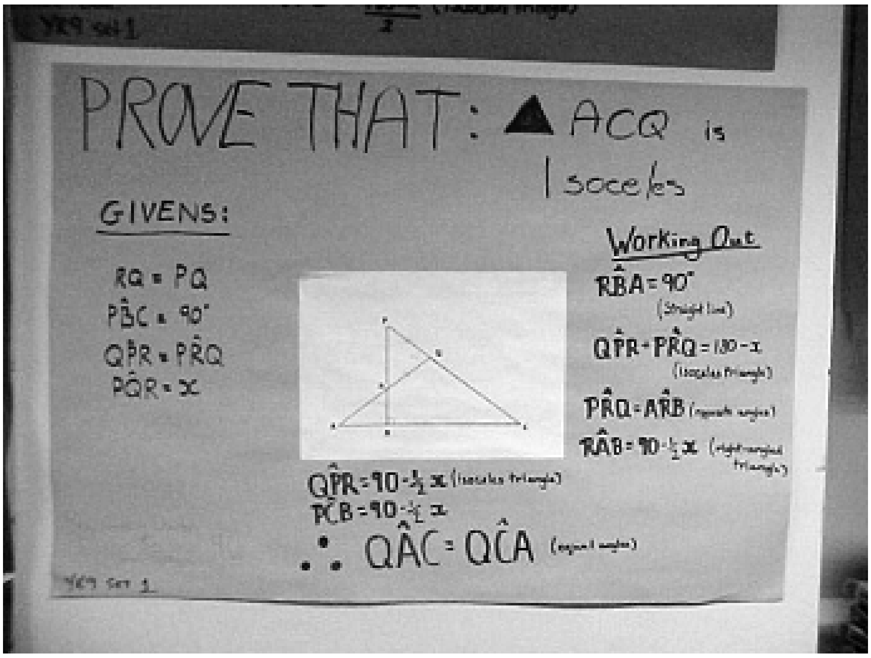
Developing geometrical reasoning in the secondary school: outcomes of trialling teaching activities in classrooms



A Report from the Southampton/Hampshire Group to the Qualifications and Curriculum Authority

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The cover illustration is from the work of a Year 9 class (pupils aged 13-14), selected as it presents a proof that QAC = QCA

Contents

Developing geometrical reasoning in the secondary school: outcomes of trialling teaching activities in classrooms

Executive Summary

This report presents the findings of the Southampton/Hampshire Group of mathematicians and mathematics educators sponsored by the Qualifications and Curriculum Authority (QCA) to develop and trial some teaching/learning materials for use in schools that focus on the development of geometrical reasoning at the secondary school level. The project ran from October 2002 to November 2003. An interim report was presented to the QCA in March 2003.

1. The Southampton/Hampshire Group consisted of five University mathematicians and mathematics educators, a local authority inspector, and five secondary school teachers of mathematics. The remit of the group was to *develop and report on teaching ideas that focus on the development of geometrical reasoning at the secondary school level*.

2. In reviewing the existing geometry curriculum, the group endorsed the RS/ JMC working group conclusion (RS/ JMC geometry report, 2001) that the current mathematics curriculum for England contains sufficient scope for the development of geometrical reasoning, but that it would benefit from some clarification in respect of this aspect of geometry education. Such clarification would be especially helpful in resolving the very odd separation, in the programme of study for mathematics, of 'geometrical reasoning' from 'transformations and co-ordinates', as if transformations, for example, cannot be used in geometrical reasoning.

3. The group formulated a rationale for designing and developing suitable teaching materials that support the teaching and learning of geometrical reasoning. The group suggests the following as guiding principles:
   - Geometrical situations selected for use in the classroom should, as far as possible, be chosen to be useful, interesting and/or surprising to pupils;
   - Activities should expect pupils to explain, justify or reason and provide opportunities for pupils to be critical of their own, and their peers', explanations;
   - Activities should provide opportunities for pupils to develop problem solving skills and to engage in problem posing;
   - The forms of reasoning expected should be examples of *local* deduction, where pupils can utilise *any* geometrical properties that they know to deduce or explain other facts or results.
   - To build on pupils' prior experience, activities should involve the properties of 2D and 3D shapes, aspects of position and direction, and the use of transformation-based arguments that are about the geometrical situation being studied (rather than being about transformations *per se*);
   - The generating of data or the use of measurements, while playing important parts in mathematics, and sometimes assisting with the building of conjectures, should not be an end point to pupils' mathematical activity. Indeed, where sensible, in order to build geometric reasoning and discourage over-reliance on empirical verification, many classroom activities might use contexts where measurements or other forms of data are *not* generated.

4. In designing and trialling suitable classroom material, the group found that the issue of how much structure to provide in a task is an important factor in maximising the opportunity for geometrical reasoning to take place. The group also found that the

role of the teacher is vital in helping pupils to progress beyond straightforward descriptions of geometrical observations to encompass the reasoning that justifies those observations. Teacher knowledge in the area of geometry is therefore important.

5. The group found that pupils benefit from working collaboratively in groups with the kind of discussion and argumentation that has to be used to articulate their geometrical reasoning. This form of organisation creates both the need and the forum for argumentation that can lead to mathematical explanation. Such development to mathematical explanation, and the forms for collaborative working that support it, do not, however, necessarily occur spontaneously. Such things need careful planning and teaching.
6. Whilst pupils can demonstrate their reasoning ability orally, either as part of group discussion or through presentation of group work to a class, the transition to individual recording of reasoned argument causes significant problems. Several methods have been used successfully in this project to support this transition, including “fact cards” and “writing frames”, but more research is needed into ways of helping written communication of geometrical reasoning to develop.
7. It was found possible in this study to enable pupils from all ages and attainments within the lower secondary (Key Stage 3) curriculum to participate in mathematical reasoning, given appropriate tasks, teaching and classroom culture. Given the finding of the project that many pupils know more about geometrical reasoning than they can demonstrate in writing, the emphasis in assessment on individual written response does not capture the reasoning skills which pupils are able to develop and exercise. Sufficient time is needed for pupils to engage in reasoning through a variety of activities; skills of reasoning and communication are unlikely to be absorbed quickly by many students.
8. The study suggests that it is appropriate for all teachers to aim to develop the geometrical reasoning of all pupils, but equally that this is a non-trivial task. Obstacles that need to be overcome are likely to include uncertainty about the nature of mathematical reasoning and about what is expected to be taught in this area among many teachers, lack of exemplars of good practice (although we have tried to address this by lesson descriptions in this report), especially in using transformational arguments, lack of time and freedom in the curriculum to properly develop work in this area, an assessment system which does not recognise students’ oral powers of reasoning, and a lack of appreciation of the value of geometry as a vehicle for broadening the curriculum for high attainers, as well as developing reasoning and communication skills for all students.
9. Areas for further work include future work in the area of geometrical reasoning, include the need for longitudinal studies of how geometrical reasoning develops through time given a sustained programme of activities (in this project we were conscious that the timescale on which we were working only enabled us to present “snapshots”), studies and evaluation of published materials on geometrical reasoning, a study of “critical experiences” which influence the development of geometrical reasoning, an analysis of the characteristics of successful and unsuccessful tasks for geometrical reasoning, a study of the transition from verbal reasoning to written reasoning, how overall perceptions of geometrical figures (“gestalt”) develops as a component of geometrical reasoning (including how to create the links which facilitate this), and the use of dynamic geometry software in any (or all) of the above.

10. As this group was one of six which could form a model for part of the work of regional centres set up like the IREMs in France, it seems worth recording that the constitution of the group worked very well, especially after members had got to know each other by working in smaller groups on specific topics. The balance of differing expertise was right, and we all felt that we learned a great deal from other group members during the experience. Overall, being involved in this type of research and development project was a powerful form of professional development for all those concerned. In retrospect, the group could have benefited from some longer full-day meetings to jointly develop ideas and analyse the resulting classroom material and experience rather than the pattern of after-school meetings that did not always allow sufficient time to do full justice to the complexity of many of the issues the group was tackling.

Full report:
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Full report available online at:
http://www.crme.soton.ac.uk/research/geomreason.html

1. Geometrical reasoning in the secondary classroom

1.1 Aims

Important objectives in teaching mathematics at the secondary school level include developing a knowledge and understanding of, and the ability to use, geometrical properties and theorems and encouraging the development and use of conjecture, deductive reasoning and proof (a more complete discussion of the aims of geometry teaching in the secondary school is given in the Royal Society/Joint Mathematical Council Report, 2001). While such objectives are relatively easy to state, developing a suitable geometry curriculum that can be taught successfully to the majority of pupils remains an elusive goal (see Jones, 2001). As the writers of one major curriculum development project observed:

> *Of all the decisions one must make in a curriculum development project with respect to choice of content, usually the most controversial and the least defensible is the decision about geometry.*
>
> (Chicago School Mathematics Project staff 1971, p281)

The reasons for these difficulties are myriad and, as a result, as a recent comparative study of geometry curricula found, considerable variation exists around the world in current approaches to the design of the school geometry curriculum (for a survey see, Hoyles, Foxman and Küchemann, 2002). For example, a 'realistic' or practical approach is apparent in Holland, whereas a predominantly theoretical approach is evident in France and Japan. Most countries, although not all, include elements of proof and proving in their curricula specifications for geometry. Here, too, there are variations, with some countries favouring an approach with congruence as a central element, while other used similarity and transformations. The review concludes by noting "there is evidence of a state of flux in the geometry curriculum, with most countries looking to change" (*ibid* p.121).

What is agreed is the central importance of geometry in mathematics. As the renowned mathematician Sir Michael Atiyah writes:

> *… spatial intuition or spatial perception is an enormously powerful tool and that is why geometry is actually such a powerful part of mathematics - not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool...*
>
> (Atiyah, 2001, p50)

While agreement about the importance of deductive reasoning in geometry teaching is widespread (Royal Society/Joint Mathematical Council; 2001), there are considerable problems in devising a successful school geometry curriculum that successfully develops such reasoning skills. A range of research across a number of countries has documented that, even after considerable teaching input, many students fail to see a need for deductive proving and/or are unable to distinguish between different forms of mathematical reasoning such as explanation, argument, verification and proof (for reviews of this research, see Hanna and Jahnke, 1996; Dreyfus, 1999). Such difficulties with proof are not restricted to the school level but are apparent at University level too (Almeida, 2000; Jones, 2000).

This report examines what might be promising ways of developing classroom approaches that provide a suitable foundation for the development of deductive reasoning and proof in geometry at the secondary school level. The next section provides a rationale for the development of classroom teaching ideas that are reported on in the main part of this report.

1.2 Classroom approaches

Hiebert *et al* (1997) offers a useful framework for considering different "dimensions" of the classroom and the links between them. The key five dimensions noted by the authors are:

- The nature of classroom tasks
- The role of the teacher
- The social culture of the classroom
- Mathematical tools as learning supports
- Equity and accessibility.

In what follows in this section, as asked by the QCA, we discuss the nature and selection of appropriate classroom tasks and offer some guidance. We hope that the description of classroom activities in the main part of the report will give insight into some of the remaining aspects. However, since language and communication emerged as a central concern in many of the lessons, we also include a section on these elements at the end of this discussion.

At this point it is worth noting that the Royal Society/Joint Mathematical Council (2001) report on the teaching and learning of geometry for pupils aged 11-19 makes a number of recommendations about suitable approaches to the teaching of deductive reasoning in geometry. It argues (in section 5) for classroom approaches that incorporate the use of logical arguments that build on what is already known by pupils in order to demonstrate the truth of some geometrical result, possibly a theorem or other result conjectured by pupils after conducting a well-chosen experiment. The report also suggests that:

- Geometrical situations (i.e. the theorems) should be chosen, as far as possible, to be useful, interesting and/or surprising to pupils.
- The level of sophistication expected in the logical argument will depend upon the age and attainment of the pupils concerned, and the proof produced might equally be called an 'explanation' or 'justification' or 'reason' for the result.
- Pupils should be encouraged to be critical of their own, and their peers', explanations as this should help them to develop the sophistication and rigour of their arguments.

Such an approach, the RS/JMC report suggests, should mean that all pupils come to understand that deductive reasoning is more than simply stating a belief or checking that the result is valid in a number of specific cases. However, the report observes, "we accept that it is not an easy matter to determine how to achieve this with each pupil and each result and that a careful choice of approach will be needed" (p. 9).

The process of deductive reasoning and proving must begin somewhere. The starting point for abstract mathematics is a *minimal* collection of initial reasonable assumptions, called axioms. In the context of school mathematics, as the Royal Society/Joint Mathematical Council report explains (in appendix 9 of the report), experience has shown that this is not a sensible approach. Rather, the report suggests, one should start with some well-known or 'obvious' facts. These need to be carefully chosen and, in a sense, be explicit. Then, the report goes on, through using deductive reasoning, a collection of related results, of a less obvious nature, can be built up. This is what is sometimes called *local* deduction, where pupils can utilise *any* geometrical properties that they know to deduce or explain other facts or results. The idea is that fluency with local deduction should provide the right form of foundation on which to successfully build knowledge of systematic axiomatisation at the appropriate stage of pupils' mathematical education. At such a later stage, facts or theorems taken as "obvious" at the earlier stage can be revisited with a view to asking whether or not they can be proved. This leads to the question of what, in say plane geometry, might constitute a minimal collection of initial reasonable assumptions (ie towards an axiomatic approach).

As Turnau (2002) explains, fluency with local deduction is best developed through pupils' problem solving activity, something that accords with advice from the QCA (2002) to strengthen the development of pupils' problem-solving skills. That reasoning and problem-solving are often under-emphasised in secondary mathematics teaching has also been highlighted by Her Majesty's Chief Inspector of Schools (2003, para 123).

Local deduction, an activity in which new theorems are deduced from accepted theorems, is an important aim of a mathematics curriculum. However deductive geometrical reasoning can be more widely interpreted to also include:

- Deriving a specific value of a variable (e.g. the size of an angle) using both known theorems and known properties of shapes.
- Deducing a specific result in relation to a figure with given properties which does not have the generality or status of a theorem (e.g. proving that two sides of a quadrilateral with a particular set of properties are equal). This type of problem used to be known as a 'rider'.
- Considering alternative definitions of geometrical shapes, deciding which of these are necessary, sufficient and minimal, and becoming familiar with the differences between the meanings of these terms.

Some pedagogically-important classroom activities include inductive reasoning (such as empirical generalisation or verification) in geometrical contexts. Examples might be measuring angles of several figures in order to generalise or to confirm a result. Inductive reasoning is an important mathematical process and can often lead to interesting conjectures. Activities aimed at developing deductive reasoning may legitimately involve inductive reasoning, recognising that this is a step (albeit often a vital one in the classroom) on the way towards forming such a conjecture, with the purpose of explaining why it is true by deductive reasoning.

When designing teaching activities to build on what pupils in secondary school already know, it is clearly vital to ascertain what they already know and are likely to have been previously taught. The geometrical priorities in the National Numeracy Strategy in primary schools (see: DfEE, 1999a, chapter 7; Jones and Mooney, 2003) are stated as:

- 2D and 3D shapes and their properties
- Position and direction
- Transformations

It is probably worth noting at this point that the mathematician Felix Klein revolutionised the study of geometry by defining it as the study of the properties of configurations that are invariant under a set of transformations. As Schuster (1971 p82) explains, "invariance is one of the most important ideas in all of mathematics, and geometry is unquestionably the most natural subject for the demonstration and use of this idea". In teaching, the use of transformations can be a means by which ideas of invariance can be studied most easily and by which the formal definitions of congruence, similarity and symmetry can be related to learners' previous intuitive ideas.

It is probably unfortunate that transformations were first introduced into the school mathematics curricula (during the reforms of the 1960s) at the same time as the overall amount of geometry was reduced and when much more of school mathematics came to be devoted to the idea of function. The consequence of this timing was that the study of transformations was less related to pupils' previous geometrical experience and intuitive knowledge and more related to transformations as mathematical operations. In some ways this has fuelled an apparent ongoing dichotomy between an approach to geometry that is based on congruent triangles (in the Euclidean tradition) and the use of transformation-based arguments. This dichotomy is captured in the current specification of the Mathematics National Curriculum for England (DfEE, 1999b) where there is a very odd separation, in the programme of study for Shape, Space and Measures, of 'geometrical reasoning' from 'transformations and co-ordinates', as if transformations, for example, cannot be used in geometrical reasoning. Mathematically, as Willson (1977), Barbeau (1988) and Nissen (2000) amply demonstrate, there is nothing to choose between methods based on congruent triangles and those based on transformations. Taking an isometry as a transformation that preserves congruence, any proof by congruence can be translated into a proof by transformations, and *vice versa*. One version may be neater or shorter than the other, but, in practice, neither approach is the sole purveyor of elegant proofs. Experience of more than one approach is, according to Meserve (1967), "a necessary step in the obtaining of sufficient understanding to apply geometrical concepts effectively to mathematical problems".

Pupil understanding of proof and proving in the UK has been the subject of a major research study, carried out prior to the implementation of the current version of the National Curriculum. The project found that, even for high attaining Year 10 pupils (aged 14-15), "empirical verification was the most popular form of argumentation used by students in their attempts to construct proofs, and in problems where empirical examples were not easily generated, the majority of students were unable to engage in the process of proving" (Healy and Hoyles, 1998). Yet Healy and Hoyles found that students, when involved in carefully-designed teaching experiments, could "respond positively to the challenge of attempting more rigorous and formal proofs alongside informal argumentation". They concluded that "developing approaches where this might be accomplished in the context of geometry, as well as of algebra, would be a useful way forward". That such developmental work is necessary is emphasised in very recent work, designed as a follow-up, which is finding that the evidence from Year 9 students suggests that "only a minority of our students have a clear and consistent grasp of what is meant by a geometric reason" (Küchemann and Hoyles, 2002, p48).

Drawing from these considerations in developing suitable teaching materials for mathematical reasoning, the following are suggested as guiding principles:

- The geometrical situations selected should be chosen, as far as possible, to be useful, interesting and/or surprising to pupils;
- Activities should expect pupils to explain, justify or reason and provide opportunities for pupils to be critical of their own, and their peers', explanations;
- Activities should provide opportunities for pupils to develop problem solving skills and to engage in problem posing;
- The forms of reasoning expected should be examples of *local* deduction, where pupils can utilise *any* geometrical properties that they know to deduce or explain other facts or results.
- To build on pupils' prior experience, activities should involve the properties of 2D and 3D shapes, aspects of position and direction, and the use of transformation-based arguments that are about the geometrical situation being studied (rather than being about transformations *per se*);
- The generating of data or the use of measurements, while playing important parts in mathematics, and sometimes assisting with the building of conjectures, should not be an end point to pupils' mathematical activity. Indeed where sensible, in order to build geometric reasoning and discourage over-reliance on empirical verification, many classroom activities might use contexts where measurements or other forms of data are *not* generated.

1.3 Language, communicating and reasoning

Enabling pupils to communicate their mathematics to their peers and others is an essential aim within the mathematics classroom. Moving from informal discussions, with imprecise oral descriptions and explanations, to precise, unambiguous and concise communication (including that involving written explanations and, for example, the invention of new symbols and diagrams) is one of the most challenging tasks for a teacher.

Giving pupils the opportunity to discuss their mathematics with their peers or teacher, rather than engaging solely in short 'question and answer' interactions, helps them to clarify their thinking and improve their understanding. It also helps them to organise their thoughts in preparation to presenting a cogent argument or line of reasoning.

Communication is particularly important in deductive reasoning. A recent collection of discussion papers by QCA on the use of spoken English in different curriculum subjects is an illuminating resource on the role of language to support pupils' learning, and in particular the section on learning to think through conversation:

> "*More challenging tasks… solving complex mathematical problems, critical analysis…, designing an experiment to test a hypothesis…all need a different approach…We develop conceptual understanding, we learn to classify and process information more efficiently, but an important part of this understanding is the increase in our knowledge about language and its uses… as we become more competent in using this 'discourse knowledge' we improve our ability to communicate our understanding of a principle or an idea to other people. In reaching the point where it can be said that a person understands, we therefore construct and reconstruct our ideas through talk with others….*"
>
> *(Galton, 2023, p.48)*

We have thus, where possible, tried to quote students' own words, sometimes oral and sometimes written, in the following sections of the report so as to allow some insight into their reasoning and how it can be elicited in the classroom. Not surprisingly, some of these quotations are not quite mathematically correct, but they have been selected to show important steps towards logical reasoning. Deductive reasoning is not easy; we believe that most students will need many opportunities and much encouragement to try out their explanations before they understand the nature of the process.

2. Activities developed and trialled

A large number of activities were designed, selected and developed during and between the group meetings, and tried out in schools by the five teachers and other colleagues. We only include details here of those which are reported in later sections of our report. Lesson plans for some of these are included in Appendix B. In the order in which these are mentioned in sections 3 to 6 include:

Name of activity	*Groups with whom trialled*
Shape properties	Jo: Y7 (set 4 of 5) Y8 (set 4 of 5)
Property sorter	Jill: Y7 (mixed ability) Y8 (set 3 of 4) Y9 (set 3 of 4)
Diagonals (folded paper)	Carol: Y9 (top set)
Diagonals (Geometer's Sketch Pad)	Carol: Y8 (bottom set) Y9 (top set) Y9 (middle set)
Using flow charts to classify	Jill: Y7 (mixed ability, set 4 of 5) Y9 (set 2)
2 piece Tangrams	Peter: Y9 (set 4 of 5) Y8 (set 5 of 5) Peter & colleague: Y9 (set 1) Jill: Y7 (mixed ability) Jill: Year 7 (set 4 of 5) (as starter activities)
Working with angles	Jill: Y9 (set 2) Paul: Y7 & Y8
Isosceles triangles	Jo: Y9 (top set)
Pythagoras theorem (and associated problems)	Jo: Y9 (top set)

This classroom trialling was the central part of the work of the group. Members reported in several ways, using video, students' written work, computer recorded work and verbal reports. What is clear from all the reports is that the materials by themselves will always be insufficient to develop geometrical reasoning to the extent that is desirable. It is clearly important that teachers use materials in ways that prompt pupils to engage in reasoning activities. The reports of the trials suggest several strategies, including:

- The questions which are posed in the materials themselves;
- Teacher intervention during classroom activity to elicit reasons for pupils' conclusions;
- Structured student recording;
- Group work where students are required to give explanations to each other, and
- Presentations by one or more students to the remainder of the class.

The following sections contain reports on the trials, with some parts (those shown in boxes) written by the teachers who used the materials, and other parts summarising discussions held at meetings of the group.

3. Classroom reports: reasoning about properties and definitions of shapes

3.1 Introduction

One aim of the 11-16 geometry curriculum is to help children to progress in sophistication in their ways of thinking about shapes and their properties. This means not only learning about the properties of shapes (e.g. conjecturing, empirically or intuitively, that rhombuses have 2 lines of symmetry) but will for many also lead on to local deduction (e.g. trying to reason why rhombuses must have 2 lines of symmetry, starting from a definition of a rhombus as a shape with four equal sides). Geometrical reasoning could include the notion of alternative definitions, and of sufficient definitions (e.g. could a rhombus be defined as a quadrilateral with 2 lines of symmetry? What extra would be needed to pin down only the set of rhombuses?). It might also lead to the idea of minimal definitions (e.g. rhombuses have two pairs of parallel sides and diagonals which bisect each other at right angles – which bits of this definition could be left out? Is there more than one answer?). This necessarily includes examining relationships between sets of shapes (e.g. rhombuses are themselves a subset of parallelograms, and include squares as a subset).

The van Hieles (1986) have described a progression in geometrical reasoning that might underlie a geometry curriculum. The levels are

1) Identification or production of a shape by visual recognition (e.g. recognising or drawing a rhombus);
2) Awareness of properties of classes of shapes (observing that a rhombus has four equal sides but no right angles);
3) Beginning to clarify definitions or relations between different shapes and between properties, and to make some logical connections;
4) Developing deductive reasoning, deriving new theorems from one or more axioms or theorems accepted as true;
5) Appreciation of the abstract structure of an axiomatic system, with axioms as an initial set of statements accepted as true, and a network of theorems which can be derived from these.

A considerable amount of international research has been carried out on this proposed progression; much of this research has broadly validated its usefulness but some has shown that it is a limited tool (see Jones, 2001, for an overview of this research). Nevertheless, it seems helpful to refer to it in discussing different types of reasoning which occurred in some of the activities used in classrooms in this study.

3.2 Listing properties of shapes

Shape Property Activity

Jo gave the task to Year 7 pupils of writing down all they knew about two shapes represented by pictures, an equilateral triangle and a square. This activity, in addition to raising issues of logic and reasoning, was particularly useful in demonstrating the problems of written communication among some pupils (especially boys in set 4 out of 5). A way of helping deal with them with spelling, which was used by Jo as a follow-up, is described in the later section about 'language and communication'. Some examples which include issues in both mathematics and language skills are given below; the numbers by each name indicate assessed levels of mathematical attainment.

- Jason (3a~ Y7) *triangle. 3-sides. No angeles.*

- Dale (4b~ Y7) *triangle. 3 corners and 3 sides the same.*
- Tom (4c~ Y7) *equarltral. 3 equale sides. 3 ponits 60º each.*
- Scott (3a~ Y7) *squir. 4 side all the same. 4 right angles the same length.*
- Lee (4b~ Y7) *Square. 4 sides all the same length. 2 sets of parallel lines. 4 right angles (90º).*
- Jason (5c ~ Y7) *Square. Has all sides parallel and equal. Has 4 right angles 4 corners (vortexes) and 4 sides. Like a rectangle but different*.
- Daryl (5c ~ Y7) *It is a square with 4 equal sides, all the angles are 90º. They are called right angles. It is a quadrilateral*.

These answers potentially provide the basis for some interesting discussions that highlight alternative definitions. The properties are correct (the intentions are clear even where the expressions are imprecise e.g. 'all sides parallel' and 'angles the same length'). Are there any others that could have been chosen? Do any of the answers provide redundant information? A square is a quadrilateral but is it 'different' from a rectangle?

So, rather than just marking and returning the answers, Jo decided to use a form of 'peer assessment' which involved students commenting on and comparing each other's work. Because doing this directly might have been upsetting for some in this class, Jo extracted some of the properties referred to by different children without indicating their origin, and asked which gave the best description.

- *It is called a square.*
- *A square is a quadrilateral.*
- *It has 4 sides and 4 corners.*
- *It has 4 right angles.*
- *It has 2 pairs of parallel lines.*
- *All the sides are the same length.*
- *That is what a square is.*

Note that the first and last of these quotations, each attached to a rough drawing of a square, could be used to illustrate van Hiele level 1 thinking which does not go beyond matching shapes with names.

The same task was given to the equivalent set (4 out of 5) in Year 8. It was clear that the Year 8 pupils had better vocabulary and some were able to use more precise language. For example:

- Caroline (4b) *Square. Each of the sides are the same length and parallel. Angles are all 90º.*
- Kabita(4b).*A square. All of the sides are the same length. Opposite sides are parallel. 4 right angles 360º.*
- Ryan (4c). *Square. 4 right angles. 4 sides the same.*

(There are some further examples of lists of properties of shapes from a mixed ability group of Year 7 children in the 2-Piece Tangram Activity in section 5 of this report)

3.3 Finding shapes with given properties

The inverse of listing properties of a single shape is the identification of shapes that possess one or more given properties. This was the intention of an activity carried out by Jill which is an adaptation of an activity shown being used with a mixed ability Year 7 class in the video for the Key Stage 3 strategy training, which is in turn modelled on an activity in Fielker (1981/1983). The worksheet is reproduced overleaf, with an account, below, of its use in the classroom.

Property Sorter Activity (as described by Jill)

> This was done with year 8 set 3 of 4, year 7 (mixed ability) and year 9 set 3 of 4.
>
> It was first attempted in the form in which it was given as a two- way table and a set of shape outlines and properties. Pupils found this very difficult to do. They chose properties to go in each of the outside boxes but then found it difficult and in some cases impossible to put the shapes into the table. Pupils often found when approaching the problem this way that they had entered the properties in such a way that they had to find a shape to fill a space that had two mutually exclusive properties. This led to some useful discussion on which properties could go together and which could definitely not go together. Some students then attempted to group the properties into categories. They still found the task really difficult and got several pieces in before realising that they couldn't go any further and had to start again.
>
> We discussed the problems and came up with the idea that instead of trying to fill in the shapes we would first decide what properties applied to each shape. So the pupils made a list of the shapes and then used the property cards to decide whether a particular shape had a particular property. They then looked for shapes sharing pairs of properties and when they did this several of the groups managed to complete the grid. The pupils liked doing it this way as although it initially took longer it seemed much more logical to them and made them think about the properties of each shape. The first method was rather hit and miss and although it raised some issues about mutually exclusive properties many pupils found it frustrating to get so far and then have to start again.

.
Sections of work shown on the pages following the copy of the worksheet were taken from two students in Year 8 (set 3 out of 4) who were in the first trial.

The first student progressed from matching one of the given properties with each given shape to identifying which sets of shapes satisfy each property. It is interesting that Jill's decision to include a square in two different orientations has led to the naming of one of these separately as a 'diamond' (again this is a characteristic of van Hiele level 1 where perception is more important than properties). The inclusion of an arrowhead under 'opposite sides equal' may suggest an alternative meaning of 'opposite' as 'reflected', which may also be used by the second student in considering the angles of an isosceles trapezium.

The second student has progressed from matching (not always correctly) many properties on the worksheet to each shape, to filling in properties around the table and identifying single shapes that fit in the cells.

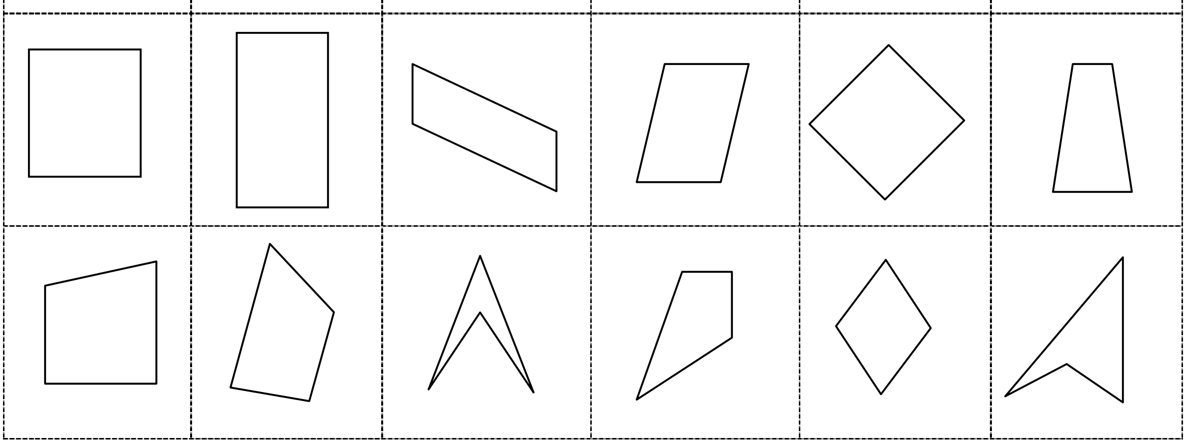
Worksheet 1 Properties Game

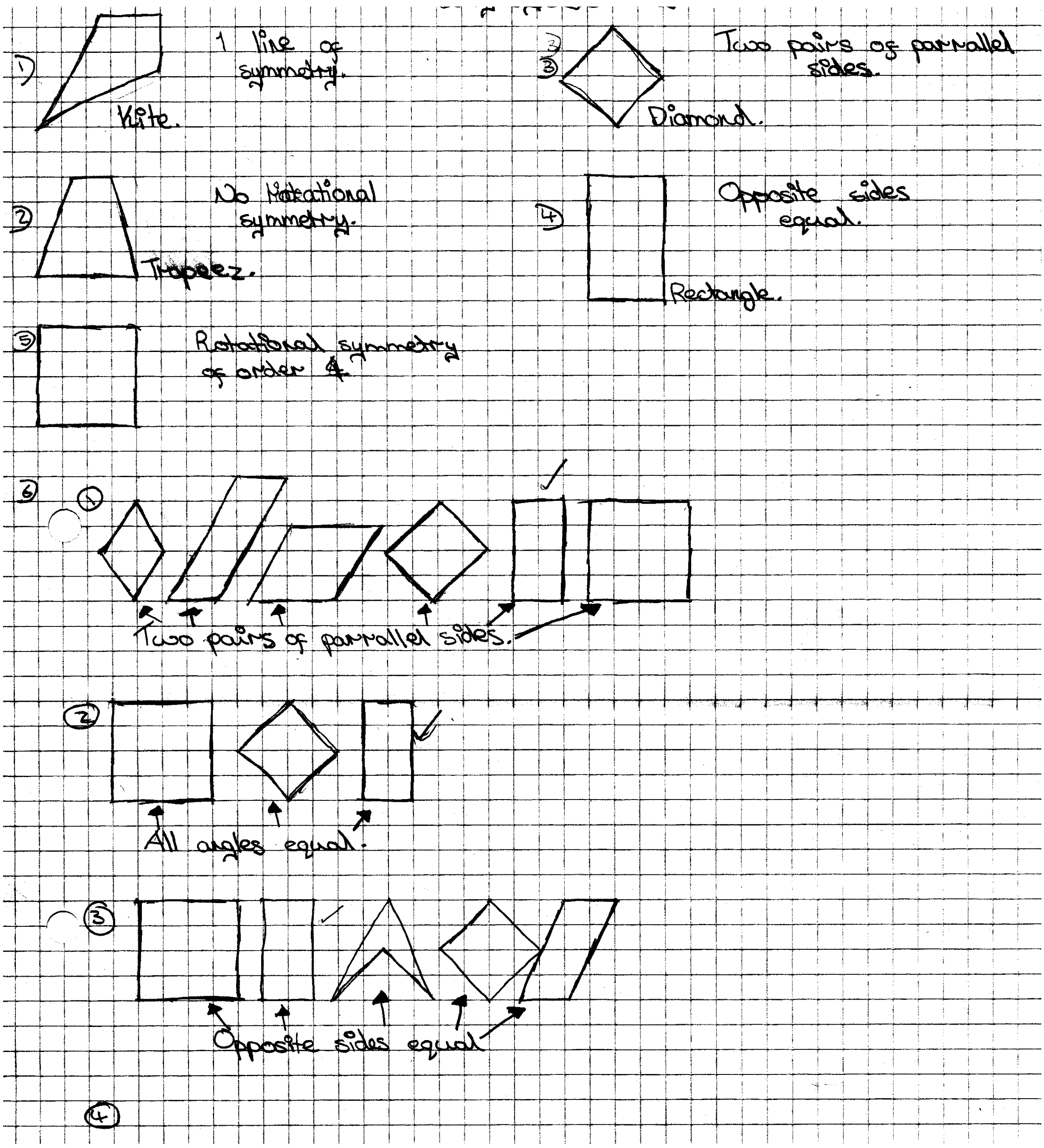
Cut out the shape and (along the dotted lines) the shape and property cards so as to leave the board intact.

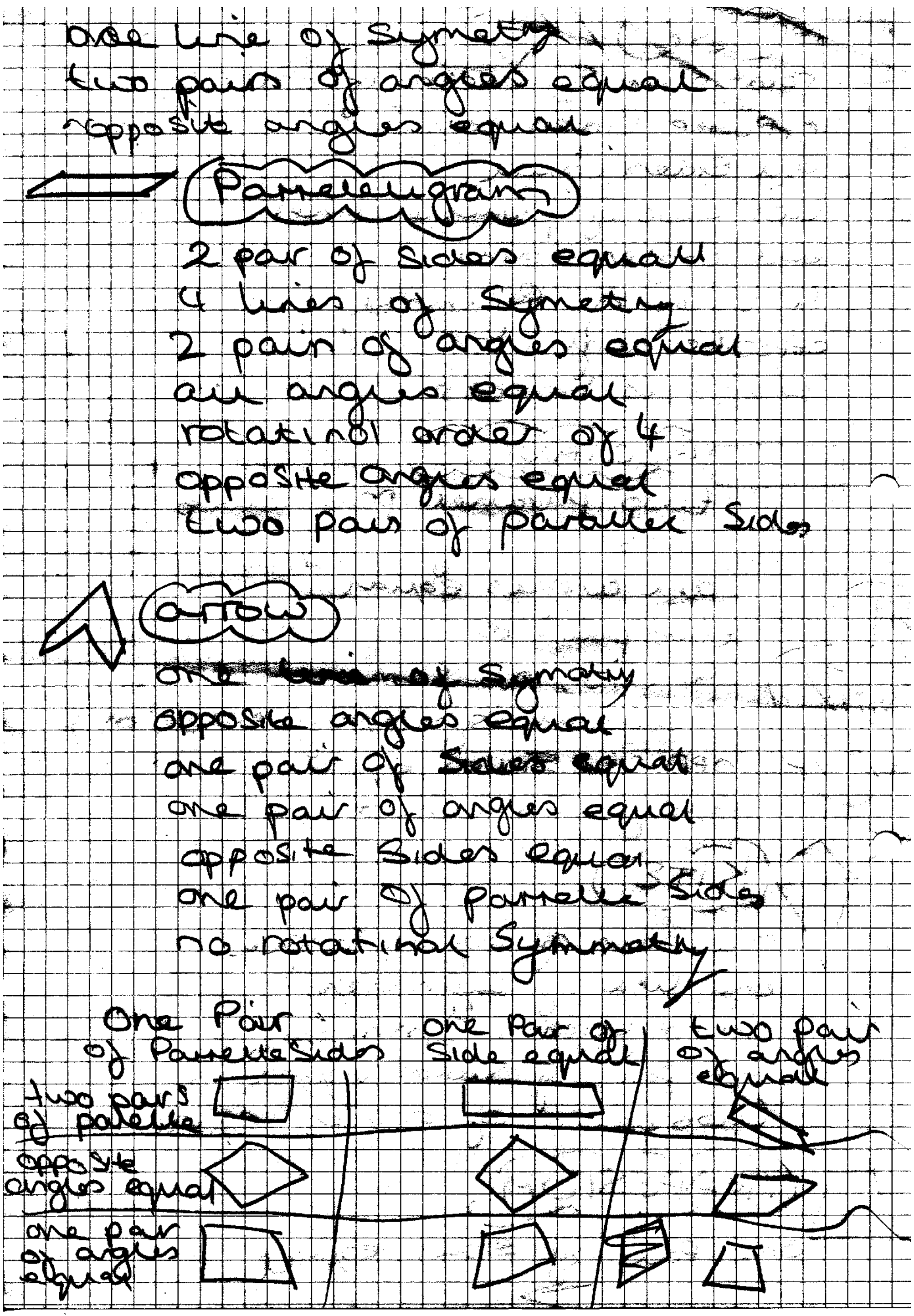
Put a property card on each shaded section of the board.

Put as many of the shape cards on the board, satisfying the row and column properties, as you can.

				one pair of parallel sides	two pairs of parallel sides
				no rotational symmetry	rotational symmetry of order 2
				rotational symmetry of order 4	no lines of symmetry
diagonals bisect	diagonals bisect at right angles	one pair of sides equal	1 line of symmetry	2 lines of symmetry	4 lines of symmetry
all sides equal	one pair of angles equal	all angles equal	two pairs of angles equal	opposite sides equal	opposite angles equal




1) 1 line of symmetry.
Kite.
3) Two pairs of parallel sides.
Diamond.
2) No rotational symmetry.
Trapeez.
4) Opposite sides equal.
Rectangle.
5) Rotational symmetry of order 4.
6) 1
Two pairs of parallel sides.
2
All angles equal.
3
Opposite sides equal
4



Clearly this is a rich task that can give rise to a great deal of reasoning activity – for example deciding for different placings of the property cards which cells must have multiple entries and which can have no entries, deciding whether e.g. ‘one pair of parallel sides’ should mean ‘only one’ or ‘at least one’, etc. However the trials suggest that it is necessary to think in advance about how much structure is appropriate for different sets of pupils.

3.4 Investigating diagonal properties of quadrilaterals

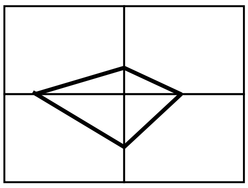
In some ways this is similar to the previous activity, in that students are not asked to list diagonal properties of each type of quadrilateral, but a task related to the inverse of this, identifying shapes with given diagonal properties, giving reasons. However the activity was made more complex as, although in each case the angle between the diagonals was given, the lengths of the diagonals were allowed to vary. Thus one might expect it to be at least as difficult as the shape sorter (van Hiele level 3), and it certainly turned out to be quite challenging. Again it built on an idea from Fielker (1981,1983). A full lesson plan is given in the Appendix to this report.

Diagonals Activity I (as described by Carol)

This task was trialled with a Year 9 top set. The starter for the lesson was a review of the angle and line properties of quadrilaterals. A quadrilateral was drawn step by step on the board. Students had to put their hands up when they were certain that they had enough information to identify the quadrilateral. This encouraged argument and counter-argument, and set the tone for the lesson. The starter was selected to avoid focusing on the diagonals. With a weaker group it may be more appropriate to focus on symmetry of quadrilaterals.

The first part of the main task was explained - a pair of perpendicular diagonals, although this was not made explicit:

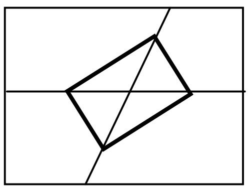
Fold a piece of A4 paper in half vertically and horizontally
Put a point on each half of each of the folds.
Join the 4 points to form a quadrilateral.
Investigate which quadrilaterals can be made and which ones can't.



The students accessed the task easily and were able to construct a range of quadrilaterals. They were asked to explain to a neighbour how they knew that the shape they constructed was a square by thinking about the triangles between the diagonals and sides of the shape. (The lesson plans in the appendix give examples of some key questions to help pupils develop their ideas.)

Students then moved on to the second part of the task - a pair of non-perpendicular diagonals.

Fold a second piece of paper in a different way and repeat the task to investigate which quadrilaterals can now be made.
Explain why certain quadrilaterals can be made.



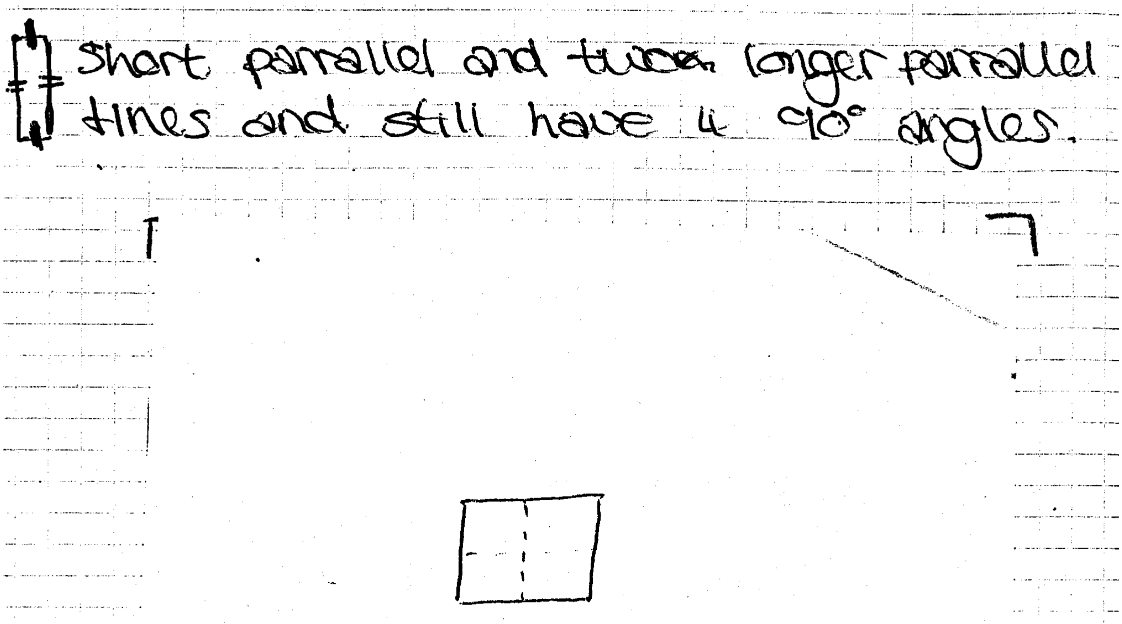
Again students were very competent in carrying out the task and correctly identifying the quadrilaterals constructed.

The final part of the task was to try and explain why certain quadrilaterals were not possible with each pair of diagonals.

Many students were certain that a rectangle could not be constructed using the first pair of diagonals, and that a square could not be constructed using the second pair of diagonals. They had a really strong feel for the task and many were quite frustrated that they could visualise why certain things weren't possible but struggled to explain verbally. Many explained what happened to the sides and angles as points were moved. The students' frustration provided an excellent opportunity to illustrate the usefulness of geometric proof.

In the plenary for the lesson (15 minutes), pairs of students used the OHP to explain their ideas to the rest of the class. This again stimulated discussion and forced students to explain carefully.

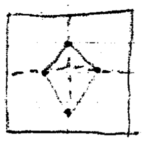
Some copies of student work are shown below:



Friday 17th January

Investigation of Quadrilaterals.

Convince yourself
Convince a friend
Convince teacher



- what quadrilaterals can you obtain.
- Are there any you can't get
- why not? explain.

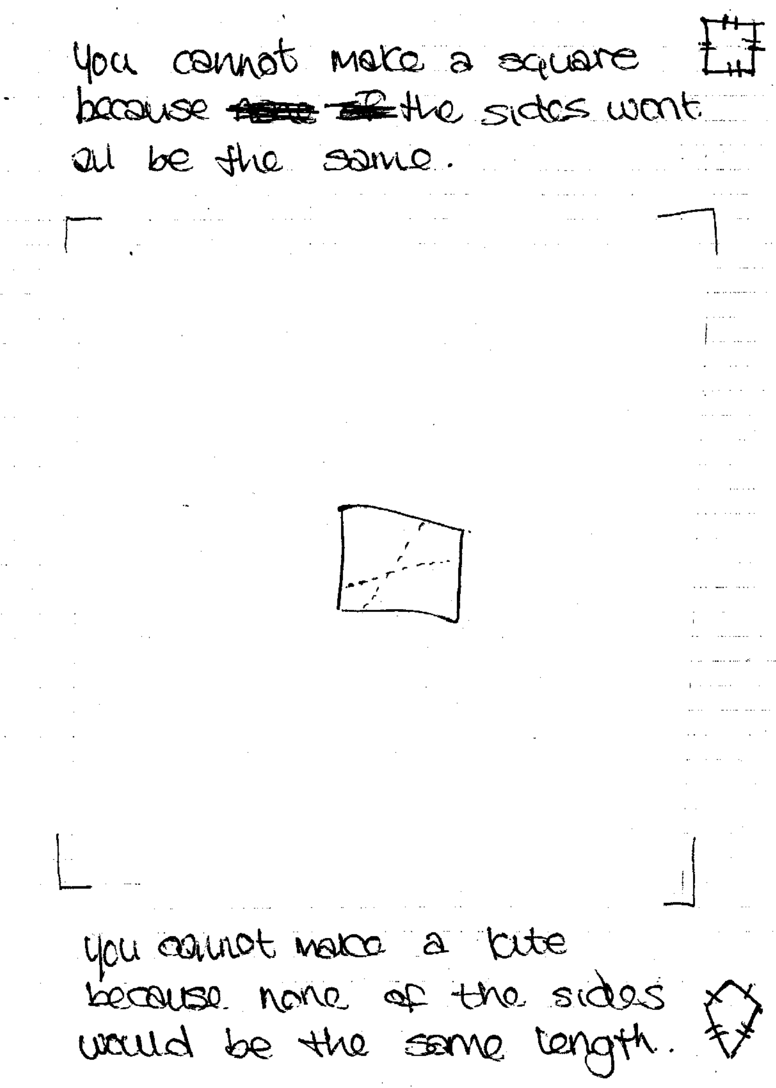
predictions

Shapes you can make:
- ~~Diamond~~
- kite
- square
- trapezium.

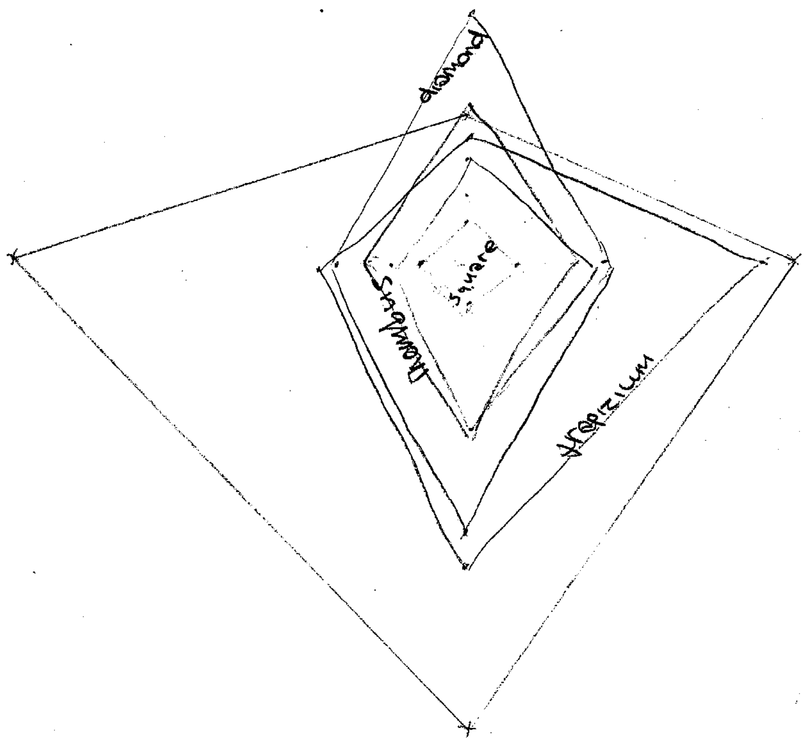
Shapes you cant make:
- Parrallelograms
- rectangle
- rhombus you can make a rhombus

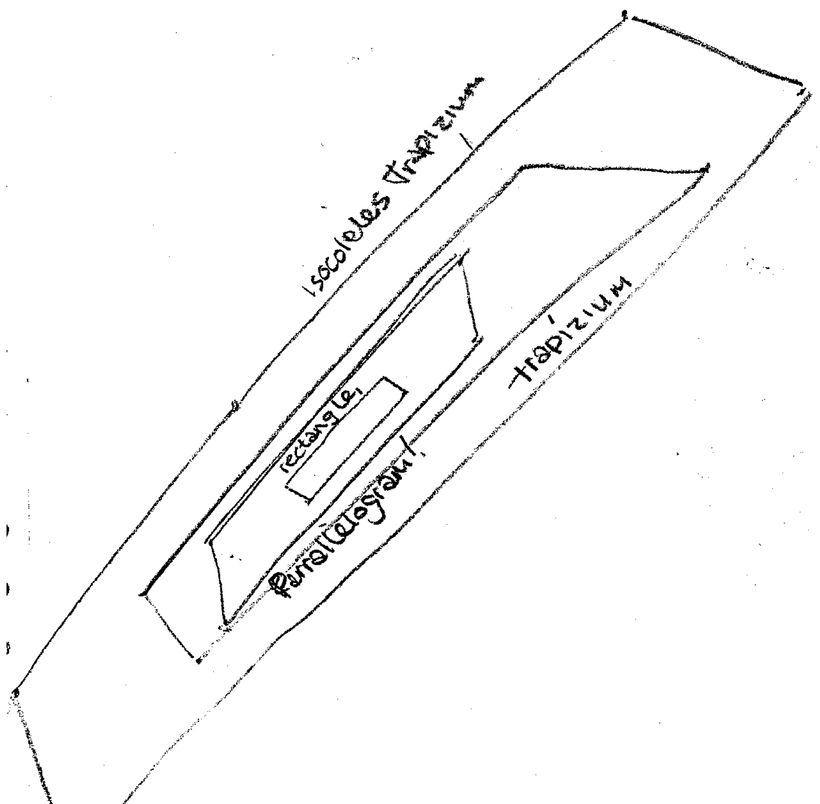
you cannot make a parrallelogram because

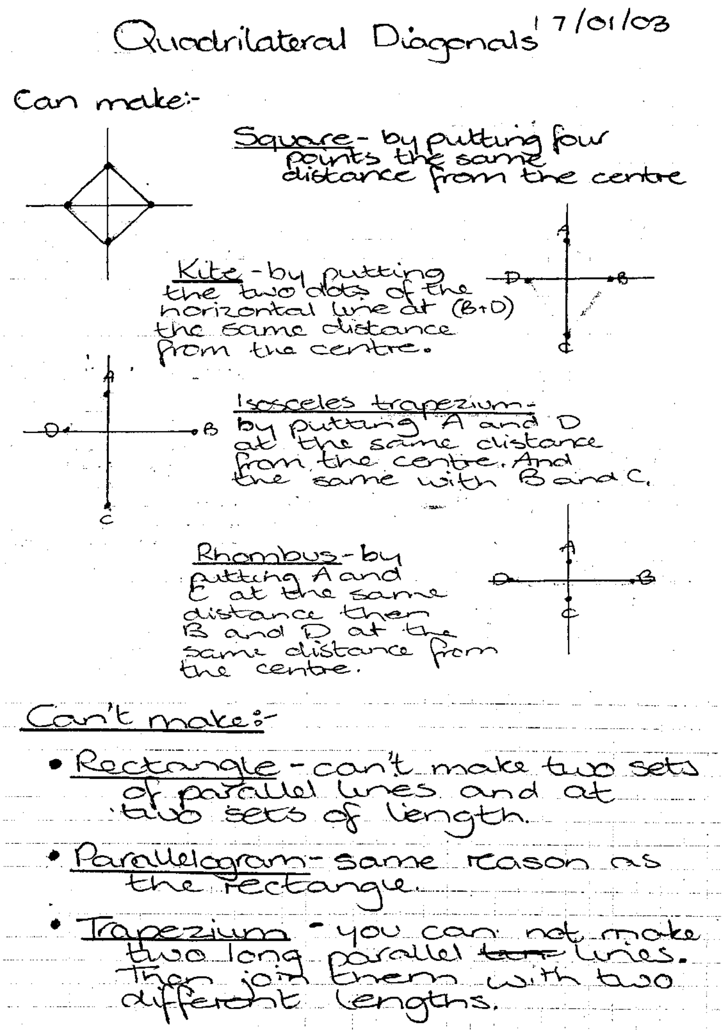
you cannot make a rectangle because you cannot make two



The start of the second piece of student work illustrated above shows the problems of trying to explain reasons clearly when trying to start with how the shapes were constructed rather than by reversing the reasoning to work with shape properties. However it finishes as shown below with an innovative (and almost correct) method of presenting the results for which shapes can be obtained in each case. This also almost manages to incorporate inclusion relations correctly. There are a lot of potential class questions here; for example is an isosceles trapezium constructable in both cases? What can you say about the symmetries of shapes with perpendicular diagonals?


diamond
square
rhombus
trapizium


iscoleles trapizium
trapizium
rectangle
parralelogram


Quadrilateral Diagonals 17/01/03

Can make:-

Square - by putting four points the same distance from the centre

Kite - by putting the two dots of the horizontal line at (B+D) the same distance from the centre.

Isosceles trapezium - by putting A and D at the same distance from the centre. And the same with B and C.

Rhombus - by putting A and C at the same distance then B and D at the same distance from the centre.

Can't make:-

- Rectangle - can't make two sets of parallel lines and at two sets of length.
- Parallelogram - same reason as the rectangle.
- Trapezium - you can not make two long parallel lines. Then join them with two different lengths.

The second piece of work is more systematic about way the shapes were constructed and is getting closer in respect of the reasons why some shapes cannot be obtained. The error in relation to trapezia perhaps arises because of a limited idea of the shape as necessarily including ‘two long parallel lines’ (perhaps a regression to van Hiele level 1, or maybe, as a theme in the research suggests, pupils operate at different “levels” in different contexts and at different times such that the “levels” are more a way or categorising activity rather than labelling pupils).

The task thus was successful in revealing some misconceptions and provided plenty of discussion points. However it proved to be challenging for this Year 9 set 1 to reason why some shapes could or couldn’t be generated. Carol felt that the activity had perhaps been too open-ended in terms of pupil expectations. As in the case of the quadrilateral property sorter, the structure and information presented up front may have been too complex to enable enough progress to be made. It is clear that the issue of how much structure to provide in a task is an important factor in maximising the opportunity for geometrical reasoning to take place. Of course it may be with more time available and greater experience of doing geometrical reasoning tasks like these, students could learn to manage more open tasks.

In order to try to make the task a bit easier, Carol decided to do a parallel version using the Geometers’ Sketch Pad program to generate a more structured set of activities.

These were designed to guide the students through the investigation at their own pace, with an exercise book and pencil to hand.

On the computer screen, different coloured text was used for instructions, for information that pupils could copy into their books if required, and questions that the pupils are required to discuss and record an answer to.

The activities were:
Activity1): Students were presented with the diagonals of a square (i.e. two intersecting lines which bisected each other at right angles). They were asked to check by measuring that the four diagonal segments were equal and perpendicular. They were then asked to join the vertices and measure the sides and angles, and to name the shape. The figure was programmed so that one vertex could move freely along its diagonal allowing the square shape to be maintained but enlarged or reduced. Students were asked to move the vertex and to describe the result.
Activities 2), 3), 4) and 5) were all similar in form except that the configuration of the diagonals was different. Progressively more degrees of freedom were also allowed in the movement of the vertices, and rotation of the diagonals, so that students could investigate special cases.
Activity 2) had bisecting diagonals that were neither perpendicular nor equal.
Activity 3) had only one diagonal the perpendicular bisector of the other.
Activity 4) had equal bisecting diagonals that were not perpendicular.
Activity 5) had unequal bisecting diagonals that were perpendicular.

Diagonals Activity II (as described by Carol)

This task was trialled with a Year 8 bottom set, and Year 9 middle set and top set. To enable the students to access this task, they needed to be able to use basic tools in Geometers' Sketchpad and understand 3 letter angle notation.

The diagonals were created to allow changes to be made but for some properties to be fixed. For example, in number 5 the diagonals are perpendicular bisectors, but not necessarily of equal length. The task requires students to investigate the properties of the diagonals and then connect the vertices to determine what the shape is.

Students were able to access this task very easily and worked at their own pace through the problems. They were given the option to make notes in their exercise books or to answer the questions on screen and then save the document to their area. This helped increase their motivation.

When asked to measure the angles around the centre point, several students enquired if this was compulsory because they were able to work out all the angles by measuring just one of them.

Students were able to make connections between the shapes that were possible with each set of diagonals. They easily determined that to turn a rectangle into a square they needed to make the diagonals cross at right angles.

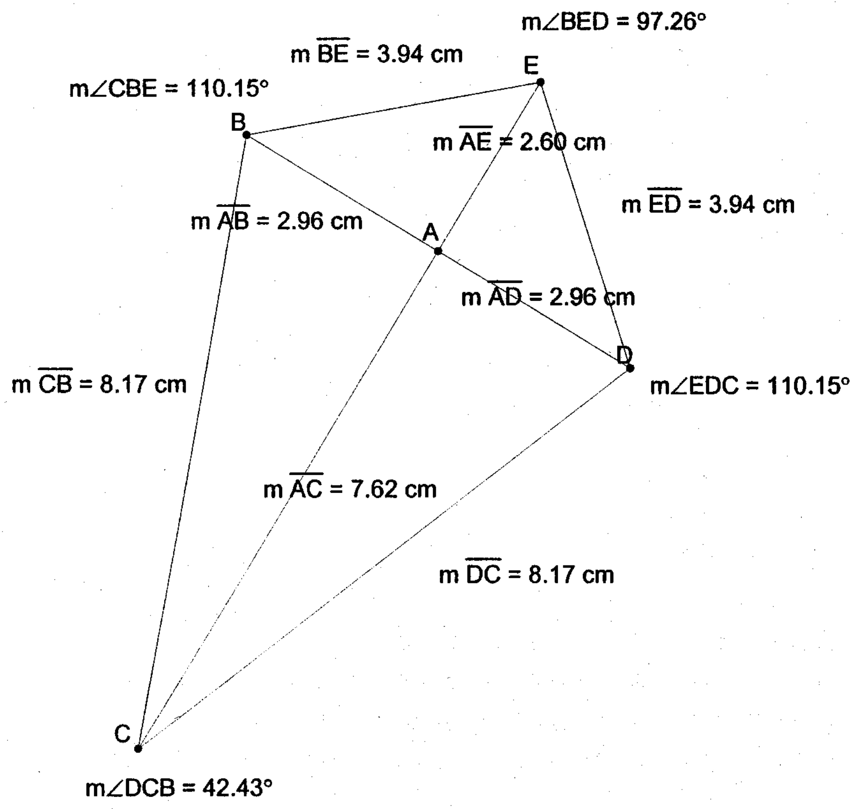
As a plenary with Year 9 set 1, two students showed their answers to numbers 3 and 5, explaining the properties of the diagonals and the shapes that were feasible with each.

This task did enhance students' understanding of properties of quadrilaterals and did encourage them to think about their reasoning.

Examples of students’ work on Activities 3 and 4 are shown below.

Number 3

Trystan Glaze



What happens to the lengths?
They can all change
Which ones are always the same?
AD & AB

Are the lines perpendicular?
Yes
Do they both bisect the other?
No

Which lines are equal in length?

CB and DC
BE and ED
CE is a line of symmetry

Are any angles equal?

CBE and CDE are because of the symmetry

Number 4

$m\angle BAE = 123.93°$

$m\angle BAC = 56.07°$

CE = 12.28 cm

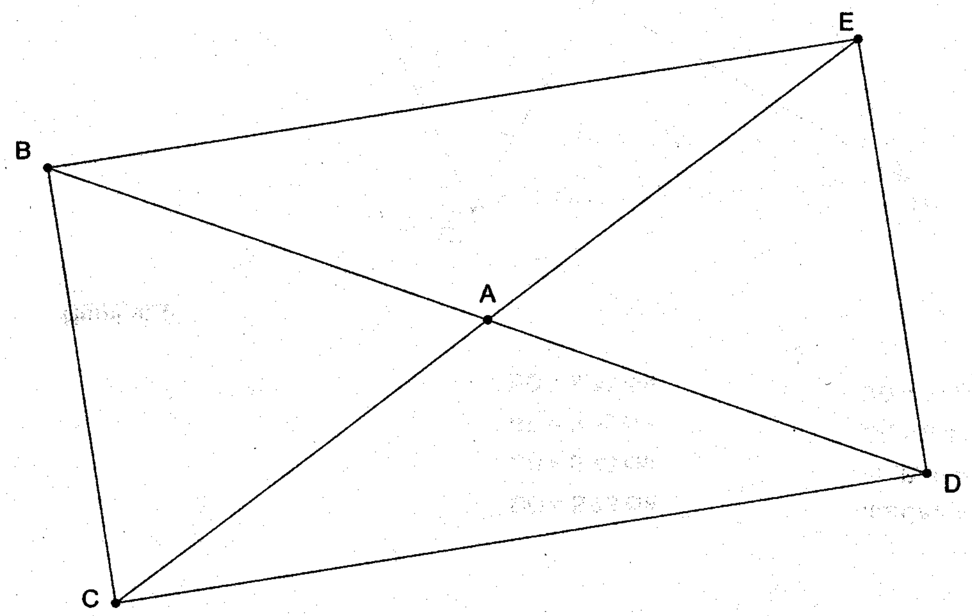
AC = 6.14 cm

AE = 6.14 cm

BD = 12.28 cm

AB = 6.14 cm

AD = 6.14 cm



Measure the lines and angles

Are the lines bisectors or perpendicular bisectors?

They are bisectors

Join the points to form the quadrilateral. Measure the sides and angles to confirm what shape it is.

BC = 5.77 cm	$m\angle BCD = 90.00°$
CD = 10.84 cm	$m\angle CDE = 90.00°$
DE = 5.77 cm	$m\angle DEB = 90.00°$
EB = 10.84 cm	$m\angle EBC = 90.00°$

Move the points.

What do you have to do to the diagonals to turn it into a square?

Make the angles on the diagonals 90 degrees

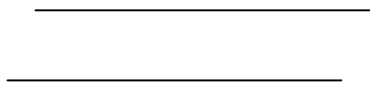
The use of Geometers' Sketch Pad enabled students to become familiar with the diagonal properties of quadrilaterals. Since the earlier task had proved a little too difficult, the Geometers' Sketch Pad version was created to be more structured and inductive in the type of reasoning required.

3.5 Classifying quadrilaterals

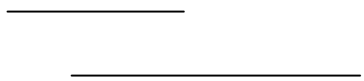
Flow charts were used in the in the main section of lessons within one week with a set 4 out of 5 in Year 7, following starters aimed at reminding about names and properties of triangles and quadrilaterals (see lesson plan in Appendix) . They were also used with set 2 in Year 9.

> Flowchart Activity (as described by Jill)
> The flow charts proved useful but had some problems. They appeared to work when used on the OHP as a framework for classification, but the children found it more difficult to construct their own. They were able to make progress with classifying triangles, and this gave rise to a realisation that for example an equilateral triangle is also isosceles. With other shapes the construction of flowcharts proved more problematic.
>
> Flowchart activities were also undertaken with a Y9 second set. They were shown a simple flowchart which classified polygons by their number of sides. The pupils then worked together to construct their own flowcharts. An important outcome was their development of a planning strategy. Those who started immediately trying to draw a flowchart realised that this was not working, and so they developed a plan involving listing the variety of shapes (triangles for example), a variety of properties and a variety of questions. They tried various classifications and it was clear that this brought out relationships between shapes and between properties. They began to find common properties appearing in different parts of their flowcharts.
>
> The classification activity provided some insight into common misconceptions, for example that parallel lines have to be equal in length, and that a trapezium can't have a right angle. It seemed likely that these could arise from the standardised way in which we present diagrams.

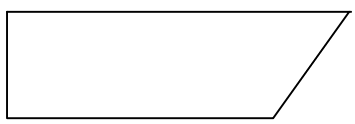
For example:



rather than



rather than



It was clear that as a result of this activity they had developed a far greater knowledge about the shapes and the relationships between them. Their thinking and reasoning developed and began to become explicit. They were discussing things such as necessary and sufficient conditions. Two examples of pupils work are reproduced below. They worked in pairs.

In the first example we can see that some pairs preferred to use the computer. They found it made changing the flowchart much easier than having to continually re-draw it.

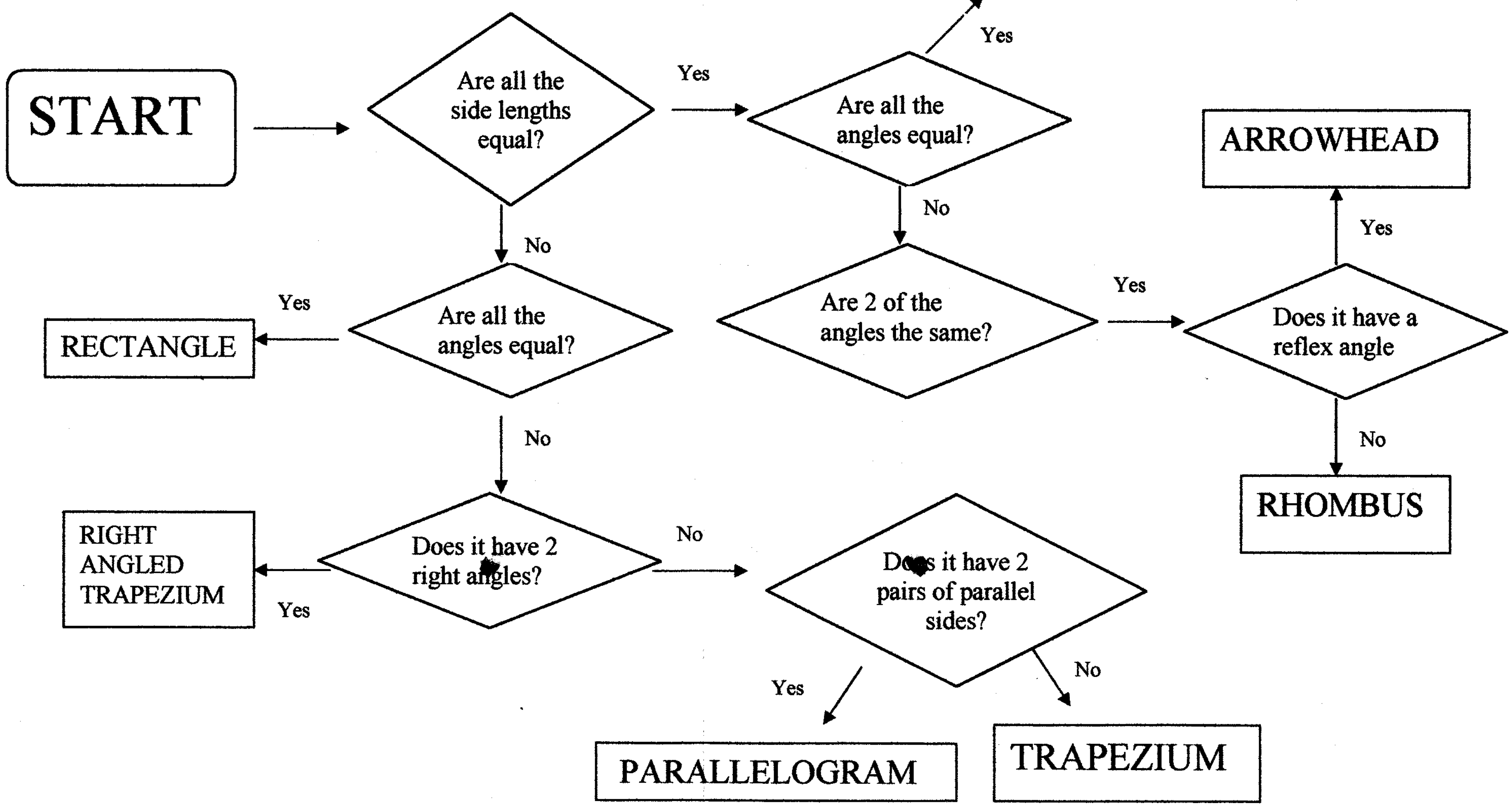
Students' work from Pair 1

Report on the Flowchart that we had to design for quadrilaterals

Our task was to design a flow chart for all the quadrilaterals there are. The idea of the flowchart is so you can define a shape by the properties it has by following the chart until you end up with a shape that resembles yours.

We came across many difficulties in doing this task. The single most frustrating one was the fact that if you had finally thought you had completed the chart, then realise that you made one minor mistake that mucked up the whole thing up so you had to start again. There also was the problem that you have chosen all the questions but you can not think of another question that tells apart two shapes. For example we could not think of a question which we hadn't yet used to tell apart an arrowhead and a rhombus. In the end we came up with the question, 'does it have a reflex angle,' which an arrowhead has but a rhombus does not.

Quadrilateral Flowchart



Students' work from Pair 2 (the original handwritten flowchart was A3 size)

How we did our Flowchart?

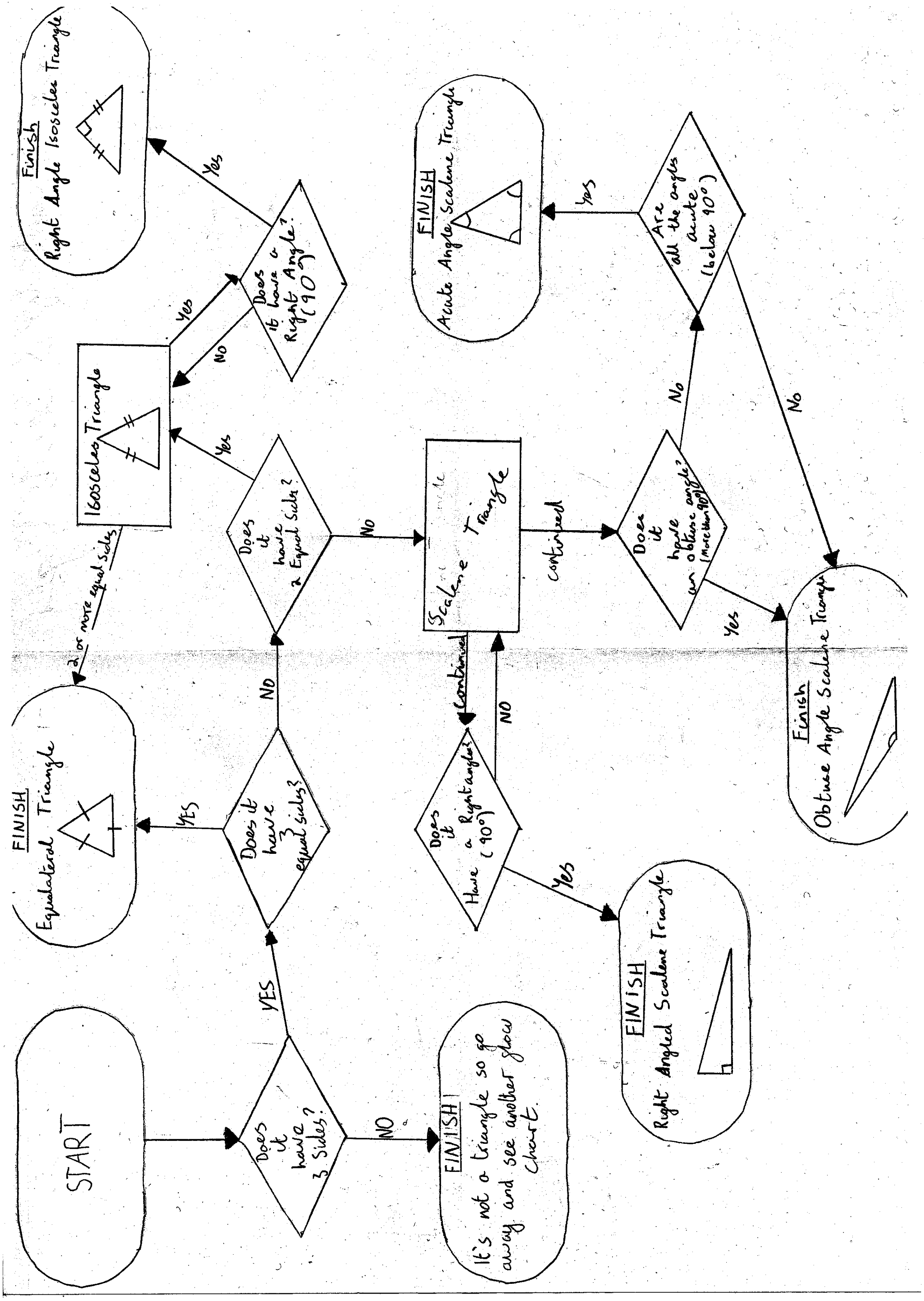
First of all we looked at every type of triangle there was, this was quite tricky as we had to draw some of them out to make sure we were right. We also looked at quadrilaterals and other polygons, this was so that we could get a better understanding of describing our flowchart. Once we started to do the flowcharts we had to change the order of things several times as we put in the same question twice. Also we had to make sure none of the answers could be two different triangles, this started to get harder and harder as we progressed because we had already used a lot of questions to do with triangles so our words became much more complex.

Describing Polygons – Triangle Flowchart

We started the topic geometric reasoning by looking at different polygons and trying to class them. We started to put them into number of side groups. This was OK until we started to find that there were quite a few quadrilaterals and triangles. We then started to try and classify them in there own groups and find a way of accurately defining them. I was allocated triangles so the first thing I did was write down the 4 groups of basic triangles, isosceles, scalene, right angled and equilateral and then simply define them. However I then started to notice that some triangles weren't just isosceles, scalene, right angled or equilateral, they were a new triangle using 2 groups, for example, a right angled isosceles and a right angled scalene triangle. I then started to think about the angles in a triangle. With equilateral and isosceles triangles, 2 or more of the sides are fixed, therefore 2 or more of the angles have to be fixed. So that left me with right angled and scalene triangles. Right angled triangles obviously have a right angle in them, therefore this angle is fixed. So finally I was left with a scalene triangle. I started to play around and came up with the idea that I could have an obtuse angle scalene triangle and an acute scalene triangle. The final list of triangles I decided to classify came down to, isosceles, right angled isosceles, right angled scalene, obtuse angled scalene, acute angled scalene and equilateral triangles.

The way we decided to get our accurate description was by using a flowchart. The first question in the flowchart was "Does it have 3 sides?" we thought this was quite important because it's no good looking for a specific quadrilateral on a triangle flowchart. We then asked questions about the sides and angles of the triangles until we had reached each one of our triangles in our list and obtained an accurate description of them all too. This was harder than it sounds but we only really came across one real problem, which was that we had repeated one of the questions, all that we had to do to sort this problem was just change a few arrows round and do some rubbing out them we were all sorted again.

Using this flowchart we have now got quite a good description of a variety of triangles, no just the basic isosceles, scalene, right angled or equilateral. I have learnt about how to lay out a very good flowchart and that flowcharts can give a lot of information quite simply. I have also learnt many things about triangles, which previous to doing the flowchart I did not know. It has been very useful to display the triangle information as well as a very good learning aid, all in all very pleasing work.


START
Does it have 3 Sides?
NO
FINISH! It's not a triangle so go away and see another flow chart.
YES
Does it have 3 equal sides?
YES
FINISH Equilateral Triangle
NO
Does it have 2 Equal Sides?
Yes
Isosceles Triangle
2 or more equal sides
Yes
Does it have a Right Angle? (90°)
Yes
Finish Right Angle Isosceles Triangle
NO
NO
Scalene Triangle
continued
Does it Have a Right angle? (90°)
Yes
FINISH Right Angled Scalene Triangle
NO
continued
Does it have an obtuse angle? (More than 90°)
Yes
Finish Obtuse Angle Scalene Triangle
No
Are all the angles acute (below 90°)
Yes
FINISH Acute Angle Scalene Triangle
No

3.6 Conclusions to this section

This section has shown examples of four diverse activities tried out with different age and attainment groups within Key Stage 3 aimed at developing mathematical reasoning about the properties and classification of triangles and quadrilaterals. None of these were about proof in any formal sense but all were about logical reasoning and some included informal deduction, for example in asking whether having two pairs of parallel sides implied having one pair of parallel sides, deciding that equilateral triangles must be a subset of isosceles triangles.

Aspects in common between activities were:

- All activities were successful in that they engaged students in developing their reasoning skills. This was true across the full attainment range in KS3.
- Some activities, especially the longer, more open, group activities, were found quite absorbing by students, with some students able to express clearly what they had learned from them.
- In all activities students found logical reasoning difficult. In some cases teachers were able to either adapt activities or to provide an intermediate step to make them easier to tackle.
- Most activities took longer than expected, and it appeared that quite long periods were needed to enable student to arrive at well-reasoned results.
- All involved some degree of openness, even in very short activities tried with lower attaining Year 7 students
- All activities were very helpful in exposing students' difficulties with reasoning, but also with language, misconceptions and/or insecure knowledge. In many cases students did not achieve correct answers on first attempts, but these gave a stimulus for whole class or peer discussion.
- Student responses across all the classes and activities taken together demonstrated each of the van Hiele levels of sophistication from 1 to 4
- Some activities successfully involved students in using ICT to support and develop their reasoning.

4. Classroom reports: from explaining to verifying to proving

4.1 Introduction

As noted above, learning to reason deductively is not easy. Amongst the reasons put forward for pupil difficulties in this area are that learning to use deductive reasoning requires the co-ordination of a range of competencies each of which, even individually, is far from trivial (Hoyles, 1997), that teaching approaches most often tend to concentrate on the verification of theorems or geometric facts and thereby devalue, or omit, exploration and explanation (de Villiers, 1999), and that learning to prove involves students making the difficult transition from a computational view of mathematics to a view that conceives of mathematics as a field of intricately related structures (Dreyfus, 1999).

One promising approach to tacking these problems is that being developed by de Villiers (see, de Villiers 1999). As de Villiers points out, in addition to explanation, mathematical proof has a range of functions, including communication, discovery, intellectual challenge, verification, systematisation, and so on. These various functions, de Villiers argues, have to be communicated to pupils in an effective way if proof and proving are to be meaningful activities for them. In fact, de Villiers suggests that it is likely to be meaningful to introduce the various functions of proof to students in more or less the sequence shown in the figure below.

The likely learning sequence of functions of mathematical proof (de Villiers 1999)

Devoting classroom time to focusing on explanation (before pupils move on the discovering what for them might be new theorems or facts that need verifying), de Villiers argues, should counteract pupils becoming accustomed to seeing geometry as *either* an imposed hierarchy of rules to memorise (and often confuse) *or* just an accumulation of empirically discovered pieces of information.

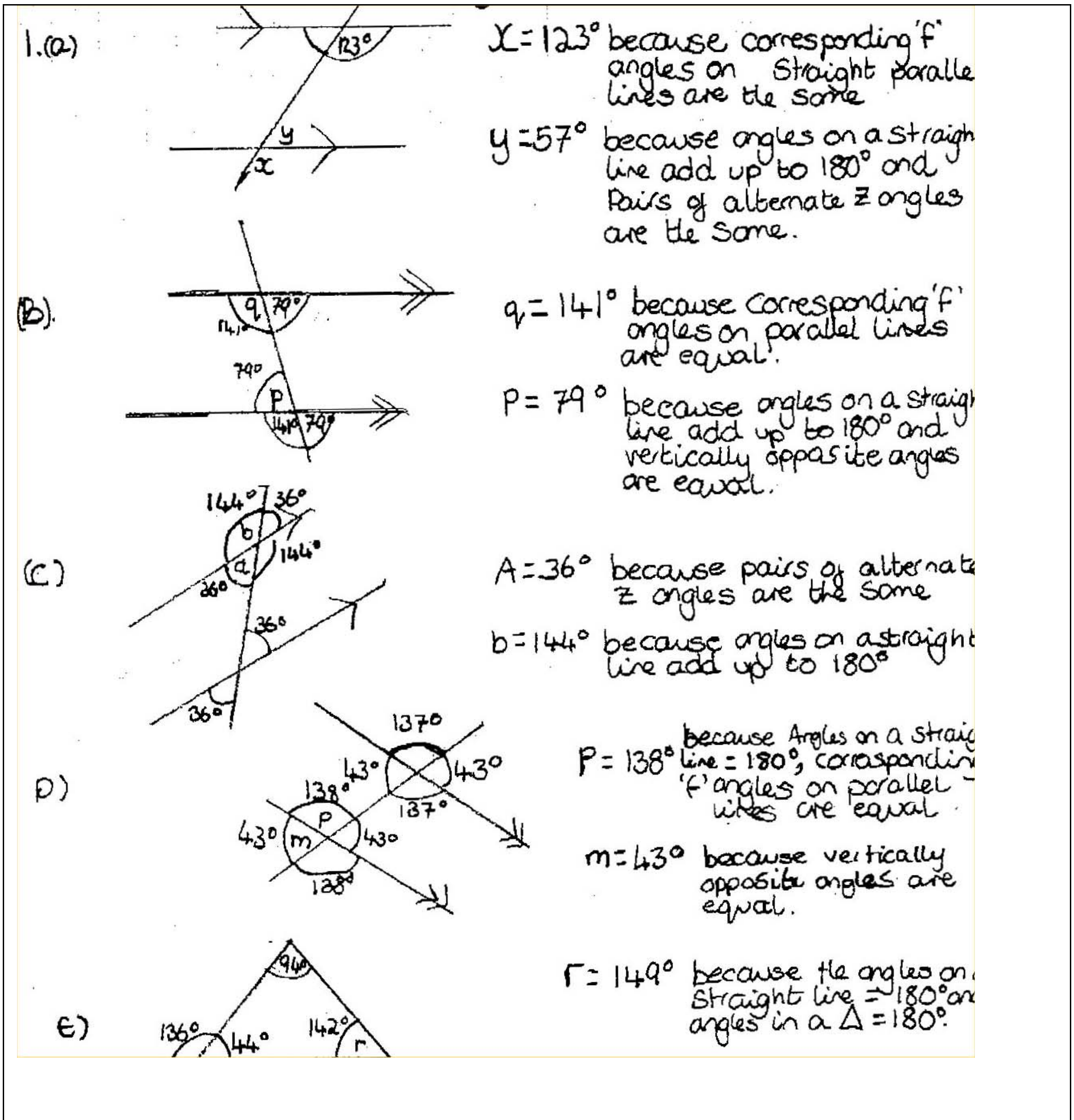
Another critical aspect of learning to reason deductively is learning to appreciate the relationship between inductive and deductive reasoning. Here we take “inductive reasoning” (not to be confused with “mathematical induction” or “inductive proofs”, which are quite different) to be the process of reasoning that argues that a general principle is true because some special cases are true. This is very different from “deductive reasoning” which refers to the process of concluding that something must be true because it is a special case of a more general principle that is known to be true. Appreciating the relationship between inductive and deductive reasoning entails realising that both are necessary parts of mathematical thinking (since inductive reasoning plays a central part in the *discovery* of mathematical truths, while deductive reasoning is the method used to demonstrate with logical certainty that something is true). Given that both forms of reasoning are central to mathematical thinking, getting the relationship right in the classroom involves understanding the complexity of learning to appreciate the relationships between these forms of reasoning. Overall, research has found that only a minority of students, even by the time they graduate from University, can fully distinguish between deductive and inductive forms of argument and thereby experience deductively derived conclusions as necessary and inductively derived conclusions as uncertain (see, for example, Morris, 2002).

Our experience on this project is that most students need many opportunities and much encouragement to test conjectures produced inductively and to try out their attempts at explanations before they begin to develop a fuller understanding of the nature of deductive reasoning in geometry.

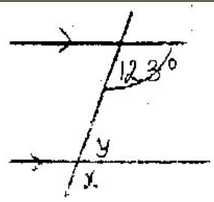
In what follows we try to illustrate some of these issues by selecting results from our classroom trails of various teaching approaches, materials and ideas.

4.2 Working with angles

Lessons involving work with the measures of angle are frequently solely about that. In such cases, whereas pupils have to use their geometrical knowledge, this is not the focus of the lesson. In the examples below, pupils in Jill's classes are expected to explain/justify their calculations.



a.

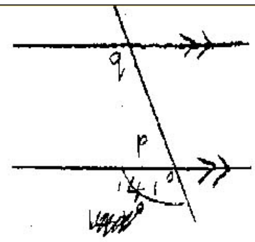


y = 57°

x = 123°

The y angle is 57° because as an angle on a straight lines are equal that means the angle next to angle y is also 123° and as angles on a straight line add up to 180° that means angle y is 57°. The angle x is 123° because the angle next to angle y is 123° the angle vertically opposite has to be the same, so that also makes it 123°.

b.

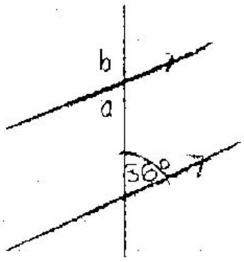


p = 39°

q = 141°

The p angle is 39° because as vertically opposite angles are equal that means the angle next to p is 141° and as angles on a straight line add up to 180° that makes angle p, 39°. As pairs of alternate angles on parallel lines are equal that makes angle q, 141°.

c.

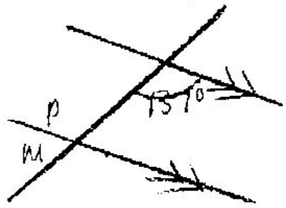


a = 36°

b = 144°

~~The~~ As pairs of corresponding angles on parallel lines are equal that makes A = 36° and as vertically opposite angles are equal that makes the angle next to b also 36° so angles on a straight line equal 180° that makes b = 144°

d.

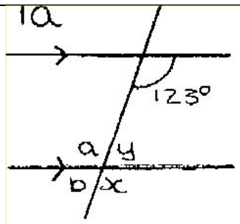


p = 137°

m = 43°

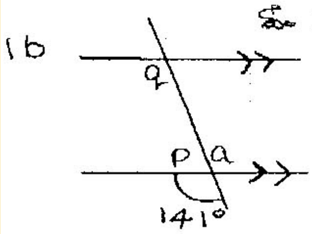
As ~~~~ vertically opposite angles are equal that makes angle p = 137° and that also makes the angle next to m 137° so as angles on a straight line equal 180° m must equal 43°.

1a



I have to Find out what angles y and x are.
pairs of alternate (Z) angles on parallel lines are equal, ∴ angle is going to be 123°. Angles on a straight line add up to 180° so you take 123° of 180° (180°-123°=57°) and that gives you 57° so angle y = 57°. ✓
Pairs of corresponding (F) angles on parallel lines are equal.
so angle b is going to be the same as angle y and angle a is going to be the same as x,
Angle a = 123°
x = 123°

1b



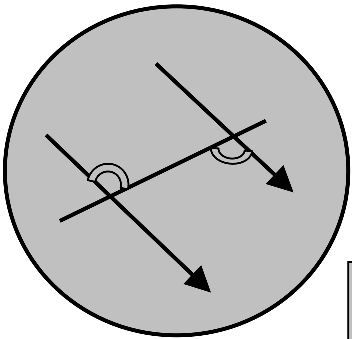
I have to Find out what angles p and q are.
Pairs of corresponding (F) angles on parallel lines are equal.
so angle a is going to be 141°, because it's oposite, then you take 141° of of 180° because Angles on a straight line add up to 180°, (180°-141°=39°)
so angle p = 39° ✓
And angle q is going to be 141°

In these examples, pupils are using known facts (such as the angle sum of a triangle is 180°) to explain/justify their solutions to numeric problems. Not all pupils take to such reasoning easily. The class described in the next section is one such class.

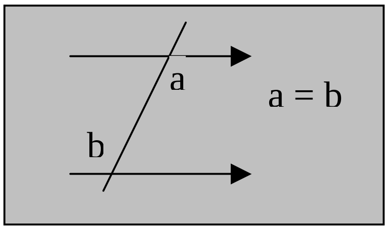
4.3 Working with angles: using "fact cards"

Paul had noticed that many of his pupils, when faced with reasonably simple diagrams, could find the "missing angle" but were unable to explain how they had done it. Some could use the diagram and point at various aspects of it and offer an oral explanation (including lots of vague pointing), but were unable to recognise and use the appropriate angle properties and geometrical facts. If angles or lengths looked the same then pupils assumed they were equal; if they didn't, they were not. Orientation of the diagram was also important. The properties of parallel lines were not recognised if the lines were drawn obliquely to the edge of the page.

Paul produced a set of "fact cards" to help pupils recognise a few simple facts and to use these to present a reasoned explanation to deduce other facts. The cards were in 'sets' of three consisting of a simple diagram; an angle fact (or property) and an interpretation of how the fact might be used (statement). Round cards were used for diagrams in an attempt to avoid giving it any particular orientation. Statements and angle facts were printed on rectangular cards of different colours, although diagrams and statements were sometimes combined. Examples are given below:



Alternate angles are equal

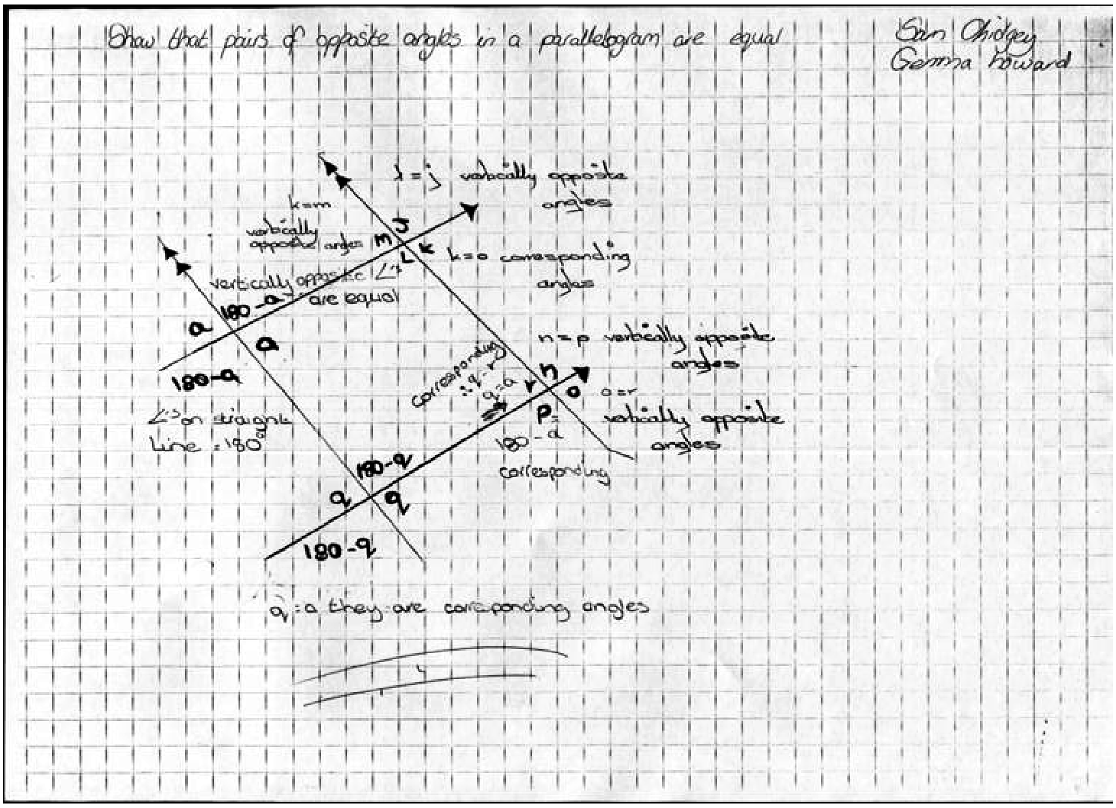


$a + b + c + \ldots\ldots = 360^o$
(a complete turn)

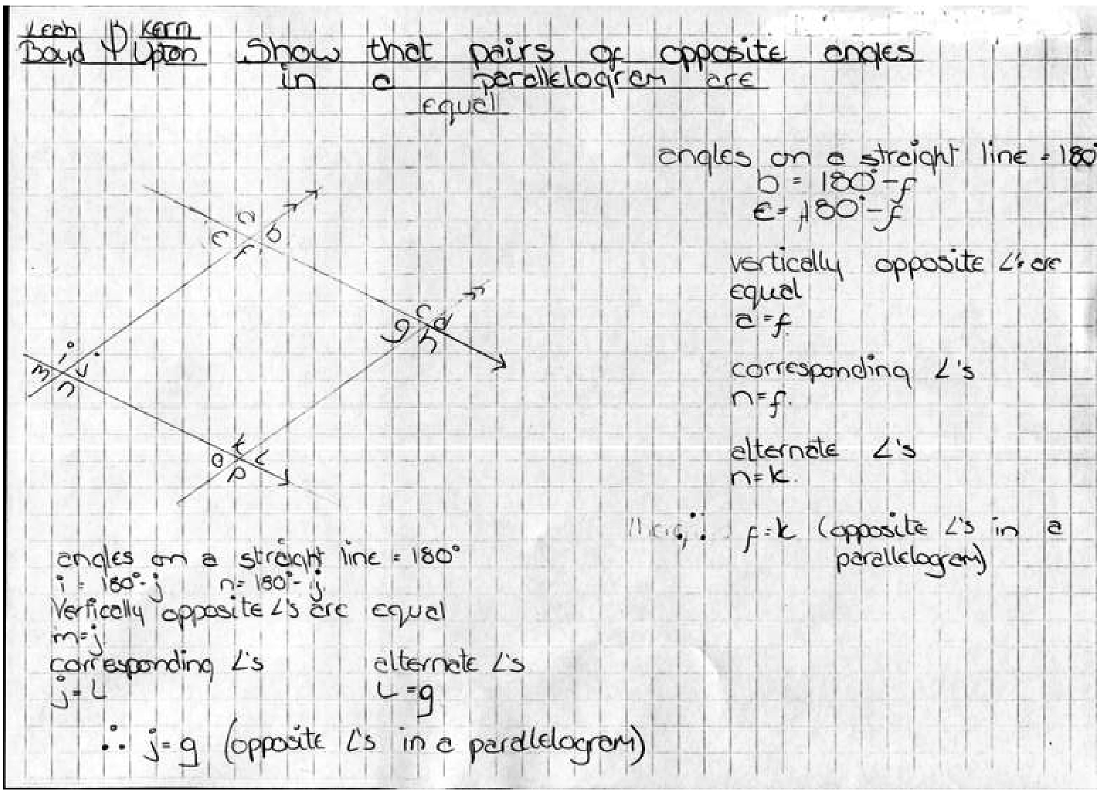
These cards were used with both Year 7 (level 3 and below in KS2 tests) and Year 8 pupils (levels 4/5 in KS2 tests). In his report Paul noted that:

> The majority of pupils did improve their ability to present an explanation including some form of justification in terms of the angle facts presented on the cards. Many were able to produce a reasoned argument both orally and in written form. They also successfully produced similar sets of angle fact/property cards sometimes based other given facts. One group of pupils created a card with: "Vertically opposite angles are equal", using one of the givens "angles on a straight line total 180^o".

Paul's report outlines three stages in which pupils became familiar with the cards and begin to use them to solve more complex problems. E.g. " Show that pairs of opposite angles in a parallelogram are equal". Work of two pairs of pupils is shown below.



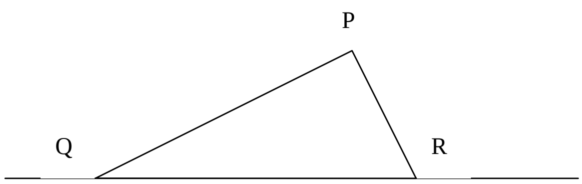
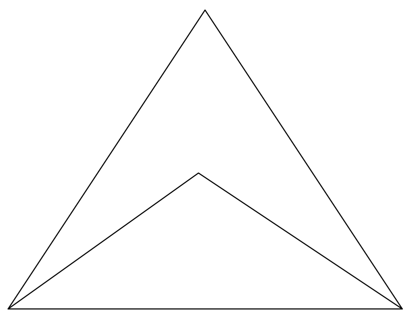
Sam and Gemma (above) tended to write all the facts that they knew on the diagram. Kerri and Leah (below) produced what looked like a more formal proof. However, this still included "redundant" lines of reasoning. This, of course, is part of the problem solving process.



With support in the form of “fact cards”, pupils can be supported in producing a reasoned argument about why a particular geometrical fact is so (such as, in this case, that the opposite angles in a parallelogram are equal. The next example illustrates what more able pupils at the top end of Key Stage 3 can do given appropriate teacher input and suitable tasks.

4.4 Isosceles triangles

The following lesson was taught by Jo to her top-set year 9 class:

Lesson Objective: pupils will be able to construct a simple proof using known geometric facts	
Starter 	This diagram is presented on an OHT (could be a laptop & projector). As a group, the task is to prove that the sum of the exterior angles of the triangle at Q and R is 270º, given that the interior angle at P is 90º. *Note: It is important for the less confident pupils to have this activity modelled by more confident peers and by the teacher to enable them to access the main task. Discussion of labelling conventions is also important here*.
Main task 	Using this diagram (with no labels), give each pair of pupils a piece of A3 paper with a copy of the diagram (note: in the diagram the top apex of the inner triangle is at the centre of a circle which passes through the apexes of the outer triangle). The task is: ‘Given that both triangles are isosceles and that the interior angle at the top apex of the outer triangle is x and the interior angle at the top apex of the inner triangle is $2x$, prove that angle a is half the size of angle x, where angle a is the angle between the “sloping” sides of each triangle’ (ie a is the size of the two equal angles in the arrowhead in the diagram). Initial discussion may be required to establish the correct position of angles

	a and *x* ~ has the teacher been explicit enough with her language? When everyone is happy as to the positions of the two relevant angles, ask each pair to construct their proof.
Plenary	Select students to share their proofs with the class (use blank OHTs for this or ask selected pairs to prepare their proof on a laptop, using PowerPoint or similar).

Below are examples of pupils' orals responses to the main activity.

Three pupils G, D and R, plus their teacher (T):

G: These angles are 180 – 2x all divided by 2. Do you get why?
D: Yes
T: Explain to R, please D.
D: OK!! Tell me again G!
G: There are 180 degrees in a triangle, so we take the 2x away from 180. Because the triangle is isosceles, the two bottom angles are the same, you know, half of 180 – 2x.
D: Does that mean each angle is 90 take x?
G: Yes!
D: Let's put that on the diagram so we don't forget.
D: If we drop a perpendicular line from the top of the big triangle, will it cut the angles in half?
G: Yeah, we can bisect x at the top to make half x. How can we make this the same as angle a?
D: We could draw the new triangle we've made and put the angles in to see what we've got.
G: We've got x/2 at the top, a right angle and this one is 90-x + a?
D: Yeah, that's right!
G: They all add up to 180 don't they?
D: Write it out like an equation.
G: So, we've got 180 = x/2 + 90 + (90 – x +a). Do we need the brackets?
D: No, lose the brackets, they're all adds and takes. Add the 90s together and the xs together. What does that look like?
G: 180 = 180 + a – x/2. Cool!
D: Get rid of the 180s … now we've got 0 = a – x/2 ! Add x/2 to both sides G.
G: Ha! a = x/2 …. We've done it!!
T: What were the key facts you used in your proof?
D: We needed to know that there were 180 degrees in a triangle and that an isosceles triangle has two equal angles. Plus, how to make the right angled triangle that cut the top angle in half.

Note: G is expected to get level 8 in the 2003 KS3 SATs; D, a strong level 7. There was no contribution from R, who is a weaker member of the class, targeted a level 7c in the SATs. When asked to explain the proof from the diagrams the group had *produced, he was able to follow the other pupils' reasoning. This indicates that he gained value from the work, even though he was rather passive.*

Two other pupils from the class, J & L, plus their teacher (T):

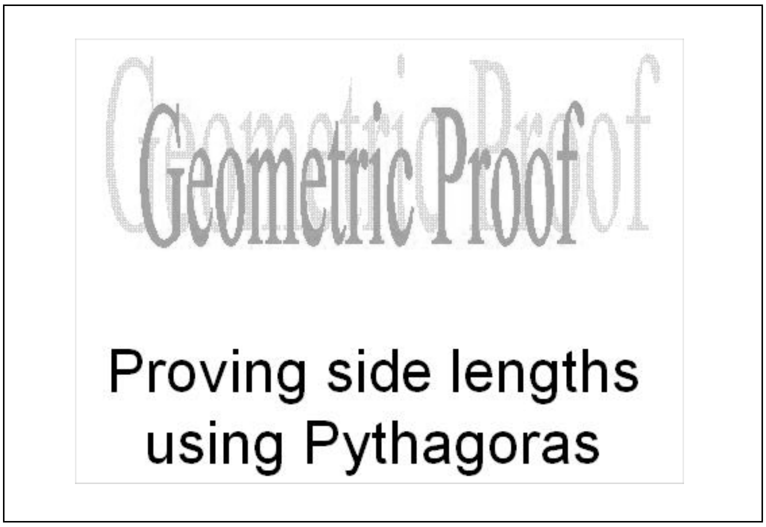
T: OK L, can you explain your proof to me?
J: I only needed 'angles round a point' and 'angles in a quadrilateral'
T: That sounds interesting. Explain how you used these ideas L.
L: The angle in the middle is 360 degrees all the way around.
J: Angles round a point sum to 180 degrees.
L: Yeah, so the bit left after the 2x is 360 – 2x.
J: That gave us terms for all the angles in an arrowhead.
L: We knew they had to add up to 360 again.
J: Because angles in a quadrilateral always sum to 360 degrees.
L: Because the big triangle is isosceles, we can label the two little angles both 'a'. So we've got 360 – 2x + x + a + a all adds up to 360.
J: The 360s cancel each other out and -2x plus x is minus x.
T: Well done! L, can you finish off for us?
L: Yeah, we've got 2a – x equals zero, so *a* equals half of x to make this true.
T: Excellent! Are you both happy with your proof? Will you explain it to the class during the plenary?
L: J can!

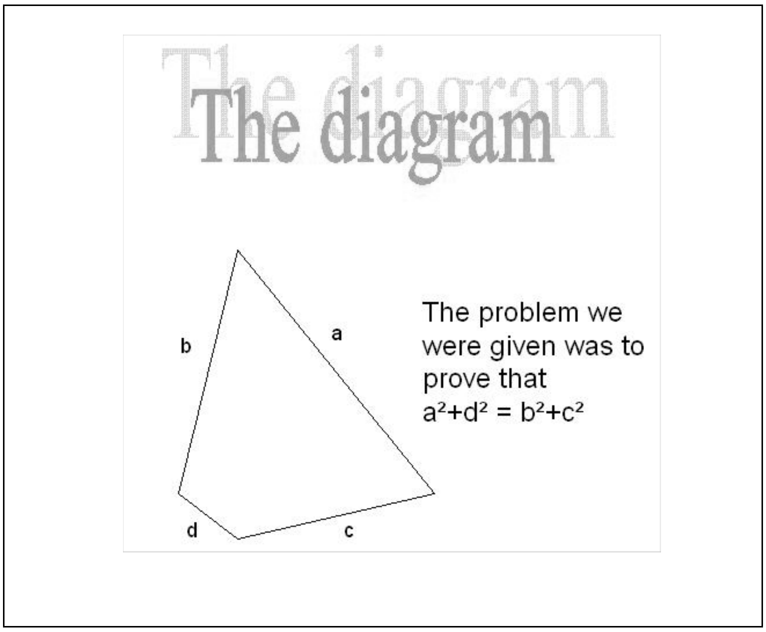
Note: J is particularly able (target level 8). L is very quiet but still able (target mid-level 7). Although J drove this idea, L was clearly able to follow the proof. She was happy to let J do all the talking but was able to write up the proof afterwards.

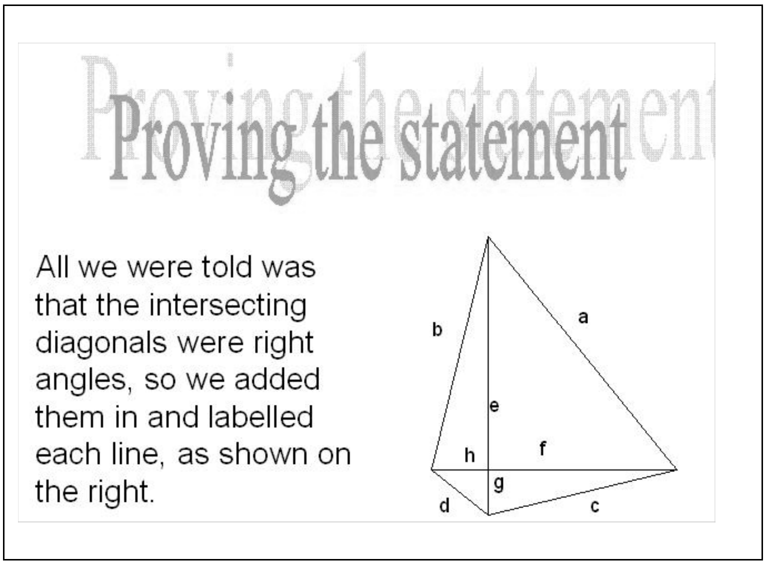
While this top set class were able to tackle this problem, the idea of using "fact cards" was used by pupils to set challenging problems for each other, as this next classroom example illustrates.

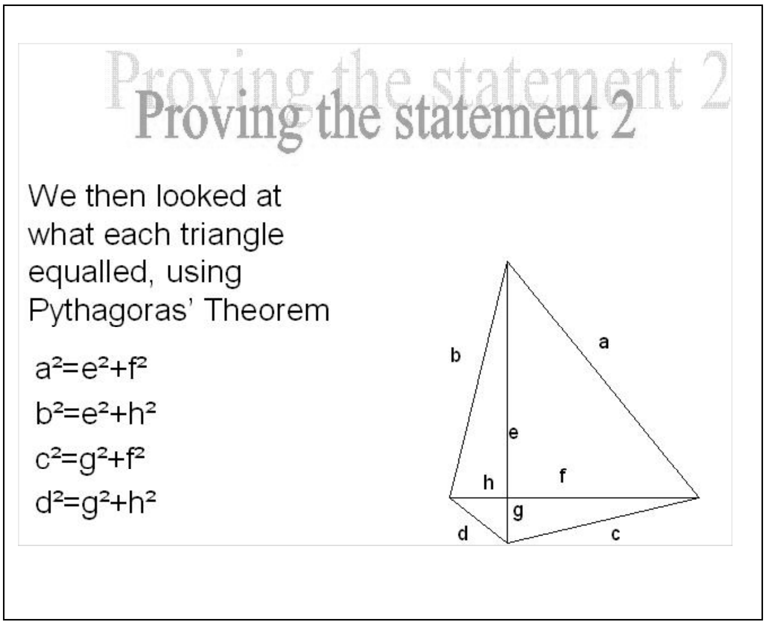
4.5 Pythagoras' theorem, additional problems and pupil-created problems

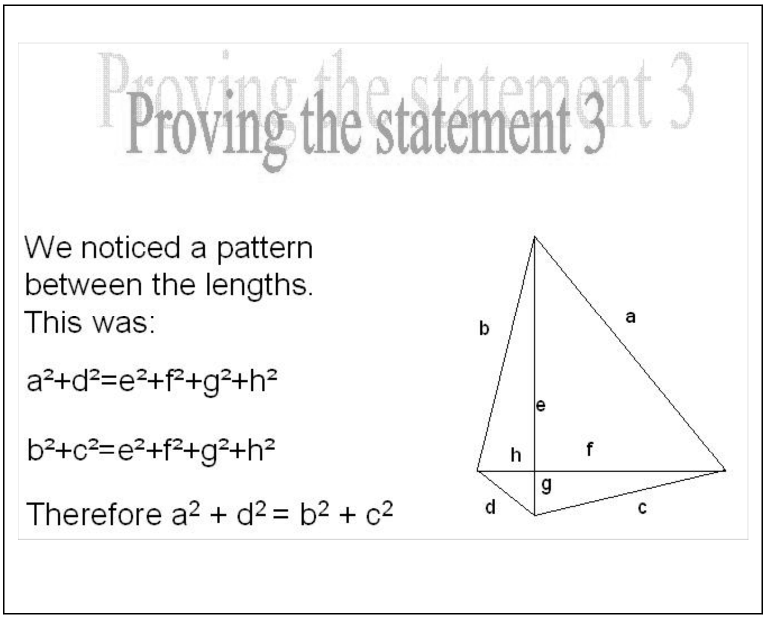
Here, the top set Year 9 class taught by Jo describe their work on Pythagoras' theorem, and beyond (taken from a presentation of the work):




The diagram
The problem we were given was to prove that
a²+d² = b²+c²
b
a
d
c


Proving the statement
All we were told was that the intersecting diagonals were right angles, so we added them in and labelled each line, as shown on the right.
b
a
e
h
f
g
d
c





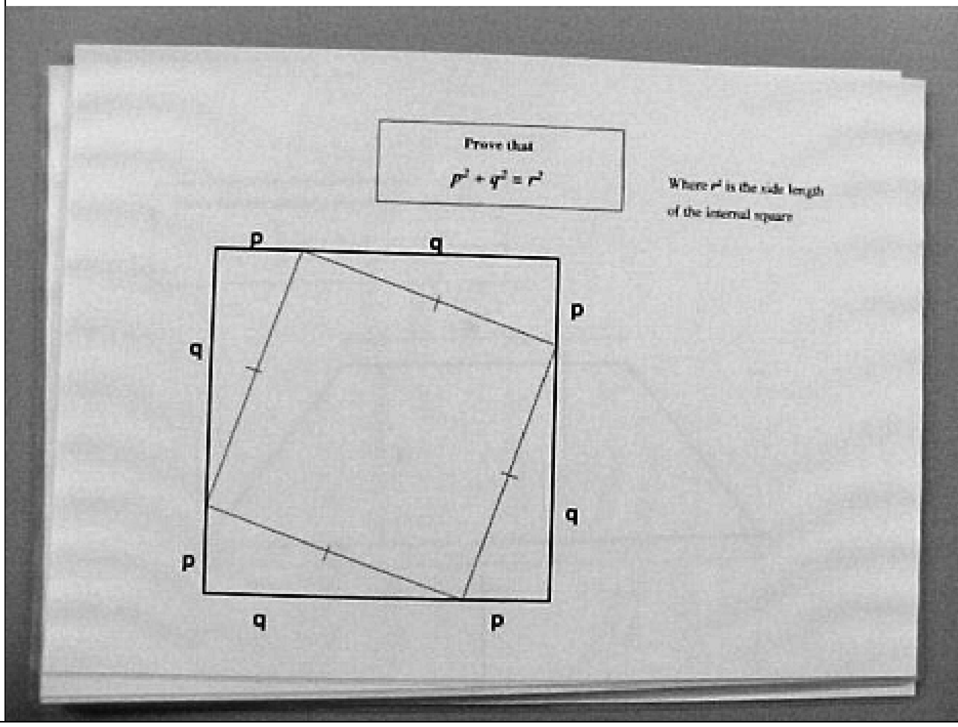
Later, the pupils selected problems for each other to solve. The one shown below is based around one of the proofs for Pythagoras' theorem, others, shown in appendix B are problems based on specific figures.

The problem statement is:

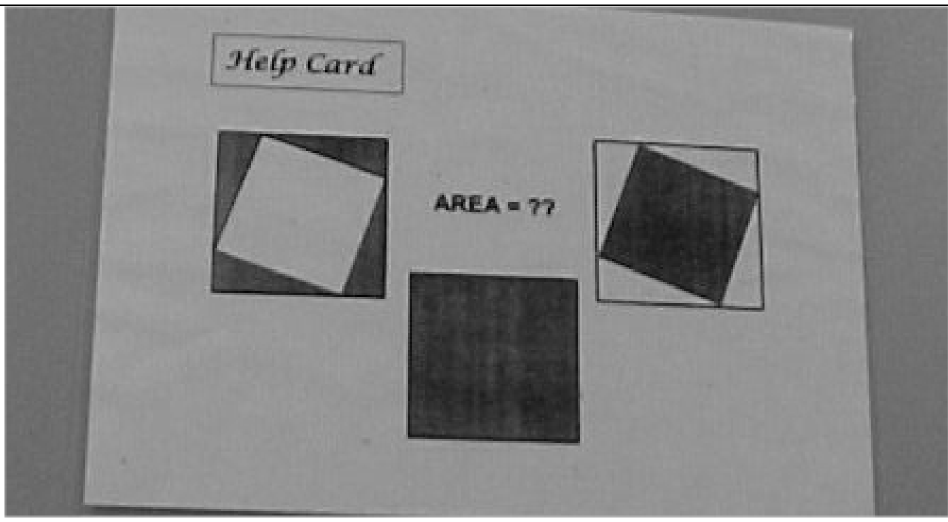
Prove that

$p^2 + q^2 = r^2$

Where r is the side length of the internal square.

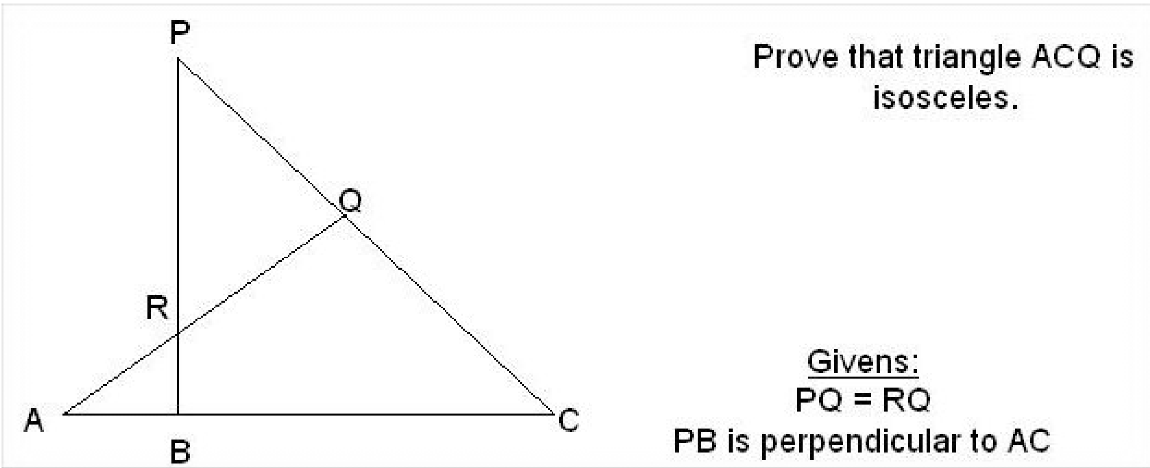


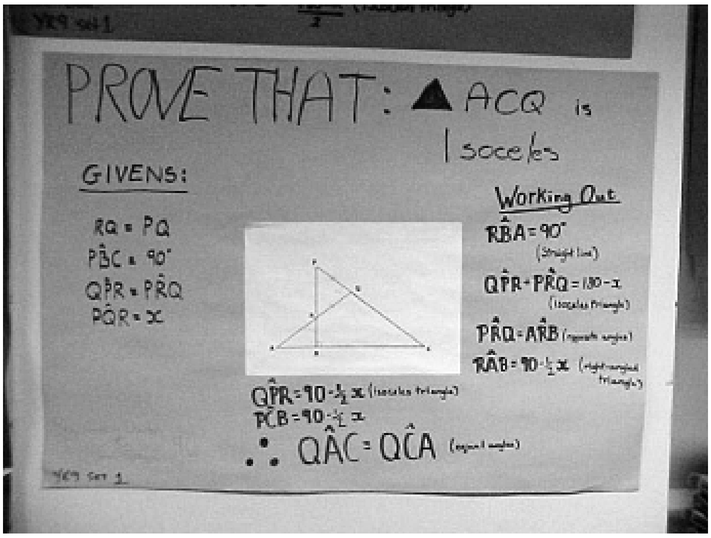
To assist their classmates, the pupils provided visual hints in the form of poster-sized “help cards”.



Other tasks that the pupils worked on can be found in appendix B.

The pupils made display posters of their work, such as the response to the problem below.





4.6 Conclusions to this section

This section contains a sample of activities tried out with different age and attainment groups within Key Stage 3. All the activities aimed at developing pupils' use of geometrical facts and theorems within their mathematical reasoning. The activities range from ones requiring solutions to be justified in terms of the geometrical facts and theorems utilised in the solution to ones requiring a proof in a more formal sense. All were about logical reasoning and included some informal or formal deduction.

Aspects to highlight include:

- All activities were successful in that they engaged students in developing their reasoning skills. This was true across the attainment range.

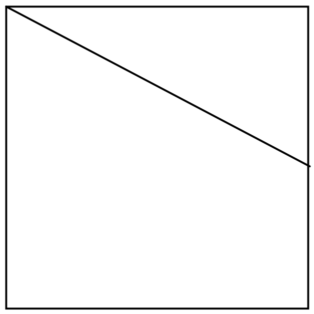
- In all the activities some students had difficulties and needed support. Techniques such as providing “fact cards” or “help cards” proved useful.
- Activities involving calculating with angle measures can be turned to focus not solely on the calculation but also on the reasoning that underpins the calculation.
- The more able pupils at the end of Key Stage 3 are more than capable of developing and recording short chains of *locally* deductive reasoning in geometry given appropriate pedagogical approaches and suitable teaching materials. This illustrates the value of geometry as a vehicle for broadening the curriculum for high attainers.

In line with what was noted in the previous section, activities tended to take longer than expected. It appears that quite long periods may be needed to enable student to arrive at well-reasoned results. Given the timescale within which we were working we are only able to present “snapshots” of student reasoning. Nevertheless, the evidence that we have does support the view that focussing classroom approaches on logical *explanations* is helpful in supporting students’ development of the various functions of proof. This evidence also suggests that more can be done to help students experience deductively derived conclusions as necessary and inductively derived conclusions as uncertain.

5. Classroom reports: Reasoning using the 2-piece tangram activity

5.1 The 2-piece tangram activity

This activity has been developed by Peter and was first trialled by him and by colleagues in his school, at many different levels in Key Stage 3. The two piece jigsaw is made from a square with a cut being made from one vertex to the midpoint of a side.

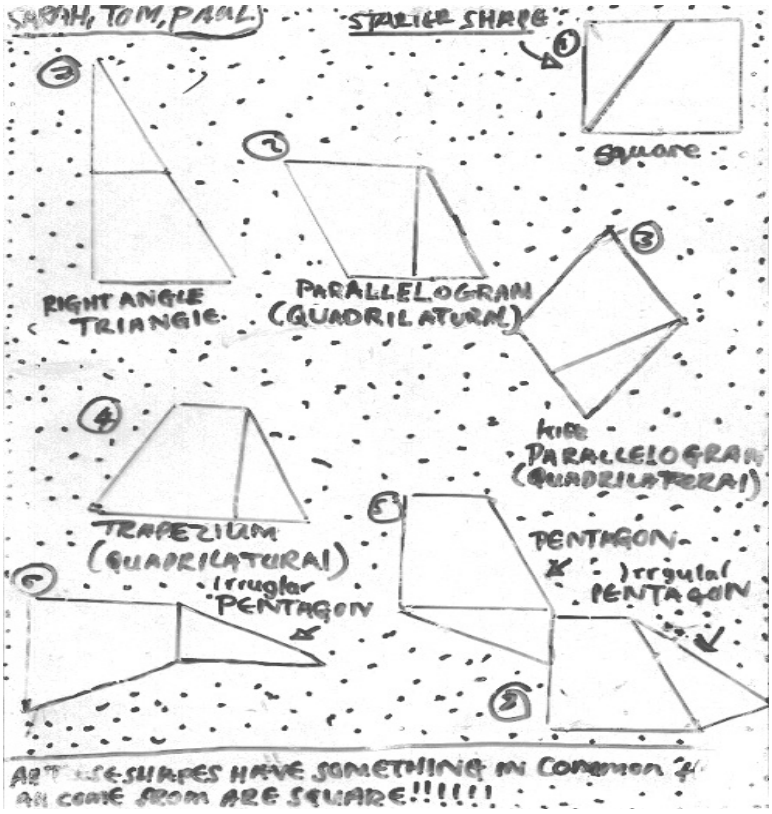


> Peter: For all attainers I give them a square piece of card from which they produce the two pieces. This means that they all work with a standard size square (with right angles!). Getting them to draw their own square can come later - it spoils it for low attainers if they struggle early on to get a reasonable start.

Students are then asked how many shapes they can make, what the shapes are, and how they can be sure that the pieces make these shapes. (A full lesson plan with suggested extensions extracted from the school documentation is given in the appendix.)

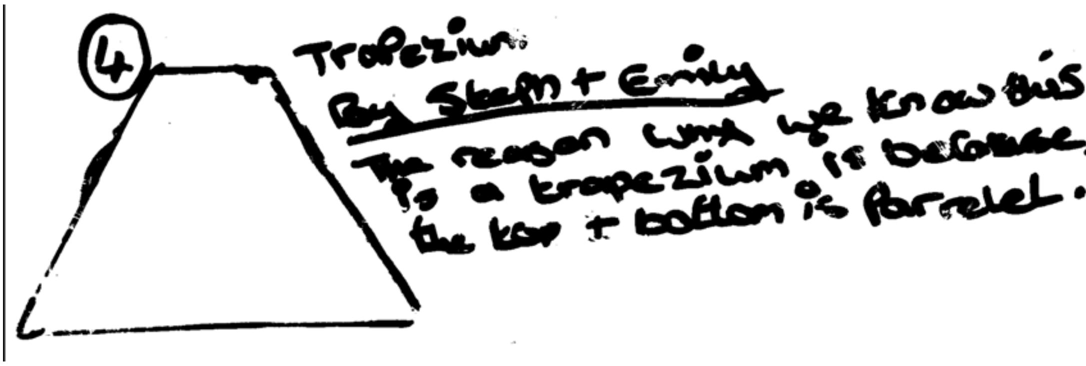
The first session with each of two classes (Year 9, set 4 out of 5, Year 8 set 5 out of 5) was videoed and viewed afterwards by two separate Heads of Mathematics groups in Hampshire as well as by the geometrical reasoning group. The sessions were also observed by a teacher from another school who wished to see work that involved group discussion. Peter reported:

> There was lots of very productive activity in each lesson. What surprised me was the mathematical vocabulary that was used by the pupils when explaining their results to each other both in pairs and as a larger group. There was much kinaesthetic work taking place – pupils explained with their hands as well as their voices. The lesson worked very well as presented in the lesson notes (*see Appendix*). Pupils enjoyed using OHTs and explaining to their peers. In the Year 8 class I was delighted to see two of the lowest attainers who lack a lot of confidence volunteer to present their work to the class. The plenary session involved two groups explaining the reasoning to the others. It was noticeable that although the OHT did not include much written reasoning, the pupils spent most time explaining their reasoning to the class. This pleased me since that was the key objective of the lesson.
>
> Here (*on the next page*) is an overhead projector transparency made by Sarah, Tom and Paul from the Year 9 class. Considering they had only an hour's lesson to explore the task and then speak about their work in the plenary session it is perhaps not surprising that no written reasoning appears!



The written work did not reflect the high level of geometrical understanding used in the class. I found that pupils are unwilling to write down a lot of what they explain verbally and with their hands. I felt that the year 9 pupils were working at the Van Hiele level 4 in the class, though this was not obvious in their written work. A colleague used a writing framework with a Year 7 class and this proved to be more successful in producing written work.

Here is a bit of written reasoning from an OHT in the Year 9 class.

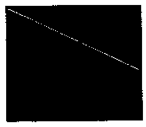


For homework I asked the pupils to write about what they did so that another pupil who has missed the task could understand what they had missed. With hindsight this did not really address the objective of using geometrical reasoning and so I felt that I'd rather wasted an opportunity. However the following report from Sarah gave me a buzz that you don't get outside teaching!

Sarah's report:

<u>Math's Shapes</u>

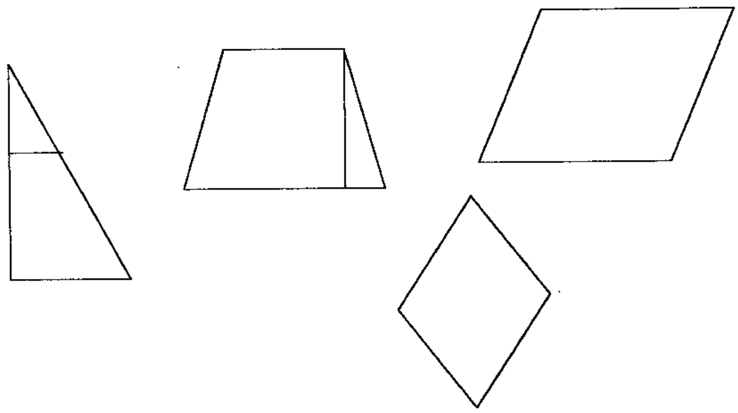
In Math's we where given a square that looked like this. ⟶



We had to cut it like shown on the square. Then we were told That we had to find as many shapes as possible with it, we worked out that we could only make 8 shapes these were:

A Right Angle Triangle
2 irregular Pentagons
A Pentagon
A Kite (Quadrilateral)
A Parallelogram
A Quadrilateral

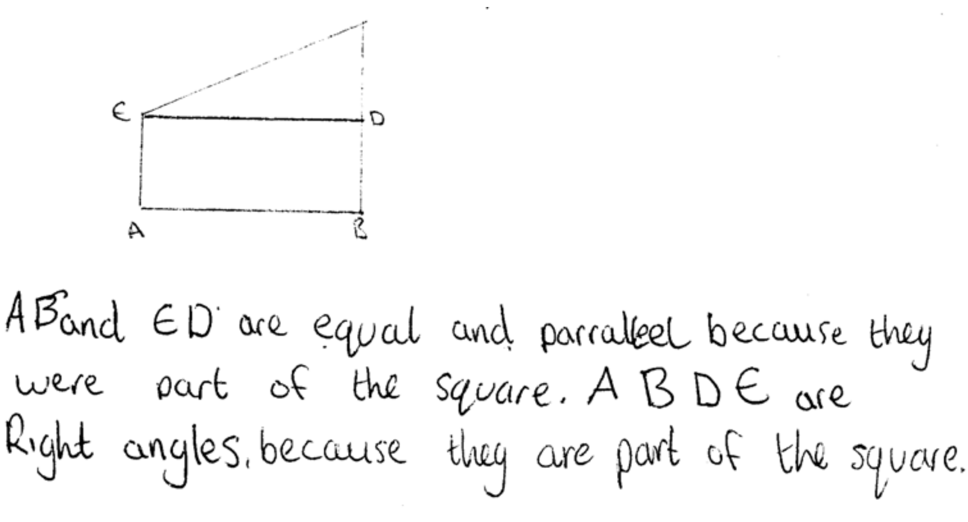
Here are some of the Shapes we made:



Then we had to explain how we found some of the shapes,
We found all the Quadrilaterals because they all had 4 sides.
We found the right angle triangle (shown above) because the two sides of the little shape were both 90° and we know that two 90 equals 180 and this is a straight line.
The kite (Parallelogram, Quadrilateral) was just a quadrilateral turned a different way.

I found that the work helped me identify some of the misconceptions that pupils had. (*The transcript of a discussion which revealed that one pupil thought that parallel lines had to be equal is given in the later section on language.*) Without this type of activity and the discussion generated I'm not sure whether I would have picked up on this misconception.

Two weeks later I followed up this lesson with the Year 9 class by working with labelling conventions and extending the task to overlaying the trapezium piece with the triangle and looking at the shape formed by just one thickness of card. There was a bit too much for the class to take in at first and I wished I hadn't started! However, after 15 minutes of hectic work the lesson turned a corner and gave the pupils plenty of opportunity for more geometrical reasoning. Here's just one piece from Paul.



This task allowed all attainers the opportunity to explain and explore at their own level and to learn from others. The opportunity for discussion and explanation was tremendous. Recently the work was explained with the video to groups of mathematics teachers and they were given the opportunity to work on the task themselves. In each case something new evolved. One group identified a cyclic quadrilateral and when prompted went on to find the diameter of the circumscribed circle. Another group took this further and found what fraction of the circle the cyclic quadrilateral covered! Clearly the task allows one to pose your own questions and could be used in Key Stage 4 as well as Key Stage 3.

When the video was shown to the whole group we felt that what both Year 8 and Year 9 classes achieved was impressive, especially in terms of articulate presentation and verbal reasoning. One girl in particular was using “because” as a natural part of her discourse about the shapes being discussed, and seemed to be operating towards van Hiele level 4.

The strategy of getting the students working in groups with the goal of preparing an overhead transparency to present their findings to the class gave them a purpose for engaging in written communication, and helped them to codify their verbal reasoning.

The video made it clear that for an activity such as this to lead to geometrical reasoning as an outcome the teacher has to pose questions which elicit reasoning, and constantly go beyond simply acknowledging empirical observations. The teacher’s own knowledge of the logical structure of the area of geometry being investigated is clearly very important here. The video also raised many interesting language issues, for example where the use of the phrase “which of the sides are parallel” appeared to elicit a different response to “which two sides are parallel”.

One boy accompanied oral description by signalling (hand gestures or pointing with a pen). The corresponding aspect in a written account would be static labelling, using letters to label points and signs to denote equal angles or parallel lines for example. It became more obvious to us that the two modes of communication (verbal and written) have significantly different characteristics, and that the transition between them needs careful management. Pupils need to be clear about the purpose of communication (in this case communication to the rest of the class).

Because there appeared to be so much more evidence of oral than written reasoning in these two classes, we asked Peter to transcribe parts of the video record and to try the activity with a higher attaining class too.

5.2 Oral reasoning in the Year 9 class

Some clips of dialogue are given below which were transcribed from the video of the Year 9 class (set 4 out of 5, most expected to reach level 5 in the Year 9 SATs). T denotes the teacher and P1, P2 etc various pupils. The first clip shows some of the initial difficulties in eliciting reasoning based on shapes and their properties.

Clip 1

The two piece tangram task has just been introduced to the pupils and they have about ten minutes to make a shape and talk about why it is that shape to a partner.

T: So what shape have you got down there lads?
P1: A quadrilateral.
P2: It's a quadrilateral because it has four sides.
T: Right. OK. Can you say anything else about it?
P2: It has four sides: two sides the same and two sides parallel. Two sides are parallel and two are not.
T: So which are the parallel sides?
P1: (*Indicates first the equal sides then taps the parallel sides when no confirmation is forthcoming.*) Them.

T: That's right. Those two are parallel. How do you know they are parallel?
P1: Because they are straight.
T: Can you think of any other reasons why they are parallel?
P1: Ermm …

T: It looks parallel doesn't it?
P1: Yeah.
T: Can you give some reasons of exactly why it's parallel?
P1: I've got a ruler (*Pupil puts ruler below the longer parallel side.*). The ruler's straight and it's straight (*Puts ruler against line.*). Don't know.

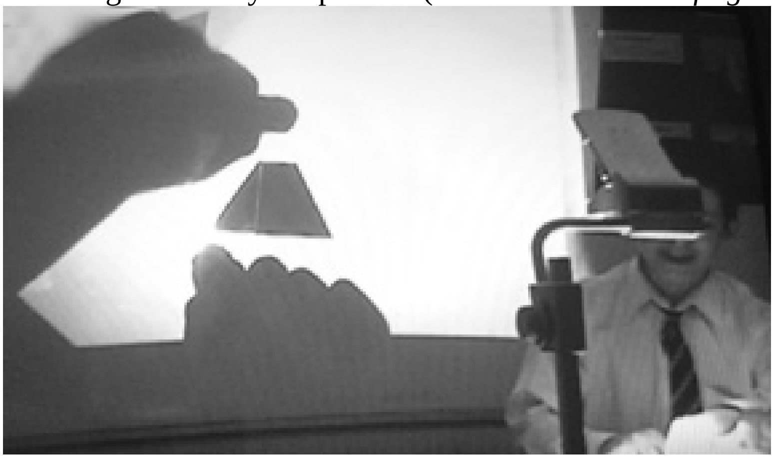
At this point I was trying to elicit the fact that since opposite sides of a square are parallel, and one of those sides together with part of the opposite side form the parallel sides of the trapezium the sides must still be parallel. This was too much for me to expect from the pupil at this stage so I left it.

After ten minutes of exploratory work in pairs I asked for some volunteers to explain to others about one of the shapes they had found. Here are two explanations, with reasoning, of why the shapes are what they are.

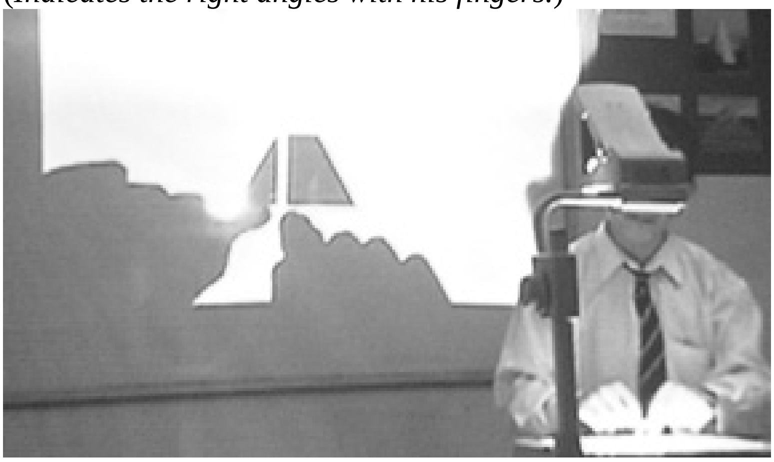
Clip 2

The OHP is set up at the front, so the first pupil decides to put his pieces on it and use it to show the others.

P3: Erm. Right. Well. This is a, well, that trapezium thing where, er, both of these sides are straight and they are parallel (*Indicates sides with fingers.*).



We know they are straight because of the parts of the shape. These are both right angles (*Indicates the right angles with his fingers.*)



and are ninety degrees and both add up to one hundred and eighty degrees which makes a straight line and that's all we have for today.

It is noticeable that a lot of explanation is accompanied by hand gestures. Pupils are very willing to talk about their shapes and learn from each demonstration – the bit about a straight line coming from two right angles was often mentioned. The pupils seemed to appreciate that this guaranteed the straightness of the line.

Clip 3

Here we find P4's line of deduction relates back to the properties of the square. He has probably learnt from P3 since P4 relates his comments back to the original square when he talks about the 90° angles which fit together to make the straight line.

P4: And you, erm, know it's a straight line (*Indicates the straight line.*) because it's a, erm, a right angle (*Takes top triangle off the other piece and indicates the right angle.*).
You can see it's still a square (*Puts the two pieces together to make the square.*).
In that corner there its, erm, ninety degrees, a right angle (*Indicates the right angles at top of triangle.*)

and in that corner there it's ninety degrees.That adds to one hundred and eighty degrees so you know it's a straight line. With right angles so you call it a right-angled triangle. (*He rearranges pieces back to make a square.*) And there's a square.

This was the explanation that followed P3. Pupils learn from hearing others and refine the arguments, using what they have heard and trying to improve.

Clip 4

I saw the right angle triangle result (*shown above, where the small right angled triangle fitted together with the other piece to make a larger congruent right angled triangle*) as an opportunity for enlargement practice with the area scale factor being the square of the linear scale factor. P4 is working on the right-angled triangle. I needed to prompt him to notice that the small triangle fitted four times into the large one and that the sides of the large triangle were each twice the size of the small triangle …

P4: …one, two, three. (*P4 puts the triangle into the three positions that do not include the one over the right angle.*) I think I'd be able to fit …

T: So how many of those red ones do you think you can fit into that big one?

P4: One, two, (*P4 draws round the red triangle in the big triangle.*) three, that's it, yeh, four.

T: Yeah, OK, so you can get all those red ones in. OK. What do you notice about the lengths of the sides?

P4: It's a straight line, the same.

T: I know we've got some straight lines there, but I'm thinking about the <u>lengths</u> of the sides.

P4: Yes.

T: You look at the lengths of your big triangle.

P4: Yes. I know it's a straight line because it's got two there which add up to one hundred and eighty degrees and it's got … and if I turn it round there that's ninety degrees which is one hundred and eighty degrees which is a straight line. (*P4 has become a bit obsessed by proving the straight line part.*)

T: Yes, that's right.
So you've got the straight line. OK.
What about the length of <u>that</u> line (*Indicates the line of medium length of the red triangle.*) and the length of <u>that</u> line (*Indicates corresponding line of the large drawn triangle.*).

P4: The length of that line and the length of that line?

T: Yes.

P4: It's the same.

T: I'm looking at that part there. (*T puts the red triangle over the right angle.*)

P4: This line is not the same as this line.

T: No

P4: This line?

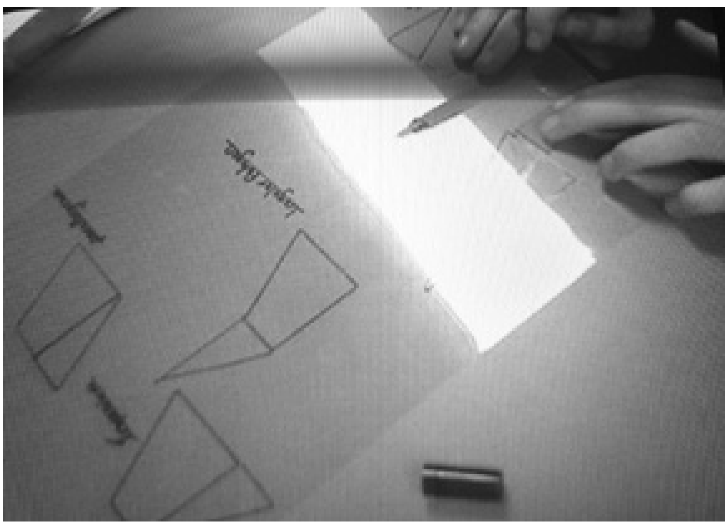
T: That red line. How many of those red lines could you get from there to there? (*T indicates the medium length of the red triangle and the medium length of the drawn triangle.*)

P4: One, two.

T: You can get two of them, right?

P4: Yes.

T: Two, that's right. Can you get two of the length of those lines (*Indicates shortest line on red triangle.*) that way (*Indicates shortest line of drawn triangle.*)?



P4: One, two. (*P4 moves the red triangle so that the short side fits twice onto the drawn triangle.*)
T: Yes, brilliant, right.

At this point I did not take this any further. This is as far as it went. I felt that I had sown the seeds of are scale factor being the square of the length scale factor, but in my opinion as the conversation progressed it was too laboured and I decided to leave it. I felt as though I was unlikely to get the true result and rather than get a reply that the area scale factor was the length scale factor plus two or length scale factor times two it would be best to leave it at this point.

This clip again demonstrates the opportunities that an experienced teacher can spot and follow up during the activity, but also the need for fine judgements over how far to take them.

5.3 Written reasoning with a Year 9 top set

Since the Year 9 set 4 students had found written reasoning very difficult, this same activity was repeated with a higher Year 9 set to see how they managed.

My colleague Maureen, with her Year 9, set 1 (out of 5) class, did the following work. About half this class achieved level 8 in the SATs; the rest got level 7. Maureen and I talked through the lesson the previous day for about 10 minutes. I observed the lesson.

The class settled very quickly and Maureen explained what she wanted them to do. They had 5 minutes to make a shape and explain to a partner why it was what it looked like. Then there were two volunteers who explained why to the rest of the class. It was apparent at this point that very little proof was being offered, explanations were based more on intuition. Maureen was very good in helping the pupils when they got a bit tongue-tied by suggesting a few appropriate bits of vocabulary (vertex, parallel etc.). We felt it might have been a mistake that we asked some deep questions of the pupils in front of the class (Why is that a straight line?), but in retrospect that was not the case - they realised that their assumptions were not water-tight.

So 15 minutes into the lesson we asked them to work in 3s or 4s to produce an A3 poster about some of the shapes, explaining why they were what they called them. Maureen and I circulated prompting them with questions:

What can you tell me about a parallelogram?

Opposite sides are equal and parallel, opposite angles equal.

Can you show any or both of those properties here?

(Some went for sides, some for angles.)

Why are those two angles equal?

Because they came from the angles made when we cut the triangle off the square.

How can you be sure they are equal?

The sides of the square are parallel and the cut hits them both. They are Z angles!

Do you know of any other name for Z angles?

Pause … *Alternate angles*

This part of the lesson was magic. You could see the pleasure obtained when pupils figured out why a shape was what it was.

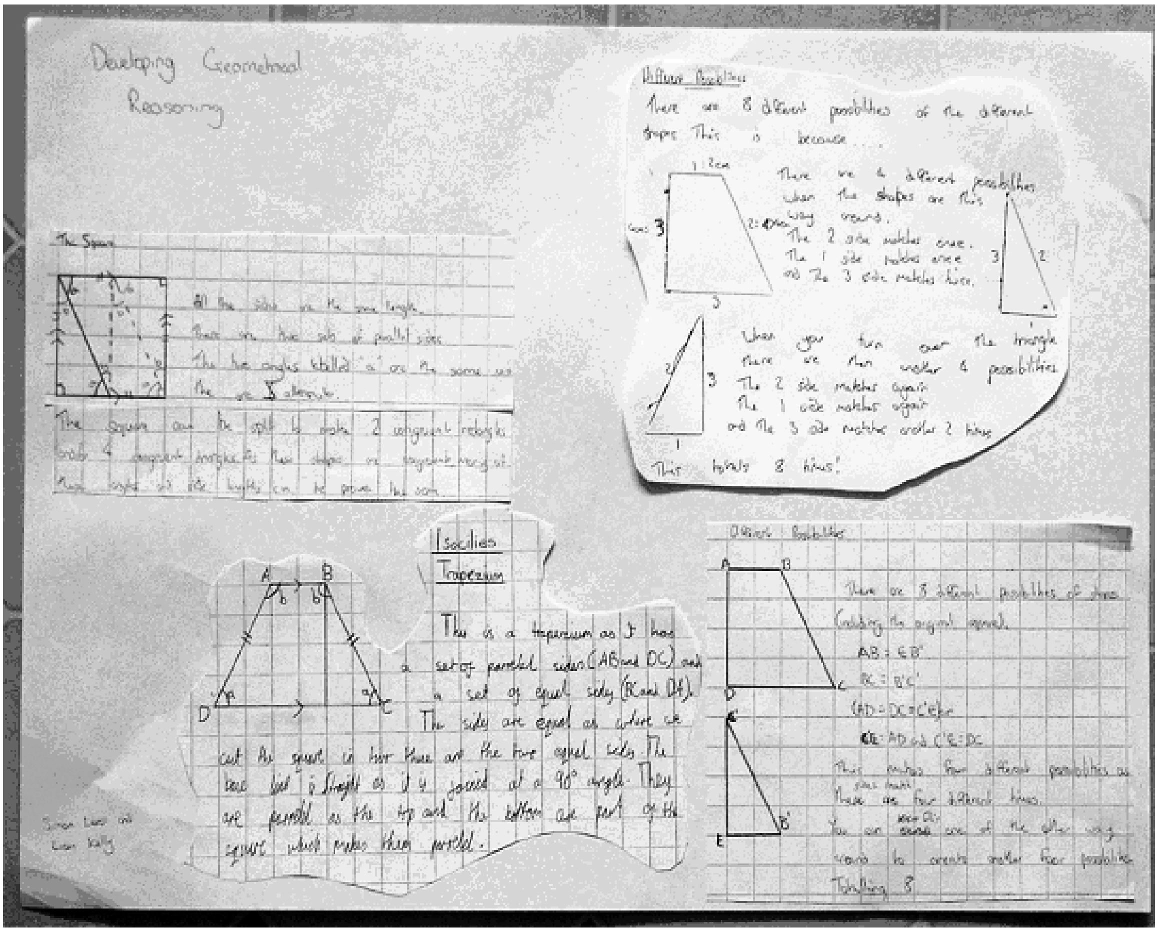
We felt that we needed another lesson on this, as pupils were getting into it and challenging themselves. For a plenary we asked them to write a couple of sentences about what they felt they had learned.

For homework we asked them to make just one shape and be ready to prove why that shape was what they said it was for next time.

I found it interesting reflecting on this lesson, comparing it with the Year 9 set 4 class taught last term. The set 1 class were far better writing down their reasons than verbally explaining them. Set 4 did very little writing but the quality of verbal explanation was very good (see video transcript). Perhaps set 1 are more used to sitting quietly and writing work in mathematics than using verbal skills. I know I have a problem shutting Set 4 up!

Maureen and I discussed what should happen in the second lesson. I suggested that the correct spellings be put up for pupils and that they should start to use some of the conventional ways of referring to lines and angles.
The work was collected in after this second lesson, and the teacher noted improvements in written presentation.

Reproductions of two contrasting pieces of work completed by pairs of pupils are shown to give a broad impression. As it is not distinct enough to read, samples of the details are transcribed below. The first piece of work was done by a pair which contained the most able student in the class.



The square
All the sides are the same length.
There are two sets of parallel sides. The two angles labelled 'a' are the same as the are Z alternate.
The square can be split to make 2 congruent rectangles and/or 4 congruent triangles. As these shapes are congruent, many of these angles and side lengths can be proven the same.

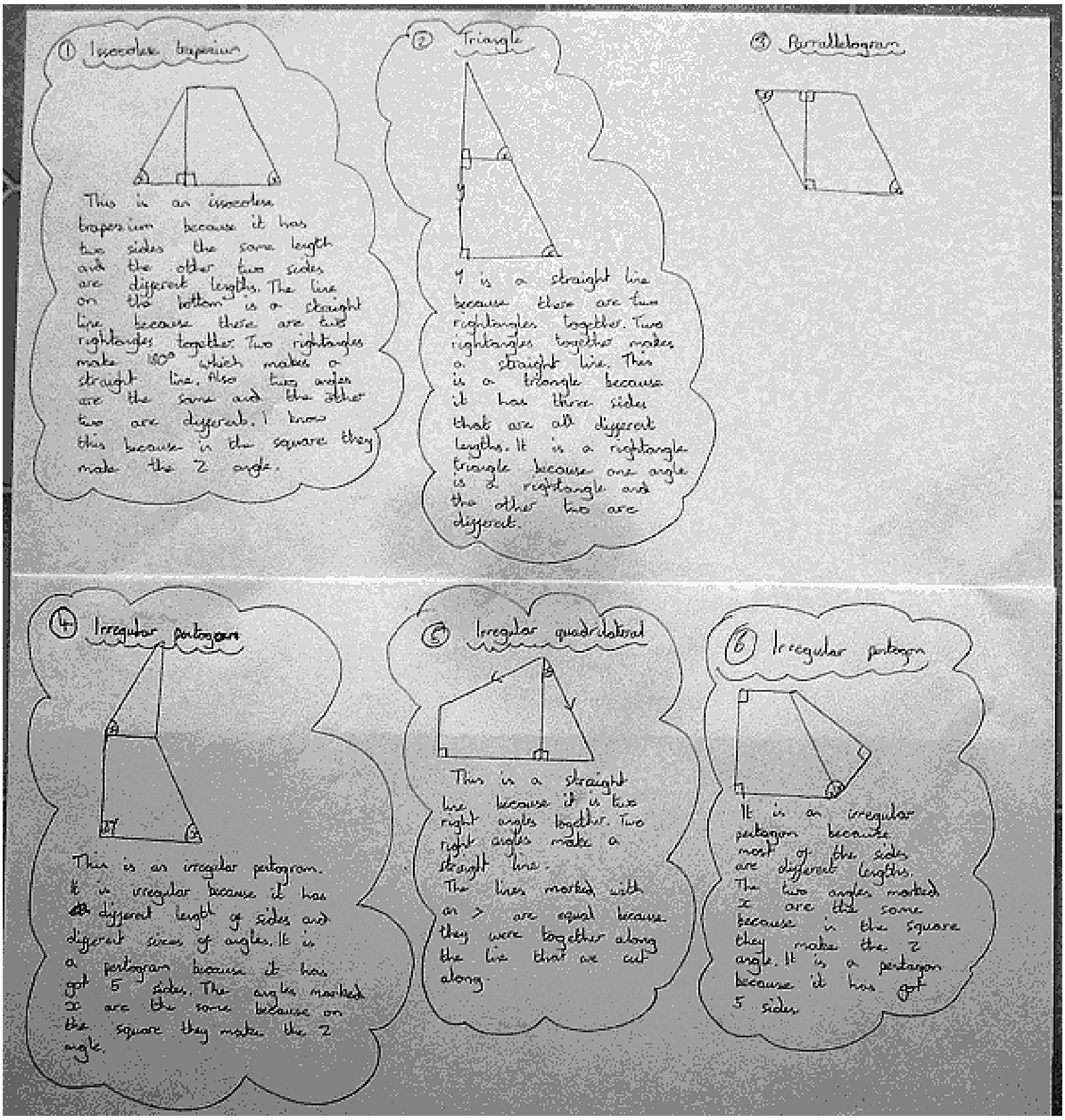
Isocilies Trapezium
This is a trapezium as it has a set of parallel sides (AB and DC) and a set of equal sides (BC and DA). The sides are equal as where we cut the square in two these are the two equal sides. The base line is straight as it is joined at a 90˚ angle. They are parallel as the top and bottom are part of the square which makes them parallel.

Different possibilities
There are 8 different possibilities of shapes (including the original square).
AB = EB'
BC = B'C'
(AD = DC = C'E) or
C'E = AD and C'E = DC
This makes four different possibilities as there sides match four different times.
You can flip one of the other way around to create another four possibilities. Totalling 8

This first piece of work uses notions like congruency of shapes and conventional ways of referring to lines, and the reasoning is complete and logical.

The second piece of work, below, contrasts with this and although neatly presented shows reasoning which is much less complete; indeed it is mathematically hardly advanced on the oral reasoning of students in set 4 in the same year (the reproduction below maintains the pupils' idiosyncratic spelling and language).

> *1 Issocolese trapezium*
> *This is an issocolese trapezium because it has two sides the same length and the other two sides are different lengths. The line on the bottom is a straight line because there are two right angles together. Two right angles make 180˚ which makes a straight line. Also two angles are the same and the other two are different. I know this because in the square they make the Z angle*
>
> *2 Triangle*
> *y is a straight line because there are two right angles together. Two right angles together makes a straight line. This is a triangle because it has three sides that are all different lengths. It is a right angle triangle because one angle is a right angle and the other two are different.*



…

<u>5 Irregular quadrilateral</u>

This is a straight line because it is two right angles together. Two right angles make a straight line.

The lines marked with an > are equal because they were together along the line that we cut along.

<u>6 Irregular pentogon</u>

It is an irregular pentagon because most of the sides are different lengths. The two angles marked x are the same because in the square they make the Z angle. It is a pentagon because it has got 5 sides.

The remainder of the work in the class fell between these two extremes.
The statements students wrote after the first lesson included:

We have learnt how to justify a certain shape and determine wether a line is parellel or just straight. We have learned to not just accept that a shape is a certain way because of the way it looks but how to explain why mathematicly and accurately.

We have leaned that we can make many different from a 4x4 square. The properties of the square contributed to all of the properties of the shapes we made. We found it hard to describe the properties of the shapes relating to the original square.

You can make anything from anything so life is worth living. That you can make different shapes out of a square + prove what they are using the original square.

Thus the trials of this task in Peter's school showed that the task itself was very rich and could lead on to many different ideas. It could also with appropriate adaptation be used across a wide range of attainment. It was particularly important with the lower attaining sets to allow the opportunity for students to express themselves orally, whereas the emphasis with a higher attaining set was appropriately to assist them with clear written presentation including the use of conventions.

5.4 An alternative approach to the 2-piece tangram

Jill decided to also use the 2-piece tangram activity with a Year 7 mixed ability group the emphasis on supporting students in developing written reasoning. The lesson plan and blank worksheet are contained in the appendix.

After a starter which included revision of angles, the tangram activity was introduced, using squares of card and an accompanying worksheet. Pupils came together as a class after about 5 minutes' individual activity and then worked through at their own pace, while the teacher went round discussing their findings with each table. The plenary focused on just one shape constructed, the parallelogram, and how the properties of the square could be used to deduce that it was a parallelogram.

Completed worksheets, shown below are from two different pupils.

Jill's commentary on the lesson was:

Just listing properties of a square raised some interesting debate among the pupils. Most pupils got four sides the same length and four right angles. No reference to the diagonals. Pupils got very involved in the tangram activity and it was accessible for all levels of ability. Although the original task was on a fairly simple level – finding the number of possible shapes - it then opened up lots of discussion about what shapes have what properties and how the properties of the original shape could be used to deduce information about the new shapes formed. At the end of the lesson we focused on the parallelogram and used the properties of the square and other geometrical facts that we used to show that the allied angles in a parallelogram added up to 180º. For the more able pupils we introduced the idea of labelling angles using algebra to show relationships.

YEAR 7 TANGRAM WORKSHEET

You have been given a square made of card. In the space below write down any facts you know about this shape.

all lines straight. 8 angles. sides same length.
A square has 4 right angles, 4 reflex angles. ~~4 sets~~ 4x of perpendicular lines 2 sets of parrelal lines. It's 2D. It has 4 sides. Its called a square.
Symetrical.

Compare your list with that of the person next to you and others on your table. Do you have any facts that they don't ? Do they have any facts that you don't ? If there are any you have missed add them to your list above in a different colour pen.

We're going to make a two piece jigsaw using your square. To do this you need to draw a straight line from one vertex to the midpoint of a side.

What do we mean by vertex ? ~~[illegible]~~ Corner

What do we mean by midpoint ? Middle of a side......

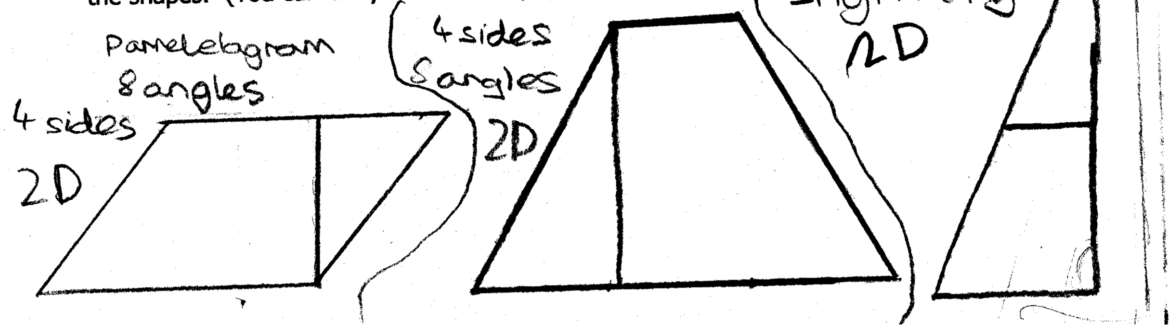
Draw the line on your shape and check with the teacher that it is correct before cutting. Now put your ruler away and cut along the line.

Look at the two shapes you have cut out.
What are the correct mathematical names for these two shapes ?

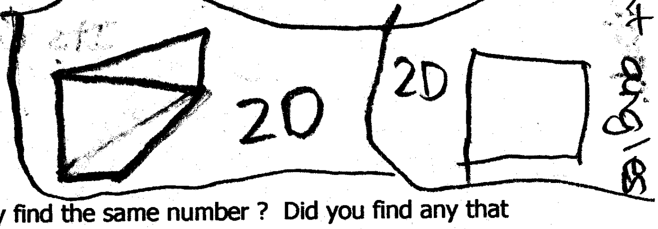
right angle triangle / scaline triangle / Trapierziem.......................................

Write down any facts you already know about these shapes and any you can work out from the shape we had before There's a right angle in the right angle triangle & scaline.
6 angles 2D 3 sides
4 sides 8 angles 2D

How many different shapes can you make using your two pieces if you put them together along sides that are the same length ? Sketch the shapes below and give the shapes their proper mathematical names if you can. Also write any facts you know or can deduce about the shapes. (You can carry on over the page)



How many different shapes did you find ?

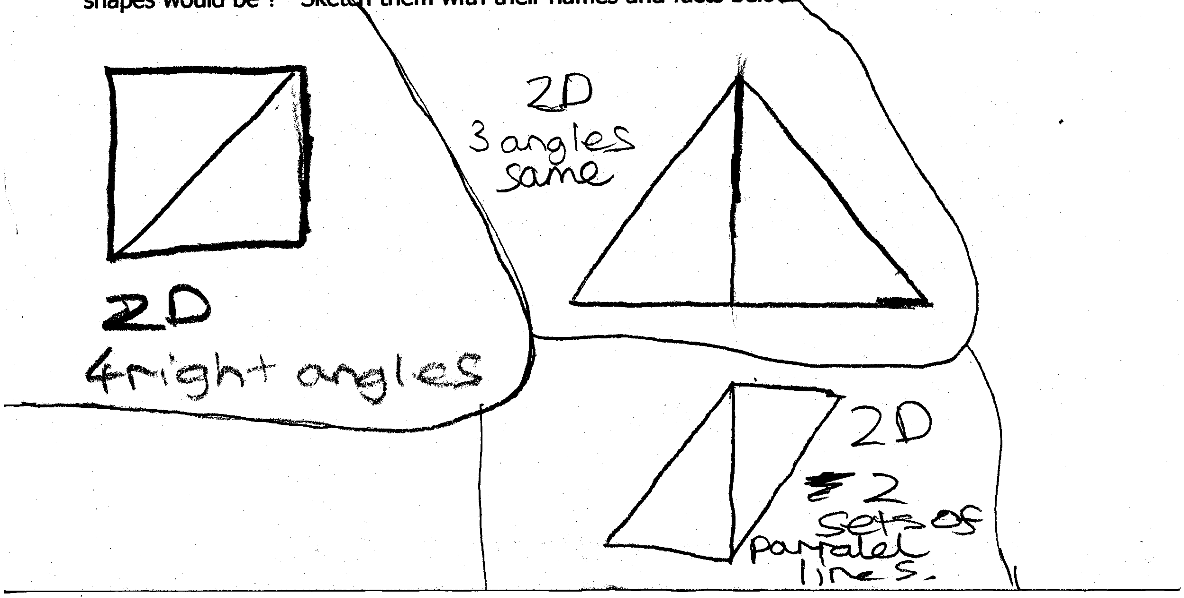


Talk to the people on your table – did they find the same number ? Did you find any that other people didn't ? Add any shapes that you didn't find here ?

Why do you think you can't make any more shapes ?

~~Bec~~ Because you can only get 2 shapes per side and square has four sides so you get 8 shapes.

What would happen if you were to make a cut from vertex to vertex instead ? Without cutting this out can you work out how many shapes you would be able to make and what the shapes would be ? Sketch them with their names and facts below



YEAR 7 TANGRAM WORKSHEET

You have been given a square made of card. In the space below write down any facts you know about this shape.

The 4 sides are the same length. It has 4 perpendicular points.
The 4 angles are the same = 90° = (Right angle)
It has 2 sets of parallel lines.
It has 4 lines of symmetry.
It has 1 face.

Compare your list with that of the person next to you and others on your table. Do you have any facts that they don't ? Do they have any facts that you don't ? If there are any you have missed add them to your list above in a different colour pen.

We're going to make a two piece jigsaw using your square. To do this you need to draw a straight line from one vertex to the midpoint of a side.

What do we mean by vertex ? ...The corner...

What do we mean by midpoint ? The middle of a line...

Draw the line on your shape and check with the teacher that it is correct before cutting. Now put your ruler away and cut along the line.

Look at the two shapes you have cut out.
What are the correct mathematical names for these two shapes ?

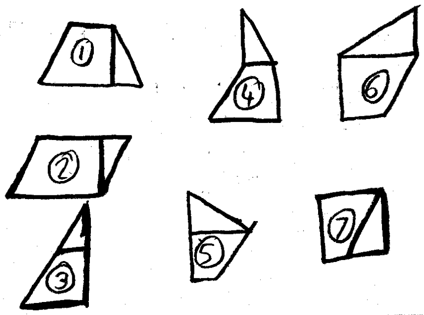
...Right-angle triangle...   ...Thapezium...

Write down any facts you already know about these shapes and any you can work out from the shape we had before

Right-angle triangle
Has 1 right-angle and 2 acute angles.
3 sides
Add up to 180°.

Trapezium
1 set of parallel lines.
4 sides.
Add up to 360°.

How many different shapes can you make using your two pieces if you put them together along sides that are the same length ? Sketch the shapes below and give the shapes their proper mathematical names if you can. Also write any facts you know or can deduce about the shapes. (You can carry on over the page)



③: Right-angle triangle = 1 right-angle.
3 sides
3 angles.

②: Parallelogram = 2 sets of parallel lines.
4 sides
4 angles.

①: (Look at trapezium)

④: Irregular pentagon =
5 sides
5 corners.

⑤ : Quadrilateral :
4 sides
4 corners

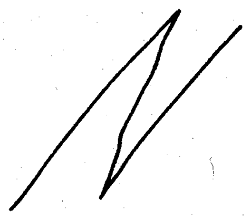
⑦ = Square : 4 sides
4 right angles.
4 straight lines.

⑥ = Irregular pentagon :
5 lines
5 corres

⑧ = Irregular pentagon :
5 lines
5 corners.

How many different shapes did you find ?
8

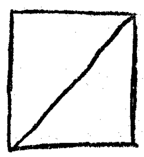
Talk to the people on your table – did they find the same number ?√ Did you find any that other people didn't ? X Add any shapes that you didn't find here ?



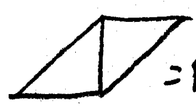
Why do you think you can't make any more shapes ?

Because with the quadrilateral you can put the triangle on each side twice so you do 2 X 4 = 8 shapes.

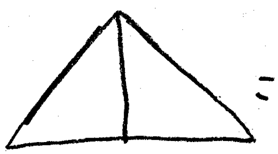
What would happen if you were to make a cut from vertex to vertex instead ? Without cutting this out can you work out how many shapes you would be able to make and what the shapes would be ? Sketch them with their names and facts below.



= Square = 4 sides
4 corners.



= Paralellogram : 4 sides
4 Corners.



= Triange = 3 sides
3 corners

Although this time with a younger group there was less emphasis in the worksheet on deriving the properties of the resulting shapes using those of the original square, this did form part of the discussion, and again the task was diagnostic (for example the first set of pupil work suggests some ambiguity about the number of angles perceived in a shape with diagonals).
Jill's worksheet led on to the notion of providing writing frameworks to help structure pupils' argument; these are discussed in the next section on language issues.

Finally Jill also used the 2-piece tangram as a starter activity.

> With a Y7 group (set 4 out of 5), the 2 piece tangram was used as a starter on a daily basis to provide the children with the opportunity to make a new shape each day. Naming and classifying was the main outcome.

5.5 Conclusions to this section
This section has dealt only with one activity, but it was used with several classes from a lower set in Year 7 to a top set in Year 9. The focus in these different groups developed from a listing of names (van Hiele level 1) and properties (level 2), which were similar to those activities included in section 4, to one on deductive thinking (van Hiele level 4). This was more similar to those activities described in section 5 and used the properties of the original square to derive properties of the constructed shapes and thus to justify the claims that were made about these shapes. However all these aspects were included in all lessons to some extent. Within most classes there were a range of pupil responses almost equivalent to the range of variation between the classes.

Alongside this development there was a changing focus from oral reasoning to written reasoning. Different methods were found of encouraging students to express their reasoning clearly, first in informal group discussions, then in oral class presentations, and finally in writing

The activity itself was found to be engaging for students and rich in mathematical opportunities, including leads into work on area and scale factors, and deriving and justifying the total number of shapes that could be made. Extensions were suggested.

Again in all these descriptions it is clear that the teacher's ability to catch the moment to ask stimulating and appropriately difficult questions and to spot opportunities for making connections is key. In several cases it would have been easy to simply accept students' work without noting that descriptions or reasoning were incomplete and using this as a basis for further probing.

6. *Teaching strategies to encourage mathematical reasoning*

6.1 Language, communication and reasoning

Mathematical reasoning is a form of argument. It is a process of providing justification (warrants) for conclusions by demonstrating that these conclusions follow logically from already accepted results (givens).

Formal written mathematical argument can seem to many school students very abstract and lacking in purpose. To make argument a more authentic process for students, it seems useful to create an audience in the classroom, so that students are presenting their reasons to peers, to the whole class, or to the teacher, before they come to record their reasoning on paper. This oral communication has the benefit of persuading students to provide reasons for their conclusions not just to obey the requirements of some distant examiners but in order to convince others. It also encourages refinement of an argument where the logical steps are faulty or incomplete and others can explain why they are not entirely convinced.

A further advantage of oral communication was that it readily exposed students to reasons used by others that they sometimes decided to adopt for themselves.

We have noted in section 6 a progression in communication of mathematical reasoning. We might illustrate this by a diagram:

Oral, informal demonstration e.g. moving shapes	Verbal jottings, notes, diagrams	Written, conventional notation, labelled diagrams, formal vocabulary

|——————————————————————————►|

In this section we aim to illustrate the different strategies that have been employed by the teaching members of the team to develop their pupils' ability to communicate their findings and reasoning, first orally and then in writing.

6.2 Oral communication in pairs/groups

Many of the lessons described here have involved students working in pairs or small groups so that they could develop their reasoning together and try it out on each other. Sometimes the discussion is then presented to the class and recorded in writing.

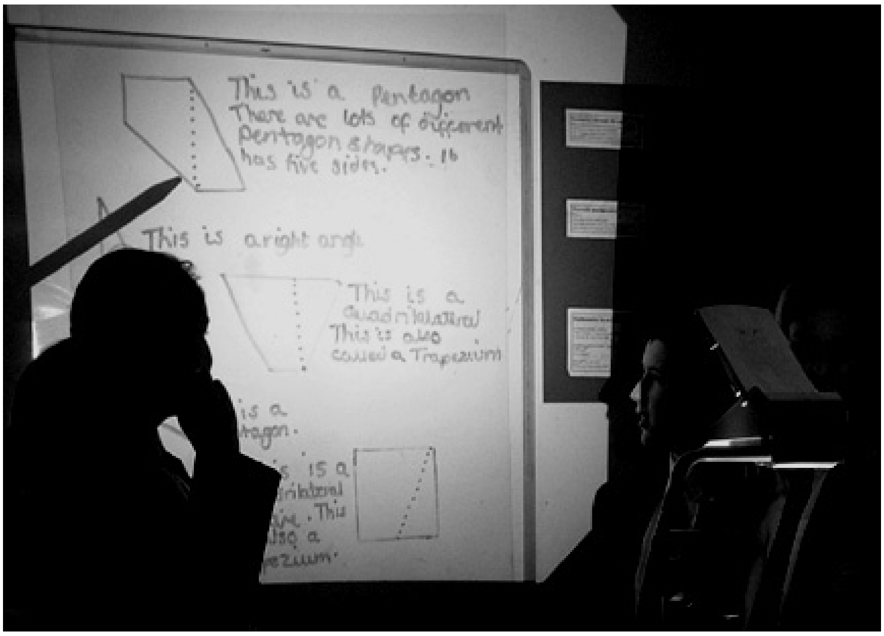
For example in Peter's work on the 2-piece tangram with Year 9 set 4 (see section 6), at one point in the lesson pupils were working in pairs or threes to produce an OHT of their work in answer to the question 'What shapes can you make? Explain why they are what they are.'

In the picture below one student is writing then the other takes the shapes to draw round. Both pupils cooperated well and were intent on the task.

6.3 Oral communication to the class

Peter's report of the "2- piece tangram" task with a low attaining Year 8 class illustrates how he helped to move pupils from informal, imprecise descriptions to well organised explanations based on the properties of the original square i.e. local deduction. To achieve this, after time for paired group discussions he included a requirement for pupils to present a succinct report to the class. He also employed skilful questioning throughout to probe pupils thinking and understanding.

Requiring pupils to present their findings and reasoning to the others in the class helped to clarify their own thinking and acted as a useful model to their peers.



In the situation shown in the picture three pupils were explaining their reasoning using an OHP and acetate they had prepared together.

6.4 Peer assessment

Jo (section 4) combined pupil presentations with peer assessment. One method of doing this was to select (anonymous) responses from among those that different students had given and ask the whole class to work in groups to decide, and later explain, which answers were the best, and why others were inadequate. Managed appropriately these enabled pupils to develop clearer and more "rigorous" lines of argument.

6.5 Probing questioning

Giving pupils time for discussion with peers and teacher seemed to be important for conceptual development at all levels. The following dialogue between a pupil and Peter illustrates just how much time and skilful questioning is required to address a pupil's misconception about parallel lines. Initially Peter thought the pupil knew what parallel lines were. However, it became clear that "parallel" meant "line segments of the same length" to this particular pupil.

> T: OK Kriss, can you just tell me about the shapes that you've found?
>
> P6: I've got a parallelogram and a trapezium.
>
> T: OK. Which one's the parallelogram?
>
> P6: The parallelogram. This one. (*P6 picks up his book and indicates which one it is.*)
>
> T: Right. Now can you show me how you made that up from the pieces? (*P6 puts the pieces together to make a square which does not fit onto his diagram.*) Right, that was the square you started with. Right. OK. So can you now separate the two pieces and put them onto that parallelogram you made? (*P6 demonstrates this correctly.*) Oh, well done. So how do you know it's a parallelogram? What is special about a parallelogram?
>
> P6: It's got four sides which makes it a quadrilateral as well.
>
> T: Right.
>
> P6: It's got four sides so that will be a trapezium and a quadrilateral.
>
> T: That's right. So let's have a look at the parallelogram again. Right.
>
> Which of the sides are parallel on that shape? (*P6 does nothing until P7 (next to him) indicates two of the parallel sides.*)
>
> P6: Those (*Indicates and smiles*.)
>
> T: And which other two sides are parallel?
>
> (P7 *indicates top and bottom of the parallelogram. P6 then indicates them also.*)
>
> T: That's right. What's special about parallel sides?
>
> P6: What's special about parallel sides? Is it that they are parallel to each other?
>
> T: That's right. Right. Can you say a bit more about what you know about 'parallel?'
>
> P6: Er …
>
> T: Can you give me another example of two things in the class that are parallel?
>
> P6: The tables are parallel.
>
> T: Which parts of the table?
>
> P6: (*Indicates the side of the table.*) This (*Indicates the side nearest him.*) and the other sides.

T: That edge and which other edge?

P6: (*Indicates the top edge of the table*.)

T: OK. Which other edges are parallel?

P6: (*Indicates other two sides and P7 pushes a ruler over to him.*)

T: Those two, yes, OK.

So they seem to go in the same direction don't they?

(*P6 takes the ruler.)*

What's that you've got on the table?

P6: A ruler and that's a parallelogram as well. The two sides there (*Indicates long sides.*) and the two sides there (*Indicates short sides.*)

T: Right. Brilliant. Well done, nice explanation. Right. Now tell me about the trapezium which you've got.

P6: It's … different. It's got a smaller, like, top and it's different. It's like longer lines down each side.

T: Right. Are there any sides on the trapezium which are parallel?

P6: Erm … (*He thinks, and then indicates the two sloping sides.*) The two sides are parallel. The two sides. (*He indicates by pointing down each line with his pencil.*)

T: No. There are not quite parallel. They are the same length aren't they?

P6: I, it. The bottom one.

T: Yes. The bottom one and which side's parallel to that?

P6: Parallel to the bottom … (*Looks and thinks with pencil in his mouth.*) … I think it's the top, but that's much too small.

T: No, that's right!

P6: It is?

T: Yes, the top's going in the same direction isn't it?

P6: I know that now!

T: Yes, OK, so you haven't got to worry about the lengths of the sides have you? What's the crucial thing about something which is parallel?

P6: It's straight.

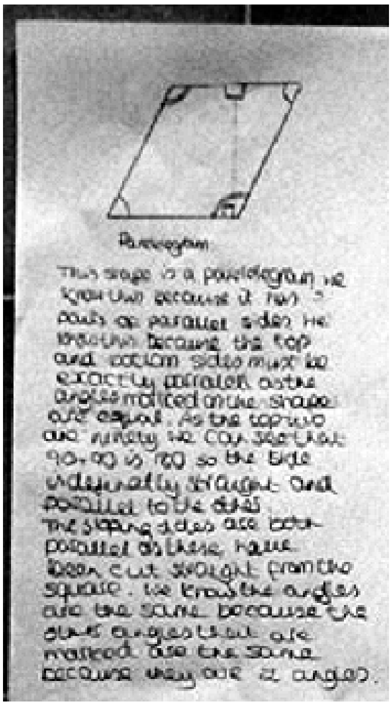
T: It's straight and they go in the …

P6: … in the same direction.

T: Same direction. Brilliant. OK. Thank you very much.

6.6 Written explanations

When the "tangram" task was given to a high attaining Year 9 group (section 6), pupils gave written explanations for their answers. These are reproduced overleaf.



Pupil’s writing to justify why a shape made is a parallelogram (retaining pupil spelling):

This shape is a parelelogram, we know this because it has 2 pairs of parallel sides. We know this because the top and bottom sides must be exactly parralell as the angles marked on the shape are equal. As the top two are ninety we can see that 90 + 90 is 180 so the side is definitely straight and parallel to the other. The sloping sides are both parallel as these have been cut straight from the square. We know the angles are the same because the other angles that are marked are the same because they are Z angles.

Although these Year 9 pupils were able to produce written explanations few of these made use of labelling on their diagrams to produce more succinct explanations. In addition, few were prepared to make oral presentations to the whole class. Clearly, the previous experiences of pupils and classroom culture play an important part in approaches that might prove successful in different classrooms.

6.7 Fact cards and Word banks

As described in section 4.3, Paul used “fact cards” to support pupils who were unable to recognise and use the angle properties and geometrical facts needed to solve a series of problems involving the calculation of various angle measures. The “fact cards” he produced helped pupils to recognise a few salient geometrical facts and to use these to present a reasoned explanation to deduce other facts.

Jo made use of the geometry vocabulary checklist in the KS3 framework to improve low attaining Year 7 pupils’ (working at levels 3b, 3a) use of language and precision when describing properties of shapes. These words were split into two alphabetical groups and pasted into the back of their exercise books. Similar lists were found by Maureen to be needed even for the top set in Year 9 in order to assist with spelling and use of appropriate vocabulary.

Geometry Word Bank (1) a~j

- Adjacent
- Angle: acute, obtuse , right , reflex
- Angles at a point
- Angles on a straight line
- Base (of a plane shape or a solid)
- Base angles
- Centre
- Circle
- Concave, convex

- Degree (º)
- Diagonal
- Diagram
- Edge (of a solid)
- Equal (sides, angles)
- Face
- Given
- Horizontal, vertical
- Identical (shapes)
- Intersect, intersection

6.8 Writing frames

Jill was keen to support Year 9 and Year 7 (extension) students in their written reasoning and introduced a writing frame. This was described in section 5, together with how it was developed and used.

7.9 ICT

In some cases teachers in the group used ICT, sometimes to help structure reasoning (Carol with the diagonals work in section 3) and sometimes to assist in recording (Jill with the flowchart work in section 3). This provided additional support and/or motivation for students' reasoning. Investigation using ICT in programs like *Cabri* or *Geometers' Sketch Pad* would effectively throw up conjectures that required proof. However, here, as elsewhere, ICT did not provide a substitute for logical reasoning by the student.

7.10 Conclusions to this section

The different parts to this section have referred to different teaching strategies used by the teachers in the team to encourage students to reason mathematically in the context of geometry activities.

The aim in lessons developed and taught by the teacher members of the group was to create both a need and a forum for geometric reasoning which seemed plausibly authentic. The description of many of the activities in earlier sections suggested that expressing their reasoning clearly and logically, whether in oral or written form, is a process that students do not find particularly easy. Teachers have used a number of strategies described here to provide additional support, for example by giving feedback and support from partners, greater structure in the tasks where appropriate, or providing cues in the form of word banks or fact cards. In some way each of these was successful in helping students move towards more sophisticated and accurate forms of expression.

7. Findings, implications and further work

7.1 Findings

The findings of this study are grouped under the headings introduced in section 2:

- The nature of classroom tasks
- The role of the teacher
- The social culture of the classroom
- Mathematical tools as learning supports
- Equity and accessibility

The nature of classroom tasks

1. The existing geometry curriculum contains scope for the development of geometrical reasoning, but would benefit from some clarification of this aspect of geometry.
2. There are good resources in existence, providing materials and ideas for teachers to use to help develop geometrical reasoning.
3. The issue of how much structure to provide in a task is an important factor in maximising the opportunity for geometrical reasoning to take place.
4. The outcome of using a given task can vary between different classes and levels of attainment – the way the task is presented needs to be carefully tailored to the group;
5. In developing suitable teaching materials, the following are suggested as guiding principles:
   - The geometrical situations selected should be chosen, as far as possible, to be useful, interesting and/or surprising to pupils and teachers;
   - Activities should expect pupils to explain, justify or reason and provide opportunities for pupils to be critical of their own, and their peers', explanations;
   - Activities should be open enough to allow pupils some freedom in generating their own responses;
   - The forms of reasoning expected should be examples of *local* deduction, where pupils can utilise *any* geometrical properties that they know to deduce or explain other facts or results;
   - To build on pupils' prior experience, activities should involve the properties of 2D and 3D shapes, aspects of position and direction, and the use of transformation-based arguments that are about the geometrical situation being studied (rather than being about transformations *per se*).

The role of the teacher

6. The role of the teacher is vital in helping pupils to progress beyond straightforward descriptions of geometrical observations to encompass the reasoning that justifies those observations. Teacher knowledge in the area of geometry is therefore important.
7. There are many talented teachers able to encourage their pupils to develop and articulate their geometrical reasoning. They are able to use existing ideas to develop resources and materials suited to the particular needs of their own pupils.
8. Teachers' skills in managing classroom intervention are an important component of the development of geometrical reasoning. Pupils need help to focus on the analysis of the logical geometrical relationships underlying their observations when they are working on problems.

9. Being involved in the type of research and development project reported herein is a powerful form of professional development for all those concerned.

The social culture of the classroom

10. Children appear to enjoy working collaboratively in groups with the kind of discussion and argumentation that has to be used to articulate their geometrical reasoning. This form of organisation creates both the need and the forum for argumentation.

11. Presentation of group work to the rest of the class, and peer assessment, provide an encouragement to improve the clarity and rigour of argument.

Mathematical tools as learning supports

12. Whilst pupils can demonstrate their reasoning ability orally, either in group discussion or through presentation of group work to a class, the transition to individual recording of reasoned argument causes significant problems. Several methods have been used successfully in this study to support this transition, including fact cards and writing frameworks, but more research is needed into ways of helping written communication of geometrical reasoning to develop.

13. Even when pupils can demonstrate verbal reasoning skills in geometry they still need help to learn and understand the norms and conventions of written mathematics.

Equity and accessibility

14. It was found possible in this study to enable pupils from all ages and attainments within Key Stage 3 to participate in mathematical reasoning, given appropriate tasks, teaching and classroom culture.

15. Many pupils know more about reasoning than they can demonstrate in writing; the emphasis in assessment on individual written response does not capture the reasoning skills which pupils are able to develop and exercise.

16. Sufficient time is needed for pupils to engage in reasoning through a variety of activities; skills of reasoning and communication are unlikely to be absorbed quickly by many students.

7.2 Implications

The work done by this group suggests that it is appropriate for all teachers to aim to develop the mathematical reasoning of all pupils in the context of geometry, but equally that this is a non-trivial task.

Obstacles that will need to be overcome are likely to include:

- Uncertainty about the nature of mathematical reasoning and about what is expected to be taught in this area among many teachers;
- Lack of exemplars of good practice (although we have tried to address this by lesson descriptions in this report), especially in using transformational arguments;
- Lack of time and freedom in the curriculum to properly develop work in this area;
- An assessment system which does not recognise students' oral powers of reasoning;
- Lack of appreciation of the value of geometry as a vehicle for broadening the curriculum for high attainers, as well as developing reasoning and communication skills for all students.

The ideas and practice included in the report will not, however, necessarily translate directly to other classrooms. Galton (2003) offers a cautionary note about translating approaches across different cultures/countries but this also applies, to some extent, when offering advice on practice where "classroom cultures" may differ, to quote:

> *"…therefore teachers should master the principles that empirical research has shown promote higher-order thinking. These include using open-ended questions, allowing suitable waiting times between asking questions and persuading pupils to respond, and encouraging pupils to explain or elaborate their answers. In the classroom, however, each teacher must use his or her own judgement, based on previous experience, as to the best way to make these principles work in practice. To attempt to operate these principles slavishly is to reduce teaching to a mere technical activity."*
>
> *Galton, 2003, p.51*

7.3 Future Work

The group generated a number of ideas for future work in the area of geometrical reasoning, including:

1. Longitudinal studies of how geometrical reasoning develops through time given a sustained programme of activities (we were conscious that the timescale on which we were working only enabled us to present "snapshots");
2. Studies and evaluation of published materials on geometrical reasoning.
3. A study of "critical experiences" which influence the development of geometrical reasoning;
4. An analysis of the characteristics of successful and unsuccessful tasks for geometrical reasoning;
5. A study of the transition from verbal reasoning to written reasoning;
6. Developing overall perceptions of geometrical figures ("gestalt") as a component of geometrical reasoning, including how to create the links which facilitate this;
7. The use of dynamic geometry software in any of the above areas.

This group was one of six set up under the overall project initiated by the QCA which could form a model for part of the work of regional centres set up like the IREMs in France. Overall, the constitution of the group worked very well, especially after members had got to know each other by working in smaller groups on specific topics. The balance of differing expertise was right, and all the group members felt that they learned a great deal from other group members during the experience. Overall, being involved in this type of research and development project was a powerful form of professional development for all those concerned. In retrospect, the group could have benefited from some longer full-day meetings to jointly develop ideas and analyse the resulting classroom material and experience rather than the pattern of after-school meetings that did not always allow sufficient time to do full justice to the complexity of many of the issues the group was tackling. There is a great deal of expertise among the mathematics teachers in Hampshire, and Ron Taylor, the mathematics inspector, has developed a very productive network of Leading Mathematics Teachers. The working group felt that this expertise could be harnessed for the continuing development of the work of QCA in the area of geometry, and were pleased that a start has been made since some of the teachers in the group are continuing to work in the area of geometry and ICT.

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Appendix A: The work of the Group

Background, constitution and aim

In 2000, following a remit from Government ministers, the Qualifications and Curriculum Authority (QCA) embarked on a three-year project to consider whether the algebra and geometry components of the national curriculum for mathematics need strengthening at key stages 3 and 4. This has involved the formation of an Algebra and Geometry Advisory Group to oversee analysis and research into, and development of, current practices in England, and best practice internationally. The outcome is likely to be advice and support for schools so they can develop and strengthen the place of algebra and geometry within the national mathematics curriculum framework.

As part of this project six small groups have been sponsored by the QCA to develop and trial some teaching/learning materials for use in schools. This report to the QCA summarises the work of one of the three geometry groups. The group was based at Southampton and included teachers and a local authority officer from Hampshire LEA, together with mathematicians and mathematics educators from the University of Southampton, Chichester University College and King's College London. The group was charged with *developing and reporting on teaching ideas that focus on the development of geometrical reasoning at the secondary school level*. The remit given to the group encouraged it to explore what is possible both within and beyond the current requirements of the National Curriculum and the Key Stage 3 strategy, and to consider the whole ability range.

The group members were:

Jill Barton, Cams Hill School, Fareham
Margaret Brown, King's College London/SMP (member of QCA advisory Group)
Ann Hirst, University of Southampton
Keith Hirst, University of Southampton (to March, 2003)
Keith Jones, University of Southampton
Carol Knights, Applemore College, Dibden Purlieu, Southampton
Jo Lees, Crestwood Community School, Eastleigh
Paul Morris, Brune Park Community School, Gosport
Adrian Oldknow, Chichester University College (member of QCA Advisory Group)
Peter Ransom, The Mountbatten School, Romsey
Ron Taylor, Mathematics Inspector for Hampshire (member of QCA Advisory Group)

Keith Hirst chaired the group until the first report in March 2003. At that stage he had to withdraw from membership of the group due to pressure of other work. After the QCA Advisory Group asked the group to extend its work until Autumn 2003, the leadership of the group was taken over by Margaret Brown, Ron Taylor and Keith Jones, with assistance from Ann Hirst.

Ways of working

Following a briefing meeting held at the QCA at the beginning of October 2002 between Keith Hirst, Margaret Brown and Jack Abramsky (the QCA Officer with responsibility for the QCA Advisory Group), a preliminary meeting of the Southampton/Hampshire group was held later that month involving also Keith Jones, Ann Hirst and Ron Taylor. At that meeting Ron Taylor this QCA sponsored project with the work of a group of Leading Mathematics Teachers (LMTs) in Hampshire. It became clear that the task the QCA had asked the group to undertake fitted very well with this, and, as a result the LMTs were invited to join the group to help devise materials on geometrical reasoning and trial them in their schools early in the Spring Term 2003. Five teachers eventually became involved. A meeting in mid-November was attended by three of the teachers together with Adrian Oldknow, a locally-based member of the Advisory Committee and chair of the Royal Society/Joint Mathematical Council Geometry Working Group. At that meeting there was a detailed discussion about the various facets of geometrical reasoning, and how various kinds of activities might contribute. As a result, the group of teachers agreed to meet in late November to formulate some activities for use with their pupils. These were discussed at a meeting in mid-December 2002, and some of the teachers agreed to meet to undertake further refinements to prepare for trialling. The group also agreed that it was important to formulate a well-constructed rationale for the kinds of activities we considered appropriate, and this forms the next section of this report.

Further meetings took place in early February 2003 to consider preliminary outcomes of the trials, and in early March to discuss the outcomes and the interim report.

After the encouragement of the QCA Advisory Group to continue, two further meetings took place, in the Summer term 2003 to consider how best to extend the work and to report on progress respectively. Following a meeting between three of the four editors in August 2003 to sort through the trailing reports a thematic structure was produced for the final report that, together with possible conclusions and implications, was discussed with the whole group in September. Finally another editorial group took place in October to review the writing arrangements.

Evaluation

As this group was one of six which could form a model for part of the work of regional centres set up like the IREMs in France, it seems worth making a few comments on the constitution of the group and ways of working.

The constitution of the group worked very well, especially after members had got to know each other by working in smaller groups on specific topics. The balance of differing expertise seemed about right, and all involved felt that they learned a great deal from other group members during the experience.

In retrospect the group could have benefited from some longer full-day meetings to jointly develop ideas and analyse the resulting classroom material. All the meetings of the full group took place between 3.00 and 5.30pm, which meant that the agendas were too full to allow sufficient time to do full justice to the complexity of many of the issues. It was also difficult to get complete attendance because of varying after-school commitments.

Appendix B: Lesson plans and/or pupil tasks

DIAGONALS OF A QUADRILATERAL I

Objective: Develop geometric reasoning and mathematical communication

Possible Starters

- Identifying symmetry properties of quadrilaterals.
- Using mini-whiteboards ask class to draw a shape/quadrilateral that has 2 lines of symmetry, rotational symmetry order 2 and no lines of symmetry etc
- Placing shapes on a two way table labelled with 2 properties

Main Task

(Can be done in pairs or small groups. A possible worksheet is attached on the following page.)
Fold a piece of A4 paper in half vertically and horizontally
Put a point on each half of each of the folds.
Join the 4 points to form a quadrilateral
Investigate which quadrilaterals can be made and which ones can't.
Use the worksheet or pose key questions to help pupils develop their ideas, e.g.

- What shapes can be made?
- Are the diagonals always lines of symmetry?
- What happens if all the dots are the same distance from the central point?
- What happens if a pair of dots are the same distance from the central point?
- Or 2 pairs of dots? (1 pair not necessarily the same distance as the other pair)

Then fold a new piece of A4 paper so that the two folds are not perpendicular. Again these lines will become the diagonals of quadrilaterals crossing at a central point.
Repeat the investigation constructing quadrilaterals using a point on each of the 'semi-diagonals' and using the same key questions.
Pupils could experiment with pairs of lines at different angles.
Possible key questions:

- With the second pair of diagonals, if a rectangle can be obtained why can't a square be obtained?
- Why can a kite be made with the first pair of diagonals but not the second?
- Why are the diagonals in the first example sometimes lines of symmetry, but the diagonals in the second example never lines of symmetry?

Plenary

Selected students or groups share their ideas with the class.

Diagonals of Quadrilaterals Worksheet

For the first pair of folds fill in this table

Quadrilateral	Can be made ✓ or ✗
Square	
Rectangle	
Parallelogram	
Rhombus	
Kite	
Trapezium	
Isosceles Trapezium	

For the second pair of folds fill in this table

Quadrilateral	Can be made ✓ or ✗
Square	
Rectangle	
Parallelogram	
Rhombus	
Kite	
Trapezium	
Isosceles Trapezium	

A square can be made with one pair of folds and a rectangle with the other pair of folds. What angle do the diagonals have to cross at to be able to draw a square?

What about a rectangle?

By thinking about symmetry or the angles formed, can you explain why this is?

Which quadrilaterals that have a ✓ in both tables or a ✗ in both tables?

Again, by considering the symmetry, can you explain why this is?

B) FLOWCHART ACTIVITY

Using Flow Charts to Classify Polygons

- Use the flow charts to classify your 2 D shapes.
- Design a similar set of flow charts using angle properties rather than side properties as much as possible. E.g. For an equilateral triangle you could ask ‘Are there three equal angles in your shape?’ rather than ‘Are all the sides the same length?’

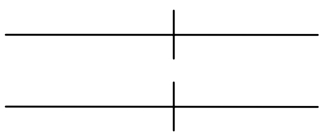
Possible Lesson Plan

Objectives: Children will:~
Appreciate the need for precise mathematical language.
Know labelling conventions for parallel, perpendicular and equal length lines.
Be able to use a flow chart to classify 2D shapes.
Be able to use properties of 2D shapes to design a flow chart.

Starter: Visualisation using white boards. ‘Imagine a rectangle. Cut it in half along one of its diagonals. Imagine the two pieces can move around any way you like and then rejoin along two edges of equal length. What shapes can you make? Draw and name one and show on the whiteboard. (see NNS Y.7 POS for this and other ideas)

(Triangle, parallelogram, kite)

Main:
Resources: 2D shape sheet, Flow charts (could be put onto A3 so they are visible together)
Introduction: Draw a parallelogram on the board. Ask for its name. Ask for any properties. When someone says pairs of equal length sides, label the diagram to show the two pairs using the appropriate conventions. Repeat for parallel lines using the arrow convention.



Show how to label perpendicular lines and discuss the fact that this means they are at right angle to each other.

Now demonstrate using the flow chart to classify a square. Point out that we are only using side properties as much as possible.

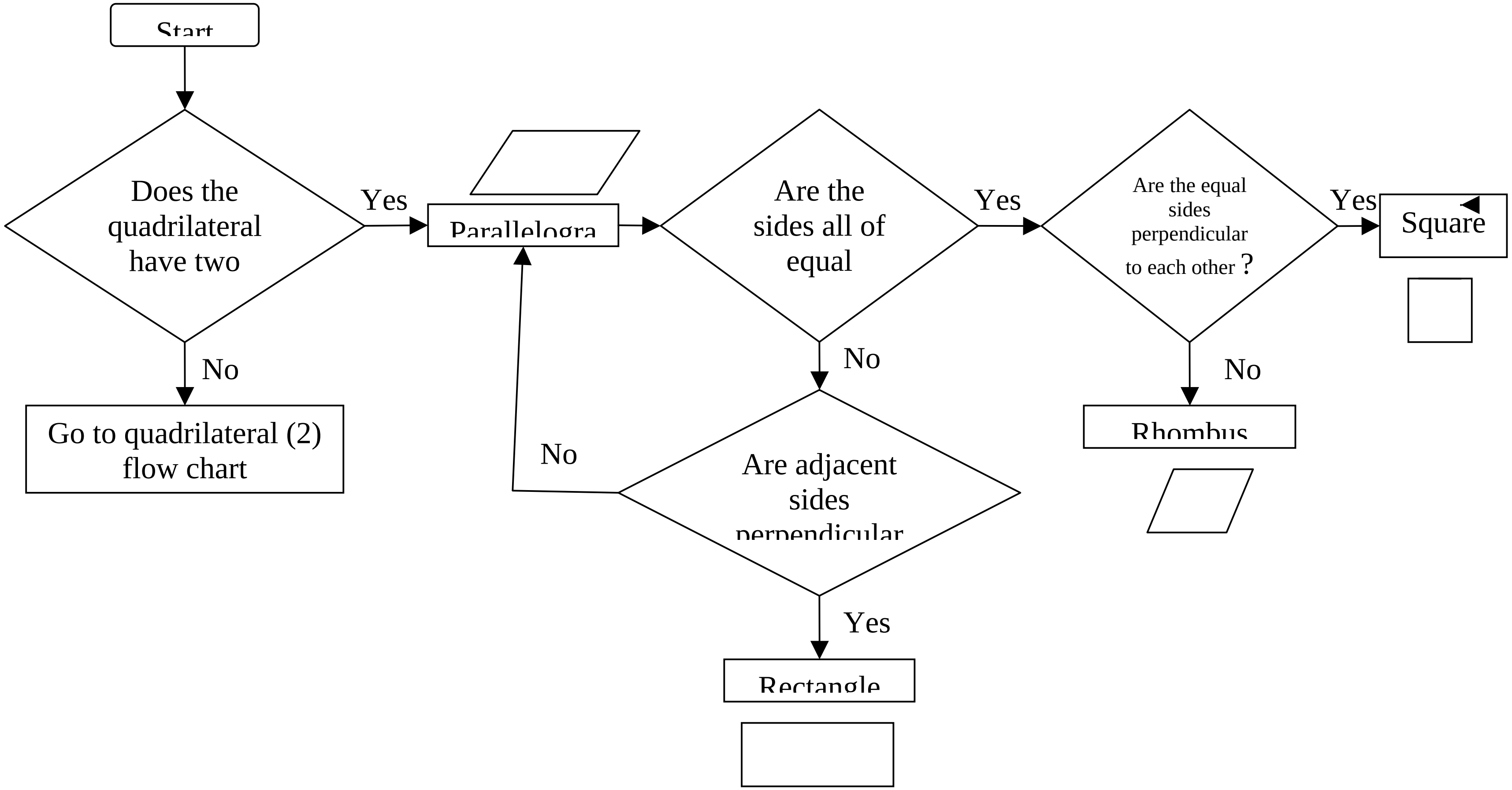
Give the children a set of 2D shapes (these can be cut out, on card or on the sheet for labelling as appropriate to the group) and ask them to name each shape according to the Flow Chart classification.

Task: Ask the children to design their own flow charts based on angle properties as far as possible. When do we have to use side properties? You can start with just triangles if the group find this daunting or let them untangle the whole lot if appropriate. This makes excellent display work…. especially if you can use ICT to present the final draft.

Plenary: Ask the children to report back so far (they won't have finished!) Discuss problems they are having and the need for precise mathematical language to ensure that there are no misunderstandings.

Homework: Complete and present their flow-chart.

Quadrilateral Flow Chart (1)

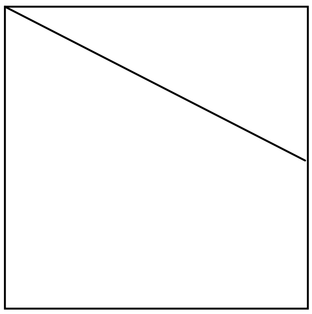


C) 2-PIECE TANGRAMS

Introduction: This piece of work using geometrical reasoning has been used in one teacher's classroom for a few years, mainly with Year 7 pupils. It is written here with an emphasis on the geometrical reasoning side for Year 8, though it can be used in Years 7 or 9.

This activity may appear simple but should not be dismissed on those grounds, as it is a very rich source of geometrical reasoning across the whole attainment spectrum. The task is simple and rich, and through skilful and appropriate classroom management and interaction between pupils and teacher can be used to diagnose and develop pupils' geometrical reasoning.

Equipment needed: A 4cm square piece of card for each student. The two piece jigsaw is made from a square with a cut being made from one vertex to the midpoint of a side. (I)



Ruler, pencil, scissors (to make just one cut)
OHT and pens to write on them
OHP.

Objectives: The activity can be used to address many of the geometrical objectives in the Framework specified for Years 7, 8 or 9. The two-piece jigsaw gives an opportunity to see how well pupils follow instructions, how accurate they are at measuring, how systematic they are and the comprehension of their shape vocabulary, as well as their reasoning skills.

A starter for 5 to 15 minutes depending on how much you want to do.
Work in pairs.
Using the two pieces of card ask them to put the pieces together along sides that are the same length to make a different shape. They should name the shape if they can and try to convince their partner why it is the shape they named. This helps to establish what they know, the precision of the language they use and helps inform the teacher so they can use the descriptions and explanations given by the pupils to develop more formal language/vocabulary in the main part of the lesson. (5 minutes)

I've found it useful for demonstration purposes to cut the two pieces from an old cake board as these are thick and generally covered in shiny paper. I can then attach the pieces to the board with Blutack, and pupils can move them round to get different shapes.

If the extended option is chosen then you can ask one or more pupils out to convince the class. This can be done by demonstration on a white board using large pieces, or using their small pieces (or colour transparencies) on an OHP.

The teacher may wish to introduce the conventional way of labelling vertices and of describing angles at this point.

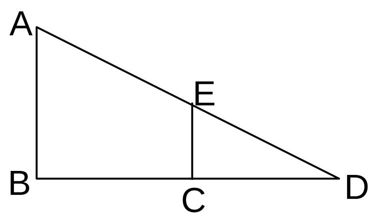
The main activity - 35 to 45 minutes: Pupils should now make some different shapes, drawing and naming them (if they can) in their books. This is a useful exercise on seeing how systematic they are (there are 8 different shapes) and what shape names they know. For a plenary I get them to put their shapes on the OHP in turn until all the possibilities have been found (and named).

Ask what you feel is appropriate from the following questions.

- What shapes can you make?
- Why are there only 8 different shapes?
- What if
  - the cut goes from corner to corner?
  - the cut goes from a corner to a point 1/3 of the way along?
  - the cut goes from a corner to a point 1/4 of the way along?

- Why do you still get an isosceles trapezium?
- Does the place where you make the cut affect the number and type of shapes you can make and if so, why?

A key part of this work is their reasoning and the preciseness of their vocabulary, e.g. “the triangle is rotated 180˚ about E”. It is hoped that the teacher will give an example like this or by using alternate angles when proving that this shape is a right-angled triangle.



After the example(s) demonstrated in the starting activity pupils should now be producing better justifications for why their shape is what they say it is.
It is very difficult to see written work on proof and justification written down by pupils. One way of encouraging this is to give each pair a sheet of plain paper and ask them to make a mini-poster for display in their classroom describing why one of the shapes is what they say it is. After producing their first draft they should swap it with a different pair for their comments on what they have done. You might find that you need to do some direct teaching here such as some work with alternate angles or how to label vertices and how to refer to angles.

If used with Year 9 we felt that some work using Pythagoras could be done, e.g. if the side of the square is 2 units, what are the lengths of the sides of each piece? What about the perimeters of the shapes?

Extensions: What is the ratio of the two triangles shown above?

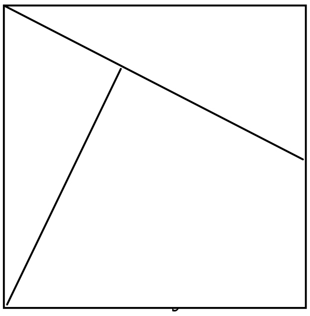
What is the ratio of the area of the two pieces?

Explain how you get your result.

What happens if you use a rectangle with sides in the ratio 1:2? 1:3? 1:n? (I)

What happens if you overlay one piece on top of another? (I)

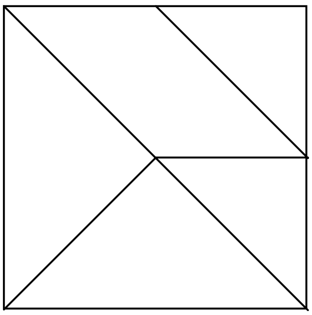
Revisit this work with the three-piece jigsaw. This is the same as the two-piece with an extra bit cut off by joining the vertex shown to the mid-point of the side shown, stopping at the existing cut. A diagram makes it clear.



Can you be more specific?
Where are there right angles?
Which sides are equal?

Then they can draw and name the shapes they get by putting two or three pieces together along equal sides (or non-equal sides for the more ambitious).

We felt that the work could be extended to use a 5 piece tangram as shown, or even the 7 piece tangram.



Plenary - 10 minutes: Another opportunity for pupils to come up and explain why the shape is what it is to their peers. This encourages mathematical communication. You know your pupils well enough to choose those who will adopt their work in the light of comments.

Points to consider: Asking open questions is sometimes harder for pupils to answer.
E,g. "Why is this a square?" can get more response than "What shape do you have and why is it that shape?"

Homework: One possible homework is to ask them to make a mini-poster for display in their classroom describing why one of the shapes is what they say it is. Another is to

ask them to explain why one of the shapes is what they say it is to a parent. (Prove it to a Parent).
Those with ICT access could ask for a Powerpoint presentation.

D) 2-PIECE TANGRAMS: ALTERNATIVE VERSION

Resources: Cardboard squares, rulers, scissors, tangram worksheet (see next page), angle sheet.

Starter (10 mins): Have word angle on (interactive) whiteboard. Ask pupils to come and circle any right angles, then any acute angles, any obtuse angles and any reflex angles. Give pupils copy of sheet with word angle on it and ask them to show the different types of angles in different colours and stick in their books making sure they use a key.

Main activity (50 mins): Give each pupil a cardboard square and introduce the tangram activity. Each pupil to have a copy of the tangram activity writing frame to fill in. Get pupils to work on introduction on square and after about five minutes get pupils to share their answers with the rest of the class. Point out any properties that are missing from the pupils' lists. Pupils can then work through the writing frame at their own pace. Discuss findings with groups at tables.

Plenary (10 mins): Discussion of findings. Focus on parallelogram (which was one of shapes constructed with two piece tangram). Show/discuss how we can use the properties of the square to show that allied angles in a parallelogram add up to 180º. Introduce the idea of calling one angle of a right angled triangle x and then being able to label the other angle (which is not a right angle) of the triangle in terms of x.

Worksheet:
(Note: Additional space needs to be left where indicated for some of the student responses)

Name: ………………………………………….
Tutor Group: ……………………….

YEAR 7 TANGRAM WORKSHEET

You have been given a square made of card. In the space below write down any facts you know about this shape.
(SPACE)

Compare your list with that of the person next to you and others on your table. Do you have any facts that they don't ? Do they have any facts that you don't ? If there are any you have missed add them to your list above in a different colour pen.

We're going to make a two piece jigsaw using your square. To do this you need to draw a straight line from one vertex to the midpoint of a side (which is marked on your card).

What do we mean by vertex ? …………………………………

What do we mean by midpoint ? …………………………………………………………

Draw the line on your shape and check with the teacher that it is correct before cutting.

Look at the two shapes you have cut out.
What are the correct mathematical names for these two shapes ?

……………………………………………………………………………………………
…

Write down any facts you already know about these shapes and any you can work out from the shape we had before
(SPACE)

How many different shapes can you make using your two pieces if you put them together along sides that are the same length ? Sketch the shapes below and give the shapes their proper mathematical names if you can. Also write any facts you know or can deduce about the shapes. (You can carry on over the page)
(SPACE)

How many different shapes did you find ?

Talk to the people on your table – did they find the same number? Did you find any that other people didn't ? Add any shapes that you didn't find here?
(SPACE)

Why do you think you can't make any more shapes ?

(SPACE)

What would happen if you were to make a cut from vertex to vertex instead ? Without cutting this out can you work out how many shapes you would be able to make and what the shapes would be ? Sketch them with their names and facts below.

(SPACE)

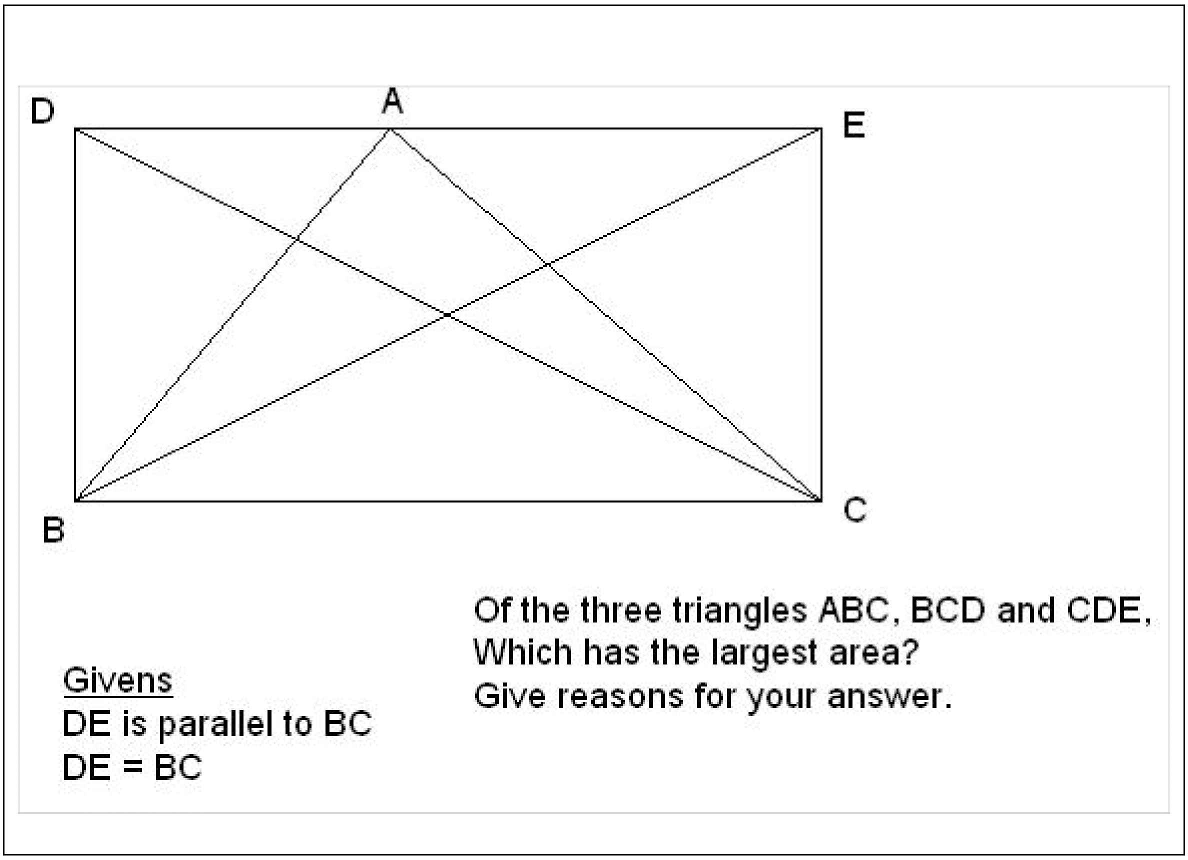
What would happen if the cut goes from a vertex to a point one third of the way along one side? Without cutting this out can you work out how many shapes you would be able to make and what the shapes would be ? Sketch them with their names and facts below.

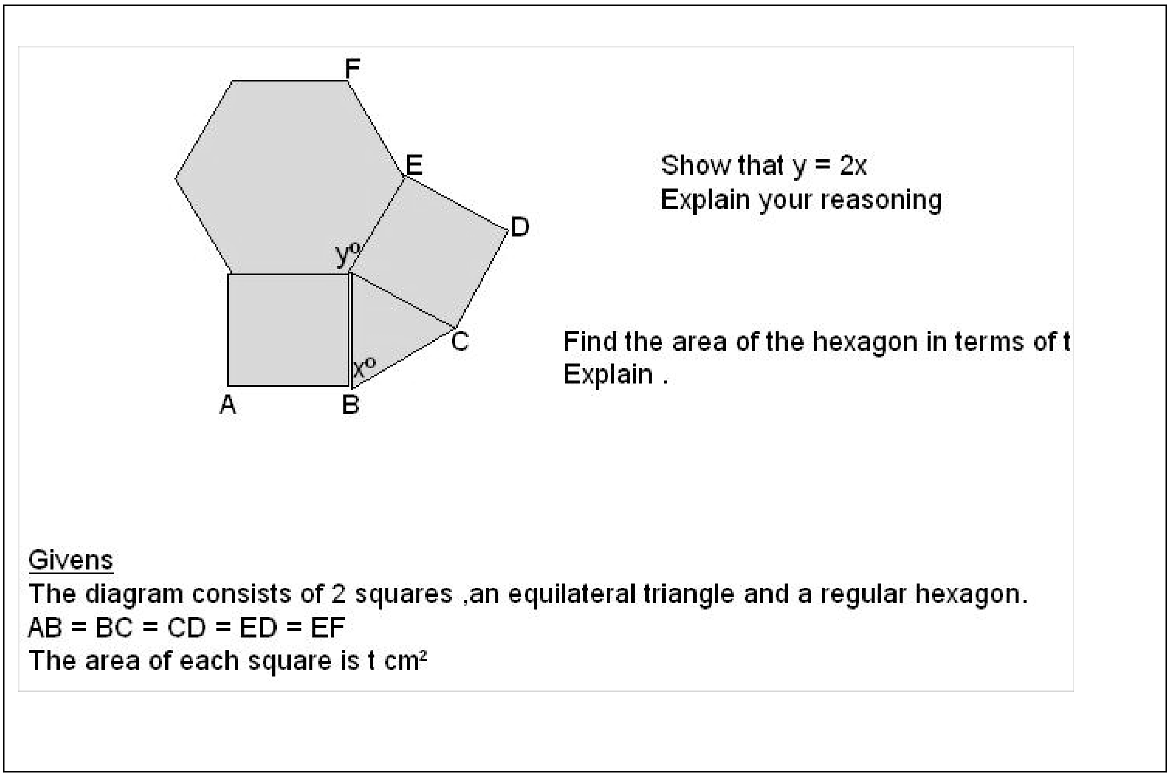
(SPACE)

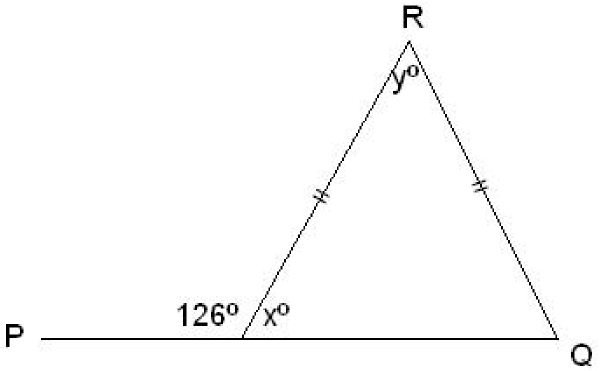
What would happen if the cut goes from a vertex to a point one quarter of the way along one side? Without cutting this out can you work out how many shapes you would be able to make and what the shapes would be ? Sketch them with their names and facts below.

(SPACE)

Proof activities







Given
PQ is a straight line

- Work out the size of the angle marked x°
- Work out the size of the angle marked y°
- Give reasons for your answer

Developing geometrical reasoning in the secondary school

Margaret Brown
Keith Jones
Ron Taylor

This is the final report of the Southampton/Hampshire Group of mathematicians and mathematics educators sponsored by the Qualifications and Curriculum Authority (QCA) to develop and trial some teaching/learning materials for use in schools that focus on the development of geometrical reasoning at the secondary school level.

The report illustrates how it is possible to enable pupils from all ages and attainments within the lower secondary (Key Stage 3) curriculum to participate in mathematical reasoning, given appropriate tasks, teaching and classroom culture. The study suggests that it is appropriate for all teachers to aim to develop the geometrical reasoning of all pupils, but equally that this is a non-trivial task. Obstacles that need to be overcome are likely to include uncertainty about the nature of mathematical reasoning and about what is expected to be taught in this area among many teachers, lack of exemplars of good practice (although this report attempts to address this by providing a range of lesson descriptions), especially in using transformational arguments, lack of time and freedom in the curriculum to properly develop work in this area, an assessment system which does not recognise students' oral powers of reasoning, and a lack of appreciation of the value of geometry as a vehicle for broadening the curriculum for high attainers, as well as developing reasoning and communication skills for all students.

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