ON PROPERTY (FA) FOR WREATH PRODUCTS

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ABSTRACT. We prove that the standard wreath product $A \wr B$ has Property (FA) if and only if B has Property (FA) and A is a finitely generated group with finite abelianisation. We also prove an analogous result for hereditary Property (FA). On the other hand, we prove that many groups with hereditary Property (FA) are not quotients of finitely presented groups with the same property.

1. Introduction

Property (FA) was introduced by Serre in his monograph [14]: a group G is said to have *Property (FA)* if every isometric action of G on a (simplicial) tree has a fixed point. Serre's fundamental result [14, Theorem I.6.15] about Property (FA) says that a denumerable group G has Property (FA) if and only if G is not an amalgam, G has no quotient isomorphic to $\mathbf{Z}$ and G is finitely generated. Traditional examples of groups with Property (FA) include

(1) finitely generated torsion groups;
(2) Coxeter groups such that the associated Coxeter matrix has no occurrence of ∞;
(3) special linear groups over the integers, $\mathrm{SL}_n(\mathbf{Z})$, for $n \geq 3$;
(4) more generally, groups with Kazhdan's Property (T);
(5) also more generally than (3), in another direction, irreducible lattices in semisimple Lie groups of real rank at least two, e.g. $\mathrm{SL}_2(\mathbf{Z}[\sqrt{2}])$.

The first three of these examples are demonstrated in Serre's original account [14]; (4) was proved by Watatani in [18], using the characterisation of property (T) in terms of affine actions on Hilbert spaces; and finally (5) is due to Margulis [13].

The aim of this article is to investigate Property (FA) for wreath products. We recall that the *(standard) wreath product* of two groups A, B is defined as the group

$$A \wr B := \oplus_{b \in B} A_b \rtimes B,$$

where A_b denote isomorphic copies of A. If A and B are finitely generated then so is $A \wr B$.

Theorem 1. *Consider the wreath product $G = A \wr B$ of two countable groups A and B, with B non-trivial. The following are equivalent*

- *G has Property (FA);*
- *B has Property (FA) and A is a finitely generated group with finite abelianisation.*

Contrast with the following result on property (T) groups [6, Proposition 2.8.2]: the wreath product $A \wr B$ of two non-trivial groups A, B has property (T) if and only if A has property (T) and B is finite.

The following is a well-known problem (it appears for instance as [3, Question 7] and in [17]).

Question 2 (fg versus fp)**.** Is every finitely generated group with Property (FA) the quotient of a finitely presented group with property (FA)?

It can also be restated as "is Property (FA) open in the space of marked groups?" (see [7, Section 2.6(h)]). The analogous question for some other fixed point properties has a positive answer

- for Property (F$\mathbf{R}$) (fixed point property on $\mathbf{R}$-trees), a result of Culler and Morgan [8, Proposition 4.1].
- Property (FH) (fixed point property on Hilbert spaces, also known as Kazhdan's Property (T)), a result independently due to Shalom and Gromov ([16, Theorem 6.7] and [10, 3.8.B])
- more generally, again by Gromov [10, 3.8.B], the fixed point property on any class of metric spaces which is stable under "scaling ultralimits", e.g. the class of all CAT(0)-spaces.

It is an old open question [15, Question A, p.286] whether Property (FA) implies the *a priori* stronger Property (F$\mathbf{R}$). Of course a positive answer would imply a positive answer to Question 2.

Some evidence for a positive answer for Question 2 is given by the case of wreath products, as the proof of Theorem 1 actually yields

Proposition 3. *If A, B are finitely presented groups, A has finite abelianisation and B has Property (FA), then $A \wr B$ is the quotient of a finitely presented group with Property (FA).*

Note that Baumslag [1] proved that a wreath product of non-trivial finitely presented groups $A \wr B$ is finitely presented only when B is finite.

Definition 4. A group G has *hereditary Property (FA)* if G and all its finite index subgroups have Property (FA).

It is natural to address Question 2 when we replace Property (FA) by hereditary (FA). Then the answer turns out to be negative, and wreath products provide a large class of elementary examples.

Theorem 5. *Let $G = A \wr B$ be the wreath product of two finitely generated groups. Assume that B is infinite and residually finite, and that A has at least a non-trivial finite quotient. Then every finitely presented group mapping onto G has a finite index subgroup with a surjective homomorphism onto a non-abelian free group.*

Examples in Theorem 5 where G has hereditary (FA) are provided by the following theorem, which relies on Theorem 1 but also requires further arguments.

Theorem 6. *Let $G = A \wr B$ be a wreath product of finitely generated groups, with B infinite. The following are equivalent*

- *G has hereditary Property (FA);*
- *B has hereditary Property (FA) and A is a finitely generated group with finite abelianisation.*

Example 7. If $G = F \wr \mathrm{SL}_3(\mathbf{Z})$ with F any non-trivial finite group, then G has hereditary Property (FA) by Theorem 6, but is not the quotient of any finitely presented group with the same property, by Theorem 5.

Remark 8. Despite the analogy between Theorems 1 and 6, Theorem 5 shows that Proposition 3 is false when (FA) is replaced by hereditary (FA).

Remark 9. Let Γ be the first Grigorchuk group [12, Chap. VIII]. This is a finitely generated group every proper quotient of which is finite; in particular it cannot be expressed as a non-trivial wreath product with an infinite quotient. Also, it is a finitely generated torsion group and therefore has hereditary Property (FA) as well as its finite index subgroups. It follows however from [9] (see also [2, Corollary 8]) that every finitely presented group mapping onto Γ has a finite index subgroup mapping onto the free group.

Remark 10. There are several possible variants or extensions of Theorem 1, with the same proof:

- it is true with Property (FA) replaced by (F$\mathbf{R}$);
- it is true for permutational wreath products

$$A \wr_X B = \oplus_{x \in X} A_x \rtimes B,$$

  where X is a B-set with finitely many B-orbits, and without any B-fixed point.

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2. Standard wreath products

In this part, we prove Theorem 1 and Proposition 3. We begin by two classical lemmas.

Lemma 11. *Suppose that a group H acts on a tree X without inversions. Let A and B be subgroups of H such that X^A and X^B are non-empty. If $[A, B] = 1$ then $X^A \cap X^B \neq \emptyset$.*

Proof of Lemma. On the contrary, suppose that $X^A \cap X^B$ is empty. Then, there is a unique geodesic α in X that realises the minimum distance between X^A and X^B. However, as A and B commute, A preserves the set X^B. This means that A fixes the geodesic α. Similarly B fixes α and the path α is common to both X^A and X^B. This proves the Lemma. □

Lemma 12 (I.6.5.10 in [14])**.** *Let $X_1, \ldots, X_m$ be subtrees of a tree X. If the X_i meet pairwise then their intersection is non-empty.*

Proof of Theorem 1. Let $G := A \wr B$ be the wreath product of A and B. If G has Property (FA) then clearly B, being a quotient of G has Property (FA). Moreover, G is finitely generated and this forces A to be finitely generated. Finally G has finite abelianisation; but $G^{\mathrm{ab}} = A^{\mathrm{ab}} \times B^{\mathrm{ab}}$. Hence the abelianisation of A is also finite. It therefore, suffices to prove the converse.

Suppose A is a finitely generated group with finite abelianisation and B is a group with Property (FA). Let G act without inversions on a tree X. We need to prove that X^G is non-empty. As B is a subgroup of G with Property (FA), it is clear that X^B is non-empty.

Case $X^A \neq \emptyset$. Recall that $A \wr B$ is the semidirect product of $\oplus_{b\in B} A$ with B, where B acts by permuting the components of $\oplus_{b\in B} A$. We will write the b-th copy of A in the direct sum as A_b. As G acts on X, each of the groups A_b, for $b \in B$ acts on X. As $X^A \neq \emptyset$ and $b.X^A = X^{A_b}$, the set $X^{A_b} \neq \emptyset$ for each $b \in B$. Moreover, since A and A_b , $b \neq 1$ commute, by Lemma 11, $X^A \cap X^{A_b} \neq \emptyset$. In fact, by Lemma 12, every finite subcollection of $\left\{X^{A_b} \ : \ b \in B\right\}$ has non-empty intersection. To prove the claim it suffices to show that $\cap_{b\in B} X^{A_b} \neq \emptyset$.

As B acts on the direct sum $\oplus_{b\in B} A$ by permuting the components, any vertex common to X^B and one of the trees X^{A_b} is a global fixed point for G. Therefore we may assume that for every $b \in B$, $X^B \cap X^{A_b}$

is empty. For each $b \in B$, let v_b be the vertex in X^{A_b} which is closest to the tree X^B.

Consider b and b', two distinct elements of B. Let v be a vertex common to both X^{A_b} and $X^{A'_b}$. Any path joining v to a vertex in X^B must pass through v_b. But, any such path must also pass through $v_{b'}$. This forces v_b to be the same vertex as v'_b. We deduce from here that there is a vertex common to all X^{A_b} and so $\cap_{b\in B} X^{A_b} \neq \emptyset$. The unique point in $\cap_{b\in B} X^{A_b}$ that is closest to X^B must be fixed by B. In other words if both A and B have fixed points in a tree on which the wreath product is acting then their fixed sets have to intersect and produce a global fixed point for G.

Case $X^A = \emptyset$. Observe that if G is a finitely generated group acting on a tree without inversions and the action is such that every element of G is elliptic on X, then X^G is not empty. Therefore the hypothesis that $X^A \neq \emptyset$ implies the existence of an element a of A such that a acts by translations on a line ℓ. Take any $1 \neq b$ from B. Then A_b commutes with a and so every element of A_b acts as a translation on ℓ. This implies there exists a non-trivial homomorphism of A_b to $\mathbf{R}$. As A_b is finitely generated, such a homomorphism is given precisely by a surjective map from A_b onto the integers. But then A_b and thus A cannot have finite abelianisation. This completes the proof. □

Proof of Proposition 3. Consider the group K generated by A and B along with the additional relations: $[a^b, a'] = 1$, for each pair of generators a and a' for A and all generators b for B. If A has finite abelianisation and B has Property (FA) then the proof of Theorem 1 implies that K has Property (FA). The group $A \wr B$ is clearly a quotient of the finitely presented group K. □

3. Hereditary Property (FA)

Proof of Theorem 5. Let $(u_k)_{k\geq 1}$ be an enumeration of $B - \{1\}$. Define $G_0 = A * B$ and for $k \leq \infty$, define G_k as the quotient of G_0 by the "relators" $[A, u_j A u_j^{-1}]$ for $j \leq k$. Note that $G_\infty = A \wr B$.

Replacing G by a quotient if necessary, we can suppose that A is finite. Let H be a finitely presented group having G as a quotient. Then H has G_k as a quotient for some k. So we only have to prove that G_k has a finite index subgroup mapping onto a free group. We borrow a construction from [4, Section 2]. The group G_k has a natural semidirect product decomposition $M \rtimes B$, where M is a "graph product", namely it is the free product of copies A_b of A, indexed by $b \in B$, quotiented by the relations $[A_b, A_{bs}] = 1$ for all $s = u_1, \dots, u_k$, and $u \in B$ shifts A_b to A_{ub}.

There exists a normal finite index subgroup C of B such that $u_i \notin C$ for all $i := 1, \dots k$. Let N be the normal subgroup of M generated by all A_b for $b \in B - C$. It is immediate from the presentation of M that the quotient of M by N is the free product of all copies A_b for $b \in C$, because of the choice of C. Moreover, N is normalized by C. So the semidirect product $G_k/N = M/N \rtimes C$ is the free product of A and C. So if C' is a normal subgroup of C, of finite index at least three, then G_k has the free product $A * C'$ as a quotient. This is a free product of two non-trivial finite groups with one of order at least three, and therefore has a finite index non-abelian free subgroup (for instance the kernel of the natural map onto $A \times C'$). □

Proposition 13. *Consider the short exact sequence of groups:*

$$1 \to A \to G \to B \to 1$$

Assume that A does not contain F_2, the non-abelian free group of rank 2. Then, G has Property (FA) if and only if B has Property (FA) and G is a finitely generated group that does not map onto the integers or the infinite dihedral group.

Proof of Proposition 13. If G is a group, define $\mathrm{NF}(G)$ as the largest normal subgroup of G without free subgroups (this is always well-defined).

Suppose that G fails to have Property (FA). Then either G maps onto $\mathbf{Z}$, or G splits as a non-trivial amalgam $H *_K L$. In the latter case, if the amalgam is degenerate (K has index two in both H and L), then G maps onto the infinite dihedral group. Otherwise, we can apply [5, Proposition 7], which says in particular that $\mathrm{NF}(G)$ is contained in K. Since A is by definition contained in $\mathrm{NF}(G)$, this shows that $G/A = B$ splits as a non-trivial amalgam $(H/A) *_{K/A} L/A$, and therefore fails to have Property (FA). □

Proof of Theorem 6. The fact that the first condition implies the second one is as straightforward as the analogous implication for Theorem 1, so we do not repeat the argument.

So assume that A has finite abelianisation and B has hereditary Property (FA).

We first prove the implication when A has *trivial* abelianisation, as the proof is then easier. In this case, by Gruenberg [11] every finite index subgroup of G contains the normal subgroup $A^{(B)}$ and is therefore of the form $A^{(B)} \rtimes C$ where C has finite index in B; since B is supposed to be infinite, C is non-trivial. This group $A^{(B)} \rtimes C$ is a permutational wreath product (with a non-transitive free action), so a straightforward extension of Theorem 1 applies (see Remark 10).

Before passing to the general case, we need to consider the special case when A is abelian (and thus finite). Every finite index subgroup H of G then lies in an extension where the kernel is torsion (and abelian) and the quotient is a finite index subgroup of B. So we can apply Proposition 13 and H has Property (FA).

Suppose now, in general, that the derived subgroup D of A has finite index in A, and let H have finite index in G. Then [11] now says that H contains $D^{(B)}$. Arguing as above with the subgroup $D^{(B)} \rtimes C$ (where $C = H \cap B$), we see that $D^{(B)}$ has a fixed point. Acting on the set of points fixed by $D^{(B)}$, we are reduced to the case when A is finite and abelian, which was considered before, so H has a fixed point. Thus H has Property (FA). □

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