta_{-k} = -\beta_k < 0$ and their modal fields have simple relations with those of forward [8]. We have four terms in the right-hand side of (13), which imply different mechanisms resulting in couplings between modes. The first term in the right-hand side denote the coupling caused by the change of modal fields. Since we use the same reference fibre for all local points along the helical fibre, it vanishes. The coupling expressed by the second and third terms is resulted from bending. Since in the weakly guiding step fibres the z-components of the modal fields for guided modes are small of an order $(\Delta)^{1/2}$ compared to the transverse components, the third term can be neglected. The last term is the coupling caused by the difference of the refractive indices between the reference fibre and the helical fibre, it has the same form as for a straight fibre [10]. The refractive index difference may result from geometrical deformation and/or induced stress, which have been studied in more detail elsewhere [11].

It is noted that the torsion of the helix does not result in coupling between the modes defined in Tang's coordinate system.

III. EIGEN-MODES IN SINGLE-MODE HELICAL FIBRE.

For a single-mode helical fibre, only the two orthogonally linearly-polarized fundamental modes $\underset{\sim}{e}_m$ and $\underset{\sim}{e}_p$ are taken into account in (12), we have

$$\begin{aligned} \frac{d}{ds}A_m &= -j\beta A_m - \sum_{i=m,p} j\,\omega\varepsilon_0 A_i \int_{A_\infty} h_s (n^2 - n_0^2)(\underset{\sim}{e}_m^* \cdot \underset{\sim}{e}_i)\, d\Omega \\ \frac{d}{ds}A_p &= -j\beta A_p - \sum_{i=m,p} j\,\omega\varepsilon_0 A_i \int_{A_\infty} h_s (n^2 - n_0^2)(\underset{\sim}{e}_p^* \cdot \underset{\sim}{e}_i)\, d\Omega \end{aligned} \qquad (14)$$

where A_m and A_p are the amplitudes of the modes e_m and e_p, respectively, β is the propagation constant of the mode in reference fibre.

Since $\nabla\zeta = -\chi(\underset{\sim}{i}_m\cos\varphi - \underset{\sim}{i}_p\sin\varphi)$, $\underset{\sim}{i}_m$ and $\underset{\sim}{i}_p$ being unit vectors in m- and p-directions, respectively, the second and third terms on the right-hand side of (13) are equal to zero. This implies that bending does not result in linear birefringence and coupling between the two modes geometrically. However bending causes lateral internal stress, which modifies the refractive index of the fibre material [12]. Consequently, the bending-induced birefringence can be described by the last term in the (13), by which the linear birefringence resulted from elliptical deformation of the cores is also obtained.

These two linear birefringences caused by the induced-stress and geometrical deformation usually do not change along the helical fibre. They are intrinsic linear birefringences. However, their principal axes are, in general, rotated following the (n, b, s) system and not Tang's coordinate system. It is therefore convenient to start from the coupled-mode equations in the (n, b, s) system. To do this, we make following transformation:

$$\begin{bmatrix} A_n \\ A_b \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} A_m \\ A_p \end{bmatrix} \tag{15}$$

and (14) can be rewritten as

$$\begin{aligned} \frac{d}{ds} A_n &= -j(\beta - \Delta\beta/2) A_n + \tau A_b \\ \frac{d}{ds} A_b &= -j(\beta + \Delta\beta/2) A_b - \tau A_n \end{aligned} \tag{16}$$

where A_n and A_b are amplitudes of the two linearly polarised modes defined in (n,b,s) system, $\Delta\beta$ is the total intrinsic birefringence. For simplicity we assume that the fast axis of the birefringence coincide with the n-axis without losing the popularity of the description.

Using the following matrix transformation [13]

$$\begin{bmatrix} A_n \\ A_b \end{bmatrix} = \begin{bmatrix} \cos\psi & j\sin\psi \\ j\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} A_r \\ A_l \end{bmatrix} \tag{17}$$

it is found that (16) is converted to:

$$\begin{aligned} \frac{d}{ds} A_r &= -j(\beta - g) A_r \\ \frac{d}{ds} A_l &= -j(\beta + g) A_l \end{aligned} \tag{18}$$

where

$$\psi = \frac{1}{2} \arctan \frac{2\tau}{\Delta\beta} \tag{19}$$

$$g = \frac{\Delta\beta}{2}\sqrt{1+(2\tau/\Delta\beta)^2} = \sqrt{\tau^2+(\Delta\beta/2)^2} \tag{20}$$

and where A_r and A_l are the amplitudes of the eigen-modes in the helical fibre, both modes are elliptically polarized with ellipticity (minor/major axis) equal to $\tan\psi$ [13]. It is not difficult to find from (17) that A_r and A_l are corresponding to right-rotated and left-rotated elliptically polarized modes, the electric field vectors of which rotate around the s-axis according to the right-hand and the left-hand rule, respectively. Equation (18) implies that the two eigen-modes in the helical fibre with linear birefringence have their propagation constant equal to $(\beta-g)$ and $(\beta+g)$ in the (n, b, s) system.

IV BIREFRINGENCE OF DIELECTRIC HELICAL WAVEGUIDES

It is apparent from Fig.1 that (n, b, s) system is always rotated around the helix axis ζ with the rotation rate α ($\alpha=2\pi/S$) per unit length of the helical fibre. Relative to the fixed coordinate system, the propagation constant $(\beta\mp g)$ for both elliptically polarized mode are therefore modified. As we are here considering a right-handed system, the right elliptically polarized mode propagates faster, i.e. its propagation constant is $(\beta-g+\alpha)$ and the left one advances slower, i.e. its propagation constant is $(\beta+g-\alpha)$ than in the original system. Thus the resulting elliptical birefringence of the whole dielectric helical waveguide formed by the helical fibre is $2(g-\alpha)(S/P)$, where the factor (S/P) is due to the fact that the helix axis is taken as the reference axis. Then we have the corresponding normalized beat-length.

$$\frac{L_t'}{P} = \frac{L_t/P}{\left|\frac{2L_t}{P} - \sqrt{\csc^2\sigma + \left(\frac{2L_t}{P}\right)^2 \sin^2\sigma}\right|} \tag{21}$$

where σ is the pitch angle (σ = arcsinP/S), $L_p = 2\pi/\Delta\beta$.

For the helical fibre without intrinsic linear birefringence, i.e. $\Delta\beta$=o, then from (21), we have

$$L_t'/P = \frac{1}{2}\,\frac{1}{(1-\sin\sigma)} \tag{22}$$

this is the special case for an isotropic helical fibre [14], and the whole waveguide is pure birefringent.

The case of $\sigma=90^{\circ}$ corresponds to spinning a fibre with intrinsic birefringence $\Delta\beta$, then from (21)

$$\frac{L_t'}{P} = \frac{L_t/P}{\sqrt{1+(2L_t/P)^2} - 2L_t/P} \tag{23}$$

this is the same result derived in Ref.[13].

It is interesting to note that the dielectric helical waveguides may have zero birefringence when the denominator in (21) equal to zero, namely

$$\frac{L_t'}{P} = \frac{1}{\sin 2\sigma} \tag{24}$$

Thus in this case, the two eigen-modes propagating along the ζ-axis (not the s-axis) are degenerate, the propagation constant difference resulted from the intrinsic birefringence and the torsion is cancelled out by that caused by the opposite rotation direction around the ζ-axis.

However, since the mode degeneracy or low-birefringence associates with packing problems, the geometrical parameters usually have to be chosen to avoid the relation (24) to be satisfied.

Equation (21) is plotted and the normalized beat-length of the waveguides is shown as a function of $2L_p/P$. The degeneracies can also be seen when the values of L_p'/P approach infinity.

Now we study the effect of external perturbations on the helical waveguides. External perturbation such as bending, winding and applied stress usually induce external linear birefringence, the principal axes of which are fixed, not following the helix. Then the same method can be used as we study the external effect on the spun fibres [13].

When the pitch angle of the helix is near 90^o, the results obtained in Ref.[13] are also valid, i.e.

$$\begin{aligned} Q &= \frac{b}{4(g-\alpha)}(1+\sin\psi)\frac{P}{S} \\ &= \frac{bL_p'}{4\pi}(1+\sin\psi) \end{aligned} \tag{25}$$

where b is the external linear birefringence, Q is the coupling capacity, which is a measure of the power transfer between the two eigen-modes in helical waveguide.

It is clear from (25), that the beat-length of a dielectric helical waveguide is still an important parameter to govern the resistence to external effect. Special care has to be taken when using the cut-back method or observing the Rayleigh scatter to measure the beat-length. However, if the circular birefrengence is predominant in the waveguides, the cut-back method is still valid.

V CONCLUSION

The propagation of a coherent light in a helical fibre can be described by coupled-mode equations. A dielectric helical waveguide formed by a helical single-mode fibre is in general elliptically birefringent. The linear part of the birefringence comes from the intrinsic linear birefringence of the helical fibre; the circular part is from the torsion of the helix and the rotation of the axes of the linear birefringence. A waveguide with isotropic helical fibre is pure circularly birefringent, and the birefringence is only from the torsion.

Theory predicts that a new degeneracy or zero birefringence case may appear, if the helical waveguide is specially designed.

ACKNOWLEDGEMENTS

The author wishes to express thanks to Professor W.A.Gambling for providing a stimulating research atmosphere, and would like to acknowledge Dr.C.D.Hussey and Mr.L.S.Li for helpful suggestions.

REFERENCES

1. A.Papp and H.Harms: "Polarisation optics of liquid core optical fibres", Appl. Opt., 5, pp. 1315-1319, 1977.
2. J.M.Ross: "The rotation of polarisation in low

birefringent monomode optical fibres due to geometric effects", Opt.& Qantum Electron., 16, pp. 455-461, 1984.

3. M.P.Varnham, R.D.Birch, and D.N.Payne: "Helical-core circularly-birefringent fibres", Proc. IOOC-ECOC, 1985, Venice.

4. M.P.Varnham, R.D.Birch, D.N.Payne, and J.D.Love: "Design of helical-core circularly-birefringent fibres", Proc.OFC/IGWO, 1986, Atlanta.

5. X.S.Fang and Z.Q.Lin: "Field in single-mode helically-wound optical fibres", Trans.IEEE, MTT-33, pp. 1150-1154, 1985.

6. C.H.Tang: "An orthogonal coordinate system for curved pipes", Trans.IEEE, MTT-18, pp. 69-70, 1970.

7. C.D.Hussey, R.D.Birch, and Y.Fujii: "Circularly birefringent single-mode optical fibres", Electron.Lett., 22, pp. 129-130, 1986.

8. A.W.Snyder and J.D.Love: Optical waveguide Theory, London, 1983, Chapman & Hall.

9. J.R.Qian and W.P.Huang: "Coupled-mode theory for LP modes", IEEE J.Lightwave Technol., Vol.LT-4, pp , 1986.

10. D.Marcuse, Theory of Dielectric waveguide. New York: Academic, 1974.

11. W.Eikhoff: "Stress-induced single-polarization single-mode fibre", Opt.Lett., vol.7, pp. 629-631, 1982.

12. R.Ulrich, S.C.Rashleigh, and W.Eickhoff: "Bending-induced birefringence in single-mode fibre", Opt.Lett., vol.5, pp. 273-275, 1980.

13. J.R.Qian and L.S.Li: "Spun linearly highly-birefringent fibre for current sensors" (to be published).

14. J.R.Qian and C.D.Hussey: "Circular birefringence in helical-core fibre", Electron.Lett., 22, pp. 515-517, 1986.

Figure Captions

Figure 1. The geometry of a helical optical fibre and one of the reference fibres.

Figure 2. The normalized beat-length of the dielectric helical waveguide.

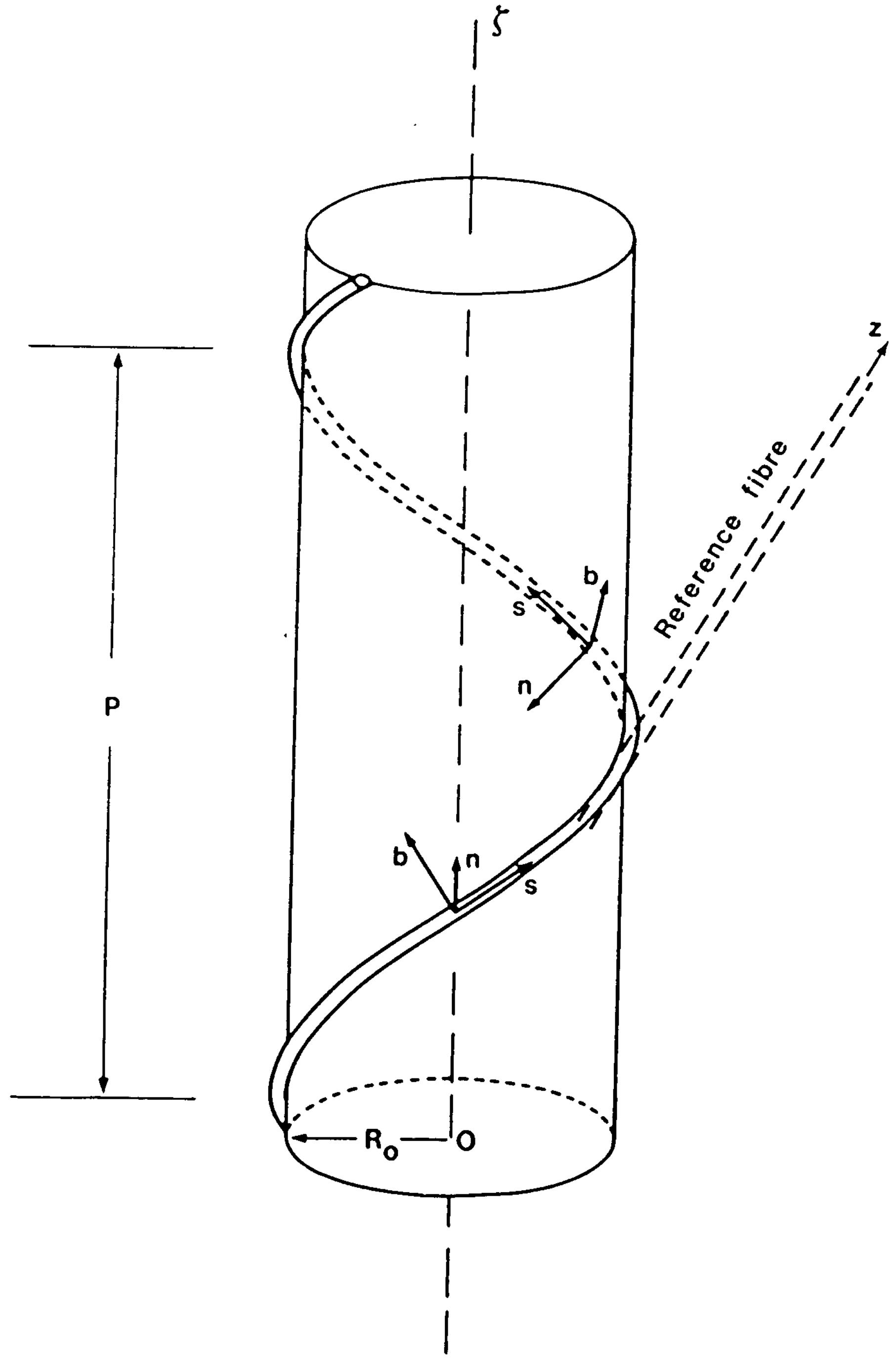

Fig 1

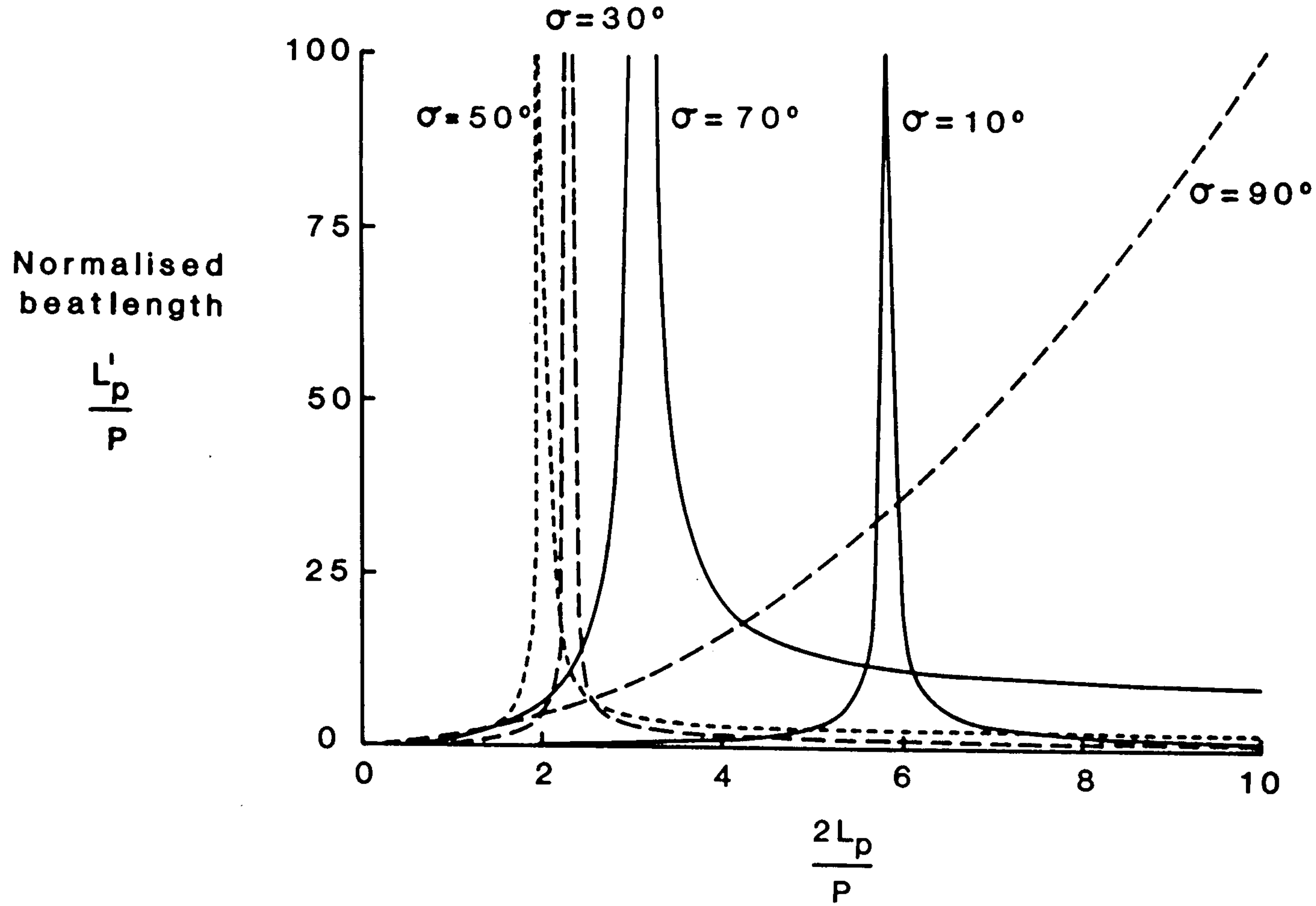
Normalised beatlength
$\frac{L'_p}{P}$
100
75
50
25
0
0
2
4
6
8
10
$\frac{2L_p}{P}$
σ=30°
σ=50°
σ=70°
σ=10°
σ=90°

Fig 2