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Ray theory of graded non-circular optical fibres

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Both bound and tunnelling rays exist in multimode graded fibres which do not have circular symmetry; however, the fraction of energy in the tunnelling rays is generally less than that in a similar circular fibre, especially for near-parabolic profiles. For a large class of non-symmetric profiles, no tunnelling rays can propagate.

1. Introduction

Usually analyses of graded-index optical fibres assume that the refractive index profile within the core is of the form $n = n(r)$, i.e. that the contours of constant index are circles. In practice, however, the fibre manufacturing process often introduces some deviation from circular symmetry. It is important to understand the effect this has on propagation, and, in particular, on the loss of leaky modes. The latter can influence the calculation of refractive indices in the near-field method of profile measurement [1].

The leaky mode loss has been investigated for particular profiles using mode theory [2, 3]. In this paper we use geometric optics to analyse light propagation in fibres with refractive index profile $n(x, y)$, where x and y are cartesian co-ordinates in the cross-section. This method is very accurate [4] for multimode fibres with typical parameters.

We shall study the propagation of the various classes of rays in non-circular fibres. Our approach emphasizes the physical mechanisms involved, whereas the mode approach tends to obscure them. We shall see that the proportion of tunnelling rays is often relatively low in fibres which lack circular symmetry.

The ray path $x = x(z), y = y(z)$ must satisfy the ray equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s}\right) = \nabla n \tag{1}$$

where $\mathbf{R}$ is the position vector of a point on the ray path, and s is the length along the path. If we let $\theta(x, y)$ be the angle a ray makes with the z (fibre) axis, we see that

$$\tilde{\beta} = n\frac{\mathrm{d}z}{\mathrm{d}s} = n(x, y)\cos\theta \tag{2}$$

is an invariant. Note that $\beta = k\tilde{\beta} = 2\pi\tilde{\beta}/\lambda$ is the modal propagation constant, where λ is the wavelength. If we denote differentiation with respect to z by a dot, then at any point on a ray with invariant $\tilde{\beta}$ we have

$$\tilde{\beta}^2 = \frac{n^2(x, y)}{1 + \dot{x}^2 + \dot{y}^2}.$$

From Equations 1 and 2

$$\ddot{x} = \frac{\mathrm{d}^2x}{\mathrm{d}z^2} = \frac{1}{2\tilde{\beta}^2}\frac{\partial}{\partial x}n^2 \tag{3a}$$

$$\ddot{y} = \frac{1}{2\tilde{\beta}^2}\frac{\partial}{\partial y}n^2. \tag{3b}$$

Let us define n_0, n_{cl} such that $n(x, y)$ decreases from a value n_0 on the axis to a value n_{cl} at the core–cladding interface; in the cladding $n(x, y) = n_{cl}$. For convenience we take

$$\gamma^2 = n_0^2 - n_{cl}^2 \tag{4}$$

and introduce a normalized $\tilde{\beta}$

$$B = \frac{n_0^2 - \tilde{\beta}^2}{\gamma^2}. \tag{5}$$

It is now useful to specify the three types of rays:

(a) bound rays, which, in a lossless fibre, propagate unattenuated. They are exactly those having $n_{cl} \leqslant \tilde{\beta} \leqslant n_0$, i.e. $0 \leqslant B \leqslant 1$.

(b) tunnelling rays, which also remain in the core, but must have $\tilde{\beta} < n_{cl}$, i.e. $B > 1$. The field associated with a tunnelling ray extends into the cladding, and it loses energy from a radiation surface outside the core [5, 6].

(c) refracting rays, which are lost when they impinge upon the core–cladding interface.

2. Separable cases

For the class of profiles of form

$$\begin{aligned} n^2(x, y) &= n_0^2 - [f(x) + g(y)]; \quad f(x) + g(y) \leqslant \gamma^2 \\ &= n_{cl}^2 \qquad\qquad\qquad ; \quad f(x) + g(y) \geqslant \gamma^2 \end{aligned} \tag{6}$$

we have

$$\ddot{x} = -\frac{1}{2\tilde{\beta}^2}\frac{\partial f}{\partial x} \quad \text{and} \quad \ddot{y} = -\frac{1}{2\tilde{\beta}^2}\frac{\partial g}{\partial y}.$$

Hence the differential equations are uncoupled, and we find $x(z)$ and $y(z)$ from

$$z = \tilde{\beta}\int \frac{dx}{[f(x_m) - f(x)]^{1/2}}$$

and

$$z = \tilde{\beta}\int \frac{dy}{[g(y_m) - g(y)]^{1/2}} \tag{7}$$

where x_m and y_m are the maximum values of x and y, respectively. If we define z_{px} as the change in z corresponding to a change in x from 0 to x_m then

$$z_{px} = \tilde{\beta}\int_0^{x_m} [f(x_m) - f(x)]^{-1/2}\, dx \tag{8}$$

z_{py} is similarly defined.

For example, if $f(x) = e^2 x^p$, where e is a constant, then

$$z_{px} = \frac{\tilde{\beta}}{pe} x_m^{1-(p/2)} \frac{\pi^{1/2}\Gamma(1/p)}{\Gamma(1/p + \frac{1}{2})}. \tag{9}$$

Thus, unless z_{px}/z_{py} is the ratio of two small integers, the motions in the x- and y-directions are independent, and the ray projection in the cross-section will fill the rectangle $|x| \leqslant x_m$, $|y| \leqslant y_m$.

Each time $(x, y) = (x_m, y_m)$ we have $\dot{x} = \dot{y} = 0$ (i.e. $\theta = 0$) and so $\tilde{\beta} = n(x_m, y_m)$. The lowest possible value of $n(x_m, y_m)$ is clearly n_{cl}, so in structures of the form of Equation 6, all rays with $\tilde{\beta} < n_{cl}$ eventually hit the core–cladding interface, and are lost by refraction. They are 'slowly refracting'; this contrasts with refracting rays in circular structures, which are lost in the first cycle. Thus, strictly speaking, there are no tunnelling rays in 'separable' profiles, as tunnelling rays must have $\tilde{\beta} < n_{cl}$.

3. Parabolic-index fibres with elliptical contours

Here we have

$$n^2(x, y) = n_0^2 - \frac{\gamma^2}{\rho^2}(x^2 + b^2 y^2); \quad x^2 + b^2 y^2 \leqslant \rho^2$$

$$= n_0^2 - \gamma^2 \frac{r^2}{\rho^2}[1 + (b^2 - 1)\sin^2\psi] \tag{10}$$

so the path is

$$x = x_m \sin\left(\frac{\gamma z}{\rho\tilde{\beta}} + \psi_1\right)$$

$$y = y_m \sin\left(\frac{\gamma b z}{\rho\tilde{\beta}} + \psi_2\right) \tag{11}$$

and

$$z_{px} = \frac{\pi\rho\tilde{\beta}}{2\gamma} = b z_{py}.$$

x_m, y_m, ψ_1, ψ_2 can be determined from the initial conditions. If $b \neq 1$, then each bound ray maps out a rectangle. If $b = 1$, then the ray path is an ellipse with semi-minor and major axes r_{min} and r_{tp} where

$$r_{min}^2 + r_{tp}^2 = x_m^2 + y_m^2.$$

In fact

$$\left(\frac{r_{min}}{\rho}\right)^2 + \left(\frac{r_{tp}}{\rho}\right)^2 = B \tag{12}$$

where B is defined in Equation 5. Clearly, a ray is bound if and only if the left-hand side of Equation 12 is less than 1. The further b is from 1, the more quickly the projèction ellipse appears to rotate.

For b close to unity, we find that after a length

$$z_e = \frac{\pi\rho n_0}{\gamma|b-1|} \tag{13}$$

all the 'slowly refracting' rays have been lost, and only bound rays remain. For typical values ($n_0 = 1.5$, $\gamma = 0.15$ are used in numerical results in this paper), $z_e = 15$ cm for 1% ellipticity. Thus after a short length, propagation effectively involves bound rays only. Note that this loss mechanism differs from that suggested in [3].

4. Ray classes in circularly symmetric profiles

For comparison purposes, we briefly describe ray types in circularly symmetric fibres. There is a second invariant [7]

$$\tilde{l} = (r/\rho)n(r)\sin\theta(r)\cos\phi(r).$$

Here $\phi(r)$ is the angle between the ray projection through a point P in the cross-section, and the line in the cross-section perpendicular to the line joining P to the centre. The ray path projection is contained within the annulus bounded by inner and outer caustics r_{min} and r_{tp}, which, in graded fibres, are the roots $(r < \rho)$ of

$$n^2(r) - \tilde{\beta}^2 - \tilde{l}^2(\rho^2/r^2) = 0. \tag{14}$$

In the step fibre, every ray reaches the interface; angles θ and $\phi(\rho)$ are conserved, and $r_{min} = \rho\cos\phi(\rho)$.

Rays with $\tilde{l} = 0$ have $n(r_{tp}) = \tilde{\beta}$ and are called meridional since they cross the fibre axis ($r_{min} = 0$). These cannot be tunnelling, because $n(r_{tp}) > n_{cl}$. As we shall see in Sections 5 and 6, there is a related class of rays in elliptical fibres. Tunnelling rays have $\tilde{\beta} < n_{cl}$, but remain entirely within the core; they have a third root $r_{rad} (> \rho)$ of Equation 14.

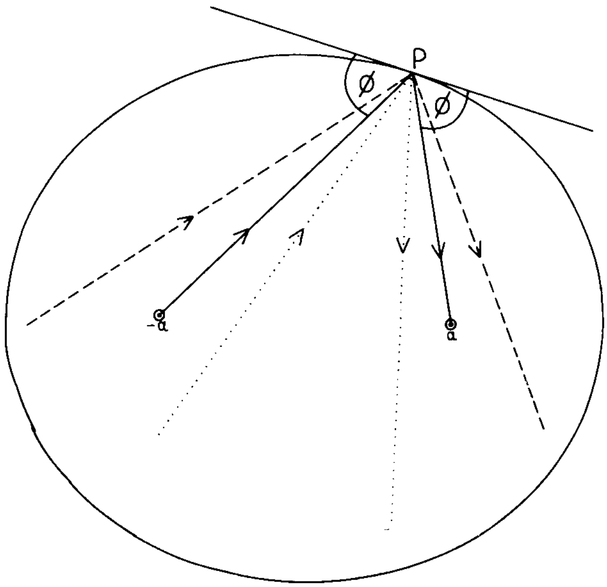

Figure 1 Ray classes in step ellipse fibre. Rays passing through both foci (solid line) form the borderline between the 'elliptic' (E) type (dashed) and the 'hyperbolic' (H) type (dotted). Rays of the H type always cross the major axis between the foci, and have no inner caustics. Rays of the E type cross the major axis between the foci and the interface; their inner caustics are ellipses.

For 'power law' profiles

$$n^2(r) = n_0^2 - \gamma^2(r/\rho)^q = n_0^2[1 - 2\Delta(r/\rho)^q]; \quad r \leqslant \rho \tag{15}$$

the maximum possible B for a tunnelling ray is $(1 + \frac{1}{2}q)$. For a Lambertian source, the ratio of tunnelling to bound power is initially 1/3 for a parabolic fibre, and 1 for a step fibre.

5. Elliptical step fibre

Ray congruences in an elliptic domain have been studied in connection with eigenvalue problems [8].

Consider the projections of rays reflected from an arbitrary point on an elliptical step fibre, as shown in Fig. 1. Let ϕ_1 be the angle the ray makes with the tangent at P and let a and $-a$ be the focal points or generators of the ellipse. If the ellipse is represented by

$$x^2 + b^2y^2 = \rho^2$$

then

$$\frac{a^2}{\rho^2} = 1 - \frac{1}{b^2}. \tag{16}$$

The tangent at any point P makes equal angles with the lines joining it to the foci; thus any ray passing through a focus point also passes through the other focus after reflection, and in general will ultimately cover the whole ellipse. A ray which crosses the major axis at some point between the two foci, such as the dotted one in Fig. 1, makes an angle greater than ϕ at P, and so, after reflection, again crosses the major axis between the foci. This 'hyperbolic' type ray fills the region delineated by the ellipse on top and bottom, and by two hyperbolae on the sides. At each intersection of a hyperbola and an ellipse, $\phi_1 = \pi/2$, indicating that $\tilde{\beta} > n_{cl}$. Thus these rays are analogous to meridional rays in the circular case. A ray of the 'elliptic' type (shown dashed in Fig. 1) makes an angle less than ϕ at P, and so always crosses the major axis at a point $|x| > a$. It is contained between the interface ellipse and an inner caustic ellipse. In this case, ϕ_1 is never $\pi/2$, so this type can be tunnelling or bound. For any degree of ellipticity, both ray classes can exist in the step, because the foci are always within the core.

6. Power law ellipse profiles

We now investigate power law profiles where the index contours are ellipses:

$$n^2(x,y) = n_0^2 - \gamma^2 \left(\frac{x^2}{\rho^2} + b^2 \frac{y^2}{\rho^2} \right)^{q/2} . \tag{17}$$

For $b = 1$ (circular) this reduces to Equation 15. Note that b is the length of the major (x-) axis divided by that of the minor (y-) axis for any index contour. Given the fibre parameters, the system of coupled differential equations obtained using Equations 17 and 3 together with the initial values of x, y and their derivatives is solved numerically, in cartesian co-ordinates, using a fourth order method for accuracy. We find that there are two classes of rays:

(a) 'elliptic' (E), where the inner and outer caustics are ellipses (see Fig. 2). As with skew rays in circular fibres, θ is never zero, so these can be bound or tunnelling. For a given ray, the ellipses are generally almost confocal.

(b) 'hyperbolic' (H), where there is no inner caustic; the outer boundary consists of arcs of hyperbolae and ellipses, as shown in Figs. 3 and 4. Again this type is analogous to meridional rays in circular graded fibres; at a corner point (x_1, y_1) we have $\theta = 0$ and so

$$\tilde{\beta} = n(x_1, y_1) \geqslant n_{cl}$$

thus $0 \leqslant B \leqslant 1$, and this type can only be bound. The ellipse and hyperbolae intersect roughly at right angles, implying that they are approximately confocal. The intersection corners can be seen in Figs. 3 and 4. If B is increased to 1, then the corners touch the core–cladding interface; if $B > 1$ the corners would enter the cladding, indicating that such an H-type ray cannot propagate.

When q is close to 2, and b is not too close to 1 (see Fig. 5), only the hyperbolic-type rays can exist. This means that essentially only bound rays propagate, because the 'slowly refracting' rays are lost within a short distance of the source.

As the fibre approaches circularity ($b = 1$ line), the two caustics become more circular (Section 4), whereas as we approach the line $q = 2$, the outer caustic becomes more rectangular (Section 3). Below

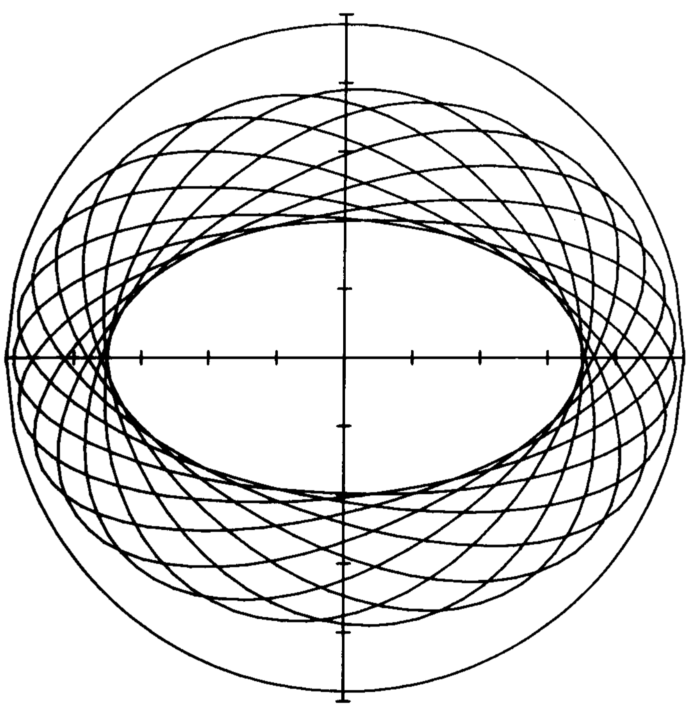

Figure 2 Example of elliptic type ray in profile of the form of Equation 17. The ray traces out its path in the region between the two (almost confocal) caustic ellipses. Here $b = 1.03$, $q = 2.4$ and $B = 1.12$. Note that this ray is tunnelling because $B > 1$ (see Section 1).

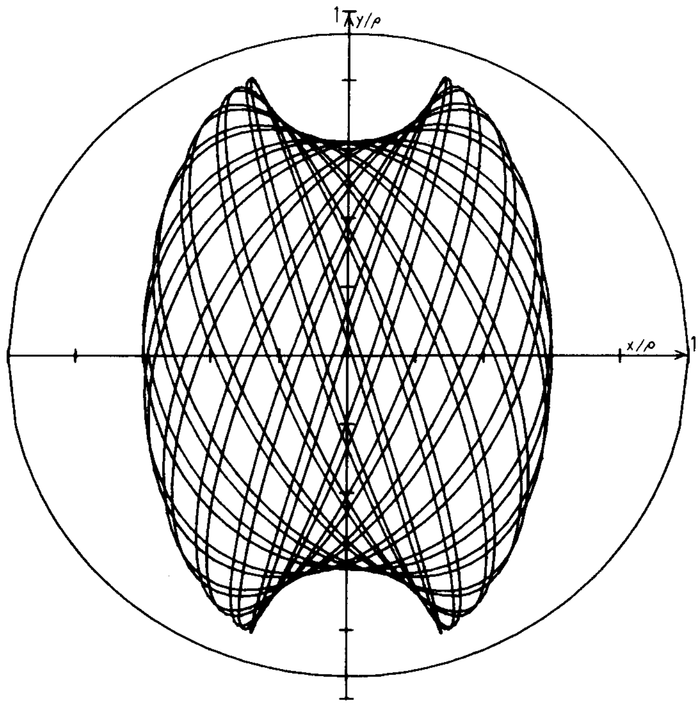

Figure 3 Example of hyperbolic type ray ($q < 2$). The ray occupies the region delineated by hyperbolae and an ellipse. There is no inner caustic. This type can only be bound. Here $b = 1.07$, $q = 1.7$ and $B = 0.852$.

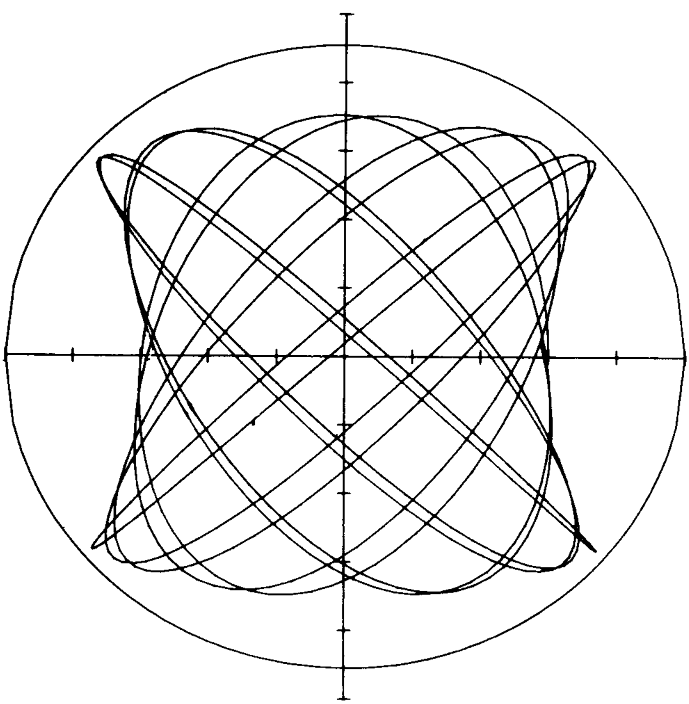

Figure 4 **Example of hyperbolic type ray ($q > 2$). Here $b = 1.1$, $q = 2.3$ and $B = 0.927$.**

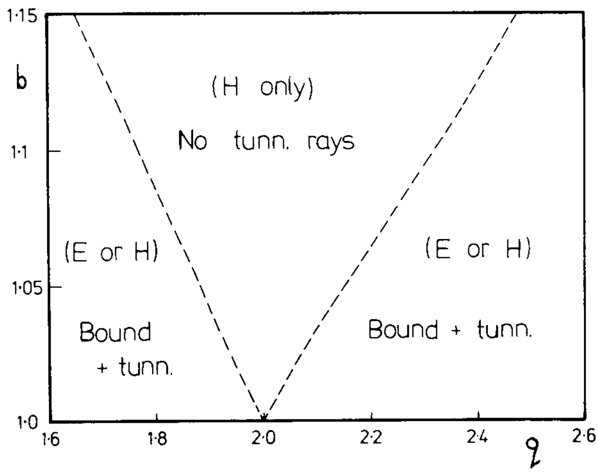

Figure 5 Possible ray types for various (b, q) values; b is the ratio of major to minor axis length for any contour, and q is the exponent (see Equation 17). Above the dividing curve only hyperbolic (H) type rays exist, and hence no tunnelling rays. Below this curve, elliptic (E) type rays can also exist.

the dividing line in Fig. 5, both ray types can exist, although just below this line there are relatively few of the elliptic type, indicating that the fraction of power in tunnelling rays is quite low. For $b > 1$ the maximum B is less than $(1 + \frac{1}{2}q)$.

7. Discussion and implications

The loss of circular symmetry tends to reduce the ratio of tunnelling to bound power. The effect is most significant in profiles of 'separable' form (Equation 6), as these do not support tunnelling rays, and also in profiles close to parabolic ($q = 2$); a 1% or 2% deviation from circularity can then be sufficient to eliminate all tunnelling rays. This is just the main region of interest in practice, as the optimum profile for the minimization of pulse dispersion is close to parabolic [7]. When further from $q = 2$, tunnelling rays cease to exist only for greater deviations.

By using the Hermite–Gaussian modes, it has been found [2] that the leaky mode attenuation in the $q = 2$ case is extremely high; coupled mode theory shows that, when q is very close to 2, this attenuation is also quite high [3]. The ray explanation presented here is more physically intuitive, as well as being more general.

In conclusion, the presence of leaky modes in non-circular fibres depends critically on the specific form of the index profile. The near-field scanning technique can be used to determine index profiles [1]. In cases where the leaky mode effect is reduced in comparison with the circular case, the leaky mode correction factor [9] used should reflect this. However, it should be noted that experimental evidence indicates that the full correction factor is required in many fibres.

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