An introduction to the π-calculus

Model, Variations, Semantics

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The π calculus

The syntax

- An infinite set of names: $\mathcal{N} = \{x, y, z, \ldots\}$.
- Action prefixes

$$\pi ::= x(y) \mid \bar{x}\langle y \rangle \mid \tau$$

- Processes

$$P ::= \sum_{i \in I} \pi_i.P_i \mid P|P \mid (\nu a)P \mid !P$$

Reductions

$$(\nu z)(\bar{x}\langle y\rangle + z(w).\bar{w}\langle y\rangle \mid x(u).\bar{u}\langle v\rangle \mid \bar{x}\langle z\rangle)$$

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Reductions

$$\begin{tikzcd}[column sep=small]
& (\nu z)(\bar{x}\langle y\rangle + z(w).\bar{w}\langle y\rangle \mid x(u).\bar{u}\langle v\rangle \mid \bar{x}\langle z\rangle) \arrow[dl] \arrow[dr] & \\
(\nu z)(0 \mid \bar{y}\langle v\rangle \mid \bar{x}\langle z\rangle) & & (\nu z)(\bar{x}\langle y\rangle + z(w).\bar{w}\langle y\rangle \mid \bar{z}\langle v\rangle \mid 0) \arrow[d] \\
& & (\nu z)(\bar{v}\langle y\rangle \mid 0 \mid 0)
\end{tikzcd}$$

Contexts and Congruences

Process Contexts

$$C ::= [\ ] \mid \pi.C + P \mid (\nu a)C \mid C|P \mid P|C \mid !C$$

Conguences

A relation $\cong$ is a congruence if it is preserved by all contexts, that is $P \cong Q$ implies:

$$\pi.P + R \cong \pi.Q + R \qquad (\nu a)P \cong (\nu a)Q$$

$$P \mid R \cong Q! \mid P \qquad R \mid P \cong R \mid Q$$

$$!P \cong !Q$$

Structural Congruence

$P \equiv Q$ if they can be transformed into each other using

- alpha-conversion: for z not free in P

$$x(y).P \equiv x(z).\{z/y\}P$$
$$(\nu a)P \equiv (\nu z)\{z/a\}P;$$

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$$
\begin{aligned}
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\end{aligned}
$$

- rearrangement of terms in summations;

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- $(\nu z)(P \mid Q) \equiv (\nu z)P \mid Q$, if z not free in Q;
  $(\nu z)0 \equiv 0$, $(\nu x)(\nu y)P \equiv (\nu y)(\nu x)P$.

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- $!P \equiv P \mid !P$

Standard Form

A process

$$(\nu\vec{a})(M_1 \mid \cdots \mid M_m \mid !Q_1 \mid \cdots \mid !Q_n)$$

is in standard form if

1. each M_i is a sum and
2. each Q_i is itself in standard form.

Thm: Every processes is structurally congruent to a process in standard form.

Reaction Rules

Reaction Rules

$$\text{Tau}: (\tau.P + M) \to P$$

$$\text{React}: (M + x(y).P) \mid (\bar{x}\langle z\rangle.Q + M') \to \{z/y\}P \mid Q$$

$$\text{Par}: \frac{P \to P'}{P \mid Q \to P' \mid Q} \qquad \text{Res}: \frac{P \to P'}{(\nu a)P \to (\nu a)P'}$$

$$\text{Struct}: \frac{P \equiv P' \quad P' \to Q' \quad Q' \equiv Q}{P \to Q}$$

An example

Although it may appear not obvious, the term

$$x(z).\bar{y}\langle z\rangle \mid !(\nu y)\bar{x}\langle y\rangle.Q$$

has a redex. Let us use $\equiv$ to uncover it.

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&\equiv \quad x(z).\bar{y}\langle z\rangle \mid (\nu a)\bar{x}\langle a\rangle.Q \mid !(\nu y)\bar{x}\langle y\rangle.Q \\
&\equiv \quad (\nu a)(x(z).\bar{y}\langle z\rangle \mid \bar{x}\langle a\rangle.Q) \mid !(\nu y)\bar{x}\langle y\rangle.Q
\end{aligned}$$

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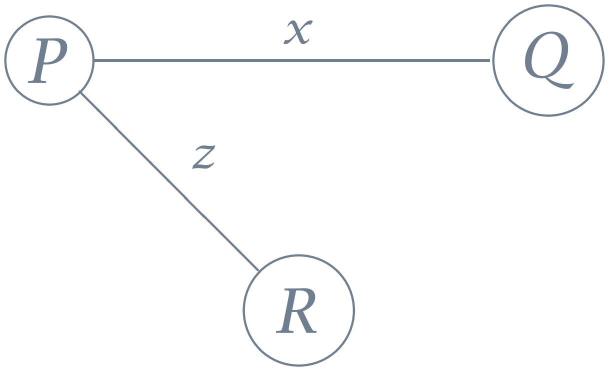
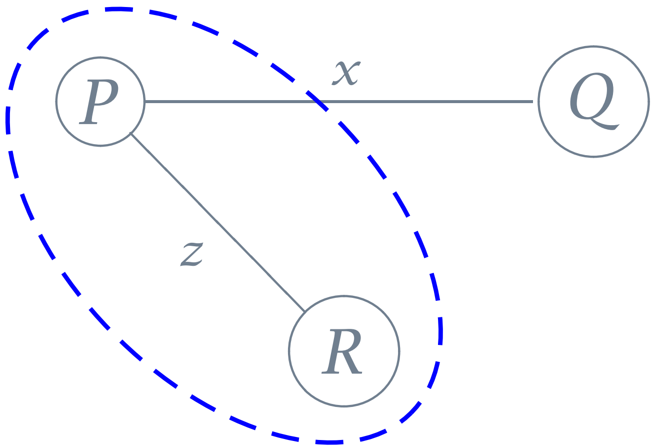
\equiv & x(z).\bar{y}\langle z\rangle \mid (\nu a)\bar{x}\langle a\rangle.Q \mid !(\nu y)\bar{x}\langle y\rangle.Q \\
\equiv & (\nu a)(x(z).\bar{y}\langle z\rangle \mid \bar{x}\langle a\rangle.Q) \mid !(\nu y)\bar{x}\langle y\rangle.Q \\
\rightarrow & (\nu a)(\bar{y}\langle a\rangle \mid Q) \mid !(\nu y)\bar{x}\langle y\rangle.Q
\end{array}$$

Mobility



$$P \mid Q \mid R$$

Mobility



$$(\nu z)(P \mid Q) \mid R$$

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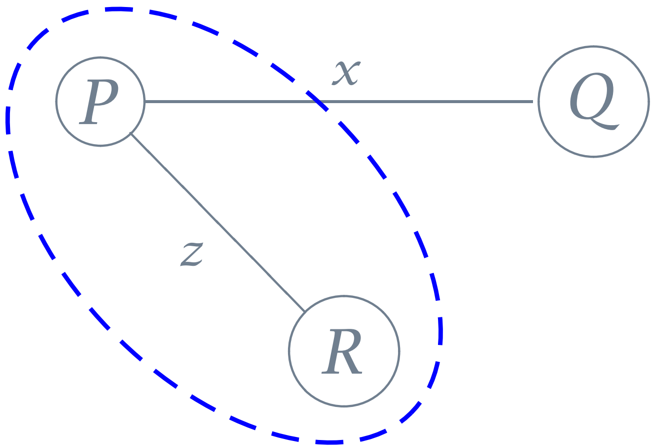

$$(\nu z)(P \mid Q) \mid R$$

Suppose $P = \bar{x}\langle z\rangle.P'$ (x not free in P') and $Q = x(y).Q'$

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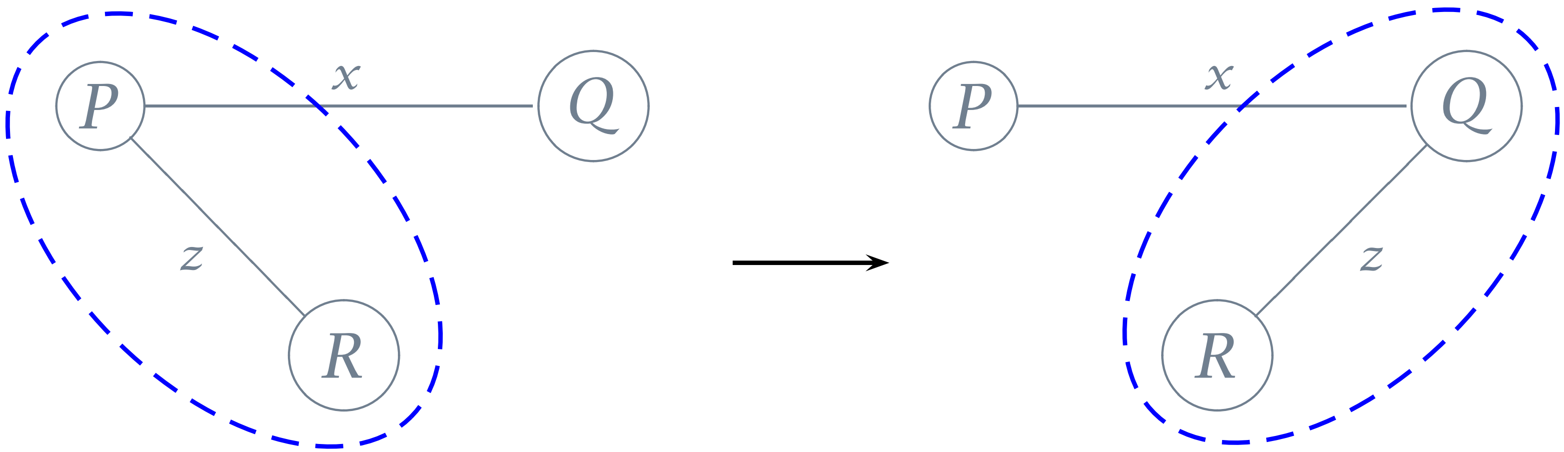

$$(\nu z)(P \mid Q) \mid R \qquad\qquad P' \mid (\nu z)(Q' \mid R)$$

New scope of z

Polyadic π

The idea:

$$x(y_1,\ldots,y_n).P \mid \bar{x}\langle z_1,\ldots,z_n\rangle.Q \to \{\vec{z}/\vec{y}\}P \mid Q$$

Is it a more expressive paradigm? Or can it be encoded?

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Encoding? $\llcorner \bar{x}\langle z_1, z_2\rangle.P \lrcorner = \bar{x}\langle z_1\rangle.\bar{x}\langle z_2\rangle.\llcorner P \lrcorner.$

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Wrong idea:

$$\llcorner \bar{x}\langle z_1,z_2\rangle.P \mid x(y_1,y_2).Q_1 \mid x(y_1,y_2).Q_2 \lrcorner$$
$$\longrightarrow^*$$
$$\llcorner P \lrcorner \mid x(y_2).\{z_1/y_1\}\llcorner Q_1 \lrcorner \mid x(y_2).\{z_2/y_2\}\llcorner Q_2 \lrcorner$$

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Encoding? $\llcorner \bar{x}\langle z_1, z_2\rangle.P \lrcorner = \bar{x}\langle z_1\rangle.\bar{x}\langle z_2\rangle.\llcorner P \lrcorner$.
Right idea:

$$\begin{aligned} \ulcorner \bar{x}\langle z_1, z_2\rangle.P \urcorner &= (\nu z)\bar{x}\langle z\rangle.\bar{z}\langle z_1\rangle.\bar{z}\langle z_2\rangle.\ulcorner P \urcorner \\ \ulcorner x(z_1, z_2).P \urcorner &= x(y).y(y_1).y(y_2).\ulcorner P \urcorner \end{aligned}$$

Thm. The translation is ‘fully abstract’.

Summation

The original calculus has unguarded sums:

$$P ::= \dots \mid P + P$$

This makes the theory more complex while producing little gains in expressiveness.

Input guarded sum: $\sum_{i \in I} x_i(y).P$

Rejects things like $(x.P + \bar{y}.Q) \mid (\bar{x}.P' + y.Q')$.

No choice:
Still, purely internal choice $\tau.P + \tau.Q$ can be defined:

$$(\nu a)(\bar{a} \mid a.P \mid a.Q)$$

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Matching and Mismatching

The original calculus has

$$P ::= \ldots \mid [x = y]P \mid [x \neq y]P$$

$$[x = x]P \to P \qquad\qquad [x \neq y]P \to P$$

Useful in programming, it somehow complicates the theory. A general encoding is not known, but in some interesting cases its effect can be recovered to a certain extent:

$$\llcorner a(x).(\sum_i [x = k_i]P_i)\lrcorner = a(x).(\bar{x}\langle\rangle \mid \sum_i k_i().\llcorner P_i \lrcorner)$$

This works provided nobody interferes with k_i.

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Recursive Definitions

It is useful to be able to write

$$A(\vec{x}) \stackrel{def}{=} Q_A, \quad \text{where } Q_A = \cdots A\langle\vec{w}\rangle \cdots A\langle\vec{w}\rangle \cdots$$

It can be obtained using replication as follows

1. Choose a_A to stand for A;
2. $\widehat{R}$ = replace $A\langle\vec{w}\rangle$ with $\bar{a}_A\langle\vec{w}\rangle$ in R;
3. $\widehat{\widehat{P}} = (\nu a_A)(\widehat{P} \mid !a_A(\vec{x}).\widehat{Q_A})$.

On the other hand, replication can be defined from recursive defs.

Asynchronous π

No continuation on output: $\bar{x}\langle y\rangle\cancel{.P}$

The closest you can simulate it: $P \equiv \tau.(P' \mid \bar{x}\langle y\rangle)$.

How can P know when and if the output is received?

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Explicit continuations

$$\text{sender: } \tau.(\bar{x}\langle y\rangle \mid x.P') \qquad \text{receiver: } x(y).(Q' \mid \bar{x}).$$

But the synchronisation is now much looser... This is widely accepted to be a better base for distribution.

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No sums

Asynchronous calculi often do not consider sums: those compatible with the idea of asynchrony are encodable.

Other variants

- Local π: Disallow inputs on x in the body of $a(x).P$.
- Internal π: Disallow external mobility (output of free names): Processes can only pass names their own private names.
- Distributed π: Several interesting calculi based on π_A: Dpi, Join, Blue, Seal, Nomadic Pict,. . .

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Higher Order π

The most natural suggestion for process mobility:

$$\pi ::= \cdots \mid x(X) \mid \bar{x}\langle P\rangle$$
$$P ::= \cdots \mid X$$

$$x(X).P \mid \bar{x}\langle R\rangle.Q \rightarrow \{R/X\}P \mid Q$$

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$$\bar{a}\langle \bar{b}\langle u\rangle\rangle.b(x) \mid a(X).(X \mid \bar{c}\langle v\rangle) \to b(x) \mid \bar{b}\langle u\rangle \mid \bar{c}\langle v\rangle.$$

This is very expressive: A general purpose replicator

$$D \stackrel{def}{=} a(X).(X \mid \bar{a}\langle X\rangle)$$

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Thm. Encoding 'fully abstract'. Assume a name an unused name x associated to each X.

$$\begin{aligned} [\![\bar{a}\langle P\rangle.Q]\!] &= (\nu p)\bar{a}\langle p\rangle.([\![Q]\!] \mid !p.[\![P]\!]),\ p \text{ fresh} \\ [\![a(X).P]\!] &= a(x).[\![P]\!]) \\ [\![X]\!] &= x \end{aligned}$$

Observations and Bisimulation

Observations: $P{\downarrow_a}$ if P can engage in action a

$$P \equiv (\nu\vec{x})(\alpha.P + \cdots) \quad \alpha \in \{\bar{a}\langle z\rangle, a(y)\},\ a \notin \vec{x}$$

Barbed Bisimulation $\dot{\sim}_B$: is the largest equivalence relation $\pitchfork$ s.t. for all $P \pitchfork Q$

1. $P \to P'$ then $Q \to Q'$ for some such that $P' \pitchfork Q'$;
2. $P{\downarrow_x}$ then $Q{\downarrow_x}$

Observations and Contexts

Barbed equivalence is very weak:

$$\bar{a}\langle u\rangle \quad \dot{\sim}_B \quad \bar{a}\langle v\rangle \quad \dot{\sim}_B \quad (\nu u)\bar{a}\langle u\rangle$$

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But contexts are very powerful enquirers:
$C = (\nu a)([\ ] \mid a(x).x)$. Then

$$C[\bar{a}\langle u\rangle] \rightarrow (\nu a)u\downarrow_u \qquad C[\bar{a}\langle v\rangle] \rightarrow (\nu a)v\downarrow_v$$
$$C[(\nu u)\bar{a}\langle u\rangle] \rightarrow (\nu au)u\not\downarrow$$

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Therefore

$$C[\bar{a}\langle u\rangle] \quad \dot{\not\sim}_B \quad C[\bar{a}\langle v\rangle] \quad \dot{\not\sim}_B \quad C[(\nu u)\bar{a}\langle u\rangle]$$

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$$C[\bar{a}\langle u\rangle] \to (\nu a)u\downarrow_u \qquad C[\bar{a}\langle v\rangle] \to (\nu a)v\downarrow_v$$
$$C[(\nu u)\bar{a}\langle u\rangle] \to (\nu a u)u\not\downarrow$$

Barbed Congruence $\sim_B$: Is the largest congruence in $\dot{\sim}_B$, that is $P \sim_B Q$ iff $C[P] \dot{\sim}_B C[Q]$, for all C.

Labelled Transitions

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- Congruences are brought about by engineering, mathematical and logical considerations.
- But barbed congruence is hard to understand and work with.
- Desiderata: Characterise it in terms of:
  1. bisimulations (easy to reason with – coinduction)
  2. labelled transition systems (describe interactions with environment explicitly, help intuition, bag of tools, . . .)

Abstractions and Concretions

$x(y).P$ has the shape xF, for $F = (y).P$

$\bar{x}\langle z\rangle.Q$ has the shape $\bar{x}.C$, for $C = \langle z\rangle.Q$

For bound output: $C = (\nu a)\langle z\rangle.Q$ ($a = z$ or ν absent).

$$(y).P \bullet (\nu a)\langle z\rangle.Q = (\nu a)(\{z/y\}P \mid Q), \qquad a \text{ not free}$$

$$(y).P \mid R = (y).(P \mid R), \qquad y \text{ not free}$$

$$(\nu c)(y).P = (y).(\nu c)P, \qquad c \neq y$$

$$(\nu a)\langle z\rangle.Q \mid R = (\nu a)\langle z\rangle.(Q \mid R), \qquad z \text{ not free}$$

$$(\nu c)(\nu a)\langle z\rangle.Q = \begin{cases} (\nu a)\langle z\rangle.(\nu c)Q, & c \neq z \\ (\nu c)\langle z\rangle.Q, & c = z (= a) \end{cases}$$

Commitment Relation

Commitment Rules

$$\textsc{Sum}: M + \alpha A \xrightarrow{\alpha} A$$

$$\textsc{React}_{L,R}: \frac{P \xrightarrow{x} F \quad Q \xrightarrow{\bar{x}} C}{P \mid Q \xrightarrow{\tau} F \bullet C} \qquad \frac{P \xrightarrow{\bar{x}} C \quad Q \xrightarrow{x} F}{P \mid Q \xrightarrow{\tau} F \bullet C}$$

$$\textsc{Par}_{L,R}: \frac{P \xrightarrow{\alpha} A}{P \mid Q \xrightarrow{\alpha} A \mid Q} \qquad \frac{Q \xrightarrow{\alpha} A}{P \mid Q \xrightarrow{\alpha} A \mid P}$$

$$\textsc{Res}: \frac{P \xrightarrow{\alpha} A}{(\nu a)P \xrightarrow{\alpha} (\nu a)A} \alpha \notin \{a, \bar{a}\} \qquad \textsc{Rep}: \frac{P \mid !P \xrightarrow{\alpha} A}{!P \xrightarrow{\alpha} A}$$

Thm. $P \xrightarrow{\tau} \equiv Q$ iff $P \rightarrow Q$.

An example

Let us show how to prove that

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The role of $\bullet$:

$$\begin{aligned} (y).P \bullet \langle a\rangle.Q &= \quad\;\; (\{a/y\}P \mid Q) \\ (y).P \bullet (\nu a)\langle a\rangle.Q &= (\nu a)(\{a/y\}P \mid Q) \end{aligned}$$

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Bisimulation

Strong (late) Bisimulation $\dot{\sim}_L$: is the largest equivalence relation $\pitchfork$ s.t. for all $P \pitchfork Q$

1. $P \xrightarrow{x} (y).P'$ then $Q \xrightarrow{x} (y').Q'$ for some $(y').Q'$ such that $\{z/y\}P' \pitchfork \{z/y'\}Q'$, for all $z \in \mathcal{N}$;
2. $P \xrightarrow{\bar{x}} (\nu a)\langle z\rangle.P'$ then $Q \xrightarrow{\bar{x}} (\nu a)\langle z\rangle.Q'$ for some Q' such that $P' \pitchfork Q'$;
3. $P \xrightarrow{\tau} P'$ then $Q \xrightarrow{\tau} Q'$ for some Q' such that $P' \pitchfork Q'$.

Congruence

$\dot\sim_L$ is not preserved by substitution:

$$\bar{x} \mid y \;\dot\sim_L\; \bar{x}.y + y.\bar{x}$$
$$\bar{x} \mid x \;\dot\sim_L\; \bar{x}.x + x.\bar{x} + \tau \;\dot{\not\sim}_L\; \bar{x}.x + x.\bar{x}$$

Similarly for matching.

$$[x = y]c \;\dot\sim_L\; 0 \quad \text{but} \quad [x = x]c \;\dot{\not\sim}_L\; 0$$

Thm. Let $\sim_L$ be the largest congruence in $\dot\sim_L$. Then, $P \sim_L Q$ iff $\sigma P \;\dot\sim_L\; \sigma Q$ for all substitutions σ.

Thm. $\dot\sim_L$ is a congruence for πA

Early bisimulation

Strong early Bisimulation $\dot{\sim}_E$: is defined as $\dot{\sim}_L$ replacing

if $P \xrightarrow{x} (y).P'$ then $\exists Q \xrightarrow{x} (y').Q'.\forall z.\{z/y\}P' \pitchfork \{z/y'\}Q'$

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Thm. Let $\sim_L$ be the largest congruence in $\dot{\sim}_L$. Then, $P \sim_L Q$ iff $\sigma P \dot{\sim}_L \sigma Q$ for all substitutions σ.
In practice the difference is minimal, but in theory:

Thm. $\sim_L = \sim_B$

Summary Strong Equivalences

$$
\begin{tikzcd}
 & \dot{\sim}_B \\
 & \dot{\sim}_E \arrow[u, hook] \\
\sim_B \;=\; \sim_E \arrow[uur, hook] \arrow[ur, hook] \arrow[dd, hook] & \\
 & \dot{\sim}_L \arrow[uu, hook] \\
\sim_L \arrow[ur, hook] &
\end{tikzcd}
$$

Weak bisimulations

Ignore internal steps (τ)

Weak Barbed $\dot{\approx}_B$ and $\approx_B$: Replace

$$\downarrow_x \text{ with } \Downarrow_x \stackrel{def}{=} \rightarrow^* \downarrow_x, \text{ and}$$
$$\rightarrow \text{ with } \Rightarrow \stackrel{def}{=} \rightarrow^*$$

Weak Bisimulations $\dot{\approx}_L$ and $\approx_L$: Replace

$$\xrightarrow{\tau} \text{ with } \Longrightarrow \stackrel{def}{=} \xrightarrow{\tau}{}^*,$$
$$\xrightarrow{x} \text{ with } \stackrel{x}{\Longrightarrow} \stackrel{def}{=} \Longrightarrow \xrightarrow{x} \Longrightarrow \text{ and}$$
$$\xrightarrow{\bar{x}} \text{ with } \stackrel{\bar{x}}{\Longrightarrow} \stackrel{def}{=} \Longrightarrow \xrightarrow{\bar{x}}$$

Summary Weak Equivalence

The theory parallels that of the strong equivalences.

$$
\begin{tikzcd}
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\approx_B = \approx_E \arrow[uur, hook] \arrow[ur, hook] & \\
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- $(\nu x) \sum_i \pi_i.P_i = \sum_i \pi_i.(\nu x)P_i$, if no π_i has name x
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$$\begin{aligned} P \mid Q = \quad & \sum\{\alpha.(P'|Q) \mid \alpha.P' \text{ a summand of } P\} \\ & + \sum\{\alpha.(P|Q') \mid \alpha.Q' \text{ a summand of } Q\} \\ & + \sum\{\tau.(F \bullet C) \mid xF, \bar{x}C \text{ summands of } P, Q\} \end{aligned}$$

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Thm. For finite agents, $P \stackrel{\cdot}{\sim}_L Q$ iff $P = Q$.

Examples

Expansion law:

$$x(y).P \mid \bar{x}\langle z\rangle.Q = \bar{x}\langle z\rangle.(x(y).P \mid Q) + x(y).(P \mid \bar{x}\langle z\rangle.Q) \\ + \tau.(\{z/y\}P \mid Q)$$

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Congruence and Weak Bisimulation

In order to extend '$=$' to capture $\sim_L$

- replace the two congruence rule by '$=$' is preserved by all contexts.
- the expansion law must include terms

$$[x = y]\tau.(F \bullet C)$$

for xF and $\bar{y}C$ summands of P and Q.

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In order to capture $\approx_L$ add the three tau laws:

- $\alpha.\tau.P = \alpha.P$
- $P + \tau.P = \tau.P$
- $\alpha.(P + \tau.Q) = \alpha.(P + \tau.Q) + \alpha.Q$

Modal Logics

$$\varphi ::= \top \mid \varphi \wedge \varphi \mid \neg\varphi \mid x = y \mid \langle\alpha\rangle\varphi \mid (\sum x)\varphi \mid (\prod x)\varphi$$

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$A \models \varphi \wedge \psi$	iff	$A \models \varphi$ and $A \models \psi$
$A \models \neg\varphi$	iff	not $A \models \varphi$
$A \models x = y$	iff	x and y are the same name
$A \models \langle\alpha\rangle\varphi$	iff	exists $A \xrightarrow{\alpha} B$ and $B \models \varphi$
$A \models (\sum x)\varphi$	iff	$A \equiv \langle z\rangle.P$ or $A \equiv (\nu z)\langle z\rangle.P_{(z \notin \mathrm{fn}(\varphi)\setminus\{x\})}$ and $P \models \varphi\{z/x\}$
$A \models (\prod x)\varphi$	iff	$A \equiv (y).P$ and $\{z/y\}P \models \varphi\{z/x\}$ for all z

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Thm. $P \mathrel{\dot{\sim}_L} Q$ iff $P \models \varphi \iff Q \models \varphi$, for all φ.

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Let $P = \bar{x}\langle y \rangle$ and $\varphi = \langle \bar{x} \rangle (\sum z)(z = y)$, and let us prove that $P \models \varphi$.

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So, we can conclude that $P \not\sim_L Q$.

Conclusion

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The π calculus is . . .

Well, I'll leave it to you to complete the sentence as you wish.