23

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Pulse dispersion in a lens-like medium

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An analysis is given of dispersion in an optical fibre waveguide having a continuous radial variation of refractive index using the scalar wave approximation. Solutions are presented for the particular case of Selfoc fibre taking into account mode dispersion, material dispersion and group delay. It is shown that for a correctly matched input Gaussian beam the pulse dispersion is small although in practice it is likely to be ~ 1 ns over a length of 1 km.

1. Introduction

Computation of the bandwidth to be expected in single-mode cladded fibres [1, 2] predicts values of tens of gigahertz over tens of kilometres although experimental measurements have not yet been made. On the other hand for multimode cladded fibres dispersions corresponding to bandwidths approaching a gigahertz per kilometre have been measured [3] with a mode-locked helium/neon laser and are in agreement with theory when the lossy nature of the cladding is taken into account [4, 5]. Another type of fibre is that having a continuous variation in refractive index, of which Selfoc [6] is the best known variety, for which some experimental measurements have been made [7, 8]. This paper presents the corresponding theoretical analysis.

2. Field distribution in a lens-like medium

Consider a continuous cylindrical medium (r, θ, z) in which the dielectric constant varies only in the transverse (radial) direction r as

$$\epsilon(r) = \epsilon(0)[1 - (gr)^2 + b(gr)^4] \tag{1}$$

where $\epsilon(0)$ is the dielectric constant on the axis and b, g are constants. Using a scalar wave approximation the wave equation for the transverse electric field E in an axially symmetric system is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E}{\partial r}\right) + \frac{\partial^2 E}{\partial z^2} + k^2(0)\,[1 - (gr)^2 + b(gr)^4]\,E = 0 \tag{2}$$

where $k(0) = \omega[\mu\epsilon(0)]^{\frac{1}{2}}$.

If bg^4 in Equation 2 is small then the equation can be solved using the stationary perturbation theory. By putting $bg^4 = 0$ and after separation of the variables it is found that the characteristic mode is of Laguerre-Gaussian form and may be written

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23

$$E'_n(r, z) = \frac{1}{\sqrt{(\pi)}\, w_0} L_n\left(\frac{r^2}{w_0^2}\right) \exp\left(-\frac{r^2}{2w_0^2}\right) \exp\left(-j\beta_n^{(1)} z\right) \tag{3}$$

where

$$\beta_n^{(1)} = [k^2(0) - 2(2n+1)\, gk(0)]^{\frac{1}{2}}\,. \tag{4}$$

$w_0 = [gk(0)]^{-\frac{1}{2}}$ is the characteristic spot size for the fibre and $L_n(r)$ is a Laguerre polynomial of degree n. The three lowest degree polynomials are

$$L_0(x) = 1$$
$$L_1(x) = 1 - x$$
$$L_2(x) = 1 - 2x + \tfrac{1}{2}x^2\,.$$

The transverse field distribution of mode n in the perturbed medium can now be expressed in terms of the normal mode of the unperturbed medium. It is assumed in what follows that the perturbation changes only the phase constant and that the transverse field distribution remains the same as that for $bg^4 = 0$. For the first order perturbation the characteristic modes $E_n(r, z)$ of Equation 2 become (see for example, reference [9]):

$$E_n(r, z) = \pi^{-\frac{1}{2}} w_0^{-1} [L_n(r^2/w_0^2) + v_n] \exp(-r^2/2w_0^2) \exp(-j\beta_n z) \tag{5}$$

where

$$\beta_n = k(0)[1 - 2(2n+1)\, g/k(0) + 2(3n^2 + 3n + 1)\, bg^2/k^2(0)]^{\frac{1}{2}} \tag{6}$$

$$v_n = \sum_{p=0,\, p\neq n}^{\infty} A'_{np} L_p \left(\frac{r^2}{w_0^2}\right)$$

$$A'_{np} = \frac{b}{4(n-p)\, k(0)\, g^3} \left[\delta_{n,p-2} - 2(3p-1)\, \delta_{n,p-1} - \frac{2(3p+2)\,[(p+1)!]^2}{(p!)^2} \delta_{n,p+1} + 2\,\frac{(p+2)!^2}{(p!)^2}\,\delta_{n,p+2} \right]$$

and δ_{np} = Kronecker delta.

It can be shown that v_n is small compared with the corresponding Laguerre polynomial and may therefore be neglected. Thus in terms of the normal mode the total field amplitude in the perturbed medium may be written

$$E(r, z) = \sum_{n=0}^{\infty} A_n E_n(r, z)\,. \tag{7}$$

Here A_n defines the amplitudes of the various modes and is determined by the radial intensity distribution of the input beam.

At the input end of the fibre $z = 0$ and

$$E(r, 0) = \sum_{n=0}^{\infty} A_n E_n(r, 0)\,. \tag{8}$$

Assuming that the input beam is launched centrally along the axis of the fibre then only the axially symmetric modes need be considered. If, as is normally the case, the input beam has a Gaussian spatial distribution then

$$E(r, 0) = \left(\frac{1}{\pi^{\frac{1}{2}} w_i}\right) \exp\left(-\frac{r^2}{2w_i^2}\right) \tag{9}$$

where w_i is the spot size of the beam. Since the normal modes form an orthogonal set it is possible to solve for A_n from Equations 8 and 9,

$$A_n = \frac{2\pi^{\frac{1}{2}}}{w_0} \int_0^\infty E(r, 0)\, L_n\left(\frac{r^2}{w_0^2}\right) \exp\left(-\frac{r^2}{2w_0^2}\right) r\, dr = \frac{2W}{1 + W^2}\left(\frac{W^2 - 1}{W^2 + 1}\right)^n \tag{10}$$

where $W = w_0/w_i$.

If the input beam is correctly matched to the fibre then $w_0 = w_i$ so that $W = 1$ and only the $n = 0$ mode in the fibre is excited.

3. Pulse broadening of mode of order *n*

The field distribution of mode n is given by Equation 5 which can be re-written as follows:

$$E_n(r, z) = E_n(r, 0) \,.\, S_n(\omega) \tag{11}$$

where

$$S_n(\omega) = \exp\left[-j\, \beta_n\, z\right]. \tag{12}$$

$S_n(\omega)$ may be treated as a transfer function which completely characterizes the performance of the linear system. Thus when an input pulse, which may be expressed in terms of the electric field as $f(t)$ is incident upon the guide, then the output pulse $q_n(t)$ is given by

$$q_n(t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F(\omega)\, S_n(\omega) \exp(j\omega t)\, d\omega \tag{13}$$

where

$$F(\omega) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(t) \exp(-j\omega t)\, dt \tag{14}$$

and ω is the circular frequency of the light wave. Measurements on mode-locked pulses from a helium/neon laser show that they have very nearly a Gaussian distribution, of frequency 5×10^{14} Hz and of about 1.5×10^9 Hz half-width so that $f(t)$ may be taken as the Gaussian function:

$$f(t) = \frac{\sqrt{(2)}}{a} \exp\left(-t^2/a^2 + j\omega_c t\right). \tag{15}$$

The Fourier transform $F(\omega)$ is

$$F(\omega) = \exp\left[-a^2(\omega - \omega_c)^2/4\right]. \tag{16}$$

Substituting (12) and (16) in (13) gives

$$q_n(t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \exp\left[\frac{-a^2(\omega-\omega_c)^2}{4}\right] \exp(-j\beta_n z + j\omega t)\, d\omega. \tag{17}$$

As it is very difficult to solve (17) directly, the propagation constant β_n is expanded as a series; see Equation 6

$$\beta_n = k(0) - (2n+1)\,g - (g^2/k(0))\,[2(1 - 3/2b)(n^2+n) + (\tfrac{1}{2} - b)]\,. \tag{18}$$

In practice for Selfoc fibre g is of the order of 1 to 10^{-1} mm^{-1} and $k(0)$ is of the order of 10^4 mm^{-1}, so that the fourth and higher-order terms in Equation 18, which are not shown, are small compared to the unperturbed value and can be neglected in the calculation. The first component of the third-order term is proportional to n^2 and gives rise to mode conversion. The reason for this is that when the power distribution at a distance z from the input is calculated by summing the amplitudes of the individual modes using Equation 7, the terms in n^2 cause a deviation from a Gaussian distribution. Equation 18 is expanded as a Taylor series about ω_c to give

$$\beta_n = \sum_{m=0}^{\infty} \frac{(\omega-\omega_c)^m}{m!} \frac{d^m}{d\omega^m} [k(0) - (2n+1)\,g - (g^2/k(0))$$
$$\{2(1 - 3/2b)(n^2+n) + (\tfrac{1}{2} - b)\}]_{\omega=\omega_c}\,. \tag{19}$$

As can be seen from Fig. 1 the refractive index is *nearly* a linear function of ω so that

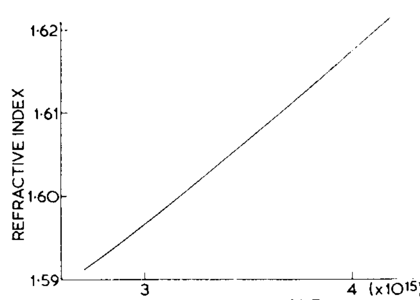

Figure 1 The variation of refractive index with angular frequency for Selfoc glass.

values of $m > 2$ can be neglected in Equation 19. Thus the phase constant may be written

$$\beta_n = \sum_{m=0}^{2} \frac{(\omega-\omega_c)^m}{m!} \alpha_{m,n} \tag{20}$$

where

$$\alpha_{m,n} = \frac{d^m}{d\omega^m}\left[k(0) - (2n+1)g - \frac{g^2}{k(0)} \{2(1 - \tfrac{3}{2}b)(n^2+n) + (\tfrac{1}{2} - b)\}\right]_{\omega=\omega_c} \tag{21}$$

Substitution of (20) into (17) gives

$$q_n(t) = \frac{1}{\sqrt{\left(\frac{a^2}{2} + jz\alpha_{2n}\right)}} \exp[-jz\alpha_{0n} + j\omega_c t] \exp\left\{-\frac{(z\alpha_{1n} - t)^2}{4\left(\frac{a^2}{4} + jz\frac{\alpha_{2n}}{2}\right)}\right\} \cdot \quad (22)$$

The shape of the detected output pulse contributed by mode n is therefore

$$|q_n(t)|^2 = \frac{2}{a^2} \cdot \frac{1}{\sqrt{\left[1 + \left(\frac{2z\alpha_{2n}}{a^2}\right)^2\right]}} \exp\left[-\frac{2(t - z\alpha_{1n})^2}{a^2\left(1 + \left(\frac{2z\alpha_{2n}}{a^2}\right)^2\right)}\right] \cdot \quad (23)$$

Equation 23 shows that during transmission the pulse is reduced in amplitude by the factor $1/\sqrt{[1 + (2z\alpha_{2n}/a^2)^2]}$ and the delay time $\Delta\tau$ relative to a plane wave is

$$\Delta\tau = +z\alpha_{1n} - \frac{n_0 z}{c} \quad (24)$$

where n_0 is the refractive index at ω_c and c is the velocity of light in a vacuum.

4. Total pulse dispersion

Using Equation 23 the pulse dispersion of the energy propagating in a particular mode can be calculated. However, when light is launched into a fibre the energy is, in general, distributed between many different modes. A number of factors therefore have to be considered. If we assume a laser (monochromatic) source is used, it is necessary to take into account, even for a single mode, both mode dispersion and also the dispersion of the bulk fibre material. Then, when more than one mode is present, it is necessary to determine the dispersion due to group delay differences between the modes. In order to do this the distribution of energy between modes must be found. For a single-mode laser this can conveniently be obtained from Equations 9 and 10 via the mode amplitude A_n.

Thus the total pulse temporal distribution is given by

$$|q(t)|^2 = \sum_{n=0}^{\infty} |A_n|^2 |q_n(t)|^2$$

$$= \frac{4W^2}{(1 + W^2)^2} \sum_{n=0}^{\infty} \left[\frac{W^2 - 1}{W^2 + 1}\right]^{2n} \frac{2}{a^2} \cdot \frac{1}{\sqrt{\left[1 + \left(\frac{2z\alpha_{2n}}{a^2}\right)^2\right]}} \cdot \exp\left[-\frac{2(t - z\alpha_{1n})^2}{a^2\left(1 + \left(\frac{2z\alpha_{2n}}{a^2}\right)^2\right)}\right] \quad (26)$$

where

$$\alpha_{1n} = \left\{\frac{dk}{d\omega} - \frac{dg}{d\omega}(2n + 1)\right\}_{\omega = \omega_c} + \{2(1 - 3/2b)(n^2 + n) + (\tfrac{1}{2} - b)\}_{\omega = \omega_c}$$
$$\{g\,k(0)^{-1}\}_{\omega = \omega_c} \left\{g\,k(0)^{-1}\frac{dk}{d\omega} - 2\frac{dg}{d\omega}\right\}_{\omega = \omega_c} \quad (27)$$

and

$$\alpha_{2n} = \left\{\frac{\mathrm{d}^2k}{\mathrm{d}\omega^2} - \frac{\mathrm{d}^2g}{\mathrm{d}\omega^2}(2n+1)\right\}_{\omega=\omega_c} + \{2(1-3/2b)(n^2+n) + (\tfrac{1}{2}-b)\}_{\omega=\omega_c}$$

$$\left\{g^2k^{-2}\frac{\mathrm{d}^2k}{\mathrm{d}\omega^2} - 2gk^{-1}\frac{\mathrm{d}^2g}{\mathrm{d}\omega^2} + 4gk^{-2}\frac{\mathrm{d}k}{\mathrm{d}\omega}\frac{\mathrm{d}g}{\mathrm{d}\omega} - 2g^2k^{-3}\left(\frac{\mathrm{d}k}{\mathrm{d}\omega}\right)^2 - 2k^{-1}\left(\frac{\mathrm{d}g}{\mathrm{d}\omega}\right)^2\right\}_{\omega=\omega_c} \quad (28)$$

As it is very difficult to calculate this summation we have to use the computer.

The foregoing equations have been used to compute pulse dispersions for a variety of launching conditions and fibre parameters.

5. Computed pulse dispersions

Results are presented for the case of a monochromatic source, namely the helium/neon laser operating at an angular frequency $\omega_0 = 2.979 \times 10^{15}$ and with the beam waist at the entry face of the fibre which is assumed flat and perpendicular to the axis. The beam is launched in an axial direction and is concentric with the fibre axis. The half-width of the input pulses is taken as $2a = 0.65$ ns and the parameters for Selfoc glasses are given by $g = 1.77$ mm^{-1} and Fig. 1. It will be seen that general conclusions can also be drawn on the pulse dispersions to be expected from any distribution of input energy.

5.1. Optimum launching

If the spot size w_i of the input Gaussian beam is equal to the characteristic spot size w_0 of the lens-like medium then $W = 1$ and only the single fundamental mode $E_0(r, z)$ is excited in the fibre. Defining pulse duration in terms of the intensity at half maximum the pulse dispersion Δ for this case is given by

$$\Delta = 2a\left\{\left[1 + \left(\frac{2z\alpha_{20}}{a^2}\right)^2\right]^{\frac{1}{2}} - 1\right\}. \quad (29)$$

Equations 23 and 29 show that for the case $a^2 = 2z\alpha_{20}$ the output pulse width has the minimum value $2\sqrt{(2)}a \simeq 2.8a$. This result has also been obtained for single-mode cladded fibres [1].

Fig. 2 shows the pulse dispersion in the single-mode (Selfoc) case $W = 1$ as a function of the radial variation parameter b for zero and actual bulk material dispersion (from Fig. 1). The amount of pulse broadening in the former case depends strongly on b and is zero when $b = \frac{1}{2}$. However, this value of b is not the ideal one since it gives rise to a certain amount of mode conversion and from this rather more important point of view the optimum value is $b \simeq 0.67$. Nevertheless the argument is perhaps an academic one in that for the wide range of values of b shown the pulse dispersion is, in any case, negligible and is completely dominated in practice by the effects of material dispersion.

5.2. Input Gaussian beam of non-optimum diameter

When the input beam width is not matched to that of the fibre then $W \neq 1$ and higher-order modes are excited. The pulse dispersion is now determined by the group delay

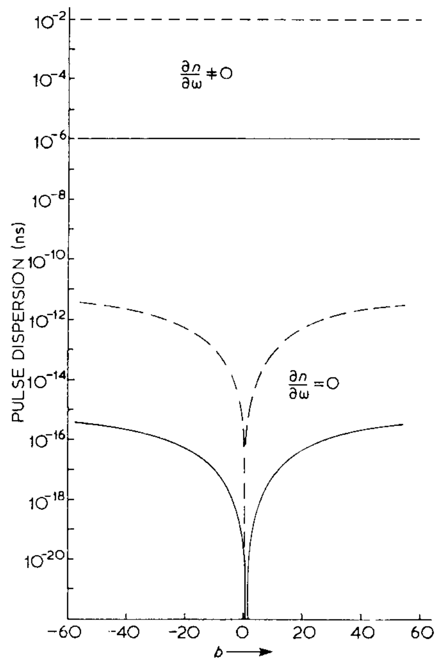

Figure 2 Variation of pulse dispersion with parameter 'b'. The lower, and upper pairs of curves are for zero and finite bulk dispersion respectively. The solid lines are for a length of 1 km and the dashed lines are for 100 km.

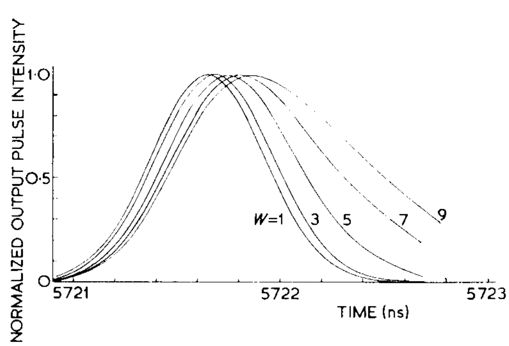

Figure 3 Normalized output pulse shapes for a length of 1 km and $b = 1$ and including the effect of material dispersion.

between modes and Fig 3 shows, for a fibre length of 1 km and $b = 1$, the rapid broadening of the output pulse that occurs as W increases. The curve for $W = 1$ shows, effectively, the shape of the input pulse. It may be seen that broadening occurs predominantly on the trailing pulse edge as in the case of multimode cladded fibres [4]. The variation of pulse dispersion with W is given in Fig. 4 for various values of b in a dispersionless fibre. The effect of the larger numbers of modes for departures of W from unity and increasing values of b is clearly evident. The effect of material dispersion in

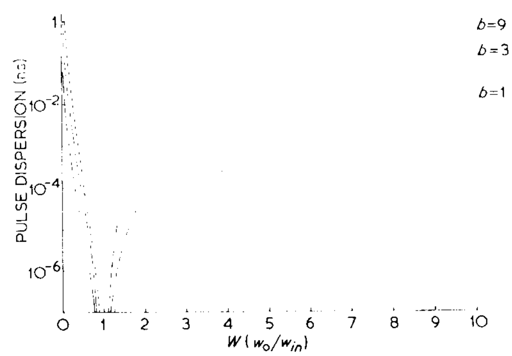

Figure 4 Pulse dispersion as a function of W for a length of 1 km assuming zero material dispersion for various values of 'b'.

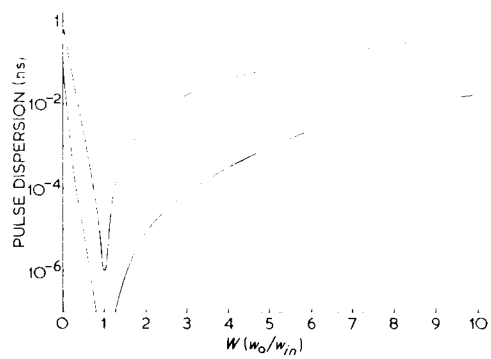

Figure 5 Pulse dispersion as a function of W for $b = 1$ and a length of 1 km. The solid curve assumes zero material dispersion.

Selfoc glass, Fig. 5, is to increase the pulse dispersion over the whole range of W values. It may be seen that any departure of W from the optimum causes a rapid increase in pulse dispersion although it is still less than 10^{-2} ns for the range $0.5 \leqslant W \leqslant 2$.

The effect of the fibre parameter b is further illustrated in Fig. 6 by the effect it has on the relative group delay between the modes. The delay time of each mode compared with the dominant TEM_{00} mode is large compared with the pulse spreading of individual modes.

It is clear therefore that for maximum bandwidth an input beam having a Gaussian distribution and matched to the fibre should be used.

On the other hand for large departures from $W = 1$, as would be caused by non-optimum sources such as multimode (including the GaAs) lasers the pulse dispersion becomes asymptotic to a value between 0.5 and 1 ns over 1 km. It is interesting to note that experimental values of pulse dispersion in Selfoc fibre corresponding to 0.6 ns km^{-1}, and therefore in encouraging agreement with Fig. 4, have been reported [7, 8].

6. Conclusion

It may be concluded that the dispersion of an individual mode is dominated by that of

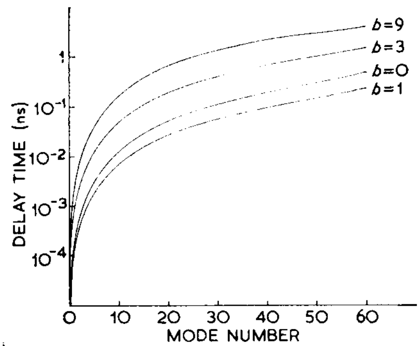

Figure 6 Relative propagation delay as a function of mode number for various values of '*b*' assuming zero material dispersion.

the bulk material and that for maximum bandwidth the input beam must have a Gaussian distribution and be matched to the fibre. However, the group delay between modes greatly exceeds the pulse spreading of individual modes. Thus if the input beam is not ideally matched or if mode conversion occurs in the fibre due to other factors such as, for example, bends, inhomogeneities or non-axial launching then the pulse dispersion is likely to exceed 1 ns km^{-1}.

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