# Astigmatic Gaussian Beams Produced by Axially Asymmetric Laser Cavities

DAVID C. HANNA

*Abstract*—**Gaussian beams generated within astigmatic resonators are themselves astigmatic, having wavefront curvatures and spot sizes that are different when measured in two orthogonal directions. Expressions are derived for the confocal parameters, spot sizes, waist positions, and stability conditions of beams formed within spherical mirror resonators that contain one or more inclined plates (or Brewster-ended laser rods). An expression is also derived for the resonant frequencies of the $\text{TEM}_{mn}$ mode in such a resonator. It is shown that the frequency degeneracy between modes of the same $m + n$ is lifted and the frequency splittings are calculated. The astigmatism produced by prisms used for dispersion is also considered and precautions are described for avoiding serious astigmatism in resonators containing such prisms.**

## Introduction

Several papers have discussed the properties of Hermite–Gaussian modes produced by spherical mirror resonators and also the propagation characteristics of these beams after they emerge from the resonator. A review of the results of these analyses and an extensive bibliography is given by Kogelnik and Li [1]. Although these papers have included a study of the effects of inserting elements such as lenses or more general lens-like devices (e.g., gas lenses and thermally distorted laser rods) into the beam either inside or outside the resonator, the analysis has been mainly restricted to devices having axial symmetry with respect to the beam axis. As a result the beams considered have axial symmetry as far as beam spot size and wavefront curvature is concerned.

In practice, however, many lasers are operated with resonators containing elements that lack axial symmetry, such as inclined plates (e.g., Brewster windows, Brewster-end laser rods) or dispersive prisms for wavelength selection [2]. Recently, a beam waveguide using Brewster-oriented lenses was reported [3]. The effect of these asymmetric devices is to produce astigmatic Gaussian beams, that is, beams that have different spot sizes, wavefront curvatures, and beam-waist positions in two orthogonal directions. This paper investigates the magnitude of some of these astigmatic effects. One effect of the astigmatism is to lift the frequency degeneracy between the $\text{TEM}_{mn}$ and $\text{TEM}_{nm}$ modes and an expression is derived for the resonance frequencies of $\text{TEM}_{mn}$ modes in a spherical mirror resonator containing an inclined plate.

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The author is with the Department of Electronics, University of Southampton, Southampton, England.

The first reference to astigmatic Gaussian beams was made by Collins [4], [5] in an analysis of a ring resonator with spherical mirrors and more complete analyses of such resonators have since appeared [6]–[8]. An analysis has also been made of the astigmatism produced in a Gaussian beam that propagates as an extraordinary wave in anisotropic media [9]. In a ring resonator, the effective curvature of the mirrors in the plane of the ring is different from the curvature perpendicular to the ring on account of the oblique incidence, and Collins' analysis shows that the spot sizes and beam-waist positions are different in these two planes. It can also be shown that the usual stability conditions for the resonator are modified [7], [10], [11]. Rigrod [7] showed this by first describing the optical properties of the resonator by a ray-transfer matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and then using the $ABCD$ law of propagation for Gaussian beams [1]. He showed that for the ring resonator there exist two such ray-transfer matrices, one for the complex beam parameter in the plane of the ring (the $x$ plane say) and one for the complex beam parameter in the plane perpendicular to this (the $y$ plane say) and the beam properties in each plane are calculated separately. The justification for this separate analysis of the $x$ and $y$ behavior of the beam propagation is readily seen. First, the expression for the Gaussian beam is separable in the $x$ and $y$ coordinates. Second, the kernel of the integral equation that describes the beam propagation (using the Fresnel–Kirchoff formulation of Huygens' principle) is separable in the $x$ and $y$ coordinates.

In this paper it is shown that an inclined plate can be described by two ray-transfer matrices, one for the plane of incidence and one for the plane perpendicular to this. Using the $ABCD$ law, it then becomes a simple matter to calculate the beam curvatures, spot sizes, waist positions, and resonance frequencies for a resonator containing an inclined plate.

## Modes of a Resonator Containing an Inclined Plate

In the Appendix a brief description is given of the derivation of the two ray-transfer matrices that describe separately the effect of an inclined plate on the spot sizes and wavefront curvatures in the $yz$ and $zx$ planes (see

Fig. 1). They are, respectively,

$$\begin{bmatrix} 1 & t/(\mu^2 - \sin^2 \theta_1)^{1/2} \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & t\mu^2(1 - \sin^2 \theta_1)/(\mu^2 - \sin^2 \theta_1)^{3/2} \\ 0 & 1 \end{bmatrix}$$

where $t$ is the plate thickness, $\mu$ the plate refractive index, and $\theta_1$ the angle of incidence of the beam at the plate.

Consider the spherical mirror resonator shown in Fig. 2. $R_1$ and $R_2$ are the mirror curvatures (both taken positive if concave, negative if convex ) and the mirror separation is $d$. When the resonator is empty (i.e., the plate is absent,) the Gaussian beam has a waist that is the same for the $xz$ and $yz$ planes, at a distance $t_1$ from mirror 1, and distance $t_2$ from mirror 2 where $t_1$ and $t_2$ are given by [1]

$$t_1 = \frac{d(R_2 - d)}{R_1 + R_2 - 2d} \quad (1)$$

$$t_2 = \frac{d(R_1 - d)}{R_1 + R_2 - 2d}. \quad (2)$$

The confocal parameter $b$ is given by

$$b^2 = \frac{4d(R_1 - d)(R_2 - d)(R_1 + R_2 - d)}{(R_1 + R_2 - 2d)^2} \quad (3)$$

and the spot sizes $\omega_1$ on mirror 1, $\omega_2$ on mirror 2 are given by

$$\omega_1^4 = \left(\frac{\lambda R_1}{\pi}\right)^2 \frac{(R_2 - d)}{(R_1 - d)} \cdot \frac{d}{(R_1 + R_2 - d)} \quad (4)$$

$$\omega_2^4 = \left(\frac{\lambda R_2}{\pi}\right)^2 \frac{(R_1 - d)}{(R_2 - d)} \cdot \frac{d}{(R_1 + R_2 - d)}. \quad (5)$$

The effect of the plate insertion is to constrain the beam to have different beam-waist sizes in the $xz$ and $yz$ planes and also to shift the positions of the beam waists (by different amounts for the $xz$ and $yz$ planes) from the beam-waist position for the empty resonator. The new spot sizes and waist positions are readily calculated by considering the equivalent "unfolded" resonator in which the mirrors are replaced by lenses. The overall ray-transfer matrix for a round trip through the resonator is then calculated separately for the $xz$ and $yz$ planes and the spot sizes and wavefront curvatures at any point in the resonator can then be calculated in terms of the elements of these "round-trip" matrices as shown by Kogelnik and Li [1]. It can easily be shown that these round-trip matrices are independent of the plate position in the resonator (provided the plate is the only element in the resonator); hence the parameters of the beam are also independent of the plate position.

In fact, the form of the ray-transfer matrices for the plate is the same as for the ray-transfer matrix of an

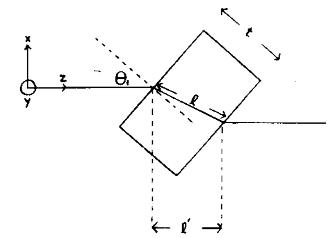

Fig. 1. Gaussian beam propagates in the $z$ direction and makes an angle of incidence $\theta$ at the plate.

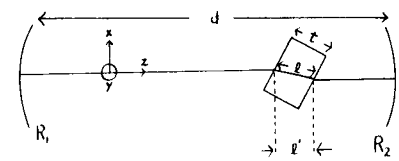

Fig. 2. General spherical mirror resonator containing an inclined plate.

empty space of length $t\mu^2(1 - \sin^2\theta_1)/(\mu^2 - \sin^2\theta_1)^{3/2}$ for the $xz$ plane and an empty space of length $t/(\mu^2 - \sin^2\theta_1)^{1/2}$ for the $yz$ plane. Thus although the beam on passing through the plate from left to right is actually closer to mirror 2 by an amount $l'$ (see Fig. 2), where

$$l' = t(1 - \sin^2 \theta_1)^{1/2} + \frac{t \sin^2 \theta_1}{(\mu^2 - \sin^2 \theta_1)^{1/2}},$$

the effect of the plate in the $yz$ plane is the same as if the beam has only traveled a distance in empty space of $t/(\mu^2 - \sin^2 \theta_1)^{1/2}$ towards mirror 2. The effective length of the resonator in Fig. 2 is thus $d - L_2$ where $L_2 = l' - t/(\mu^2 - \sin^2\theta_1)^{1/2}$ for the properties of the beam in the $yz$ plane, and $d - L_1$ where

$$L_1 = l' - \frac{t\mu^2(1 - \sin^2 \theta_1)}{(\mu^2 - \sin^2 \theta_1)^{3/2}}$$

for the properties of the beam in the $xz$ plane. This means that the usual formulas for spot sizes, beam-waist positions, etc., as given by Kogelnik and Li [1] for the empty resonator are modified in the $yz$ plane simply by replacing $d$ by $d - L_2$ and in the $xz$ plane by replacing $d$ by $d - L_1$.

These results are presented below with the following notation. $(\omega_x)_1$, $(\omega_y)_1$ are the spot sizes in the $xz$, $yz$ planes, respectively, on the surface of mirror 1, and $(\omega_x)_2$, $(\omega_y)_2$ are the spot sizes on mirror 2. $b_x$ and $b_y$ are the confocal parameters in the $xz$, $yz$ planes, respectively, of both the beam to the left of the plate ("left beam") and of the beam to the right of the plate ("right beam"). $(t_x)_1$, $(t_y)_1$ are the distances from mirror 1 to the waists in the $xz$, $yz$ planes, respectively, of the "left beam," $(t_x)_2$ and $(t_y)_2$ are the distances from mirror 2 to the waists of the "right beam."

It is found that

$$(t_x)_1 = \frac{(d - L_1)(R_2 - d + L_1)}{(R_1 + R_2 - 2d + 2L_1)} \quad (6)$$

$$(t_y)_1 = \frac{(d - L_2)(R_2 - d + L_2)}{(R_1 + R_2 - 2d + 2L_2)} \quad (7)$$

where $L_1$ and $L_2$ are given by

$$L_1 = L_2 + t \sin^2 \theta_1 \frac{(\mu^2 - 1)}{(\mu^2 - \sin^2 \theta_1)^{3/2}} \tag{8}$$

$$L_2 = t\left(\cos \theta_1 - \frac{\cos^2 \theta_1}{(\mu^2 - \sin^2 \theta_1)^{1/2}}\right). \tag{9}$$

When $\mu = 1$, that is, when the resonator is empty, both $L_1$ and $L_2$ become 0 and the expressions of (6) and (7) coincide with $t_1$ in (1).

$(t_x)_2$ and $(t_y)_2$ are given by

$$(t_x)_2 = \frac{(d - L_1)(R_1 - d + L_1)}{(R_1 + R_2 - 2d + 2L_1)} \tag{10}$$

$$(t_y)_2 = \frac{(d - L_2)(R_1 - d + L_2)}{(R_1 + R_2 - 2d + 2L_2)}. \tag{11}$$

When $\mu = 1$, (10) and (11) reduce to (2).

$b_x$ and $b_y$ are given by

$$(b_x)^2 = \frac{4(d-L_1)(R_1-d+L_1)(R_2-d+L_1)(R_1+R_2-d+L_1)}{(R_1+R_2-2d+2L_1)^2} \tag{12}$$

$$(b_y)^2 = \frac{4(d-L_2)(R_1-d+L_2)(R_2-d+L_2)(R_1+R_2-d+L_2)}{(R_1+R_2-2d+2L_2)^2}. \tag{13}$$

When $\mu = 1$, (12) and (13) reduce to (3).

$(\omega_x)_1$, $(\omega_y)_1$, $(\omega_x)_2$, and $(\omega_y)_2$ are given by

$$(\omega_x)^4 = \left(\frac{\lambda R_1}{\pi}\right)^2 \frac{(d - L_1)(R_2 - d + L_1)}{(R_1 - d + L_1)(R_1 + R_2 - d + L_1)} \tag{14}$$

$$(\omega_y)_1^4 = \left(\frac{\lambda R_1}{\pi}\right)^2 \frac{(d - L_2)(R_2 - d + L_2)}{(R_1 - d + L_2)(R_1 + R_2 - d + L_2)} \tag{15}$$

$$(\omega_x)_2^4 = \left(\frac{\lambda R_2}{\pi}\right)^2 \frac{(d - L_1)(R_1 - d + L_1)}{(R_2 - d + L_1)(R_1 + R_2 - d + L_1)} \tag{16}$$

$$(\omega_y)_2^4 = \left(\frac{\lambda R_2}{\pi}\right)^2 \frac{(d - L_2)(R_1 - d + L_2)}{(R_2 - d + L_2)(R_1 + R_2 - d + L_2)}. \tag{17}$$

When $\mu = 1$, (14) and (15) reduce to (4), and (16) and (17) reduce to (5).

These results may be readily extended to resonators containing more than one plate provided the plates are not skew with respect to each other (i.e., provided the planes of the plates are parallel or intersect in a line perpendicular to the resonator axis). This follows since the round-trip ray-transfer matrix is then independent of the plate positions. The result is particularly simple for parallel plates of the same refractive index since (6)–(17) are still valid provided $t$ is interpreted as the sum of the plate thicknesses. The new stability conditions can be found from (14)–(17). First, we review the stability conditions for an empty resonator. Thus, suppose the resonator of Fig. 2 is empty and $R_2 > R_1$, then the mirror separation $d$ must lie in the ranges $0 < d < R_1$, and $R_2 < d < R_1 + R_2$. This can be seen to follow from the requirement that $\omega_1$ and $\omega_2$ be real and hence that the right-hand side of (4) and (5) be positive. Similarly, by requiring that the right-hand sides of (14)–(17) be positive and noting that $L_1 > L_2$, the ranges of stable values of $d$ for a cavity containing a plate are

$$L_1 < d < R_1 + L_2, \quad \text{and} \quad R_2 + L_1 < d < R_1 + R_2 + L_2. \tag{18}$$

Just as for the empty cavity, there are two distinct ranges of stability although the extent of these ranges is now reduced (since $L_1 > L_2$) relative to the empty resonator case. The lower limit on $d$ is in fact trivial since the presence of the plate automatically ensures that the mirrors be separated by a distance greater than $L_1$. To give some idea of the magnitude of these astigmatic effects, a few examples are given. Generally the effects are rather small since spot size is not a rapidly changing function of mirror separation except when approaching the unstable condition (as for example in the commonly used hemispherical resonator). Since the stability conditions are different for the $xz$ and $yz$ planes it is then possible to obtain quite different spot sizes for the $xz$ and $yz$ planes.

*Example 1:* A gas laser resonator has mirrors $R_1 = \infty$, $R_2 = 1.5$ meters, operates at wavelength $\lambda = 633$ m$\mu$, and contains a laser tube with Brewster windows 4 mm thick of refractive index 1.5. Table I shows the spot sizes $\omega_1$, $\omega_2$ for the empty resonator, and $(\omega_x)_1$, $(\omega_y)_1$, $(\omega_x)_2$, $(\omega_y)_2$ for the resonator containing the tube for various values of $d$. This example shows that close to the limit of stability there can be appreciable astigmatism even with quite thin Brewster windows. The magnitude of the astigmatism increases with increasing window thickness and can be quite appreciable in resonators containing such devices as Brewster-oriented ultrasonic molulators (e.g., Hargrove *et al.* [12]) dispersive prisms (e.g., Zitter *et al.* [13]), or long Brewster-ended laser rods.

*Example 2:* A solid-state laser resonator has mirrors $R_1 = \infty$, $R_2 = 1.5$ meters, separated by a distance of 1.4 meters, operates at wavelength $\lambda = 1060$ m$\mu$, and contains a Brewster-ended laser rod of length $l$ and of refractive index 1.6. In Table II, the spot sizes are given for the empty resonator and for the resonator with two different lengths of laser rod. Even if the astigmatism of the beam outside the Brewster-angle rod is negligible, the astigmatism within the rod will be large, since inside the rod, the spot size in the $xz$ plane is then $\mu \times$ greater than the spot size in the $yz$ plane. This can be an advantage in a solid-state laser designed to operate on the $TEM_{00}$ mode since the mode volume within the laser rod is $\mu \times$ larger than for a plane-ended rod of the same dimensions in the same resonator. The output energy in the $TEM_{00}$ mode should therefore be enhanced by a factor of $\mu$.

## Mode Frequencies for a Resonator Containing an Inclined Plate

The field $E(x, y, z)$ of a nonastigmatic $TEM_{mn}$ mode propagating in the $z$ direction with confocal parameter $b$ is given by [14]

TABLE I

| $d$ | $\omega_1$ | $\omega_2$ | $(\omega_x)_1$ | $(\omega_y)_1$ | $(\omega_x)_2$ | $(\omega_y)_2$ |
|---|---|---|---|---|---|---|
| 1200 | 0.3477 | 0.7775 | 0.3489 | 0.3482 | 0.7725 | 0.7754 |
| 1300 | 0.3205 | 0.8778 | 0.3225 | 0.3218 | 0.8706 | 0.8748 |
| 1400 | 0.2746 | 1.064 | 0.2783 | 0.2762 | 1.047 | 1.056 |
| 1450 | 0.2329 | 1.276 | 0.2394 | 0.2357 | 1.239 | 1.260 |
| 1475 | 0.1967 | 1.524 | 0.2073 | 0.2013 | 1.442 | 1.487 |
| 1500 | 0 | ∞ | 0.1381 | 0.1106 | 2.182 | 2.73 |

All dimensions in millimeters.

TABLE II

| $l$ | $\omega_1$ | $\omega_2$ | $(\omega_x)_1$ | $(\omega_y)_1$ | $(\omega_x)_2$ | $(\omega_y)_2$ |
|---|---|---|---|---|---|---|
| 75 | 0.3553 | 1.376 | 0.3943 | 0.3737 | 1.214 | 1.296 |
| 300 | 0.3553 | 1.376 | 0.4585 | 0.4140 | 0.9734 | 1.139 |

All dimensions in millimeters.

$$E(x, y, z) = E_0\left[\frac{2}{1+\xi^2}\right]^{1/2} H_m\left[x\left\{\frac{2K}{b(1+\xi^2)}\right\}^{1/2}\right] \cdot H_n\left[y\left\{\frac{2K}{b(1+\xi^2)}\right\}^{1/2}\right] \exp\left\{\frac{-K(x^2+y^2)}{b(1+\xi^2)}\right\} \cdot \exp\left(-j\left\{K\left[\frac{b(1+\xi)}{2} + \frac{\xi(x^2+y^2)}{b(1+\xi^2)}\right] - (1+m+n)\left(\frac{\pi}{2} - \phi\right)\right\}\right) \tag{19}$$

where $z$ is the distance from the beam waist, $\xi = 2z/b$, $\tan\phi = (1-\xi)/(1+\xi)$, and $H_m$ is the Hermite polynomial of order $m$. This expression can be readily generalized to describe an astigmatic $TEM_{mn}$ mode having confocal parameters $b_x$, $b_y$ in the $xz$ and $yz$ planes, respectively. The field at a point described by coordinates $x$, $y$, $z_x$, $z_y$ (where $z_x$ is the distance along the axis from the beam waist in the $xz$ plane, $z_y$ the distance from the waist in the $y_z$ plane), is given by

$$E(x, y, z_x, z_y) = E_0\left[\frac{2}{1+\xi_x^2}\right]^{1/4}\left[\frac{2}{1+\xi_y^2}\right]^{1/4} \cdot H_m\left[x\left\{\frac{2K}{b_x(1+\xi_x^2)}\right\}^{1/2}\right] H_n\left[y\left\{\frac{2K}{b_y(1+\xi_y^2)}\right\}^{1/2}\right] \cdot \exp\left\{\frac{-Kx^2}{b_x(1+\xi_x^2)}\right\} \exp\left\{\frac{-Ky^2}{b_y(1+\xi_y^2)}\right\} \cdot \exp\left(-j\left\{K\left[\frac{b_x(1+\xi_x)}{4} + \frac{\xi_x x^2}{b_x(1+\xi_x^2)}\right] - \left(\frac{1}{2}+m\right)\left(\frac{\pi}{2} - \phi_x\right)\right\}\right) \cdot \exp\left(-j\left\{K\left[\frac{b_y(1+\xi_y)}{4} + \frac{\xi_y y^2}{b_y(1+\xi_y^2)}\right] - \left(\frac{1}{2}+n\right)\left(\frac{\pi}{2} - \phi_y\right)\right\}\right) \tag{20}$$

where $\xi_x = 2z_x/b_x$, $\xi_y = 2z_y/b_y$, $\tan\phi_x = 1 - \xi_x/1 + \xi_x$ and $\tan\phi_y = (1-\xi_y)/(1+\xi_y)$.

The phase terms are separable in $x$ and $y$ so the calculation of the phase shift from mirror 1 to mirror 2 can be performed separately for the $xz$ plane and $yz$ plane and these then added to find the total phase shift. By equating this total phase shift to an integral multiple of $\pi$, the resonant frequencies of the resonator containing an inclined plate are found. An equivalent method, but with considerable simplification of the algebra, is to use a generalization of the results obtained by Kogelnik [15] and by Arnaud [8] in which the resonance frequencies are expressed directly in terms of the round-trip ray-transfer matrices.

After some manipulation it is found that the resonant frequencies (which are independent of plate position) are given by

$$\frac{\nu}{\nu_0} = (q+1) + \frac{1}{\pi}\left(m+\tfrac{1}{2}\right) \cdot \cos^{-1}\left\{\left[1 - \frac{d-L_1}{R_1}\right]^{1/2}\left[1 - \frac{d-L_1}{R_2}\right]^{1/2}\right\} + \frac{1}{\pi}\left(n+\tfrac{1}{2}\right)\cos^{-1}\left\{\left[1 - \frac{d-L_2}{R_1}\right]^{1/2} \cdot \left[1 - \frac{d-L_2}{R_2}\right]^{1/2}\right\} \tag{21}$$

where $\nu_0 = c/2$ (total optical path length between mirrors 1 and 2).

Since the frequencies are independent of plate position, if two or more parallel plates of the same material are placed at any position in a resonator, the oscillation frequencies will be the same as for a single plate whose thickness is the sum of the plate thicknesses.

Equation (21) has been cast in a form that can be seen to reduce to the form given by Kogelnik and Li [1] for an empty resonator, i.e.,

$$\frac{\nu}{\nu_0} = (q+1) + \frac{1}{\pi}(m+n+1) \cdot \cos^{-1}\left[\left(1 - \frac{d}{R_1}\right)^{1/2}\left(1 - \frac{d}{R_2}\right)^{1/2}\right]. \tag{22}$$

The form of (21) shows clearly that, since $L_1 \neq L_2$, the frequencies of the $TEM_{mn}$ and $TEM_{pq}$ modes are not the same even if $m + n = p + q$. For the empty resonator, however, (22) shows that these modes have the same frequencies.

To give some idea of the magnitude of the frequency splitting, (21) was used to calculate that for the resonator of Example 1 with $d = 1.2$ meters the splitting between the $TEM_{01}$ and $TEM_{10}$ modes is ~110 kHz. For small splittings such as in the example above the beat frequency is proportional to the plate thickness. Photomixing beats of this order have been observed by Uchida [16] in the output of a He–Ne laser using a Brewster-ended plasma tube. He showed conclusively that these beat signals could be assigned to the mixing of modes having the same value of $m + n$. Using (21) to calculate the splitting between $TEM_{01}$ and $TEM_{10}$ modes in Uchida's resonator, a splitting of 136 kHz is found (assuming the Brewster windows to be 4 mm thick and of refractive index 1.5). However, an exact comparison with Uchida's result is not possible since Uchida found that the splitting could be varied between 0 and 1 MHz by tilting the mirrors and thus introducing additional astigmatism.

## Additional Sources of Astigmatism

It has already been mentioned that astigmatism will arise from the use of intracavity elements such as prisms used for dispersion. To reduce losses and for compactness, the laser mirror is sometimes made an integral part of the prism by placing the mirror coating on one face of the prism [2] [e.g., as in Fig. 3(a)]. Also dye $Q$-switch cells are sometimes constructed with the mirror forming one window of the cell and a Brewster-angle entrance prism forming the other window. Provided the mirror surface is plane, the astigmatism of the beam within the resonator is generally not significant. Even so, it is shown below that the output beam from the prism end of such a resonator has considerable astigmatism. If, in addition, the mirror surface is spherical, then the astigmatism within the resonator can become severe and the stability condition of the resonator drastically modified. White [2] has shown that in such an arrangement the astigmatism in the resonator can be corrected by putting an appropriate cylindrical curvature on the Brewster face of the prism although the beam emerging from the prism end of the resonator is still highly astigmatic. Thus, as a general rule, such devices consisting of an integral prism and mirror should either incorporate a plane mirror, or if a curved mirror surface is used, the Brewster face must have an appropriate correcting curvature. In either case, the other resonator mirror should be used as the output mirror if an astigmatism-free output beam is required.

Consider the resonator of Fig. 3(a). The prism has its entrance face at the Brewster angle, has refractive index $\mu$, and the length of the beam path in the prism is $l$. By simple extension of the calculations described in the Appendix, it can be shown that for rays entering the prism being reflected at mirror 1 (radius of curvature $R_1$), and then leaving the prism, the ray-transfer matrix for the $yz$ plane is

$$\begin{bmatrix} 1 - 2l/R_1 & (2l - 2l^2/R_1)/\mu \\ -2\mu/R_1 & 1 - 2l/R_1 \end{bmatrix}$$

and for the $xz$ plane

$$\begin{bmatrix} 1 - 2l/R_1 & (2l - 2l^2/R_1)/\mu^3 \\ -2\mu^3/R_1 & 1 - 2l/R_1 \end{bmatrix}.$$

In the resonator of Fig. 3(a), $R_1$ is plane and these matrices are then the same as the matrices for a double pass through a plate inclined at the Brewster angle where $l$ is the length of the single-pass beam path within the plate. Then, provided the resonator is not operated close to instability, it can be seen from Example 1 that the astigmatism of the beam to the right of the prism is small. The spot size for the $yz$ plane just inside the prism

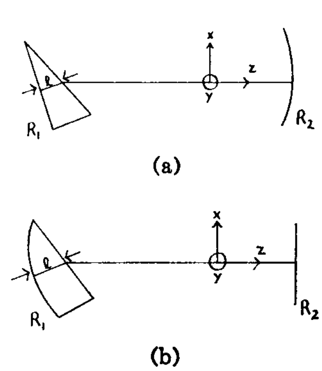


Fig. 3. Resonators using dispersive prisms. The resonator of (a) is relatively free of astigmatism whereas that of (b) produces a severely astigmatic beam.

face is the same as the spot size just to the right of the prism entrance face. However, it follows from Huygens' construction that in the $xz$ plane the spot size just inside the prism is greater by a factor $\mu$ (for Brewster-angle operation) than the spot size just to the right of the entrance face. Thus, instead of a beam of circular cross section, the beam as it emerges from mirror 1 has elliptical contours of constant intensity, the ratio of the $x$ major axis to the $y$ major axis being $\mu$. On the other hand, the output from mirror 2 shows only little astigmatism.

Consider now the resonator of Fig. 3(b) where mirror 2 is plane. The round-trip matrices for the unfolded resonator can be calculated using the prism matrices given above and the spot sizes on mirror 2 can then be calculated in terms of the elements of these round-trip matrices. If $l$ is taken to be 0 (i.e., the prism is thin) and if $d$ is the separation between mirrors 1 and 2, the spot sizes of the waist formed on mirror 2 are given by

$$(\omega_x)_2^4 = \left(\frac{\lambda}{\pi}\right)^2 d\left(\frac{R_1}{\mu^3} - d\right) \tag{23}$$

$$(\omega_y)_2^4 = \left(\frac{\lambda}{\pi}\right)^2 d\left(\frac{R_1}{\mu} - d\right). \tag{24}$$

Besides showing that the spot sizes on the plane mirror are quite different in the $xz$ and $yz$ planes, (23) shows that the stability condition is that $d < R_1/\mu^3$, instead of the usual stability condition for the empty resonator, i.e., $d < R_1$. As shown by White [2] this astigmatism can be removed by putting a cylindrical curvature on the Brewster face (concave, of radius $R_1/\mu \cos(\tan^{-1}\mu)$ in the $xz$ plane). Equation (23) for $(\omega_x)_2$ then becomes the same as (24) for $(\omega_y)_2$ and the stability condition is then that $d < R_1/\mu$.

Even if a laser resonator produces a beam without astigmatism it must be remembered that devices such as prisms placed in the beam outside the resonator will introduce significant astigmatism for beams having different angles of incidence at the exit and entrance faces. An example of this problem may be seen in the use of calcite prisms as laser-beam combiners [17]. In general, the combined beams will not have the same spot sizes or waist positions when combined and this can seriously affect the efficiency of processes such as optical parametric oscillation and optical mixing. The unwanted astigmatism can be corrected however by the use of astigmatic lenses, cylindrical lenses, or tilted lenses [7].

Astigmatism can also be produced in optically pumped lasers if the pump-induced thermal distortions are not cylindrically symmetrical as for example in a focal-ellipse type of pumping reflector [18]. In pulsed operation at low repetition rates, the thermal distortion, and hence, the astigmatism also are rather small. However, in continuously pumped lasers the thermal distortions are much greater and significant astigmatism may be present.

## Conclusion

The modes of a resonator containing an inclined plate have been analyzed and expressions derived for the spot sizes, waist positions, and resonance frequencies. The astigmatism has been shown to be quite significant for resonators containing inclined plates several centimeters thick (e.g., Brewster-ended laser rods). In resonators operating close to the stability limits, the astigmatism can be pronounced even in resonators containing inclined plates of quite modest thickness. The astigmatism lifts the frequency degeneracy of $\text{TEM}_{mn}$ modes having the same value of $m + n$, and the frequency splittings are typically in the 100-kHz range for visible gas lasers using Brewster-ended plasma tubes. Large astigmatism can be produced by devices that use an integral combination of mirror and prism although with suitable precautions this astigmatism can be reduced to insignificant proportions.

## Appendix

### Ray-Transfer Matrices for an Inclined Plate

Consider an astigmatic Gaussian beam propagating with its axis in the $z$ direction and having wavefront curvatures $R_x$, $R_y$ and spot sizes $\omega_x$, $\omega_y$ in the $x$, $y$ directions, respectively, where $x$, $y$, $z$ form a right-handed Cartesian triad. Suppose the beam is incident at an angle $\theta_1$ on a transparent parallel-sided plate of thickness $t$ and refractive index $\mu$, the plane of the plate being perpendicular to the $z$, $x$ plane (see Fig. 1). The spot sizes and curvatures of the beam emerging from the plate are calculated separately for the $xz$ and $yz$ planes. Huygens' construction is applied to find the discontinuous changes of spot size and curvature at the entrance and exit faces, and the changes in these parameters due to propagation within the plate are calculated using the well-known laws of propagation for Gaussian beams [1]. In this way it is found that for the $xz$ plane, apart from a sideways displacement, the exit-beam parameters differ from the entrance-beam parameters in exactly the same way as if the beam had traveled a free-space distance of

$$t\mu^2(1 - \sin^2\theta_1)/(\mu^2 - \sin^2\theta_1)^{3/2}.$$

As for the $yz$ plane parameters, the beam appears to have traveled a free-space distance of $t/(\mu^2 - \sin^2\theta_1)^{1/2}$.

Hence for the $xz$ plane the ray-transfer matrix is

$$\begin{bmatrix} 1 & t\mu^2(1 - \sin^2\theta_1)/(\mu^2 - \sin^2\theta_1)^{3/2} \\ 0 & 1 \end{bmatrix}$$

and for the $yz$ plane, the ray-transfer matrix is

$$\begin{bmatrix} 1 & t/(\mu^2 - \sin^2\theta_1)^{1/2} \\ 0 & 1 \end{bmatrix}.$$

If $\theta_1$ is the Brewster angle, the matrices are

$$\begin{bmatrix} 1 & t/\mu^3 \sin(\tan^{-1}\mu) \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & t/\mu \sin(\tan^{-1}\mu) \\ 0 & 1 \end{bmatrix}.$$

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